BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(100), 2022, Pages 3–14
ISSN 1024–7696, E-ISSN 2587–4322

# Some integrals for groups of bounded linear operators on finite-dimensional non-Archimedean Banach spaces

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**Abstract.** In this paper, we extend the Volkenborn integral and Shnirelman integral for groups of bounded linear operators on finite-dimensional non-Archimedean Banach spaces over $\mathbb{Q}_p$ and $\mathbb{C}_p$ respectively. When the ground field is a complete non-Archimedean valued field, which is also algebraically closed, we give some functional calculus for groups of infinitesimal generator $A$ such that $A$ is a nilpotent operator on finite-dimensional non-Archimedean Banach spaces.

**Mathematics subject classification:** 47D03, 47S10, 46S10.
**Keywords and phrases:** Volkenborn integral, Shnirelman integral, groups of bounded linear operators, $p$-adic theory.

## 1 Introduction and Preliminaries

Throughout this paper, $\mathbb{K}$ is a non-Archimedean non trivially complete valued field with valuation $|\cdot|$, $X$ is a non-Archimedean Banach space over $\mathbb{K}$, $\mathbb{Q}_p$ is the field of $p$-adic numbers ($p \geq 2$ being a prime) equipped with $p$-adic valuation $|\cdot|_p$ and $\mathbb{Z}_p$ denotes the ring of $p$-adic integers (the ring of $p$-adic integers $\mathbb{Z}_p$ is the unit ball of $\mathbb{Q}_p$). We denote the completion of algebraic closure of $\mathbb{Q}_p$ under the $p$-adic valuation $|\cdot|_p$ by $\mathbb{C}_p$ and $B(X)$ denotes the set of all bounded linear operators on $X$.
The study of Archimedean $C_0$-semigroup or $C_0$-group of bounded linear operators was first attempted by Yosida and Hille [8]. From [8], Corollary 2.5, if $A$ is the infinitesimal generator of a $C_0$-semigroup then it is closed and $\overline{D(A)} = X$. By [8], $(b)$ of Theorem 2.4:

$$\text{For } x \in X,\ t \in \mathbb{R}^+,\ \int_0^t T(s)ds \in D(A),$$

and

$$\text{for } x \in X,\ T(t)x - T(s)x = \int_s^t T(u)Axdu = \int_s^t AT(u)xdu.$$

This is thanks to the Haar measure on the topological group $(\mathbb{R}, +)$.
In the non-Archimedean analysis, there is no Haar measure on a subset of $\mathbb{Q}_p$ into $\mathbb{Q}_p$, see Theorem 5. When $\mathbb{K} = \mathbb{C}_p$, it is useful to use the Shnirelman integral defined


DOI: https://doi.org/10.56415/basm.y2022.i3.p3

as: let $f(z)$ be a $\mathbb{C}_p$-valued function defined for all $z \in \mathbb{C}_p$ such that $|z-a|_p = r$ where $a \in \mathbb{C}_p$ and $r > 0$ with $r \in |\mathbb{C}_p|_p$. Let $\Gamma \in \mathbb{C}_p$ such that $|\Gamma|_p = r$. Then the Shnirelman integral of $f$ is defined as the following limit, if it exists,

$$\int_{a,\Gamma} f(z)dz = {\lim_{n\to\infty}}' \frac{1}{n} \sum_{\zeta^n=1} f(a+\zeta\Gamma), \tag{1}$$

where $\lim'$ indicates that the limit is taken over $n$ such that $gcd(n,p) = 1$. For more details, we refer to [2], [4] and [9]. But there is a different results in non-Archimedean analysis, by [2], Theorem 1, we have:

$$\int_{a,\Gamma} e^z dz = e^a,$$

and

$$\int_{a,\Gamma} (z-a)e^z dz = 0.$$

Recently, Diagana [3] introduced the notion of $C_0$-groups of bounded linear operators on a free non-Archimedean Banach space, for more details we refer to [3] and [5]. In [5], A. El Amrani, A. Blali, J. Ettayb and M. Babahmed introduced the notions of $C$-groups and cosine families of bounded linear operators on non-Archimedean Banach space. Let $r > 0$, $\Omega_r = \{t \in \mathbb{K} : |t| < r\}$ [5]. We have the following definition.

**Definition 1.** [5] Let $r > 0$ be a real number. A one-parameter family $(T(t))_{t\in\Omega_r}$ of bounded linear operators from $X$ into $X$ is a group of bounded linear operators on $X$ if

(i) $T(0) = I$, where $I$ is the unit operator of $X$.

(ii) For all $t, s \in \Omega_r$, $T(t+s) = T(t)T(s)$.

The group $(T(t))_{t\in\Omega_r}$ will be called of class $C_0$ or strongly continuous if the following condition holds:

- For each $x \in X$, $\lim\limits_{t\to 0} \|T(t)x - x\| = 0$.

A group of bounded linear operators $(T(t))_{t\in\Omega_r}$ is uniformly continuous if and only if $\lim\limits_{t\to 0} \|T(t) - I\| = 0$.
The linear operator $A$ defined by

$$D(A) = \{x \in X : \lim_{t\to 0} \frac{T(t)x - x}{t} \text{ exists}\},$$

and

$$Ax = \lim_{t\to 0} \frac{T(t)x - x}{t}, \text{ for each } x \in D(A),$$

is called the infinitesimal generator of the group $(T(t))_{t\in\Omega_r}$ .

In this paper, we extend to Volkenborn integral and Shnirelman integral for studying the $C_0$-groups of bounded linear operators on some non-Archimedean Banach spaces and we show some results about it. Now, we assume that $\mathbb{K}=\mathbb{C}_p$. We have the following definition.

**Definition 2.** [4] Let $f(z)$ be a $\mathbb{C}_p$-valued function defined for all $z\in\mathbb{C}_p$ such that $|z-a|_p=r$ where $a\in\mathbb{C}_p$ and $r>0$ with $r\in|\mathbb{C}_p|_p$. Let $\Gamma\in\mathbb{C}_p$ such that $|\Gamma|_p=r$. Then the Shnirelman integral of $f$ is defined as the following limit, if it exists,

$$\int_{a,\Gamma} f(z)dz=\lim_{n\to\infty}{}' \frac{1}{n}\sum_{\zeta^n=1} f(a+\zeta\Gamma),$$

where $\lim'$ indicates that the limit is taken over $n$ such that $gcd(n,p)=1$.

**Theorem 1.** [1] *Let $f(z)=\sum_{n\in\mathbb{N}} a_nf_n(z)$ where the series on the right converges uniformly to $f(z)$ for all points $z\in\mathbb{C}_p$ such that $|z-a|_p=|\gamma|_p$. Suppose that for all $n\in\mathbb{N}$, $\int_{a,\gamma} f_n(z)dz$ exists.*
*Then $\int_{a,\gamma} f(z)dz$ exists and $\int_{a,\gamma} f(z)dz=\sum_{n\in\mathbb{N}} a_n\int_{a,\gamma} f_n(z)dz$.*

**Lemma 1.** [1] *Let $p$ be any integer such that $0<|p|<n$. Then*

$$\sum_{i=1}^{n}\xi_i^{(n)p}=0.$$

Now, let $f(z)=a_0+a_1z+a_2z^2+\cdots$ be a power series converging for all $z\in\mathbb{C}_p$ such that $|z|_p<R\ \Big(R>0\Big)$, we have the following:

**Theorem 2.** [1] *If $|a|_p<R$ and $|\gamma|_p<R$, then*

$$\int_{a,\gamma} f(z)dz=f(a).$$

**Corollary 1.** [1] *With the same hypothesis as in Theorem 2, we have:*

$$\int_{a,\gamma}(z-a)f(z)dz=0.$$

**Theorem 3.** [1] *Let $f(z)=\sum_{k=0}^{\infty} a_kz^k$ be a power series converging for all $z\in\mathbb{C}_p$ such that $|z|_p<R$ $(R>0)$. Suppose that $x,r\in\mathbb{C}_p$ such that $|x|_p,|r|_p<R$. Then,*

$$\int_{0,r}\frac{zf(z)}{z-x}dz=\begin{cases} f(x) & if\ |x|_p<|r|_p,\\ 0 & if\ |x|_p>|r|_p.\end{cases}$$

**Theorem 4.** [1] *With the same hypothesis as in Theorem 3, we have:*

$$\int_{0,r} \frac{zf(z)}{(z-x)^{n+1}}dz = \frac{f^n(x)}{n!} \text{ for } |x|_p < |r|_p.$$

In the next, we assume that $\mathbb{K} = \mathbb{Q}_p$. There is no Newton-Leibniz formula in the $p$-adic analysis. There is no $\mathbb{Q}_p$-valued Lebesgue measure $\int_{\mathbb{Q}_p} f(t)dt$ is not well defined as usual.

**Theorem 5.** *[7] Additive, translation invariant and bounded $\mathbb{Q}_p$-valued measure on clopens of $\mathbb{Z}_p$ is the zero measure.*

We denote by $C(\mathbb{Z}_p, \mathbb{Q}_p)$ the space of all functions defined and continuous from $\mathbb{Z}_p$ into $\mathbb{Q}_p$.

**Theorem 6.** *[7] Let $f \in C(\mathbb{Z}_p, \mathbb{Q}_p)$. The function defined on $\mathbb{N}$ by*

$$F(0) = 0, \ F(n) = f(0) + f(1) + \cdots + f(n-1)$$

*is uniformly continuous. The extended function is denoted by $Sf(x)$ (called indenite sum of $f$). If $f$ is strictly differentiable, so is $Sf$.*

We denote by $C_s^1(\mathbb{Z}_p, \mathbb{Q}_p)$ the space of all functions defined and strictly differentiable in $\mathbb{Z}_p$ taking values in $\mathbb{Q}_p$. For more details, we refer to [7].

**Definition 3.** [7] The Volkenborn integral of $f \in C_s^1(\mathbb{Z}_p, \mathbb{Q}_p)$ is defined by

$$\int_{\mathbb{Z}_p} f(t)dt = \lim_{n\to\infty} p^{-n} \sum_{j=0}^{p^n-1} f(j) = \lim_{n\to\infty} \frac{Sf(p^n) - Sf(0)}{p^n} = (Sf)'(0).$$

**Lemma 2.** *[7] For all $t \in \Omega^*_{p^{\frac{-1}{p-1}}}$,*

$$\int_{\mathbb{Z}_p} e^{tu}du = \frac{t}{e^t - 1}.$$

## 2 Integral for $C_0$-groups on finite-dimentional Banach space over $\mathbb{C}_p$

In this section, let $\mathbb{K} = \mathbb{C}_p$ and let $\Omega_r$ be the open ball of $\mathbb{K}$ centered at 0 with radius $r > 0$. We always assume that $r$ is suitably chosen such that $t \in \Omega_r \mapsto T(t)$ is well-defined, we have the following definition.

**Definition 4.** Let $r > 0$ be a real number. A one-parameter family $(T(t))_{t\in\Omega_r}$ of bounded linear operators from $\mathbb{C}_p^n$ into $\mathbb{C}_p^n$ is said to be analytic group of bounded linear operators on $\mathbb{C}_p^n$ if

(i) $T(0) = I$, where $I$ is the unit operator of $\mathbb{C}_p^n$.

(ii) For all $t, s \in \Omega_r$, $T(t+s) = T(t)T(s)$.

(iii) For all $x \in X$, $t \to T(t)x$ is analytic on $\Omega_r$.

We extend the following definition.

**Definition 5.** Let $(T(t))_{t\in\Omega_r}$ be analytic group of bounded linear operators on $\mathbb{C}_p^n$. The group $(T(t))_{t\in\Omega_r}$ is said to be integrable in the sense of Schnirelman if for all $a \in \Omega_r$ and $\gamma \in \Omega_r\backslash\{0\}$, the sequence $(S_n)_n \subset B(\mathbb{C}_p^n)$ defined by

$$S_n = \sum_{\zeta^n=1} T(a+\zeta\gamma),$$

converges strongly as $n \to \infty$ $\Big($the limit is taken over $n$ such that $gcd(n,p) = 1\Big)$ to a bounded linear operator. More precisely

$$\int_{a,\gamma} T(t)dt = \lim_{n\to\infty}{}' \frac{1}{n} \sum_{\zeta^n=1} T(a+\zeta\gamma),$$

where $\lim'$ indicates that the limit is taken over $n$ such that $gcd(n,p) = 1$.

**Lemma 3.** *Let $(T(t))_{t\in\Omega_r}$ be analytic group on $\mathbb{C}_p^n$ such that $\displaystyle\int_{a,\gamma} T(t)dt$ exists and $\sup\limits_{t\in\Omega_r}\|T(t)\|\leq M$ where $a \in \Omega_r$ and $\gamma \in \Omega_r\backslash\{0\}$. Then*

*(i) For all $x \in \mathbb{C}_p^n$, $\displaystyle\|\int_{a,\gamma} T(t)xdt\| \leq M\|x\|$.*

*(ii) For all $a \in \Omega_r$, $x \in \mathbb{C}_p^n$, $\displaystyle\int_{a,\gamma} T(t)xdt = T(a)\int_{0,\gamma} T(t)xdt$.*

*Proof.* Let $(T(t))_{t\in\Omega_r}$ be analytic group on $\mathbb{C}_p^n$ such that $\int_{a,\gamma} T(t)dt$ exists, then

(i) It suffices to apply Definition 5.

(ii) By Definition 5, for all $x \in \mathbb{C}_p^n$ and for each $a \in \Omega_r$, then

$$\begin{aligned}
\int_{a,\gamma} T(t)xdt &= \lim_{n\to\infty}{}' \frac{1}{n} \sum_{\zeta^n=1} T(a+\zeta\gamma)x \\
&= T(a) \lim_{n\to\infty}{}' \frac{1}{n} \sum_{\zeta^n=1} T(\zeta\gamma)x \\
&= T(a)\int_{0,\gamma} T(t)xdt.
\end{aligned}$$

□

**Definition 6.** [6] Let $A \in B(\mathbb{C}_p^n)$. $A$ is said to be nilpotent of index $d$, if there is an integer number $d \leq n$ such that $A^n = 0_{\mathbb{C}_p^n}$ and $A^{n-1} \neq 0_{\mathbb{C}_p^n}$ $\Big($where $0_{\mathbb{C}_p^n}$ denotes the null operator from $\mathbb{C}_p^n$ into $\mathbb{C}_p^n\Big)$.

**Example 1.** Let $A \in B(\mathbb{C}_p^4)$ be defined by

$$\begin{pmatrix} 0 & a & b & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ where } a,\ b,\ c \in \mathbb{C}_p.$$

It is easy to see that $A$ is nilpotent of index 4.

**Proposition 1.** *Let $A$ be a nilpotent operator of index $n$ on $\mathbb{C}_p^n$ such that $\|A\| < p^{\frac{-1}{p-1}}$. Then $e^{tA} = \sum_{k=0}^{n-1} \frac{t^k A^k}{k!}$.*

*Proof.* Since $A$ is nilpotent operator of index $n$ on $\mathbb{C}_p^n$. Then,

$$\begin{aligned} e^{tA} &= \sum_{k\in\mathbb{N}} \frac{t^k A^k}{k!} \\ &= \sum_{k=0}^{n-1} \frac{t^k A^k}{k!}. \end{aligned}$$

□

**Theorem 7.** *Let $e^{tA}$ be a $C_0$-group of infinitesimal generator $A$ on $\mathbb{C}_p^n$ such that $A$ is nilpotent operator of index $n$ on $\mathbb{C}_p^n$. Then for all $x \in \mathbb{C}_p^n$, $\int_{a,\gamma} e^{tA} x dt = e^{aA} x$.*

*Proof.* Let $e^{tA} = \sum_{k=0}^{n-1} \frac{t^k A^k}{k!}$. Using Proposition 1 and Theorem 2, we have for all $x \in \mathbb{C}_p^n$,

$$\begin{aligned} \int_{a,\gamma} e^{tA} x dt &= \sum_{k=0}^{n-1} \frac{A^k}{k!} \int_{a,\gamma} t^k x dt \\ &= \sum_{k=0}^{n-1} \frac{a^k A^k}{k!} x = e^{aA} x. \end{aligned}$$

□

**Corollary 2.** *Under the hypothesis of Theorem 7, for all $x \in \mathbb{C}_p^n$,*

$$\int_{a,\gamma} (t-a)e^{tA}x dt = 0.$$

*Remark* 1. Let $A \in B(\mathbb{C}_p^n)$ be a nilpotent operator, then $e^{tA}$ is integrable in the sense of Shnirelman.

Set for all $\lambda \in \rho(A)$, $R(\lambda, A) = (\lambda I - A)^{-1}$ where $\rho(A)$ is the resolvent set of the linear operator $A$ defined on $X$, we have the following:

**Proposition 2.** *Let $A \in B(\mathbb{C}_p^n)$. If $A$ is a nilpotent operator of index $n$, then for all $\lambda \in \mathbb{C}_p^*$, $R(\lambda, A)$ exists. Furthermore, for each $\lambda \in \mathbb{C}_p^*$, we have*

$$R(\lambda, A) = \sum_{k=0}^{n-1} \frac{A^k}{\lambda^{k+1}}.$$

*Proof.* Let $\lambda \in \mathbb{C}_p^*$, then

$$\begin{aligned}(\lambda I - A)\Big(\sum_{k=0}^{n-1} \frac{A^k}{\lambda^{k+1}}\Big) &= \sum_{k=0}^{n-1} \frac{A^k}{\lambda^k} - \sum_{k=0}^{n-1} \frac{A^{k+1}}{\lambda^{k+1}} \\ &= I.\end{aligned}$$

On the other hand,

$$\begin{aligned}\Big(\sum_{k=0}^{n-1} \frac{A^k}{\lambda^{k+1}}\Big)(\lambda I - A) &= \sum_{k=0}^{n-1} \frac{A^k}{\lambda^k} - \sum_{k=0}^{n-1} \frac{A^{k+1}}{\lambda^{k+1}} \\ &= I.\end{aligned}$$

Consequently, for all $\lambda \in \mathbb{C}_p^*$, $R(\lambda, A) = \sum_{k=0}^{n-1} \frac{A^k}{\lambda^{k+1}}$. □

**Proposition 3.** *Let $A$ be a nilpotent operator of index $n$ on $\mathbb{C}_p^n$ and $r = \frac{-1}{p-1}$. Then*

$$\textit{for all } t \in \Omega_r,\ e^{tA} = \int_{0,\gamma} \lambda e^{\lambda t} R(\lambda, A) d\lambda, \textit{ where } \gamma \in \Omega_r \backslash \{0\}.$$

*Proof.* By Proposition 2, for all $\lambda \in \Omega_{\frac{-1}{p-1}} \backslash \{0\}$, $R(\lambda, A)$ has a polynomial function form on $\mathbb{C}_p^n$, hence it is analytic on $\Omega_{\frac{-1}{p-1}} \backslash \{0\}$. Using Theorem 4, we obtain

$$\begin{aligned}\int_{0,\gamma} \lambda e^{\lambda t} R(\lambda, A) &= \int_{0,\gamma} \sum_{k=0}^{n-1} \lambda^{-k} e^{t\lambda} A^k d\lambda \\ &= \sum_{k=0}^{n-1} A^k \int_{0,\gamma} \lambda^{-k} e^{t\lambda} d\lambda\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{n-1} A^k \frac{\left(e^{t\lambda}\right)^{(k)}(0)}{k!} \\
&= \sum_{k=0}^{n-1} A^k \frac{t^k}{k!} = e^{tA}.
\end{aligned}
$$

□

We have the following proposition.

**Proposition 4.** *Let $A$ and $B$ be nilpotent operators on $\mathbb{C}_p^n$ and let $e^{tA}$ and $e^{tB}$ be two $C_0$-groups of infinitesimal generators $A$ and $B$ respectively. If $R(\lambda, A)$ and $R(\lambda, B)$ commute, then $e^{tA}$ and $e^{tB}$ commute.*

*Proof.* By Proposition 3, we have

$$e^{tA} = \int_{0,\gamma} \lambda e^{\lambda t}(\lambda I - A)^{-1} d\lambda \text{ and } e^{tB} = \int_{0,\gamma} \lambda e^{\lambda t}(\lambda I - B)^{-1} d\lambda.$$

Asumme that $R(\lambda, A)$ and $R(\lambda, B)$ commute, then

$$
\begin{aligned}
e^{tA}e^{tB} &= \int_{0,\gamma} \lambda e^{\lambda t}(\lambda I - A)^{-1} d\lambda \int_{0,\gamma} \lambda e^{\lambda t}(\lambda I - B)^{-1} d\lambda \\
&= \int_{0,\gamma}\int_{0,\gamma} \lambda e^{\lambda t}(\lambda I - A)^{-1} \lambda e^{\lambda t}(\lambda I - B)^{-1} d\lambda d\lambda \\
&= \int_{0,\gamma}\int_{0,\gamma} \lambda e^{\lambda t}(\lambda I - B)^{-1} \lambda e^{\lambda t}(\lambda I - A)^{-1} d\lambda d\lambda \\
&= e^{tB}e^{tA}.
\end{aligned}
$$

□

We have the following:

**Proposition 5.** *Let $A$ and $(A_k)_{k\in\mathbb{N}}$ be nilpotent operators on $\mathbb{C}_p^n$. If, $R(\lambda, A_k) \to R(\lambda, A)$ as $k \to \infty$, then $e^{tA_k}$ converges to $e^{tA}$ as $k \to \infty$.*

*Proof.* From Proposition 3, we have

$$\text{for all } t \in \Omega_r,\ e^{tA} = \int_{0,\gamma} \lambda e^{\lambda t} R(\lambda, A) d\lambda,$$

where $\gamma \in \Omega_r \backslash \{0\}$ and $r = \frac{-1}{p-1}$ and

$$\text{for all } t \in \Omega_r,\ k \in \mathbb{N},\ \ e^{tA_k} = \int_{0,\gamma} \lambda e^{\lambda t} R(\lambda, A_k) d\lambda.$$

Moreover,

$$e^{tA_k} - e^{tA} = \int_{0,\gamma} \lambda e^{t\lambda}[R(\lambda, A_k) - R(\lambda, A)]d\lambda$$

is well-defined. Set

$$M = \max_{|\lambda|_p = |\gamma|_p} |\lambda e^{t\lambda}|_p < \infty.$$

Since $R(\lambda, A_k) \to R(\lambda, A)$ as $k \to \infty$, it follows that for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $k \geq N$, $\|R(\lambda, A_k) - R(\lambda, A)\| \leq \frac{\varepsilon}{M}$. Consequently

$$\begin{aligned} \|e^{tA_k} - e^{tA}\| &\leq \left\| \int_{0,\gamma} \lambda e^{t\lambda}[R(\lambda, A_k) - R(\lambda, A)]d\lambda \right\| \\ &\leq \max_{|\lambda|_p = |\gamma|_p} |\lambda e^{t\lambda}|_p \|R(\lambda, A_k) - R(\lambda, A)\| \\ &\leq M \cdot \frac{\varepsilon}{M} \\ &= \varepsilon, \end{aligned}$$

whenever $k \geq N$, then $e^{tA_k}$ converges to $e^{tA}$ as $k \to \infty$. □

## 3 Integral of groups of linear operators on $\mathbb{Q}_p^n$

From now on we assume that $\mathbb{K} = \mathbb{Q}_p$, we extend the Volkenborn integral to some non-Archimedean Banach spaces.

**Definition 7.** Let $f \in C_s^1(\mathbb{Z}_p, \mathbb{Q}_p^n)$. The sequence $(S_m)_m \subset B(\mathbb{Q}_p^n)$ defined by

$$S_m = p^{-m} \sum_{j=0}^{p^m - 1} f(j),$$

converges strongly as $m \to \infty$ to a bounded linear operator. More precisely

$$\int_{\mathbb{Z}_p} f(t)dt = \lim_{m \to \infty} p^{-m} \sum_{j=0}^{p^m - 1} f(j).$$

Set $B_r(\mathbb{Q}_p^n) = \{A \in B(\mathbb{Q}_p^n) : 0 < \|A\| < r\}$ where $r = p^{\frac{-1}{p-1}}$.

**Proposition 6.** *Let $A \in B_r(\mathbb{Q}_p^n)$ be invertible diagonal operator, then $(e^{tA})_{t \in \mathbb{Z}_p}$ is $C^1$ function and $(e^A - I)^{-1} \in B(\mathbb{Q}_p^n)$.*

*Proof.* Let $A \in B_r(\mathbb{Q}_p^n)$ be invertible diagonal operator, then

$$\text{for all } i \in \{1, \cdots, n\},\ Ae_i = a_i e_i,$$

where $a_i \in \mathbb{Q}_p^*$ such that $|a_i|_p < r$ and $(e_i)_{1 \leq i \leq n}$ is the canonical basis of $\mathbb{Q}_p^n$. Hence, for all $t \in \Omega_r$, $e^{tA}$ exists and is given by

$$\text{for all } i \in \{1, \cdots, n\},\ e^{tA}e_i = e^{ta_i}e_i.$$

Hence $e^{tA}$ is $C^{\infty}$ that is $C^1$. Moreover,

$$\text{for all } i \in \{1, \cdots, n\},\ \Big(e^A - I\Big)e_i = \Big(e^{a_i} - 1\Big)e_i.$$

We have for all $i \in \{1, \cdots, n\},\ 1 - e^{a_i} \neq 0$. Consequently, $\det(e^A - I) \neq 0$, then $e^A - I$ is invertible. Moreover,

$$\text{for all } i \in \{1, \cdots, n\},\ \Big(e^A - I\Big)^{-1} e_i = \Big(\tfrac{1}{e^{a_i}-1}\Big)e_i.$$

Hence $\|(e^A - I)^{-1}\| = \sup_{1\leq i\leq n} \left|\dfrac{1}{e^{a_i} - 1}\right|_p = \dfrac{1}{\inf\limits_{1\leq i\leq n} |e^{a_i} - 1|_p} < \infty$. Consequently, $(e^A - I)^{-1} \in B(\mathbb{Q}_p^n)$. □

**Proposition 7.** *Let $A \in B_r(\mathbb{Q}_p^n)$ be invertible diagonal operator such that $\displaystyle\int_{\mathbb{Z}_p} e^{tA}dt$ exists. Then for all $x \in \mathbb{Q}_p^n$, $(e^A - I)\displaystyle\int_{\mathbb{Z}_p} e^{tA}xdt = Ax$.*

*Proof.* Let $A \in B_r(\mathbb{Q}_p^n)$ be invertible diagonal operator. By Proposition 6, the $C_0$-group $(e^{tA})_{t\in\mathbb{Z}_p}$ is locally analytic function and $(e^A - I)^{-1} \in B(\mathbb{Q}_p^n)$. Let $x \in \mathbb{Q}_p^n$, set $S_m x = p^{-m} \displaystyle\sum_{j=0}^{p^m-1} e^{jA}x$. Hence for all $x \in \mathbb{Q}_p^n$, we have

$$\begin{aligned}(e^A - I)S_m x &= S_m(e^A - I)x \\ &= \frac{e^{p^m A}x - x}{p^m}.\end{aligned}$$

By assumption, for all $x \in \mathbb{Q}_p^n$, we have

$$\int_{\mathbb{Z}_p} e^{tA}xdt = \lim_{m\to\infty} S_m x.$$

Then, for all $x \in \mathbb{Q}_p^n$, we have

$$\begin{aligned}(e^A - I)\int_{\mathbb{Z}_p} e^{tA}xdt &= (e^A - I)\lim_{m\to\infty} S_m x \\ &= \lim_{m\to\infty} \frac{e^{p^m A}x - x}{p^m} \\ &= Ax.\end{aligned}$$

□

**Example 2.** Let $r = \frac{-1}{p-1}$ and let $A \in B(\mathbb{Q}_p^2)$ defined by

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \text{ where } a,\ b \in \Omega_r^*.$$

Then, for all $t \in \mathbb{Z}_p$, we have

$$e^{tA} = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{pmatrix}.$$

Hence,

$$\int_{\mathbb{Z}_p} e^{tA} dt = \begin{pmatrix} \displaystyle\int_{\mathbb{Z}_p} e^{at} dt & 0 \\ 0 & \displaystyle\int_{\mathbb{Z}_p} e^{bt} dt \end{pmatrix}.$$

Thus, for all $x = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{Q}_p^2$, we have

$$\begin{aligned} \int_{\mathbb{Z}_p} e^{tA} x dt &= \begin{pmatrix} \frac{a}{e^a - 1} & 0 \\ 0 & \frac{b}{e^b - 1} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\ &= \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{e^a - 1} & 0 \\ 0 & \frac{1}{e^b - 1} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\ &= (e^A - I)^{-1} A x. \end{aligned}$$

**Definition 8.** Let $A \in B(\mathbb{Q}_p^n)$. $A$ is said to be scalar multiple of identity operator on $\mathbb{Q}_p^n$, if $A = aI$ for some $a \in \mathbb{Q}_p$ and $I$ is the identity operator on $\mathbb{Q}_p^n$.

**Example 3.** Let $A$ be invertible scalar multiple of identity operator on $\mathbb{Q}_p^n$ such that $A = aI$, where $a \in \Omega_r^*$ with $r = \frac{-1}{p-1}$. Hence for all $t \in \mathbb{Z}_p$, $T(t) = e^{ta} I$, then for all $x \in \mathbb{Q}_p^n$ and $a \in \Omega_r^*$, we have

$$\int_{\mathbb{Z}_p} T(tu) x du = \frac{a}{e^a - 1} x = (T(1) - I)^{-1} A x. \tag{2}$$

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*Received April 06, 2022*

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(100), 2022, Pages 15–21
ISSN 1024–7696, E-ISSN 2587–4322

# A fixed point theorem for $p$-contraction mappings in partially ordered metric spaces and application to ordinary differential equations

Ahmed Chaouki Aouine

**Abstract.** In this paper, we prove a fixed point theorem for $p$-contraction mappings in partially ordered metric spaces. As an application, we investigate the possibility of optimally controlling the solution of the ordinary differential equations.

**Mathematics subject classification:** 47H10, 54H25.
**Keywords and phrases:** Fixed point, p-contraction type maps, partially ordered metric spaces, ordinary differential equation.

## 1 Introduction and Preliminaries

The applications of fixed point theorems are very important in diverse disciplines of mathematics, statistics, engineering and economics in dealing with problems arising in: approximation theory, potential theory, game theory, mathematical economics, theory of differential equations, theory of integral equations, etc.

In this paper, we prove a fixed point theorem for $p$-contraction mappings in partially ordered metric spaces and we apply this theorem to ordinary differential equation. For this aim we need the following definitions.
First of all, we define the fixed point of mapping $A$.

**Definition 1.** [2] Let $A, S : X \to X$ be two mappings. A point $u \in X$ is said to be
i) a fixed point of $A$ if $Au = u$,
ii) a coincidence point of $A$ and $S$ if $Au = Su$. The point $z = Au = Su$ is called a point of coincidence of $A$ and $S$.
iii) a common fixed point of $A$ and $S$ if $Au = Su = u$.
iv) $A$ and $S$ are weakly compatible iff they commute at their coincidence point.

Also, we must mention the famous Banach contraction.

**Definition 2.** [4] A mapping $T : X \longrightarrow X$ is said to be a Banach contraction mapping if it satisfies the following inequality:

$$d(T(x), T(y)) \leq \lambda d(x, y),$$

for all $x, y \in X$, where $0 < \lambda < 1$. It is well known that a Banach contraction mapping $T$ on a complete metric space X has a unique fixed point.
Let $X$ be a topological space and $Y \subset X$ be equipped with relativized topology.


DOI: https://doi.org/10.56415/basm.y2022.i3.p15

**Definition 3.** A mapping $T : Y \subset X \longrightarrow X$ is said to be a weak topological contraction if $Y$ is $T$-invariant and T is continuous and closed such that for each non-empty closed subset A of Y with $T(A) = A$, A is a singleton set. Further, if the diameter $\delta(T^n(Y)) \rightarrow 0$ as $n \rightarrow \infty$ then the mapping $T$ is said to be a strong topological contraction.

*Remark* 1. If $X$ is a bounded metric space (i.e., $\delta(X)$, the diameter of $X$, is finite) and $T$ is a Banach contraction, then clearly $T$ is a weak topological contraction. In 2008, H. K. Pathak and N. Shahzad introduced in the following definition the notion of $p$-contraction which is more general than the Banach contraction principle.

**Definition 4.** [8] Let $(X, d)$ be a metric space. A mapping $T : Y \subset X \longrightarrow X$ is said to be a metric $p$-contraction (or simply $p$-contraction) mapping if $Y$ is $T$-invariant and it satisfies the following inequality:

$$d(T(x), T^2(x)) \leq p(x)d(x, T(x)) \tag{1}$$

for all $x$ in $Y$, where $p : Y \longrightarrow [0, 1]$ is a function such that $p(x) < 1$ for all $x \in Y$ and $\sup_{x \in Y} p(Tx) = \alpha < 1$. Further, if $\cap_{n=0}^{\infty} T^n(Y)$ is a singleton set, where $T^n(Y) = T(T^{n-1}(Y))$ for each $n \in \mathbb{N}$ and $T^0(Y) = Y$, then $T$ is said to be a strong $p$-contraction.

*Remark* 2. 1) If $p(x) \leq 1$ for all $x \in Y$, then the $p$-contraction mapping is said to be a fundamental contraction which is also known as a Banach operator.

2) If $Y = X$ and $y = T(x)$, then a Banach contraction mapping is a fundamental contraction.

3) If $p(x) \leq 1$ for all $x \in Y$ and $\sup_{x \in Y} p(Tx) = 1$, then the $p$-contraction mapping is said to be fundamentally $p$-non-expansive. In particular when $p(x) = 1$ for all $x \in Y$, then the fundamentally $p$-non-expansive mapping is said to be fundamentally non-expansive.

4) If $\sup_{x \in Y} p(x) < 1$ (*or* $\leq 1$), then $\sup_{x \in Y} p(Tx) < 1$ (or $\leq 1$) since $T(Y) \subset Y$.

*Remark* 3. The concept of $p$-contraction is more general than the Banach contraction principle, see [8], example 2.1.

*Remark* 4. A $p$-contraction mapping is not continuous in general, see [8], example 2.2.

In 1976, Caristi [3] proved the following theorem.

**Theorem 1.** *Let $(X, d)$ be complete and $\varphi : X \longrightarrow \mathbb{R}$ a lower semi-continuous function with a finite lower bound. Let $T : X \longrightarrow X$ be any (not necessarily continuous) function such that*

$$d(y, Ty)(y) \leq \varphi(y) - \varphi(Ty)$$

*for each $y \in X$. Then $T$ has a fixed point.*

*Remark* 5. In general, a $p$-contraction does not satisfy Caristi's condition but every fundamental contraction does.

**Definition 5.** [6] A metric space $(X, d)$ is said to be $T$-orbitally complete if $T$ is a self-mapping of $X$ and if any Cauchy subsequence $\{T^{n_i}x\}$ in orbit $O(x, T)$, $x \in X$, converges in $X$.

**Definition 6.** [6] An operator $T : X \longrightarrow X$ on $X$ is said to be orbitally continuous if $T^{n_i}x \longrightarrow u$, then $T(T^{n_i}x) \longrightarrow Tu$ as $i \longrightarrow \infty$.

**Definition 7.** [6] An operator $T : X \longrightarrow X$ on $X$ is said to be weakly orbitally continuous if $T^{n_i}x \longrightarrow u$, then $d(T^{n_i}x, T(T^{n_i}x)) \longrightarrow d(u, Tu)$ as $i \longrightarrow \infty$.

*Remark* 6. It is clear that a complete metric space is orbitally complete with respect to any self-mapping of $X$ and that a continuous mapping is always orbitally continuous and an orbitally continuous mapping is always weakly orbitally continuous, but the converse implications are not true in general, see [8], example 2.3.

The aim of this paper is to prove a fixed point result for self-mapping which satisfies $p$-contraction condition in partially ordered metric spaces. For this purpose we need the following definitions.

**Definition 8.** [1] Let $X$ be a nonempty set. Then $(X, d, \preceq)$ is called an ordered metric space iff:
(i) $(X, d)$ is a metric space; (ii) $(X, \preceq)$ is partially ordered set.

**Definition 9.** [6] Let $(X, \preceq)$ be a partially ordered set. $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.

## 2 Main Results

**Theorem 2.** *Let $(X, \leq)$ be a partially ordered set. Suppose that there exists a metric $d$ in $X$ such that $(X, d)$ is $T$-orbitally complete, $T$ is a non-decreasing mapping such that there exists $x_0 \in X$ with $x_0 \leq T(x_0)$ and $T$ is a strongly fundamental contraction mapping with $T(x) \leq x$. Assume that either $T$ is orbitally continuous or $X$ is such that*

$$\text{if a sequence } x_n \to x \text{ in } X \text{ is non-decreasing, then } x_n \leq x. \tag{2}$$

*Then $T$ has a fixed point. Moreover, if*

$$\text{for each } x \in X, \text{ there exists } z \in X \text{ which is comparable to } x \text{ and } T(x), \tag{3}$$

*therefore, the fixed point is unique.*

*Proof.* First, we show that $T$ has a fixed point. Let $x_0$ be an arbitrary point of $X$. We construct an iterative sequence $\{x_n\}$ defined by $x_{n+1} = T(x_n) = T^n(x_0)$. Since $x_0 \leq T(x_0)$ and $T$ is a nondecreasing function, we have by induction

$$x_0 \leq T(x_0) \leq T^2(x_0) \leq T^3(x_0) \leq ... \leq T^n(x_0) \leq T^{n+1}(x_0) \leq ....$$

As $x_n \leq x_{n+1}$ for each $n \in \mathbb{N}$, applying (1) we get

$$d(x_1, x_2) \leq p(x_0)d(x_0, x_1),$$

$$\begin{aligned} d(x_2, x_3) &\leq p(x_1)d(x_1, x_2) \\ &\leq p(x_0)p(x_1)d(x_0, x_1). \end{aligned}$$

By induction we obtain

$$d(x_n, x_{n+1}) \leq \prod_{i=1}^{n} p_i d(x_0, x_1), \tag{4}$$

where $p_i = p(x_{i-1}) = p(T^{i-1}(x_0)),\ i \in \mathbb{N}$. Since $\max\{p(x_0), \sup p(Tx)\} \leq \lambda < 1$, using (4) we have

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1), \forall n \in \mathbb{N}.$$

For $m > n$, $m, n \in \mathbb{N}$, we get

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq [\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}]d(x_0, x_1) \\ &= \frac{\lambda^n(1-\lambda^{m-n})}{1-\lambda} d(x_0, x_1) \\ &< \frac{\lambda^n}{1-\lambda} d(x_0, x_1). \end{aligned}$$

Therefore, the sequence $\{x_n\}$ is a Cauchy sequence in $X$. Since $X$ is $T$-orbitally complete, it follows that there exists a Cauchy subsequence $\{T^{n_i}(x_0)\}$ of $\{x_n\}$ in the orbit $O(x, Tx)$, $x \in X$, which converges to a point $z \in X$. Suppose that $T$ is orbitally continuous. Then

$$\begin{aligned} z &= \lim_{n\to\infty} x_{n_i} = \lim_{n\to\infty} T^{n_i}(x_0) \\ &= \lim_{n\to\infty} T^{n_i+1}(x_0) = \lim_{n\to\infty} T(T^{n_i}(x_0)) \\ &= T(z), \end{aligned}$$

which shows that $z$ is a fixed point of $T$. Hence $T(z) = z$. If case (2) holds, then

$$\begin{aligned} d(T(z), z) &\leq d(T(z), T(x_n)) + d(T(x_n), z) \\ &\leq p(x_n)d(z, x_n) + d(x_{n+1}, z) \\ &\leq d(z, x_n) + d(x_{n+1}, z). \end{aligned}$$

Since $d(z, x_n) \to 0$ then we obtain $T(z) = z$. To prove the uniqueness of the fixed point, let $w$ be another fixed point of $T$. From (3) there exists $x \in X$ which is comparable to $w$ and $z$. Monotonicity implies that $T^n(x)$ is comparable to $T^n(w) = w$ and $T^n(z) = z$ for $n = 1, 2, ....$ Then

$$\begin{aligned} d(z, T^n(x)) &= d(T^n(z), T^n(x)) \\ &\leq p(T^{n-1}(z))d(T^{n-1}(z), T^{n-1}(x)). \end{aligned} \tag{5}$$

Therefore

$$d(z, T^n(x)) \leq d(z, T^{n-1}(x)).$$

Consequently, the sequence $\{\gamma_n\}$ defined by $\gamma_n = d(z, T^n(x))$ is nonnegative and nonincreasing and so $\lim_{n\to\infty} d(z, T^n(x)) = \gamma \geq 0$. Now, we show that $\gamma = 0$. On the contrary, assume that $\gamma > 0$. By passing to the limit in (5), we get

$$\gamma \leq \sup_{x\in X} p(Tx)\gamma < \gamma,$$

which is a contradiction and so $\gamma = 0$. Simillarly, it can be proved that $\lim_{n\to\infty} d(w, T^n(x)) = 0$. Finally,

$$d(z, w) \leq d(z, T^n(x)) + d(T^n(x), w),$$

and taking the limit as $n \to \infty$, we obtain $z = w$. □

## 3 Application to ordinary differential equations

Inspired by the papers of Pathak and Shahzad [8] and Aouine and Aliouche [3], we investigate the possibility of optimally controlling the solution of the ordinary differential equation (6) via dynamic programming.

Let $A$ be a compact subset of $\mathbb{R}^m$ and for each given $a \in A$, $F_a : \mathbb{R}^n \to \mathbb{R}^n$ be a strong $p$-contraction mapping such that $F_a(x) = f(x, a)$, $\forall x \in \mathbb{R}^n$, where $f : \mathbb{R}^n \times A \to \mathbb{R}^n$ is a given bounded function which satisfies the following generalized contractive condition:

$$|f(x, a) - f(y, a)| \leq q(|x - y|)|x - y|,\ x, y \in \mathbb{R}^n, a \in A, \tag{*}$$

where $q : \mathbb{R}_+ \to [0, 1]$ is a function with $\sup_{t\geq 0} q(t) \leq \lambda < 1$. We will now study the possibility of optimally controlling the solution $x(\cdot)$ of the ordinary differential equation

$$\begin{cases} \dot{x}(s) = f(x(s), \alpha(s)) \quad (t < s < T) \\ x(t) = x_0. \end{cases} \tag{6}$$

Here $T > 0$ is a fixed terminal time, and $x \in \mathbb{R}^n$ is a given initial point, taken on by our solution $x(\cdot)$ at the starting time $t \geq 0$. At later times $t < s < T$, $x(\cdot)$ evolves according to the ODE (6). The function $\alpha(\cdot)$ appearing in (6) is a control, that is, some appropriate scheme for adjusting parameters from the set $A$ as time evolves, there by affecting the dynamics of the system modelled by (6). Let us write

$$A = \{\alpha : [0, T] \to A | \alpha(\cdot) \text{ is measurable}\}, \tag{7}$$

to denote the set of admissible controls. Then since

$$|f(x, a)| \leq C,\ |f(x, a) - f(y, a)| \leq q(|x - y|)|x - y|,\ x, y \in \mathbb{R}^n, a \in A, \tag{8}$$

where $q$ is defined in $(*)$, we have

$$|F_a(x) - F_a(y)| \quad \leq \quad q(|x - y|)|x - y| \tag{9}$$

$$\leq \quad p(x)|x-y|,$$

where

$$p(x) = \sup_{y\in\mathbb{R}^n} q(|x-y|),$$

for all x in $\mathbb{R}^n$, where $p : \mathbb{R}^n \to [0,1]$ is a function such that $\sup\limits_{x\in\mathbb{R}^n} p(x) = \lambda < 1$. We see that for each control $\alpha(\cdot) \in A$. We have proved that the ODE (6) has a unique generalized contractive continuous solution $x(\cdot) = x^{\alpha(\cdot)}(\cdot)$, existing on the time interval $[t,T]$ and solving the ODE for a.e. time $t < s < T$. We call $x(\cdot)$ the response of the system to the control $\alpha^*(\cdot)$ and $x(s)$ the state of the system at time s.

Our aim is to find control $\alpha^*(\cdot)$ which optimally steers the system. In order to define what "optimal" means however, we must first introduce a cost criterion. Given $x \in \mathbb{R}^n$ and $0 \leq t \leq T$, let us define for each admissible control $\alpha(\cdot) \in A$ the corresponding cost

$$C_{x,t}[\alpha(\cdot)] := \int_t^T h(x(s), \alpha(s))ds + g(x(T)), \tag{10}$$

where $x(\cdot) = x^{\alpha(\cdot)}(\cdot)$ solves the ODE (6) and $h : \mathbb{R}^n \times\ A \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$ are given functions. We call $h$ the running cost per unit time and $g$ the terminal cost and will henceforth assume

$$\left\{ \begin{array}{c} |H_a(x)|, |g(x)| \leq C, \\ |H_a(x) - H_a(y)| \leq p(x)|x-y|,\ |g(x) - g(y)| \leq p(x)|x-y|, \\ x, y \in \mathbb{R}^n, a \in A, \end{array} \right. \tag{11}$$

for some constant $C$, where $p : \mathbb{R}^n \to [0,1]$ is a function such that $\sup\limits_{x\in\mathbb{R}^n} p(x) = \lambda < 1$ and for each given $a \in A$, $H_a : \mathbb{R}^n \to \mathbb{R}^n$ is a strongly fundamental contraction mapping such that $H_a(x) = h(x,a)$ for all $x \in \mathbb{R}^n$.

Given now $x \in \mathbb{R}^n$ and $0 \leq t \leq T$, we would like to find if a control $\alpha^*(\cdot)$ is possible which minimizes the cost functional (10) among all admissible controls.
To investigate the above problem we shall apply the method of dynamic programming. We now turn our attention to the value function $u(x,t)$ defined by

$$u(x,t) := \inf_{\alpha(\cdot)\in A} C_{x,t}[\alpha(\cdot)](x \in \mathbb{R}^n, 0 \leq t \leq T). \tag{12}$$

The plan is this: having defined $u(x,t)$ as the least cost given we start at the position $x$ at time $t$, we want to study $u$ as a function of $x$ and $t$. We are therefore embedding our given control problem (6) and (10) into the larger class of all such problems, as $x$ and $t$ vary. This idea then can be used to show that $u$ solves a certain Hamilton–Jacobi type PDE, and finally to show conversely that a solution of this PDE helps us to synthesize an optimal feedback control. Let us fix $x \in \mathbb{R}^n$, $0 \leq t \leq T$. Following the technique of Evans [7], p. 553, we can obtain the optimality conditions in the form given below: For each $\xi > 0$ so small that $t + \xi \leq T$,

$$u(x,t) := \inf_{\alpha(\cdot)\in A} \left\{ \int_t^{t+\xi} h(x(s), \alpha(s))ds + u(x(t+\xi), t+\xi) \right\}, \tag{13}$$

where $x(\cdot) = x^{\alpha(\cdot)}$, solves the ODE (6) for the control $\alpha(\cdot)$.

**Acknowledgement.** The author would like to thank the referee for his valuable comments and suggestions.

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*Received April 15, 2022*

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(100), 2022, Pages 22–29
ISSN 1024–7696, E-ISSN 2587–4322

# Optimal control of jump-diffusion processes with random parameters

Mario Lefebvre

**Abstract.** Let $X(t)$ be a controlled jump-diffusion process starting at $x \in [a, b]$ and whose infinitesimal parameters vary according to a continuous-time Markov chain. The aim is to minimize the expected value of a cost function with quadratic control costs until $X(t)$ leaves the interval $(a, b)$, and a termination cost that depends on the final value of $X(t)$. Exact and explicit solutions are obtained for important processes.

**Mathematics subject classification:** 93E20.
**Keywords and phrases:** Brownian motion, Poisson process, first-passage time, jump size, differential-difference equation.

## 1 Introduction

Let $\{Y(t), t \geq 0\}$ be a continuous-time Markov chain with state space $E = \{1, 2, \ldots, k\}$. We consider the two-dimensional process $\{(X(t), Y(t)), t \geq 0\}$, where $X(t)$ is defined by the stochastic differential equation

$$\mathrm{d}X(t) = \mu[X(t), Y(t)]\,\mathrm{d}t + b_0\,u[X(t), Y(t)]\,\mathrm{d}t + \sigma[X(t), Y(t)]\,\mathrm{d}B(t) + \epsilon\,\mathrm{d}N(t), \quad (1)$$

in which $b_0$ and $\epsilon$ are positive constants, $\{B(t), t \geq 0\}$ is a standard Brownian motion and $\{N(t), t \geq 0\}$ is a time-homogeneous Poisson process with rate $\lambda > 0$. That is, $\{X(t), t \geq 0\}$ is a controlled jump-diffusion process with random infinitesimal mean $\mu(\cdot, \cdot)$ and variance $\sigma^2(\cdot, \cdot)$. We assume that $\{B(t), t \geq 0\}$ and $\{N(t), t \geq 0\}$ are independent processes.

Jump-diffusion processes are very important in mathematical finance, where they are used as models for the evolution of stock or commodity prices. Moreover, because of frequent regime changes, the fact that the parameters $\mu(\cdot, \cdot)$ and $\sigma^2(\cdot, \cdot)$ are random is more realistic.

In this paper, we are looking for the control $u^*(x, i)$ that minimizes the expected value of the cost function

$$J(x, i) := \int_0^{T(x,i)} \left\{ \frac{1}{2} q_{0,i}\, u^2[X(t), i] + \theta_i \right\} \mathrm{d}t + K_i[X(T(x, i))], \quad (2)$$

where $q_{0,i} > 0$, $\theta_i \in \mathbb{R}$ and $T(x, i)$ is the first-passage time

$$T(x, i) = \inf\{t \geq 0 : X(t) \notin (a, b) \mid X(0) = x \in [a, b], Y(0) = i\}, \quad (3)$$


DOI: https://doi.org/10.56415/basm.y2022.i3.p22

for $i = 1, 2, \ldots, k$. If the constant $\theta_i$ is positive (respectively, negative), then the optimizer wants the process to leave the continuation region as soon (respectively, late) as possible. Furthermore, we assume that the termination cost is of the form

$$K_i[X(T(x,i))] = a_i X^2(T(x,i)) + b_i X(T(x,i)) + k_i, \tag{4}$$

where $a_i$, $b_i$ and $k_i$ are constants, for $i = 1, 2, \ldots, k$. Depending on the values of these constants (and the other parameters in the problem), the aim can be to try to leave the interval $(a, b)$ through $a$ rather than $b$, or vice versa.

The problem set up above is an extension of the so-called *LQG homing* problem studied by Whittle [7] for $n$-dimensional diffusion processes. He showed that it is sometimes possible to obtain the exact optimal control by computing a mathematical expectation for the corresponding uncontrolled process. Lefebvre [3] extended Whittle's results to the case of one-dimensional jump-diffusion processes with deterministic infinitesimal parameters. The optimal control then becomes approximate, rather than exact. At any rate, even if one is able to reduce the stochastic optimal control problem to a purely probabilistic problem, computing the mathematical expectation needed to obtain the optimal control is usually very difficult, especially in two or more dimensions.

For applications of LQG homing problems, see in particular Ionescu *et al.* [1] and [2], as well as Lefebvre [4] and [5]. See also Makasu [6] for the solution to a two-dimensional problem.

In the next section, we will give the system of non-linear differential-difference equations that we must solve to determine the optimal controls $u^*(x, i)$, for $i = 1, 2, \ldots, k$. In Section 3, exact solutions to particular problems for important processes will be found explicitly.

## 2 Dynamic programming

We define the value function

$$F(x,i) = \inf_{u[X(t),i],\ 0 \le t \le T(x,i)} E[J(x,i)], \tag{5}$$

for $i = 1, 2, \ldots, k$. To obtain the dynamic programming equation satisfied by the function $F(x, i)$, we will use the following results: first, by definition, a Poisson process starts at $N(0) = 0$, and the number $N(t)$ of events in the interval $(0, t]$ has a Poisson distribution with parameter $\lambda t$, which implies that

$$P[N(\Delta t) = 0] = e^{-\lambda \Delta t} = 1 - \lambda \Delta t + o(\Delta t) \tag{6}$$

and

$$P[N(\Delta t) = 1] = \lambda \Delta t e^{-\lambda \Delta t} = \lambda \Delta t + o(\Delta t). \tag{7}$$

Next, we assume that $B(0) = 0$; then, as is well known, we can write that

$$E[B(\Delta t)] = 0 \tag{8}$$

and

$$E[B^2(\Delta t)] = V[B(\Delta t)] = \Delta t. \tag{9}$$

Finally, in the case of the continuous-time Markov chain $\{Y(t), t \geq 0\}$, when it enters state $i$, it remains there for a random time $\tau_i$ having an exponential distribution with parameter denoted by $\nu_i$. Then, it will move to state $j \neq i$ with probability $p_{i,j}$, with $\sum_{j \neq i} p_{i,j} = 1$. Therefore, when $Y(0) = i$,

$$P[Y(\Delta t) = i] = P[\tau_i > \Delta t] = e^{-\nu_i \Delta t} = 1 - \nu_i \Delta t + o(\Delta t) \tag{10}$$

and

$$P[Y(\Delta t) = j] = \left(1 - e^{-\nu_i \Delta t}\right) p_{i,j} = \nu_i p_{i,j} \Delta t + o(\Delta t), \tag{11}$$

for $j \neq i$.

Using the standard arguments, we obtain the following dynamic programming equation (DPE):

$$\begin{aligned} 0 \quad = \quad \inf_{u(x,i)} \Bigg\{ & \frac{1}{2} q_{0,i} u^2(x,i) + \theta_i + [\mu(x,i) + b_0 u(x,i)] F'(x,i) \\ & + \frac{1}{2} \sigma^2(x,i) F''(x,i) + \sum_{j \neq i} \nu_i p_{i,j} \left[F(x,j) - F(x,i)\right] \\ & + \lambda \left[F(x+\epsilon, i) - F(x,i)\right] \Bigg\}. \end{aligned} \tag{12}$$

From Eq. (12), we deduce at once that, in terms of $F(x,i)$, the optimal control $u^*(x,i)$ is

$$u^*(x,i) = -\frac{b_0}{q_{0,i}} F'(x,i). \tag{13}$$

We can now state the following proposition, obtained by substituting the above expression into the DPE.

**Proposition 2.1.** *The value functions $F(x,i)$, $i = 1, \ldots, k$, satisfy the system of non-linear second-order differential-difference equations*

$$\begin{aligned} 0 \quad = \quad & \theta_i + \mu(x,i) F'(x,i) - \frac{1}{2} \frac{b_0^2}{q_{0,i}} \left[F'(x,i)\right]^2 \\ & + \frac{1}{2} \sigma^2(x,i) F''(x,i) + \sum_{j \neq i} \nu_i p_{ij} \left[F(x,j) - F(x,i)\right] \\ & + \lambda \left[F(x+\epsilon, i) - F(x,i)\right]. \end{aligned} \tag{14}$$

*The system is valid for $a < x < b$. Moreover, because the jump size is a positive constant $\epsilon$, the boundary conditions are*

$$F(a,i) = K_i(a) \quad \textit{and} \quad F(x,i) = K_i(x) \quad \textit{if } b \leq x < b + \epsilon. \tag{15}$$

In the next section, we will find exact solutions to the above system in important particular cases.

## 3 Particular cases

For the sake of simplicity, we assume that the state space of the Markov chain $\{Y(t), t \geq 0\}$ is the set $\{1, 2\}$; that is, $k = 2$ in Proposition 2.1. Then, we have that $p_{i,j} = 1$.

First, making use of Taylor's formula, we can write that

$$F(x + \epsilon, i) = F(x, i) + \epsilon F'(x, i) + \frac{1}{2}\epsilon^2 F''(x, i) + o(\epsilon^2), \tag{16}$$

which implies that the system (14) can be rewritten as follows:

$$\begin{aligned} 0 \quad &\approx \quad \theta_i + \mu(x,i)F'(x,i) - c_i^2\left[F'(x,i)\right]^2 + \frac{1}{2}\sigma^2(x,i)F''(x,i) \\ &\quad + \nu_i\left[F(x,j) - F(x,i)\right] + \lambda\left[\epsilon F'(x,i) + \frac{1}{2}\epsilon^2 F''(x,i)\right] \end{aligned} \tag{17}$$

for $i = 1, 2$, where

$$c_i^2 := \frac{1}{2}\frac{b_0^2}{q_{0,i}}. \tag{18}$$

If $\epsilon$ is small, then the solution to the above system should be a good approximation to the exact solution that we are looking for. Furthermore, if $F(x, i)$ is a polynomial of degree 1 or 2, then the solution to (17) is actually the *exact* solution to our problem.

**Case I.** Assume that the interval $[a, b]$ is $[0, 1]$, and that $\mu(x, i) \equiv \mu_i \in \mathbb{R}$, for $i = 1, 2$. If $\sigma^2(x, i) \equiv \sigma_i^2$, then the continuous part of the process $\{X(t), t \geq 0\}$ is a Wiener process with random infinitesimal parameters. The Wiener process is surely among the most important diffusion processes.

Suppose that

$$K_i[X(T(x,i))] = b_i X(T(x,i)) + k_i, \tag{19}$$

where $b_i \neq 0$, for $i = 1, 2$. So, we take $a_i = 0$ in Eq. (4). The boundary conditions are therefore

$$F(0, i) = k_i \quad \text{and} \quad F(x, i) = b_i x + k_i \quad \text{if } 1 \leq x < 1 + \epsilon. \tag{20}$$

Hence, in general, the optimizer should try to make the controlled process leave the interval $(0, 1)$ through the origin, so that we expect $u^*(x, i)$ to be negative. However the sign of the optimal control also depends, in particular, on the value of $\theta_i$. If $\theta_i > 0$ is large and $x$ is close to 1, it might be better to leave the interval through $x \geq 1$ and accept the larger termination cost.

Now, let us try a value function $F(x, i)$ of the same form as $K_i(x)$. Substituting this expression into the system (17), we find that

$$0 \quad = \quad \theta_1 + \mu_1 b_1 - c_1^2 b_1^2 + \nu_1\left[(b_2 - b_1)x + (k_2 - k_1)\right] + \lambda\epsilon b_1, \tag{21}$$

$$0 \quad = \quad \theta_2 + \mu_2 b_2 - c_2^2 b_2^2 - \nu_2\left[(b_2 - b_1)x + (k_2 - k_1)\right] + \lambda\epsilon b_2. \tag{22}$$

We deduce from the above equations that a necessary condition for the solution to be valid is that we must have $b_1 = b_2$, so that the constants $b_1 \neq 0$, $k_1$ and $k_2$ must be such that

$$\begin{aligned} 0 &= \theta_1 + \mu_1 b_1 - c_1^2 b_1^2 + \nu_1 (k_2 - k_1) + \lambda \epsilon b_1, && (23) \\ 0 &= \theta_2 + \mu_2 b_1 - c_2^2 b_1^2 - \nu_2 (k_2 - k_1) + \lambda \epsilon b_1. && (24) \end{aligned}$$

Let us choose the following parameters:

$$\mu_1 = -1, \; \mu_2 = 0, \; \lambda = \epsilon = \theta_i = \nu_i = b_0 = q_{0,i} = 1, \text{ for } i = 1, 2.$$

Then, one can check that the system (23), (24) is satisfied if $b_1 = 2$, $k_1 = 0$ and $k_2 = 1$. Thus, we have that

$$F(x, 1) = 2x \quad \text{and} \quad F(x, 2) = 2x + 1 \quad \text{if } 0 < x < 1. \tag{25}$$

Furthermore, the functions $F(x, i)$ satisfy the boundary conditions in (20) with $b_1 = b_2 = 2$, for $i = 1, 2$.

From Eq. (13), we obtain that the optimal control is given by

$$u^*(x, 1) = u^*(x, 2) \equiv -2. \tag{26}$$

For other choices of the parameters $q_{0,1}$ and $q_{0,2}$, $u^*(x, 1)$ and $u^*(x, 2)$ could be different, but they are always constant in this first example.

**Remarks.** (i) If instead of $\mu_2 = 0$ above, we rather have $\mu_2 = -2$, then we find that the system is satisfied if $b_1 = -2$ (together with $k_1 = 0$, $k_2 = 1$). Therefore, we have a second explicit solution to the problem considered. Moreover, notice that the optimal control $u^*(x, i)$ would then be positive.

(ii) Since the solution to our problem does not depend on $\sigma_1^2(x, i)$ and $\sigma_2^2(x, i)$, it is valid, in particular, in the case of a Wiener process with random parameters and Poissonian jumps, as mentioned above.

(iii) We can easily find other particular solutions. For instance, if $\mu_1 = 0$, $\mu_2 = 1/3$, $k_1 = 0$ and $k_2 = 1/2$, then $b_1 = 3$, etc.

**Case II.** Assume again that the continuation region is the interval $(0, 1)$. This time, we take $\mu(x, i) = -\gamma_i x$, for $i = 1, 2$. If the constant $\gamma_i$ is positive and if $\sigma^2(x, i) \equiv \sigma_i^2$, $\{X(t), t \geq 0\}$ is then an Ornstein-Uhlenbeck process with random parameters and Poissonian jumps. The Ornstein-Uhlenbeck process is also among the most important diffusion processes for applications.

We choose the termination cost function in Eq. (19), and we try a solution $F(x, i) = K_i(x)$ of the system (17). We obtain the following system:

$$\begin{aligned} 0 &= \theta_1 - \gamma_1 b_1 x - c_1^2 b_1^2 + \nu_1 \left[ (b_2 - b_1) x + (k_2 - k_1) \right] + \lambda \epsilon b_1, && (27) \\ 0 &= \theta_2 - \gamma_2 b_2 x - c_2^2 b_2^2 - \nu_2 \left[ (b_2 - b_1) x + (k_2 - k_1) \right] + \lambda \epsilon b_2. && (28) \end{aligned}$$

Therefore, we must have (for the terms in $x$)

$$0 = -\gamma_1 b_1 + \nu_1 (b_2 - b_1), \tag{29}$$

$$0 = -\gamma_2 b_2 - \nu_2(b_2 - b_1) \tag{30}$$

and (for the constant terms)

$$0 = \theta_1 - c_1^2 b_1^2 + \nu_1(k_2 - k_1) + \lambda\epsilon b_1, \tag{31}$$
$$0 = \theta_2 - c_2^2 b_2^2 - \nu_2(k_2 - k_1) + \lambda\epsilon b_2. \tag{32}$$

Let us choose the parameters $\gamma_1 = 1$, $\gamma_2 = -1/2$, $\lambda = \epsilon = \nu_1 = \nu_2 = b_0 = 1$, $\theta_1 = -1/2$, $\theta_2 = -1$, $q_{0,1} = 1$ and $q_{0,2} = 2$. We find that a solution of the above systems (that also satisfies the appropriate boundary conditions) is

$$F(x,1) = x + k_1 \quad \text{and} \quad F(x,2) = 2x + k_2 \quad \text{if } 0 < x < 1, \tag{33}$$

for any choice of the constants $k_1$ and $k_2$ such that

$$k_2 - k_1 = 0. \tag{34}$$

Furthermore, the optimal control is

$$u^*(x,1) = u^*(x,2) \equiv -1. \tag{35}$$

**Case III.** Finally, we take $\mu(x,i) = \mu_i x$, where $\mu_i \in \mathbb{R}$, and $\sigma^2(x,i) = \sigma_i^2 x^2$, for $i = 1, 2$. Therefore, the continuous part of the process $\{X(t), t \geq 0\}$ is a geometric Brownian motion, which is widely used in financial mathematics. Because this diffusion process is always positive (if it starts at $X(0) > 0$), we assume that $a > 0$ in the interval $[a, b]$. We choose the interval $[1, 2]$ and the termination cost function in (4), with $a_1 = a_2$ and $b_1 = b_2$. The boundary conditions are

$$F(1,i) = a_1 + b_1 + k_i \quad \text{and} \quad F(x,i) = a_1 x^2 + b_1 x + k_i \quad \text{if } 2 \leq x < 2 + \epsilon. \tag{36}$$

Proceeding as in the previous cases, we try a solution of the same form as the function $K_i(\cdot)$. We then obtain the system

$$\begin{aligned} 0 &= \theta_1 + \mu_1 x(2a_1 x + b_1) - c_1^2(2a_1 x + b_1)^2 + \sigma_1^2 x^2 a_1 \\ &\quad + \nu_1(k_2 - k_1) + \lambda\left[\epsilon(2a_1 x + b_1) + \epsilon^2 a_1\right], \end{aligned} \tag{37}$$
$$\begin{aligned} 0 &= \theta_2 + \mu_2 x(2a_1 x + b_1) - c_2^2(2a_1 x + b_1)^2 + \sigma_2^2 x^2 a_1 \\ &\quad - \nu_2(k_2 - k_1) + \lambda\left[\epsilon(2a_1 x + b_1) + \epsilon^2 a_1\right]. \end{aligned} \tag{38}$$

For the sake of simplicity, let us choose $\lambda = \epsilon = 1$. We then deduce that we must have (for the terms in $x^2$)

$$0 = 2\mu_1 a_1 - 4c_1^2 a_1^2 + \sigma_1^2 a_1, \tag{39}$$
$$0 = 2\mu_2 a_1 - 4c_2^2 a_1^2 + \sigma_2^2 a_1, \tag{40}$$

(for the terms in $x$)

$$0 = \mu_1 b_1 - 4c_1^2 a_1 b_1 + 2a_1, \tag{41}$$

$$0 = \mu_2 b_1 - 4c_2^2 a_1 b_1 + 2a_1 \tag{42}$$

and (for the constant terms)

$$0 = \theta_1 - c_1^2 b_1^2 + \nu_1 (k_2 - k_1) + b_1 + a_1, \tag{43}$$

$$0 = \theta_2 - c_2^2 b_1^2 - \nu_2 (k_2 - k_1) + b_1 + a_1. \tag{44}$$

We can check that the function

$$F(x,i) = x^2 + x + k_i \quad \text{for } 1 < x < 2 \tag{45}$$

is a solution to our problem if

$$\mu_1 = 0,\ \mu_2 = -1,\ \sigma_1^2 = 2,\ \sigma_2^2 = 3,\ b_0 = \nu_1 = \nu_2 = q_{0,1} = 1$$

and $q_{0,2} = 2$, together with

$$\theta_1 = -\frac{3}{2} - (k_2 - k_1) \quad \text{and} \quad \theta_2 = -\frac{7}{4} + (k_2 - k_1). \tag{46}$$

It follows that the optimal controls are affine functions of $x$:

$$u^*(x,1) = -(2x+1) \quad \text{and} \quad u^*(x,2) = -\frac{1}{2}(2x+1). \tag{47}$$

## 4 Conclusion

In this paper, we considered a difficult problem, namely an optimal control problem for jump-diffusion processes with random parameters, when in addition the final time is a first-passage time random variable. The aim was to obtain exact and explicit solutions to such problems.

In Section 3, we were able to solve three particular problems for very important diffusion processes. Wiener processes, Ornstein-Uhlenbeck processes and geometric Brownian motions appear in numerous applications.

For the discrete part of the jump-diffusion processes, we assumed that jumps occurred according to a time-homogeneous Poisson process and that the jump size was a positive constant. This enabled us, making use of Taylor's formula, to transform a system of differential-difference equations into an approximate system of differential equations. However, this approximate system becomes an exact one in the case when the value function is a polynomial of degree equal to 1 or 2.

It would be interesting to try to generalize the results obtained in this paper to the case of a random jump size. We could also assume that there can be both positive and negative jumps that are generated by two independent Poisson processes.

Finally, as mentioned above, the aim was to obtain analytical solutions to the problem set up in Section 1. When the state space $E$ of the Markov chain contains many values, it should at least be possible to use numerical methods to determine the value functions and the optimal controls.

### Acknowledgments

This research was supported by the Natural Sciences and Engineering Research Council of Canada.

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*Received September 28, 2022*

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(100), 2022, Pages 30–40
ISSN 1024–7696, E-ISSN 2587–4322

# A self-similar solution for the two-dimensional Broadwell system via the Bateman equation

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**Abstract.** A self-similar solution of the Broadwell system is found. Here the solution is sought using a reduction that transforms the given system into a system of differential equations. Further, the solution is constructed using the Painlevé series. Here the system already passes the Painlevé test and it is possible to find the solution if the equations in resonance satisfy the solution of the two-dimensional Bateman equation. Exact solution of the Bateman equation is established, allowing to find new explicit solution for the original system. In the process of calculations, we use the Wolfram Mathematica program. The proof of these results is carried out at a rigorous mathematical level.

**Mathematics subject classification:** 35L40, 35Q20, 35C06.
**Keywords and phrases:** Broadwell system, self-similar solution, Painlevé test, Bateman equation.

## 1 Introduction

We consider the well-known two-dimensional Broadwell model [7,10,11,15,17,20]

$$\begin{aligned}
&\partial_t u + \partial_x u = \frac{1}{\varepsilon}(wz - uv),\\
&\partial_t v - \partial_x v = \frac{1}{\varepsilon}(wz - uv), \quad x, y \in \mathbb{R},\ t > 0,\\
&\partial_t w + \partial_y w = \frac{1}{\varepsilon}(uv - wz),\\
&\partial_t z - \partial_y z = \frac{1}{\varepsilon}(uv - wz).
\end{aligned} \tag{1}$$

Here $u(x,y,t), v(x,y,t), w(x,y,t), z(x,y,t)$ are the densities of particle groups, $\varepsilon$ is the Knudsen parameter. It is required to find a self-similar solution of the system (1). As is known, most of the equations of mathematical physics describe various physical processes, for example, the Burgers equation, the Korteweg-de Vries equation, the Allen-Kahn equation, etc. One of such equations is the discrete kinetic Boltzmann equation [22] (see p.1). We consider the so-called Broadwell model [7, 11, 20], which is a consequence in the discrete case when the collision integral on the right side of the Boltzmann equation is replaced by a finite sum. From here, the given system of equations is directly obtained. The physical interpretation of the


DOI: https://doi.org/10.56415/basm.y2022.i3.p30

system can be found in [7,20]. Works [1,5,11] are devoted to finding exact solutions of kinetic systems by means of the Bateman equation [3,6,13,14]. These systems are non-integrable (kinetic systems Carleman [1], Godunov-Sultangazin [9,11] (one-dimensional model of Broadwell), McKean [5,12], two-dimensional model of Broadwell). As a result, the Painlevé test fails. Here, in resonance, the author obtained the Bateman equation and, knowing its implicit solution, constructed a solution for the original system. Stationary solutions of systems were found in [16,18]. In the works [7,9,19] it is proved that the solution of systems tends to a positive equilibrium state exponentially fast. Also recently in [8,12,18,20], solutions were found that can take both positive and negative values. Nevertheless, ones produce interesting results. In our work, a self-similar solution of the system is presented.

## 2 Bateman equation

The two-dimensional Bateman equation is an equation of the form [4,6,11]

$$\frac{\partial^2\varphi}{\partial\eta^2}\Big(\frac{\partial\varphi}{\partial\xi}\Big)^2-2\frac{\partial\varphi}{\partial\eta}\frac{\partial\varphi}{\partial\xi}\frac{\partial^2\varphi}{\partial\xi\partial\eta}+\Big(\frac{\partial\varphi}{\partial\eta}\Big)^2\frac{\partial^2\varphi}{\partial\xi^2}=0. \tag{2}$$

This equation has an implicit solution

$$\xi f(\varphi)+\eta g(\varphi)=c, \tag{3}$$

where $f, g$ are arbitrary smooth functions, $c\in\mathbb{R}$. The proof is carried out by direct computation

$$\begin{aligned}
\frac{\partial\varphi}{\partial\xi}&=-\frac{f(\varphi)}{\xi f'(\varphi)+\eta g'(\varphi)},\\
\frac{\partial^2\varphi}{\partial\xi^2}&=-\frac{f(\varphi)\Big(-2\xi f'(\varphi)^2-2\eta f'(\varphi)g'(\varphi)+f(\varphi)(\xi f''(\varphi)+\eta g''(\varphi))\Big)}{(\xi f'(\varphi)+\eta g'(\varphi))^3},\\
\frac{\partial\varphi}{\partial\eta}&=-\frac{g(\varphi)}{\xi f'(\varphi)+\eta g'(\varphi)},\\
\frac{\partial^2\varphi}{\partial\eta^2}&=-\frac{g(\varphi)\Big(-2\xi f'(\varphi)g'(\varphi)-2\eta g'(\varphi)^2+g(\varphi)(\xi f''(\varphi)+\eta g''(\varphi))\Big)}{(\xi f'(\varphi)+\eta g'(\varphi))^3},\\
\frac{\partial^2\varphi}{\partial\xi\partial\eta}&=\frac{fg'(\xi f'+\eta g')+g\Big(\xi f'(\varphi)^2+\eta f'(\varphi)g'(\varphi)-f(\xi f''+\eta g'')\Big)}{(\xi f'(\varphi)+\eta g'(\varphi))^3}.
\end{aligned} \tag{4}$$

Substituting (4) into (2), we are convinced of the equality.

## 3 A self-similar solution for the Broadwell system

We look for a self-similar solution in the form (see [21], S.3.3., p. 708)

$$u(x,y,t)=x^{\alpha}U(\xi,\eta), v(x,y,t)=x^{\beta}V(\xi,\eta),$$

$$w(x,y,t)=x^{\gamma}W(\xi,\eta), z(x,y,t)=x^{s}Z(\xi,\eta),$$

where $\xi=tx^{A},\eta=yx^{B}$. System is scale invariant under

$$t=C^{k}\bar{t}, x=C\overline{x}, y=C^{l}\overline{y}, u=C^{m}\overline{u}, v=C^{n}\overline{v}, w=C^{p}\overline{w}, z=C^{q}\overline{z}, C>0. \tag{5}$$

The scaling transformation (5) converts system (1) into

$$\begin{aligned}
C^{m-k}\frac{\partial\bar{u}}{\partial\bar{t}}+C^{m-1}\frac{\partial\bar{u}}{\partial\bar{x}}&=\frac{1}{\varepsilon}(C^{p+q}\bar{w}\bar{z}-C^{m+n}\bar{u}\bar{v}),\\
C^{n-k}\frac{\partial\bar{v}}{\partial\bar{t}}-C^{n-1}\frac{\partial\bar{v}}{\partial\bar{x}}&=\frac{1}{\varepsilon}(C^{p+q}\bar{w}\bar{z}-C^{m+n}\bar{u}\bar{v}),\\
C^{p-k}\frac{\partial\bar{w}}{\partial\bar{t}}+C^{p-l}\frac{\partial\bar{w}}{\partial\bar{y}}&=\frac{1}{\varepsilon}(C^{m+n}\bar{u}\bar{v}-C^{p+q}\bar{w}\bar{z}),\\
C^{q-k}\frac{\partial\bar{z}}{\partial\bar{t}}-C^{q-l}\frac{\partial\bar{z}}{\partial\bar{y}}&=\frac{1}{\varepsilon}(C^{m+n}\bar{u}\bar{v}-C^{p+q}\bar{w}\bar{z}),
\end{aligned}$$

Equating the powers of $C$ yields the following system of linear algebraic equations for the constants $m,k,p,q,l$ and $n$:

$$m-1-m+k=0, p+q-m+k=0, m+n-m+k=0,$$

$$n-1-n+k=0, p+q-n+k=0, m+n-n+k=0,$$

$$p-l-p+k=0, m+n-p+k=0, p+q-p+k=0,$$

$$q-l-q+k=0, m+n-q+k=0, p+q-q+k=0.$$

This system has a unique solution

$$k=l=1, n=p=m=q=-1.$$

In this case according to the formulas from (see [21], S.3.3., p. 708)

$$\alpha=\beta=\gamma=s=-1, A=B=-1.$$

Then we have

$$u(x,y,t)=\frac{1}{x}U(\xi,\eta), v(x,y,t)=\frac{1}{x}V(\xi,\eta), \tag{6}$$

$$w(x,y,t)=\frac{1}{x}W(\xi,\eta), z(x,y,t)=\frac{1}{x}Z(\xi,\eta), \tag{7}$$

where $\xi=t/x,\eta=y/x$. Substituting expressions (6)-(7) into (1), we obtain for the first equation

$$\frac{1}{x}U'_{\xi}\frac{1}{x}-\frac{U}{x^2}+\frac{1}{x}(U'_{\xi}(-\frac{t}{x^2})+U'_{\eta}(-\frac{y}{x^2}))=\frac{1}{\varepsilon}(\frac{1}{x^2}WZ-\frac{1}{x^2}UV). \tag{8}$$

The rest of the equations are obtained similarly. Hence, we have system

$$\begin{aligned} U'_\xi(1-\xi) - U'_\eta\eta &= U + \frac{1}{\varepsilon}(WZ - UV), \\ V'_\xi(1+\xi) + V'_\eta\eta &= -V + \frac{1}{\varepsilon}(WZ - UV), \\ W'_\xi + W'_\eta &= \frac{1}{\varepsilon}(UV - WZ), \\ Z'_\xi - Z'_\eta &= \frac{1}{\varepsilon}(UV - WZ). \end{aligned} \tag{9}$$

We apply the Painlevé expansion [2]:

$$\begin{aligned} U(\xi,\eta) = \varphi^{-p}\sum_{j=0}^{\infty} U_j(\xi,\eta)\varphi^j, V(\xi,\eta) = \varphi^{-\beta}\sum_{j=0}^{\infty} V_j(\xi,\eta)\varphi^j, \\ W(\xi,\eta) = \varphi^{-\gamma}\sum_{j=0}^{\infty} W_j(\xi,\eta)\varphi^j, Z(\xi,\eta) = \varphi^{-q}\sum_{j=0}^{\infty} Z_j(\xi,\eta)\varphi^j, \end{aligned} \tag{10}$$

where $\varphi = \varphi(\xi,\eta)$ is an analytic function in a neighborhood of the manifold $\varphi(\xi,\eta) = 0$. Firstly, we find the dominant terms

$$U = U_0\varphi^{-p}, V = V_0\varphi^{-\beta}, W = W_0\varphi^{-\gamma}, Z = Z_0\varphi^{-q}, \tag{11}$$

where $p, \beta, \gamma, q$ are positive integers. Substituting the leading terms of (11) into our original system, we have

$$\begin{aligned} (1-\xi)(U'_{0\xi}\varphi^{-p} - p\varphi^{-p-1}\varphi'_\xi U_0) - \eta(U'_{0\eta}\varphi^{-p} - p\varphi^{-p-1}\varphi'_\eta U_0) = \\ = U_0\varphi^{-p} + \frac{1}{\varepsilon}(W_0Z_0\varphi^{-\gamma-q} - U_0V_0\varphi^{-p-\beta}), \\ (1+\xi)(V'_{0\xi}\varphi^{-\beta} - \beta\varphi^{-\beta-1}\varphi'_\xi V_0) + \eta(V'_{0\eta}\varphi^{-p} - p\varphi^{-p-1}\varphi'_\eta V_0) = \\ = -V_0\varphi^{-\beta} + \frac{1}{\varepsilon}(W_0Z_0\varphi^{-\gamma-q} - U_0V_0\varphi^{-p-\beta}), \\ W'_{0\xi}\varphi^{-\gamma} - \gamma\varphi^{-\gamma-1}\varphi'_\xi W_0 + W'_{0\eta}\varphi^{-\gamma} - \gamma\varphi^{-\gamma-1}\varphi'_\eta W_0 = \\ = \frac{1}{\varepsilon}(U_0V_0\varphi^{-p-\beta} - W_0Z_0\varphi^{-\gamma-q}), \\ Z'_{0\xi}\varphi^{-q} - q\varphi^{-q-1}\varphi'_\xi Z_0 - Z'_{0\eta}\varphi^{-q} + q\varphi^{-q-1}\varphi'_\eta Z_0 = \\ = \frac{1}{\varepsilon}(U_0V_0\varphi^{-p-\beta} - W_0Z_0\varphi^{-\gamma-q}). \end{aligned} \tag{12}$$

Multiplying the first equation of (12) by $\varphi^{p+1}$ and taking into account that $\varphi(\xi,\eta)=0$, we have $p=\beta=\gamma=q=1$. From here

$$\begin{aligned}
-(1-\xi)\varphi'_\xi U_0+\varphi'_\eta U_0\eta&=\frac{1}{\varepsilon}(W_0Z_0-U_0V_0),\\
-(1+\xi)\varphi'_\xi V_0-\varphi'_\eta V_0\eta&=\frac{1}{\varepsilon}(W_0Z_0-U_0V_0),\\
-\varphi'_\xi W_0-\varphi'_\eta W_0&=\frac{1}{\varepsilon}(U_0V_0-W_0Z_0),\\
-\varphi'_\xi Z_0+\varphi'_\eta Z_0&=\frac{1}{\varepsilon}(U_0V_0-W_0Z_0).
\end{aligned}\tag{13}$$

Solving the system (13), we obtain solution

$$\begin{aligned}
U_0(\xi,\eta)&=-\frac{\varepsilon(\eta\varphi_\eta+(\xi+1)\varphi_\xi)(\varphi_\eta^2-\varphi_\xi^2)}{(\eta^2-1)\varphi_\eta^2+2\xi\eta\varphi_\xi\varphi_\eta+\xi^2\varphi_\xi^2},\\
V_0(\xi,\eta)&=\frac{\varepsilon(\eta\varphi_\eta+(\xi-1)\varphi_\xi)(\varphi_\eta^2-\varphi_\xi^2)}{(\eta^2-1)\varphi_\eta^2+2\xi\eta\varphi_\xi\varphi_\eta+\xi^2\varphi_\xi^2},\\
W_0(\xi,\eta)&=-\frac{\varepsilon(\varphi_\eta-\varphi_\xi)(\eta\varphi_\eta+(\xi-1)\varphi_\xi)(\eta\varphi_\eta+(\xi+1)\varphi_\xi)}{(\eta^2-1)\varphi_\eta^2+2\xi\eta\varphi_\xi\varphi_\eta+\xi^2\varphi_\xi^2},\\
Z_0(\xi,\eta)&=\frac{\varepsilon(\varphi_\eta+\varphi_\xi)(\eta\varphi_\eta+(\xi-1)\varphi_\xi)(\eta\varphi_\eta+(\xi+1)\varphi_\xi)}{(\eta^2-1)\varphi_\eta^2+2\xi\eta\varphi_\xi\varphi_\eta+\xi^2\varphi_\xi^2}.
\end{aligned}\tag{14}$$

The truncated Painlevé expansion has the form

$$\begin{aligned}
U=U_0\varphi^{-1}+U_1, V=V_0\varphi^{-1}+V_1,\\
W=W_0\varphi^{-1}+W_1, Z=Z_0\varphi^{-1}+Z_1,
\end{aligned}\tag{15}$$

where $U_0,V_0,W_0,Z_0$ are defined by (14) and $U_1,V_1,W_1,Z_1$ are arbitrary functions.

Substituting (15) into (9), we have

$$\begin{aligned}
\varphi^{-1}\Big(U'_{0\xi}(1-\xi)-\eta U'_{0\eta}-U_0-\frac{1}{\varepsilon}(W_0Z_1+W_1Z_0-U_0V_1-U_1V_0)\Big)+\\
+\varphi^{-2}\Big(-U_0\varphi'_\xi(1-\xi)+\eta\varphi'_\eta U_0-\frac{1}{\varepsilon}(W_0Z_0-U_0V_0)\Big)+\\
+\varphi^{0}\Big(U'_{1\xi}(1-\xi)-\eta U'_{1\eta}-U_1-\frac{1}{\varepsilon}(W_1Z_1-U_1V_1)\Big)=0,
\end{aligned}$$

$$\begin{aligned}
\varphi^{-1}\Big(V'_{0\xi}(1+\xi)+\eta V'_{0\eta}+V_0-\frac{1}{\varepsilon}(W_0Z_1+W_1Z_0-U_0V_1-U_1V_0)\Big)+\\
+\varphi^{-2}\Big(-V_0\varphi'_\xi(1+\xi)-\eta\varphi'_\eta V_0-\frac{1}{\varepsilon}(W_0Z_0-U_0V_0)\Big)+\\
+\varphi^{0}\Big(V'_{1\xi}(1+\xi)+\eta V'_{1\eta}+V_1-\frac{1}{\varepsilon}(W_1Z_1-U_1V_1)\Big)=0,
\end{aligned}$$

$$\varphi^{-1}\Big(W'_{0\xi}+W'_{0\eta}-\frac{1}{\varepsilon}(U_0V_1+U_1V_0-W_0Z_1-W_1Z_0)\Big)+$$
$$+\varphi^{-2}\Big(-W_0\varphi'_\xi-\varphi'_\eta W_0-\frac{1}{\varepsilon}(U_0V_0-W_0Z_0)\Big)+$$
$$+\varphi^{0}\Big(W'_{1\xi}+W'_{1\eta}-\frac{1}{\varepsilon}(U_1V_1-W_1Z_1)\Big)=0,$$

$$\varphi^{-1}\Big(Z'_{0\xi}-Z'_{0\eta}-\frac{1}{\varepsilon}(U_0V_1+U_1V_0-W_0Z_1-W_1Z_0)\Big)+$$
$$+\varphi^{-2}\Big(-Z_0\varphi'_\xi+\varphi'_\eta Z_0-\frac{1}{\varepsilon}(U_0V_0-W_1Z_1)\Big)+$$
$$+\varphi^{0}\Big(Z'_{1\xi}-Z'_{1\eta}-\frac{1}{\varepsilon}(U_1V_1-W_1Z_1)\Big)=0.$$

The coefficients at $\varphi^{-2}$ give the well-known expressions defined by (14). Assuming that $U_1=V_1=W_1=Z_1=0$, the coefficients at $\varphi^0$ are satisfied. It remains to consider at $\varphi^{-1}$. Equating each coefficient of $\varphi^{-1}$ to zero, we have

$$\begin{aligned}
U'_{0\xi}(1-\xi)-\eta U'_{0\eta}-U_0&=\frac{1}{\varepsilon}(W_0Z_1+W_1Z_0-U_0V_1-U_1V_0),\\
V'_{0\xi}(1+\xi)+\eta V'_{0\eta}+V_0&=\frac{1}{\varepsilon}(W_0Z_1+W_1Z_0-U_0V_1-U_1V_0),\\
W'_{0\xi}+W'_{0\eta}&=\frac{1}{\varepsilon}(U_0V_1+U_1V_0-W_0Z_1-W_1Z_0),\\
Z'_{0\xi}-Z'_{0\eta}&=\frac{1}{\varepsilon}(U_0V_1+U_1V_0-W_0Z_1-W_1Z_0).
\end{aligned}\tag{16}$$

We rewrite the system (16) as

$$U'_{0\xi}(1-\xi)-\eta U'_{0\eta}-U_0=V'_{0\xi}(1+\xi)+\eta V'_{0\eta}+V_0,\tag{17}$$
$$U'_{0\xi}(1-\xi)-\eta U'_{0\eta}-U_0=-W'_{0\xi}-W'_{0\eta},\tag{18}$$
$$U'_{0\xi}(1-\xi)-\eta U'_{0\eta}-U_0=-Z'_{0\xi}+Z'_{0\eta},\tag{19}$$
$$U'_{0\xi}(1-\xi)-\eta U'_{0\eta}-U_0=0.\tag{20}$$

Substituting (14) into (17), we have

$$\frac{4\eta\varepsilon(\eta\varphi_\eta^2+(-1+\eta^2+\xi^2)\varphi_\eta\varphi_\xi+\eta\xi\varphi_\xi^2)(\varphi_{\eta\eta}\varphi_\xi^2+\varphi_\eta(-2\varphi_\xi\varphi_{\xi\eta}+\varphi_\eta\varphi_{\xi\xi}))}{((-1+\eta^2)\varphi_\eta^2+2\eta\xi\varphi_\eta\varphi_\xi+\xi^2\varphi_\xi^2)^2}=0,$$

which contains one of the equations – the Bateman equation. Similarly, the equations (18), (19) also yield the given equation. Finally, substituting (14) into (20), we have condition using the Wolfram Mathematica

$$
\begin{aligned}
&-2\eta\varphi_\eta^5+\varphi_\eta^4\Big(\eta^2(-1+\eta^2)\varphi_{\eta\eta}-2\xi\varphi_\xi-2\eta\xi\varphi_{\xi\eta}+\varphi_{\xi\xi}-\eta^2\varphi_{\xi\xi}+\\
&+2\eta^2\xi\varphi_{\xi\xi}-\xi^2\varphi_{\xi\xi}-\eta^2\xi^2\varphi_{\xi\xi}\Big)+\xi\varphi_\xi^4\Big(\eta^2(2+\xi)\varphi_{\eta\eta}-\\
&-2\varphi_\xi-2\eta\varphi_{\xi\eta}+\xi\varphi_{\xi\xi}-\xi^3\varphi_{\xi\xi}\Big)-2\eta\varphi_\eta^3\varphi_\xi\Big(-2\eta^2\xi\varphi_{\eta\eta}-2\varphi_\xi-\eta\varphi_{\xi\eta}+\\
&+\eta^3\varphi_{\xi\eta}+2\eta\xi\varphi_{\xi\eta}-\eta\xi^2\varphi_{\xi\eta}+\varphi_{\xi\xi}-\eta^2\varphi_{\xi\xi}-\xi\varphi_{\xi\xi}+\eta^2\xi\varphi_{\xi\xi}-\\
&-\xi^2\varphi_{\xi\xi}+\xi^3\varphi_{\xi\xi}\Big)+\varphi_\eta^2\varphi_\xi^2\Big(\eta^2(-1+\eta^2+2\xi+5\xi^2)\varphi_{\eta\eta}+4\xi\varphi_\xi+4\eta\varphi_{\xi\eta}-\\
&-4\eta^3\varphi_{\xi\eta}-4\eta^3\xi\varphi_{\xi\eta}-4\eta\xi^2\varphi_{\xi\eta}+4\eta\xi^3\varphi_{\xi\eta}-3\varphi_{\xi\xi}+3\eta^2\varphi_{\xi\xi}+\\
&+2\eta^2\xi\varphi_{\xi\xi}+4\xi^2\varphi_{\xi\xi}-5\eta^2\xi^2\varphi_{\xi\xi}-\xi^4\varphi_{\xi\xi}\Big)+\\
&2\varphi_\eta\varphi_\xi^3\Big(\eta(1+\xi)(-1+\eta^2+\xi^2)\varphi_{\eta\eta}-\eta\varphi_\xi-\\
&-(1+\xi)\Big((\eta^2-(-1+\xi)^2)(1+\xi)\varphi_{\xi\eta}+2\eta(-1+\xi)\xi\varphi_{\xi\xi}\Big)\Big)=0.
\end{aligned}
\tag{21}
$$

Equating coefficients to zero at the same degrees, we obtain

$$\xi: -4f^3g^2(f')^2+2fg^4(f')^2+2f^4gf'g'+f^4g^2f''-f^2g^4f''=0,$$

$$\eta: -4f^3g^2f'g'+2fg^4f'g'+2f^4g(g')^2+f^4g^2g''-f^2g^4g''=0,$$

$$\xi^2: 0,$$

$$
\begin{aligned}
\xi^2\eta: 8f^4g(f')^2+4f^5f'g'-8f^3g^2f'g'-4f^4g(g')^2-&\\
-2f^5gf''+2fg^5f''-f^6g''+f^2g^4g''=0,&
\end{aligned}
$$

$$\xi^2\eta^2: 0, \eta^2: 0,$$

$$
\begin{aligned}
\eta^2\xi: 4f^3g^2(f')^2+8f^4gf'g'-4f^2g^3f'g'-8f^3g^2(g')^2-&\\
-f^4g^2f''+g^6f''-2f^5gg''+2fg^5g''=0,&
\end{aligned}
$$

$$\xi^3: 4f^5(f')^2-4f^4gf'g'-f^6f''+f^2g^4f''=0,$$

$$\xi^3\eta: 0,$$

$$\xi^3\eta^2: -12f^3g^2(f')^2-8f^4gf'g'+12f^2g^3f'g+8f^3g^2(g')^2+$$

$$+6f^4g^2f'' - 6f^2g^4f'' + 4f^5gg'' - 4f^3g^3g'' = 0,$$

$$\xi^3\eta^3 : 0,$$

$$\eta^3 : 4f^3g^2f'g' - 4f^2g^3(g')^2 - f^4g^2g'' + g^6g'' = 0,$$

$$\eta^3\xi : 0,$$

$$\eta^3\xi^2 : -8f^2g^3(f')^2 - 12f^3g^2f'g' + 8fg^4f'g' + 12f^2g^3(g')^2 + \\ + 4f^3g^3f'' - 4fg^5f'' + 6f^4g^2g'' - 6f^2g^4g'' = 0,$$

$$\xi^4 : 0,$$

$$\xi^4\eta : -8f^4g(f')^2 - 2f^5f'g' + 8f^3g^2f'g' + 2f^4g(g')^2 + \\ + 4f^5gf'' - 4f^3g^3f'' + f^6g'' - f^4g^2g'' = 0,$$

$$\xi^4\eta^2 : 0, \xi^4\eta^3 : 0, \xi^4\eta^4 : 0, \eta^4 : 0,$$

$$\eta^4\xi : -2fg^4(f')^2 - 8f^2g^3f'g' + 2g^5f'g' + 8fg^4(g')^2 + \\ + f^2g^4f'' - g^6f'' + 4f^3g^3g'' - 4fg^5g'' = 0,$$

$$\eta^4\xi^2 : 0; \eta^4\xi^3 : 0; \eta^4\xi^4 : 0,$$

$$\xi^5 : -2f^5(f')^2 + 2f^4gf'g' + f^6f'' - f^4g^2f'' = 0,$$

$$\xi^5\eta : 0,$$

$$\eta^5 : -2fg^4f'g' + 2g^5(g')^2 + f^2g^4g'' - g^6g'' = 0.$$

This system of equations is satisfied for $g(\varphi) = \pm f(\varphi)$. Taking this equality into (3), we have for $g(\varphi) = f(\varphi)$

$$\varphi = F\Big(\frac{c}{\xi + \eta}\Big), \tag{22}$$

where $F$ is an arbitrary invertible function. And finally, to get the final solution of our system, we substitute (22) in (15) and take into account the formula (6).

We can formulate a proposition.

**Proposition.** *A self-similar solution of (1) is*

$$u(x,y,t) = \frac{1}{x}\frac{U_0}{\varphi}, v(x,y,t) = \frac{1}{x}\frac{V_0}{\varphi}, w(x,y,t) = \frac{1}{x}\frac{W_0}{\varphi}, z(x,y,t) = \frac{1}{x}\frac{Z_0}{\varphi},$$

*where $U_0, V_0, W_0, Z_0$ are defined by (14) and $\varphi$ satisfies the two-dimensional Bateman equation (2) and (21). Solution for $\varphi$ is*

$$\varphi(\xi,\eta) = F\Big(\frac{c}{\xi \pm \eta}\Big), c \in \mathbb{R}.$$

*Here $F$ is an arbitrary invertible function.*

Solutions of system (1) are for $g(\varphi) = f(\varphi)$

$$\begin{aligned} u(x,y,t) = 0, v(x,y,t) = 0, \\ w(x,y,t) = 0, z(x,y,t) = -\frac{1}{x}\frac{2c\varepsilon F'(\frac{c}{\xi+\eta})}{(\xi+\eta)^2 F(\frac{c}{\xi+\eta})} \end{aligned} \tag{23}$$

and for $g(\varphi) = -f(\varphi)$

$$\begin{aligned} u(x,y,t) = 0, v(x,y,t) = 0, \\ w(x,y,t) = -\frac{1}{x}\frac{2c\varepsilon F'(\frac{c}{\xi-\eta})}{(\xi-\eta)^2 F(\frac{c}{\xi-\eta})}, z(x,y,t) = 0, \end{aligned} \tag{24}$$

where $\xi = t/x, \eta = y/x$ are self-similar variables.

Comment. We give an example of a solution that is not described in the work [11]:

$$\begin{aligned} u(x,y,t) &= -\frac{14}{15} - \frac{16}{5(-\frac{71}{60} + \frac{7}{60}\sqrt{191}\tan(\frac{7}{120}\sqrt{191}(1+t+2x+3y)))}, \\ v(x,y,t) &= 3 + \frac{48}{5(-\frac{71}{60} + \frac{7}{60}\sqrt{191}\tan(\frac{7}{120}\sqrt{191}(1+t+2x+3y)))}, \\ w(x,y,t) &= 2 + \frac{12}{5(-\frac{71}{60} + \frac{7}{60}\sqrt{191}\tan(\frac{7}{120}\sqrt{191}(1+t+2x+3y)))}, \\ z(x,y,t) &= 1 - \frac{24}{5(-\frac{71}{60} + \frac{7}{60}\sqrt{191}\tan(\frac{7}{120}\sqrt{191}(1+t+2x+3y)))}. \end{aligned}$$

This solution is taken from [20].

## 4 Conclusion

We investigated the two-dimensional Broadwell system. We found the self-similar solutions for this system using the Bateman equation.

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*Received October 20, 2022*

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(100), 2022, Pages 41–55
ISSN 1024–7696, E-ISSN 2587–4322

# Construction of medial ternary self-orthogonal quasigroups

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**Abstract.** Algorithms for checking if a medial ternary quasigroup has a set of six triple-wise orthogonal principal parastrophes and a set of six triple-wise strongly orthogonal principal parastrophes are found. It is proved that $n$-ary strongly self-orthogonal linear (including medial) quasigroups do not exist when $n > 3$.

**Mathematics subject classification:** 05B15, 20N05, 20N15.
**Keywords and phrases:** medial quasigroup, orthogonal quasigroups, self-orthogonal quasigroup, strongly self-orthogonal quasigroup, central quasigroup, determinant, polynomial.

A quasigroup algebra is a universal algebra whose operations are invertible. Quasigroup algebras satisfying orthogonality problems have wide applications in algebra, combinatorics, cryptography, geometry, coding theory, etc., but the problem of their construction is still open, especially when their arities are greater than two and also for the case when the number of operations is larger than their arities.

Recall that the Cayley table of an $n$-ary operation of order $m$ is a hypercube of order $m$, i.e. an $\underbrace{m \times m \times \cdots \times m}_{n}$ array on $m$ distinct symbols. An operation is invertible (in other words, a quasigroup operation) if each line of the hypercube contains all symbols. An $n$-element set of $n$-ary operations are orthogonal if superimposing the corresponding hypercubes all possible $n$-tuples of the symbols are obtained. Another interpretation of orthogonal quasigroups as an MDS code is given, for instance, in [1].

In [2], the authors proposed an algorithm for constructing a big number of tuples of $n$-ary operations which generalizes the recursive algorithm introduced by G. Belyavskaya and G. Mullen [3] and improved by S. Markovsky and A. Mileva [4]. However, operations of these sets are not necessarily invertible. There are some algorithms for constructing orthogonal Latin hypercubes, for example, T. Evans [5] proposed a method for constructing using two sets of orthogonal Latin hypercubes of less dimensions, later M. Trenkler [6] suggested a method by a pair of orthogonal Latin squares.

Each $n$-ary quasigroup operation has $(n + 1)!$ parastrophes and $n!$ of them are principal. A quasigroup is called:

- *asymmetric* if all parastrophes are different;


DOI: https://doi.org/10.56415/basm.y2022.i3.p41

- *parastrophically orthogonal* if it has $n$ orthogonal parastrophes;
- *self-orthogonal* if it has $n$ orthogonal principal parastrophes;
- *totally parastrophically orthogonal* (briefly, *top*) if each set of $n$ different parastrophes is orthogonal.

*Note* that throughout the article we consider maximal sets of principal parastrophes, so by *self-orthogonality* we understand self-orthogonality with the additional condition that all principal parastrophes are different and triple-wise orthogonal.

There are a number of papers concerned with the parastrophic orthogonality of quasigroups. For example, V. D. Belousov [7] described all minimal identities which define varieties of parastrophically orthogonal quasigroups; G. Belyavskaya and T. Popovich [8] described conditions when a binary central asymmetric quasigroup is a top quasigroup. Consequently, they gave a method for constructing six pairwise orthogonal binary quasigroups (Latin squares).

P. Syrbu [9, 10] was the first who described series of self-orthogonal identities. Much later, in a joint paper, P. Syrbu and D. Cheban [11, 12] found a series of identities for ternary self-orthogonal quasigroups.

The main goal of this paper is to study methods for constructing orthogonal ternary quasigroups and Latin cubes. Throughout the article, we focus on medial asymmetric self-orthogonal ternary quasigroups with the restriction that all principal parastrophes are orthogonal and we found conditions under which these parastrophes are triple-wise orthogonal. Thus, having such a self-orthogonal ternary quasigroup we have 6 triple-wise orthogonal ternary quasigroups.

Here, Section 2 contains some introductory statements about medial quasigroups. The necessary and sufficient conditions for a medial ternary quasigroup to be self-orthogonal are given in Section 3, in particular these conditions are reduced to conditions of invertibility of 3 polynomials under a set of decomposition authomorphisms of the quasigroup. In Section 4, the necessary and sufficient conditions for a medial ternary quasigroup to be strongly self-orthogonal are found, these conditions are reduced to conditions of invertibility of 5 polynomials under a set of decomposition authomorphisms of the quasigroup. Also, we give some conclusions for self-orthogonal $n$-ary quasigroups.

## 1 Preliminaries

We should mention some necessary notions reformulating them for ternary case. Throughout the article, all operations are defined on a fixed set $Q$ called a *carrier* and $|Q| =: m < \infty$.

A ternary operation $f$ defined on $Q$ is called *invertible* and the pair $(Q; f)$ is called a *quasigroup* of the order $m$ if for every $a$, $b$ of $Q$ each of the terms $f(x, a, b)$, $f(a, x, b)$, $f(a, b, x)$ defines a permutation of $Q$.

To each ternary quasigroup $(Q; f)$ of order $m$ there corresponds a Latin cube of order $m$, i.e., a 3-dimensional array on $m$ distinct symbols from $Q$, each of which occurs exactly once in any line of the array.

**Orthogonality.** Orthogonality of ternary operations [3] is defined as follows:

1) a triplet of ternary operations $f_1$, $f_2$, $f_3$ is called *orthogonal* if for all $a_1, a_2, a_3 \in Q$, the system of equations

$$\begin{cases} f_1(x_1, x_2, x_3) = a_1, \\ f_2(x_1, x_2, x_3) = a_2, \\ f_3(x_1, x_2, x_3) = a_3 \end{cases}$$

has a unique solution;

2) a pair of ternary operations $f_1$, $f_2$ is called *orthogonal* if for all $a_1, a_2 \in Q$, the system of equations

$$\begin{cases} f_1(x_1, x_2, x_3) = a_1, \\ f_2(x_1, x_2, x_3) = a_2 \end{cases}$$

has $m$ solutions;

3) an operation $f$ is called *complete* if the equation

$$f_1(x_1, x_2, x_3) = a_1$$

has $m^2$ solutions for all $a_1 \in Q$.

Orthogonality of three (two) operations means that under superimposition of the corresponding cubes each triplet (respectively pair) of elements from $Q$ occurs exactly once (resp. $m$ times). Completeness of an $m$-ordered ternary operation means that each element of the carrier occurs exactly $m^2$ times in this cube. Therefore, completeness can be considered as a partial case of orthogonality.

A set of ternary operations $\Sigma = \{f_1, f_2, \ldots, f_s\}$ is called

– *orthogonal* if each triplet of distinct operations from $\Sigma$ is orthogonal, where $s \geqslant 3$;

– *pairwise orthogonal* if each pair of distinct operations from $\Sigma$ is orthogonal, where $s \geqslant 2$.

A set of ternary operations $f_1, f_2, f_3$ on a set $Q$ is called *strongly orthogonal* if the set of operations $\{f_1, f_2, f_3, e_1, e_2, e_3\}$ is triple-wise orthogonal, where $e_1$, $e_2$, $e_3$ are defined by the equalities

$$e_1(x_1, x_2, x_3) = x_1, \quad e_2(x_1, x_2, x_3) = x_2, \quad e_3(x_1, x_2, x_3) = x_3.$$

The operations $e_1$, $e_2$, $e_3$ are called the *1st selector*, the *2nd selector* and the *3rd selector* respectively.

Recall some definitions from [2] specifying it for ternary case.

Let $f$ be a ternary operation defined on a set $Q$. Binary operations $f_{(c,\{1,2\})}$, $f_{(b,\{1,3\})}$ and $f_{(a,\{2,3\})}$ which are defined by

$$f_{(c,\{1,2\})} := f(x_1, x_2, c), \quad f_{(b,\{1,3\})} := f(x_1, b, x_3), \quad f_{(a,\{2,3\})} := f(a, x_2, x_3)$$

are called $\{1,2\}$-, $\{1,3\}$-, $\{2,3\}$-retracts of $f$ by $a$, $b$, $c \in Q$.

Let $\delta := \{i, j\} \subset \{1, 2, 3\}$, operations $f$ and $g$ be ternary operations defined on $Q$ and $a, b \in Q$. Binary operations $f_{(a,\delta)}$ and $g_{(b,\delta)}$ are called *similar* $\delta$-retracts of $f$ and $g$ if $a = b$.

**Definition 1.** [2] Let $\delta \subseteq \{1, 2, 3\}$. A set of ternary operations is called *$\delta$-retractly orthogonal* if all tuples of similar $\delta$-retracts of these operations are orthogonal.

If $\delta = \{i\}$, then $\delta$-retract orthogonality of $f$ degenerates into its $i$-invertibility. If $\delta = \{1, 2, 3\}$, then retract orthogonality of ternary operations $f_1, \dots, f_n$ is orthogonality.

The next statement is another form of Theorem 3 [13] for ternary quasigroups.

**Theorem 1.** *An orthogonal set of ternary quasigroups $f_1, f_2, \dots, f_t$ defined on a set $Q$, where $t \geqslant 1$, is strongly orthogonal if and only if it is $\{i, j\}$-retractly orthogonal for each $i, j \in \{1, 2, 3\}$, where $i \neq j$.*

Let $f_1$, $f_2$, $f_3$ be ternary operations defined on $Q$ and

$$\bar\theta(x_1, x_2, x_3) := (f_1(x_1, x_2, x_3);\ f_2(x_1, x_2, x_3);\ f_3(x_1, x_2, x_3)).$$

Therefore, the mapping $\bar\theta \mapsto (f_1, f_2, f_3)$ defines a one-to-one correspondence between the set of all transformations of the set $Q^3$ and the set of all triplets of ternary operations defined on $Q$. Since a transformation $\bar\theta$ is a permutation of $Q^3$ if and only if the corresponding triplet $(f_1, f_2, f_3)$ of operations are orthogonal, there are $(m^3)!$ ordered triplets of orthogonal ternary operations of order $m$.

Let $f$ be an operation defined on $Q$ and $\gamma$ be a permutation of $Q$. Then the operation $\alpha f$ being defined by

$$(\alpha f)(x, y, z) := \alpha\left(f(x, y, z)\right)$$

is called a *torsion* of $f$.

**Proposition 1.** *If a set of operations is orthogonal, then their torsions are also orthogonal.*

*Proof.* If a triplet $(a, b, c)$ takes each value in the set $Q^3$ and $\alpha$, $\beta$, $\gamma$ are permutations of $Q$, then the triplet $(\alpha^{-1}(a), \beta^{-1}(b), \gamma^{-1}(c))$ also takes each value in the set $Q^3$. Therefore, the statement '*for all $a$, $b$, $c$ the triplet of operations $f_1$, $f_2$, $f_3$ is orthogonal*' means that '*for all $a$, $b$, $c$ of $Q$ the system*

$$\begin{cases} f_1(x_1, x_2, x_3) = \alpha^{-1}(a), \\ f_2(x_1, x_2, x_3) = \beta^{-1}(b), \\ f_3(x_1, x_2, x_3) = \gamma^{-1}(c) \end{cases}$$

*has a unique solution*'. Since this system is equivalent to the system

$$\begin{cases} (\alpha f_1)(x_1, x_2, x_3) = a, \\ (\beta f_2)(x_1, x_2, x_3) = b, \\ (\gamma f_3)(x_1, x_2, x_3) = c, \end{cases}$$

the triplet $\alpha f_1$, $\beta f_2$, $\gamma f_3$ is orthogonal. □

**Parastrophes.** For every permutation $\sigma \in S_4$, a *$\sigma$-parastrophe* ${}^{\sigma}f$ of an invertible ternary operation $f$ is defined by

$${}^{\sigma}f(x_{1\sigma}, x_{2\sigma}, x_{3\sigma}) = x_{4\sigma} :\Longleftrightarrow f(x_1, x_2, x_3) = x_4.$$

This relationship is equivalent to

$${}^{\sigma}f(x_1, x_2, x_3) = x_4 :\Longleftrightarrow f(x_{1\sigma^{-1}}, x_{2\sigma^{-1}}, x_{3\sigma^{-1}}) = x_{4\sigma^{-1}}. \tag{1}$$

It is easy to verify that the formula

$${}^{\sigma}({}^{\tau}f) = {}^{\sigma\tau}f \tag{2}$$

holds for all permutations $\sigma, \tau \in S_4$ and for each invertible operation $f$.

A $\sigma$-parastrophe is called:

- an *$i$-th division* if $\sigma = (i4)$ for $i = 1, 2, 3, 4$, where $(44) := \iota$ and ${}^{(44)}f := {}^{\iota}f = f$;
- an *identical division* if $\sigma = \iota$;
- a *principal parastrophe* if $4\sigma = 4$.

The formula (1) implies that any principal $\sigma$-parastrophe can be defined by

$${}^{\sigma}f(x_1, x_2, x_3) = f(x_{1\sigma^{-1}}, x_{2\sigma^{-1}}, x_{3\sigma^{-1}}). \tag{3}$$

Therefore, a ternary operation is invertible if and only if it has four divisions. Each ternary operation has at most $4! = 24$ parastrophes. If all parastrophes are pairwise different, the operation is called *asymmetric*. Four of the parastrophes are divisions, one of them is *identical*. Each operation has $3! = 6$ principal parastrophes.

Consider the subgroup $S_3 := \{\sigma \mid 4\sigma = 4\}$ of the symmetric group $S_4$ and the right cosets of $S_3$:

$$S_4 = S_3(14) \cup S_3(24) \cup S_3(34) \cup S_3(44).$$

If $\tau \in S_3(i4)$, i.e. $\tau = \sigma(i4)$ for some $\sigma \in S_3$, then

$$i = 4(i4) = 4\sigma(i4) = 4\tau$$

and

$$\begin{aligned} {}^{\tau}\!f(x_1,x_2,x_3) &= {}^{\sigma(i4)}\!f(x_1,x_2,x_3) \overset{(2)}{=} {}^{\sigma}\big({}^{(i4)}\!f\big)(x_1,x_2,x_3) = \\ &\overset{(3)}{=} {}^{(i4)}\!f(x_{1\sigma^{-1}},x_{2\sigma^{-1}},x_{3\sigma^{-1}}) \end{aligned}$$

and so

$${}^{\tau}\!f(x_1,x_2,x_3) = {}^{(i4)}\!f(x_{1\sigma^{-1}},x_{2\sigma^{-1}},x_{3\sigma^{-1}}).$$

Hence,

$${}^{(i4)}\!f(x_1,x_2,x_3) = {}^{\tau}\!f(x_{1\sigma},x_{2\sigma},x_{3\sigma}).$$

Therefore, $i$-th division ${}^{(i4)}\!f$ exists, i.e. the operation $f$ is $i$-invertible (note that each ternary operation is 4-invertible). Moreover, for every $\kappa \in S_3(i4)$ there exists $\kappa$-parastrophe.

Thus, the following theorem has been proved.

**Theorem 2.** *Let $\tau \in S_4$ and $\tau \in S_3(i4)$, where $i := 4\tau$ and $\sigma := \tau(i4) \in S_3$. Then $\tau$-parastrophe of a ternary invertible operation is its principal $\sigma$-parastrophe of $i$-th division of the operation.*

## 2 Medial quasigroups

A pair $(Q;\Omega)$ is called a *ternary quasigroup algebra* if all elements from $\Omega$ are ternary invertible operations defined on $Q$.

A ternary quasigroup algebra $(Q;\Omega)$ is called:

- *medial* [14] if each pair $f$, $g$ of operations from $\Omega$ satisfies the *identity of mediality*:

$$\begin{aligned} &f(g(x_{11},x_{12},x_{13}),g(x_{21},x_{22},x_{23}),g(x_{31},x_{32},x_{33})) = \\ &\qquad g(f(x_{11},x_{21},x_{31}),f(x_{12},x_{22},x_{32}),f(x_{13},x_{23},x_{33})). \end{aligned} \tag{4}$$

- *abelian* [14] if it is medial and has a one-element subalgebra, i.e. it has an element $0 \in Q$ such that $f(0,0,0) = 0$ for all operations from $\Omega$. The abelian algebra is denoted by $(Q;\Omega,0)$.

The theorem given below follows from more general statement [14, Theorem 3].

**Theorem 3.** *A ternary quasigroup algebra $(Q;\Omega)$ is medial if and only if there exists an abelian group $(Q;+,0)$, a set $E$ of pairwise commuting automorphisms of the group and a set $A \subseteq Q$ of elements such that for each operation $g \in \Omega$ there exist automorphisms $\psi_1$, $\psi_2$, $\psi_3$ from $E$ and elements $a_g \in A$ such that*

$$g(y_1,y_2,y_3) = \psi_1 y_1 + \psi_2 y_2 + \psi_3 y_3 + a_g,$$

$$\mu_g(a_h) = \mu_h(a_g) \tag{5}$$

*for all $h \in \Omega$, where $\mu_g := \psi_1 + \psi_2 + \psi_3 - \iota$.*

Hence, a ternary quasigroup $(Q; f)$ is called medial if the identity (4) with $f = g$ holds. Since (5) with $f = g$ is evident, the following assertion holds.

**Corollary 1** ([15]). *A quasigroup $(Q; f)$ is medial if and only if there exists an abelian group $(Q; +)$ such that*

$$f(x_1, x_2, x_3) = \varphi_1 x_1 + \varphi_2 x_2 + \varphi_3 x_3 + a, \tag{6}$$

*where $\varphi_1$, $\varphi_2$, $\varphi_3$ are pairwise commuting automorphisms of $(Q; +)$ and $a \in Q$.*

These automorphisms are called *coefficients*, the element $a$ is a *free term*, and $(Q; +)$ is a *decomposition group* of $f$.

**Corollary 2.** *A ternary quasigroup algebra $(Q; \Omega, 0)$ is abelian if and only if there exists an abelian group $(Q; +, 0)$ and a set $E$ of pairwise commuting automorphisms of $(Q; +, 0)$ such that for every operation $g \in \Omega$ there exist automorphisms $\psi_1$, $\psi_2$, $\psi_3$ from $E$ such that*

$$g(y_1, y_2, y_3) = \psi_1 y_1 + \psi_2 y_2 + \psi_3 y_3.$$

**Lemma 1.** *Let $(Q; f)$ be a medial quasigroup with (6) and $J(x) := -x =: \varphi_4(x)$. Then for each $\tau \in S_4$*

$${}^{\tau}f(x_1, x_2, x_3) = J\varphi_{4\tau}^{-1}\varphi_{1\tau}(x_1) + J\varphi_{4\tau}^{-1}\varphi_{2\tau}(x_2) + J\varphi_{4\tau}^{-1}\varphi_{3\tau}(x_3) + b, \tag{7}$$

*where $b := J\varphi_{4\tau}^{-1}(a)$.*

*Proof.* Suppose $f$ is a medial quasigroup and is defined by (6). By virtue of (1),

$$\varphi_1(x_{1\tau^{-1}}) + \varphi_2(x_{2\tau^{-1}}) + \varphi_3(x_{3\tau^{-1}}) + a = x_{4\tau^{-1}},$$

i.e.,

$$\varphi_1(x_{1\tau^{-1}}) + \varphi_2(x_{2\tau^{-1}}) + \varphi_3(x_{3\tau^{-1}}) + \varphi_4(x_{4\tau^{-1}}) + a = 0.$$

As the group $(Q; +)$ is commutative, the equality is equivalent to

$$\varphi_{1\tau}(x_1) + \varphi_{2\tau}(x_2) + \varphi_{3\tau}(x_3) + \varphi_{4\tau}(x_4) + a = 0.$$

Therefrom,

$$x_4 = J\varphi_{4\tau}^{-1}\varphi_{1\tau}(x_1) + J\varphi_{4\tau}^{-1}\varphi_{2\tau}(x_2) + J\varphi_{4\tau}^{-1}\varphi_{3\tau}(x_3) + J\varphi_{4\tau}^{-1}(a).$$

Thus, (7) holds. $\square$

**Corollary 3.** *Any parastrophe of a medial ternary quasigroup is medial.*

**Corollary 4.** *Let $(Q; f, 0)$ be an abelian quasigroup with (6) and $J(x) := -x =: \varphi_4(x)$. Then for each $\tau \in S_4$,*

$${}^{\tau}f(x_1, x_2, x_3) = J\varphi_{4\tau}^{-1}\varphi_{1\tau}(x_1) + J\varphi_{4\tau}^{-1}\varphi_{2\tau}(x_2) + J\varphi_{4\tau}^{-1}\varphi_{3\tau}(x_3). \tag{8}$$

**Lemma 2.** *Let $(Q; f)$ be a medial ternary quasigroup $(Q; f)$ with* (6) *and $\tau_1$, $\tau_2$, $\tau_3 \in S_4$. The parastrophes ${}^{\tau_1}f$, ${}^{\tau_2}f$, ${}^{\tau_3}f$ are orthogonal if and only if the determinant*

$$\begin{vmatrix} \varphi_{1\tau_1} & \varphi_{2\tau_1} & \varphi_{3\tau_1} \\ \varphi_{1\tau_2} & \varphi_{2\tau_2} & \varphi_{3\tau_2} \\ \varphi_{1\tau_3} & \varphi_{2\tau_3} & \varphi_{3\tau_3} \end{vmatrix} \tag{9}$$

*is an automorphism of the group $(Q; +)$, where $\varphi_4 := J$.*

*Proof.* According to Proposition 1, orthogonality of the parastrophes ${}^{\tau_1}f$, ${}^{\tau_2}f$, ${}^{\tau_3}f$ is equivalent to orthogonality of their torsions

$$L_a^{-1}\varphi_{4\tau_1}J\,({}^{\tau_1}f)\,, \qquad L_a^{-1}\varphi_{4\tau_2}J\,({}^{\tau_2}f)\,, \qquad L_a^{-1}\varphi_{4\tau_3}J\,({}^{\tau_3}f)\,.$$

By Lemma 1,

$$\begin{aligned} L_a^{-1}\varphi_{4\tau_1}J\,({}^{\tau_1}f)\,(x_1,x_2,x_3) &= \varphi_{1\tau_1}(x_1)+\varphi_{2\tau_1}(x_2)+\varphi_{3\tau_1}(x_3), \\ L_a^{-1}\varphi_{4\tau_2}J\,({}^{\tau_2}f)\,(x_1,x_2,x_3) &= \varphi_{1\tau_2}(x_1)+\varphi_{2\tau_2}(x_2)+\varphi_{3\tau_2}(x_3), \\ L_a^{-1}\varphi_{4\tau_3}J\,({}^{\tau_3}f)\,(x_1,x_2,x_3) &= \varphi_{1\tau_3}(x_1)+\varphi_{2\tau_3}(x_2)+\varphi_{3\tau_3}(x_3), \end{aligned}$$

where $L_a(x) := x+a$. Thus, the parastrophes ${}^{\tau_1}f$, ${}^{\tau_2}f$, ${}^{\tau_3}f$ are orthogonal if and only if the system of equations

$$\begin{cases} \varphi_{1\tau_1}(x_1)+\varphi_{2\tau_1}(x_2)+\varphi_{3\tau_1}(x_3) = b_1, \\ \varphi_{1\tau_2}(x_1)+\varphi_{2\tau_2}(x_2)+\varphi_{3\tau_2}(x_3) = b_2, \\ \varphi_{1\tau_3}(x_1)+\varphi_{2\tau_3}(x_2)+\varphi_{3\tau_3}(x_3) = b_3 \end{cases}$$

has a unique solution for all $b_1$, $b_2$, $b_3$ in $Q$. Since the automorphisms $\varphi_1$, $\varphi_2$, $\varphi_3$, $\varphi_4$ of the commutative group $(Q; +)$ pairwise commute, they generate a commutative ring $K$. Therefore, this system has a unique solution if and only if the determinant (9) is invertible, i.e. it is an automorphism of the group $(Q; +)$. □

## 3 Self-orthogonal medial ternary quasigroups

A self-orthogonal ternary operation $f$ has 6 triple-wise orthogonal operations, i.e. 20 triplets of orthogonal principal parastrophes of $f$. Therefore according to Lemma 2, to check self-orthogonality of an invertible medial operation $f$, we have to examine invertibility of 20 determinants, which can be described by polynomials with some conditions.

**Definition 2.** A polynomial $p$ over a commutative ring $K$ will be called *invertible-valued* over a subset $H \subseteq K$ if $p(a, b, c)$ is invertible in $K$ whenever $a$, $b$, $c$ are in $H$.

**Lemma 3.** *A ternary medial quasigroup* $(Q, f)$*, where* $f$ *is defined by* (6)*, is self-orthogonal if and only if the polynomials*

$$\begin{gathered}\gamma_1-\gamma_2, \qquad \gamma_1+\gamma_2+\gamma_3,\\ \gamma_1^2+\gamma_2^2+\gamma_3^2-\gamma_1\gamma_2-\gamma_1\gamma_3-\gamma_2\gamma_3\end{gathered} \tag{10}$$

*are invertible-valued over the set of automorphisms* $\{\varphi_1,\, \varphi_2,\, \varphi_3\}$ *of the group* $(Q, +)$.

*Proof.* Suppose that $(Q; f)$ is a medial ternary quasigroup and (6) is its decomposition which exists by Corollary 1. Since the automorphisms $\varphi_1$, $\varphi_2$, $\varphi_3$ of the group $(Q; +)$ pairwise commute, they generate a subring $K$ in the ring of all endomorphisms of the commutative group $(Q; +)$. According to Lemma 2, orthogonality of principal parastrophes ${}^{\tau_1}f$, ${}^{\tau_2}f$, ${}^{\tau_3}f$ of the operation $f$ is equivalent to invertibility of the determinant (9) in the ring $K$, i.e. the determinant should be an automorphism of the group $(Q; +)$. Self-orthogonality means that all 20 determinants of the form (9) with conditions $\tau_1$, $\tau_2$, $\tau_3 \in S_3$ should be invertible. In other words, the polynomial

$$d := d_{\vec{\tau}}(\gamma_1,\gamma_2,\gamma_3) := \begin{vmatrix} \gamma_{1\tau_1} & \gamma_{2\tau_1} & \gamma_{3\tau_1}\\ \gamma_{1\tau_2} & \gamma_{2\tau_2} & \gamma_{3\tau_2}\\ \gamma_{1\tau_3} & \gamma_{2\tau_3} & \gamma_{3\tau_3} \end{vmatrix} \tag{11}$$

is invertible-valued over the set $\{\varphi_1,\varphi_2,\varphi_3\}$ in the ring $K$, where $\gamma_1$, $\gamma_2$, $\gamma_3$ are variables, $\vec{\tau} := (\tau_1,\tau_2,\tau_3)$.

Two polynomials are supposed to be equivalent if they are invertible simultaneously and we will denote this fact by $\sim$.

Now permute columns in (11) to get sequence $\gamma_1$, $\gamma_2$, $\gamma_3$ in the first row and permute the second and third rows to get the determinant

$$d \sim \begin{vmatrix} \gamma_1 & \gamma_2 & \gamma_3\\ \gamma_{1\nu} & \gamma_{2\nu} & \gamma_{3\nu}\\ \gamma_{1\tau} & \gamma_{2\tau} & \gamma_{3\tau} \end{vmatrix}$$

with $1 \leqslant 1\nu \leqslant 1\tau$, where $\nu, \tau \in S_3$. Add the first and second columns to the third one:

$$d \sim (\gamma_1+\gamma_2+\gamma_3) \begin{vmatrix} \gamma_1 & \gamma_2 & \iota\\ \gamma_{1\nu} & \gamma_{2\nu} & \iota\\ \gamma_{1\tau} & \gamma_{2\tau} & \iota \end{vmatrix}$$

Therefore,

$$d \sim \begin{vmatrix} \gamma_1 & \gamma_2 & \iota\\ \gamma_{1\nu} & \gamma_{2\nu} & \iota\\ \gamma_{1\tau} & \gamma_{2\tau} & \iota \end{vmatrix} \tag{12}$$

under the condition that the polynomial $\gamma_1+\gamma_2+\gamma_3$ is invertible.

No column has three repetitions of a variable, otherwise $d$ has two equal rows and consequently $d = 0$. If $d$ has two repetitions of a variable, then rename it by $\gamma_1$. Permuting rows and columns, we obtain (12) with $1\nu = 1$. In the second row of this determinant, $2\nu = 3$, otherwise the first and second rows coincide. Multiply the first row by $-\iota$ and add it to the second row:

$$d \sim \begin{vmatrix} \gamma_1 & \gamma_2 & \iota \\ 0 & \gamma_3 - \gamma_2 & 0 \\ \gamma_{1\tau} & \gamma_{2\tau} & \iota \end{vmatrix} = (\gamma_3 - \gamma_2)(\gamma_1 - \gamma_{1\tau}).$$

Thus, invertibility of $d$ is equivalent to the fact that both polynomials of the form $\gamma_1 + \gamma_2 + \gamma_3$ and $\gamma_1 - \gamma_2$ are invertible-valued over the set $\{\varphi_1, \varphi_2, \varphi_3\}$.

At last, suppose the variables are different in each row and in each column. Then we obtain

$$d \sim \begin{vmatrix} \gamma_1 & \gamma_2 & \iota \\ \gamma_2 & \gamma_3 & \iota \\ \gamma_3 & \gamma_1 & \iota \end{vmatrix} = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 - \gamma_1\gamma_2 - \gamma_1\gamma_3 - \gamma_2\gamma_3.$$

Consequently, the lemma has been proved. □

**Theorem 4.** *A ternary medial quasigroup $(Q, f)$, where $f$ is defined by* (6)*, is self-orthogonal if and only if the mappings*

$$\begin{gathered} \varphi_1 - \varphi_2, \qquad \varphi_1 - \varphi_3, \qquad \varphi_2 - \varphi_3, \qquad \varphi_1 + \varphi_2 + \varphi_3, \\ (\varphi_1 + \varphi_2 + \varphi_3)^2 - 3(\varphi_1\varphi_2 + \varphi_1\varphi_3 + \varphi_2\varphi_3) \end{gathered} \tag{13}$$

*are automorphisms of the group $(Q, +)$.*

*Proof.* The polynomial $\gamma_1 - \gamma_2$ is invertible-valued over the automorphisms $\varphi_1$, $\varphi_2$, $\varphi_3$ if and only if the endomorphisms

$$\varphi_1 - \varphi_2, \qquad \varphi_1 - \varphi_3, \qquad \varphi_2 - \varphi_3$$

are automorphisms. The polynomials

$$\gamma_1 + \gamma_2 + \gamma_3, \qquad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 - \gamma_1\gamma_2 - \gamma_1\gamma_3 - \gamma_2\gamma_3$$

are symmetric and contain three variables, then they are invertible-valued if and only if the endomorphisms $\varphi_1 + \varphi_2 + \varphi_3$ and

$$\varphi_1^2 + \varphi_2^2 + \varphi_3^2 - (\varphi_1\varphi_2 + \varphi_1\varphi_3 + \varphi_2\varphi_3) = (\varphi_1 + \varphi_2 + \varphi_3)^2 - 3(\varphi_1\varphi_2 + \varphi_1\varphi_3 + \varphi_2\varphi_3)$$

are automorphisms of the group $(Q; +)$. □

**Corollary 5.** *If at least two coefficients of a central quasigroup coincide, then the quasigroup can not be self-orthogonal.*

## 4 Strongly self-orthogonal medial quasigroups

The concept of strong orthogonality of the given ternary quasigroups which follows from Theorem 1 with the restriction of mediality is: a triplet of ternary medial quasigroups is strongly orthogonal if for all $s \in \{1, 2, 3\}$, all minors of order $s$ of the corresponding determinant are invertible.

**Lemma 4.** *A ternary medial quasigroup* $(Q, f)$*, where* $f$ *is defined by* (6)*, is strongly self-orthogonal if and only if the polynomials* (10) *and the polynomials*

$$\gamma_1\gamma_2 - \gamma_3^2, \qquad \gamma_1 + \gamma_2 \tag{14}$$

*are invertible-valued over the set of automorphisms* $\{\varphi_1, \varphi_2, \varphi_3\}$ *of the group* $(Q, +)$*.*

*Proof.* According to Theorem 1, the necessary and sufficient condition for a set of quasigroups to be strongly orthogonal is that each its subset is retractly orthogonal. For the ternary quasigroups it means that

1. Each operation has to be a quasigroup;
2. Each pair of quasigroups has to be $\{1, 2\}$-, $\{1, 3\}$-, $\{2, 3\}$-retractly orthogonal;
3. Each triplet of quasigroups has to be orthogonal.

Conditions for satisfying item 3 are found in Lemma 3. Consequently, it remains to consider item 2, i.e. invertibility conditions for minors of order 2 of determinant (11).

If every column contains each of the variables $\gamma_1$, $\gamma_2$, $\gamma_3$, then the determinant is a Latin square of order 3. Therefore permuting rows and columns, we obtain the determinant

$$\begin{vmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_2 & \gamma_3 & \gamma_1 \\ \gamma_3 & \gamma_1 & \gamma_2 \end{vmatrix}$$

which is invertible simultaneously with (11). It is obvious that this determinant contains only one form of minors of order 2 up to sign $J$ and relabeling of the variables: namely, it is $\gamma_1\gamma_2 - \gamma_3^2$.

Suppose that not every column contains each of the variables $\gamma_1$, $\gamma_2$, $\gamma_3$. Therefore, one of the variables repeats. Let us label it by means of $\gamma_1$. Then permuting rows and columns, we obtain the determinant

$$d = \begin{vmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_1 & \gamma_i & \gamma_j \\ \gamma_{1\nu} & \gamma_{2\nu} & \gamma_{3\nu} \end{vmatrix}$$

each row of which contains different variables. If $i = 2$, then $j = 3$ and thence the first and second rows are equal and so $d = 0$. It means that $i = 3$ and $j = 2$. If $1\nu = 1$, then two rows coincide and so the determinant is 0. Therefore, $1\nu \neq 1$.

Let $1\nu = 3$. Now relabel variable $\gamma_2$ by $\gamma_3$, $\gamma_3$ by $\gamma_2$ and then permute the second and third columns. As a result, we obtain determinants with $1\nu = 2$. There are two such determinants:

$$d_1 = \begin{vmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_1 & \gamma_3 & \gamma_2 \\ \gamma_2 & \gamma_1 & \gamma_3 \end{vmatrix}, \qquad d_2 = \begin{vmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_1 & \gamma_3 & \gamma_2 \\ \gamma_2 & \gamma_3 & \gamma_1 \end{vmatrix}.$$

They are equivalent. Indeed, relabel variable $\gamma_1$ by $\gamma_3$, $\gamma_3$ by $\gamma_1$ in $d_2$ and then permute the rows and the columns:

$$d_2 \sim \begin{vmatrix} \gamma_3 & \gamma_2 & \gamma_1 \\ \gamma_3 & \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_1 & \gamma_3 \end{vmatrix} \sim \begin{vmatrix} \gamma_3 & \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_1 & \gamma_3 \\ \gamma_3 & \gamma_2 & \gamma_1 \end{vmatrix} \sim \begin{vmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_1 & \gamma_3 & \gamma_2 \\ \gamma_2 & \gamma_1 & \gamma_3 \end{vmatrix} = d_1.$$

Thus, it is enough to consider all minors of the determinant $d_1$.

The minors of the first and second rows are

$$\begin{vmatrix} \gamma_1 & \gamma_2 \\ \gamma_1 & \gamma_3 \end{vmatrix} = \gamma_1(\gamma_3 - \gamma_2), \qquad \begin{vmatrix} \gamma_1 & \gamma_3 \\ \gamma_1 & \gamma_2 \end{vmatrix} = \gamma_1(\gamma_2 - \gamma_3),$$

$$\begin{vmatrix} \gamma_2 & \gamma_3 \\ \gamma_3 & \gamma_2 \end{vmatrix} = \gamma_2^2 - \gamma_3^2 = (\gamma_2 + \gamma_3)(\gamma_2 - \gamma_3).$$

The minors of the first and third rows are

$$\begin{vmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_1 \end{vmatrix} = \gamma_1^2 - \gamma_2^2 = (\gamma_1 + \gamma_2)(\gamma_1 - \gamma_2), \qquad \begin{vmatrix} \gamma_1 & \gamma_3 \\ \gamma_2 & \gamma_3 \end{vmatrix} = \gamma_3(\gamma_1 - \gamma_2),$$

$$\begin{vmatrix} \gamma_2 & \gamma_3 \\ \gamma_1 & \gamma_3 \end{vmatrix} = \gamma_3(\gamma_2 - \gamma_1).$$

The minors of the second and third rows are

$$\begin{vmatrix} \gamma_1 & \gamma_3 \\ \gamma_2 & \gamma_1 \end{vmatrix} = \gamma_1^2 - \gamma_2\gamma_3, \qquad \begin{vmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_3 \end{vmatrix} = \gamma_1\gamma_3 - \gamma_2^2,$$

$$\begin{vmatrix} \gamma_3 & \gamma_2 \\ \gamma_1 & \gamma_3 \end{vmatrix} = \gamma_3^2 - \gamma_1\gamma_2.$$

Consequently, strong self-orthogonality of $f$ is equivalent to the fact that polynomials (10) and (14) are invertible-valued over $\{\varphi_1, \varphi_2, \varphi_3\}$. □

**Theorem 5.** *A ternary medial quasigroup $(Q, f)$, where $f$ is defined by* (6), *is strongly self-orthogonal if and only if the mappings* (13) *and*

$$\begin{array}{lll} \varphi_2\varphi_3 - \varphi_1^2, & \varphi_1\varphi_3 - \varphi_2^2, & \varphi_1\varphi_2 - \varphi_3^2, \\ \varphi_1 + \varphi_2, & \varphi_1 + \varphi_3, & \varphi_2 + \varphi_3 \end{array} \tag{15}$$

*are automorphisms of the group $(Q, +)$.*

*Proof.* The first part of the theorem follows from Theorem 4.

The polynomial $\gamma_1 + \gamma_2$ is invertible-valued over the automorphisms $\varphi_1$, $\varphi_2$, $\varphi_3$ if and only if the endomorphisms

$$\varphi_1 + \varphi_2, \qquad \varphi_1 + \varphi_3, \qquad \varphi_2 + \varphi_3$$

are automorphisms. The polynomial $\gamma_1\gamma_2 - \gamma_3^2$ is invertible-valued if and only if the endomorphisms

$$\varphi_1\varphi_2 - \varphi_3^2, \qquad \varphi_1\varphi_3 - \varphi_2^2, \qquad \varphi_2\varphi_3 - \varphi_1^2$$

are automorphisms of the group $(Q; +)$. □

## Conclusion

Let $\mathbb{Z}_m$ be a ring of integers modulo $m$. Consider a ternary operation $f$ with decomposition

$$f(x, y, z) := x + 2y + 3z.$$

If $m$ is relatively prime to 6, then $(\mathbb{Z}_m; f)$ is a quasigroup.

Let us now consider conditions (13) and (15) for $f$. Conditions (13) are

$$2 - 1 = 1, \quad 3 - 1 = 2, \quad 3 - 2 = 1, \quad 1 + 2 + 3 = 6,$$
$$6^2 - 3 \cdot (1 \cdot 2 + 1 \cdot 3 + 2 \cdot 3) = 36 - 33 = 3.$$

Conditions (15) are

$$1 \cdot 2 - 3^2 = -7, \quad 1 \cdot 3 - 2^2 = -1, \quad 2 \cdot 3 - 1^2 = 5,$$
$$1 + 2 = 3, \quad 1 + 3 = 4, \quad 2 + 3 = 5.$$

According to Theorem 4 and Theorem 5, we have three conclusions:

1. $(\mathbb{Z}_m; f)$ is a self-orthogonal ternary quasigroup if $m$ is not divisible by 6;
2. $(\mathbb{Z}_m; f)$ is a self-orthogonal ternary quasigroup, but it is not strongly self-orthogonal if $m$ is not divisible by 6 and $m$ is divisible by 5 or 7;
3. $(\mathbb{Z}_m; f)$ is a strongly self-orthogonal ternary quasigroup if $m$ is not divisible by 2, 3, 5 and 7.

**Corollary 6.** *$n$-ary strongly self-orthogonal linear quasigroups do not exist if $n > 3$.*

*Proof.* Suppose that $(G; h)$ is an $n$-ary strongly self-orthogonal linear quasigroup, where $n > 3$, i.e. there exists a group $(G; +)$, its automorphisms $\varphi_1, \ldots, \varphi_n$ and an element $a \in G$ such that

$$h(x_1, x_2, \ldots, x_n) = \varphi_1 x_1 + \varphi_2 x_2 + \ldots + \varphi_n x_n + a.$$

A decomposition of its principal $\sigma$-parastrophe ${}^{\sigma}h$ with the conditions $1\sigma = 1$, $2\sigma = 2$ is

$$ {}^{\sigma}h(x_1, x_2, x_3, \dots, x_n) = \varphi_1 x_1 + \varphi_2 x_2 + \varphi_3 x_{3\sigma^{-1}} + \dots + \varphi_n x_{n\sigma^{-1}} + a. $$

Strong self-orthogonality of $h$ implies that, in particular, any $\{1, 2\}$-retracts of $h$ and ${}^{\sigma}h$ are orthogonal, i.e. for every $b_1$, $b_2$ and for all $a_3$, …, $a_n$, the system

$$ \begin{cases} \varphi_1 x_1 + \varphi_2 x_2 + \varphi_3 a_3 + \dots + \varphi_n a_n + a = b_1, \\ \varphi_1 x_1 + \varphi_2 x_2 + \varphi_3 a_{3\sigma^{-1}} + \dots + \varphi_n a_{n\sigma^{-1}} + a = b_2 \end{cases} $$

has a unique solution. Therefore, the system

$$ \begin{cases} \varphi_1 x_1 + \varphi_2 x_2 = c_1, \\ \varphi_1 x_1 + \varphi_2 x_2 = c_2 \end{cases} $$

has a unique solution for all $c_1$ and $c_2$ from $G$, in particular, when $c_1 \neq c_2$, which is a contradiction. This contradiction shows that an $n$-ary $(n > 3)$ linear quasigroup $(G; h)$ with the property of strong self-orthogonality does not exist. □

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*Received November 12, 2022*

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BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(100), 2022, Pages 56–94
ISSN 1024–7696, E-ISSN 2587–4322

# Poisson Stable Motions and Global Attractors of Symmetric Monotone Nonautonomous Dynamical Systems

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**Abstract.** This paper is dedicated to the study of the problem of existence of Poisson stable (Bohr/Levitan almost periodic, almost automorphic, almost recurrent, recurrent, pseudo-periodic, pseudo-recurrent and Poisson stable) motions of symmetric monotone non-autonomous dynamical systems (NDS). It is proved that every precompact motion of such system is asymptotically Poisson stable. We give also the description of the structure of compact global attractor for monotone NDS with symmetry. We establish the main results in the framework of general non-autonomous (cocycle) dynamical systems. We apply our general results to the study of the problem of existence of different classes of Poisson stable solutions and global attractors for a chemical reaction network and nonautonomous translation-invariant difference equations.

**Mathematics subject classification:** 39A24, 37B05, 37B20, 37B55, 34C12, 34C27.
**Keywords and phrases:** Poisson stable motions, compact global attractor, monotone nonautonomous dynamical systems, translation-invariant dynamical systems.

## 1 Introduction

This article continues the author's series of works [13]-[18] devoted to the study of Poisson stable motions and global attractors of monotone nonautonomous dynamical systems.

In present work we study a class of monotone nonautonomous dynamical systems with symmetry. The writing of this article was motivated by works D. Angeli and E. Sontag [1,2], D. Angeli, P. Leenheer and E. Sontag [3] (for autonomous systems), H. Hu and J. Jiang [22, 23] (for periodic and almost periodic systems) and Q. Liu and Y. Wang [28] (for almost periodic and almost automorphic systems). We study these problems within the framework of general non-autonomous dynamical systems (cocycles).

## 2 NDS: some general properties

In this section we collect some notions and facts for non-autonomous dynamical systems which we will use below; the reader may refer to [9],[12, Ch. IX],[31] for details.


DOI: https://doi.org/10.56415/basm.y2022.i3.p56

Throughout the paper, we assume that $X$ and $Y$ are metric spaces and for simplicity we use the same notation $\rho$ to denote the metrics on them, which we think would not lead to confusion. Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$, $\mathbb{S} = \mathbb{R}$ or $\mathbb{Z}$, $\mathbb{S}_+ := \{s \in \mathbb{S} |\ s \geq 0\}$, $\mathbb{S}_- := \{s \in \mathbb{S} |\ s \leq 0\}$ and $\mathbb{T} \subseteq \mathbb{S}$ be a sub-semigroup of $\mathbb{S}$ such that $\mathbb{S}_+ \subseteq \mathbb{T}$. For given dynamical system $(X, \mathbb{T}, \pi)$ and given point $x \in X$, we denote by $\Sigma_x$ (respectively, $\Sigma_x^+$) its *trajectory* (respectively, *semi-trajectory*), i.e. $\Sigma_x := \{\pi(t, x) : t \in \mathbb{T}\}$ (respectively, $\Sigma_x^+ := \{\pi(t, x) : t \in \mathbb{T}_+\}$), and call the mapping $\pi(\cdot, x) : \mathbb{T} \to X$ the *motion* through $x$ at the moment $t = 0$. For given set $A \subseteq X$, we denote $\Sigma_A := \{\pi(t, x) : t \in \mathbb{T}, x \in A\}$; $\Sigma_A^+$ is defined similarly. We denote the *hull* (respectively, *semi-hull*) of a point $x$ by $H(x) := \overline{\Sigma}_x$ (respectively, $H^+(x) := \overline{\Sigma}_x^+$), where by bar we mean closure. A point $x \in X$ is called *Lagrange stable*, "st. $L$" in short, (respectively, *positively Lagrange stable*, "st. $L^+$" in short) if $H(x)$ (respectively, $H^+(x)$) is compact.

Let $(Y, \mathbb{S}, \sigma)$ be a two-sided dynamical system on $Y$ and $E$ be a metric space.

**Definition 1.** (Cocycle on the state space $E$ with the base $(Y, \mathbb{S}, \sigma)$.). A triplet $\langle E, \phi, (Y, \mathbb{S}, \sigma) \rangle$ (or briefly $\phi$ if no confusion) is said to be a *cocycle* on state space (or fibre) $E$ with base $(Y, \mathbb{S}, \sigma)$ (or driving system $(Y, \mathbb{S}, \sigma)$) if the mapping $\phi : \mathbb{S}_+ \times Y \times E \to E$ satisfies the following conditions:

1. $\phi(0, u, y) = u$ for all $u \in E$ and $y \in Y$;
2. $\phi(t + \tau, u, y) = \phi(t, \phi(\tau, u, y), \sigma(\tau, y))$ for all $t, \tau \in \mathbb{S}_+, u \in E$ and $y \in Y$;
3. the mapping $\phi$ is continuous.

**Definition 2.** (Skew-product dynamical system.) Let $\langle E, \phi, (Y, \mathbb{S}, \sigma) \rangle$ be a cocycle on $E$, $X := E \times Y$ and $\pi$ be a mapping from $\mathbb{S}_+ \times X$ to $X$ defined by $\pi := (\phi, \sigma)$, i.e., $\pi(t, (u, y)) = (\phi(t, u, y), \sigma(t, y))$ for all $t \in \mathbb{S}_+$ and $(u, y) \in E \times Y$. The triplet $(X, \mathbb{S}_+, \pi)$ is an autonomous dynamical system and is called *skew-product dynamical system.*

**Definition 3.** (Nonautonomous dynamical system.) Let $\mathbb{T}_1 \subseteq \mathbb{T}_2$ be two sub-semigroups of the group $\mathbb{S}$, $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ be two autonomous dynamical systems and $h : X \to Y$ be a homomorphism from $(X, \mathbb{T}_1, \pi)$ on $(Y, \mathbb{T}_2, \sigma)$ (i.e., $h(\pi(t, x)) = \sigma(t, h(x))$ for all $t \in \mathbb{T}_1$ and $x \in X$, and $h$ is continuous and surjective), then the triplet $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ is called a *nonautonomous dynamical system* (NDS) with basis $(Y, \mathbb{T}_2, \sigma)$.

**Example 1.** (The nonautonomous dynamical system generated by cocycle $\phi$.) An important class of NDS are generated from cocycles. Indeed, let $\langle E, \phi, (Y, \mathbb{S}, \sigma) \rangle$ be a cocycle, $(X, \mathbb{S}_+, \pi)$ be the associated skew-product dynamical system $(X = E \times Y, \pi = (\phi, \sigma))$ and $h = pr_2 : X \to Y$ (the natural projection mapping), then the triplet $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h \rangle$ is a NDS.

Lagrange stable (or called "compact") motions have been studied comprehensively, but it is not the case for non-Lagrange stable motions. The following concept of conditional compactness introduced in [9] is important for our study of noncompact motions and NDS with non-compact base (driving system).

**Definition 4.** Let $(X,h,Y)$ be a fiber space [24]. A set $M \subseteq X$ is said to be *conditionally precompact* if its intersection with the preimage of any precompact subset $Y' \subseteq Y$, i.e. the set $h^{-1}(Y') \bigcap M$, is a precompact subset of $X$. A set $M$ is called *conditionally compact* if it is closed and conditionally precompact.

*Remark* 1. 1. Let $K$ be a compact space, $Y$ is a noncompact metric space, $X := K \times Y$ and $h = pr_2 : X \to Y$. Then the triplet $(X,h,Y)$ is a fiber space. The space $X$ is conditionally compact, but it is not compact.

2. If $Y$ is a compact set and $M \subseteq X$ is conditionally precompact, then $M$ is a precompact set.

Let $x_0 \in X$. Denote by $\Sigma^+_{x_0} := \{\pi(t,x_0) : \ t \geq 0\}$ the positive semi-trajectory of point $x_0$ and $H^+(x_0) := \overline{\Sigma}^+_{x_0}$ the semi-hull of $x_0$, where by bar the closure of $\Sigma^+_{x_0}$ in $X$ is denoted.

The following result provides a useful criterion for conditional compactness in applications.

**Lemma 1** ([7]). *Let $\langle E, \phi, (Y,\mathbb{S},\sigma)\rangle$ be a cocycle and $\langle (X,\mathbb{S}_+,\pi),(Y,\mathbb{R}S\sigma),h\rangle$ be the NDS generated by the cocycle $\phi$ (cf. Example 1).*
*Assume that $x_0 := (u_0,y_0) \in X = E\times Y$ and the set $Q^+_{(u_0,y_0)} := \overline{\{\phi(t,u_0,y_0) : t \in \mathbb{S}_+\}}$ is compact. Then the semi-hull $H^+(x_0)$ is conditionally compact.*

Denote by $C(\mathbb{T},X)$ the family of all continuous functions $f : \mathbb{T} \to X$ equipped with the compact-open topology. This topology can be generated by Bebutov distance (see, e.g.[4],[41, ChIV])

$$d(f,g) := \sup_{L>0} \min\{\max_{|t|\leq L} \rho(f(t),g(t)), 1/L\}.$$

Denote by $(C(\mathbb{T},X),\mathbb{T},\sigma)$ the shift dynamical system (or called Bebutov dynamical system), i.e. $\sigma(\tau,f) := f^{\tau}$, where $f^{\tau}(t) := f(t+\tau)$ for $t \in \mathbb{T}$. Note that the function $f \in C(\mathbb{T},X)$ is positively Lagrange stable (respectively, Lagrange stable) if and only if the function $f$ is bounded and uniformly continuous on $\mathbb{T}$ (see, e.g.[34],[41, ChIV]).

Let $(Y,\mathbb{S},\sigma)$ be a two-sided dynamical system.

**Definition 5.** A point $y \in Y$ is called *positively* (respectively, *negatively*) *Poisson stable* if there exists a sequence $t_n \to +\infty$ (respectively, $t_n \to -\infty$) such that $\sigma(t_n,y) \to y$ as $n \to \infty$. If $y$ is Poisson stable in both directions, it is called *Poisson stable.*

**Definition 6.** Let $\langle E,\phi,(Y,\mathbb{S},\sigma)\rangle$ (respectively, $(X,\mathbb{S}_+,\pi)$) be a cocycle (respectively, one-sided dynamical system). A continuous mapping $\nu : \mathbb{S} \to E$ (respectively, $\gamma : \mathbb{S} \to X$) is called an *entire trajectory* of cocycle $\phi$ (respectively, of dynamical system $(X,\mathbb{S}_+,\pi)$) passing through the point $(u,y) \in E \times Y$ (respectively, $x \in X$) at $t = 0$ if $\phi(t,\nu(s),\sigma(s,y)) = \nu(t+s)$ and $\nu(0) = u$ (respectively, $\pi(t,\gamma(s)) = \gamma(t+s)$ and $\gamma(0) = x$) for all $t \in \mathbb{S}_+$ and $s \in \mathbb{S}$.

Denote by

– $C(\mathbb{T},X)$ the space of all continuous functions $f:\mathbb{T}\to X$ equipped with the compact-open topology;

– $\Phi_x$ the family of all entire trajectories of $(X,\mathbb{S}_+,\pi)$ passing through the point $x\in X$ at the initial moment $t=0$ and $\Phi:=\bigcup\{\Phi_x:\ x\in X\}$.

*Remark* 2. Note that:

1. if $\gamma\in\Phi_x$ then $\gamma^\tau\in\Phi_{\gamma(\tau)}$, where $\gamma^\tau(t):=\gamma(t+\tau)$ for $t\in\mathbb{T}$, and consequently $\Phi$ is a translation invariant subset of $C(\mathbb{T},X)$;
2. if $\gamma_n\in\Phi_{x_n}$ and $\gamma_n\to\gamma$ in $C(\mathbb{T},X)$ as $n\to\infty$, then $\gamma\in\Phi_x$ with $x:=\lim\limits_{n\to\infty}x_n$ and consequently $\Phi$ is a closed subset of $C(\mathbb{T},X)$.

By Remark 2 $\Phi$ is a closed and invariant (with respect to shifts) subset of $C(\mathbb{T},X)$, and consequently on $\Phi$ a shift dynamical system $(\Phi,\mathbb{T},\lambda)$ induced from $\big(C(\mathbb{T},X),\mathbb{T},\lambda\big)$ is defined.

Let $M$ be a subset of $X$. We denote the $\omega$-limit set of $M$ by

$$\omega(M):=\bigcap_{t\ge0}\overline{\bigcup\{\pi(\tau,M):\ \tau\ge t\}};$$

for a singleton set, for simplicity we also write $\omega(x)$ or $\omega_x$ for $\omega(\{x\})$ and denote $\omega_q(M):=\omega(M)\bigcap h^{-1}(q)$. Note that $x\in\omega(M)$ if and only if there exist sequences $\{x_n\}\subset M$ and $\{t_n\}\subset\mathbb{R}$ such that $t_n\to+\infty$ as $n\to\infty$ and $\lim\limits_{n\to\infty}\pi(t_n,x_n)=x$.

**Definition 7.** Let $\langle(X,\mathbb{S}_+,\pi),(Y,\mathbb{S},\sigma),h\rangle$ be an NDS. A subset $A\subseteq X$ is said to be *(positively) uniformly stable* if for arbitrary $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that $\rho(x,a)<\delta$ ($a\in A,\ x\in X$ and $h(a)=h(x)$) implies $\rho(\pi(t,x),\pi(t,a))<\varepsilon$ for any $t\ge0$. In particular, a point $x_0\in X$ is called uniformly stable if the singleton set $\{x_0\}$ is so.

*Remark* 3. Let $A\subseteq X$ be uniformly stable and $B\subseteq A$, then $B$ is also uniformly stable.

**Lemma 2.** *([5, ChIV],[6]) If the set $A\subseteq X$ is uniformly stable and the mapping $h:X\to Y$ is open, then the closure $\overline{A}$ of $A$ is uniformly stable.*

**Corollary 1.** *If $\Sigma^+_{x_0}$ is uniformly stable and $h$ is open, then:*

1. *$H^+(x_0)$ is uniformly stable;*
2. *$\omega_{x_0}$ is uniformly stable, because $\omega_{x_0}\subseteq H^+(x_0)$.*

*Remark* 4. Note that if an NDS is generated by a skew-product dynamical system (or equivalently by a cocycle) in which case the homomorphism $h$ is given by the natural projection mapping, then clearly $h$ is open.

# 3 Poisson stable motions and their comparability by character of recurrence

## 3.1 Classes of Poisson stable motions

Let $(X, \mathbb{S}, \pi)$ be a dynamical system. Let us recall the classes of Poisson stable motions we study in this paper, see [31, 34, 37, 41] for details.

**Definition 8.** A point $x \in X$ is called *stationary* (respectively, *$\tau$-periodic*) if $\pi(t, x) = x$ (respectively, $\pi(t + \tau, x) = \pi(t, x)$) for all $t \in \mathbb{S}$.

**Definition 9.** A point $x \in X$ is called *quasi-periodic* with the base of frequency $\nu = (\nu_1, \nu_2, \dots, \nu_k)$ if the associated function $f(\cdot) := \pi(\cdot, x) : \mathbb{S} \to X$ satisfies the following conditions:

1. the numbers $\nu_1, \nu_2, \dots, \nu_k$ are rationally independent;
2. there exists a continuous function $\Phi : \mathbb{R}^k \to X$ such that $\Phi(t_1 + 2\pi, t_2 + 2\pi, \dots, t_k + 2\pi) = \Phi(t_1, t_2, \dots, t_k)$ for all $(t_1, t_2, \dots, t_k) \in \mathbb{R}^k$;
3. $f(t) = \Phi(\nu_1 t, \nu_2 t, \dots, \nu_k t)$ for $t \in \mathbb{R}$.

**Definition 10.** For given $\varepsilon > 0$, a number $\tau \in \mathbb{R}$ is called an *$\varepsilon$-shift of $x$* (respectively, *$\varepsilon$-almost period of $x$*) if $\rho(\pi(\tau, x), x) < \varepsilon$ (respectively, $\rho(\pi(\tau + t, x), \pi(t, x)) < \varepsilon$ for all $t \in \mathbb{R}$).

**Definition 11.** A point $x \in X$ is called *almost recurrent* (respectively, *Bohr almost periodic*) if for any $\varepsilon > 0$ there exists a positive number $l$ such that any segment of length $l$ contains an $\varepsilon$-shift (respectively, $\varepsilon$-almost period) of $x$.

**Definition 12.** If a point $x \in X$ is almost recurrent and its trajectory $\Sigma_x$ is precompact, then $x$ is called *(Birkhoff) recurrent*.

Denote $\mathfrak{N}_y := \{\{t_n\} \subset \mathbb{S} : \sigma(t_n, y) \to y\}$, $\mathfrak{N}_y^{+\infty} := \{\{t_n\} \in \mathfrak{N}_y : t_n \to +\infty\}$, $\mathfrak{N}_y^{-\infty} := \{\{t_n\} \in \mathfrak{N}_y : t_n \to -\infty\}$, and $\mathfrak{N}_y^{\infty} := \{\{t_n\} \in \mathfrak{N}_y : t_n \to \infty\}$.

**Definition 13.** A point $x \in X$ is called *Levitan almost periodic* [27] (see also [5, 10, 26]) if there exists a dynamical system $(Y, \mathbb{T}, \sigma)$ and a Bohr almost periodic point $y \in Y$ such that $\mathfrak{N}_y \subseteq \mathfrak{N}_x$.

**Definition 14.** A point $x \in X$ is called *almost automorphic* if it is Lagrange stable and Levitan almost periodic.

**Definition 15.** A point $x \in X$ is said to be *uniformly Poisson stable* or *pseudo-periodic* in the positive (respectively, negative) direction if for arbitrary $\varepsilon > 0$ and $l > 0$ there exists an $\varepsilon$-almost period $\tau > l$ (respectively, $\tau < -l$) of $x$. The point $x$ is said to be uniformly Poisson stable or pseudo-periodic if it is so in both directions.

**Definition 16** ([32, 33])**.** A point $x \in X$ is said to be *pseudo-recurrent* if for any $\varepsilon > 0$, $p \in \Sigma_x$ and $t_0 \in \mathbb{R}$ there exists $L = L(\varepsilon, t_0) > 0$ such that

$$B(p, \varepsilon) \bigcap \pi([t_0, t_0 + L], p) \neq \emptyset,$$

where
$B(p, \varepsilon) := \{x \in X : \rho(p, x) < \varepsilon\}$ and $\pi([t_0, t_0 + L], p) := \{\pi(t, p) : t \in [t_0, t_0 + L]\}$.

**Definition 17.** A point $x \in X$ is said to be [16, ChI] *strongly Poisson stable* (in the positive direction) if $p \in \omega_p$ for any $p \in H(x)$.

*Remark* 5. It is known that:

1. a strongly Poisson stable point is Poisson stable, but the converse is not true in general;

2. all the motions introduced above (Definitions 8–16) are strongly Poisson stable.

**Definition 18** ([11, 38])**.** A point $x \in X$ is said to be asymptotically stationary (respectively, asymptotically $\tau$-periodic, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotic recurrent, asymptotically Levitan almost periodic, asymptotically almost recurrent, asymptotically pseudo-periodic, asymptotically pseudo-recurrent, asymptotically Poisson stable) if there exists a stationary (respectively, $\tau$-periodic, Bohr almost periodic, almost automorphic, recurrent, Levitan almost periodic, almost recurrent, pseudo-periodic, pseudo-recurrent, Poisson stable) point $p \in X$ such that $\lim\limits_{t \to +\infty} \rho(\pi(t, x), \pi(t, p)) = 0$.

### 3.2 Comparability of motions by their character of recurrence

In this subsection we present some notions and results stated and proved by B. A. Shcherbakov [34]–[37].

Let $(X, \mathbb{S}, \pi)$ and $(Y, \mathbb{S}, \sigma)$ be two dynamical systems.

**Definition 19.** A point $x \in X$ is said to be *comparable with $y \in Y$ by character of recurrence* if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that every $\delta$-shift of $y$ is an $\varepsilon$-shift for $x$, i.e., $\rho(\sigma(\tau, y), y) < \delta$ implies $\rho(\pi(\tau, x), x) < \varepsilon$.

**Theorem 1.** *The following conditions are equivalent:*

1. *the point $x$ is comparable with $y$ by character of recurrence;*

2. $\mathfrak{N}_y \subseteq \mathfrak{N}_x$*;*

3. $\mathfrak{N}_y^\infty \subseteq \mathfrak{N}_x^\infty$*;*

4. *from any sequence $\{t_n\} \in \mathfrak{N}_y$ we can extract a subsequence $\{t_{n_k}\} \in \mathfrak{N}_x$;*

5. *from any sequence $\{t_n\} \in \mathfrak{N}_y^\infty$ we can extract a subsequence $\{t_{n_k}\} \in \mathfrak{N}_x^\infty$.*

**Theorem 2.** *Let $x \in X$ be comparable with $y \in Y$. If the point $y$ is stationary (respectively, $\tau$-periodic, Levitan almost periodic, almost recurrent, Poisson stable), then so is the point $x$.*

**Definition 20.** A point $x \in X$ is called *uniformly comparable with $y \in Y$ by character of recurrence* if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that every $\delta$-shift of $\sigma(t, y)$ is aN $\varepsilon$-shift of $\pi(t, x)$ for all $t \in \mathbb{S}$, i.e., $\rho(\sigma(t+\tau, y), \sigma(t, y)) < \delta$ implies $\rho(\pi(t+\tau, x), x) < \varepsilon$ for all $t \in \mathbb{R}$ (or equivalently: $\rho(\sigma(t_1, y), \sigma(t_2, y)) < \delta$ implies $\rho(\pi(t_1, x), \pi(t_2, x)) < \varepsilon$ for all $t_1, t_2 \in \mathbb{S}$).

Denote $\mathfrak{M}_x := \{\{t_n\} \subset \mathbb{S} : \ \{\pi(t_n, x)\}$ converges$\}$, $\mathfrak{M}_x^{+\infty} := \{\{t_n\} \in \mathfrak{M}_x : t_n \to +\infty$ as $n \to \infty\}$ and $\mathfrak{M}_x^{\infty} := \{\{t_n\} \in \mathfrak{M}_x : t_n \to \infty$ as $n \to \infty\}$.

**Definition 21** ([8,11])**.** A point $x \in X$ is said to be *strongly comparable with $y \in Y$ by character of recurrence* if $\mathfrak{M}_y \subseteq \mathfrak{M}_x$.

**Theorem 3.** *(i) If $\mathfrak{M}_y \subseteq \mathfrak{M}_x$, then $\mathfrak{N}_y \subseteq \mathfrak{N}_x$, i.e. strong comparability implies comparability.*

*(ii) Let $X$ be a complete metric space. If the point $x$ is uniformly comparable with $y$ by character of recurrence, then $\mathfrak{M}_y \subseteq \mathfrak{M}_x$, i.e. uniform comparability implies strong comparability.*

**Theorem 4.** *Let $y$ be Lagrange stable. Then $\mathfrak{M}_y \subseteq \mathfrak{M}_x$ holds if and only if the point $x$ is Lagrange stable and uniformly comparable with $y$ by character of recurrence.*

**Theorem 5.** *Let $X$ and $Y$ be two complete metric spaces. Let the point $x \in X$ be uniformly comparable with $y \in Y$ by character of recurrence. If $y$ is quasi-periodic (respectively, Bohr almost periodic, almost automorphic, Birkhoff recurrent, Lagrange stable, pseudo-periodic, pseudo-recurrent), then so is $x$.*

## 4 Global Attractors of Non-Autonomous Dynamical Systems

**Definition 22.** A family $\{A_y|\ y \in Y\}$ of subsets $A_y$ of $W$ indexed by $y \in Y$ is called a non-autonomous set.

Let $\{A_y|\ y \in Y\}$ be a non-autonomous set. Denote by $\mathbb{A}$ the subset of $X := W \times Y$ defined by equality

$$\mathbb{A} := \bigcup\{A_p \times \{y\}|\ y \in Y\}\} = \{(w, y) \in X|\ w \in A_y,\ y \in Y\}.$$

*Remark* 6. 1. Let $\mathcal{A}$ be a subset of $X = W \times Y$, $\mathcal{A}_y := \mathbb{A} \bigcap pr_2^{-1}(y)$ and $A_y := pr_1(\mathcal{A}_y)$, then $\{A_y|\ y \in Y\}$ is a non-autonomous set.

2. Denote by $\mathfrak{A} = \bigcup\{A_y \times \{y\}|\ y \in Y\}$, then $\mathcal{A} \subseteq \mathfrak{A}$.

**Definition 23.** A non-autonomous set $\{A_y|\ y \in Y\}$is said to be

1. precompact (respectively, uniformLY precompact) if for every $y \in Y$ the set $A_y$ (respectively, $\bigcup\{A_y|\ y \in Y\}$) is a precompact subset of $W$;

2. bounded (respectively, uniformLY bounded) if for every $y \in Y$ the set $A_y$ (respectively, $\bigcup\{A_y |\ y \in Y\}$) is a bounded subset of $W$.

Let $W$ be a complete metric space.

**Definition 24.** A cocycle $\varphi$ over $(Y, \mathbb{T}, \sigma)$ with the fiber $W$ is said to be compactly dissipative if there exits a nonempty compact $K \subseteq W$ such that

$$\lim_{t\to+\infty} \sup\{\beta(U(t,y)M, K) \mid y \in Y\} = 0 \tag{1}$$

for any $M \in C(W)$, where $\beta(A,B) := \sup\{\rho(a,B) : a \in A\}$ is a semi-distance of Hausdorff.

**Definition 25.** The family $\{I_y \mid y \in Y\}(I_y \subset W)$ of nonempty compact subsets is called a compact (forward) global attractor of the cocycle $\varphi$ if the following conditions are fulfilled:

1. the set $I := \bigcup\{I_y \mid y \in Y\}$ is relatively compact;
2. the family $\{I_y \mid y \in Y\}$ is invariant with respect to the cocycle $\varphi$;
3. the equality
$$\lim_{t\to+\infty} \sup_{y\in Y} \beta(\varphi(t,K,y), I) = 0$$
holds for every $K \in C(W)$.

Let $M \subseteq W$ and

$$\omega_y(M) := \bigcap_{t\geq 0} \overline{\bigcup_{\tau\geq t} \varphi(\tau, M, \sigma(-\tau,y))}$$

for any $y \in Y$.

**Theorem 6.** *[12, ChII] Let $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be compactly dissipative and $K$ be the nonempty compact subset of $W$ appearing in the equality (1), then:*

1. *$I_y = \omega_y(K) \neq \emptyset$, is compact, $I_y \subseteq K$ and*
$$\lim_{t\to+\infty} \beta(U(t,\sigma(-t,y))K, I_y) = 0$$
*for every $y \in Y$;*
2. *$U(t,y)I_y = I_{yt}$ for all $y \in Y$ and $t \in \mathbb{T}_+$;*
3. $$\lim_{t\to+\infty} \beta(U(t,\sigma(-t,y))M, I_y) = 0$$
*for all $M \in C(W)$ and $y \in Y$ ;*
4. *the set $I$ is relatively compact, where $I := \cup\{I_y \mid y \in Y\}$.*

**Theorem 7.** *[14] Let $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be compactly dissipative and $K$ be the nonempty compact subset of $W$ appearing in the equality (1), then the family of subsets $\{I_y |\ y \in Y\}$ is a maximal family possessing the properties 2.–4.*

**Definition 26.** Let $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be compactly dissipative, $K$ be the nonempty compact subset of $W$ appearing in the equality (1) and $I_y := \omega_y(K)$ for any $y \in Y$. The family of compact subsets $\{I_y |\ y \in Y\}$ is said to be a Levinson center (compact global attractor) of non-autonomous (cocycle) dynamical system $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$.

*Remark* 7. According to Theorem 7 by Definition 26 the notion Levinson center (compact global attractor) for non-autonomous (cocycle) dynamical system $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ is well defined.

**Corollary 2.** *Let $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be compactly dissipative non-autonomous dynamical system, $\{I_y |\ y \in Y\}$ be its Levinson center and $\nu : \mathbb{T} \mapsto W$ be a relatively compact full trajectory of $\varphi$ (i.e., $\nu(\mathbb{S})$ is relatively compact and there exists a point $y_0 \in Y$ such that $\nu(t + s) = \varphi(t, \nu(s), \sigma(s, y_0))$ for any $t \geq 0$ and $s \in \mathbb{S}$), then $\nu(0) \in I_{y_0}$.*

**Theorem 8.** *[12, ChII] Under the conditions of Theorem 6 $w \in I_y$ $(y \in Y)$ if and only if there exits a whole trajectory $\nu : \mathbb{S} \to W$ of the cocycle $\varphi$, satisfying the following conditions: $\nu(0) = w$ and $\nu(\mathbb{S})$ is relatively compact.*

**Definition 27.** A family of subsets $\{I_y |\ y \in Y\}$ $(I_y \subseteq W$ for any $y \in Y)$ is said to be upper semicontinuous if for any $y_0 \in Y$ and $y_n \to y_0$ as $n \to \infty$ we have

$$\lim_{n\to\infty} \beta(I_{y_n}, I_{y_0}) = 0.$$

**Lemma 3.** *[14] The following statements hold:*

1. *the family $\{I_y |\ y \in Y\}$ is invariant if and only if the set $J := \bigcup\{J_y |\ y \in Y\}$, where $J_y := I_y \times \{y\}$, is invariant with respect to skew-product dynamical system $(X, \mathbb{S}_+, \pi)$ $(X := W \times Y$ and $\pi := (\varphi, \sigma))$;*

2. *if $\bigcup\{I_y |\ y \in Y\}$ is relatively compact, then the family $\{I_y |\ y \in Y\}$ is upper semicontinuous if and only if the set $J$ is closed in $X$.*

**Definition 28.** A non-autonomous set $\mathbb{K} = \{K_y :\ y \in Y\}$ with $K_y \subseteq W$ for any $y \in Y$ is said to be positively Lyapunov stable (respectively, uniformly stable) if for arbitrary $\varepsilon > 0$ and $y \in Y$ there exists a positive number $\delta = \delta(\varepsilon, y, \mathbb{K}) > 0$ (respectively, $\delta = \delta(\varepsilon, \mathbb{K}) > 0$) such that $\rho(\varphi(t_0, u, y), \varphi(t_0, u_0, y)) < \delta$ $(u_0 \in K_y$ and $u \in W)$ implies $\rho(\varphi(t, u, y), \varphi(t, u_0, y)) < \varepsilon$ for any $t \geq t_0$ and $y \in Y$.

**Definition 29.** A trajectory $\varphi(t, u_0, y_0)$ of the point $(u_0, y_0) \in W \times Y$ is said to be positively uniformly Lyapunov stable if the set $K_0 := \varphi(\mathbb{T}_+, u_0, y_0) = \{\varphi(t, u_0, y_0) |\ t \in \mathbb{S}_+\}$ is uniformly Lyapunov stable.

**Definition 30.** A cocycle $\varphi$ is said to be positively Lyapunov stable (respectively, uniformly stable) if for arbitrary $\varepsilon > 0$ and non-autonomous uniformly precompact and upper semicontinuous set $\mathbb{K} = \{K_y : \ y \in Y\}$ there exists a number $\delta = \delta(\varepsilon, y, \mathbb{K}) > 0$ (respectively, $\delta = \delta(\varepsilon, \mathbb{K}) > 0$) such that $\rho(u, u_0) < \delta$ with $u_0 \in K_y$ and $u \in W$ implies $\rho(\varphi(t, u, y), \varphi(t, u_0, y)) < \varepsilon$ for any $t \geq 0$.

**Theorem 9.** *[12, Ch.IX] Let $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be a cocycle with the following properties:*

1. *It admits a conditionally relatively compact invariant set $\{I_y \mid y \in Y\}$ (i.e. $\bigcup\{I_y \mid y \in Y'\}$ is relatively compact subset of $W$ for any relatively compact subset $Y'$ of $Y$).*

2. *The cocycle $\varphi$ is positively uniformly stable on $\{I_y| \quad y \in Y\}$.*

*Then all motions on $J := \bigcup\{J_y| \quad y \in Y\} \quad (J_y := I_y \times \{y\})$ may be continued uniquely to the left and on $J$ a two-sided dynamical system $(J, \mathbb{S}, \pi)$ is defined , i.e., the skew-product system $(X, \mathbb{S}_+, \pi)$ generates on $J$ a two-sided dynamical system $(J, \mathbb{S}, \pi)$.*

## 5 Monotone NDS: existence and convergence to Poisson stable motions

Let $E$ be a real Banach space with a closed convex cone $P \subset E$ such that $P \bigcap (-P) = \{0\}$. Assume that $Int(P) \neq \emptyset$. For $u_1, u_2 \in E$, we write $u_1 \leq u_2$ if $u_2 - u_1 \in P$; $u_1 < u_2$ if $u_2 - u_1 \in P \setminus \{0\}$; $u_1 \ll u_2$ if $u_2 - u_1 \in Int(P)$.

Assume that $E$ is an ordered space.

**Definition 31.** A subset $U$ of $E$ is said [25] to be order convex if for any $a, b \in U$ with $a < b$, the segment $\{a + s(b - a) : \ s \in [0, 1]\}$ is contained in $U$.

Let $V = [0, b]_E$ with $b \gg 0$ or $V = P$, or furthermore, $V$ be an order convex subset of $E$.

**Definition 32.** A subset $U$ of $E$ is called lower-bounded (respectively, upper-bounded) if there exists an element $a \in E$ such that $a \leq U$ (respectively, $a \geq U$). Such an $a$ is said to be a lower bound (respectively, upper bound) for $U$.

**Definition 33.** A lower bound $\alpha$ is said to be the *greatest lower bound* (g.l.b.) or *infimum*, if any other lower bound $a$ satisfies $a \leq \alpha$. Similarly, we can define the *least upper bound* (l.u.b.) or *supremum*.

A bundle $(X, h, Y)$ is said to be *ordered* if each fiber $X_y$ is ordered. Note that only points on the same fiber may be order related: if $x_1 \leq x_2$ or $x_1 < x_2$, then it implies $h(x_1) = h(x_2)$. We assume that the order relation and the topology on $X$ are compatible in the sense that $x \leq \tilde{x}$ if $x_n \leq \tilde{x}_n$ for all $n$ and $x_n \to x$, $\tilde{x}_n \to \tilde{x}$ as $n \to \infty$.

**Definition 34.** For given bundle $(X, h, Y)$, an NDS $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ defined on it is said to be *monotone* (respectively, *strictly monotone*) if $x_1 \le x_2$ (respectively, $x_1 < x_2$) implies $\pi(t, x_1) \le \pi(t, x_2)$ (respectively, $\pi(t, x_1) < \pi(t, x_2)$) for any $t > 0$.

For given NDS $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$, let $\mathcal{S} \subseteq X$ be a nonempty closed ordered subset possessing the following properties:

1. $h(\mathcal{S}) = Y$;
2. $\mathcal{S}$ is positively invariant with respect to $\pi$, i.e., $\langle (\mathcal{S}, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ is an NDS.

Below we will use the following assumptions:

(**C1**) For every conditionally compact subset $K$ of $\mathcal{S}$ and $y \in Y$ the set $K_y := h^{-1}(y) \bigcap K$ has both infimum $\alpha_y(K)$ and supremum $\beta_y(K)$.

(**C2**) For every $x \in \mathcal{S}$, the semi-trajectory $\Sigma_x^+$ is conditionally precompact, $\omega_x \neq \emptyset$ and the set $\omega_x$ is positively uniformly stable.

(**C2.1**) For every $x \in \mathcal{S}$, the semi-trajectory $\Sigma_x^+$ is conditionally precompact and $\omega_x \neq \emptyset$.

(**C3**) The NDS
$$\langle (\mathcal{S}, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$$
is monotone.

Let $X, Y$ be two complete metric spaces and
$$\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle \tag{2}$$
be a non-autonomous dynamical system.

**Definition 35.** A closed subset $M$ of $X$ is said to be a minimal set of non-autonomous dynamical system (NDS) (2) if it possesses the following properties:

a. $h(M) = Y$;

b. $M$ is positively invariant, i.e., $\pi(t, M) \subseteq M$ for any $t \in \mathbb{T}_1$;

c. $M$ is a minimal subset of $X$ possessing properties a. and b..

**Theorem 10.** *[16, Ch.IV] Let $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ be a non-autonomous dynamical system and $M \subset X$ be a nonempty, conditionally compact and positively invariant set. If the dynamical system $(Y, \mathbb{T}_2, \sigma)$ is minimal, then the subset $M$ is a minimal subset of NDS (2) if and only if $H(x) = M$ for any $x \in M$.*

**Theorem 11.** *[16, Ch.IV] Suppose that $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is a non-autonomous dynamical system, $(Y, \mathbb{T}_2, \sigma)$ is minimal and the space $X$ is conditionally compact, then there exists a minimal subset $M$ of NDS (2).*

**Lemma 4.** *Let $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ be an NDS with the following properties:*

*a. there exists a point $x_0 \in X$ such that the positive semi-trajectory $\Sigma^+_{x_0}$ is conditionally precompact;*

*b. the point $y_0 := h(x_0)$ is Poisson stable, i.e., $y_0 \in \omega_{y_0}$.*

*Then the following statements hold:*

*1. there are a Poisson stable point $p \in \omega_{x_0}$ and a sequence $\{t_k\} \in \mathfrak{N}^{+\infty}_{y_0}$ such that*

$$\lim_{k\to\infty} \rho(\pi(t_k, x_0), \pi(t_k, p)) = 0;$$

*2. if the dynamical system $(Y, \mathbb{S}, \sigma)$ is minimal, then there are a minimal subset $M \subseteq \omega_{x_0}$ of non-autonomous dynamical system $\langle (\omega_{x_0}, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$, a point $p \in M \bigcap X_{y_0}$ and a sequence $\{t_k\} \in \mathfrak{N}^{+\infty}_{y_0}$ such that (3) is fulfilled.*

**Corollary 3.** *Let $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ be an NDS with the following properties:*

*a. there exists a point $x_0 \in X$ such that the positive semi-trajectory $\Sigma^+_{x_0}$ is conditionally precompact;*

*b. the point $y_0 := h(x_0)$ is Poisson stable, i.e., $y_0 \in \omega_{y_0}$;*

*c. the set $\omega_{x_0}$ is positively uniformly stable.*

*Then the following statements hold:*

*1. there is a Poisson stable point $p \in \omega_{x_0}$ such that*

$$\lim_{t\to+\infty} \rho(\pi(t, x_0), \pi(t, p)) = 0; \tag{3}$$

*2. if the dynamical system $(Y, \mathbb{S}, \sigma)$ is minimal, then there are a minimal subset $M \subseteq \omega_{x_0}$ of non-autonomous dynamical system $\langle (\omega_{x_0}, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ and a point $p \in M \bigcap X_{y_0}$ such that (3) is fulfilled, and hence, $\omega_{x_0}$ is a minimal subset of non-autonomous dynamical system $\langle (X_0, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$.*

Now fix $(x_0, y_0) \in V \times Y$, then the set $\omega_{(x_0,y_0)}$ is a nonempty , conditionally compact, positively invariant set. Assume that $(Y, \mathbb{T}, \sigma)$ is minimal and $y \in \omega_y$ for any $y \in Y$, then $h(\omega_{(x_0,y_0)}) = Y$. According to Corollary 3 the set $\omega_{(x_0,y_0)}$ is a minimal set of non-autonomous dynamical system $\langle (V \times Y, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$. We put $K := \omega_{(x_0,y_0)}$.

Let $(E_i, P_i)$, $1 \le i \le n$, be ordered Banach spaces with $Int(P_i) \neq \emptyset$. For each $I = \{j_1, \ldots, j_m\} \subseteq N := \{1, \ldots, n\}$, we define

$$E_I := \prod_{k=1}^{m} E_{j_k}, \quad P_I := \prod_{k=1}^{m} P_{j_k}.$$

Then $(E_I, P_I)$ is an ordered Banach space with

$$Int(P_I) = \prod_{k=1}^{m} Int(P_{j_k}) \neq \emptyset.$$

Let $\leq_I$ (respectively, $<_I$ and $\ll_I$) be the orders induced by $P_I$ in $E_I$. In the case where $I = N$, we use $(E, P)$ to denote the ordered Banach space $(E_N, P_N)$, and omit the order subscripts to get the orders $\leq$, $<$ and $\ll$ in $E$, respectively. For each $1 \leq i \leq n$, let $Q_i : E \times Y :\to E_i$ be the projection mapping defined by $Q_i(x, y) = x_i$.

*Condition* **(C4)**. For any two bounded full orbits $\gamma_j \in \mathcal{F}_{(x_j,y)}$ $(j = 1, 2)$ with $\gamma_1(t) \leq \gamma_2(t)$ for any $t \in \mathbb{S}$, there exists $t_0 > 0$ such that whenever $Q_i\gamma_1(s) < Q_i\gamma_2(s)$ holds for some $i \in N$ and $s \in \mathbb{S}$, then $Q_i\gamma_1(t) \ll Q_i\gamma_2(t)$ for all $t \geq s + t_0$.

**Definition 36.** A skew-product semi-flow $\pi$ on $V \times Y$ is said [25] to be componentwise strongly monotone if it is monotone and whenever $x_1 \leq x_2$ with $x_{1i} < x_{2i}$, one has $Q_i\pi(t, (x_1, y)) \ll Q_i\pi(t, (x_2, y))$ for all $t > 0$.

*Remark* 8. If the semi-flow $\pi$ is componentwise strongly monotone, then it satisfies Condition (**C4**).

**Theorem 12.** *[20] Assume that the dynamical system $(Y, \mathbb{S}, \sigma)$ is minimal and $q \in \omega_q$ for any $q \in Y$. Under conditions **(C1)-(C4)** for any $(x_0, y_0) \in V \times Y$ the following statements hold:*

1. *for any $q \in Y$ the set*
$$\omega_{(x_0,y_0)} \bigcap X_q$$
*consists of a single point $\{(x_q, q)\}$;*

2. *the point $(x_q, q)$ is strongly comparable by character of recurrence with the point $q \in Y$;*

3. $$\lim_{t\to+\infty} \rho(\varphi(t, x_0, y_0), \varphi(t, x_{y_0}, y_0)) = 0.$$

**Corollary 4.** *Under the conditions $(\boldsymbol{C1}) - (\boldsymbol{C4})$ if the point $y_0$ is $\tau$-periodic (respectively, quasi-periodic, Bohr almost periodic, recurrent, strongly Poisson stable and $H(y_0)$ is a minimal set), then:*

1. *the point $x_{y_0}$ is so;*

2. *the point $x_0$ is asymptotically $\tau$-periodic (respectively, asymptotically quasi-periodic, asymptotically Bohr almost periodic, asymptotically recurrent, asymptotically strongly Poisson stable).*

## 6 Structure of the Levinson center for monotone non-autonomous dynamical systems

**Lemma 5.** *[16, Ch. V] Suppose that $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ is a cocycle under $(Y, \mathbb{S}, \sigma)$ with the fibre $W$. If $Y$ is a compact space, then the following conditions are equivalent:*

*a) the cocycle $\varphi$ is positively uniformly Lyapunov stable;*

*b) every trajectory $\varphi(t, u_0, y_0)$ ($x_0 := (u_0, y_0) \in W \times Y$) of cocycle $\varphi$ is positively uniformly Lyapunov stable.*

**Theorem 13.** *[17] Assume that the cocycle $\langle E, \varphi, (Y, \mathbb{S}, \sigma)\rangle$*

1. *is monotone;*
2. *admits a compact global attractor $\boldsymbol{I} := \{I_y |\ y \in Y\}$;*
3. *is positively uniformly Lyapunov stable and denote by $\alpha(y)$ (respectively, by $\beta(y)$) the greatest lower bound of the set $I_y$ (respectively, the least upper bound of $I_y$) and*
4. *the point $y \in Y$ is positively Poisson stable, i.e., $y \in \omega_y$.*

*Then the following statements hold:*

1. *$\alpha(y) \le u \le \beta(y)$ for any $u \in I_y$ and $y \in Y$;*
2. *$\alpha(y), \beta(y) \in I_y$ and, consequently, $I_y \subseteq [\alpha(y), \beta(y)]$;*
3. *$\varphi(t, \alpha(y), y) = \alpha(\sigma(t, y))$ (respectively, $\varphi(t, \beta(y), y) = \beta(\sigma(t, y))$) for any $t \ge 0$;*
4. *the point $\gamma_*(y) := (\alpha(y), y) \in X = E \times Y$ (respectively, $\gamma^*(y) := (\beta(y), y) \in X$) is comparable by character of recurrence with the point $y$;*
5. *if $u \in E$ and $u \le \alpha(y)$ (respectively, $u \ge \beta(y)$), then $\omega_x \bigcap X_y = \{\gamma_*(y)\}$ (respectively, $\omega_x \bigcap X_y = \{\gamma^*(y)\}$), where $x := (u, y)$;*
6. *if $u \le \alpha(y)$ (respectively, $u \ge \beta(y)$),then*
$$\lim_{t \to +\infty} \rho(\varphi(t, u, y).\gamma_*(\sigma(t, y))) = 0$$
*(respectively,*
$$\lim_{t \to +\infty} \rho(\varphi(t, u, y).\gamma^*(\sigma(t, y))) = 0);$$
7. *if $y$ is strongly Poisson stable, then the point $\gamma_*(y) := (\alpha(y), y) \in X = E \times Y$ (respectively, $\gamma^*(y) := (\beta(y), y) \in X$) is strongly comparable by character of recurrence with the point $y$.*

**Corollary 5.** *Under the conditions of Theorem 13 the following statements take place:*

1. *if the point $y$ is $\tau$-periodic (respectively, Levitan almost periodic, almost recurrent, almost automorphic, recurrent, Poisson stable), then the full trajectory $\gamma_y$ passing through the point $(\alpha(y), y)$ (respectively, through the point $(\beta(y), y)$) is so;*

2. *if the point $y$ is quasi-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent, pseudo-recurrent and Lagrange stable, uniformly Poisson stable and stable in the sense of Lagrange), then the full trajectory $\gamma_y$ passing through the point $(\alpha(y), y)$ (respectively, through the point $(\beta(y), y)$) is so.*

*Remark* 9. Corollary 5 generalizes and refines the results of the work [14] which give as the positive answer for I. U. Bronshtein's conjecture [5, ChIV, p.273] for monotone Bohr almost periodic systems.

## 7 Translation-invariant monotone systems

**Definition 37.** Let $E$ be a real Banach space. A cone $P$ is said to be normal if the norm $|\cdot|$ in $E$ is semi-monotone, i.e., there exists a constant $k > 0$ such that the property $0 \leq u \leq v$ implies that $|u| \leq k|v|$.

Assume that $E$ is a strongly ordered Banach space with normal cone $P$.

Fix $v \in Int(P)$ with $|v| = 1$. Let $G$ be the group of phase-translations $T_a : E \to E$; $T_a(u) := u + av$, by a scalar $a \in \mathbb{R}$.

**Definition 38.** The phase-translation group $G = \{T_a |\ a \in \mathbb{R}\}$ commutes with the skew-product dynamical system $(X, \mathbb{T}, \pi)$ ($X = E \times Y$, $\pi = (\varphi, \sigma)$) if

$$\pi(t, (T_a(u), y)) = (T_a(\varphi(t, u, y)), \sigma(t, y))$$

for any $x = (u, y) \in X = E \times Y$, $t \in \mathbb{T}$ and $T_a \in G$.

For such $v$ above, the Banach space $E$ has a direct sum decomposition

$$E = E_0 \oplus Span(v),$$

where $E_0$ is the null space of a bounded linear functional $f$ on E with $\langle f, v\rangle = 1$.

Let $v \in P$ be a strongly positive unit vector, i.e., $v \in Int(P)$ and $|v| = 1$. Define the $v$-norm as follows

$$||u||_v := \inf\{\alpha > 0 :\ -\alpha v \leq u \leq \alpha v\}.$$

*Remark* 10. If the cone $P$ is normal, then the norms $|\cdot|$ and $||\cdot||_v$ are equivalent.

Let $V$ be an ordered convex subset of $E$.

**Definition 39.** A cocycle $\langle V, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ is said to be translation invariant with respect to $v$ if

$$\varphi(t, u + \lambda v, y) = \varphi(t, u, y) + \lambda v$$

for any $(t, u, y, \lambda) \in \mathbb{S}_+ \times V \times Y \times \mathbb{R}$.

**Lemma 6.** *[22, 28] Assume that the following conditions are fulfilled:*

1. *$(X, \mathbb{S}_+, \pi)$ is the skew-product dynamical system on $X := V \times Y$ generated by cocycle $\varphi$;*
2. *the subset $\mathcal{E} \subset V$ is invariant with respect to translation by a strongly positive vector $v$, i.e., $v \gg 0$ and $u + \lambda v \in \mathcal{E}$ for any $u \in \mathcal{E}$ and $\lambda \in \mathbb{R}$;*
3. *the set $\mathcal{E} \times Y$ is invariant with respect to skew-product dynamical system $(X, \mathbb{S}_+, \pi)$.*

*Then*

1. *the cocycle $\varphi$ is positively uniformly stable;*
2. *every positive semitrajectory of the skew-product dynamical system $(\mathcal{E} \times Y, \mathbb{S}_+, \pi)$ is uniformly positively stable;*
3. *$\|\varphi(t, u_1, y) - \varphi(t, u_2, y)\|_v \le \|u_1 - u_2\|_v$ for any $u_1, u_2 \in E$, $y \in Y$ and $t \ge 0$.*

*Proof.* Let $(u_1, y) \in X \times Y$. Then we shall prove that

$$\|\varphi(t, u_1, y) - \varphi(t, u_2, y)\|_v \le \|u_1 - u_2\|_v, \tag{4}$$

for all $t >$. By the definition of $v$-norm,

$$-\|u_1 - u_2\|_v v \le u_1 - u_2 \le \|u_1 - u_2\|_v v,$$

for all $u_1, u_2 \in E$, that is,

$$u_2 - \|u_1 - u_2\|_v v \le u_1 \le u_2 + \|u_1 - u_2\|_v v,$$

for all $u_1, u_2 \in E$. This inequality implies together with monotonicity and positive translation invariance that for all $u_1, u_2 \in E, t > 0$,

$$\varphi(t, u_2, y) - \|u_1 - u_2\|_v v \le \varphi(t, u_1, y) \le \varphi(t, u_2, y) + \|u_1 - u_2\|_v v.$$

Equivalently, for all $u_1, u_2 \in E, t > 0$,

$$-\|u_1 - u_2\|_v v \le (\varphi(t, u_1, y) - \varphi(t, u_2, y) \le \|u_1 - u_2|_v v. \tag{5}$$

(4) immediately follows from (5) and the definition of $v$-norm and, consequently, $\varphi$ is positively uniformly stable. By the cocyle property, we have that

$$\varphi(t + \tau, u, y) = \varphi(t, \varphi(\tau, u, y), \sigma(\tau, y)) \ \text{ for all } u \in E,\ y \in Y,\ \ t, \tau > 0.$$

From this cocyle property together with (4), we conclude that

$$\|\varphi(t + \tau, u_1, y) - \varphi(t + \tau, u_2, y)\|_v \le \|\varphi(\tau, u_1, y) - \varphi(\tau, u_2, y)\|_v, \ \text{ for all } \ t, \tau > 0.$$

This proves that every forward orbit of $(X, \mathbb{R}_+, \pi)$ is uniformly stable in the order-norm. The normality of the cone $P$ implies that every forward orbit of $(X, \mathbb{R}_+, \pi)$ is uniformly stable. $\square$

**Theorem 14.** *Let $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be a cocycle over dynamical system $(Y, \mathbb{S}, \sigma)$ with the fiber $W$. Assume that the dynamical system $(Y, \mathbb{S}, \sigma)$ is minimal, $q \in \omega_q$ for any $q \in Y$ and the cocycle $\varphi$ is translation invariant with respect to $v \in Int(P)$.*

*Under the conditions **(C1)**, **(C2.1)** and **(C3)**-**(C4)** for any $(x_0, y_0) \in V \times Y$ the following statements hold:*

1. *for any $q \in Y$ the set*
$$\omega_{(x_0,y_0)} \bigcap X_q$$
*consists of a single point $\{(x_q, q)\}$;*
2. *the point $(x_q, q)$ is strongly comparable by character of recurrence with the point $q \in Y$;*
3. $$\lim_{t\to+\infty} \rho(\varphi(t, x_0, y_0), \varphi(t, x_{y_0}, y_0)) = 0.$$

*Proof.* Since the cocycle $\varphi$ is monotone and translation invariant with respect to $v \in Int(P)$, then by Lemma 6

1. every trajectory $\varphi(t, u, y)$ ($(u, y) \in W \times Y$) is positively uniformly Lyapunov stable;
2. every semi-trajectory $\Sigma_x^+$ of skew-product dynamical system $(X, \mathbb{S}_+, \pi)$ ($X := W \times Y$ and $\pi := (\varphi, \sigma)$) is conditionally precompact and $\omega_x \neq \emptyset$.

Now to finish the proof of Theorem it is sufficient to apply Theorem 12. □

**Corollary 6.** *Under the conditions of Theorem 14 if the point $y_0$ is $\tau$-periodic (respectively, quasi-periodic, Bohr almost periodic, recurrent, strongly Poisson stable and $H(y_0)$ is a minimal set), then:*

1. *the point $x_{y_0}$ is so;*
2. *the point $x_0$ is asymptotically $\tau$-periodic (respectively, asymptotically quasi-periodic, asymptotically Bohr almost periodic, asymptotically recurrent, asymptotically strongly Poisson stable).*

**Definition 40.** A non-autonomous set $\{A_y |\ y \in Y\}$ is said to be

1. positively (respectively, negatively) invariant (with respect to cocycle $\varphi$) if $\varphi(t, A_y, y) \subseteq A_{\sigma(t,y)}$ (respectively, $\varphi(t, A_y, y) \supseteq A_{\sigma(t,y)}$) for any $y \in Y$ and $t \geq 0$;
2. invariant if it is positively and negatively invariant.

**Lemma 7.** *[19] Assume that the set $Y$ is invariant, that is, $\sigma(t, Y) = Y$ for any $t \in \mathbb{T}$. The non-autonomous set $\{A_y |\ y \in Y\}$ is positively invariant (respectively, negatively invariant or invariant) if and only if the set $\mathbb{A}$ is a positively invariant (respectively, negatively invariant or invariant) subset of skew-product dynamical system $(X, \mathbb{T}, \pi)$.*

**Lemma 8.** *[19] The following statements are equivalent:*

1. *for any compact subset $K \subseteq Y$ the set $\bigcup\{A_y |\ y \in K\}$ is precompact in $W$;*
2. *the set $\mathbb{A} \subseteq X$ is conditionally precompact in $(X, h, Y)$ ($X = W \times Y$ and $h := pr_2 : X \to Y$).*

**Corollary 7.** *Let $\{A_y |\ y \in Y\}$ be a uniformly precompact non-autonomous set, then the set $\mathbb{A}$ is a conditionally compact subset of $X$ with respect to $(X, h, Y)$, where $h = pr_2$.*

**Lemma 9.** *[19] Let $\{I_y |\ y \in Y\}$ be a non-autonomous set. Assume that the set $\mathbb{J} = \bigcup\{J_y = I_y \times \{y\} |\ y \in Y\}$ is conditionally precompact, then the following statements are equivalent:*

1. *the mapping $y \to I_y$ is upper semicontinuous;*
2. *the set $\mathbb{J}$ is closed in $X$.*

**Definition 41.** A trajectory $\varphi(t, u_0, y_0)$ ($x_0 := (u_0, y_0) \in W \times Y$) of cocycle $\varphi$ is said to be precompact if $Q_{(u_0,y)} := \overline{\varphi(\mathbb{T}_+, u_0, y_0)}$ is a compact subset of $W$.

**Lemma 10.** *Suppose that $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ is a cocycle under $(Y, \mathbb{S}, \sigma)$ with the fibre $W$ and $\mathbb{K} = \{K_y :\ y \in Y\}$ is a non-autonomous set with $K_y \subset W$ for any $y \in Y$. Assume that the following conditions hold:*

1. *$Y$ is a compact space;*
2. *$\mathbb{K} = \{K_y :\ y \in Y\}$ is uniformly precompact;*
3. *$\mathbb{K} = \{K_y :\ y \in Y\}$ is upper semicontinuous.*

*Then the following conditions are equivalent:*

*a) the non-autonomous set $\mathbb{K} = \{K_y :\ y \in Y\}$ is positively uniformly Lyapunov stable;*

*b) every precompact trajectory $\varphi(t, u_0, y_0)$ ($x_0 := (u_0, y_0) \in \mathcal{K} := \bigcup\{K_y \times \{y\} :\ y \in Y\}$) of cocycle $\varphi$ is positively uniformly Lyapunov stable.*

*Proof.* Taking into consideration that the implication $a) \Rightarrow b)$ is evident it is sufficient to show that b) implies a). If we suppose that it is not true then there are a positive number $\varepsilon_0$, sequences $\{y_k\} \subseteq Y$, $\{u_k^0\}$ ($u_k^0 \in K_{y_k}$), and $\{u_k\}$ ($u_k \in W$), $\{t_k\} \subset \mathbb{S}_+$ such that

$$\rho(u_k, u_k^0) < \delta_k \quad \text{and} \quad \rho(\varphi(t_k, u_k, y_k), \varphi(t_k, u_k^0, y_k)) \geq \varepsilon_0. \tag{6}$$

Since $\mathbb{K}$ is uniformly precompact and the space $Y$ is compact, then without loss of generality we can suppose that the sequences $\{u_k^0\}$ $\{u_k\}$ and $\{y_k\}$ are convergent.

Denote by $u_0 := \lim\limits_{k\to\infty} u_k^0 = \lim\limits_{k\to\infty} u_k$ and $y_0 := \lim\limits_{k\to\infty} y_k$. Since $\{K_y : \ y \in Y\}$ is upper semicontinuous, then $u_0 \in K_{y_0}$. By condition b) for $y_0 \in Y$, $u_0 \in K_{y_0}$ and $\varepsilon_0$ there exists a positive number $\delta_0 := \delta(\varepsilon_0/3, u_0, y_0) > 0$ such that

$$\rho(\varphi(t_0, u, y_0), \varphi(t_0, u_0, y_0)) < \delta$$

implies

$$\rho(\varphi(t, u_0, y_0), \varphi(t, u_0, y_0)) < \varepsilon_0/3$$

for any $t \geq t_0 \geq 0$. Let $k_0 = k_0(\varepsilon_0/3)$ be a natural number such that $\rho(u_k, u_0) < \delta_0$ for any $k \geq k_0$ and, consequently,

$$\rho(\varphi(t, u_k, y_0), \varphi(t, u_0, y_0)) < \varepsilon_0/3 \tag{7}$$

for $t \geq 0$. Then from (7) we obtain

$$\begin{aligned} \rho(\varphi(t, u_k, y_0), \varphi(t, u_k^0, y_0)) \leq \rho(\varphi(t, u_k, y_0), \varphi(t, u_0, y_0)) + \\ \rho(\varphi(t, u_0, y_0), \varphi(t, u_k^0, y_0)) < \tfrac{\varepsilon_0}{3} + \tfrac{\varepsilon_0}{3} < \varepsilon_0 \end{aligned} \tag{8}$$

for any $t \geq 0$. Inequalities (6) and (8) are contradictory. The obtained contradiction proves our statement. □

Let $E$ be a real Banach space and $P \subset E$ be a cone in E with $Int(P) \neq \emptyset$ and $W \subseteq E$.

**Theorem 15.** *Let $\langle W, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ (shortly $\varphi$) be a cocycle over dynamical system $(Y, \mathbb{T}, \sigma)$ with the fiber $W$. Assume that the following conditions are fulfilled:*

1. *the cone $P$ is normal;*
2. *the space $Y$ is compact and $(Y, \mathbb{T}, \sigma)$ is minimal;*
3. *the cocycle $\varphi$ is monotone and translation invariant with respect to $v \in Int(P)$;*
4. *the cocycle $\varphi$ satisfies (**C1**);*
5. *the cocycle $\varphi$ admits a uniformly precompact global attractor $\boldsymbol{I} = \{I_y : \ y \in Y\}$.*

*Then the following statements hold:*

1. *$\alpha(y) \leq u \leq \beta(y)$ for any $u \in I_y$ and $y \in Y$;*
2. *$\alpha(y), \beta(y) \in I_y$ and, consequently, $I_y \subseteq [\alpha(y), \beta(y)]$;*
3. *$\varphi(t, \alpha(y), y) = \alpha(\sigma(t, y))$ (respectively, $\varphi(t, \beta(y), y) = \beta(\sigma(t, y))$) for any $t \geq 0$;*
4. *the point $\gamma_*(y) := (\alpha(y), y) \in X = W \times Y$ (respectively, $\gamma^*(y) := (\beta(y), y) \in X$) is strongly comparable by character of recurrence with the point $y$;*

5. *if $u \in W$ and $u \le \alpha(y)$ (respectively, $u \ge \beta(y)$), then $\omega_x \bigcap X_y = \{\gamma_*(y)\}$ (respectively, $\omega_x \bigcap X_y = \{\gamma^*(y)\}$), where $x := (u, y)$;*

6. *if $u \le \alpha(y)$ (respectively, $u \ge \beta(y)$),then*

$$\lim_{t \to +\infty} \rho(\varphi(t, u, y).\gamma_*(\sigma(t, y))) = 0$$

*(respectively,*

$$\lim_{t \to +\infty} \rho(\varphi(t, u, y).\gamma^*(\sigma(t, y))) = 0).$$

*Proof.* Since the cocycle $\varphi$ is monotone and translation invariant with respect to $v \in Int(P)$, then by Lemma 6

1. the cocycle $\varphi$ is positively uniformly stable;

2. every trajectory $\varphi(t, u, y)$ ($(u, y) \in W \times Y$) is positively uniformly Lyapunov stable;

3. the uniformly compact global attractor $\mathbf{I} = \{I_y : \ y \in Y\}$ is positively uniformly Lyapunov stable.

Now to finish the proof of Theorem it is sufficient to apply Theorem 13. □

**Corollary 8.** *Under the conditions of Theorem 15 the following statements take place:*

1. *if the point $y$ is $\tau$-periodic (respectively, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent), then the full trajectory $\gamma_y$ passing through the point $(\alpha(y), y)$ (respectively, through the point $(\beta(y), y)$) is so.*

*Proof.* This statement follows from Theorem 15 and Corollary 5. □

# 8 Application

## 8.1 Time-dependent chemical reaction networks

In the works of Angeli and Sontag [1, 2] and Angeli, Leenheer and Sontag [3] the authors have contributed a new type of global convergence condition, named positive translation invariance, which is motivated by a chemical reaction network. A standard form for representing (well-mixed and isothermal) chemical reactions by ordinary differential equations is

$$S' = \Gamma R(S), \tag{9}$$

evolving on the nonnegative orthant $\mathbb{R}^m_+$, where $S$ is an $m$-vector specifying the concentrations of $m$ chemical species, $\Gamma : \mathbb{R}^n \to \mathbb{R}^m$ is the stoichiometry matrix, and

$R : \mathbb{R}^m_+ \to \mathbb{R}^n$ is a function which provides the vector of reaction rates for any given vector of concentrations. Choosing $\mu \in \mathbb{R}^m_+$ and using the reaction coordinates $x$,

$$S = \mu + \Gamma x,$$

instead of the traditional species coordinates $S$, the authors of [1]–[3] have investigated the monotonicity and global behavior of systems in the reaction coordinates,

$$x' = f_\mu(x) = R(\mu + \Gamma x), \tag{10}$$

evolving on the state space $X_\mu := \{x \in \mathbb{R}^n | \ \mu + \Gamma x \geq 0\}$.

Suppose that the matrix $\Gamma$ has rank exactly $n - 1$ and its kernel is spanned by a strongly positive vector $v$. Then the state space is invariant with respect to translation by $v$, namely,

$$x \in X_\mu \Rightarrow x + \lambda v \in X, \ \forall \lambda \in \mathbb{R},$$

and the solution $\varphi(t, \xi)$ generated by (10) enjoys positive translation invariance:

$$\varphi(t, \xi + \lambda v) = \varphi(t, \xi) + \lambda v, \ \forall \ x \in X_\mu \text{ and } \lambda \in \mathbb{R}.$$

Motivated by the study of Angeli and Sontag [1,2], Angeli, Leenheer and Sontag [3] and Hongxiao Hu and Jifa Jiang [22], we shall investigate the nonautonomous chemical reaction network. Suppose that the reaction rates depend on time

$$S' = \Gamma R(t, S), \tag{11}$$

where $R(t, S)$ is almost periodic (respectively, quasi-periodic, Bohr almost periodic, automorphic, Birkhoff recurrent, Levitan almost periodic, Bebutov almost recurrent, Poisson stable) in $t$. Choosing $\mu \in \mathbb{R}^m_+$ and using the reaction coordinates $x : S = \mu + \Gamma x$, we transform (11) into a system in the reaction coordinates:

$$x' = F_\mu(t, x) := R(t, \mu + \Gamma x) \tag{12}$$

evolving on the state space $X_\mu$.

*Remark* 11. Note that $X_\mu$ is an ordered convex closed subset of $\mathbb{R}^n$.

Let $U$ be a subset of $\mathbb{R}^m$. Denote by $C(\mathbb{T} \times U, \mathbb{R}^n)$ the space of all continuous functions $F : \mathbb{T} \times U \to \mathbb{R}^n$ equipped with the compact-open topology. This topology can be generated, for example, by the following distance $d$:

$$d(F, G) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_k(F, G)}{1 + d_k(F, G)},$$

where $d_k(F, G) := \max\{(t, x) \in \mathbb{T} \times U | \ |t| \leq k, \ |x| \leq k\}$ $(k \in \mathbb{N})$. For $F \in C(\mathbb{T} \times U, \mathbb{R}^n)$ and $\tau \in \mathbb{T}$ we denote by $F^\tau$ the $\tau$-translation of $F$ with respect to time $t$, i.e., $F^\tau(t, x) := F(t+\tau, x)$ for any $(t, x) \in \mathbb{T} \times U$ and by $(C(\mathbb{T} \times U, \mathbb{R}^n), \mathbb{T}, \lambda)$ the shift (Bebutov's) dynamical system on the space $C(\mathbb{T} \times U, \mathbb{R}^n)$. Let $f \in C(\mathbb{T} \times U, \mathbb{R}^n)$

and denote by $H(f)$ its hull, i.e., $H(f) := \overline{\{f^\tau | \ \tau \in \mathbb{T}\}}$, where by bar the closure in the space $C(\mathbb{T} \times U, \mathbb{R}^n)$ is denoted.

*Condition* (**A1**). A function $F_\mu \in C(\mathbb{R} \times X_\mu, \mathbb{R}^n)$ is regular, that is, for any $u \in X_\mu$ and $G \in H(F_\mu)$ there exists a unique solution $\varphi(t, u, G)$ of equation

$$u' = G(t, u) \tag{13}$$

defined on $\mathbb{R}_+$.

Assume that the function $F_\mu \in C(\mathbb{R} \times X_\mu, \mathbb{R}^n)$ is regular. Let $\phi(t, v, G)$ denote the solution of (13) passing through $v$ at $t = 0$. Then it enjoys positive translation invariance:

$$\phi(t, \xi + \lambda v, G) = \varphi(t, v, G) + \lambda v, \ \forall \ \xi \in X_\mu, \ \lambda \in \mathbb{R} \text{ and } G \in H(F_\mu), \tag{14}$$

where $H(F_\mu)$ is the hull of $F_\mu$. So the skew-product flow induced by $H$-class

$$v' = G(t, v) \ \ (G \in H(F_\mu)) \tag{15}$$

of system (12) has positive translation invariance.

**Lemma 11.** *[22] Suppose that the function $F_\mu$ is regular. If the positive orthant $\mathbb{R}^m_+$ is positively invariant for (11), then $X_\mu$ is invariant under(12).*

Recall that the set $\mathbb{R}^m_+$ (respectively, the set $X_\mu$) is positively invariant with respect to (11) (respectively, with respect to (12)) if for any $(S_0, \tilde{R}) \in \mathbb{R}^m_+ \times H(R)$ (respectively, $(u_0, G) \in X_\mu \times H(F_\mu)$) $\phi(t, S_0, \tilde{R}) \in \mathbb{R}^m_+$
(respectively, $\phi(t, u_0, G) \in X_\mu$) for any $t \in \mathbb{R}_+$.

The purpose of this paper is to study the periodic (respectively, quasi-periodic, Bohr almost periodic, automorphic, recurrent in the sense of Birkhoff, Levitan almost periodic, Bebutov almost recurrent, Poisson stable) solutions and compact global attractors of monotone equation (12) (respectively, (15)) possessing a positively translation-invariant property (14).

With the above notation, a chemical reaction network is described by the following differential equations:

$$S' = \Gamma R(t, S), \ t > 0, \ S(0) = S_0 \in \mathbb{R}^m_+, \tag{16}$$

where $\mathbb{R}^m_+ := \{S \in \mathbb{R}^m | \ S_i \geq 0\}$. Of course, $S_0$ is the initial concentration of all species and R(t,S) is a time-dependent vector-valued function. Given a chemical reaction network (16), following [28] we introduce the so-called associate system in reaction coordinates. For any $\mu \in \mathbb{R}^m_+$, such a system in reaction coordinates is defined as the following nonautonomous system:

$$u' = F_\mu(t, u), \ t > 0, \ u(0) = u_0 \in X_\mu, \tag{17}$$

where $F_\mu(t, u) := f(t, \mu + \Gamma u)$. Here $u = (u_1, \ldots, u_n)$ is called the extent of the reaction [29]. For systems (16) and (17), let $H(F_\mu)$ and $H(f)$ be the hull of $F_\mu$ and $f$, respectively.

Denote by $X_{(\mu,\Gamma)} := \{\mu + \Gamma(u) | \ u \in X_\mu\}$.

**Lemma 12.** *If $rank(\Gamma) = n-1$ and its kernel is spanned by a strongly positive vector $v$, then $X_{(\mu,\Gamma)}$ is a closed subset of $\mathbb{R}^m_+$.*

*Proof.* To prove this statement it is sufficient to show that $\overline{X_{(\mu,\Gamma)}} \subseteq X_{(\mu,\Gamma)}$, where by bar the closure of the set $X_{(\mu,\Gamma)}$ in the space $\mathbb{R}^m$ is denoted.

Let $U \in \overline{X_{(\mu,\Gamma)}}$, then there exists a sequence $\{U_k\} \subset X_{(\mu,\Gamma)}$ such that $U_k \to U$ as $k \to \infty$. Since $U_k \in X_{(\mu,\Gamma)}$, then there exists an element $u_k \in X_\mu$ such that $U_k = \mu + \Gamma u_k$. On the other hand we have $\mathbb{R}^n = \mathfrak{B}_1 \bigoplus \mathfrak{B}_2$, where $\mathfrak{B}_2 = Span\{v\}$. Since $rank(\Gamma) = n-1$ and $\mathfrak{B}_2 = Span\{v\}$, then the subspaces $rank(\Gamma)$ and $\mathfrak{B}_1$ are isomorphic. Thus there exists a unique element $u^i_k \in \mathfrak{B}_i$ $(i = 1,2)$ such that $u_k = u^1_k + u^2_k$. Note that $\Gamma(u_k) = \Gamma(u^1_k) + \Gamma(u^2_k) = \Gamma(u^1_k)$ and, consequently, $\mu+\Gamma(u^1_k) = \mu+\Gamma(u_k) \geq 0$, i.e., $u^1_k \in X_\mu$. Since $U_k - \mu = \Gamma(u^1_k) \to U - \mu$ as $k \to \infty$, then the sequence $\{u^1_k\}$ is convergent in $\mathbb{R}^n$. Denote by $u^1 = \lim\limits_{k\to\infty} u^1_k$, then $u^1 \in X_\mu$ and $U = \mu + \Gamma(u^1) \in X_{(\mu,\Gamma)}$. Lemma is proved. □

Let $W$ be a subset of $\mathbb{R}^m_+$.

**Definition 42.** A function $f \in C(\mathbb{T} \times W, \mathbb{R}^n)$ is said to be Lagrange stable if the set $\Sigma_f := \bigcup\{f^\tau |\ \tau \in \mathbb{T}\}$ is precompact.

Let $\Phi$ be the mapping from $C(\mathbb{T} \times X_{(\mu,\Gamma)}, \mathbb{R}^n)$ into $C(\mathbb{T} \times X_\mu, \mathbb{R}^n)$ defined by equality

$$\Phi(f) := F_\mu,$$

where $F_\mu(t,u) := f(t, \mu + \Gamma u)$ for any $(t,u) \in \mathbb{T} \times X_\mu$.

**Lemma 13.** *The following statements hold:*

1. *the mapping $\Phi$ is continuous;*
2. *$\Phi(f_1) \neq \Phi(f_2)$ for any $f_1, f_2 \in C(\mathbb{T} \times X_{(\mu,\Gamma)}, \mathbb{R}^n)$ with $f_1 \neq f_2$;*
3. *$\Phi(\lambda(\tau, f)) = \lambda(\tau, \Phi(f))$ for any $(\tau, f) \in \mathbb{T} \times C(\mathbb{T} \times X_{(\mu,\Gamma)}, \mathbb{R}^n)$, i.e., $\Phi$ is a homomorphism of dynamical system $(C(\mathbb{T} \times X_{(\mu,\Gamma)}, \mathbb{R}^n), \mathbb{T}, \lambda)$ into $(C(\mathbb{T} \times X_\mu, \mathbb{R}^n), \mathbb{T}, \lambda)$;*
4. *$\Phi(H(f)) \subseteq H(\Phi(f))$ for any $f \in C(\mathbb{T} \times X_{(\mu,\Gamma)}, \mathbb{R}^n)$;*
5. *if the function $f$ is Lagrange stable, then $\Phi(H(f)) = H(\Phi(f))$.*

*Proof.* The first three statements of Lemma are evident.

Let now $g \in H(f)$, then there is a sequence $\{\tau_n\} \subset \mathbb{T}$ such that $f^{\tau_n} \to g$ in $C(\mathbb{T} \times X_{(\mu,\Gamma)}, \mathbb{R}^n)$ as $n \to \infty$. Then taking into consideration the second and third statements of Lemma 13 we obtain

$$\Phi(g) = \Phi(\lim_{n\to\infty} f^{\tau_n}) = \lim_{n\to\infty} \Phi(f^{\tau_n}) = \lim_{n\to\infty} \Phi^{\tau_n}(f) := G \in H(\Phi(f)),$$

i.e., $\Phi(H(f)) \subseteq H(\Phi(f))$.

If the function $f$ is Lagrange stable, then we will establish that the converse inclusion $H(\Phi(f)) \subseteq \Phi(H(f))$ is also true. In fact if $G \in H(\Phi(f))$, then there exists a sequence $\{\tau_n\} \subseteq \mathbb{T}$ such that $\Phi^{\tau_n}(f) \to G$ as $n \to \infty$. Note that $\Phi^{\tau_k}(t,u) = f(t+\tau_k, \mu + \Gamma u)$ for any $(t,u) \in \mathbb{T} \times X_\mu$. Since $f$ is Lagrange stable, then there exists a subsequence $\{\tau_{k_m}\} \subset \{\tau_k\}$ such that the sequence $\{f^{\tau_{k_m}}\}$ converges in the space $C(\mathbb{T} \times X_{(\mu,\Gamma)}, \mathbb{R}^n)$. Denote by $h(t,x) = \lim\limits_{m\to\infty} f(t+\tau_{k_m}, x)$ for any $(t,x) \in \mathbb{T} \times X_{(\mu,\Gamma)}$, then $h \in H(f)$ and $\Phi(h) = \lim\limits_{n\to\infty} \Phi^{\tau_{k_m}}(f)$ and, consequently, $\Phi(h) \in H(\Phi(f))$. Notice that

$$G := \lim_{m\to\infty} \Phi^{\tau_{k_m}}(f) = \lim_{m\to\infty} \Phi(f^{\tau_{k_m}}) = \Phi(\lim_{m\to\infty} f^{\tau_{k_m}}) = \Phi(h) \in H(\Phi(f)),$$

i.e., $H(\Phi(f)) \subseteq \Phi(H(f))$. Lemma is completely proved. □

**Corollary 9.** *The following statements hold:*

1. *$\mathfrak{N}_f \subseteq \mathfrak{N}_{\Phi(f)}$ and, consequently if the function $f$ is stationary (respectively, $\tau$-periodic, Levitan almost periodic, almost recurrent, Poisson stable) in $t \in \mathbb{T}$ uniformly with respect to $x$ on every compact subset from $X_{(\mu,\Gamma)}$ then the function $\Phi(f)$ is so;*

2. *$\mathfrak{M}_f \subseteq \mathfrak{M}_{\Phi(f)}$ and, consequently if the function $f$ is stationary (respectively, $\tau$-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent, Poisson stable) in $t \in \mathbb{T}$ uniformly with respect to $x$ on every compact subset from $X_{(\mu,\Gamma)}$ then the function $\Phi(f)$ is so;*

3. *If the function $f$ is Lagrange stable, then $\Phi$ is a homomorphism of dynamical systems $(H(f), \mathbb{T}, \lambda)$ onto $(H(\Phi(f)), \mathbb{T}, \lambda)$ and, consequently, $\mathfrak{M}_f = \mathfrak{M}_{\Phi(f)}$.*

*Proof.* This statement follows from Lemma 13. □

*Remark* 12. According to Corollary 9 (item ii) the function $\Phi(f)$ is strongly comparable by character of recurrence with the function $f$. Moreover, there is a stronger statement. Namely, the function $\Phi(f)$ is uniformly comparable with the function $f$, i.e., for any $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ such that $d(\lambda(\tau_1, f), \lambda(\tau_2, f)) < \delta$ implies $d(\lambda(\tau_1, \Phi(f)), \lambda(\tau_2, \Phi(f))) < \varepsilon$.

As a consequence from Lemma 13 (item (v)) if the function $f$ is Lagrange stable, then for any $G \in H(F_\sigma)$ there exists a unique $h \in H(f)$ $(G = \Phi(h))$ such that

$$G(t,u) = h(t, \mu + \Gamma u).$$

In particular, $G = F_\sigma$ if and only if $h = f$. For every $G \in H(F_\mu)$ and $h \in H(f)$ in (18), let $\varphi(t, x_0, h)$ and $\phi(t, u_0, G)$ be the solutions of

$$x' = \Gamma h(t,x),\ t > 0,\ x(0) = x_0 \in \mathbb{R}^m_+ \tag{18}$$

and

$$u' = G(t,u),\ t > 0,\ u(0) = u_0 \in X_\mu, \tag{19}$$

respectively.

**Lemma 14.** *[28] Let the function $f \in C(\mathbb{T} \times X_{(\mu,\Gamma)}, \mathbb{R}^n)$ be Lagrange stable and $\varphi(t, x_0, h), \phi(t, u_0, G)$ be the solutions of (18) and (19), respectively. Then we have*

$$\varphi(t, \mu + \Gamma u_0, h) = \mu + \Gamma\phi(t, u_0, \Phi(h))$$

*for any $t \in \mathbb{T}$, $u_0 \in X_\mu$ and $h \in H(f)$.*

*Condition* (**A2**). Equation (17) is monotone (respectively, strongly monotone). This means that the cocycle $\langle \mathbb{R}^n, \varphi, (H(F_\mu), \mathbb{R}, \sigma)\rangle$ generated by (17) is monotone (respectively, strongly monotone), i.e. if $u, v \in \mathbb{R}^d$ and $u \le v$ (respectively, $u < v$) then $\varphi(t, u, G) \le \varphi(t, v, G)$ (respectively, $\varphi(t, u, G) \ll \varphi(t, v, G)$) for all $t \ge 0$ and $G \in H(F_\mu)$.

**Definition 43.** Let $f \in C(\mathbb{R} \times W, \mathbb{R}^n)$. The set $H(f)$ is said to be minimal if it is a minimal set of shift dynamical system $(C(\mathbb{R} \times W, \mathbb{R}^n), \mathbb{R}, \sigma)$.

**Definition 44.** A function $f \in C(\mathbb{R} \times W, \mathbb{R}^n)$ is said to be strongly Poisson stable if every function $g \in H(f)$ is Poisson stable.

*Remark* 13. If the function $f \in C(\mathbb{R} \times W, \mathbb{R}^n)$ is time almost periodic, then

1. the set $H(f)$ is minimal;
2. every function $h \in H(f)$ is almost recurrent and, consequently, $f$ is strongly Poisson stable.

**Theorem 16.** *Suppose that the following conditions hold:*

1. *$\mu \in \mathbb{R}^m$ is such that the system (17) is strongly monotone;*
2. *the set $H(F_\mu)$ is minimal and $F_\mu$ is strongly Poisson stable;*
3. *the matrix $\Gamma$ has rank exactly $n-1$ whose kernel is spanned by a strongly positive vector $v$;*
4. *for any $G \in H(F_\mu)$ all forward solutions of equation (19) are bounded.*

*Then for any $U_0 \in X_{(\mu,\Gamma)}$ the following statements hold:*

1. *the set $\omega_{(U_0,f)} \bigcap X_f$ consists of a single point $p_0 = (V_0, f)$, where $\omega_{(U_0,f)}$ is the $\omega$-limit set of the motion $\pi(t,(U_0,f))$ of the skew-product dynamical system $(X, \mathbb{R}_+, \pi)$ ($X := X_{(\mu,\Gamma)} \times H(f)$, $\pi := (\varphi, \sigma)$) and $X_f := X_{(\mu,\Gamma)} \times \{f\}$;*
2. *the solution $\varphi(t, V_0, f)$ of equation (18) is defined on $\mathbb{R}$, $\overline{\varphi(\mathbb{R}, v_0, f)} \subseteq Q_+^{(U_0,f)}$ and it is strongly compatible;*
3. $$\lim_{t\to\infty} |\varphi(t, U_0, f) - \varphi(t, V_0, f)| = 0.$$

*Proof.* Let $(H(F_\mu),\mathbb{R},\sigma)$ (respectively, $(H(f),\mathbb{R},\sigma)$) be the shift dynamical system on $H(F_\mu)$ (respectively, on $H(f)$). Denote by $\langle X_\mu,\phi,(H(F_\mu),\mathbb{R},\sigma)\rangle$ (respectively, $\langle X_{(\mu,\Gamma)},\varphi,(H(f),\mathbb{R},\sigma)\rangle$, where $X_{(\mu,\Gamma)} = \mu+\Gamma(X_\mu) := \{\mu+\Gamma u|\ u\in X_\mu\}$) the cocycle generated by family of equations (19) (respectively, (18)). Note that under the conditions of Theorem 16 Conditions (**C1**), (**C3**) and (**C4**) are fulfilled for the cocycle $\langle X_\mu,\phi,(H(F_\mu),\mathbb{R},\sigma)\rangle$. Let $U_0\in X_{(\mu,\Gamma)}$, then there exists a point $u_0\in X_\mu$ such that $U_0=\mu+\Gamma(u_0)$. By equality (14) the cocycle $\phi$ is translation invariant with respect to vector $v\gg 0$. According to Theorem 14 for given $u_0\in X_\mu$ the following statements are fulfilled:

a. the set $\omega_{(u_0,F_\mu)}\bigcap X_{F_\mu}$ consists of a single point $q_0=(v_0,F_\mu)$, where $\omega_{(u_0,F_\mu)}$ is the $\omega$-limit set of the motion $\pi(t,(u_0,F_\mu))$ of the skew-product dynamical system $(X,\mathbb{R}_+,\pi)$ ($X:=X_\mu\times H(F_\mu)$, $\pi:=(\phi,\sigma)$) and $X_{F_\mu}:=X_{(\mu,\Gamma)}\times\{F_\mu\}$;

b. the solution $\phi(t,V_0,F_\mu)$ of equation (17) is defined on $\mathbb{R}$, $\overline{\phi(\mathbb{R},v_0,F_\mu)}\subseteq Q_+^{(u_0,F_\mu)}$ and it is strongly compatible;

c.
$$\lim_{t\to\infty}|\phi(t,u_0,F_\mu)-\phi(t,v_0,F_\mu)|=0.$$

Denote by $V_0:=\mu+\Gamma(v_0)\in X_{(\mu,\Gamma)}$ and consider the solutions $\varphi(t,U_0,f)$ and $\varphi(t,V_0,f)$ of equation (18) ($h=f$). Since $\phi(\cdot,v_0,F_\mu)$ is a strongly compatible solution of equation (19) ($G=F_\mu$), then

$$\mathfrak{M}_{F_\mu}\subseteq\mathfrak{M}_{\phi(\cdot,v_0,F_\mu)}. \tag{20}$$

By Lemma 14 we have $\varphi(t,V_0,f)=\mu+\phi(t,v_0,F_\mu)$ for any $t\in\mathbb{R}$. Note that

$$\mathfrak{M}_{\phi(\cdot,v_0,F_\mu)}\subseteq\mathfrak{M}_{\varphi(\cdot,V_0,f)}. \tag{21}$$

Indeed if $\{t_k\}\in\mathfrak{M}_{\phi(\cdot,v_0,F_\mu)}$ then we have

$$\varphi(t+t_k,V_0,f)-\bar{\varphi}(t)=\mu+\Gamma\phi(t+t_k,v_0,F_\mu)-(\mu+\Gamma\bar{\phi}(t))=$$
$$\Gamma(\phi(t+t_k,v_0,F_\mu)-\bar{\phi}(t))\to 0$$

as $k\to\infty$ uniformly with respect to $t$ on every compact subset from $\mathbb{R}$, where $\bar{\phi}=\lim\phi(\cdot+t_k,v_0,F_\mu)$ in the space $C(\mathbb{R},\mathbb{R}^n)$. This means that $\{t_k\}\in\mathfrak{M}_{\varphi(\cdot,V_0,f)}$.

From (20) and (21) we have

$$\mathfrak{M}_{F_\mu}\subseteq\mathfrak{M}_{\varphi(\cdot,V_0,f)}. \tag{22}$$

Finally, from Corollary 9 (item (ii)) we have

$$\mathfrak{M}_f\subseteq\mathfrak{M}_{\Phi(f)}. \tag{23}$$

In virtue of (22)–(23) and taking into consideration the equality $\Phi(f)=F_\mu$ we obtain

$$\mathfrak{M}_f\subseteq\mathfrak{M}_{\varphi(\cdot,V_0,f)},$$

i.e., $\varphi(t, V_0, f)$ is a strongly compatible solution of equation (18) (for $h = f$).

To finish the proof of Theorem it is sufficient to note that

$$|\varphi(t, U_0, f) - \varphi(t, V_0, f)| = |(\mu + \Gamma\phi(t, u_0, F_\mu)) - (\mu + \Gamma\phi(t, v_0, F_\mu))| =$$
$$|\Gamma(\phi(t, u_0, F_\mu) - \phi(t, v_0, F_\mu))| \leq \|\Gamma\||\phi(t, u_0, F_\mu) - \phi(t, v_0, F_\mu)| \to 0$$

as $t \to \infty$. □

**Corollary 10.** *Under the conditions of Theorem 16 if the function $f \in C(\mathbb{R} \times X_{(\mu,\Gamma)}, \mathbb{R}^n)$ is stationary (respectively, $\tau$-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff, strongly Poisson stable) in time $t \in \mathbb{R}$, then for any $U_0 \in X_{(\mu,\Gamma)}$ the following statements hold:*

1. *the set $\omega_{(U_0,f)} \bigcap X_f$ consists of a single point $p_0 = (V_0, f)$;*
2. *$\varphi(t, V_0, f)$ is a stationary (respectively, $\tau$-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff, strongly Poisson stable) solution of equation (17);*
3. *$\lim\limits_{t\to+\infty} |\varphi(t, U_0, f) - \varphi(t, V_0, f)| = 0$, i.e., $\varphi(t, u_0, f)$ is asymptotically stationary (respectively, asymptotically $\tau$-periodic, asymptotically quasi-periodic, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotically recurrent in the sense of Birkhoff, asymptotically strongly Poisson stable).*

*Proof.* This statement follows from Theorem 16 and Corollary 6. □

Let $Y$ be a compact metric space, $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be a cocycle on the state space $\mathbb{R}^n$ and $(X, \mathbb{S}_+, \pi)$ be the corresponding skew-product dynamical system, where $X := \mathbb{R}^n \times Y$ and $\pi := (\varphi, \sigma)$.

**Definition 45.** The cocycle $\langle \mathbb{R}^n\varphi, (Y, \mathbb{S}, \sigma)\rangle$ is said to be dissipative if for any $y \in Y$ there is a positive number $r_y$ such that

$$\limsup_{t\to+\infty} |\varphi(t, u, y)| < r_y$$

for any $y \in Y$ and $u \in \mathbb{R}^n$, i.e., for all $u \in \mathbb{R}^n$ and $y \in Y$ there exists a positive number $L(u, y)$ such that $|\varphi(t, u, y)| < r_y$ for any $t \geq L(u, y)$.

**Theorem 17.** *[12, ChIII] Let $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be a cocycle over the dynamical system $(Y, \mathbb{S}, \sigma)$ with the fiber $\mathbb{R}^n$. Then the following statements are equivalent:*

1. *There exists a positive number $R$ such that*

$$\limsup_{t\to+\infty} |\varphi(t, u, y)| < R$$

*for all $u \in \mathbb{R}^n$ and $y \in Y$.*

2. *There is a positive number $r_1$ such that for all $u \in \mathbb{R}^n$ and $y \in Y$ there exists $\tau = \tau(u, y) > 0$ for which $|\varphi(\tau, u, y)| < r_1$.*

3. *There is a positive number $r_2$ such that*
$$\liminf_{t \to +\infty} |\varphi(t, u, y)| < r_2$$
*for all $u \in \mathbb{R}^n$ and $y \in Y$.*

4. *There exists a positive number $R_0$ and for all $R > 0$ there is $l(R) > 0$ such that $|\varphi(t, u, y)| \le R_0$ for all $t \ge l(R)$, $u \in \mathbb{R}^n$, $|u| \le R$ and $y \in Y$.*

*Remark* 14. 1. Note that every condition 1.-4. that figures in Theorem 17 is equivalent to the (compact) dissipativity of the non-autonomous dynamical system $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h \rangle$ associated by the cocycle $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{S}, \sigma) \rangle$ over $(Y, \mathbb{S}, \sigma)$ with the fiber $\mathbb{R}^n$.

2. Note that Theorem 17 remains true if we replace the space $\mathbb{R}^n$ by a closed subset $W$ of $\mathbb{R}^n$.

Consider the differential equation
$$u' = f(t, u), \tag{24}$$
where $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$. Along with the equation (24) we consider its $H$-class [5, 21, 27, 34, 37], i.e., the family of the equations
$$v' = g(t, v), \tag{25}$$
where $g \in H(f) = \overline{\{f_\tau : \tau \in \mathbb{R}\}}$ and $f_\tau(t, u) = f(t + \tau, u)$, with the bar indicating closure in the compact-open topology.

We will suppose that the function $f$ is regular. Denote by $\varphi(\cdot, v, g)$ the solution of (25) passing through the point $v \in \mathbb{R}^n$ for $t = 0$. Then the mapping $\varphi : \mathbb{R}_+ \times \mathbb{R}^n \times H(f) \to \mathbb{R}^n$ satisfies the following conditions (see, for example,[5, 31]):

1) $\varphi(0, v, g) = v$ for all $v \in \mathbb{R}^n$ and $g \in H(f)$;

2) $\varphi(t, \varphi(\tau, v, g), g_\tau) = \varphi(t + \tau, v, g)$ for each $v \in \mathbb{R}^n$, $g \in H(f)$ and $t, \tau \in \mathbb{R}_+$;

3) $\varphi : \mathbb{R}_+ \times \mathbb{R}^n \times H(f) \to \mathbb{R}^n$ is continuous.

Denote by $Y := H(f)$ and $(Y, \mathbb{R}, \sigma)$ a dynamical system of translations on $Y$, induced by the dynamical system of translations $(C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n), \mathbb{R}, \sigma)$. The triple $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{R}, \sigma) \rangle$ is a cocycle over $(Y, \mathbb{R}_+, \sigma)$ with the fiber $\mathbb{R}^n$. Hence, the equation (24) generates a cocycle $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{R}, \sigma) \rangle$ and the non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$, where $X := \mathbb{R}^n \times Y$, $\pi := (\varphi, \sigma)$ and $h := pr_2 : X \to Y$.

**Definition 46.** Recall that the equation (24) is called dissipative [21, 30, 39, 40] if for all $t_0 \in \mathbb{R}$ and $u_0 \in \mathbb{R}^n$ there exists a unique solution $x(t; u_0, t_0)$ of the equation (24) passing through the point $(u_0, t_0)$ and if there exists a number $R > 0$ such that

$\lim_{t\to+\infty} \sup |x(t;u_0,t_0)| < R$ for all $u_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}$. In other words, for every solution $x(t;u_0,t_0)$ there is an instant $t_1 = t_0 + l(t_0,u_0)$ such that $|x(t;u_0,t_0)| < R$ for any $t \geq t_1$. If for any $r > 0$ the number $l(t_0,u_0)$ can be chosen independently on $t_0$ and $u_0$ with $|u_0| \leq r$, then the equation (24) is called uniformly dissipative [21].

**Lemma 15.** *[12, ChIII] Let $W \subseteq \mathbb{R}^m$ and $f \in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$ be regular. If $H(f)$ is compact, then equation (24) is uniformly dissipative if and only if there is a positive number $r$ such that*

$$\limsup_{t\to+\infty} |\varphi(t,u_0,g)| < r \quad (u_0 \in W,\ g \in H(f))\ .$$

*Remark* 15. If $f \in C(\mathbb{R}\times W,\mathbb{R}^n)$ is regular, $H(f)$ is compact and then equation (24) is uniformly dissipative, then the cocycle $\varphi$ generated by equation (24) admits a compact global attractor.

**Theorem 18.** *Suppose that the following assumptions are fulfilled:*

- *$\mu \in \mathbb{R}^m$ is such that the system (17) is monotone;*
- *the matrix $\Gamma$ has rank exactly $n-1$ whose kernel is spanned by a strongly positive vector $v$;*
- *the function $F_\mu \in C(\mathbb{R}\times X_\mu,\mathbb{R}^n)$ is recurrent in $t \in \mathbb{R}$ uniformly with respect to $u$ on every compact subset from $X_\mu$;*
- *the cocycle $\phi$ generated by equation (17) admits a compact global attractor and $\boldsymbol{I} := \{I_G|\ G \in H(F_\mu)\}$ is its Levinson center.*

*Then under conditions $(\boldsymbol{A1})-(\boldsymbol{A2})$ the following statements hold:*

1. *$\alpha(G),\beta(G) \in I_G$ for any $G \in H(F_\mu)$ and, consequently, $I_G \subseteq [\alpha(G),\beta(G)]$;*
2. *$\phi(t,\alpha(G),G) = \alpha(\sigma(t,G))$ (respectively, $\phi(t,\beta(G),G) = \beta(\sigma(t,G))$) for any $t \geq 0$ and $G \in H(F_\mu)$;*
3. *the point $\gamma_*(F_\mu) := (\alpha(F_\mu),F_\mu) \in X = X_\mu \times Y$ (respectively, $\gamma^*(F_\mu) := (\beta(F_\mu),F_\mu) \in X$) is strongly comparable by character of recurrence with the point $F_\mu$;*
4. *for any $h \in H(f)$ equation (19) has at least two solutions $\varphi(t,U_0(h),h)$ ($U_0(h) = \mu + \Gamma\alpha(\Phi(h))$) and $\varphi(t,V_0(h),h)$ ($V_0(h) = \mu + \Gamma\beta(\Phi(h))$) defined and bounded on $\mathbb{R}$ which are strongly compatible and belong to Levinson center of (17);*
5. *if the function $F_\mu \in C(\mathbb{R}\times X_{(\mu,\Gamma)},\mathbb{R}^n)$ is quasi-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent, pseudo-recurrent and Lagrange stable, uniformly Poisson stable and stable in the sense of Lagrange) in $t \in \mathbb{R}$ uniformly with respect to $u$ on every compact subset from $X_{(\mu,\Gamma)}$,*

*then $\varphi(t, U_0(f), f)$ and $\varphi(t, V_0(f), f)$ are quasi-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent, pseudo-recurrent and Lagrange stable, uniformly Poisson stable and stable in the sense of Lagrange).*

*Proof.* Let $(H(F_\mu), \mathbb{R}, \sigma)$ (respectively, $(H(f), \mathbb{R}, \sigma)$) be the shift dynamical system on $H(F_\mu)$ (respectively, on $H(f)$). Denote by $\langle X_\mu, \phi, (H(F_\mu), \mathbb{R}, \sigma)\rangle$ (respectively, $\langle X_{(\mu,\Gamma)}, \varphi, (H(f), \mathbb{R}, \sigma)\rangle$, where $X_{(\mu,\Gamma)} = \mu + \Gamma(X_\mu) := \{\mu + \Gamma u |\ u \in X_\mu\}$) the cocycle generated by family of equations (19) (respectively, (18)). By equality (14) the cocycle $\phi$ is translation invariant with respect to vector $v \gg 0$. Applying Theorem 15 to nonautonomous dynamical system $\langle X_\mu, \phi, (H(F_\mu), \mathbb{R}, \sigma)\rangle$ we obtain the following statements:

1. $\alpha(G), \beta(G) \in I_G$ for any $G \in H(F_\mu)$ and, consequently, $I_G \subseteq [\alpha(G), \beta(G)]$, where $\alpha(G) := \inf I_G$ (respectively, $\beta(G) := \sup I_G$);

2. $\phi(t, \alpha(G), G) = \alpha(\sigma(t, G))$ (respectively, $\phi(t, \beta(G), G) = \beta(\sigma(t, G))$) for any $t \geq 0$ and $G \in H(F_\sigma)$;

3. the point $\gamma_*(F_\mu) := (\alpha(F_\mu), F_\mu) \in X = X_\mu \times Y$ (respectively, $\gamma^*(F_\mu) := (\beta(F_\mu), F_\mu) \in X$) is strongly comparable by character of recurrence with the point $F_\mu$.

Note that the nonautonomous dynamical system $\langle X_{(\mu,\Gamma)}, \varphi, (H(f), \mathbb{R}, \sigma)\rangle$ is compactly dissipative because $\varphi(t, U, h) = \mu + \Gamma\phi(t, u, \Phi(h))$ for any $h \in H(f)$ ($U = \mu + \Gamma u$) and the cocycle $\phi$ is so. Let $\mathbf{A} = \{A_h |\ h \in H(f)\}$ be the Levinson center for the compact dissipative cocycle $\varphi$ generated by equation (18). Denote by $U(h) := \mu + \Gamma\alpha(\Phi(h))$ and $V(h) := \mu + \Gamma\beta(\Phi(h))$. Then by Lemma 14 for any $h \in H(f)$

$$\varphi(t, U(h), h) = \mu + \Gamma\phi(t, \alpha(\Phi(h)), \Phi(h)) \tag{26}$$

(respectively,

$$\varphi(t, V(h), h) = \mu + \Gamma\phi(t, \beta(\Phi(h)), \Phi(h))) \tag{27}$$

is a bounded on $\mathbb{R}$ solution of equation (19). By Theorem 8 we have $U(h), V(h) \in A_h$, i.e., $U(h)$ and $V(h)$ belong to the Levinson center of the cocycle $\varphi$. Finally, from (26) (respectively, (27)) it follows that $\varphi(t, U(f), f)$ (respectively, $\varphi(t, V(f), f)$) is a strongly compatible solution of equation (18) for $h = f$, because $\phi(t, \alpha(\Phi(h), \Phi(h))$ (respectively, $\phi(t, \beta(\Phi(h), \Phi(h)))$ is a strongly compatible solution of equation (19), $\Phi : H(f) \to H(F_\mu)$ is a homeomorphism and $\Phi(f) = F_\mu$. Theorem is proved. $\square$

## 8.2 Translation-Invariant Discrete Monotone Systems

Consider the discrete version of chemical reactions by ordinary differential equations (9), i.e.,

$$\Delta S(k) = \Gamma R(S(k)),\ (\Delta S(k) := S(k+1) - S(k)\ \forall\ t \in \mathbb{T})$$

evolving on the nonnegative orthant $\mathbb{R}^m_+$. Choosing $\mu \in \mathbb{R}^m_+$ and using the reaction coordinates $x$,

$$S = \mu + \Gamma u,$$

instead of the traditional species coordinates $S$, we will have investigated the monotonicity and global behavior of systems in the reaction coordinates,

$$\Delta u(k) = f_\mu(u(k)) = R(\mu + \Gamma u(k)), \tag{28}$$

evolving on the state space $X_\mu := \{u \in \mathbb{R}^n |\ \mu + \Gamma u \geq 0\}$.

Suppose that the matrix $\Gamma$ has rank exactly $n - 1$ and its kernel is spanned by a strongly positive vector $v$. Then the state space is invariant with respect to translation by $v$, namely,

$$u \in X \Rightarrow u + \lambda v \in X,\ \forall\ \lambda \in \mathbb{R},$$

and the solution $\varphi(t, \xi)$ generated by (28) enjoys positive translation invariance:

$$\varphi(k, \xi + \lambda v) = \varphi(k, \xi) + \lambda v,\ \forall\ \xi \in X_\mu \text{ and } \lambda \in \mathbb{R}.$$

Suppose that the reaction rates depend on time

$$\Delta S(k) = \Gamma R(k, S(k)), \tag{29}$$

where $R(k, S)$ is almost periodic (respectively, Bohr almost periodic, almost automorphic, Birkhoff recurrent, Levitan almost periodic, Bebutov almost recurrent, Poisson stable) in $k$. Choosing $\mu \in \mathbb{R}^m_+$ and using the reaction coordinates $x : S = \mu + \Gamma x$, we transform (29) into a system in the reaction coordinates:

$$\Delta u(k) = F_\mu(k, u(k)) := R(k, \mu + \Gamma u(k)) \tag{30}$$

evolving on the state space $X_\mu := \{u \in \mathbb{R}^n |\ \mu + \Gamma u \geq 0\}$.

Let $\varphi(k, \xi, f)$ denote the unique solution of (30) passing through $\xi$ at $k = 0$. Then it enjoys positive translation invariance:

$$\varphi(k, \xi + \lambda v, f) = \varphi(k, \xi, f) + \lambda v,\ \forall\ \xi \in X_\mu,\ \lambda \in \mathbb{R}. \tag{31}$$

It can be checked that the solution for every $h \in H(f)$ possesses the positive translation invariance property (31), where $H(f)$ is the hull of $f$. So the skew-product flow induced by $H$-class

$$\Delta u(k) = h(k, u(k)) \quad (h \in H(f))$$

of system (30) has positive translation invariance.

According to Corollary 9 (item (iii)) for any $G \in H(F_\mu)$ there exists a unique $h \in H(f)$ $(G = \Phi(h))$ such that

$$G(k, u) = h(k, \mu + \Gamma u), \tag{32}$$

for any $(k,u) \in \mathbb{Z} \times X_\mu$. In particular, $G = F_\mu$ if and only if $h = f$. For every $G \in H(F_\mu)$ and $h \in H(f)$ in (32), let $\varphi(t, x_0, h)$ and $\phi(t, u_0, G)$ be the solutions of

$$\Delta U(k) = \Gamma h(k, U(k)),\ k > 0,\ U(0) = U_0 \in \mathbb{R}^m_+ \tag{33}$$

and

$$\Delta u(k) = G(k, u(k)),\ k > 0,\ u(0) = u_0 \in X_\mu, \tag{34}$$

respectively.

**Lemma 16.** *Let $\varphi(k, x_0, h)$ and $\phi(k, u_0, G)$ be the solutions of (33) and (34), respectively. Then we have*

$$\varphi(k, \mu + \Gamma u_0, h) = \mu + \Gamma\phi(k, u_0, \Phi(h))$$

*for any $k \in \mathbb{Z}_+$, $u_0 \in X_\mu$ and $h \in H(f)$.*

*Proof.* Denote by

$$\psi(k) := \mu + \Gamma\phi(k, u_0, G)\ (G = \Phi(h)) \tag{35}$$

for any $(k, u_0, G) \in \mathbb{Z}_+ \times X_\mu \times H(F_\mu)$. Then from (32), (34) and (35) we obtain

$$\Delta\psi(k) = \Gamma\Delta\phi(k, u_0, G) = \Gamma G(k, \phi(k, u_0, G)) =$$
$$\Gamma h(k, \mu + \Gamma\phi(k, u_0, G)) = \Gamma h(k, \psi(k))$$

for any $k \in \mathbb{Z}_+$. Thus $\psi$ is a solution of equation (33). Taking into consideration that $\psi(0) = \mu + \Gamma u_0$, then we will have $\psi(k) = \varphi(k, \mu + \Gamma u_0, h)$ and, consequently, $\varphi(k, \mu + \Gamma u_0, h) = \mu + \Gamma\phi(k, u_0, \Phi(h))$ for any $(k, u_0, h) \in \mathbb{Z}_+ \times X_\mu \times H(f)$. Lemma is proved. □

*Condition* **(D).** Equation (34) is monotone (respectively, strongly monotone). This means that the cocycle $\langle \mathbb{X}_\mu, \phi, (H(F_\mu), \mathbb{Z}, \sigma)\rangle$ generated by (34) is monotone (respectively, strongly monotone), i.e. if $u, v \in \mathbb{X}_\mu$ and $u \le v$ (respectively, $u < v$) then $\phi(k, u, g) \le \phi(k, v, g)$ (respectively, $\phi(k, u, g) \ll \phi(k, v, g)$) for all $k \ge 0$ and $G \in H(F_\mu)$.

**Definition 47.** Let $F_\mu \in C(\mathbb{Z} \times X_\mu, \mathbb{R}^n)$. The set $H(F_\mu)$ is said to be minimal if it is a minimal set of shift dynamical system $(C(\mathbb{Z} \times X_\mu, \mathbb{R}^n), \mathbb{Z}, \sigma)$.

**Definition 48.** A function $F_\mu \in C(\mathbb{Z} \times X_\mu, \mathbb{R}^n)$ is said to be strongly Poisson stable if every function $G \in H(F_\mu)$ is Poisson stable.

**Theorem 19.** *Suppose that the following conditions hold:*

1. *$\mu \in \mathbb{R}^m$ is such that the system (30) is strongly monotone;*
2. *the set $H(F_\mu)$ is minimal and $F_\mu$ is strongly Poisson stable;*
3. *the matrix $\Gamma$ has rank exactly $n - 1$ whose kernel is spanned by a strongly positive vector $v$;*

*4. for any $G \in H(F_\mu)$ all forward solutions of equation (30) are bounded.*

*Then for any $U_0 \in X_{(\mu,\Gamma)}$ the following statements hold:*

1. *the set $\omega_{(U_0,f)} \bigcap X_f$ consists of a single point $p_0 = (V_0, f)$, where $\omega_{(U_0,f)}$ is the $\omega$-limit set of the motion $\pi(k,(U_0,f))$ of the skew-product dynamical system $(X,\mathbb{Z}_+,\pi)$ ($X := X_{(\mu,\Gamma)} \times H(f)$, $\pi := (\varphi,\sigma)$) and $X_f := X_{(\mu,\Gamma)} \times \{f\}$;*

2. *the solution $\varphi(k,V_0,f)$ of equation (30) is defined on $\mathbb{Z}$, $\overline{\varphi(\mathbb{Z},v_0,f)} \subseteq Q_+^{(U_0,f)}$ and it is strongly compatible;*

3. $$\lim_{k\to\infty} |\varphi(k,U_0,f) - \varphi(k,V_0,f)| = 0.$$

*Proof.* Let $(H(F_\mu),\mathbb{Z},\sigma)$ (respectively, $(H(f),\mathbb{Z},\sigma)$) be the shift dynamical system on $H(F_\mu)$ (respectively, on $H(f)$). Denote by $\langle X_\mu,\phi,(H(F_\mu),\mathbb{Z},\sigma)\rangle$ (respectively, $\langle X_{(\mu,\Gamma)},\varphi,(H(f),\mathbb{Z},\sigma)\rangle$, where $X_{(\mu,\Gamma)} = \mu + \Gamma(X_\mu) := \{\mu + \Gamma u |\ u \in X_\mu\}$) the cocycle generated by family of equations (34) (respectively, (33)). Note that under the conditions of Theorem 19 Conditions (**C1**), (**C3**) and (**C4**) are fulfilled for the cocycle $\langle X_\mu,\phi,(H(F_\mu),\mathbb{R},\sigma)\rangle$. Let $U_0 \in X_{(\mu,\Gamma)}$, then there exists a point $u_0 \in X_\mu$ such that $U_0 = \mu + \Gamma(u_0)$. By equality (30) the cocycle $\phi$ is translation invariant with respect to vector $v \gg 0$. According to Theorem 14 for given $u_0 \in X_\mu$ the following statements are fulfilled:

a. the set $\omega_{(u_0,F_\mu)} \bigcap X_{F_\mu}$ consists of a single point $q_0 = (v_0,F_\mu)$, where $\omega_{(u_0,F_\mu)}$ is the $\omega$-limit set of the motion $\pi(k,(u_0,F_\mu))$ of the skew-product dynamical system $(X,\mathbb{Z}_+,\pi)$ ($X := X_\mu \times H(F_\mu)$, $\pi := (\phi,\sigma)$) and $X_{F_\mu} := X_{(\mu,\Gamma)} \times \{F_\mu\}$;

b. the solution $\phi(k,V_0,F_\mu)$ of equation (30) is defined on $\mathbb{Z}$, $\overline{\phi(\mathbb{Z},v_0,F_\mu)} \subseteq Q_+^{(u_0,F_\mu)}$ and it is strongly compatible;

c. $$\lim_{k\to\infty} |\phi(k,u_0,F_\mu) - \phi(k,v_0,F_\mu)| = 0.$$

Denote by $V_0 := \mu + \Gamma(v_0) \in X_{(\mu,\Gamma)}$ and consider the solutions $\varphi(k,U_0,f)$ and $\varphi(k,V_0,f)$ of equation (33) ($h = f$). Since $\phi(\cdot,v_0,F_\mu)$ is a strongly compatible solution of equation (34) ($G = F_\mu$), then

$$\mathfrak{M}_{F_\mu} \subseteq \mathfrak{M}_{\phi(\cdot,v_0,F_\mu)}. \tag{36}$$

By Lemma 16 we have $\varphi(k,V_0,f) = \mu + \phi(k,v_0,F_\mu)$ for any $k \in \mathbb{Z}$. Note that

$$\mathfrak{M}_{\phi(\cdot,v_0,F_\mu)} \subseteq \mathfrak{M}_{\varphi(\cdot,V_0,f)}. \tag{37}$$

Indeed if $\{k_m\} \in \mathfrak{M}_{\phi(\cdot,v_0,F_\mu)}$ then we have

$$\varphi(k+k_m,V_0,f) - \bar{\varphi}(k) = \mu + \Gamma\phi(k+k_m,v_0,F_\mu) - (\mu + \Gamma\bar{\phi}(k)) =$$

$$\Gamma(\phi(k+k_m, v_0, F_\mu) - \bar{\phi}(k)) \to 0$$

as $m \to \infty$ uniformly with respect to $k$ on every compact subset from $\mathbb{Z}$, where $\bar{\phi} = \lim_{m\to\infty} \phi(\cdot + k_m, v_0, F_\mu)$ in the space $C(\mathbb{Z}, \mathbb{R}^n)$. This means that $\{k_m\} \in \mathfrak{M}_{\varphi(\cdot, V_0, f)}$.

From (36) and (37) we have

$$\mathfrak{M}_{F_\mu} \subseteq \mathfrak{M}_{\varphi(\cdot, V_0, f)}. \tag{38}$$

Finally, from Corollary 9 (item (ii)) we have

$$\mathfrak{M}_f \subseteq \mathfrak{M}_{\Phi(f)}. \tag{39}$$

In virtue of (38)–(39) and taking into consideration the equality $\Phi(f) = F_\mu$ we obtain

$$\mathfrak{M}_f \subseteq \mathfrak{M}_{\varphi(\cdot, V_0, f)},$$

i.e., $\varphi(k, V_0, f)$ is a strongly compatible solution of equation (33) (for $h = f$).

To finish the proof of Theorem it is sufficient to note that

$$|\varphi(k, U_0, f) - \varphi(k, V_0, f)| = |(\mu + \Gamma\phi(k, u_0, F_\mu)) - (\mu + \Gamma\phi(k, v_0, F_\mu))| =$$
$$|\Gamma(\phi(k, u_0, F_\mu) - \phi(k, v_0, F_\mu))| \leq \|\Gamma\| |\phi(k, u_0, F_\mu) - \phi(k, v_0, F_\mu)| \to 0$$

as $k \to \infty$. □

**Corollary 11.** *Under the conditions of Theorem 19 if the function $f \in C(\mathbb{Z} \times X_{(\mu,\Gamma)}, \mathbb{R}^n)$ is stationary (respectively, $\tau$-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff, strongly Poisson stable) in time, then for any $U_0 \in X_{(\mu,\Gamma)}$ the following statements hold:*

1. *the set $\omega_{(U_0,f)} \bigcap X_f$ consists of a single point $p_0 = (V_0, f)$;*
2. *$\varphi(k, V_0, f)$ is a stationary (respectively, $\tau$-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff, strongly Poisson stable) solution of equation (30);*
3. *$\lim_{k\to+\infty} |\varphi(k, U_0, f) - \varphi(k, V_0, f)| = 0$, i.e., $\varphi(k, u_0, f)$ is asymptotically stationary (respectively, asymptotically $\tau$-periodic, asymptotically quasi-periodic, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotically recurrent in the sense of Birkhoff, asymptotically strongly Poisson stable).*

*Proof.* This statement follows from Theorem 19 and Corollary 6. □

Consider the difference equation

$$\Delta u(k) = f(k, u(k)), \tag{40}$$

where $f \in C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n)$. Along with the equation (40) we consider its $H$-class [5, 21, 27, 34, 37], i.e., the family of the equations

$$\Delta v(k) = g(k, v(k)), \tag{41}$$

where $g \in H(f) = \overline{\{f_\tau : \tau \in \mathbb{Z}\}}$ and $f_\tau(k, u) = f(k + \tau, u)$, with the bar indicating closure in the compact-open topology.

We will suppose that the function $f$ is regular. Denote by $\varphi(\cdot, v, g)$ the solution of (41) passing through the point $v \in \mathbb{R}^n$ for $k = 0$. Then the mapping $\varphi : \mathbb{Z}_+ \times \mathbb{R}^n \times H(f) \to \mathbb{R}^n$ satisfies the following conditions (see, for example,[5, 31]):

1) $\varphi(0, v, g) = v$ for all $v \in \mathbb{R}^n$ and $g \in H(f)$;

2) $\varphi(k, \varphi(\tau, v, g), g_\tau) = \varphi(k + \tau, v, g)$ for each $v \in \mathbb{R}^n$, $g \in H(f)$ and $k, \tau \in \mathbb{Z}_+$;

3) $\varphi : \mathbb{Z}_+ \times \mathbb{R}^n \times H(f) \to \mathbb{R}^n$ is continuous.

Denote by $Y := H(f)$ and $(Y, \mathbb{Z}, \sigma)$ a dynamical system of translations on $Y$, induced by the dynamical system of translations $(C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n), \mathbb{Z}, \sigma)$. The triple $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{Z}, \sigma)\rangle$ is a cocycle over $(Y, \mathbb{Z}_+, \sigma)$ with the fiber $\mathbb{R}^n$. Hence, the equation (40) generates a cocycle $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{Z}, \sigma)\rangle$ and the non-autonomous dynamical system $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{Z}, \sigma), h\rangle$, where $X := \mathbb{R}^n \times Y$, $\pi := (\varphi, \sigma)$ and $h := pr_2 : X \to Y$.

**Definition 49.** Recall that the difference equation (40) is called dissipative if for all $t_0 \in \mathbb{R}$ and $u_0 \in \mathbb{R}^n$ there exists a unique solution $x(k; u_0, k_0)$ of the equation (40) passing through the point $(u_0, k_0)$ and if there exists a number $R > 0$ such that $\lim\limits_{k\to+\infty} \sup |x(k; u_0, k_0)| < R$ for all $x_0 \in \mathbb{R}^n$ and $k_0 \in \mathbb{Z}$. In other words, for every solution $x(k; u_0, k_0)$ there is an instant $k_1 = k_0 + l(k_0, u_0)$, such that $|x(k; u_0, k_0)| < R$ for any $k \geq k_1$. If for any $r > 0$ the number $l(k_0, u_0)$ can be chosen independently on $k_0$ and $u_0$ with $|u_0| \leq r$, then the equation (40) is called uniformly dissipative.

**Lemma 17.** *Let $f \in C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n)$ be regular. If $H(f)$ is compact, then equation (40) is uniformly dissipative if and only if there is a positive number $r$ such that*

$$\limsup_{k\to+\infty} |\varphi(k, u_0, g)| < r \quad (u_0 \in \mathbb{R}^n,\ g \in H(f)) \ .$$

*Proof.* This statement can be proved using the same arguments as in the proof of Lemma 15 (see [12, ChIII]). □

*Remark* 16. If $f \in C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n)$ is regular, $H(f)$ is compact and then equation (40) is uniformly dissipative, then the cocycle $\varphi$ generated by equation (40) admits a compact global attractor.

**Theorem 20.** *Suppose that the following assumptions are fulfilled:*

– *$\mu \in \mathbb{R}^m$ is such that the system (30) is monotone;*

– *the matrix* $\Gamma$ *has rank exactly* $n-1$ *whose kernel is spanned by a strongly positive vector* $v$*;*

– *the function* $F_\mu \in C(\mathbb{Z} \times X_\mu, \mathbb{R}^n)$ *is recurrent in* $k \in \mathbb{Z}$ *uniformly with respect to* $u$ *on every compact subset from* $X_\mu$*;*

– *the cocycle* $\phi$ *generated by equation (30) admits a compact global attractor and* $\boldsymbol{I} := \{I_G |\ G \in H(F_\mu)\}$ *is its Levinson center.*

*Then under the condition* ($\boldsymbol{D}$) *the following statements hold:*

1. $\alpha(G), \beta(G) \in I_G$ *for any* $G \in H(F_\mu)$ *and, consequently,* $I_G \subseteq [\alpha(G), \beta(G)]$*;*
2. $\phi(k, \alpha(G), G) = \alpha(\sigma(k, G))$ *(respectively,* $\phi(k, \beta(G), G) = \beta(\sigma(k, G))$*) for any* $k \geq 0$ *and* $G \in H(F_\mu)$*;*
3. *the point* $\gamma_*(F_\mu) := (\alpha(F_\mu), F_\mu) \in X = X_\mu \times Y$ *(respectively,* $\gamma^*(F_\mu) := (\beta(F_\mu), F_\mu) \in X$*) is strongly comparable by character of recurrence with the point* $F_\mu$*;*
4. *for any* $h \in H(f)$ *equation (34) has at least two solutions* $\varphi(k, U_0, h)$ *(*$U_0 = \mu + \Gamma\alpha(\Phi(h))$*) and* $\varphi(k, V_0, h)$ *(*$V_0 = \mu + \Gamma\beta(\Phi(h))$*) defined and bounded on* $\mathbb{Z}$ *which are strongly compatible and belong to Levinson center of (30);*
5. *if the function* $f \in C(\mathbb{Z} \times X_{(\mu,\Gamma)}, \mathbb{R}^n)$ *is quasi-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent, pseudo-recurrent and Lagrange stable, uniformly Poisson stable and stable in the sense of Lagrange) in* $k \in \mathbb{Z}$ *uniformly with respect to* $u$ *on every compact subset from* $X_{(\mu,\Gamma)}$*, then* $\varphi(k, u_0, f)$ *and* $\varphi(k, V_0, f)$ *are quasi-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent, pseudo-recurrent and Lagrange stable, uniformly Poisson stable and stable in the sense of Lagrange).*

*Proof.* Let $(H(F_\mu), \mathbb{Z}, \sigma)$ (respectively, $(H(f), \mathbb{Z}, \sigma)$) be the shift dynamical system on $H(F_\mu)$ (respectively, on $H(f)$). Denote by $\langle X_\mu, \phi, (H(F_\mu), \mathbb{Z}, \sigma)\rangle$ (respectively, $\langle X_{(\mu,\Gamma)}, \varphi, (H(f), \mathbb{Z}, \sigma)\rangle$, where $X_{(\mu,\Gamma)} = \mu + \Gamma(X_\mu) := \{\mu + \Gamma u |\ u \in X_\mu\}$) the cocycle generated by family of equations (34) (respectively, (33)).

By equality (30) the cocycle $\phi$ is translation invariant with respect to vector $v \gg 0$. Applying Theorem 15 to nonautonomous dynamical system $\langle X_\mu, \phi, (H(F_\mu), \mathbb{Z}, \sigma)\rangle$ we obtain the following statements:

1. $\alpha(G), \beta(G) \in I_G$ for any $G \in H(F_\mu)$ and, consequently, $I_G \subseteq [\alpha(G), \beta(G)]$, where $\alpha(G) := \inf I_G$ (respectively, $\beta(G) := \sup I_G$);
2. $\phi(k, \alpha(G), G) = \alpha(\sigma(k, G))$ (respectively, $\phi(k, \beta(G), G) = \beta(\sigma(k, G))$) for any $k \geq 0$ and $G \in H(F_\mu)$;
3. the point $\gamma_*(F_\mu) := (\alpha(F_\mu), F_\mu) \in X = X_\mu \times Y$ (respectively, $\gamma^*(F_\mu) := (\beta(F_\mu), F_\mu) \in X$) is strongly comparable by character of recurrence with the point $F_\mu$.

Note that the nonautonomous dynamical system $\langle X_{(\mu,\Gamma)}, \varphi, (H(f), \mathbb{Z}, \sigma)\rangle$ is compactly dissipative because $\varphi(k, U, h) = \mu + \Gamma\phi(k, u, \Phi(h))$ for any $h \in H(f)$ ($U = \mu + \Gamma u$) and the cocycle $\phi$ is so. Let $\mathbf{A} = \{A_h |\ h \in H(f)\}$ be the Levinson center for the compact dissipative cocycle $\varphi$ generated by equation (33). Denote by $U(h) := \mu + \Gamma\alpha(\Phi(h))$ and $V(h) := \mu + \Gamma\beta(\Phi(h))$. Then by Lemma 16 for any $h \in H(f)$

$$\varphi(k, U(h), h) = \mu + \Gamma\phi(k, \alpha(\Phi(h)), \Phi(h)) \tag{42}$$

(respectively,

$$\varphi(k, V(h), h) = \mu + \Gamma\phi(k, \beta(\Phi(h)), \Phi(h))) \tag{43}$$

is a bounded on $\mathbb{Z}$ solution of equation (34). By Theorem 8 we have $U(h), V(h) \in A_h$, i.e., $U(h)$ and $V(h)$ belongs to the Levinson center of the cocycle $\varphi$. Finally, from (42) (respectively, (43)) it follows that $\varphi(k, U(f), f)$ (respectively, $\varphi(k, V(f), f)$) is a strongly compatible solution of equation (33) for $h = f$, because $\phi(k, \alpha(\Phi(h)), \Phi(h))$ (respectively, $\phi(k, \beta(\Phi(h)), \Phi(h)))$ is a strongly compatible solution of equation (34), $\Phi : H(f) \to H(F_\mu)$ is a homeomorphism and $\Phi(f) = F_\mu$. Theorem is proved. □

## 9 Acknowledgement

This research was supported by the State Program of the Republic of Moldova "Multivalued dynamical systems, singular perturbations, integral operators and non-associative algebraic structures (20.80009.5007.25)".

## 10 Conflict of Interest

The author declares that he does not have conflict of interest.

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*Received November 23, 2022*