# Semi-symmetric isotopic closure of some group varieties and the corresponding identities 

Halyna Krainichuk, Olena Tarkovska


#### Abstract

Four families of pairwise equivalent identities are given and analyzed. Every identity from each of these families defines one of the following varieties: 1) the semi-symmetric isotopic closure of the variety of all Boolean groups; 2) the semisymmetric isotopic closure of the variety of all Abelian groups; 3) the semi-symmetric isotopic closure of the variety of all groups; 4) the variety of all semi-symmetric quasigroups. It is proved that these varieties are different and form a chain. Quasigroups belonging to these varieties are described. In particular, quasigroups from 1) and $2)$ varieties are medial and in addition, they are either groups or non-commutative semi-symmetric quasigroups.


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## 1 Introduction

It is well known that the class of all semi-symmetric quasigroups is described by

$$
\begin{equation*}
x y \cdot x=y \tag{1}
\end{equation*}
$$

According to A. Sade [26], a groupoid or a quasigroup $(Q ; \cdot)$ satisfying the identity (1) for all $x, y$ of $Q$ is called semi-symmetric. He also established properties and structure of semi-symmetric quasigroups. Semi-symmetric quasigroups have also been described as '3-cyclic'. They were studied by J. M. Osborn [21], A. Sade [2629], N. S. Mendelsohn [19], G. Grätzer and R.Padmanabhan [15], A. Mitschke and H. Werner [20], J. W. DiPaola and E. Nemeth [9]. The use of semi-symmetric quasigroups for reducing homotopies to homomorphisms first appeared in [32], inspired by work of Gvaramiya and Plotkin that interpreted homotopies as homomorphisms of heterogeneous algebras [32]. The classical approach to studying properties of a quasigroup invariant under isotopy was geometrical, through the concept of a 3-net, as presented in A. A. Albert [2], V.D. Belousov [6], H. O. Pflugfelder [23], V. A. Shcherbacov [30], J. D. H. Smith [33] and A. B. Romanowska [34].
F. Sokhatsky [38] proposed a symmetry concept for parastrophes of quasigroup varieties and their quasigroups. This concept is used for the investigation of the parastrophes of quasigroup varieties and, in particular, quasigroups and their parastrophes. F. Sokhatsky's symmetry concept generalizes the symmetry known as triality which was investigated by J.D.H.Smith [31]. If a $\sigma$-parastrophe coincides
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with a quasigroup itself, then $\sigma$ is called a symmetry of the quasigroup. The set of all symmetries of a binary quasigroup forms a group, which is a subgroup of the symmetry group $S_{3}$. According to the symmetry group, there are six classes of quasigroups: commutative (middle symmetric), left-, right-, semi-, totally symmetric and asymmetric (which consists of quasigroups with a unitary symmetry group).

We consider semi-symmetric isotopic closures of some group varieties. The necessary and sufficient conditions for a group isotope to be semi-symmetric are wellknown. For example, F. Radó [25] found the necessary and sufficient conditions for existence of the semi-symmetric group isotopes of prime order. The first author of the article [16] established the criterion for the semi-symmetry of group isotopes. The second author [42] gave a variety of Abelian group isotopes containing semisymmetric medial quasigroups. I. M. H. Etherington [11] and A. Sade [26] showed that every semi-symmetric groupoid is necessarily a semi-symmetric quasigroup. V.V.Iliev [14] studied a construction of the semi-symmetric algebras over a commutative ring with the unit. V.D. Belousov [5] has found a quadratic identity in five variables describing the isotopic closure of all groups. F. M. Sokhatsky [36] has established an identity in four variables which also describes this variety but his identity is not quadratic. The isotopic closure of some group varieties was studied by G. B. Belyavskaya [7], A. Drapal [10], A. Kh. Tabarov [41].

In this article, we have found families of identities: 1) nine quadratic identities in three variables (11); 2) nine quadratic identities in four variables (12); 3) one non-quadratic identity in four variables $(15) ; 4)$ ten quadratic identities in two variables (Corollary 11). Identities (11) are pairwise equivalent (Lemma 2) and describe the variety $\mathfrak{B}_{\mathrm{ss}}$ of the semi-symmetric isotopic closure of all Boolean groups (Corollary 14 from Theorem 8). Identities (12) are pairwise equivalent (Theorem 9) and describe the variety $\mathfrak{A}_{\mathrm{ss}}$ of the semi-symmetric isotopic closure of all Abelian groups (Corollary 18). The identity (15) describes the variety $\mathfrak{G}_{\text {ss }}$ of the semi-symmetric isotopic closure of all groups (Theorem 10). All identities from Corollary 11 are pairwise equivalent and describe the variety $\mathfrak{S}$ of all semi-symmetric quasigroups (Lemma 1). Every identity from (11), (12), (15) and from Corollary 11 implies semi-symmetry (see corresponding Theorems 6, 7, and Corollaries 12, 22).

The quasigroups belonging to varieties $\mathfrak{B}_{\mathrm{ss}}$ and $\mathfrak{A}_{\mathrm{ss}}$ are medial (Corollary 19). Moreover, they are either groups or non-commutative semi-symmetric quasigroups (Corollaries 16,19 ). All varieties $\mathfrak{B}_{\mathrm{ss}}, \mathfrak{A}_{\mathrm{ss}}, \mathfrak{G}_{\mathrm{ss}}$ and $\mathfrak{S}$ are totally symmetric, that is every parastrophe of a quasigroup of the variety belongs to this variety (Corollaries $10,13,17,23$ ). It is proved that these varieties are different and form a chain (Theorem 11).

## 2 Preliminaries

A quasigroup is a natural generalization of the concept of a group. Quasigroups differ from groups in that they need not be associative. A quasigroup is a group if and only if it satisfies the associativity [6].

As usual, whenever unambiguous, a term like $x \cdot y$ is shortened to $x y$. The word 'iff' stands for 'if and only if'.

An algebra $\left(Q ;,,,{ }^{\ell}, r\right)$ with identities

$$
\begin{equation*}
(x \cdot y)^{\ell} \cdot y=x, \quad\left(x^{\ell} \cdot y\right) \cdot y=x, \quad x \cdot(x \cdot y)=y, \quad x \cdot\left(x^{r} \cdot y\right)=y \tag{2}
\end{equation*}
$$

is called a quasigroup [6,12]. In [3], an equational quasigroup is defined as an algebra with three binary operations $(Q ; \cdot, \cdot, \cdot, \cdot)$ that fulfill the following six identities: (2) and $x^{\ell} \cdot\left(y^{r} \cdot x\right)=y,(x \cdot y) \stackrel{r}{\bullet} x=y$. The triples of identities composed of these six, emphasizing those that axiomatize the variety of quasigroups, are investigated in [22].

The main operation of a quasigroup is denoted by (•). A quasigroup operation $(\cdot)$ is often considered together with its inverse operations: left $(\cdot)$ ) and right $\left({ }^{r} \cdot\right)$ divisions which are defined by: $x \cdot y=z \Leftrightarrow x^{r} \cdot z=y \Leftrightarrow z^{\ell} \cdot y=x$. Both inverse operations are also quasigroups.

Such quasigroups are called equational quasigroups (equasigroups, earlier primitive quasigroups). The equational definition of quasigroups is due to T. Evans [13]. The equational definition of twisted quasigroups is due to A. Krapež [18].

The operations (2) and their duals which are defined by

$$
\begin{equation*}
x^{s} \cdot y:=y \cdot x, \quad x^{s \ell} \cdot y:=y^{\ell} \cdot x, \quad x^{s r} \cdot y:=y^{r} \cdot x \tag{3}
\end{equation*}
$$

are called parastrophes of $(\cdot)$. The defining identities (2) and (3) are called primary.

### 2.1 On symmetry of an arbitrary proposition

The relationships (3) imply that each identity of the signature $\left(\cdot,{ }^{\ell}, \stackrel{r}{r}, \stackrel{s}{ }, \stackrel{s}{\bullet},{ }^{s r}\right)$ can be written in the signature $(\cdot, \cdot, \cdot, \cdot)$. Nevertheless throughout the article, we consider identities on quasigroups of signature $(\cdot, \stackrel{\ell}{\bullet} \stackrel{r}{r}, \stackrel{s}{ }, \stackrel{s \ell}{ }, \stackrel{s r}{\bullet})$. All parastrophes of $(\cdot)$ can be defined by

$$
\begin{equation*}
x_{1 \sigma}{ }^{\sigma} \cdot x_{2 \sigma}=x_{3 \sigma}: \Leftrightarrow x_{1} \cdot x_{2}=x_{3} \tag{4}
\end{equation*}
$$

where $\sigma \in S_{3}:=\{\iota, \ell, r, s, s \ell, s r\}, \ell:=(13), r:=(23), s:=(12)$. It is easy to verify that

$$
{ }^{\sigma}(\tau)=\left({ }^{\sigma \tau}\right)
$$

holds for all $\sigma, \tau \in S_{3}$.
F. Sokhatsky $[38,39]$ has shown that a mapping $(\sigma ;(\cdot)) \mapsto\left({ }^{\sigma}\right)$ is an action on the set $\Delta$ of all quasigroup operations defined on $Q$. A stabilizer $\operatorname{Ps}(\cdot)$ is called a parastrophic symmetry of $(\cdot)$. Thus, the number of different parastrophes of a quasigroup operation (.) depends on its group of parastrophic symmetry $\operatorname{Ps}(\cdot)$. Since $\operatorname{Ps}(\cdot)$ is a subgroup of the symmetric group $S_{3}$, then there are six classes of quasigroups.

If $\operatorname{Ps}(\cdot) \supseteq A_{3}$, then a quasigroup is called semisymmetric. The class of all semisymmetric quasigroups is described by $x \cdot y x=y$. It means that

$$
\begin{equation*}
(\cdot)=\left(\cdot \frac{s \ell}{\cdot}\right)=\left({ }^{s r}\right), \quad\left({ }^{s} \cdot\right)=(\cdot)=(\cdot) . \tag{5}
\end{equation*}
$$

If $\operatorname{Ps}(\cdot)=S_{3}$, then a quasigroup is called totally symmetric. The class of all totally symmetric quasigroups is described by $x y=y x$ and $x y \cdot y=x$, it means that all parastrophes coincide.

Let $P$ be an arbitrary proposition in a class of quasigroups $\mathfrak{A}$. The proposition ${ }^{\sigma} P$ is said to be a $\sigma$-parastrophe of $P$, if it can be obtained from $P$ by replacing every $\left({ }^{\tau}\right)$ with $\left({ }^{\tau \sigma^{-1}}\right) ; \sigma_{\mathfrak{A}}$ denotes the class of all $\sigma$-parastrophes of quasigroups from $\mathfrak{A}$.

Theorem 1 (see $[38,39]$ ). Let $\mathfrak{A}$ be a class of quasigroups, then a proposition $P$ is true in $\mathfrak{A}$ iff ${ }^{\sigma}$ P is true in $\sigma_{\mathfrak{A}}$.

Corollary 1 (see $[38,39])$. Let $P$ be true in a class of quasigroups $\mathfrak{A}$, then ${ }^{\sigma} P$ is true in $\mathfrak{A}$ for all $\sigma \in S_{3}$.

Corollary 2 (see $[38,39]$ ). Let $P$ be true in a totally symmetric class $\mathfrak{A}$, then ${ }^{\sigma} P$ is true in $\mathfrak{A}$ for all $\sigma$.

Definition 1. Transition of the identity $\mathfrak{i d}$ to the identity ${ }^{\sigma} \mathfrak{i d}$ is called a parastrophic transformation ( $\sigma$-parastrophic transformation) if ${ }^{\sigma} \mathfrak{i d}$ can be obtained by replacing the main operation with its $\sigma^{-1}$-parastrophe.

Two identities are called:

1) equivalent if they define the same variety;
2) primarily equivalent if one of them can be obtained from the other in a finite number of applications of primary identities (2) - (3) (primary equivalent identities are equivalent);
3) $\sigma$-parastrophic if one of them can be obtained from the other by $\sigma$-parastrophic transformation;
4) $\sigma$-parastrophically equivalent if they define $\sigma$-parastrophic varieties (according to Theorem $1, \sigma$-parastrophically equivalent identities define $\sigma$-parastrophic varieties);
5) $\sigma$-parastrophically primarily equivalent if one of them can be obtained in a finite number of applications of primary identities and $\sigma_{1}-, \sigma_{2}-, \ldots, \sigma_{k}-$ parastrophic transformations such that $\sigma_{1} \sigma_{2} \ldots \sigma_{k}=\sigma$ for some $k \in \mathbb{N}$.

In a generalized case $\sigma$ will be omitted. For example, two identities are called parastrophically equivalent if they are $\sigma$-parastrophically equivalent for some $\sigma \in S_{3}$.

### 2.2 On group isotopes

A groupoid $(Q ; \cdot)$ is called an isotope of a groupoid ( $Q ;+$ ) iff there exists a triplet of bijections ( $\alpha, \beta, \gamma$ ), which is called an isotopism, such that the relationship $x \cdot y:=\gamma^{-1}(\alpha x+\beta y)$ holds. An isotope of a group is called a group isotope.

Definition 2 (see [36]). Let ( $Q ; \cdot$ ) be a group isotope and 0 be an arbitrary element of $Q$, then the right side of the formula

$$
\begin{equation*}
x \cdot y=\alpha x+a+\beta y \tag{6}
\end{equation*}
$$

is called a 0 -canonical decomposition if $(Q ;+)$ is a group, 0 is its neutral element and $\alpha 0=\beta 0=0$.

In this case, we say: the element 0 defines the canonical decomposition; ( $Q ;+$ ) is its decomposition group; $\alpha, \beta$ are its coefficients and $a$ is its free member.

Theorem 2 (see [36]). An arbitrary element of a group isotope uniquely defines a canonical decomposition of the isotope.

Corollary 3 (see [36]). The isotopic closure of the variety of all groups is a variety of quasigroups which is described by the following identity:

$$
\begin{equation*}
\left(x\left(u^{r} \cdot y\right)^{\ell} \cdot u\right) z=x\left(u^{r} \cdot\left(y^{\ell} \cdot u\right) z\right) \tag{7}
\end{equation*}
$$

Corollary 4 (see [35]). If a group isotope ( $Q ; \cdot$ ) satisfies the identity

$$
w_{1}(x) \cdot w_{2}(y)=w_{3}(y) \cdot w_{4}(x)
$$

and the variables $x, y$ are quadratic, then $(Q ; \cdot)$ is isotopic to a commutative group.
Recall that a variable is quadratic in an identity if it has exactly two appearances in this identity. An identity is called quadratic if all variables are quadratic. If a quasigroup $(Q ; \cdot)$ is isotopic to a parastrophe of a quasigroup $(Q ; \circ)$, then $(Q ; \cdot)$ and $(Q ; \circ)$ are called isostrophic.

Theorem 3 (see [37]). Let four pairwise isostrophic operations connected by a quadratic identity satisfy the conditions:

1) an arbitrary subterm of the length two has two different variables;
2) an arbitrary subterm of the length three has three different variables.

Then all these operations are isotopic to the same group.
Belousov's theorem on four quasigroups $[1,4,40]$ implies the following corollary.
Corollary 5. If four quasigroups are connected by the generalized associativity law, then each of these quasigroups is isotopic to the same group.

Theorem 4 and its Corollary 7 below are well known and can be found in many articles, for example, in $[6,36]$.

Theorem 4. A triple $(\alpha, \beta, \gamma)$ of permutations of $a$ set $Q$ is an autotopism of $a$ group $(Q,+)$ iff there exists an automorphism $\theta$ of $(Q,+)$ and elements $b, c \in Q$ such that

$$
\alpha x=c+\theta x-b, \quad \beta x=b+\theta x, \quad \gamma x=c+\theta x .
$$

Corollary 6. (6) is a canonical decomposition of a group iff $\alpha=\beta=\iota$.
Proof. Let (6) be a canonical decomposition of a group ( $Q ; \cdot \cdot$. Therefore, the groups $(Q ;+)$ and $(Q ; \cdot)$ are isotopic, consequently they are isomorphic and let $\varphi$ be the corresponding isomorphism. Then

$$
\varphi\left(\varphi^{-1} x+\varphi^{-1} y\right)=\alpha x+a+\beta y
$$

holds. Theorem 4 implies the existence of an automorphism $\theta$ and an element $b$ from $(Q ;+)$ such that $\varphi x=b+\theta x$. Therefore,

$$
x-b+y=\alpha x+a+\beta y
$$

holds. The left and the right sides of the equality are canonical decomposition of the same group isotope. Its uniqueness implies $\alpha=\beta=\iota$.

Corollary 7. Let $\alpha, \beta_{1}, \beta_{2}, \beta_{3}, \beta_{4}$ be permutations of a set $Q$. Besides, $\alpha$ is a unitary transformation of a group $(Q,+)$ and let

$$
\alpha\left(\beta_{1} x+\beta_{2} y\right)=\beta_{3} u+\beta_{4} v,
$$

where $\{x, y\}=\{u, v\}$ holds for all $x, y \in Q$. Then the following statements are true:

1) $\alpha$ is an automorphism of $(Q,+)$ if $u=x, v=y$;
2) $\alpha$ is an anti-automorphism of $(Q,+)$ if $u=y, v=x$.

Systematizing all criteria on symmetry, the first author [16] gave a classification of group isotopes according to their groups of parastrophic symmetry and formulated the corollary on the classification of isotopes of Abelian groups.

Theorem 5 (see [16]). Let ( $Q ; \cdot$ ) be a group isotope and (6) be its canonical decomposition, then $(Q ; \cdot)$ is

1) commutative iff $(Q ;+)$ is Abelian and $\beta=\alpha$;
2) left symmetric iff $(Q ;+)$ is Abelian and $\beta=-\iota$;
3) right symmetric iff $(Q ;+)$ is Abelian and $\alpha=-\iota$;
4) semi-symmetric iff $\alpha$ is an anti-automorphism of $(Q ;+)$, $\beta=\alpha^{-1}, \alpha^{3}=-I_{a}^{-1}, \alpha a=-a$, where $I_{a}(x):=-a+x+a ;$
5) totally symmetric iff $(Q ;+)$ is Abelian and $\alpha=\beta=-\iota$;
6) asymmetric iff $(Q ;+)$ is not Abelian or $-\iota \neq \alpha \neq \beta \neq-\iota$ and at least one of the following conditions is true: $\alpha$ is not an anti-automorphism, $\beta \neq \alpha^{-1}$, $\alpha^{3} \neq-I_{a}^{-1}, \alpha a \neq-a$.

Theorem 5 implies Corollary 8.
Corollary 8 (see [16]). Let $(Q ; \cdot)$ be an isotope of an Abelian group and (6) be its canonical decomposition, then $(Q ; \cdot)$ is

1) commutative iff $\beta=\alpha$;
2) left symmetric iff $\beta=-\iota$;
3) right symmetric iff $\alpha=-\iota$;
4) semi-symmetric iff $\alpha$ is an automorphism of $(Q ;+)$,
$\beta=\alpha^{-1}, \alpha^{3}=-\iota, \alpha a=-a ;$
5) totally symmetric iff $\alpha=\beta=-\iota$;
6) asymmetric iff $-\iota \neq \alpha \neq \beta \neq-\iota$ and at least one of the following conditions is true: $\alpha$ is not an automorphism, $\beta \neq \alpha^{-1}, \alpha^{3} \neq-\iota, \alpha a \neq-a$.

## 3 Identities implying semi-symmetry

In this section, we find the relations among identities specifying semi-symmetric quasigroups. We systematize some well-known results for identities in two variables for using them in our further investigation. A semi-symmetry can be defined by different conditions. We consider some of them. We find nine quadratic identities in three variables and nine quadratic identities in four variables each of them implies semi-symmetry.

### 3.1 Identities in two variables

A quasigroup $(Q ; \cdot)$ is called semi-symmetric if the identity (1) holds for all $x$, $y$ from $Q$. Using the definition of the left division, we have the equivalent identity $y^{\ell} \cdot x=x y$. We apply the definition of $s$-parastrophe to the left and to the right sides of the identity separately:

$$
\begin{equation*}
x^{s \ell} y=x y, \quad y^{\ell} \cdot x=y \cdot{ }^{s} \cdot x \tag{8}
\end{equation*}
$$

These identities mean that $\left({ }^{s \ell}\right)=(\cdot)$ and $\left({ }^{\ell}\right)=\left({ }^{s}\right)$ hold. That is why each identity from (8) is equivalent to (1). The equality $\left({ }^{s \ell}\right)=(\cdot)$ means that $s \ell \in \operatorname{Ps}(\cdot)$.

Similarly, one can show that the identity

$$
\begin{equation*}
x \cdot y x=y \tag{9}
\end{equation*}
$$

is equivalent to

$$
\begin{equation*}
y \cdot \stackrel{s r}{ } x=y x, \quad x \stackrel{r}{\cdot} y=x^{s} \cdot y . \tag{10}
\end{equation*}
$$

Therefore, $\left({ }^{s r}\right)=(\cdot)$ and $\left({ }^{r}\right)=\left({ }^{s}\right)$ hold and the equality $\left({ }^{s r}\right)=(\cdot)$ means that $s r \in \operatorname{Ps}(\cdot)$. As a result, we obtain the following lemma.

Lemma 1. In an arbitrary quasigroup $(Q ; \cdot)$ the following statements are equivalent:

1) ( $Q ; \cdot$ ) is semi-symmetric;
2) $A_{3}$ is a subgroup of $\operatorname{Ps}(\cdot)$;
3) $(Q ; \cdot)$ satisfies ( 9 ).

Proof. 1) $\Leftrightarrow$ 2). As we have shown above, (1) is equivalent to $s \ell \in \operatorname{Ps}(\cdot)$. But $s \ell$ generates the group $A_{3}$, then $A_{3}$ is a subgroup of $\operatorname{Ps}(\cdot)$. The inverse statement is evident. 2) $\Leftrightarrow$ 3) can be proved in the same way.

Corollary 9. If a semi-symmetric variety contains s-parastrophe of each of its quasigroups, then it is totally symmetric.

Proof. The proof follows from item 1) of Lemma 1.

Corollary 10. The variety of all semi-symmetric quasigroups is totally symmetric.
Proof. Let $\mathfrak{S}$ be the variety of semi-symmetric quasigroups. Therefore, $\mathfrak{S}$ contains $s \ell$-parastrophe of an arbitrary quasigroup from $\mathfrak{S}$. $s$-Parastrophe of a quasigroup from $\mathfrak{S}$ satisfies $s$-parastrophe of the identity (1), i. e., $\left(x^{s} \cdot y\right)^{s} \cdot x=y$. The identity is equivalent to $x \cdot y x=y$ which defines $\mathfrak{S}$. Thus, $s \ell$ and $s$ belong to the group $\operatorname{Ps}(\mathfrak{S})$, that is why $\operatorname{Ps}(\mathfrak{S})=S_{3}$. It means that $\mathfrak{S}$ is totally symmetric.

Corollary 11. The identities (1), (8), (9), (10) and $x\left(x^{\ell} \cdot y\right)=y,\left(x^{r} \cdot y\right) y=x$, $x^{\ell} \cdot x y=y, x y \cdot r \cdot x=x, x^{\ell} \cdot y=y x, x^{r} \cdot y=y x$ are equivalent.

Proof. Using the definitions of the left and right divisions, the proof is evident.

The equivalency of the identities (1), (9) and the last two identities from Corollary 11 is shown in [31, Proposition 1.2]. The equivalency of the identities (1), (8), (9), (10) and the last two identities from Corollary 11 are established in $[8,24]$.

Thus, we have the variety of all semi-symmetric quasigroups, defined by one of ten equivalent axioms from Corollary 11.

Corollary 12. The identities from Corollary 11 imply semi-symmetry.

### 3.2 Identities in three variables

In this subsection, nine quadratic identities in three variables are investigated, namely

$$
\begin{array}{lllll}
(x \cdot y z) \cdot z=y x, & \left(i_{1}\right) & x \cdot(x y \cdot z)=z y, & \left(i_{2}\right) & x y \cdot y z=z x, \\
x(y(y x \cdot z))=z, & \left(i_{4}\right) & x y \cdot(y \cdot x z)=z, & \left(i_{5}\right) & x(x y \cdot y z)=z,  \tag{11}\\
((x \cdot y z) z) y=x, & \left(i_{7}\right) & (x y \cdot z) \cdot z y=x, & \left(i_{8}\right) & (x y \cdot y z) z=x,
\end{array}
$$

In this form, these identities were among 100 identities without squares, which were listed in [17]. We establish relations among identities (11), namely, relations of equivalency and parastrophically primary equivalency. Each quasigroup satisfying one of the identities from (11) is semi-symmetric (Theorem 6).

Proposition 1. The identities $\left(i_{4}\right),\left(i_{5}\right),\left(i_{6}\right)$ are equivalent.
Proof. Multiply ( $i_{4}$ ) by $y x$ from the left: $y x \cdot(x \cdot(y \cdot(y x \cdot z)))=y x \cdot z$. Replacing $y x \cdot z$ with $z$, we have $y x \cdot(x \cdot y z)=z$. Mutually relabeling $x$ and $y$, we obtain $\left(i_{5}\right)$. Since applied transformations are invertible, then $\left(i_{4}\right)$ and $\left(i_{5}\right)$ are equivalent. Multiplying ( $i_{5}$ ) by $x$ from the left and replacing $x z$ with $z$, we obtain equivalency of $\left(i_{5}\right)$ and $\left(i_{6}\right)$.

Proposition 2. The identities $\left(i_{7}\right),\left(i_{8}\right),\left(i_{9}\right)$ are equivalent.
Proof. Multiply $\left(i_{7}\right)$ by $y z$ from the right: $(((x \cdot y z) \cdot z) \cdot y) \cdot y z=x \cdot y z$. Replacing $x \cdot y z$ with $x$, we obtain $(x z \cdot y) \cdot y z=x$. Mutually relabeling $z$ and $y$, we obtain $\left(i_{8}\right)$. Since applied transformations are invertible, then $\left(i_{7}\right)$ and $\left(i_{8}\right)$ are equivalent.

Multiplying $\left(i_{8}\right)$ by $y$ from the right and replacing $x y$ with $x$, we have $(x z \cdot z y) \cdot y=$ $x$. Mutually relabeling $z$ and $y$, we obtain the equivalency of $\left(i_{8}\right)$ and $\left(i_{9}\right)$.

Theorem 6. Every identity from (11) implies semi-symmetry.
Proof. Let $(Q, \cdot)$ be a quasigroup. Replacing $z$ with $x$ in identities $\left(i_{1}\right)$ and $\left(i_{2}\right)$, we have

$$
(x \cdot y x) \cdot x=y \cdot x, \quad x \cdot(x y \cdot x)=x \cdot y
$$

Canceling out $x$ in both sides of these identities, we obtain semi-symmetric identity in both cases.

We put $z=y^{r} \cdot x$ in $\left(i_{3}\right), z=y x \stackrel{r}{x}$ in $\left(i_{4}\right):$

$$
x y \cdot y\left(y^{r} \cdot x\right)=\left(y^{r} \cdot x\right) x, \quad x \cdot y\left(y x \cdot\left(y x x^{r} x\right)\right)=y x \stackrel{r}{r}
$$

Apply (2):

$$
x y \cdot x=\left(y^{r} \cdot x\right) \cdot x, \quad x \cdot y x=y x^{r} \cdot x
$$

Canceling out $x$ in the first identity and replacing $y x$ with $x$ in the second identity, we obtain $x y=y \stackrel{r}{\cdot} x$ in both cases. According to the right division, we obtain semi-symmetric identity $y \cdot x y=x$.

By Proposition 1, the identities $\left(i_{4}\right),\left(i_{5}\right),\left(i_{6}\right)$ are equivalent. Then the identities $\left(i_{5}\right),\left(i_{6}\right)$ imply semi-symmetry.

We replace $x$ with $x^{\ell} \cdot y z$ in $\left(i_{7}\right):\left(\left(x^{\ell} \cdot y z\right) \cdot y z\right) z \cdot y=x^{\ell} \cdot y z$.
Apply (2): $x z \cdot y=x^{\ell} \cdot y z$.
Putting $x=y$, we obtain semi-symmetric law $y z \cdot y=z$. Proposition 2 implies that semi-symmetric law follows from $\left(i_{8}\right)$ and $\left(i_{9}\right)$.

### 3.3 Identities in four variables

In this subsection, nine quadratic identities in four variables

$$
\begin{array}{llll}
(x y \cdot u) \cdot x v=y \cdot u v, & \left(m_{1}\right) & x y \cdot(u \cdot v y)=x u \cdot v, & \left(m_{2}\right) \\
(x \cdot(y u \cdot v)) \cdot y=x u \cdot v, & \left(m_{3}\right) & x \cdot((y \cdot u x) \cdot v)=y \cdot u v, & \left(m_{4}\right) \\
x y \cdot(u x \cdot v y)=u v, & \left(m_{5}\right) & (x y \cdot u v) \cdot x u=y v, & \left(m_{6}\right)  \tag{12}\\
x y \cdot(u x \cdot v)=u \cdot y v, & \left(m_{7}\right) & (x \cdot y u) \cdot v y=x v \cdot u, & \left(m_{8}\right) \\
x \cdot\left((y \cdot x u)^{\ell} \cdot v\right)=u v \cdot y & \left(m_{9}\right) & &
\end{array}
$$

are considered. It is proved that each of these identities implies semi-symmetry.
Theorem 7. Every identity from (12) implies semi-symmetry.
Proof. Put $u=x$ in $\left(m_{1}\right)$ and $u=y$ in $\left(m_{2}\right)$ :

$$
(x y \cdot x) \cdot x v=y \cdot x v, \quad x y \cdot(y \cdot v y)=x y \cdot v .
$$

Canceling out $x v$ in the first identity and $x y$ in the second one, we receive semisymmetry from each of these identities.

When we put $v=y$ in $\left(m_{3}\right)$ and $y=x$ in $\left(m_{4}\right)$, then

$$
(x \cdot(y u \cdot y)) \cdot y=x u \cdot y, \quad x \cdot((x \cdot u x) \cdot v))=x \cdot u v .
$$

Cancel out $y$ in the first identity and $x$ in the second one:

$$
x \cdot(y u \cdot y)=x u, \quad(x \cdot u x) \cdot v=u v .
$$

Canceling out $x$ and $v$ respectively in these identities, we receive semi-symmetry in both cases.

Put $v=x$ in $\left(m_{5}\right)$ and $u=y$ in $\left(m_{6}\right)$ :

$$
x y \cdot(u x \cdot x y)=u x, \quad(x y \cdot y v) \cdot x y=y v .
$$

Replace $x y$ with $y$ and $u x$ with $u$ in the first identity, $x y$ with $x$ and $y v$ with $v$ in the second one. We obtain semi-symmetric law in both cases.

Putting $u x=y$ and $u=y^{\ell} \cdot x$ in $\left(m_{7}\right)$, we have $x y \cdot y v=\left(y^{\ell} \cdot x\right) \cdot y v$. Canceling out $y v$ in both sides of the identity and using the definition of the left division, we receive the semi-symmetric identity.

Put $y u=v$ and $u=y^{r} \cdot v$ in $\left(m_{8}\right)$, then $x v \cdot v y=x v \cdot\left(y^{r} \cdot v\right)$. Divide both sides of this identity by $x v$. According to Corollary 11, the obtained identity is equivalent to semi-symmetry.

$$
\text { Put } x u=v \text { and } x=v^{\ell} \cdot u \text { in }\left(m_{9}\right):
$$

$$
\left(v^{\ell} \cdot u\right) \cdot\left(y v^{\ell} \cdot v\right)=u v \cdot y
$$

According to the first identity from (2), we have $\left(v^{\ell} \cdot u\right) \cdot y=u v \cdot y$. Divide both sides of this identity by $y$ on the right. According to Corollary 11, the obtained identity is equivalent to semi-symmetry.

## 4 The varieties of semi-symmetric isotopic closures of some groups

V. D. Belousov [5] has found a quadratic identity in five variables describing the isotopic closure of all groups:

$$
\left(x\left(y^{r} \cdot z\right)^{\ell} \cdot u\right) v=x\left(y^{r} \cdot\left(z^{\ell} \cdot u\right) v\right)
$$

F. M. Sokhatsky [36] has established an identity (7) in four variables, which also describes isotopic closure of all groups, but it is not quadratic.

In this section, we find the semi-symmetric isotopic closure of all Boolean groups, the semi-symmetric isotopic closure of all Abelian groups and the semi-symmetric isotopic closure of all groups.

### 4.1 The variety of semi-symmetric isotopes of all Boolean groups

In this subsection, we consider the semi-symmetric isotopic closure of Boolean groups. We find nine identities (11) which describe the variety of semi-symmetric isotopes of all Boolean groups. This variety is totally symmetric, that is every parastrophe of a quasigroup from the variety belongs to it. These quasigroups are medial and they are either groups or non-commutative semi-symmetric quasigroups.

Lemma 2. The identities (11) are equivalent and define a totally symmetric variety.
Proof. To obtain $\left(i_{3}\right)$ we use semi-symmetry law:

- multiply $\left(i_{9}\right)$ by $z$ from the left;
- multiply $\left(i_{6}\right)$ by $x$ from the right;
- replace $z$ with $y z$ in $\left(i_{2}\right)$ and multiply the obtained identity by $x$ from the right;
- replace $x$ with $x y$ in $\left(i_{1}\right)$ and multiply the obtained identity by $z$ from the left.

Taking into account Proposition 1 and Proposition 2, we obtain equivalency of all identities from (11).

Consider $s$-parastrophe of $\left(i_{1}\right):(x \stackrel{s}{\cdot}(y \stackrel{s}{\bullet} z))^{s} z=y{ }^{s} \cdot x$. By the definition of $s$-parastrophe of the operation $(\cdot)$, we obtain $z \cdot(z y \cdot x)=x y$. This identity coincides with $\left(i_{2}\right)$ after mutual relabeling of $x$ and $z$. This means that $s$-parastrophe of $\left(i_{1}\right)$ defines the same variety. Since the variety is semi-symmetric, then it is totally symmetric.

Theorem 8. In an arbitrary quasigroup $(Q ; \cdot)$ the following statements are equivalent:

1) ( $Q ; \cdot)$ is a semi-symmetric isotope of a Boolean group;
2) ( $Q ; \cdot$ ) satisfies an arbitrary identity from (11);
3) there exists a Boolean group ( $Q ;+$ ), its automorphism $\alpha$ and an element $a \in Q$ such that

$$
\begin{equation*}
x \cdot y=\alpha x+a+\alpha^{2} y, \quad \alpha^{3}=\iota, \quad \alpha a=a . \tag{13}
\end{equation*}
$$

Proof. Since all identities from (11) are equivalent by virtue of Lemma 2, then they define the same variety. Therefore, it is enough to prove the theorem for one of them.

1) $\Leftrightarrow 3)$. Let $(Q ; \cdot)$ be a semi-symmetric isotope of a Boolean group $(G ; *)$. Then all groups being isotopic to ( $Q ; \cdot$ ) are Boolean. Therefore, according to item 5) of Theorem 5 , item 1) and item 3) of the theorem are equivalent.
$2) \Rightarrow 1$ ). Let $(Q ; \cdot)$ be a quasigroup satisfying the identity $\left(i_{1}\right)$ from (11). By Theorem $6,(Q ; \cdot)$ is a semi-symmetric quasigroup. According to Theorem 3 and Corollary 5, this quasigroup is isotopic to a group, so $(Q ; \cdot)$ is a semi-symmetric group isotope.
2) $\Rightarrow$ 2). Let (13) hold for a quasigroup $(Q ; \cdot)$. Prove that the identity $\left(i_{1}\right)$ is true. Indeed,

$$
(x \cdot y z) \cdot z=\alpha\left(\alpha x+a+\alpha^{2}\left(\alpha y+a+\alpha^{2} z\right)\right)+a+\alpha^{2} z .
$$

Because $\alpha$ is an automorphism, then

$$
(x \cdot y z) \cdot z=\alpha^{2} x+\alpha a+\alpha y+a+\alpha^{2} z+a+\alpha^{2} z .
$$

Since $(Q ;+)$ is a Boolean group and $\alpha a=a$, then $2 a=0$ and $2 \alpha^{2} z=0$. Consequently,

$$
(x \cdot y z) \cdot z=\alpha y+a+\alpha^{2} x=y \cdot x .
$$

Theorem 8 implies several corollaries.
Corollary 13. The variety of quasigroups being defined by one of the identities (11) is totally symmetric.

Proof. The proof follows from Lemma 2 and from Theorem 8.
Corollary 14. The semi-symmetric isotopic closure of all Boolean groups is defined by pairwise equivalent identities (11).

Proof. The proof is evident, taking into account Theorem 8.
Corollary 15. The semi-symmetric isotopic closure of all Boolean groups is the intersection of the variety of all semi-symmetric quasigroups and the variety of all Boolean groups.

Proof. The proof immediately follows from Theorem 8 and Corollary 14.
Corollary 16. Every quasigroup satisfying one of the identities (11) is either a Boolean group or a non-commutative semi-symmetric quasigroup.

Proof. Let $(Q ; \cdot)$ be a quasigroup satisfying the identity $\left(i_{1}\right)$. Then by Theorem 8 , its canonical decomposition has the form (13), where $\alpha$ is some automorphism and $a \in Q$.

If $(Q ; \cdot)$ is commutative, then according to Theorem $5, \alpha^{2}=\alpha$, i.e., $\alpha=\iota$. The equality $x \cdot y=x+a+y$ means that $L_{a}$ is an isomorphism between $(Q ; \cdot)$ and $(Q ;+)$. Thus, $(Q ; \cdot)$ is a Boolean group.

If ( $Q ; \cdot$ ) is non-commutative, then according to Theorem $5 \alpha^{2} \neq \alpha$. Therefore, $\alpha \neq \iota$ and according to Corollary $6,(Q ; \cdot)$ is not a group, but by Theorem 6 , it is semi-symmetric.

Example 1. Consider the group $\mathbb{Z}_{2}^{2}:=\mathbb{Z}_{2} \times \mathbb{Z}_{2}$. Define the transformation $\alpha$ of the set $\mathbb{Z}_{2}^{2}$ :

$$
\alpha(x):=x \cdot\left(\begin{array}{ll}
0 & 1 \\
1 & 1
\end{array}\right) .
$$

Since $\alpha^{3}=\iota$, then $\alpha$ is an automorphism of the group $\mathbb{Z}_{2}^{2}$. By Theorem 8 , a quasigroup $\left(\mathbb{Z}_{2}^{2} ; \circ\right.$ ) defined by the equation $x \circ y:=\alpha x+\alpha^{2} y$ satisfies the identity $\left(i_{1}\right)$. Because $\alpha \neq \alpha^{2}$, then ( $Q, \circ$ ) is non-commutative. By Corollary 16, the quasigroup $(Q, \circ)$ is semi-symmetric and not a group.

### 4.2 The variety of semi-symmetric isotopes of all Abelian groups

In this subsection, the variety being defined by identities (12) is considered. Each of these identities determines the totally symmetric variety of all semi-symmetric medial quasigroups. This variety is the semi-symmetric isotopic closure of all Abelian groups. Quasigroups belonging to this variety are either Boolean groups or nonBoolean totally symmetric quasigroups or non-commutative semi-symmetric quasigroups.

Theorem 9. The identities (12) are equivalent and define the variety of all medial semi-symmetric quasigroups.

Proof. According to Theorem 7, semi-symmetry follows from any identity in (12). Using semi-symmetry, further it will be shown that each of the identities from (12) is equivalent to mediality.

Put $y x$ instead of $y$ in $\left(m_{1}\right)$ and $y v$ instead of $v$ in $\left(m_{2}\right)$ :

$$
((x \cdot y x) \cdot u) \cdot x v=y x \cdot u v, \quad x y \cdot(u \cdot(y v \cdot y))=x u \cdot y v .
$$

Using semi-symmetry, we receive mediality in both cases.
Replace $y$ with $u y$ in $\left(m_{3}\right)$ and use semi-symmetry: $(x \cdot y v) \cdot u y=x u \cdot v$. Replacing $v$ with $v y$ and applying semi-symmetry to the last identity, we get mediality.

Change $x$ by $x u$ in $\left(m_{4}\right)$ and apply the semi-symmetric identity: $x u \cdot(y x \cdot v)=$ $y \cdot u v$. Put $y$ instead of $x y$ in the obtained identity. Using semi-symmetry, we receive $x u \cdot y v=x y \cdot u v$, that is the medial law holds for every $x, u, y, v$.

Multiply $\left(m_{6}\right)$ by $x u$ on the left and $\left(m_{5}\right)$ by $x y$ on the right:

$$
x u \cdot((x y \cdot u v) \cdot x u)=x u \cdot y v, \quad(x y \cdot(u x \cdot v y)) \cdot x y=u v \cdot x y .
$$

Applying semi-symmetry to these identities, we obtain medial identity in the first case and $u x \cdot v y=u v \cdot x y$ in the second one. The last identity means that the mediality holds for all $u, x, v, y$.

Substitute $u$ with $x u$ in $\left(m_{7}\right), u$ with $u y$ in ( $m_{8}$ ) and apply semi-symmetry to the received identities, as a result we obtain mediality in both cases.

Consider $\left(m_{9}\right)$. Since $(\cdot)$ is semi-symmetric, then $\left({ }^{\ell}\right)=\left({ }^{s}\right)$, that is $x^{\ell} \cdot y=y x$. Then $\left(m_{9}\right)$ can be written as follows: $x \cdot(v \cdot(y \cdot x u))=u v \cdot y$. Replace $x$ with $u x$ in this identity and use semi-symmetry: $u x \cdot(v \cdot y x)=u v \cdot y$. Substituting $y$ with $x y$ and using semi-symmetry law, we have $u x \cdot v y=u v \cdot x y$. It means that the mediality holds for all $u, x, v, y$.

Thus, a quasigroup satisfying an arbitrary identity from (12) is semi-symmetric and medial simultaneously. This means that identities (12) define the same variety of semi-symmetric medial quasigroups.

Corollary 17. The variety of all semi-symmetric medial quasigroups is totally symmetric.

Proof. It is well known that the variety of all medial quasigroups is totally symmetric, according to Corollary 10, the variety of all semi-symmetric quasigroups is totally symmetric as well. Therefore, the variety of all semi-symmetric medial quasigroups is totally symmetric, since it is the intersection of two totally symmetric varieties.

Corollary 18. The semi-symmetric isotopic closure of all Abelian groups is defined by pairwise equivalent identities (12).

Proof. By virtue of Theorem 9, all identities from (12) are equivalent, then it is enough to prove this theorem for one of them. Let $(Q ; \cdot)$ be an arbitrary quasigroup. Let us prove that ( $Q ; \cdot$ ) is semi-symmetric isotope of Abelian groups iff it satisfies the identity $\left(m_{1}\right)$.

Let $(Q ; \cdot)$ satisfy $\left(m_{1}\right)$, then according to Theorem $9,(Q ; \cdot)$ is medial and ToyodaBruck theorem implies that $(Q ; \cdot)$ is an isotope of an Abelian group. By Theorem 7, $(Q ; \cdot)$ is semi-symmetric. Thus, $(Q ; \cdot)$ is semi-symmetric isotope of an Abelian group.

Vice versa, let ( $Q ; \cdot \cdot$ be an arbitrary semi-symmetric isotope of an Abelian group. Then by item 4) of Corollary 8 , its canonical decomposition is the following:

$$
\begin{equation*}
x \cdot y=\alpha x+a+\alpha^{-1} y, \quad \alpha^{3}=-\iota, \quad \alpha a=-a, \tag{14}
\end{equation*}
$$

where $(Q ;+)$ is an Abelian group, $\alpha$ is its automorphism and an element $a \in Q$. Let us show that conditions (14) satisfy the identity $\left(m_{1}\right)$.

$$
(x y \cdot u) \cdot x v \stackrel{(14)}{=} \alpha\left(\alpha\left(\alpha x+a+\alpha^{-1} y\right)+a+\alpha^{-1} u\right)+a+\alpha^{-1}\left(\alpha x+a+\alpha^{-1} v\right) .
$$

Because $\alpha$ and $\alpha^{-1}$ are automorphisms, then

$$
(x y \cdot u) \cdot x v=\alpha^{3} x+\alpha^{2} a+\alpha y+\alpha a+u+a+x+\alpha^{-1} a+\alpha^{-2} v .
$$

Since $(Q ;+)$ is an Abelian group and $\alpha^{3}=-\iota, \alpha a=-a$, then

$$
\begin{aligned}
& (x y \cdot u) \cdot x v=-x+a+\alpha y-a+u+a+x+\alpha^{-1} a+\alpha^{-2} v= \\
& =\alpha y+a+\alpha^{-1} \alpha u+\alpha^{-1} a+\alpha^{-2} v=\alpha y+a+\alpha^{-1}\left(\alpha u+a+\alpha^{-1} v\right)=y \cdot u v
\end{aligned}
$$

Corollary 19. The semi-symmetric isotopic closure of all Abelian groups is the intersection of the variety of semi-symmetric quasigroups and the variety of all medial semi-symmetric quasigroups.

Proof. The proof immediately follows from Theorem 9 and Corollary 18.
Corollary 20. Every quasigroup satisfying one of the identities (12) is either a Boolean group or a non-Boolean totally symmetric quasigroup, or a non-commutative semi-symmetric quasigroup.

Proof. Let $(Q ; \cdot)$ be a quasigroup satisfying the identity $\left(m_{1}\right)$, then according to the proof of Corollary 18, (14) is its canonical decomposition.

If $\alpha=\iota$, then Corollary 6 implies that $(Q ; \cdot)$ is a Boolean group.
If $\alpha=-\iota$, then according to item 5) of Corollary 8, the quasigroup ( $Q ; \cdot$ ) is totally symmetric. There is at least one totally symmetric quasigroup which is nonBoolean group. For example, the quasigroup $\left(\mathbb{Z}_{3} ; \bullet\right)$ defined by $x \bullet y:=-x+1-y$ is totally symmetric quasigroup and is a non-Boolean group, since $2 \cdot(-1)=-2 \neq 0$.

Consider the case $\alpha \neq \iota$ and $\alpha \neq-\iota$. Since condition $\alpha^{3}=-\iota$ from (14) implies $\alpha \neq \alpha^{-1}$, then quasigroup ( $Q ; \cdot$ ) is non-commutative. But canonical decomposition (14) satisfies semi-symmetry. Indeed,

$$
x \cdot y x \stackrel{(14)}{=} \alpha x+a+\alpha^{-1}\left(\alpha y+a+\alpha^{-1} x\right)=\alpha x+a+y+\alpha^{-1} a+\alpha^{-2} x .
$$

Since conditions $\alpha^{3}=-\iota$ and $\alpha a=-a$ imply $\alpha^{-2}=-\alpha$ and $\alpha^{-1} a=-a$, then $x \cdot y x=\alpha x+a+y-a-\alpha x=y$. The corollary has been proved.

Example 2. The quasigroup $\left(\mathbb{Z}_{9} ; *\right), x * y=2 x+3+5 y$, belongs to the variety $\mathfrak{A}_{\mathrm{ss}}$ and does not belong to the variety $\mathfrak{B}_{\mathrm{ss}}$. Indeed, this quasigroup satisfies the canonical decomposition (14), since $\alpha^{3}=2^{3}=-\iota, \alpha a=2 \cdot 3=6=-3$ and does not satisfy conditions (13), because $\alpha^{3} \neq \iota$. Thus, taking into account Corollary 20, $\left(\mathbb{Z}_{9} ; *\right)$ is a non-commutative semi-symmetric quasigroup.

### 4.3 The variety of semi-symmetric isotopes of all groups

In this subsection, we find an identity which describes the semi-symmetric isotopic closure of all groups.

Theorem 10. In an arbitrary quasigroup $(Q ; \cdot)$ the following statements are equivalent:

1) $(Q ; \cdot)$ is a semi-symmetric group isotope;
2) $(Q ; \cdot)$ satisfies

$$
\begin{equation*}
u(x \cdot y u)=z(x \cdot(u y \cdot z) u) ; \tag{15}
\end{equation*}
$$

3) there exists a group $(Q ;+)$, its anti-automorphism $\alpha$, an element $a \in Q$ such that $x \cdot y=\alpha x+a+\alpha^{-1} y$ and $\alpha^{3}=-I_{a}^{-1}, \alpha a=-a$, where $I_{a}(x):=-a+x+a$.

Proof. 2) $\Rightarrow 1$ ). Let a quasigroup ( $Q ; \cdot$ ) satisfy (15). Put $z=u$ in (15):

$$
u(x \cdot y u)=u(x \cdot(u y \cdot u) u) .
$$

Cancelling out $u, x, u$, we obtain identity (1). Hence, $(Q ; \cdot)$ is semi-symmetric.
Multiply (15) by $z$ from the right and use the identity of semi-symmetry:

$$
\begin{equation*}
u(x \cdot y u) \cdot z=x \cdot(u y \cdot z) u . \tag{16}
\end{equation*}
$$

Since ( $Q ; \cdot \cdot$ ) is semi-symmetric, then (5) hold. Replacing the operation (•) with its patasrophes in (16), we have (7). Corollary 3 implies that $(Q ; \cdot)$ is isotopic to a group.

1) $\Rightarrow$ 2). Let $(Q ; \cdot)$ be a semi-symmetric group isotope, then the equalities (5) are true and (7) can be written as (16). Multiply (16) by $z$ from the left and apply the identity (1). As a result we obtain (15).
$3) \Leftrightarrow 1$ ). It follows from item 5) of Theorem 5.
Corollary 21. The semi-symmetric isotopic closure of all groups is defined by (15).
Proof. It is evident from Theorem 10.
Corollary 22. The identity (15) implies semi-symmetry.
Proof. The proof follows from Theorem 10.
Corollary 23. The variety of quasigroups being defined by (15) is totally symmetric.

Proof. Let $\mathfrak{Q}$ be the variety defined by (15). It means that each quasigroup ( $Q ; \cdot$ ) from $\mathfrak{Q}$ satisfies the identity $x \cdot y x=y$. This identity is equivalent to $x y \cdot x=y$. Define the operation $(\circ):=\left({ }^{\circ}\right)$. Then the last identity can be written as $x \circ(y \circ x)=y$, i.e., $s$-parastrophe of an arbitrary quasigroup from $\mathfrak{Q}$ is in $\mathfrak{Q}$. Thus, for all $\sigma \in S_{3}$ the relation ${ }^{\sigma} \mathfrak{Q}=\mathfrak{Q}$. Therefore, this variety is totally symmetric.

## 5 Main results

In this article, we have found families (11), (12) and (15) of identities. Namely:

1) identities (11) are pairwise equivalent and describe the variety $\mathfrak{B}_{\mathrm{ss}}$ of the semisymmetric isotopic closure of all Boolean groups;
2) identities (12) are pairwise equivalent and describe the variety $\mathfrak{A}_{\mathrm{ss}}$ of the semisymmetric isotopic closure of all Abelian groups;
3) the identity (15) describes the variety $\mathfrak{G}_{\mathrm{ss}}$ of the semi-symmetric isotopic closure of all groups;
4) the identities from Corollary 11 are pairwise equivalent and describe the variety $\mathfrak{S}$ of all semi-symmetric quasigroups.

To establish a relationship among these varieties we give the following examples.
Example 3. In the symmetric group ( $S_{3} ; \cdot$ ), where ( $\cdot$ ) denotes the composition of permutations, we define a transformation $\alpha$ by $\alpha(x):=s \ell \cdot x^{-1} \cdot s r$. Here $\alpha$ is anti-automorphism of the group $\left(S_{3} ; \cdot\right)$ and $\alpha^{3}=I$. Indeed,

$$
\begin{gathered}
\alpha(x \cdot y)=s \ell \cdot(x y)^{-1} \cdot s r=s \ell \cdot y^{-1} x^{-1} \cdot s r=s \ell y^{-1} s r \cdot s \ell x^{-1} s r=\alpha(y) \cdot \alpha(x) \\
\alpha^{3}(x)=\alpha\left(s \ell\left(s \ell x^{-1} s r\right)^{-1} s r\right)=\alpha(s \ell s \ell x s r s r)=(s \ell)^{3} x^{-1}(s r)^{3}=I
\end{gathered}
$$

According to item 5) of Theorem 5, the groupoid ( $S_{3} ; \circ$ ) is defined by

$$
x \circ y:=\alpha(x) \cdot \alpha^{-1}(y)
$$

and it is a semi-symmetric group isotope. Therefore, $S_{3}$ is a semi-symmetric isotope of a non-commutative group.

Example 4. Let $Q:=\{1,2,3,4,5\}$. On the set $Q$ we define the operation (•):

| $(\cdot)$ | 1 | 2 | 3 | 4 | 5 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 4 | 5 | 3 | 2 |
| 2 | 5 | 2 | 4 | 1 | 3 |
| 3 | 4 | 5 | 3 | 2 | 1 |
| 4 | 2 | 3 | 1 | 5 | 4 |
| 5 | 3 | 1 | 2 | 4 | 5 |


| $(\circ)$ | 1 | 2 | 3 | 4 | 5 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 2 | 3 | 4 | 5 |
| 2 | 2 | 4 | 5 | 3 | 1 |
| 3 | 3 | 5 | 4 | 1 | 2 |
| 4 | 4 | 1 | 2 | 5 | 3 |
| 5 | 5 | 3 | 1 | 2 | 4 |

It is easy to verify that $(Q ; \cdot)$ is a semi-symmetric quasigroup. Permuting rows by the cycle (2534) and columns by the cycle (2435), we obtain the loop ( $Q ;$ ) . Suppose, the quasigroup $(Q ; \cdot)$ is isotopic to a group $(G ; \diamond)$. ( $Q ; \circ$ ) and ( $Q ; \cdot$ ) are isotopic according to construction of $(Q ; \circ)$. Then the loop $(Q ; \circ)$ and the group $(G ; \diamond)$ are isotopic, therefore they are isomorphic. $(Q ; \circ)$ is commutative as a prime order group. But this statement is false, because $4 \circ 2=1 \neq 3=2 \circ 4$. Consequently, the assumption is false and the quasigroup $(Q ; \cdot)$ is not a group isotope.

Theorem 11. The varieties $\mathfrak{B}_{\mathrm{ss}}, \mathfrak{A}_{\mathrm{ss}}, \mathfrak{G}_{\mathrm{ss}}$ and $\mathfrak{S}_{\mathrm{ss}}$ are different and form the following chain: $\mathfrak{B}_{\mathrm{ss}} \subset \mathfrak{A}_{\mathrm{ss}} \subset \mathfrak{G}_{\mathrm{ss}} \subset \mathfrak{S}$.

Proof. Nonstrict inclusion of these varieties follows from their definitions. To prove strict inclusion, we consider some examples of quasigroups which belong to a wider variety and do not belong to the smaller variety. The total symmetry of each of the varieties $\mathfrak{B}_{\mathrm{ss}}, \mathfrak{A}_{\mathrm{ss}}, \mathfrak{G}_{\mathrm{ss}}, \mathfrak{S}$ is provided by Corollaries $10,13,17,23$.

In Example 2, the groupoid $\left(\mathbb{Z}_{9} ; *\right)$ is a semi-symmetric quasigroup and it is isotopic to the cyclic group $\left(\mathbb{Z}_{9} ;+\right)$, which is not Boolean. Hence, $\left(\mathbb{Z}_{9} ; *\right)$ belongs to the variety $\mathfrak{A}_{\mathrm{ss}}$ and does not belong to $\mathfrak{B}_{\mathrm{ss}}$.

The quasigroup ( $S_{3} ; \circ$ ) from Example 3 belongs to the variety $\mathfrak{G}_{\text {ss }}$ and does not belong to $\mathfrak{A}_{\mathrm{ss}}$, because the group $S_{3}$ is non-commutative.

The quasigroup $(Q ; \cdot)$ from Example 4 belongs to the variety $\mathfrak{S}_{\mathrm{ss}}$ and does not belong to $\mathfrak{G}_{\text {ss }}$, because the quasigroup ( $Q ; \cdot$ ) is not isotopic to a group.

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Halyna Krainichuk, Olena Tarkovska
Received November 30, 2016
V. Stus Donetsk National University

Department of mathematical analysis
and differential equations
21000 Vinnytsia, Ukraine
E-mail: kraynichuk@ukr.net; tark.olena@gmail.com

# Factorizations in the rings of the block matrices 

Vasyl' Petrychkovych, Nataliia Dzhaliuk


#### Abstract

The factorizations in the rings of the block triangular and the block diagonal matrices over an integral domain of finitely generated principal ideals are described. Conditions for existence and uniqueness up to the association of the factorizations in such rings are established. The construction of the factorizations of matrices is reduced to the factorizations of diagonal blocks of the block triangular matrices and the solving of the linear Sylvester matrix equations.


Mathematics subject classification: 15A21, 15A24.
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## 1 Introduction

Let $R$ be an integral domain of finitely generated principal ideals. We will denote the ring of $n \times n$ matrices by $M(n, R)$, the set of $n \times m$ matrices by $M(n, m, R)$, the group of invertible $n \times n$ matrices over $R$ by $G L(n, R)$, the subring of the block upper triangular matrices

$$
T=\operatorname{triang}\left(T_{11}, \ldots, T_{k k}\right)=\left[\begin{array}{cccc}
T_{11} & T_{12} & \ldots & T_{1 k} \\
0 & T_{22} & \ldots & T_{2 k} \\
\ldots & \ldots & \ldots & \ldots \\
0 & 0 & \cdots & T_{k k}
\end{array}\right]
$$

where $T_{i i} \in M\left(n_{i}, R\right), i=1, \ldots, k$, by $B T\left(n_{1}, \ldots, n_{k}, R\right)$. Factorizations $T=A B$ and $T=A_{1} B_{1}$ of the matrix $T \in M(n, R)$ are called associate if $A_{1}=A V$ and $B_{1}=V^{-1} B$, where $V \in G L(n, R)$. We will consider the factorizations of matrices in the ring $M(n, R)$ and in its subring $B T\left(n_{1}, \ldots, n_{k}, R\right)$ of the block triangular matrices. We will describe the factorizations of matrices up to the association. We would like to note that the block matrices arise in various problems, such as in $[10,16]$.

The theory of factorization of the polynomial matrices, which are matrices over the polynomial ring, has been well developed. Such factorizations of the polynomial matrices have been used in the theory of matrix and differential equations $[4,7,14]$, in the theory of operator pencils [9] and in other applied problems [8]. In [1], conditions for uniqueness up to the association of the factorizations of matrices over the principal ideal rings have been formulated.

[^0]In this article, conditions for existence and uniqueness up to the association of the factorizations in the ring of the block triangular matrices have been obtained. We have established such classes of the block triangular matrices, where each factorization is associated to its factorization in the ring $B T\left(n_{1}, \ldots, n_{k}, R\right)$ of the block triangular matrices. We should note that the block matrices are connected with the matrix linear bilateral equations. It is known that such equation is solvable if and only if the block triangular and the block diagonal matrices composed of the equation coefficients are equivalent $[3,5,6,15]$. Hence, the factorization of the block triangular matrices is reduced to the factorization of the diagonal blocks and the solving of the matrix linear equations. Similar results for matrices over the ring of polynomials have been obtained in [13].

## 2 Preliminaries

Let $A \in M(n, m, R), n \leq m, d_{n}^{A} \neq 0$ and the matrix $A$ have the factorization $A=B C, \quad B \in M(n, R), C \in M(n, m, R)$. Let us write the matrices in the block form

$$
\left[\begin{array}{c}
A_{1}  \tag{1}\\
\vdots \\
A_{k}
\end{array}\right]=\left[\begin{array}{c}
B_{1} \\
\vdots \\
B_{k}
\end{array}\right] C, \quad A_{i}, B_{i} \in M\left(n_{i}, m, R\right), \quad n_{i} \geq 1, \quad i=1, \ldots, k
$$

Further, we will denote $r$-th determinantal divisor of the matrix $A$ by $d_{r}^{A}$, the greatest common divisor of elements $a$ and $b$ by $(a, b)=d$. Let $\left(d_{n_{i}}^{A_{i}}, d_{n_{j}}^{A_{j}}\right)=d^{\left(A_{i}, A_{j}\right)}$ and $\left(\operatorname{det} B, d_{n}^{C}\right)=d^{(B, C)}$.

Lemma 1. Let $\left(\operatorname{det} B, d_{n_{i}}^{A_{i}}\right)=\varphi_{i}, \quad i=1, \ldots, k$. If

$$
\begin{equation*}
\left(d^{(B, C)}, d_{n-1}^{A}\right)=1, \tag{2}
\end{equation*}
$$

then $d_{n_{i}}^{B_{i}}=\varphi_{i}, i=1, \ldots, k$.
Proof. Let $k=2$. From $A_{1}=B_{1} C$ and $\left(\operatorname{det} B, d_{n_{1}}^{A_{1}}\right)=\varphi_{1}$ it follows that $d_{n_{1}}^{B_{1}} \mid d_{n_{1}}^{A_{1}}$ and $d_{n_{1}}^{B_{1}} \mid \varphi_{1}$, that is $\varphi_{1}=d_{n_{1}}^{B_{1}} g$. We assume that $d_{n_{1}}^{B_{1}} \neq \varphi_{1}$. This means that $g \notin U(R)$, where $U(R)$ is the group of units of the ring $R$.

Let $p$ be an irreducible element from the ring $R$ such that $p \mid g$. We suppose that $p \mid d^{(B, C)}$. The matrix $B_{1}$ can be written as

$$
B_{1}=G F_{1}, \quad G \in M\left(n_{1}, R\right), \quad F_{1} \in M\left(n_{1}, n, R\right), \quad \operatorname{det} G=d_{n_{1}}^{B_{1}}, \quad d_{n_{1}}^{F_{1}}=1
$$

Hence, $A_{1}=G H_{1}, \quad H_{1} \in M\left(n_{1}, n, R\right)$. So, from (1) we obtain

$$
\left[\begin{array}{l}
H_{1} \\
A_{2}
\end{array}\right]=\left[\begin{array}{l}
F_{1} \\
B_{2}
\end{array}\right] C .
$$

For matrix $F_{1}$ there exists such a matrix $W \in G L(n, R)$ that $F_{1} W=\left[\begin{array}{ll}I_{n_{1}} & 0\end{array}\right]$, where $I_{n_{1}}$ is an identity matrix. Therefore

$$
\left[\begin{array}{l}
H_{1} \\
A_{2}
\end{array}\right]=\left[\begin{array}{cc}
I_{n_{1}} & 0 \\
\widetilde{B}_{21} & \widetilde{B}_{22}
\end{array}\right] \widetilde{C}, \quad \widetilde{C}=W^{-1} C, \widetilde{B}_{21} \in M\left(n_{2}, n_{1}, R\right), \widetilde{B}_{22} \in M\left(n_{2}, R\right)
$$

So, for the matrix

$$
V=\left[\begin{array}{cc}
I_{n_{1}} & 0 \\
-\widetilde{B}_{21} & I_{n_{2}}
\end{array}\right]
$$

we obtain

$$
V\left[\begin{array}{l}
H_{1} \\
A_{2}
\end{array}\right]=\left[\begin{array}{l}
H_{1} \\
H_{2}
\end{array}\right]=\left[\begin{array}{cc}
I_{n_{1}} & 0 \\
0 & \widetilde{B}_{22}
\end{array}\right]\left[\begin{array}{cc}
\widetilde{C}_{11} & \widetilde{C}_{12} \\
\widetilde{C}_{21} & \widetilde{C}_{22}
\end{array}\right]=\widetilde{B} \widetilde{C} .
$$

Since $p \mid \operatorname{det} \widetilde{B}_{22}$, we obtain $p \mid d_{n_{2}}^{H_{2}}$. On the other hand $p \mid d_{n_{1}}^{H_{1}}$, hence $p \mid d_{n-1}^{A}$, which contradicts the condition (2) of the lemma. Thus, $g \in U(R)$ and $d_{n_{1}}^{B_{1}}=\varphi_{1}$.

In the same way, we can prove that $d_{n_{2}}^{B_{2}}=\varphi_{2}$.
If $p \nmid d^{(B, C)}$ (does not divide), the proof of the lemma is similar.
For an arbitrary $k$, we prove the lemma by induction.
Lemma 2. Let $\left(\left(d_{n_{l}}^{A_{l}}, d_{n_{l+1}}^{A_{l+1}}\right), d^{(B, C)}\right)=1, \quad l=1, \ldots, k-1$. If $d_{n}^{A}=d_{n_{1}}^{A_{1}} \cdots d_{n_{k}}^{A_{k}}$, then $\operatorname{det} B=d_{n_{1}}^{B_{1}} \cdots d_{n_{k}}^{B_{k}}$.

Proof. Following the same procedure as in the proof of Lemma 1, we obtain that $\left(\operatorname{det} B, d_{n_{i}}^{A_{i}}\right)=d_{n_{i}}^{B_{i}}, i=1, \ldots, k$. We suppose that $\operatorname{det} B=d_{n_{1}}^{B_{1}} \cdots d_{n_{k}}^{B_{k}} f, f \notin U(R)$. The matrices $B_{i}$ and $A_{i}$ from (1) can be written as $B_{i}=G_{i} F_{i}, A_{i}=G_{i} H_{i}, \quad G_{i} \in$ $M\left(n_{i}, R\right), F_{i}, H_{i} \in M\left(n_{i}, m, R\right), \operatorname{det} G_{i}=d_{n_{i}}^{B_{i}}, d_{n_{i}}^{F_{i}}=1, i=1, \ldots, k$.

From (1) we obtain

$$
\left[\begin{array}{c}
H_{1}  \tag{3}\\
\vdots \\
H_{k}
\end{array}\right]=\left[\begin{array}{c}
F_{1} \\
\vdots \\
F_{k}
\end{array}\right] C
$$

or else $H=F C$. Let $q$ be an irreducible element from the ring $R$ such that $q \mid f$. It is obvious that $q \mid \operatorname{det} H$. Since $\operatorname{det} H=d_{n_{1}}^{H_{1}} \cdots d_{n_{k}}^{H_{k}}, q \mid d_{n_{i}}^{H_{i}}$ for a certain $i$. We assume that $q \mid d_{n_{1}}^{H_{1}}$. Then from (3) we have $H_{1}=F_{1} C$. Since $d_{n_{1}}^{F_{1}}=1, q \mid d_{n}^{C}$. So, from $H_{j}=F_{j} C$ we obtain $q \mid d_{n_{j}}^{H_{j}}, j=1, \ldots, k$. Thus $q \mid d^{\left(A_{l}, A_{l+1}\right)}$ for all $l=1, \ldots, k-1$.

Hence, we get $q \mid d^{(B, C)}$. Since $\left(\left(d_{n_{l}}^{A_{l}}, d_{n_{l+1}}^{A_{l+1}}\right), d^{(B, C)}\right)=1, \quad l=1, \ldots, k-1, \quad q=1$. So $f \in U(R)$ and thus, $\operatorname{det} B=d_{n_{1}}^{B_{1}} \cdots d_{n_{k}}^{B_{k}}$.

Corollary 1. Let $A \in M(n, R)$ and $\operatorname{det} A=\varphi_{1} \cdots \varphi_{k}$. Then the matrix $A$ is the right equivalent to the block diagonal matrix, that is $A V=\operatorname{diag}\left(D_{1}, \ldots, D_{k}\right), D_{i} \in$
$M\left(n_{i}, R\right), \operatorname{det} D_{i}=\varphi_{i}, \quad i=1, \ldots k$, if and only if the matrix $A$ can be written in the form

$$
A=\left[\begin{array}{c}
A_{1} \\
\vdots \\
A_{k}
\end{array}\right], \quad A_{i} \in M\left(n_{i}, m, R\right), d_{n_{i}}^{A_{i}}=\varphi_{i}, \quad i=1, \ldots, k
$$

Lemma 3. Let $C \in M(n, m, R), n \leq m$ and $d_{n}^{C} \neq 0$. Let $A=\left[\begin{array}{lll}\mathbf{c}_{j_{1}} & \ldots & \mathbf{c}_{j_{n}}\end{array}\right]$ be a submatrix which is composed of $j_{1}, \ldots, j_{n}$ columns of the matrix $C$ and such that $\operatorname{det} A=d_{n}^{C}$. Then there exists a matrix $Q \in G L(m, R)$ such that $C Q=\left[\begin{array}{ll}A & 0\end{array}\right]$.

Proof. Using the elementary column operations, we reduce the matrix $C$ to the form $C P=\left[\begin{array}{ll}A & B\end{array}\right]=C_{1}$, where $P \in G L(m, R)$. For the matrices $A$ and $B$ there exist matrices $V_{1} \in G L(n, R)$ and $V_{2} \in G L(m-n, R)$ such that $A V_{1}=A_{1}$, $B V_{2}=B_{1}$ and they are lower triangular matrices.

Put $m-n \geq n$. Then

$$
\begin{gathered}
{\left[\begin{array}{ll}
A & B
\end{array}\right]\left[\begin{array}{cc}
V_{1} & 0 \\
0 & V_{2}
\end{array}\right]=\left[\begin{array}{ll}
A_{1} & B_{1}
\end{array}\right]=} \\
{\left[\begin{array}{ccccccccccc}
a_{1} & 0 & \cdots & 0 & b_{1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
a_{21} & a_{2} & \cdots & 0 & b_{21} & b_{2} & \cdots & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{n 1} & a_{n 2} & \cdots & a_{n} & b_{n 1} & b_{n 2} & \cdots & b_{n} & 0 & \cdots & 0
\end{array}\right]=C_{2} .}
\end{gathered}
$$

It is obvious that $d_{n}^{C_{2}}=\operatorname{det} A_{1}=\operatorname{det} A$. Therefore all the $n$-th order minors of the matrix $C_{2}$ are divided by $\operatorname{det} A_{1}=a_{1} \cdots a_{n}$. Hence, the element $b_{i}$ of the matrix $C_{2}$ is divided by $a_{i}$ for all $i=1, \ldots, n$.

Using the elementary column operations, we reduce the matrix $C_{2}$ to the form

$$
\left[\begin{array}{cccccccccccc}
a_{1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
a_{21} & a_{2} & \cdots & 0 & b_{21}^{\prime} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
a_{n 1} & a_{n 2} & \cdots & a_{n} & b_{n 1}^{\prime} & b_{n 2}^{\prime} & \cdots & b_{n, n-1}^{\prime} & 0 & 0 & \cdots & 0
\end{array}\right]=C_{3} .
$$

Continuing this way, we obtain that $C_{1} W=\left[\begin{array}{ll}A_{1} & 0\end{array}\right]$, where $W \in G L(m, R)$. Hence, the matrix $C$ is the right equivalent to the matrix $\left[\begin{array}{ll}A & 0\end{array}\right]$.

If $m-n<n$, the proof of the lemma is similar.
Corollary 2. Let $C=\left[\begin{array}{ll}A & B\end{array}\right], C \in M(n, m, R), \quad A \in M(n, R), d_{n}^{C} \neq 0$. If $\operatorname{det} A=d_{n}^{C}$, then there exists such a unitriangular matrix $S=\left[\begin{array}{cc}I_{n} & S_{12} \\ 0 & I_{m-n}\end{array}\right]$ that

$$
\left[\begin{array}{ll}
A & B
\end{array}\right] S=\left[\begin{array}{ll}
A & 0
\end{array}\right] .
$$

## 3 Factorizations of the block matrices

We suppose that the nonsingular matrix $T=\operatorname{triang}\left(T_{11}, \ldots, T_{k k}\right)$ has the factorization in the ring $B T\left(n_{1}, \ldots, n_{k}, R\right)$ :

$$
T=B C=\left[\begin{array}{cccc}
B_{11} & B_{12} & \ldots & B_{1 k}  \tag{4}\\
0 & B_{22} & \ldots & B_{2 k} \\
\cdots & \ldots & \ldots & \ldots \\
0 & 0 & \cdots & B_{k k}
\end{array}\right]\left[\begin{array}{cccc}
C_{11} & C_{12} & \ldots & C_{1 k} \\
0 & C_{22} & \ldots & C_{2 k} \\
\cdots & \ldots & \ldots & \ldots \\
0 & 0 & \cdots & C_{k k}
\end{array}\right],
$$

where $\quad B_{i i}, C_{i i} \in M\left(n_{i}, R\right), \quad B_{i j}, C_{i j} \in M\left(n_{i}, n_{j}, R\right), i, j=1, \ldots, k, i<j$. Then the diagonal blocks $T_{i i}$ and their determinants $\operatorname{det} T_{i i}$ of the matrix $T$ have such factorizations

$$
\begin{equation*}
T_{i i}=B_{i i} C_{i i}, \quad i=1, \ldots, k \tag{5}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{det} T_{i i}=\varphi_{i} \psi_{i}, \quad \varphi_{i}=\operatorname{det} B_{i i}, \psi_{i}=\operatorname{det} C_{i i}, \quad i=1, \ldots, k \tag{6}
\end{equation*}
$$

Definition 1. We will call the factorization (4) of the matrix $T$ the corresponding one to the factorization (5) of its diagonal blocks $T_{i i}$ and the parallel one to the factorization (6) of the determinants $\operatorname{det} T_{i i}$ of their diagonal blocks or briefly, the parallel factorization of the matrix $T$ in the ring $B T\left(n_{1}, \ldots, n_{k}, R\right)$.

It should be highlighted that there does not exist the corresponding factorization of the matrix $T$, that is its factorization in the $\operatorname{ring} B T\left(n_{1}, \ldots, n_{k}, R\right)$, for every factorization (5) of the diagonal blocks $T_{i i}$.

For each factorization

$$
\begin{equation*}
\operatorname{det} T=\varphi \psi, \varphi=\prod_{i=1}^{k} \varphi_{i}, \psi=\prod_{i=1}^{k} \psi_{i}, i=1, \ldots, k \tag{7}
\end{equation*}
$$

of the determinant of the matrix $T$ there exists the parallel factorization $T=B C$ of the matrix $T$ in the ring $M(n, R)$, that is the factorization is such that $\operatorname{det} B=$ $\varphi, \operatorname{det} C=\psi$. However, there does not exist the parallel factorization (4) in the ring $B T\left(n_{1}, \ldots, n_{k}, R\right)$ for every factorization $\operatorname{det} T_{i i}=\varphi_{i} \psi_{i}$ of the determinants of the diagonal blocks $T_{i i}$ of the matrix $T$.

Further, we describe the factorizations of the matrices in the ring $B T\left(n_{1}, \ldots, n_{k}, R\right)$. We have established some conditions, under which the factorizations of the matrices $T \in B T\left(n_{1}, \ldots, n_{k}, R\right)$ are the same block triangular form up to the association, that is when they are the factorizations in the ring $B T\left(n_{1}, \ldots, n_{k}, R\right)$. We have proved the uniqueness criteria of such factorizations.

Theorem 1. Let $T \in B T\left(n_{1}, \ldots, n_{k}, R\right)$ be a nonsingular matrix and its diagonal blocks $T_{i i}, i=1, \ldots, k$, have the factorizations of the form (5). Then there exists a unique up to the association factorization of the matrix $T$ in the ring $B T\left(n_{1}, \ldots, n_{k}, R\right)$, that is $T=\operatorname{triang}\left(B_{11}, \ldots, B_{k k}\right) \operatorname{triang}\left(C_{11}, \ldots, C_{k k}\right)$ if and only if

$$
\begin{equation*}
\left(\operatorname{det} B_{s s}, \operatorname{det} C_{s+t, s+t}\right)=1, \quad \text { for all } \quad s=1, \ldots, k-1, \quad t=1, \ldots, k-s \tag{8}
\end{equation*}
$$

Proof. The matrix $T$ has the factorization (4) corresponding to the factorizations (5) of the diagonal blocks $T_{i i}, \quad i=1, \ldots, k$, if and only if the system of the linear matrix equations

$$
\begin{equation*}
B_{i i} X_{i j}+Y_{i j} C_{j j}+\sum_{l=i+1}^{j-1} Y_{i l} X_{l j}=T_{i j}, \quad 1 \leq i<j \leq k, \tag{9}
\end{equation*}
$$

has solutions. The system solutions are $X_{i j}=C_{i j}, \quad Y_{i j}=B_{i j}, \quad i<j, \quad i, j=$ $1, \ldots, k$. The solving of the system is reduced to the solving of the linear Sylvester matrix equations in the form

$$
\begin{equation*}
B_{i i} X_{i j}+Y_{i j} C_{j j}=T_{i j}, \quad 1 \leq i<j \leq k \tag{10}
\end{equation*}
$$

From (8) it follows that $\left(\operatorname{det} B_{i i}, \operatorname{det} C_{j j}\right)=1, \quad 1 \leq i<j \leq k$. Then every linear matrix equation (10) has a solution [12]. Therefore, the system of the matrix equations (9) has a solution. Consequently, the matrix $T$ has the factorization of the form (4) corresponding to the factorizations (5) of its diagonal blocks.

For the matrix $T$ there exist such invertible matrices $U$ and $V$ over $R$ that $T U=F, \quad B V=H^{B}, \quad V^{-1} C U=D$ are upper triangular matrices. The matrix $H^{B}$ has the Hermite normal form [11]. It follows from (4) that $F=H^{B} D$ :

$$
\left[\begin{array}{cccc}
F_{11} & F_{12} & \ldots & F_{1 k}  \tag{11}\\
0 & F_{22} & \ldots & F_{2 k} \\
\cdots & \cdots & \cdots & \ldots \\
0 & 0 & \cdots & F_{k k}
\end{array}\right]=\left[\begin{array}{cccc}
H^{B_{11}} & G_{12} & \ldots & G_{1 k} \\
0 & H^{B_{22}} & \ldots & G_{2 k} \\
\cdots & \ldots & \ldots & \ldots \\
0 & 0 & \cdots & H^{B_{k k}}
\end{array}\right]\left[\begin{array}{cccc}
D_{11} & D_{12} & \ldots & D_{1 k} \\
0 & D_{22} & \ldots & D_{2 k} \\
\cdots & \cdots & \cdots & \ldots \\
0 & 0 & \cdots & D_{k k}
\end{array}\right]
$$

where $H^{B_{p p}}=B_{p p} V_{p p}=\left[h_{i j}^{(p)}\right]_{1}^{n_{p}}$ is the Hermite normal form of the block $B_{p p}$. Each element of the $i$-th row of the matrix $G_{p q}=\left[g_{i j}^{(p q)}\right]_{1}^{n_{p}, n_{q}}$ lies in a prescribed complete set of residues modulo the diagonal element $h_{i i}^{(p)}$ of the matrix $H^{B_{p p}}$, that is $g_{i j}^{(p q)} \in R_{h_{i i}^{(p)}}, i=1, \ldots, n_{p}, j=1, \ldots, n_{q}, 1 \leq p<q \leq k$.

It follows from the factorization (11) that the matrices $X_{p q}=D_{p q}, Y_{p q}=$ $G_{p q}, 1 \leq p<q \leq k$, are the solutions of the system of the linear matrix equations

$$
\begin{equation*}
H^{B_{p p}} X_{p q}+Y_{p q} D_{q q}+\sum_{l=p+1}^{q-1} Y_{p l} X_{l g}=F_{p q}, \quad 1 \leq p<q \leq k \tag{12}
\end{equation*}
$$

The solving of this system of the matrix equations is reduced to the solving of the linear Sylvester matrix equations in the form

$$
\begin{equation*}
H^{B_{p p}} X_{p q}+Y_{p q} D_{q q}=F_{p q}, \quad 1 \leq p<q \leq k . \tag{13}
\end{equation*}
$$

It follows from [2] that the solution $X_{p q}=D_{p q}, Y_{p q}=G_{p q}=\left[g_{i j}^{(p q)}\right]_{1}^{n_{p}, n_{q}}$ of the equation (13), where $g_{i j}^{(p q)} \in R_{h_{i i}^{(p)}}, \quad i=1, \ldots, n_{p}, j=1, \ldots, n_{q}, \quad 1 \leq p<q \leq k$, is unique if and only if $\left(\operatorname{det} H^{B_{i i}}, \operatorname{det} D_{j j}\right)=1, \quad i, j=1, \ldots, k, i<j$. These conditions hold if the conditions (8) are true. The factorizations (11) and (4) of the matrix $T$ are associate.

Corollary 3. Let the determinants $\operatorname{det} T_{i i}$ of the diagonal blocks $T_{i i}, i=1, \ldots, k$, of the matrix $T \in B T\left(n_{1}, \ldots, n_{k}, R\right)$ have the factorizations

$$
\begin{equation*}
\operatorname{det} T_{i i}=\varphi_{i} \psi_{i}, \quad i=1, \ldots, k, \quad \text { and } \prod_{i=1}^{k} \varphi_{i}=\varphi, \prod_{i=1}^{k} \psi_{i}=\psi \tag{14}
\end{equation*}
$$

Let at least one of the following conditions hold:
(i) $\left(\prod_{i=1}^{s} \varphi_{i}, \psi_{s+1}\right)=1, \quad s=1, \ldots, k-1, \quad$ and $\quad\left((\varphi, \psi), d_{n-1}^{T}\right)=1$,
(ii) $\left(\operatorname{det} T_{i i},(\varphi, \psi)\right)=1, i=1, \ldots, k-1$.

Then there exist the factorizations

$$
\begin{equation*}
T_{i i}=B_{i i} C_{i i}, \quad \operatorname{det} B_{i i}=\varphi_{i}, \quad \operatorname{det} C_{i i}=\psi_{i}, \quad p=1, \ldots, k \tag{15}
\end{equation*}
$$

of the diagonal blocks $T_{i i}$ and the factorization of the matrix $T$

$$
\begin{equation*}
T=B C, \quad \operatorname{det} B=\varphi, \quad \operatorname{det} C=\psi, \tag{16}
\end{equation*}
$$

in the ring $B T\left(n_{1}, \ldots, n_{k}, R\right)$. This factorization of the matrix $T$ is unique up to the association.

Theorem 2. Let $T=\operatorname{triang}\left(T_{11}, \ldots, T_{k k}\right)$ be a nonsingular matrix and the determinants of the diagonal blocks $T_{i i}, i=1, \ldots, k$, have the factorizations in the form (14). If at least one of the following conditions holds:
(i) $\left(\prod_{i=1}^{s} \varphi_{i}, \psi_{s+1}\right)=1$, and $\quad\left((\varphi, \psi), d_{n-1}^{T}\right)=1, \quad s=1, \ldots, k-1$,
(ii) $\left(\operatorname{det} T_{i i},(\varphi, \psi)\right)=1, i=1, \ldots, k-1$,
then there exists the parallel factorization of the matrix $T$ in the ring $B T\left(n_{1}, \ldots, n_{k}, R\right): \quad T=B C, B, C \in B T\left(n_{1}, \ldots, n_{k}, R\right)$, that is $B=\operatorname{triang}\left(B_{11}, \ldots, B_{k k}\right), C=\operatorname{triang}\left(C_{11}, \ldots, C_{k k}\right), \quad B_{i i}, C_{i i} \in M\left(n_{i}, R\right)$ and $\operatorname{det} B_{i}=\varphi_{i}, \operatorname{det} C_{i}=\psi_{i}, i=1, \ldots, k$. Each parallel factorization $T=B C, B, C \in$ $M(n, R), \quad \operatorname{det} B=\varphi, \quad \operatorname{det} C=\psi$ of the matrix $T$ in the ring $M(n, R)$ is associate to the parallel factorization $T=\widetilde{B} \widetilde{C}$, where $\widetilde{B}=\operatorname{triang}\left(\widetilde{B}_{11}, \ldots, \widetilde{B}_{k k}\right), \widetilde{C}=$ $\operatorname{triang}\left(\widetilde{C}_{11}, \ldots, \widetilde{C}_{k k}\right)$ and $\operatorname{det} \widetilde{B}_{i i}=\varphi_{i}$, $\operatorname{det} \widetilde{C}_{i i}=\psi_{i}, \quad i=1, \ldots, k$, in the ring $B T\left(n_{1}, \ldots, n_{k}, R\right)$.

Proof. Let $k=2$, that is $T=\operatorname{triang}\left(T_{11}, T_{22}\right)$. It follows from the conditions (7) that there exists such a factorization $T=B C$ of the matrix $T$ that $\operatorname{det} B=\varphi$, $\operatorname{det} C=\psi$. We write it in an appropriate block form

$$
\operatorname{triang}\left(T_{11}, T_{22}\right)=\left[\begin{array}{ll}
B_{11} & B_{12}  \tag{17}\\
B_{21} & B_{22}
\end{array}\right] C
$$

$B_{i j} \in M\left(n_{i}, n_{j}, R\right), \quad C \in M(n, R), i, j=1,2$. It follows from the conditions (i) of the theorem that $\left(\operatorname{det} B, \operatorname{det} T_{22}\right)=\varphi_{2}$.

According to Lemma $1 d_{n_{2}}^{B_{2}}=\varphi_{2}$, where $B_{2}=\left[\begin{array}{ll}B_{21} & B_{22}\end{array}\right]$, there exists such a matrix $V \in G L(n, R)$ that $B_{2} V=\left[\begin{array}{ll}0 & \widetilde{B}_{22}\end{array}\right]$, where $\widetilde{B}_{22} \in M\left(n_{2}, R\right)$ and $\operatorname{det} \widetilde{B}_{22}=$ $\varphi_{2}$. So, from (17) we get

$$
\operatorname{triang}\left(T_{11}, T_{22}\right)=\operatorname{triang}\left(\widetilde{B}_{11}, \widetilde{B}_{22}\right) \operatorname{triang}\left(\widetilde{C}_{11}, \widetilde{C}_{22}\right)=\widetilde{B} \widetilde{C}
$$

where $\widetilde{B}=B V, \widetilde{C}=V^{-1} C, V \in G L(n, R), B_{i j}, C_{i j} \in M\left(n_{i}, n_{j}, R\right)$ and $\operatorname{det} \widetilde{B}_{i i}=$ $\varphi_{i}, \operatorname{det} \widetilde{C}_{i i}=\psi_{i}, i=1,2$.

Similarly, the theorem can be proved under the condition (ii).
For an arbitrary $k$, we prove the theorem by induction.
Corollary 4. Let the determinants $\operatorname{det} T_{i i}$ of the diagonal blocks $T_{i i}, i=1, \ldots, k$, of the matrix $T \in B T\left(n_{1}, \ldots, n_{k}, R\right)$ have the factorizations in the form (14). If at least one of the following conditions holds:
(i) $\left(\prod_{i=1}^{s} \varphi_{i}, \psi_{s+1}\right)=1, \quad s=1, \ldots, k-1, \quad$ and $\quad\left((\varphi, \psi), d_{n-1}^{T}\right)=1$,
(ii) $\left(\operatorname{det} T_{i i},(\varphi, \psi)\right)=1, i=1, \ldots, k-1$,
then there exist factorizations (15) of the diagonal blocks $T_{i i}$ and the factorization of matrix $T$ (16) in the ring $B T\left(n_{1}, \ldots, n_{k}, R\right)$. This factorization of the matrix $T$ is unique up to the association.

Theorem 3. Let the determinants of the diagonal blocks $T_{i i}, i=1, \ldots, k$, of the matrix $T \in B T\left(n_{1}, \ldots, n_{k}, R\right)$ have the factorizations in the form (14). Then there exists the factorization of the matrix $T$ parallel to the factorization (14) of the determinants of the diagonal blocks if and only if the following conditions hold:
(i) $\left(\left(\varphi_{i}, \psi_{i}\right), d_{n_{i}-1}^{T_{i i}}\right)=1, i=1, \ldots, k$,
(ii) $\left(\varphi_{s}, \psi_{s+t}\right)=1$ for all $s=1, \ldots, k-1, \quad t=1, \ldots, k-s$.

This factorization of the matrix $T$ is unique up to the association.
Proof. It follows from the factorizations (14) of the determinants $\operatorname{det} T_{i i}$ of the diagonal blocks $T_{i i}$ of the matrix $T$ that there exist the parallel factorizations of the diagonal blocks $T_{i i}$. When the condition (i) holds, these factorizations of the blocks $T_{i i}$ are parallel to the factorizations of their determinants up to the association and they are unique. From Theorem 1 we conclude that there exists the factorization of the matrix $T$ corresponding to the factorizations (5) of its diagonal blocks $T_{i i}$ and parallel to the factorizations (14) of the determinants of the diagonal blocks $T_{i i}, i=1, \ldots, k$, and it is unique up to the association.

It should be highlighted that there does not exist the parallel factorization in the ring $B T\left(n_{1}, \ldots, n_{k}, R\right)$ for every factorization of the determinants of the diagonal blocks $T_{i i}, i=1, \ldots, k$, of the matrix $T$.

We establish the matrices having such a property in the following corollary.

Corollary 5. Let the determinants of the diagonal blocks $T_{i i}, i=1, \ldots, k$, of the matrix $T \in B T\left(n_{1}, \ldots, n_{k}, R\right)$ be pairwise relatively prime, that is $\left(\operatorname{det} T_{i i}, \operatorname{det} T_{j j}\right)=1$. Then for each factorization (14) of the determinants $\operatorname{det} T_{i i}$ of the diagonal blocks $T_{i i}, i=1, \ldots, k$, there exists the parallel factorization of the matrix $T$, that is the matrix $T$ has the maximum number of the parallel factorizations.

The block diagonal matrices $D=\operatorname{diag}\left(D_{11}, \ldots, D_{k k}\right), D_{i i} \in M\left(n_{i}, R\right), i=$ $1, \ldots, k$, form the subring $B D\left(n_{1}, \ldots, n_{k}, R\right)$ of the ring of the block triangular matrices. We consider the factorizations of the matrices in the ring $B D\left(n_{1}, \ldots, n_{k}, R\right)$.

Definition 2. Let the determinants of the diagonal blocks $D_{i i} \in M\left(n_{i}, R\right), i=$ $1, \ldots, k$, of the matrix $D=\operatorname{diag}\left(D_{11}, \ldots, D_{k k}\right)$ have the factorizations

$$
\begin{equation*}
\operatorname{det} D_{i i}=\varphi_{i} \psi_{i}, \quad i=1, \ldots, k \tag{18}
\end{equation*}
$$

The factorization $D=B C, \quad B=\operatorname{diag}\left(B_{11}, \ldots, B_{k k}\right), \quad C=\operatorname{diag}\left(C_{11}, \ldots, C_{k k}\right)$, of the matrix $D$ is such that $\operatorname{det} B_{i i}=\varphi_{i}, \operatorname{det} C_{i i}=\psi_{i}, i=1, \ldots, k$, and is called the parallel factorization to the factorizations (18) of the determinants of the diagonal blocks $D_{i i}, i=1, \ldots, k$, or briefly, the parallel factorization of the matrix $D$ in the ring $B D\left(n_{1}, \ldots, n_{k}, R\right)$ of the block diagonal matrices.

Theorem 4. Let $D \in B D\left(n_{1}, \ldots, n_{k}, R\right)$, that is $D=\operatorname{diag}\left(D_{11}, \ldots, D_{k k}\right), D_{i i} \in$ $M\left(n_{i}, R\right), i=1, \ldots, k$, and the determinants of its diagonal blocks $D_{i i}$ have the factorizations:

$$
\begin{equation*}
\operatorname{det} D_{i i}=\varphi_{i} \psi_{i}, \prod_{i=1}^{k} \varphi_{i}=\varphi, \prod_{i=1}^{k} \psi_{i}=\psi, i=1, \ldots, k \tag{19}
\end{equation*}
$$

If $\left(\left(\operatorname{det} D_{i i}, \operatorname{det} D_{j j}\right),(\varphi, \psi)\right)=1, \quad i, j=1, \ldots, k, \quad i \neq j$, then for the $m a-$ trix $D$ there exists the factorization $D=B C, \quad B, C \in M(n, R), \quad \operatorname{det} B=$ $\varphi, \quad \operatorname{det} C=\psi$, in the ring $M(n, R)$ and each of such factorizations is associate to the parallel factorization of the matrix $\underset{\widetilde{\sim}}{D}$ in the $\underset{\sim}{\operatorname{B}}$ ring $B D\left(n_{1}, \ldots, n_{k}, R\right)$, that is $D=\widetilde{B} \widetilde{C}$, where $\widetilde{B}=B V=\operatorname{diag}\left(\widetilde{B}_{11}, \ldots, \widetilde{B}_{k k}\right), \quad \widetilde{C}=V^{-1} C=\operatorname{diag}\left(\widetilde{C}_{11}, \ldots, \widetilde{C}_{k k}\right)$, $V \in G L(n, R), \widetilde{B}_{i i}, \widetilde{C}_{i i} \in M\left(n_{i}, R\right), \operatorname{det} \widetilde{B}_{i i}=\varphi_{i}, \operatorname{det} \widetilde{C}_{i i}=\psi_{i}, i=1, \ldots, k$.

Proof. Let $k=2$. It follows from (19) that there exists such a factorization $D=B C$ of the matrix $D$ that $\operatorname{det} B=\varphi, \operatorname{det} C=\psi$. We write it in the block form

$$
\left[\begin{array}{cc}
D_{1} & 0  \tag{20}\\
0 & D_{2}
\end{array}\right]=\left[\begin{array}{ll}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{array}\right] C
$$

where $B_{i i} \in M\left(n_{i}, R\right), C \in M(n, R), i=1,2$. Then, from (19) we have that $\left(\operatorname{det} B, \operatorname{det} D_{i}\right)=\varphi_{i}, i=1,2$.

Based on Lemma 2, $\operatorname{det} B=d_{n_{1}}^{B_{1}} d_{n_{2}}^{B_{2}}$, where $B_{i}=\left[\begin{array}{ll}B_{i 1} & B_{i 2}\end{array}\right], i=1,2$. Since $d_{n_{i}}^{B_{i}} \mid \varphi_{i}, i=1,2$, and $\operatorname{det} B=\varphi_{1} \varphi_{2}$, it follows that $d_{n_{i}}^{B_{i}}=\varphi_{i}, i=1,2$. For the matrix
$B_{2}$ there exists such a matrix $U \in G L(n, R)$ that $B_{2} U=\left[\begin{array}{ll}0 & \widetilde{B}_{22}\end{array}\right], \quad \widetilde{B}_{22} \in$ $M\left(n_{2}, R\right), \quad \operatorname{det} \widetilde{B}_{22}=\varphi_{2}$.

Then from the equality (20) we obtain:
$\left[\begin{array}{cc}D_{1} & 0 \\ 0 & D_{2}\end{array}\right]=\left[\begin{array}{cc}\widetilde{B}_{11} & \widetilde{B}_{12} \\ 0 & \widetilde{B}_{22}\end{array}\right] \widetilde{C}$, where $\left[\begin{array}{cc}\widetilde{B}_{11} & \widetilde{B}_{12} \\ 0 & \widetilde{B}_{22}\end{array}\right]=\left[\begin{array}{ll}B_{11} & B_{12} \\ B_{21} & B_{22}\end{array}\right] U, \quad \widetilde{C}=U^{-1} C$.
According to Corollary 2 there exists such a matrix $Q=\left[\begin{array}{cc}I_{n_{1}} & Q_{12} \\ 0 & I_{n_{2}}\end{array}\right]$ that $\left[\begin{array}{ll}\widetilde{B}_{11} & \widetilde{B}_{12}\end{array}\right] Q=\left[\begin{array}{ll}\widetilde{B}_{11} & 0\end{array}\right]$.

Thus, we get $D=\widetilde{B} \widetilde{C}, \quad \widetilde{B}=B W=\operatorname{diag}\left(\widetilde{B}_{11}, \widetilde{B}_{22}\right), \quad \widetilde{C}=W^{-1} C=$ $\operatorname{diag}\left(\widetilde{C}_{11}, \widetilde{C}_{22}\right), \quad W=U Q, \quad \widetilde{B}_{i i}, \widetilde{C}_{i i} \in M\left(n_{i}, R\right), \quad \operatorname{det} \widetilde{B}_{i i}=\varphi_{i}, \quad \operatorname{det} \widetilde{C}_{i i}=$ $\psi_{i}, \quad i=1,2$.

For an arbitrary $k$, we prove the theorem by induction.
Corollary 6. If the determinants of the diagonal blocks $D_{i i}$ of the matrix $D \in B D\left(n_{1}, \ldots, n_{k}, R\right)$ are pairwise relatively prime, then each factorization $D=B C, B, C \in M(n, R)$ of the matrix $D$ in the ring $M(n, R)$ is associate to a certain parallel factorization of the matrix $D$ in the ring $B D\left(n_{1}, \ldots, n_{k}, R\right)$.

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Vasyl' Petrychkovych Received January 10, 2017
Natalita Dzhaliuk
Pidstryhach Institute for Applied Problems of Mechanics
and Mathematics of the NAS
of Ukraine Department of Algebra
3b Naukova Str., 79060, L'viv, Ukraine
E-mail: vas_petrych@yahoo.com
E-mail: nataliya.dzhalyuk@gmail.com

# Inclusion Properties of Certain Subclass of Univalent Meromorphic Functions Defined by a Linear Operator Associated with the $\lambda$-Generalized Hurwitz-Lerch Zeta Function 

H. M. Srivastava, F. Ghanim, R. M. El-Ashwah


#### Abstract

By using a linear operator associated with the $\lambda$-generalized HurwitzLerch zeta function, the authors introduce and investigate several properties of a certain subclass of meromorphically univalent functions in the open unit disk, which is defined here by means of the Hadamard product (or convolution). Mathematics subject classification: Primary 11M35, 30C45; Secondary 30C10. Keywords and phrases: Analytic functions, Meromorphic functions, Zeta functions, Linear operators, Hadamard products (or convolution)..


## 1 Main remarks

Let $\Sigma$ denote the class of meromorphic functions $f(z)$ normalized by

$$
\begin{equation*}
f(z)=\frac{1}{z}+\sum_{k=1}^{\infty} a_{k} z^{k} \tag{1}
\end{equation*}
$$

which are analytic in the punctured unit disk

$$
\mathbb{U}^{*}=\{z: z \in \mathbb{C} \quad \text { and } \quad 0<|z|<1\}=\mathbb{U} \backslash\{0\},
$$

$\mathbb{C}$ being (as usual) the set of complex numbers. We denote by $\Sigma \mathcal{S}^{*}(\beta)$ and $\Sigma \mathcal{K}(\beta)$ ( $\beta \geqq 0$ ) the subclasses of $\Sigma$ consisting of all meromorphic functions which are, respectively, starlike of order $\beta$ and convex of order $\beta$ in $\mathbb{U}^{*}$ (see also the recent works [1] and [2]).

For functions $f_{j}(z)(j=1,2)$ defined by

$$
\begin{equation*}
f_{j}(z)=\frac{1}{z}+\sum_{k=1}^{\infty} a_{k, j} z^{k} \quad(j=1,2) \tag{2}
\end{equation*}
$$

we denote the Hadamard product (or convolution) of $f_{1}(z)$ and $f_{2}(z)$ by

$$
\begin{equation*}
\left(f_{1} * f_{2}\right)(z)=\frac{1}{z}+\sum_{k=1}^{\infty} a_{k, 1} a_{k, 2} z^{k} . \tag{3}
\end{equation*}
$$

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Let us consider the function $\widetilde{\phi}(\alpha, \beta ; z)$ defined by

$$
\begin{gather*}
\widetilde{\phi}(\alpha, \beta ; z)=\frac{1}{z}+\sum_{k=0}^{\infty} \frac{(\alpha)_{k+1}}{(\beta)_{k+1}} a_{k} z^{k}  \tag{4}\\
\left(\beta \in \mathbb{C} \backslash \mathbb{Z}_{0}^{-} ; \alpha \in \mathbb{C}\right)
\end{gather*}
$$

where

$$
\mathbb{Z}_{0}^{-}=\{0,-1,-2, \cdots\}=\mathbb{Z}^{-} \cup\{0\}
$$

Here, and in the remainder of this paper, $(\lambda)_{\kappa}$ denotes the general Pochhammer symbol defined, in terms of the Gamma function, by

$$
(\lambda)_{\kappa}:=\frac{\Gamma(\lambda+\kappa)}{\Gamma(\lambda)}= \begin{cases}\lambda(\lambda+1) \cdots(\lambda+n-1) & (\kappa=n \in \mathbb{N} ; \lambda \in \mathbb{C})  \tag{5}\\ 1 & (\kappa=0 ; \lambda \in \mathbb{C} \backslash\{0\})\end{cases}
$$

it being understood conventionally that $(0)_{0}:=1$ and assumed tacitly that the $\Gamma$ quotient exists (see, for details, [3, p. 21 et seq.]), $\mathbb{N}$ being the set of positive integers.

It is easy to see that, in the case when $a_{k}=1(k=0,1,2, \cdots)$, the following relationship holds true between the function $\widetilde{\phi}(\alpha, \beta ; z)$ and the Gaussian hypergeometric function [4]:

$$
\begin{equation*}
\widetilde{\phi}(\alpha, \beta ; z)=\frac{1}{z}{ }_{2} F_{1}(1, \alpha ; \beta ; z) \tag{6}
\end{equation*}
$$

Recently, Ghanim ([5]; see also [6] and [7]) made use of the Hadamard product for functions $f(z) \in \Sigma$ in order to introduce a new linear operator $L_{a}^{s}(\alpha, \beta)$, which is defined on $\Sigma$ by

$$
\begin{align*}
L_{a}^{s}(\alpha, \beta)(f)(z) & =\widetilde{\phi}(\alpha, \beta ; z) * G_{s, a}(z) \\
& =\frac{1}{z}+\sum_{k=1}^{\infty} \frac{(\alpha)_{n+1}}{(\beta)_{n+1}}\left(\frac{a+1}{a+k}\right)^{s} a_{k} z^{k} \quad\left(z \in \mathbb{U}^{*}\right) \tag{7}
\end{align*}
$$

where

$$
\begin{align*}
G_{s, a}(z) & :=(a+1)^{s}\left[\Phi(z, s, a)-a^{s}+\frac{1}{z(a+1)^{s}}\right] \\
& =\frac{1}{z}+\sum_{k=1}^{\infty}\left(\frac{a+1}{a+k}\right)^{s} z^{k} \quad\left(z \in \mathbb{U}^{*}\right) \tag{8}
\end{align*}
$$

and the function $\Phi(z, s, a)$ is the well-known Hurwitz-Lerch zeta function defined by (see, for example, [8, p. 121 et seq.]; see also [9] and [10, p. 194 et seq.])

$$
\begin{equation*}
\Phi(z, s, a):=\sum_{n=0}^{\infty} \frac{z^{n}}{(n+a)^{s}} \tag{9}
\end{equation*}
$$

$$
\left(a \in \mathbb{C} \backslash \mathbb{Z}_{0}^{-} ; s \in \mathbb{C} \quad \text { when } \quad|z|<1 ; \Re(s)>1 \quad \text { when } \quad|z|=1\right)
$$

We recall that the following new family of the $\lambda$-generalized Hurwitz-Lerch zeta functions was introduced and investigated systematically by Srivastava [11] (see also [12-16] ):

$$
\begin{gather*}
\Phi_{\lambda_{1}, \cdots, \lambda_{p} ; \mu_{1}, \cdots, \mu_{q}}^{\left(\rho_{1}, \cdots, \rho_{p}, \sigma_{1}, \cdots, \sigma_{q}\right)}(z, s, a ; b, \lambda)=\frac{1}{\lambda \Gamma(s)} \\
\cdot \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}\left(\lambda_{j}\right)_{n \rho_{j}}}{(a+n)^{s} \cdot \prod_{j=1}^{q}\left(\mu_{j}\right)_{n \sigma_{j}}} H_{0,2}^{2,0}\left[\left.(a+n) b^{\frac{1}{\lambda}}\right|_{(s, 1),\left(0, \frac{1}{\lambda}\right)}\right] \frac{z^{n}}{n!} \quad(10)  \tag{10}\\
(\min \{\Re(a), \Re(s)\}>0 ; \Re(b)>0 ; \lambda>0) \\
\left(\lambda_{j} \in \mathbb{C}(j=1, \cdots, p) \quad \text { and } \quad \mu_{j} \in \mathbb{C} \backslash \mathbb{Z}_{0}^{-}(j=1, \cdots, q) ; \rho_{j}>0 \quad(j=1, \cdots, p) ;\right. \\
\left.\sigma_{j}>0 \quad(j=1, \cdots, q) ; 1+\sum_{j=1}^{q} \sigma_{j}-\sum_{j=1}^{p} \rho_{j} \geqq 0\right),
\end{gather*}
$$

where the equality in the convergence condition holds true for suitably bounded values of $|z|$ given by

$$
|z|<\nabla:=\left(\prod_{j=1}^{p} \rho_{j}^{-\rho_{j}}\right) \cdot\left(\prod_{j=1}^{q} \sigma_{j}^{\sigma_{j}}\right) .
$$

Definition 1. The $H$-function involved in the right-hand side of (10) is the wellknown Fox's $H$-function [17, Definition 1.1] (see also [3,18]) defined by

$$
\begin{align*}
H_{\mathfrak{p}, \mathfrak{q}}^{m, n}(z) & =H_{\mathfrak{p}, \mathfrak{q}}^{m, n}\left[z \left\lvert\, \begin{array}{c}
\left(a_{1}, A_{1}\right), \cdots,\left(a_{\mathfrak{p}}, A_{\mathfrak{p}}\right) \\
\left(b_{1}, B_{1}\right), \cdots,\left(b_{\mathfrak{q}}, B_{\mathfrak{q}}\right)
\end{array}\right.\right] \\
& =\frac{1}{2 \pi \mathrm{i}} \int_{\mathcal{L}} \Xi(s) z^{-s} \mathrm{~d} s \quad(z \in \mathbb{C} \backslash\{0\} ;|\arg (z)|<\pi), \tag{11}
\end{align*}
$$

where

$$
\Xi(s)=\frac{\prod_{j=1}^{m} \Gamma\left(b_{j}+B_{j} s\right) \cdot \prod_{j=1}^{n} \Gamma\left(1-a_{j}-A_{j} s\right)}{\prod_{j=n+1}^{\mathfrak{p}} \Gamma\left(a_{j}+A_{j} s\right) \cdot \prod_{j=m+1}^{\mathfrak{q}} \Gamma\left(1-b_{j}-B_{j} s\right)}
$$

an empty product is interpreted as $1, m, n, \mathfrak{p}$ and $\mathfrak{q}$ are integers such that

$$
1 \leqq m \leqq \mathfrak{q} \quad \text { and } \quad 0 \leqq n \leqq \mathfrak{p}
$$

$$
\begin{array}{rllll}
A_{j}>0 & (j=1, \cdots, \mathfrak{p}) & \text { and } & B_{j}>0 & (j=1, \cdots, \mathfrak{q}), \\
a_{j} \in \mathbb{C} & (j=1, \cdots, \mathfrak{p}) & \text { and } & b_{j} \in \mathbb{C} & (j=1, \cdots, \mathfrak{q})
\end{array}
$$

and $\mathcal{L}$ is a suitable Mellin-Barnes type contour separating the poles of the gamma functions

$$
\left\{\Gamma\left(b_{j}+B_{j} s\right)\right\}_{j=1}^{m}
$$

from the poles of the gamma functions

$$
\left\{\Gamma\left(1-a_{j}+A_{j} s\right)\right\}_{j=1}^{n} .
$$

We choose to mention here that, by using the fact that [11, p. 1496, Remark 7]

$$
\begin{equation*}
\lim _{b \rightarrow 0}\left\{H_{0,2}^{2,0}\left[(a+n) b^{\frac{1}{\lambda}} \left\lvert\, \overline{(s, 1),\left(0, \frac{1}{\lambda}\right)}\right.\right]\right\}=\lambda \Gamma(s) \quad(\lambda>0) \tag{12}
\end{equation*}
$$

the equation (8) reduces to the following form:

$$
\begin{align*}
\Phi_{\lambda_{1}, \cdots, \lambda_{p} ; \mu_{1}, \cdots, \mu_{q}}^{\left(\rho_{1}, \cdots, \rho_{p}, \sigma_{1}, \cdots, \sigma_{q}\right)}(z, s, a ; 0, \lambda) & :=\Phi_{\lambda_{1}, \cdots, \lambda_{p}, \mu_{1}, \cdots, \mu_{q}}^{\left(\rho_{1}, \cdots, \rho_{p}, \sigma_{1}, \cdots, \sigma_{q}\right)}(z, s, a) \\
& =\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}\left(\lambda_{j}\right)_{n \rho_{j}}}{(a+n)^{s} \cdot \prod_{j=1}^{q}\left(\mu_{j}\right)_{n \sigma_{j}}} \frac{z^{n}}{n!} . \tag{13}
\end{align*}
$$

Definition 2. The function $\Phi_{\lambda_{1}, \cdots, \lambda_{p} ; \mu_{1}, \cdots, \mu_{q}}^{\left(\rho_{1}, \cdots, \rho_{p}, \sigma_{1}, \cdots, \sigma_{q}\right)}(z, s, a)$ involved in (13) is the multiparameter extension and generalization of the Hurwitz-Lerch zeta function $\Phi(z, s, a)$ introduced by Srivastava et al.[16, p. 503, Eq. (6.2)] defined by

$$
\begin{gather*}
\Phi_{\lambda_{1}, \cdots, \lambda_{p} ; \mu_{1}, \cdots, \mu_{q}}^{\left(\rho_{1}, \cdots, \rho_{p}, \sigma_{1}, \ldots, \sigma_{q}\right)}(z, s, a):=\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}\left(\lambda_{j}\right)_{n \rho_{j}}}{(a+n)^{s} \cdot \prod_{j=1}^{q}\left(\mu_{j}\right)_{n \sigma_{j}}} \frac{z^{n}}{n!}  \tag{14}\\
\left(p, q \in \mathbb{N}_{0} ; \lambda_{j} \in \mathbb{C}(j=1, \cdots, p) ; a, \mu_{j} \in \mathbb{C} \backslash \mathbb{Z}_{0}^{-}(j=1, \cdots, q) ;\right. \\
\rho_{j}, \sigma_{k} \in \mathbb{R}^{+}(j=1, \cdots, p ; k=1, \cdots, q) ; \\
\Delta>-1 \text { when } s, z \in \mathbb{C} ; \\
\Delta=-1 \text { and } s \in \mathbb{C} \text { when }|z|<\nabla^{*} ; \\
\left.\Delta=-1 \text { and } \Re(\Xi)>\frac{1}{2} \text { when }|z|=\nabla^{*}\right)
\end{gather*}
$$

with

$$
\begin{gather*}
\nabla^{*}:=\left(\prod_{j=1}^{p} \rho_{j}^{-\rho_{j}}\right) \cdot\left(\prod_{j=1}^{q} \sigma_{j}^{\sigma_{j}}\right)  \tag{15}\\
\Delta:=\sum_{j=1}^{q} \sigma_{j}-\sum_{j=1}^{p} \rho_{j} \quad \text { and } \quad \Xi:=s+\sum_{j=1}^{q} \mu_{j}-\sum_{j=1}^{p} \lambda_{j}+\frac{p-q}{2} . \tag{16}
\end{gather*}
$$

By applying this new family of the $\lambda$-generalized Hurwitz-Lerch zeta functions, Srivastava and Gaboury [19] introduced a new linear operator which provides a generalization of the largely- (and widely-) studied Srivastava-Attiya operator [20] (see also [21-23]). This new operator contains, as its special cases, the operators investigated earlier by Prajapat and Bulboacă [24, p. 571, Eq. (1.8)], Noor and Bukhari [25, p. 2, Eq. (1.3)], Choi et al. [26], Cho and Srivastava [27], Jung et al. [28], Bernardi [1], Carlson and Shaffer [29], Owa and Srivastava [30] and by Dziok and Srivastava [31,32]. The Dziok-Srivastava convolution operator studied by Dziok and Srivastava $[31,32]$ is, in turn, a generalization of the Hohlov operator [33] and the Ruscheweyh operator [34]. In fact, the Dziok-Srivastava convolution operator is itself a special case of the Srivastava-Wright operator (see, for details, [35] and [36]; see also the other closely-related works cited in each of these recent publications).

In this paper, we consider the following linear operator:

$$
J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda, \alpha, \beta} f(z): \Sigma \rightarrow \Sigma,
$$

which is defined by

$$
J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda, \alpha, \beta} f(z)=G_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda}(z) * \widetilde{\phi}(\alpha, \beta ; z),
$$

where $*$ denotes the Hadamard product (or convolution) of analytic functions and the function $G_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda}(z)$ is given by

$$
\begin{align*}
& G_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda}(z):=(a+1)^{s} \cdot {\left[\Phi_{\lambda_{1}, \cdots, \lambda_{p} ; \mu_{1}, \cdots, \mu_{q}}^{(1, \cdots, 1,1, \cdots, 1)}(z, s, a ; b, \lambda)\right.} \\
&\left.-\frac{a^{-s}}{\lambda \Gamma(s)} \Lambda(a, b, s, \lambda)+\frac{(a+1)^{-s}}{z}\right] \\
&=\frac{1}{z}+\sum_{k=1}^{\infty} \frac{\prod_{j=1}^{p}\left(\lambda_{j}\right)_{k}}{\prod_{j=1}^{q}\left(\mu_{j}\right)_{k}}\left(\frac{a+1}{a+k}\right)^{s} \frac{\Lambda(a+k, b, s, \lambda)}{\lambda \Gamma(s)} \frac{z^{k}}{k!} \tag{17}
\end{align*}
$$

with

$$
\left.\Lambda(a, b, s, \lambda):=H_{0,2}^{2,0}\left[a b^{\frac{1}{\lambda}} \left\lvert\, \overline{(s, 1),\left(0, \frac{1}{\lambda}\right.}\right.\right)\right] .
$$

By combining (17) and (4), we obtain

$$
\begin{align*}
& J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda, \alpha, \beta} f(z)=\frac{1}{z} \\
& +\sum_{k=1}^{\infty} \frac{(\alpha)_{k+1} \prod_{j=1}^{p}\left(\lambda_{j}\right)_{k}}{(\beta)_{k+1} \prod_{j=1}^{q}\left(\mu_{j}\right)_{k}}\left(\frac{a+1}{a+k}\right)^{s} \frac{\Lambda(a+k, b, s, \lambda)}{\lambda \Gamma(s)} a_{k} \frac{z^{k}}{k!}  \tag{18}\\
& \left(z \in \mathbb{U}^{*} ; \alpha, \lambda_{j} \in \mathbb{C}(j=1, \cdots, p) ; \beta, \mu_{j} \in \mathbb{C} \backslash \mathbb{Z}_{0}^{-}(j=1, \cdots, q) ; p \leqq q+1\right)
\end{align*}
$$

with

$$
\min \{\Re(a), \Re(s)\}>0, \quad \lambda>0 \quad \text { if } \quad \Re(b)>0
$$

and

$$
s \in \mathbb{C} \quad \text { and } \quad a \in \mathbb{C} \backslash \mathbb{Z}_{0}^{-} \quad \text { if } \quad b=0
$$

Clearly, upon setting $p-1=q=0$ and $\lambda_{1}=1$ in (18) and taking the limit as $b \rightarrow 0$, we obtain the operator $L_{a}^{s}(\alpha, \beta)(f)(z)$ studied earlier by Ghanim [5].

It is easily observed from (18) that

$$
\begin{equation*}
z\left(J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda, \alpha, \beta} f(z)\right)^{\prime}=\alpha\left(J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda, \alpha+1, \beta} f(z)\right)-(\alpha+1)\left(J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda, \alpha, \beta} f(z)\right) \tag{19}
\end{equation*}
$$

and

$$
\begin{equation*}
z\left(J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda, \alpha, \beta+1} f(z)\right)^{\prime}=\beta\left(J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda, \alpha, \beta} f(z)\right)-(\beta+1)\left(J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda,, \alpha, \beta+1} f(z)\right) \tag{20}
\end{equation*}
$$

Now, with the help of the linear operator $J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda, \alpha, \beta} f(z)$, we introduce the following subclass:

$$
\Sigma_{\left(\lambda_{p}\right),\left(\delta_{q}\right), b}^{s, a, \lambda, \alpha, \beta}(\mu)=\Sigma(\alpha, \beta, \mu)
$$

of meromorphic functions as follows:
Definition 3. For fixed parameters $A, B(-1 \leqq B<A \leqq 1)$ and $0 \leqq \mu<1$, the function $f(z) \in \Sigma$ is said to be in the class $\Sigma(\alpha, \beta, \mu)$ if it satisfies the following subordination condition:

$$
\begin{equation*}
\frac{1}{1-\mu}\left(-\frac{z\left(J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda, \alpha, \beta} f(z)\right)^{\prime}}{J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \alpha, \beta} f(z)}-\mu\right) \prec \frac{1+A z}{1+B z} \quad\left(z \in \mathbb{U}^{*}\right) \tag{21}
\end{equation*}
$$

or, equivalently,


## 2 A Set of Lemmas

To establish our main results, we shall need each of the following lemmas:
Lemma 1 (see [37]). If $-1 \leqq B<A \leqq 1, \nu \neq 0$ and the complex number $\tau$ satisfies the inequality:

$$
\Re\{\tau\} \geqq-\frac{\nu(1-A)}{1-B}
$$

then the following differential equation:

$$
q(z)+\frac{z q^{\prime}(z)}{\nu q(z)+\tau} \prec \frac{1+A z}{1+B z} \quad(z \in \mathbb{U}),
$$

has a univalent solution in $\mathbb{U}$ given by

$$
q(z)= \begin{cases}\frac{z^{\nu+\tau}(1+B z)^{\nu(A-B) / B}}{\nu \int_{0}^{z} t^{\nu+\tau-1}(1+B t)^{\nu(A-B) / B} d t}-\frac{\tau}{\nu} & (B \neq 0)  \tag{23}\\ \frac{z^{\nu+\tau} \exp (\nu A z)}{\nu \int_{0}^{z} \nu^{\nu+\tau-1} \exp (\nu A t) d t}-\frac{\tau}{\nu} & (B=0)\end{cases}
$$

If the function $\phi$ given by

$$
\phi(z)=1+c_{1} z+c_{2} z+\cdots
$$

is analytic in $\mathbb{U}$ and satisfies the following subordination:

$$
\begin{equation*}
\phi(z)+\frac{z \phi^{\prime}(z)}{\nu \phi(z)+\tau} \prec \frac{1+A z}{1+B z} \quad(z \in \mathbb{U}), \tag{24}
\end{equation*}
$$

then

$$
\phi(z) \prec q(z) \prec \frac{1+A z}{1+B z} \quad(z \in \mathbb{U})
$$

and $q(z)$ is the best dominant of (24).
Lemma 2 (see [38]). Let v be a positive measure on $[0,1]$. Let $h$ be a complex-valued function defined on $\mathbb{U} \times[0,1]$ such that $h(., t)$ is analytic in $\mathbb{U}$ for each $t \in[0,1]$ and $h(z,$.$) is v$-integrable on $[0,1]$ for all $z \in \mathbb{U}$. Suppose also that $\Re\{h(z, t)\}>0$, $h(-r, t)$ is real and

$$
\Re\left\{\frac{1}{h(z, t)}\right\} \geqq \frac{1}{h(-r, t)} \quad(|z| \leqq r<1 ; t \in[0,1])
$$

If

$$
\mathfrak{h}(z)=\int_{0}^{1} h(z, t) d v(t),
$$

then

$$
\Re\left\{\frac{1}{\mathfrak{h}(z)}\right\} \geqq \frac{1}{\mathfrak{h}(-r)} \quad(|z| \leqq r<1)
$$

Lemma 3 (see [39]). For real numbers $a, b$ and $c(c \neq 0,-1,-2, \cdots)$, it is asserted that

$$
\begin{gather*}
\int_{0}^{1} t^{b-1}(1-t)^{c-b-1}(1-z t)^{-a} d t=\frac{\Gamma(b) \Gamma(c-b)}{\Gamma(c)}{ }_{2} F_{1}(a, b ; c ; z)  \tag{25}\\
(\Re\{c\}>\Re\{b\}>0 ; z \in \mathbb{U}) .
\end{gather*}
$$

Moreover,

$$
\begin{equation*}
{ }_{2} F_{1}(a, b ; c ; z)={ }_{2} F_{1}(b, a ; c ; z) \tag{26}
\end{equation*}
$$

and

$$
\begin{gather*}
{ }_{2} F_{1}(a, b ; c ; z)=(1-z)^{-\alpha}{ }_{2} F_{1}\left(a, c-b ; c ; \frac{z}{z-1}\right)  \tag{27}\\
(c \neq 0,-1,-2, \cdots ;|\arg (1-z)|<\pi) .
\end{gather*}
$$

Inclusion properties of various classes of analytic and meromorphic functions were studied earlier by several different methods (see, for example, [40-43] and [44]). In this paper, we find two inclusion theorems for the meromorphic function class $\Sigma(\alpha, \beta, \mu)$. In particular, we show that, if we increase the parameter $\alpha$ by one, the overall size of the meromorphic function class $\Sigma(\alpha, \beta, \mu)$ would get smaller. On the other hand, by increasing the parameters $\beta$ by one, the overall size of the meromorphic function class $\Sigma(\alpha, \beta, \mu)$ would get bigger.

## 3 Main Results

Unless otherwise mentioned, we assume throughout the remainder of the paper that

$$
-1 \leqq B<A \leqq 1,0 \leqq \mu<1, \alpha, \beta>0, a \in \mathbb{C} \backslash \mathbb{Z}_{0}^{-}, s \in \mathbb{C} \quad \text { and } \quad z \in \mathbb{U}
$$

We begin with some inclusion relationships concerning the parameter $\alpha$ of the class $\Sigma(\alpha, \beta, \mu)$.

## Theorem 1.

(i) If $f(z) \in \Sigma(\alpha+1, \beta, \mu)$ and

$$
\begin{equation*}
\alpha-\mu+1 \geqq \frac{(1-\mu)(1-A)}{(1-B)}, \tag{28}
\end{equation*}
$$

then

$$
\begin{align*}
\frac{1}{1-\mu}\left(-\frac{z\left(J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda, \alpha, \beta} f(z)\right)^{\prime}}{J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \alpha, \beta} f(z)}-\mu\right) & \prec \frac{1}{1-\mu}\left((\alpha-\mu+1)-\frac{1}{Q_{1}(z)}\right) \\
& =q_{1}(z) \prec \frac{1+A z}{1+B z} \quad(z \in \mathbb{U}), \tag{29}
\end{align*}
$$

where

$$
Q_{1}(z)= \begin{cases}\int_{0}^{1} u^{\alpha-1}\left(\frac{1+B z u}{1+B z}\right)^{-(1-\mu)(A-B) / B} d u & (B \neq 0) \\ \int_{0}^{1} u^{\alpha-1} e^{-(1-\mu) A(u-1) z} d u & (B=0)\end{cases}
$$

and $q_{1}(z)$ is the best dominant of (29). Moreover,

$$
\begin{equation*}
\Sigma(\alpha+1, \beta, \mu) \subseteq \Sigma(\alpha, \beta, \mu) \tag{30}
\end{equation*}
$$

(ii) If the additional constraints $0<B<1$ and

$$
\begin{equation*}
\alpha+1 \geqq \frac{(1-\mu)(A-B)}{B} \tag{31}
\end{equation*}
$$

are satisfied, then

$$
\begin{equation*}
\frac{1-|A|}{1-|B|}<\frac{1}{1-\mu}\left(-\Re\left\{\frac{z\left(J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda, \alpha, \beta} f(z)\right)^{\prime}}{J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda, \beta} f(z)}\right\}-\mu\right)<\rho_{1} \tag{32}
\end{equation*}
$$

where

$$
\begin{equation*}
\rho_{1}=\frac{1}{1-\mu}\left\{(\alpha-\mu+1)-\frac{\alpha}{{ }_{2} F_{1}\left(1, \frac{(1-\mu)(A-B)}{B} ; \alpha+1 ; \frac{B}{B-1}\right)}\right\} \tag{33}
\end{equation*}
$$

The bound $\rho_{1}$ is the best possible.
Proof. Let $f(z) \in \Sigma(\alpha+1, \beta, \mu)$ and set

$$
\begin{equation*}
\phi(z)=\frac{1}{1-\mu}\left(-\frac{z\left(J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda, \alpha, \beta} f(z)\right)^{\prime}}{J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda, \beta} f(z)}-\mu\right) \tag{34}
\end{equation*}
$$

Then it is clear that $\phi(z)$ is analytic in $\mathbb{U}$ and $\phi(0)=1$. An application of the identity (19) in (34) yields

$$
\begin{equation*}
-(1-\mu) \phi(z)+(\alpha-\mu+1)=\alpha \frac{J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda, \alpha+1, \beta} f(z)}{J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \alpha, \beta} f(z)} . \tag{35}
\end{equation*}
$$

By using the logarithmic differentiation of both sides of (35) with respect to $z$, we obtain

$$
\begin{aligned}
\phi(z)+\frac{z \phi^{\prime}(z)}{(\alpha-\mu+1)-(1-\mu) \phi(z)} & =\frac{1}{1-\mu}\left(-\frac{z\left(J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda, \alpha+1, \beta} f(z)\right)^{\prime}}{J_{\left(\lambda_{p},,,\left(\mu_{q}\right), b\right.}^{s+1, \beta} f(z)}-\mu\right) \\
& \prec \frac{1+A z}{1+B z} \quad(z \in \mathbb{U}) .
\end{aligned}
$$

Therefore, by applying Lemma 1 with

$$
\nu=-(1-\mu) \quad \text { and } \quad \tau=\alpha-\mu+1
$$

we have

$$
\phi(z) \prec q_{1}(z) \prec \frac{1+A z}{1+B z} \quad(z \in \mathbb{U}),
$$

where the best dominant $q_{1}(z)$ is defined by (29). The proof of Theorem 1 (i) is completed.

In order to establish (32) of Theorem 1 (ii), we observe that an application of the principle of subordination in (21) gives

$$
\frac{1-|A|}{1-|B|}<\frac{1}{1-\mu}\left(-\Re\left\{\frac{z\left(J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda, \alpha, \beta} f(z)\right)^{\prime}}{J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda, \alpha, \beta} f(z)}\right\}-\mu\right)
$$

which is precisely the left-hand inequality in (32). Also, by the principle of subordination in (29), we have

$$
\begin{align*}
\frac{1}{1-\mu}\left(-\Re\left\{\frac{z\left(J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda, \alpha, \beta} f(z)\right)^{\prime}}{J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda)} f(z)}\right\}-\mu\right) & \leqq \sup _{z \in \mathbb{U}^{*}} \Re\left\{q_{1}(z)\right\} \\
& =\sup _{z \in \mathbb{U}}\left[\frac{1}{1-\mu}\left(\alpha-\mu+1-\Re\left\{\frac{1}{Q_{1}(z)}\right\}\right)\right] \\
& =\frac{1}{1-\mu}\left(\alpha-\mu+1-\inf _{z \in \mathbb{U}} \Re\left\{\frac{1}{Q_{1}(z)}\right\}\right) . \tag{36}
\end{align*}
$$

The rest of the proof is devoted to find

$$
\inf _{z \in \mathbb{U}} \Re\left\{\frac{1}{Q_{1}(z)}\right\} .
$$

By hypothesis, $B \neq 0$. Therefore, by using (29), we have

$$
Q_{1}(z)=(1+B z)^{\delta} \int_{0}^{1} u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1+B z u)^{-\delta} d u
$$

where

$$
\delta=\frac{(1-\mu)(A-B)}{B} \quad \text { and } \quad \gamma=\alpha+1
$$

Also, since $\gamma>\alpha>0$, by successively using (25) to (27) of Lemma 3, we obtain

$$
\begin{equation*}
Q_{1}(z)=\frac{\Gamma(\alpha)}{\Gamma(\gamma)}{ }_{2} F_{1}\left(1, \delta ; \gamma ; \frac{B z}{B z+1}\right) . \tag{37}
\end{equation*}
$$

Furthermore, the condition:

$$
\alpha+1>\frac{(1-\mu)(A-B)}{B} \quad(0<B<1)
$$

implies that $\gamma>\delta>0$. Another application of (27) of Lemma 3 to (37) gives

$$
Q_{1}(z)=\int_{0}^{1} h(z, u) d v(u),
$$

where

$$
h(z, u)=\frac{1+B z}{1+(1-u) B z} \quad(0 \leqq u \leqq 1)
$$

and

$$
d v(u)=\frac{\Gamma(\alpha)}{\Gamma(\delta) \Gamma(\gamma-\delta)} u^{\delta-1}(1-u)^{\gamma-\delta-1} d u
$$

is a positive measure on $u \in[0,1]$. We note that

$$
\Re\{h(z, u)\}>0 \quad \text { and } \quad h(-r, u)
$$

is real for $0 \leqq r<1$ and $u \in[0,1]$. Therefore, by using Lemma 2 , we get

$$
\Re\left\{\frac{1}{Q_{1}(z)}\right\} \geqq \frac{1}{Q_{1}(-r)} \quad(|z| \leqq r<1)
$$

so that

$$
\begin{align*}
\inf _{z \in \mathbb{U}} \Re\left\{\frac{1}{Q_{1}(z)}\right\} & =\sup _{0 \leqq r<1} \frac{1}{Q_{1}(-r)} \\
& =\sup _{0 \leqq r<1} \frac{1}{\int_{0}^{1} h(-r, u) d v}=\frac{1}{\int_{0}^{1} h(-1, u) d v}=\frac{1}{Q_{1}(-1)} \\
& =\frac{\alpha}{{ }_{2} F_{1}\left(1, \frac{(1-\mu)(A-B)}{B}, \alpha+1, \frac{B}{B-1}\right)} . \tag{38}
\end{align*}
$$

Hence, in view of (36), the right-hand inequality of (32) follows from (38).
The result is the best possible as the function $q_{1}(z)$ is the best dominant of (29). This completes the proof of Theorem 1.

The next theorem gives the corresponding results involving the parameter $\beta$.

## Theorem 2.

(i) If $f(z) \in \Sigma(\alpha, \beta, \mu)$ and

$$
\begin{equation*}
\beta-\mu+1 \geqq \frac{(1-\mu)(1-A)}{(1-B)} \tag{39}
\end{equation*}
$$

then

$$
\begin{gather*}
\frac{1}{1-\mu}\left(-\frac{z\left(J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda, \alpha, \beta+1} f(z)\right)^{\prime}}{J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda, \alpha,+1} f(z)}-\mu\right) \prec \frac{1}{1-\mu}\left((\beta-\mu+1)-\frac{1}{Q_{2}(z)}\right) \\
=q_{2}(z) \prec \frac{1+A z}{1+B z} \quad(z \in \mathbb{U}) \tag{40}
\end{gather*}
$$

where

$$
Q_{2}(z)= \begin{cases}\int_{0}^{1} u^{\beta-1}\left(\frac{1+B z u}{1+B z}\right)^{-(1-\mu)(A-B) / B} d u & (B \neq 0) \\ \int_{0}^{1} u^{\beta-1} e^{-(1-\mu) A(u-1) z} d u & (B=0)\end{cases}
$$

and $q_{2}(z)$ is the best dominant of (40). It is also asserted that

$$
\begin{equation*}
\Sigma(\alpha, \beta, \mu) \subseteq \Sigma(\alpha, \beta+1, \mu) \tag{41}
\end{equation*}
$$

(ii) If the additional constraints $0<B<1$ and

$$
\begin{equation*}
\beta+1 \geqq \frac{(1-\mu)(A-B)}{B} \tag{42}
\end{equation*}
$$

are satisfied, then

$$
\begin{equation*}
\frac{1-|A|}{1-|B|}<\frac{1}{1-\mu}\left(-\Re\left\{\frac{z\left(J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda, \alpha, \beta+1} f(z)\right)^{\prime}}{J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda, \alpha, \beta+1} f(z)}\right\}-\mu\right)<\rho_{2} \tag{43}
\end{equation*}
$$

where

$$
\begin{equation*}
\rho_{2}=\frac{1}{1-\mu}\left((\beta+1-\mu)-\frac{\beta}{{ }_{2} F_{1}\left(1, \frac{(1-\mu)(A-B)}{B} ; \beta+1 ; \frac{B}{B-1}\right)}\right) \tag{44}
\end{equation*}
$$

The bound $\rho_{2}$ is the best possible.
Proof. Let $f(z) \in \Sigma(\alpha, \beta, \mu)$ and set

$$
\begin{equation*}
\phi(z)=\frac{1}{1-\mu}\left(-\frac{z\left(J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda, \alpha, \beta+1} f(z)\right)^{\prime}}{J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda, \alpha, \beta+1} f(z)}-\mu\right) \tag{45}
\end{equation*}
$$

Then, by using (17) and logarithmic differentiation for (45) with respect to $z$, we get

$$
\phi(z)+\frac{z \phi^{\prime}(z)}{-(1-\mu) \phi(z)+(\beta+1-\mu)}=\frac{1}{1-\mu}\left(-\frac{z\left(J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda, \alpha, \beta} f(z)\right)^{\prime}}{J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda, \alpha, \beta} f(z)}-\mu\right)
$$

$$
\prec \frac{1+A z}{1+B z} \quad(z \in \mathbb{U}) .
$$

Therefore, by an application of Lemma 1 with

$$
\nu=-(1-\mu) \quad \text { and } \quad \tau=\beta-\mu+1,
$$

we have

$$
\phi(z) \prec q_{2}(z) \prec \frac{1+A z}{1+B z} \quad(z \in \mathbb{U}),
$$

where the best dominant $q_{2}(z)$ is defined by (40). The proof of Theorem 2 (i) is completed.

In order to establish (43) of Theorem 2 (ii), we apply the principle of subordination in (21) and use the same technique which was used in the proof of Theorem 1. We thus find that

$$
\begin{align*}
Q_{2}(z) & =(1+B z)^{\delta} \int_{0}^{1} u^{\beta-1}(1-u)^{\gamma-\beta-1}(1+B z u)^{-\delta} d u \\
& =\frac{\Gamma(\beta)}{\Gamma(\gamma)}{ }_{2} F_{1}\left(1, \delta ; \gamma ; \frac{B z}{B z+1}\right) \tag{46}
\end{align*}
$$

where $\delta=\frac{(1-\mu)(A-B)}{B}$ and $\gamma=\beta+1$.
Furthermore, the condition:

$$
\beta+1>\frac{(1-\mu)(A-B)}{B} \quad(0<B<1)
$$

implies that $\gamma>\delta>0$. Another application of (27) of Lemma 3 to (46) gives

$$
Q_{2}(z)=\int_{0}^{1} h(z, u) d v(u)
$$

where

$$
h(z, u)=\frac{1+B z}{1+(1-u) B z}, \quad(0 \leqq u \leqq 1)
$$

and

$$
d v(u)=\frac{\Gamma(\beta)}{\Gamma(\gamma) \Gamma(\gamma-\delta)} u^{\delta-1}(1-u)^{\gamma-\delta-1} d u .
$$

Using Lemma 2 implies that

$$
\begin{equation*}
\inf _{z \in \mathbb{U}} \Re\left\{\frac{1}{Q_{2}(z)}\right\}=\frac{\beta}{{ }_{2} F_{1}\left(1, \frac{(1-\mu)(A-B)}{B} ; \beta+1 ; \frac{B}{B-1}\right)} . \tag{47}
\end{equation*}
$$

The right-hand inequality of (43) follows from (47).
The bound $\rho_{2}$ is sharp by the principle of subordination. The proof of Theorem 2 is thus completed.

## 4 Concluding Remarks and Observations

In our present sequel to an earlier work (see $[5,6,14]$ and $[15]$ ), we have investigated several further properties of the linear operator defined by (18), which is associated with Hurwitz-Lerch zeta function:

$$
J_{\left(\lambda_{p}\right),\left(\mu_{q}\right), b}^{s, a, \lambda, \alpha, \beta} f(z)=\frac{1}{z}+\sum_{k=1}^{\infty} \frac{(\alpha)_{k+1} \prod_{j=1}^{p}\left(\lambda_{j}\right)_{k}}{(\beta)_{k+1} \prod_{j=1}^{q}\left(\mu_{j}\right)_{k}}\left(\frac{a+1}{a+k}\right)^{s} \frac{\Lambda(a+k, b, s, \lambda)}{\lambda \Gamma(s)} a_{k} \frac{z^{k}}{k!},
$$

as given by (8) and with the notation used with (17). The various properties and results, which we have presented in this paper, are related to a certain subclass of the class of (normalized) meromorphically univalent functions in the punctured unit disk $\mathbb{U}^{*}$, which is defined here by means of the Hadamard product (or convolution). Many interesting results (asserted by Theorems 1 and 2 above) have also been deduced in this paper. In addition, there are more extensions and ideas that can be found based on these results.

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H. M. Srivastava

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Department of Mathematics and Statistics University of Victoria, Victoria, British Columbia V8W 3R4, Canada
Department of Medical Research, China Medical
University Hospital, China Medical University
Taichung 40402, Taiwan, Republic of China
E-mail: harimsri@math.uvic.ca
F. Ghanim

College of Sciences, University of Sharjah
Sharjah, United Arab Emirates
E-mail: fgahmed@sharjah.ac.ae
R. M. El-Ashwah

Department of Mathematics
Faculty of Science, Damietta University
New Damietta 34517, Egypt
E-mail: r_elashwah@yahoo.com

# Interpolating Bézier spline surfaces with local control 

A. P. Pobegailo


#### Abstract

This paper presents an approach to construct interpolating spline surfaces over a bivariate network of curves with rectangular patches. Patches of the interpolating spline surface are constructed by means of blending their boundaries with special polynomials. In order to ensure a necessary parametric continuity of the designed surface the polynomials of the corresponding degree are used. The constructed interpolating spline surfaces have local shape control. If the surface frame is determined by means of Bézier curves then patches of the interpolating spline surface are Bézier surfaces.


Mathematics subject classification: 65D05, 65D07, 65D17.
Keywords and phrases: Blending parametric curves, interpolating surfaces, spline surfaces, Bézier surfaces.

## 1 Introduction

Interpolating spline surfaces play important role in different geometric applications. This paper presents an approach to construction of interpolating spline surfaces which have local shape control. Patches of the interpolating spline surface are constructed by means of blending their boundaries with special polynomials. In order to ensure a necessary parametric continuity of the constructed surface the polynomials of the corresponding degree must be used. The presented approach is aimed at construction of interpolating spline surface over the bivariate network of curves with rectangular patches. Interpolation with Bézier patches over the bivariate network of Bézier curves is considered as application of the general approach. A classification of algorithms for local smooth surface interpolation with piecewise polynomials is given in the paper of Peters [14]. A survey of blending methods that use parametric surfaces can be found in the paper of Vida, Martin, Várady [20]. Construction of surface patches by linear blending of its boundaries was firstly introduced by Coons [5]. Contemporary representation of the patches was given by Forrest [8] and considered by Faux and Pratt [7]. Spline-blended surface interpolation through curve networks was proposed by Gordon [10]. The presented approach can be considered as generalization of the techniques. Another approach to surface interpolation by means of linear blending is considered in the paper of Juhásza and Hoffmann [12].
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## 2 Construction of a rectangular patch by blending its boundaries

Construction of a surface patch by means of linear blending of its boundaries was introduced by Coons [5]. The presented approach can be considered as generalization of the technique.

Consider four parametric curves $\mathbf{p}_{0}(u), \mathbf{p}_{1}(u), u \in[0,1]$, and $\mathbf{q}_{0}(v), \mathbf{q}_{1}(v), v \in$ $[0,1]$, which have the following boundary points:

$$
\begin{align*}
& \mathbf{p}_{0}(0)=\mathbf{q}_{0}(0)=\mathbf{r}_{0,0}, \mathbf{p}_{0}(1)=\mathbf{q}_{1}(0)=\mathbf{r}_{1,0},  \tag{1}\\
& \mathbf{p}_{1}(0)=\mathbf{q}_{0}(1)=\mathbf{r}_{0,1}, \mathbf{p}_{1}(1)=\mathbf{q}_{1}(1)=\mathbf{r}_{1,1} . \tag{2}
\end{align*}
$$

The problem is to construct a rectangular patch $\mathbf{r}(u, v),(u, v) \in[0,1] \times[0,1]$, which has the considered parametric curves as boundaries, that is

$$
\begin{align*}
& \mathbf{r}(u, 0)=\mathbf{p}_{0}(u), \mathbf{r}(u, 1)=\mathbf{p}_{1}(u),  \tag{3}\\
& \mathbf{r}(0, v)=\mathbf{q}_{0}(v), \mathbf{r}(1, v)=\mathbf{q}_{1}(v) \tag{4}
\end{align*}
$$

and partial derivatives of the patch $\mathbf{r}(u, v)$ satisfy the following conditions at the corner points:

$$
\begin{align*}
& \frac{\partial^{m} \mathbf{r}(u, v)}{\partial u^{m}}(0,0)=\left(\mathbf{p}_{0}^{(m)}(u)\right)(0), \frac{\partial^{m} \mathbf{r}(u, v)}{\partial v^{m}}(0,0)=\left(\mathbf{q}_{0}^{(m)}(v)\right)(0),  \tag{5}\\
& \frac{\partial^{m} \mathbf{r}(u, v)}{\partial u^{m}}(0,1)=\left(\mathbf{p}_{1}^{(m)}(u)\right)(0), \frac{\partial^{m} \mathbf{r}(u, v)}{\partial v^{m}}(0,1)=\left(\mathbf{q}_{0}^{(m)}(v)\right)(1),  \tag{6}\\
& \frac{\partial^{m} \mathbf{r}(u, v)}{\partial u^{m}}(1,0)=\left(\mathbf{p}_{0}^{(m)}(u)\right)(1), \frac{\partial^{m} \mathbf{r}(u, v)}{\partial v^{m}}(1,0)=\left(\mathbf{q}_{1}^{(m)}(v)\right)(0),  \tag{7}\\
& \frac{\partial^{m} \mathbf{r}(u, v)}{\partial u^{m}}(1,1)=\left(\mathbf{p}_{1}^{(m)}(u)\right)(1), \frac{\partial^{m} \mathbf{r}(u, v)}{\partial v^{m}}(1,1)=\left(\mathbf{q}_{1}^{(m)}(v)\right)(1), \tag{8}
\end{align*}
$$

for all $m \in\{1,2, \ldots, n\}$ where $s+r=m$ and $n \in \mathbb{N}$. In order to solve the problem define the following parametric surface:

$$
\begin{equation*}
\mathbf{r}(u, v)=\mathbf{s}(u, v)-\tilde{\mathbf{r}}(u, v), u, v \in[0,1], \tag{9}
\end{equation*}
$$

where

$$
\begin{gathered}
\mathbf{s}(u, v)=\left(1-w_{n+1}(v)\right) \mathbf{p}_{0}(u)+w_{n+1}(v) \mathbf{p}_{1}(u)+ \\
+\left(1-w_{n+1}(u)\right) \mathbf{q}_{0}(u)+w_{n+1}(u) \mathbf{q}_{1}(u), \\
\tilde{\mathbf{r}}(u, v)=\left(1-w_{n+1}(u)\right)\left(1-w_{n+1}(v)\right) \mathbf{r}_{0,0}+w_{n+1}(u)\left(1-w_{n+1}(v)\right) \mathbf{r}_{1,0}+ \\
+\left(1-w_{n+1}(u)\right) w_{n+1}(v) \mathbf{r}_{0,1}+w_{n+1}(u) w_{n+1}(v) \mathbf{r}_{1,1}
\end{gathered}
$$

and the polynomials $w_{n}(u)$ are defined as follows:

$$
w_{n}(u)=\sum_{i=n}^{2 n-1} b_{2 n-1, i}(u), u \in[0,1],
$$

where $b_{n, m}(u)$ denotes a Bernstein polynomial

$$
b_{n, m}(u)=\frac{n!}{m!(n-m)!}(1-u)^{n-m} u^{m}, u \in[0,1] .
$$

Detailed considerations of the polynomials $w_{n}(u)$ can be found in the paper of Pobegailo [15] where it is shown that the polynomials have the following boundary values:

$$
\begin{equation*}
w_{n}(0)=0, w_{n}(1)=1 \tag{10}
\end{equation*}
$$

and satisfy the following boundary conditions:

$$
\begin{equation*}
w_{n}^{(m)}(0)=w_{n}^{(m)}(1)=0 \tag{11}
\end{equation*}
$$

for all $m \in\{1,2, \ldots, n-1\}$.
Show that the parametric curves $\mathbf{p}_{0}(u), \mathbf{p}_{1}(u)$ and $\mathbf{q}_{0}(v), \mathbf{q}_{1}(v)$ are boundaries of the patch $\mathbf{r}(u, v)$. Substitution of boundary values of the polynomials $w_{n+1}(u)$ from Equations (10) and parametric curves from Equations (1) and (2) in Equation (9) yields that

$$
\begin{gathered}
\mathbf{s}(u, 0)=\mathbf{p}_{0}(u)+\left(1-w_{n+1}(u)\right) \mathbf{r}_{0,0}+w_{n+1}(u) \mathbf{r}_{1,0}, \\
\tilde{\mathbf{r}}(u, 0)=\left(1-w_{n+1}(u)\right) \mathbf{r}_{0,0}+w_{n+1}(u) \mathbf{r}_{1,0}
\end{gathered}
$$

and therefore

$$
\mathbf{r}(u, 0)=\mathbf{s}(u, 0)-\tilde{\mathbf{r}}(u, 0)=\mathbf{p}_{0}(u)
$$

Then

$$
\begin{gathered}
\mathbf{s}(0, v)=\left(1-w_{n+1}(v)\right) \mathbf{r}_{0,0}+w_{n+1}(v) \mathbf{r}_{0,1}+\mathbf{q}_{0}(v) \\
\tilde{\mathbf{r}}(0, v)=\left(1-w_{n+1}(v)\right) \mathbf{r}_{0,0}+w_{n+1}(v) \mathbf{r}_{0,1}
\end{gathered}
$$

and therefore

$$
\mathbf{r}(0, v)=\mathbf{s}(u, 0)-\tilde{\mathbf{r}}(u, 0)=\mathbf{q}_{0}(v)
$$

Analogously it can be shown that

$$
\mathbf{r}(u, 1)=\mathbf{p}_{1}(u), \mathbf{r}(1, v)=\mathbf{q}_{1}(v) .
$$

Thus Equations (3) and (4) are fulfilled.
Show that the patch $\mathbf{r}(u, v)$ has necessary partial derivatives at the corner points, that is Equations (5-8) are also fulfilled. For this purpose compute partial derivatives of the parametric surface $\mathbf{r}(u, v)$. It is obtained that

$$
\frac{\partial^{m} \mathbf{r}(u, v)}{\partial u^{m}}=\frac{\partial^{m} \mathbf{s}(u, v)}{\partial u^{m}}-\frac{\partial^{m} \tilde{\mathbf{r}}(u, v)}{\partial u^{m}}
$$

where

$$
\frac{\partial^{m} \mathbf{s}(u, v)}{\partial u^{m}}=\left(1-w_{n+1}(v)\right) \mathbf{p}_{0}^{(m)}(u)+w_{n+1}(v) \mathbf{p}_{1}^{(m)}(u)+
$$

$$
\begin{gathered}
+\left(1-w_{n+1}(u)\right)^{(m)} \mathbf{q}_{0}(v)+w_{n+1}^{(m)}(u) \mathbf{q}_{1}(v) \\
\frac{\partial^{m} \tilde{\mathbf{r}}(u, v)}{\partial u^{m}}=\left(1-w_{n+1}(u)\right)^{(m)}\left(1-w_{n+1}(v)\right) \mathbf{r}_{0,0}+w_{n+1}^{(m)}(u)\left(1-w_{n+1}(v)\right) \mathbf{r}_{1,0}+ \\
+\left(1-w_{n+1}(u)\right)^{(m)} w_{n+1}(v) \mathbf{r}_{0,1}+w_{n+1}^{(m)}(u) w_{n+1}(v) \mathbf{r}_{1,1}
\end{gathered}
$$

and analogously

$$
\frac{\partial^{m} \mathbf{r}(u, v)}{\partial v^{m}}=\frac{\partial^{m} \mathbf{s}(u, v)}{\partial v^{m}}-\frac{\partial^{m} \tilde{\mathbf{r}}(u, v)}{\partial v^{m}}
$$

where

$$
\begin{gathered}
\frac{\partial^{m} \mathbf{s}(u, v)}{\partial v^{m}}=\left(1-w_{n+1}(v)\right)^{(m)} \mathbf{p}_{0}(u)+w_{n+1}^{(m)}(v) \mathbf{p}_{1}(u)+ \\
+\left(1-w_{n+1}(u)\right) \mathbf{q}_{0}^{(m)}(v)+w_{n+1}(u) \mathbf{q}_{1}^{(m)}(v) \\
\frac{\partial^{m} \tilde{\mathbf{r}}(u, v)}{\partial v^{m}}=\left(1-w_{n+1}(u)\right)\left(1-w_{n+1}(v)\right)^{(m)} \mathbf{r}_{0,0}+w_{n+1}(u)\left(1-w_{n+1}(v)\right)^{(m)} \mathbf{r}_{1,0}+ \\
+\left(1-w_{n+1}(u)\right) w_{n+1}^{(m)}(v) \mathbf{r}_{0,1}+w_{n+1}(u) w_{n+1}^{(m)}(v) \mathbf{r}_{1,1}
\end{gathered}
$$

for all $m \in\{1,2, \ldots, n\}, n \in \mathbb{N}$. Substituting boundary values of the polynomials $w_{n+1}(u)$ and their derivatives from Equations (10) and (11) in these equations, it is obtained that

$$
\frac{\partial^{m} \mathbf{r}(0,0)}{\partial u^{m}}=\frac{\partial^{m} \mathbf{s}(0,0)}{\partial u^{m}}-\frac{\partial^{m} \tilde{\mathbf{r}}(0,0)}{\partial u^{m}}=\mathbf{p}_{0}^{(m)}(u)
$$

and

$$
\frac{\partial^{m} \mathbf{r}(u, v)}{\partial v^{m}}=\frac{\partial^{m} \mathbf{s}(u, v)}{\partial v^{m}}-\frac{\partial^{m} \tilde{\mathbf{r}}(u, v)}{\partial v^{m}}=\mathbf{q}_{0}^{(m)}(v)
$$

Thus Equations (5) are fulfilled. Analogously it can be proven that Equations (6)-(8) are also fulfilled.

Now compute mixed partial derivatives of the parametric surface $\mathbf{r}(u, v)$ at the corner points. It is obtained that

$$
\frac{\partial^{m} \mathbf{r}(u, v)}{\partial u^{r} \partial v^{s}}=\frac{\partial^{m} \mathbf{s}(u, v)}{\partial u^{r} \partial v^{s}}-\frac{\partial^{m} \tilde{\mathbf{r}}(u, v)}{\partial u^{r} \partial v^{s}}
$$

where

$$
\begin{gathered}
\frac{\partial^{m} \mathbf{s}(u, v)}{\partial u^{r} \partial v^{s}}=\left(1-w_{n+1}(v)\right)^{(s)} \mathbf{p}_{0}^{(r)}(u)+w_{n+1}^{(s)}(v) \mathbf{p}_{1}^{(r)}(u)+ \\
+\left(1-w_{n+1}(u)\right)^{(r)} \mathbf{q}_{0}^{(s)}(v)+w_{n+1}^{(r)}(u) \mathbf{q}_{1}^{(s)}(v) \\
\frac{\partial^{m} \tilde{\mathbf{r}}(u, v)}{\partial u^{r} \partial v^{s}}=\left(1-w_{n+1}(u)\right)^{(r)}\left(1-w_{n+1}(v)\right)^{(s)} \mathbf{r}_{0,0}+w_{n+1}^{(r)}(u)\left(1-w_{n+1}(v)\right)^{(s)} \mathbf{r}_{1,0}+ \\
+\left(1-w_{n+1}(u)\right)^{(r)} w_{n+1}^{(s)}(v) \mathbf{r}_{0,1}+w_{n+1}^{(r)}(u) w_{n+1}^{(s)}(v) \mathbf{r}_{1,1}
\end{gathered}
$$

for all $m \in\{1,2, \ldots, n\}$ where $s+r=m$ and $n \in \mathbb{N}$. Substituting values of derivatives which are defined by Equations (11) in these equations, it is obtained that

$$
\frac{\partial^{m} \mathbf{s}(u, v)}{\partial u^{r} \partial v^{s}}=0, \frac{\partial^{m} \tilde{\mathbf{r}}(u, v)}{\partial u^{m}}=0
$$

and therefore

$$
\frac{\partial^{m} \mathbf{r}(0,0)}{\partial u^{r} \partial v^{s}}=\frac{\partial^{m} \mathbf{s}(0,0)}{\partial u^{r} \partial v^{s}}-\frac{\partial^{m} \tilde{\mathbf{r}}(0,0)}{\partial u^{r} \partial v^{s}}=0 .
$$

Analogously it can be proven that the other mixed partial derivatives at the corners of the patch $\mathbf{r}(u, v)$ are also equal to zero. Thus it is obtained that

$$
\begin{equation*}
\frac{\partial^{m} \mathbf{r}(u, v)}{\partial u^{r} \partial v^{s}}(0,0)=\frac{\partial^{m} \mathbf{r}(u, v)}{\partial u^{r} \partial v^{s}}(0,1)=\frac{\partial^{m} \mathbf{r}(u, v)}{\partial u^{r} \partial v^{s}}(1,0)=\frac{\partial^{m} \mathbf{r}(u, v)}{\partial u^{r} \partial v^{s}}(1,1)=0 \tag{12}
\end{equation*}
$$

for all $m \in\{1,2, \ldots, n\}$ where $s+r=m$ and $n \in \mathbb{N}$. These values of mixed partial derivatives are natural because the patch $\mathbf{r}(u, v)$ is defined only by the boundary curves.

## 3 Construction of spline surfaces by blending frame curves

Spline-blended surface interpolation through curve networks was proposed by Gordon [10]. Then Gregory [11] introduced a smooth interpolation scheme without twist constraints. $G^{1}$ smoothness conditions for rectangular and triangular Gregory patches are discussed by Farin and Hansford [6]. Another approach to surface interpolation of control point mesh was proposed by Comninos [4]. The surface is generated by piecewise bicubic interpolation and is derived from a classical Coons patch. This paper presents an approach to interpolating bivariate network of curves by means of patches which are constructed by blending frame curves. The presented approach provides $C^{n}$ continuity of the constructed surface. Another approach to surface interpolation by means of linear blending is considered in the paper of Juhásza and Hoffmann [12].

Consider a rectangular grid of points $\mathbf{r}_{i, j}, i \in\{0,1, \ldots, k\}, j \in\{0,1, \ldots, l\}, k, l \in \mathbb{N}$. Suppose that the rectangular grid is framed by parametric curves $\mathbf{p}_{i, j}(u), u \in[0,1]$, and $\mathbf{q}_{i, j}(v), v \in[0,1]$, where $i \in\{0,1, \ldots, k-1\}, j \in\{0,1, \ldots, l-1\}$, which satisfy the following boundary conditions:

$$
\begin{align*}
& \mathbf{p}_{i, j}(0)=\mathbf{q}_{i, j}(0)=\mathbf{r}_{i, j}, \\
& \mathbf{p}_{i, j}(1)=\mathbf{p}_{i+1, j}(0)=\mathbf{r}_{i+1, j},  \tag{13}\\
& \mathbf{q}_{i, j}(1)=\mathbf{q}_{i, j+1}(0)=\mathbf{r}_{i, j+1} .
\end{align*}
$$

Besides the considered parametric curves are $C^{n}$ continuously joined at the common grid points, that is

$$
\begin{equation*}
\left(\mathbf{p}_{i, j}^{(m)}(u)\right)(1)=\left(\mathbf{p}_{i+1, j}^{(m)}(u)\right)(0),\left(\mathbf{q}_{i, j}^{(m)}(v)\right)(1)=\left(\mathbf{q}_{i, j+1}^{(m)}(v)\right)(0) \tag{14}
\end{equation*}
$$

for all $m \in\{1,2, \ldots, n\}, n \in \mathbb{N}$. The problem is to construct a $C^{n}$ continuous parametric surface $\mathbf{r}(u, v)$ which interpolates the points of this grid and the parametric curves $\mathbf{p}_{i, j}(u)$ and $\mathbf{q}_{i, j}(v)$ are boundaries of rectangular patches which form the surface. Using Equation (9) define rectangular patches of the surface $\mathbf{r}(u, v)$ as follows:

$$
\begin{equation*}
\mathbf{r}_{i, j}(u, v)=\mathbf{s}_{i, j}(u, v)-\tilde{\mathbf{r}}_{i, j}(u, v), \quad(u, v) \in[0,1] \times[0,1], \tag{15}
\end{equation*}
$$

where

$$
\begin{gathered}
\mathbf{s}_{i, j}(u, v)=\left(1-w_{n+1}(v)\right) \mathbf{p}_{i, j}(u)+w_{n+1}(v) \mathbf{p}_{i, j+1}(u)+ \\
+\left(1-w_{n+1}(u)\right) \mathbf{q}_{i, j}(u)+w_{n+1}(u) \mathbf{q}_{i+1, j}(u) \\
\tilde{\mathbf{r}}_{i, j}(u, v)=\left(1-w_{n+1}(u)\right)\left(1-w_{n+1}(v)\right) \mathbf{r}_{i, j}+w_{n+1}(u)\left(1-w_{n+1}(v)\right) \mathbf{r}_{i+1, j}+ \\
+\left(1-w_{n+1}(u)\right) w_{n+1}(v) \mathbf{r}_{i, j+1}+w_{n+1}(u) w_{n+1}(v) \mathbf{r}_{i+1, j+1}
\end{gathered}
$$

for all $i \in\{0,1, \ldots, k-1\}, j \in\{0,1, \ldots, l-1\}$. It follows from Equations (10) and (13) that the parametric curves $\mathbf{p}_{i, j}(u), \mathbf{p}_{i, j+1}(u), \mathbf{q}_{i, j}(v)$ and $\mathbf{q}_{i+1, j}(v)$ are boundaries of the patch $\mathbf{r}_{i, j}(u, v)$.

Show that the parametric surface $\mathbf{r}(u, v)$ is $C^{n}$ continuous. The surface $\mathbf{r}(u, v)$ is $C^{n}$ continuous at the knot points $\mathbf{r}_{i, j}$ because the frame curves are $C^{n}$ continuous at the knot points and taking into consideration Equations (12). Then it is necessary to show that the patches of the parametric surface $\mathbf{r}(u, v)$ are smoothly joined along their common boundaries. For this purpose compute partial derivatives of the adjustment patches along their common boundaries. It is obtained taking into account Equations (14) that

$$
\begin{gathered}
\frac{\partial^{m} \mathbf{r}_{i, j}(u, v)}{\partial u^{m}}(1, v)=\left(1-w_{n+1}(v)\right)\left(\mathbf{p}_{i, j}^{(m)}(u)\right)(1)+w_{n+1}(v)\left(\mathbf{p}_{i, j+1}^{(m)}(u)\right)(1)= \\
=\left(1-w_{n+1}(v)\right)\left(\mathbf{p}_{i+1, j}^{(m)}(u)\right)(0)+w_{n+1}(v)\left(\mathbf{p}_{i+1, j+1}^{(m)}(u)\right)(0)=\frac{\partial^{m} \mathbf{r}_{i+1, j}(u, v)}{\partial u^{m}}(0, v)
\end{gathered}
$$

and analogously

$$
\frac{\partial^{m} \mathbf{r}_{i, j}(u, v)}{\partial v^{m}}(u, 1)=\frac{\partial^{m} \mathbf{r}_{i, j+1}(u, v)}{\partial v^{m}}(u, 0)
$$

for all $m \in\{1,2, \ldots, n\}$. Now determine mixed partial derivatives across boundaries of the patches. It is obtained using Equation (14) that

$$
\begin{aligned}
& \frac{\partial^{m} \mathbf{r}_{i, j}(u, v)}{\partial u^{r} \partial v^{s}}(1, v)=\left(1-w_{n+1}(v)\right)^{(s)}\left(\mathbf{p}_{i, j}^{(r)}(u)\right)(1)+w_{n+1}^{(s)}(v)\left(\mathbf{p}_{i, j+1}^{(m)}(u)\right)(1)= \\
= & \left(1-w_{n+1}(v)\right)^{(s)}\left(\mathbf{p}_{i+1, j}^{(m)}(u)\right)(0)+w_{n+1}^{(s)}(v)\left(\mathbf{p}_{i+1, j+1}^{(m)}(u)\right)(0)=\frac{\partial^{m} \mathbf{r}_{i, j}(u, v)}{\partial u^{r} \partial v^{s}}(0, v)
\end{aligned}
$$

and analogously

$$
\frac{\partial^{m} \mathbf{r}_{i, j}(u, v)}{\partial u^{r} \partial v^{s}}(u, 1)=\frac{\partial^{m} \mathbf{r}_{i, j+1}(u, v)}{\partial u^{r} \partial v^{s}}(u, 0)
$$

for all $m \in\{1,2, \ldots, n\}$ where $s+r=m$. Thus the spline surface $\mathbf{r}(u, v)$ constructed by means of Equation (19) is $C^{n}$ continuous.

It is obvious that a shape of the interpolating surface constructed by the proposed method is mainly dependent on boundary curves of the patches. But two features of the interpolating surface shape which are common for all surfaces constructed by the approach can be mentioned.

Firstly it follows from Equations (12) that the twist vector $\mathbf{r}_{u, v}$ is equal to zero at all knot points of the interpolating spline surface. Therefore the proposed method can lead to local flattening of the generated surface near patch corners. There are more elaborated methods which use geometric specifications along the patch boundaries and at the corner points or the surface can be constructed with optimal twist vectors as a tool for interpolating a network of curves with a minimum energy surface, for example, see the paper of Kallay and Ravani [13]. But these methods can be used only for offline processing because it is difficult to adjust additional geometric specifications or global computation procedures for online data point processing. The old problem of specifying the mixed partial derivatives or twist vectors at the grid points for an interpolating surface over a rectangular network of curves is considered in detail by Barnhill, Brown, Klucewicz [1]; Faux, Pratt [7]; Barnhill, Farin, Fayard, Hagen [2].

Secondly it follows from the extremum property of the polynomials $w_{n}(u)$ that patches of interpolating surfaces are generated with energy minimizing polynomials. It can be seen from profiles of the polynomials that the higher degree of continuity of the interpolating surface the shape of patches closes to shape of frame curves at knot points and an inflection of the shape moves from knot points to a parametric center of the patch.

## 4 Interpolating Bézier spline surfaces with local control

A translation of the Gordon scheme into a Bézier-like form was carried out by Chiyokura and Kimura [3]. Local surface interpolation with Bézier patches for meshes of cubic curves is described by Shirman, Sequin [18-19]. The method is local and provides $G^{1}$ continuity between patches. In this section construction of spline surfaces using blending of Bézier frame curves is presented.

Suppose that frame curves of a rectangular grid are constructed by means of $C n$ continuous spline Bézier curves. Since the proposed approach is aimed at local interpolation of the framed grid, the Bézier curves must also have a local control. In order to ensure this property Bézier curves, which are segments of the curve net, must have at least $2 \mathrm{n}+1$ order. Such a net of spline curves can be constructed by the approach considered in the paper of Pobegailo [17]. In this case boundaries of
the patch $\mathbf{r}_{i, j}(u, v)$ can be described by the following Bézier curves:

$$
\begin{align*}
& \mathbf{p}_{i, j}(u)=\sum_{k=0}^{2 n+1} b_{2 n+1, k}(u) \mathbf{p}_{i, j, k}, u \in[0,1], \\
& \mathbf{q}_{i, j}(v)=\sum_{l=0}^{2 n+1} b_{2 n+1, l}(v) \mathbf{q}_{i, j, l}, v \in[0,1] \tag{16}
\end{align*}
$$

where boundary points of the Bézier curves $\mathbf{p}_{i, j}(u)$ and $\mathbf{q}_{i, j}(v)$ are knot points of the grid, that is

$$
\begin{aligned}
\mathbf{p}_{i, j}(0) & =\mathbf{p}_{i, j, 0}=\mathbf{r}_{i, j}, \quad \mathbf{p}_{i, j}(1)=\mathbf{p}_{i, j, 2 n+1}=\mathbf{r}_{i+1, j} \\
\mathbf{q}_{i, j}(0) & =\mathbf{q}_{i, j, 0}=\mathbf{r}_{i, j}, \quad \mathbf{q}_{i, j}(1)=\mathbf{q}_{i, j, 2 n+1}=\mathbf{r}_{i, j+1}
\end{aligned}
$$

Then the patch $\mathbf{r}_{i, j}(u, v)$ can be described using Equations (15) and (16) as follows:

$$
\begin{aligned}
& \mathbf{r}_{i, j}(u, v)=\mathbf{s}_{i, j}(u, v)-\tilde{\mathbf{r}}_{i, j}(u, v)= \\
& =\sum_{l=0}^{n} b_{2 n+1, l}(v) \sum_{k=0}^{2 n+1} b_{2 n+1, k}(u) \mathbf{p}_{i, j, k}+\sum_{l=n+1}^{2 n+1} b_{2 n+1, l}(v) \sum_{k=0}^{2 n+1} b_{2 n+1, k}(u) \mathbf{p}_{i, j+1, k}+ \\
& +\sum_{k=0}^{n} b_{2 n+1, k}(u) \sum_{l=0}^{2 n+1} b_{2 n+1, l}(v) \mathbf{q}_{i, j, l}+\sum_{k=n+1}^{2 n+1} b_{2 n+1, k}(u) \sum_{l=0}^{2 n+1} b_{2 n+1, l}(v) \mathbf{q}_{i+1, j, l}- \\
& \quad-\sum_{k=0}^{n} b_{2 n+1, k}(u) \sum_{l=0}^{n} b_{2 n+1, l}(v) \mathbf{r}_{i, j}-\sum_{k=n+1}^{2 n+1} b_{2 n+1, k}(u) \sum_{l=0}^{n} b_{2 n+1, l}(v) \mathbf{r}_{i+1, j}- \\
& - \\
& \sum_{k=0}^{n} b_{2 n+1, k}(u) \sum_{l=n+1}^{2 n+1} b_{2 n+1, l}(v) \mathbf{r}_{i, j+1}-\sum_{k=n+1}^{2 n+1} b_{2 n+1, k}(u) \sum_{l=n+1}^{2 n+1} b_{2 n+1, l}(v) \mathbf{r}_{i+1, j+1}
\end{aligned}
$$

Combination of the similar terms yields that

$$
\begin{aligned}
& \mathbf{r}_{i, j}(u, v)=\sum_{k=0}^{n} b_{2 n+1, k}(u) \sum_{l=0}^{n} b_{2 n+1, l}(v)\left(\mathbf{p}_{i, j, k}+\mathbf{q}_{i, j, l}-\mathbf{r}_{i, j}\right)+ \\
& +\sum_{k=0}^{n} b_{2 n+1, k}(u) \sum_{l=n+1}^{2 n+1} b_{2 n+1, l}(v)\left(\mathbf{p}_{i, j+1, k}+\mathbf{q}_{i, j, l}-\mathbf{r}_{i, j+1}\right)+ \\
& \sum_{k=n+1}^{2 n+1} b_{2 n+1, k}(u) \sum_{l=0}^{n} b_{2 n+1, l}(v)\left(\mathbf{p}_{i, j, k}+\mathbf{q}_{i+1, j, l}-\mathbf{r}_{i+1, j}\right)+ \\
& \sum_{k=n+1}^{2 n+1} b_{2 n+1, k}(u) \sum_{l=n+1}^{2 n+1} b_{2 n+1, l}(v)\left(\mathbf{p}_{i, j+1, k}+\mathbf{q}_{i+1, j, l}-\mathbf{r}_{i+1, j+1}\right) .
\end{aligned}
$$

This is a Bézier representation of the patch $\mathbf{r}_{i, j}(u, v)$ for a $C^{n}$ continuous Bézier spline surface. It can be seen from the last equation that the knot and control
points of the Bézier patch $\mathbf{r}_{i, j}(u, v)$ can be arranged in a square block matrix

$$
\mathbf{P}_{k, l}=\left[\begin{array}{ll}
\mathbf{B}_{0,0} & \mathbf{B}_{0,1} \\
\mathbf{B}_{1,0} & \mathbf{B}_{1,1}
\end{array}\right]
$$

where every internal block corresponds to a term of the patch equation.
In geometric applications surfaces of $C^{1}$ and $C^{2}$ continuity are usually used. A patch $\mathbf{r}_{i, j}(u, v)$ of the $C^{1}$ continuous surface has the following Bézier representations:

$$
\mathbf{r}_{i, j}(u, v)=\sum_{k=0}^{3} b_{2 n+1, k}(u) \sum_{l=0}^{3} b_{2 n+1, l}(v) \mathbf{p}_{k, l}
$$

where points $\mathbf{p}_{k, l}$ are corresponding elements of the following matrix:

$$
\mathbf{P}_{k, l}=\left[\begin{array}{cccc}
\mathbf{r}_{i, j} & \mathbf{q}_{i, j, 1} & \mathbf{q}_{i, j, 2} & \mathbf{r}_{i, j+1} \\
\mathbf{p}_{i, j, 1} & \mathbf{p}_{i, j, 1}+\mathbf{q}_{i, j, 1}-\mathbf{r}_{i, j} & \mathbf{p}_{i, j+1,1}+\mathbf{q}_{i, j, 2}-\mathbf{r}_{i, j+1} & \mathbf{p}_{i, j+1,1} \\
\mathbf{p}_{i, j, 2} & \mathbf{p}_{i, j, 2}+\mathbf{q}_{i+1, j, 1}-\mathbf{r}_{i+1, j} & \mathbf{p}_{i, j+1,2}+\mathbf{q}_{i+1, j, 2}-\mathbf{r}_{i+1, j+1} & \mathbf{p}_{i, j+1,2} \\
\mathbf{r}_{i+1, j} & \mathbf{q}_{i+1, j, 1} & \mathbf{q}_{i+1, j, 2} & \mathbf{r}_{i+1, j+1}
\end{array}\right] .
$$

A patch $\mathbf{r}_{i, j}(u, v)$ of the $C^{2}$ continuous surface has the following Bézier representation:

$$
\mathbf{r}_{i, j}(u, v)=\sum_{k=0}^{5} b_{2 n+1, k}(u) \sum_{l=0}^{5} b_{2 n+1, l}(v) \mathbf{p}_{k, l}
$$

where points $\mathbf{p}_{k, l}$ are corresponding elements of the following matrix blocks:

$$
\begin{gathered}
\mathbf{B}_{0,0}=\left[\begin{array}{ccc}
\mathbf{r}_{i, j} & \mathbf{q}_{i, j, 1} & \mathbf{q}_{i, j, 2} \\
\mathbf{p}_{i, j, 1} & \mathbf{p}_{i, j, 1}+\mathbf{q}_{i, j, 1}-\mathbf{r}_{i, j} & \mathbf{p}_{i, j, 1}+\mathbf{q}_{i, j, 2}-\mathbf{r}_{i, j} \\
\mathbf{p}_{i, j, 2} & \mathbf{p}_{i, j, 2}+\mathbf{q}_{i, j, 1}-\mathbf{r}_{i, j} & \mathbf{p}_{i, j, 2}+\mathbf{q}_{i, j, 2}-\mathbf{r}_{i, j}
\end{array}\right], \\
\mathbf{B}_{0,1}=\left[\begin{array}{ccc}
\mathbf{q}_{i, j, 3} & \mathbf{q}_{i, j, 4} & \mathbf{r}_{i, j+1} \\
\mathbf{p}_{i, j+1,1}+\mathbf{q}_{i, j, 3}-\mathbf{r}_{i, j+1} & \mathbf{p}_{i, j+1,1}+\mathbf{q}_{i, j, 4}-\mathbf{r}_{i, j+1} & \mathbf{p}_{i, j+1,1} \\
\mathbf{p}_{i, j+1,2}+\mathbf{q}_{i, j, 3}-\mathbf{r}_{i, j+1} & \mathbf{p}_{i, j+1,2}+\mathbf{q}_{i, j, 4}-\mathbf{r}_{i, j+1} & \mathbf{p}_{i, j+1,2}
\end{array}\right], \\
\mathbf{B}_{1,0}=\left[\begin{array}{cccc}
\mathbf{p}_{i, j, 3} & \mathbf{p}_{i, j, 3}+\mathbf{q}_{i, j, 1}-\mathbf{r}_{i+1, j} & \mathbf{p}_{i, j, 3}+\mathbf{q}_{i, j, 2}-\mathbf{r}_{i+1, j} \\
\mathbf{p}_{i, j, 4} & \mathbf{p}_{i, j, 4}+\mathbf{q}_{i+1, j, 1}-\mathbf{r}_{i+1, j} & \mathbf{p}_{i, j, 4}+\mathbf{q}_{i+1, j, 2}-\mathbf{r}_{i+1, j} \\
\mathbf{r}_{i+1, j} & \mathbf{q}_{i+1, j, 1} & \mathbf{q}_{i+1, j, 2}
\end{array}\right], \\
\mathbf{B}_{1,1}=\left[\begin{array}{ccc}
\mathbf{p}_{i, j+1,3}+\mathbf{q}_{i+1, j, 3}-\mathbf{r}_{i+1, j+1} & \mathbf{p}_{i, j+1,3}+\mathbf{q}_{i+1, j, 4}-\mathbf{r}_{i+1, j+1} & \mathbf{p}_{i, j+1,3} \\
\mathbf{p}_{i, j+1,4}+\mathbf{q}_{i+1, j, 3}-\mathbf{r}_{i+1, j+1} & \mathbf{p}_{i, j+1,4}+\mathbf{q}_{i+1, j, 4}-\mathbf{r}_{i+1, j+1} & \mathbf{p}_{i, j+1,4} \\
\mathbf{q}_{i+1, j, 3} & \mathbf{q}_{i+1, j, 4} & \mathbf{r}_{i+1, j+1}
\end{array}\right] .
\end{gathered}
$$

## 5 Rational Bézier spline surfaces with local control

Now suppose that frame curves of the rectangular grid are constructed by means of $C^{n}$ continuous rational spline Bézier curves with a local shape control. In order to ensure this property rational Bézier curves, which are segments of the net, must have at least $2 \mathrm{n}+1$ order. In this case boundaries of the patch $\mathbf{r}_{i, j}(u, v)$ can be described by the following rational Bézier curves:

$$
\begin{gathered}
\mathbf{p}_{i, j}(u)=\frac{\sum_{k=0}^{2 n+1} b_{2 n+1, k}(u) w_{i, j, k} \mathbf{p}_{i, j, k}}{\sum_{k=0}^{2 n+1} b_{2 n+1, k}(u) w_{i, j, k}}, u \in[0,1], \\
\mathbf{q}_{i, j}(v)=\frac{\sum_{l=0}^{2 n+1} b_{2 n+1, l}(u) w_{i, j, l} \mathbf{q}_{i, j, l}}{\sum_{l=0}^{2 n+1} b_{2 n+1, l}(u) w_{i, j, l}}, v \in[0,1]
\end{gathered}
$$

where boundary points of the rational Bézier curves $\mathbf{p}_{i, j}(u)$ and $\mathbf{q}_{i, j}(u)$ are knot points of the grid. Such a net of spline curves can be constructed by the approach considered in the paper of Pobegailo [16]. Introduce the following denotations for numerators and denominators of the rational Bézier curves $\mathbf{p}_{i, j}(u)$ and $\mathbf{q}_{i, j}(u)$ :

$$
\begin{aligned}
& \mathbf{P}_{i, j}(u)=\sum_{k=0}^{2 n+1} b_{2 n+1, k}(u) w_{i, j, k} \mathbf{p}_{i, j, k}, u \in[0,1] \\
& \mathbf{Q}_{i, j}(v)=\sum_{l=0}^{2 n+1} b_{2 n+1, l}(v) w_{i, j, l} \mathbf{q}_{i, j, l}, v \in[0,1] \\
& P_{i, j}(u)=\sum_{k=0}^{2 n+1} b_{2 n+1, k}(u) w_{i, j, k}, u \in[0,1] \\
& Q_{i, j}(v)=\sum_{l=0}^{2 n+1} b_{2 n+1, l}(v) w_{i, j, l}, v \in[0,1]
\end{aligned}
$$

Then by analogy with non-rational case, see Equation (15), define the following rational patches:

$$
\mathbf{r}_{i, j}(u, v)=\frac{\mathbf{S}_{i, j}(u, v)-\tilde{\mathbf{R}}_{i, j}(u, v)}{S_{i, j}(u, v)-\tilde{R}_{i, j}(u, v)},(u, v) \in[0,1] \times[0,1]
$$

where

$$
\begin{gathered}
\mathbf{S}_{i, j}(u, v)=\left(1-w_{n+1}(v)\right) \mathbf{P}_{i, j}(u)+w_{n+1}(v) \mathbf{P}_{i, j+1}(u)+ \\
\quad+\left(1-w_{n+1}(u)\right) \mathbf{Q}_{i, j}(u)+w_{n+1}(u) \mathbf{Q}_{i+1, j}(u)
\end{gathered}
$$

$$
\begin{gathered}
\tilde{\mathbf{R}}_{i, j}(u, v)=\left(1-w_{n+1}(u)\right)\left(1-w_{n+1}(v)\right) w_{i, j} \mathbf{r}_{i, j}+ \\
w_{n+1}(u)\left(1-w_{n+1}(v)\right) w_{i+1, j} \mathbf{r}_{i+1, j}+ \\
+\left(1-w_{n+1}(u)\right) w_{n+1}(v) w_{i, j+1} \mathbf{r}_{i, j+1}+ \\
w_{n+1}(u) w_{n+1}(v) w_{i+1, j+1} \mathbf{r}_{i+1, j+1}, \\
S_{i, j}(u, v)=\left(1-w_{n+1}(v)\right) P_{i, j}(u)+w_{n+1}(v) P_{i, j+1}(u)+ \\
+\left(1-w_{n+1}(u)\right) Q_{i, j}(u)+w_{n+1}(u) Q_{i+1, j}(u), \\
\\
\tilde{R}_{i, j}(u, v)=\left(1-w_{n+1}(u)\right)\left(1-w_{n+1}(v)\right) w_{i, j}+ \\
w_{n+1}(u)\left(1-w_{n+1}(v)\right) w_{i+1, j}+ \\
+\left(1-w_{n+1}(u)\right) w_{n+1}(v) w_{i, j+1}+ \\
w_{n+1}(u) w_{n+1}(v) w_{i+1, j+1}
\end{gathered}
$$

for all $i \in\{0,1, \ldots, k-1\}, j \in\{0,1, \ldots, l-1\}$. By transition to homogeneous coordinates and using Grassmann algebra of weighted points, see Goldman [9], it can be proven that the constructed rational spline surfaces are $C^{n}$ continuous. It should be noted that rational spline surface provides more opportunities for modification of its shape by changing weights of knot points.

## 6 Conclusions

The approach to construction of $C^{n}$ continuous interpolating spline surfaces by means of blending boundaries of the surface patches is introduced. The considered spline surfaces are constructed locally over bivariate networks of curves. This approach ensures local control of the interpolating surface shape. If the surface frame is determined by means of Bézier curves then patches of the interpolating spline surface are represented by Bézier surfaces. General properties of the interpolating surface shape are considered. The proposed approach can be used for sketching and fast prototyping of spline surfaces in geometric design. Besides local control of the constructed interpolating surfaces makes the approach useful in on-line geometric applications.

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A. P. Pobegailo

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Department of Applied Mathematics
and Computer Science
Belarus State University
pr. Nezavisimosty 4
Minsk, Belarus
E-mail: pobegailo@bsu.by

# A Note on the Equivalence of Control Systems on Lie Groups 

Rory Biggs, Claudiu C. Remsing


#### Abstract

We consider state space equivalence and (a specialization of) feedback equivalence in the context of left-invariant control affine systems. Simple algebraic characterizations of both local and global forms of these equivalence relations are obtained. Several illustrative examples regarding the classification of systems on lowdimensional Lie groups are discussed in some detail.


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## 1 Introduction

Invariant control systems on (real, finite dimensional) Lie groups have been a topic of interest in mathematical control theory since the early 1970's (see, e.g., $[22,27,28,34]$ ). These systems form a natural framework for various (variational) problems in mathematical physics, mechanics, elasticity, and dynamical systems (see, e.g., $[3,20,27,32]$ ).

In order to understand the local geometry of control systems, one needs to introduce some natural equivalence relations. The most natural equivalence relation is equivalence up to coordinate changes in the state space (viz. state space equivalence). Another weaker equivalence relation often considered is feedback equivalence; here state-dependent transformations of the controls are also allowed (see, e.g., $[26,33]$ ).

In this note we consider state space equivalence and feedback equivalence in the context of left-invariant control affine systems. We adapt Krener's (general) characterization of local state space equivalence [30] to this context. A global analogue is also obtained. Two examples pertaining to classification of systems on the Euclidean group SE(2) and pseudo-orthogonal group SO $(2,1)_{0}$ are provided. We specialize feedback equivalence in the context of left-invariant control affine systems by restricting to transformations compatible with the Lie group structure. This is called detached feedback equivalence. Characterizations of local (resp. global) detached feedback equivalence are obtained in terms of Lie algebra (resp. Lie group) isomorphisms. Further three examples pertaining to the classification of systems on low-dimensional Lie groups (namely SE (2), SO $(2,1)_{0}$ and the oscillator group) are provided. Some remarks conclude the paper. A detailed treatment of these equivalence relations can be found in [18].
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## 2 Invariant control systems

An $\ell$-input left-invariant control affine system $\Sigma=(\mathrm{G}, \Xi)$ takes the form

$$
\dot{g}=\Xi(g, u)=g\left(A+u_{1} B_{1}+\cdots+u_{\ell} B_{\ell}\right), \quad g \in \mathrm{G}, u \in \mathbb{R}^{\ell} .
$$

Here the state space $G$ is a connected Lie group with Lie algebra $\mathfrak{g}$, and $A, B_{1}, \ldots, B_{\ell} \in \mathfrak{g}$. For the sake of simplicity we shall assume that $\mathrm{G} \subseteq \operatorname{GL}(n, \mathbb{R})$ is a matrix Lie group. The dynamics $\Xi: G \times \mathbb{R}^{\ell} \rightarrow T \mathrm{G}$ are invariant under left translations, i.e., $\Xi(g, u)=g \Xi(\mathbf{1}, u)$ for all $g \in \mathrm{G}, u \in \mathbb{R}^{\ell}$. The parametrization map

$$
\Xi(\mathbf{1}, \cdot): \mathbb{R}^{\ell} \rightarrow \mathfrak{g}, \quad u \mapsto A+u_{1} B_{1}+\cdots+u_{\ell} B_{\ell}
$$

is assumed to be injective (i.e., $B_{1}, \ldots, B_{\ell}$ are linearly independent). The trace $\Gamma=\operatorname{im} \Xi(\mathbf{1}, \cdot) \subset \mathfrak{g}$ of the system is the affine subspace

$$
A+\Gamma^{0}=A+\left\langle B_{1}, \ldots, B_{\ell}\right\rangle
$$

A system is called homogeneous if $A \in \Gamma^{0}$ and inhomogeneous otherwise; a system has full rank if its trace $\Gamma$ generates the whole Lie algebra $\mathfrak{g}$. When $G$ is fixed, we shall specify a system $\Sigma$ by simply writing

$$
\Sigma: A+u_{1} B_{1}+\cdots+u_{\ell} B_{\ell}
$$

Remark 1. Any controllable system has full rank. On the other hand, any fullrank homogeneous system is controllable. Likewise, full-rank systems evolving on certain Lie groups, such as compact groups and Euclidean groups, are known to be controllable.

## 3 State Space Equivalence

Let $\Sigma=(\mathrm{G}, \Xi)$ and $\Sigma=\left(\mathrm{G}^{\prime}, \Xi^{\prime}\right)$ be two left-invariant control affine systems with the same input space $\mathbb{R}^{\ell}$. The systems $\Sigma$ and $\Sigma^{\prime}$ are locally state space equivalent (shortly $S_{\text {loc }}$-equivalent) if there exists a diffeomorphism $\phi: N \subseteq \mathrm{G} \rightarrow N^{\prime} \subseteq \mathrm{G}^{\prime}$ such that $T_{g} \phi \cdot \Xi(g, u)=\Xi(\phi(g), u)$ for all $g \in \mathrm{G}$ and $u \in \mathbb{R}^{\ell}$. Here $N$ and $N^{\prime}$ are some neighbourhoods of the identity elements $1 \in G$ and $1^{\prime} \in \mathrm{G}^{\prime}$, respectively, and it is assumed that $\phi(\mathbf{1})=\mathbf{1}^{\prime} . \Sigma$ and $\Sigma^{\prime}$ are state space equivalent (shortly $S$-equivalent) if this happens globally (i.e., $N=\mathrm{G}$ and $N^{\prime}=\mathrm{G}^{\prime}$ ).
Remark 2. The assumption $\phi(\mathbf{1})=\mathbf{1}^{\prime}$ can always be met by composing $\phi$ with some appropriate left translations.

Krener's result [30] states that full-rank systems $\Sigma$ and $\Sigma^{\prime}$ are $S_{\text {loc }}$-equivalent if and only if there exists a linear isomorphism $\psi: T_{1} \mathrm{G} \rightarrow T_{1} \mathrm{G}^{\prime}$ such that the equality

$$
\psi\left[\cdots\left[\Xi_{u_{1}}, \Xi_{u_{2}}\right], \ldots, \Xi_{u_{k}}\right](\mathbf{1})=\left[\cdots\left[\Xi_{u_{1}}^{\prime}, \Xi_{u_{2}}^{\prime}\right], \ldots, \Xi_{u_{k}}^{\prime}\right](\mathbf{1})
$$

holds for any $k \geq 1$ and any $u_{1}, \ldots, u_{k} \in \mathbb{R}^{\ell}$. Here $\Xi_{u}$ is the vector field specified by $\Xi_{u}(g)=\Xi(g, u)$. Hence in the context of left-invariant systems we have the following characterization.

Theorem 1. Two full-rank systems $\Sigma$ and $\Sigma^{\prime}$ are $S_{\text {loc }}$-equivalent if and only if there exists a Lie algebra isomorphism $\psi: \mathfrak{g} \rightarrow \mathfrak{g}^{\prime}$ such that $\psi \cdot \Xi(\mathbf{1}, u)=\Xi^{\prime}(\mathbf{1}, u)$ for every $u \in \mathbb{R}^{\ell}$.
Remark 3. If full-rank systems $\Sigma$ and $\Sigma^{\prime}$ are $S_{l o c}$-equivalent and G and $\mathrm{G}^{\prime}$ are simply connected, then $\Sigma$ and $\Sigma^{\prime}$ are $S$-equivalent.

On the other hand, we have the following global analogue of this result.
Theorem 2. Two full-rank systems $\Sigma$ and $\Sigma^{\prime}$ are $S$-equivalent if and only if there exists a Lie group isomorphism $\phi: \mathrm{G} \rightarrow \mathrm{G}^{\prime}$ such that $T_{\mathbf{1}} \phi \cdot \Xi(\mathbf{1}, u)=\Xi^{\prime}(\mathbf{1}, u)$ for every $u \in \mathbb{R}^{\ell}$.
Proof. Suppose $\Sigma$ and $\Sigma^{\prime}$ are $S$-equivalent. Then there exists a diffeomorphism $\phi: \mathrm{G} \rightarrow \mathrm{G}^{\prime}$ such that $\phi_{*} \Xi_{u}=\Xi_{u}^{\prime}$. Clearly $\phi$ satisfies $T_{\mathbf{1}} \phi \cdot \Xi(\mathbf{1}, u)=\Xi^{\prime}(\mathbf{1}, u)$. We have $\phi_{*}\left[\Xi_{u}, \Xi_{v}\right]=\left[\phi_{*} \Xi_{u}, \phi_{*} \Xi_{v}\right]=\left[\Xi_{u}^{\prime}, \Xi_{v}^{\prime}\right]$. As $\Sigma$ has full rank, it follows that $\phi$ preserves left-invariant vector fields and so $\phi$ is a Lie group isomorphism (see, e.g., [7]). Conversely, suppose $\phi: \mathrm{G} \rightarrow \mathrm{G}^{\prime}$ is a Lie group isomorphism such that $T_{\mathbf{1}} \phi \cdot \Xi(\mathbf{1}, u)=\Xi^{\prime}\left(\mathbf{1}^{\prime}, u\right)$. By left invariance and as $\phi$ is an isomorphism we have that $T_{g} \phi \cdot \Xi(g, u)=T_{g} \phi \cdot g \Xi(\mathbf{1}, u)=T_{\mathbf{1}} L_{\phi(g)} \cdot T_{\mathbf{1}} \phi \cdot \Xi(\mathbf{1}, u)=\phi(g) \Xi^{\prime}(\mathbf{1}, u)=\Xi^{\prime}(\phi(g), u)$.

We conclude the section with some specific examples on the classification, under local state space equivalence, of systems on some three-dimensional Lie groups.
Example 1 ( see [1]). The Euclidean group

$$
\mathrm{SE}(2)=\left\{\left[\begin{array}{ccc}
1 & 0 & 0 \\
x & \cos z & -\sin z \\
y & \sin z & \cos z
\end{array}\right]: x, y, z \in \mathbb{R}\right\}
$$

has Lie algebra $\mathfrak{s e}(2)$ given by

$$
\left\{\left[\begin{array}{ccc}
0 & 0 & 0 \\
x & 0 & -z \\
y & z & 0
\end{array}\right]=x E_{1}+y E_{2}+z E_{3}: x, y, z \in \mathbb{R}\right\}
$$

The nonzero commutator relations for the ordered basis $\left(E_{1}, E_{2}, E_{3}\right)$ are $\left[E_{2}, E_{3}\right]=$ $E_{1}$ and $\left[E_{3}, E_{1}\right]=E_{2}$.

Any two-input inhomogeneous full-rank system on $\mathrm{SE}(2)$ is $S_{\text {loc-equivalent to }}$ exactly one of the following full-rank systems:

$$
\begin{aligned}
& \Sigma_{1, \alpha \beta \gamma}^{(2,1)}: \alpha E_{3}+u_{1}\left(E_{1}+\gamma_{1} E_{2}\right)+u_{2}\left(\beta E_{2}\right) \\
& \Sigma_{2, \alpha \beta \gamma}^{(2,1)}: \beta E_{1}+\gamma_{1} E_{2}+\gamma_{2} E_{3}+u_{1}\left(\alpha E_{3}\right)+u_{2} E_{2} \\
& \Sigma_{3, \alpha \beta \gamma}^{(2,1)}: \beta E_{1}+\gamma_{1} E_{2}+\gamma_{2} E_{3}+u_{1}\left(E_{2}+\gamma_{3} E_{3}\right)+u_{2}\left(\alpha E_{3}\right)
\end{aligned}
$$

or, in matrix form
$\Sigma_{1, \alpha \beta \gamma}^{(2,1)}:\left[\begin{array}{c|cc}0 & 1 & 0 \\ 0 & \gamma_{1} & \beta \\ \alpha & 0 & 0\end{array}\right], \quad \Sigma_{2, \alpha \beta \gamma}^{(2,1)}:\left[\begin{array}{c|cc}\beta & 0 & 0 \\ \gamma_{1} & 0 & 1 \\ \gamma_{2} & \alpha & 0\end{array}\right], \quad \Sigma_{3, \alpha \beta \gamma}^{(2,1)}:\left[\begin{array}{c|cc}\beta & 0 & 0 \\ \gamma_{1} & 1 & 0 \\ \gamma_{2} & \gamma_{3} & \alpha\end{array}\right]$.

Here $\alpha>0, \beta \neq 0$ and $\gamma_{1}, \gamma_{2}, \gamma_{3} \in \mathbb{R}$ parametrize families of class representatives, each different values yielding distinct (non-equivalent) class representatives.

The group of automorphisms Aut(se (2)) is given by

$$
\left\{\left[\begin{array}{ccc}
x & y & v \\
-\varsigma y & \varsigma x & w \\
0 & 0 & \varsigma
\end{array}\right]: \begin{array}{c}
x, y, v, w \in \mathbb{R}, \\
x^{2}+y^{2} \neq 0, \varsigma= \pm 1
\end{array}\right\} .
$$

Let $\Sigma: \sum a_{i} E_{i}+u_{1} \sum b_{i} E_{i}+u_{2} \sum c_{i} E_{i}$ be a two-input inhomogeneous full-rank system; in matrix form

$$
\Sigma:\left[\begin{array}{l|ll}
a_{1} & b_{1} & c_{1} \\
a_{2} & b_{2} & c_{2} \\
a_{3} & b_{3} & c_{3}
\end{array}\right] .
$$

It straightforward to show that there exists an automorphism $\psi \in \operatorname{Aut}(\mathfrak{s e}(2))$ such that

$$
\begin{aligned}
& \psi \cdot\left[\begin{array}{l|ll}
a_{1} & b_{1} & c_{1} \\
a_{2} & b_{2} & c_{2} \\
a_{3} & b_{3} & c_{3}
\end{array}\right]=\left[\begin{array}{c|cc}
0 & 1 & 0 \\
0 & \gamma_{1} & \beta \\
\alpha & 0 & 0
\end{array}\right] \quad \text { if } b_{3}=0 \text { and } c_{3}=0 \\
& \psi \cdot\left[\begin{array}{l|ll}
a_{1} & b_{1} & c_{1} \\
a_{2} & b_{2} & c_{2} \\
a_{3} & b_{3} & c_{3}
\end{array}\right]=\left[\begin{array}{c|cc}
\beta & 0 & 0 \\
\gamma_{1} & 0 & 1 \\
\gamma_{2} & \alpha & 0
\end{array}\right] \quad \text { if } b_{3} \neq 0 \text { and } c_{3}=0 \\
& \psi \cdot\left[\begin{array}{l|ll}
a_{1} & b_{1} & c_{1} \\
a_{2} & b_{2} & c_{2} \\
a_{3} & b_{3} & c_{3}
\end{array}\right]=\left[\begin{array}{c|cc}
\beta & 0 & 0 \\
\gamma_{1} & 1 & 0 \\
\gamma_{2} & \gamma_{3} & \alpha
\end{array}\right] \quad \text { if } c_{3} \neq 0 .
\end{aligned}
$$

Thus $\Sigma$ is $S_{\text {loc }}$-equivalent to $\Sigma_{1, \alpha \beta \gamma}, \Sigma_{2, \alpha \beta \gamma}$, or $\Sigma_{3, \alpha \beta \gamma}$. It is a simple matter to verify that no two of the class representatives are equivalent.

Example 2 (see [19]). The pseudo-orthogonal group

$$
\mathrm{SO}(2,1)=\left\{g \in \mathbb{R}^{3 \times 3}: g^{\top} J g=g, \operatorname{det} g=1\right\}
$$

is a three-dimensional simple Lie group. Here $J=\operatorname{diag}(1,1,-1)$. The identity component of $\mathrm{SO}(2,1)$ is $\mathrm{SO}(2,1)_{0}=\left\{g \in \mathrm{SO}(2,1): g_{33}>0\right\}$. Its Lie algebra $\mathfrak{s o}(2,1)$ is given by

$$
\left\{\left[\begin{array}{ccc}
0 & z & y \\
-z & 0 & x \\
y & x & 0
\end{array}\right]=x E_{1}+y E_{2}+z E_{3}: x, y, z \in \mathbb{R}\right\}
$$

and has commutator relations $\left[E_{2}, E_{3}\right]=E_{1},\left[E_{3}, E_{1}\right]=E_{2}$, and $\left[E_{1}, E_{2}\right]=-E_{3}$.

Any two-input homogeneous full-rank system on $\mathrm{SO}(2,1)_{0}$ is $S_{\text {loc }}$-equivalent to exactly one of the following full-rank systems (displayed in matrix form):

$$
\left.\left.\begin{array}{c}
\Sigma_{1, \boldsymbol{\alpha} \gamma}^{(2,0)}:\left[\begin{array}{c|cc}
\gamma_{3} & \alpha_{2} & 0 \\
0 & 0 & 0 \\
\gamma_{2} & \gamma_{1} & \alpha_{1}
\end{array}\right], \quad \Sigma_{2, \beta \gamma}^{(2,0)}:\left[\begin{array}{c|cc}
\gamma_{3} & \beta+\gamma_{1} & 1 \\
0 & 0 & 0 \\
\gamma_{2} & \gamma_{1} & 1
\end{array}\right], \\
\Sigma_{3, \boldsymbol{\alpha} \beta \boldsymbol{\gamma}}^{(2,0)}: \\
\left(\beta-\frac{1}{4}\right) \gamma_{2} \\
\gamma_{3} \\
\left(\beta+\frac{1}{4}\right.
\end{array} \gamma_{2} \right\rvert\, \begin{array}{cc}
\left(\beta-\frac{1}{4}\right. & 0
\end{array}\right] .
$$

Here $\alpha_{i}>0, \beta \neq 0$ and $\gamma_{i} \in \mathbb{R}$ parametrize families of class representatives, each different values yielding distinct (non-equivalent) class representatives.

The group $\operatorname{Aut}(\mathfrak{s o}(2,1))$ of automorphisms of $\mathfrak{s o}(2,1)$ is exactly $S O(2,1)$. The (Lorentzian) product $\odot$ on $\mathfrak{s o}(2,1)$ is given by $A \odot B=a_{1} b_{1}+a_{2} b_{2}-a_{3} b_{3}$; here $A=\sum a_{i} E_{i}$ and $B=\sum b_{i} E_{i}$. Any automorphism $\psi$ preserves $\odot$, i.e., $(\psi \cdot A) \odot(\psi \cdot B)=A \odot B$. Furthermore, the group Aut $(\mathfrak{s o}(2,1))$ acts transitively on each of the hyperboloids (and punctured cone) $\mathcal{H}_{\alpha}=\{A \in \mathfrak{s o}(2,1): A \odot B=$ $\alpha, A \neq 0\}$. Hence for every $A \in \mathfrak{s o}(2,1)$, there exists $\psi \in \operatorname{Aut}(\mathfrak{s o}(2,1))$ such that $\psi \cdot A$ equals $\alpha E_{2}, \alpha E_{3}$, or $E_{1}+E_{3}$. The subgroup of automorphisms fixing these elements are $\left\{\exp \left(t E_{2}\right), \varsigma \circ \exp \left(t E_{2}\right): t \in \mathbb{R}\right\}$, where

$$
\varsigma=\left[\begin{array}{ccc}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{array}\right]
$$

$\left\{\exp \left(t E_{3}\right): t \in \mathbb{R}\right\}$, and $\left\{\exp \left(t\left(E_{1}+E_{3}\right)\right): t \in \mathbb{R}\right\}$, respectively. Moreover, any automorphism fixing at least two of $E_{1}, E_{2}, E_{3}$, and $E_{1}+E_{3}$ is the identity automorphism.

Suppose $\Sigma: A+u_{1} B_{1}+u_{2} B_{2}$ is a two-input homogeneous full-rank system on SO $(2,1)_{0}$. Then there exists an automorphism $\psi \in \operatorname{Aut}(\mathfrak{s o}(2,1))$ such that $\psi \cdot B_{2}$ equals $\alpha E_{2}, \alpha E_{3}$ or $E_{1}+E_{3}$. Hence $\Sigma$ is equivalent to $\Sigma^{\prime}: A^{\prime}+u_{1} B_{1}^{\prime}+u_{2}\left(\alpha E_{3}\right)$, $\Sigma^{\prime}: A^{\prime}+u_{1} B_{1}^{\prime}+u_{2}\left(E_{1}+E_{3}\right)$, or $\Sigma^{\prime}: A^{\prime}+u_{1} B_{1}^{\prime}+u_{2}\left(\alpha E_{2}\right)$. In each case we then further reduce the system by considering the action of the subgroup of automorphisms fixing $E_{3}, E_{1}+E_{3}$, or $E_{2}$, respectively, on the system.

## 4 Detached Feedback Equivalence

Two systems $\Sigma$ and $\Sigma^{\prime}$ are (globally) feedback equivalent if there exists a diffeomorphism $\Phi: \mathrm{G} \times \mathbb{R}^{\ell} \rightarrow \mathrm{G}^{\prime} \times \mathbb{R}^{\ell^{\prime}},(g, u) \mapsto(\phi(g), \varphi(g, u))$ transforming the first system into the second, i.e., $T_{g} \phi \cdot \Xi(g, u)=\Xi^{\prime}(\phi(g), \varphi(g, u))$. We specialize feedback equivalence, by requiring that the transformation $u^{\prime}=\varphi(g, u)$ is constant over the state space; such transformations are exactly those that are compatible with the Lie group structure (cf. [7]). More precisely, $\Sigma$ and $\Sigma^{\prime}$ are called locally detached feedback equivalent (shortly $D F_{\text {loc }}$-equivalent) if there exist diffeomorphims $\phi: N \subseteq \mathrm{G} \rightarrow N^{\prime} \subseteq \mathrm{G}^{\prime}$ and $\varphi: \mathbb{R}^{\ell} \rightarrow \mathbb{R}^{\ell^{\prime}}$ such that $T_{g} \phi \cdot \Xi(g, u)=\Xi^{\prime}(\phi(g), \varphi(u))$ for
$g \in N, u \in \mathbb{R}^{\ell}$. Here $N$ and $N^{\prime}$ are some neighbourhoods of the identity elements $\mathbf{1} \in G$ and $\mathbf{1}^{\prime} \in \mathrm{G}^{\prime}$ and it is assumed that $\phi(\mathbf{1})=\mathbf{1}^{\prime}$. On the other hand, $\Sigma$ and $\Sigma^{\prime}$ are called detached feedback equivalent (shortly $D F$-equivalent) if this happens globally (i.e., $N=\mathrm{G}$ and $N^{\prime}=\mathrm{G}^{\prime}$ ).
Theorem 3. Two full-rank systems $\Sigma$ and $\Sigma^{\prime}$ are $D F_{\text {loc-equivalent }}$ if and only if there exists a Lie algebra isomorphism $\psi: \mathfrak{g} \rightarrow \mathfrak{g}^{\prime}$ such that $\psi \cdot \Gamma=\Gamma^{\prime}$.
Proof. Suppose $\Sigma$ and $\Sigma^{\prime}$ are $D F_{l o c}$-equivalent, i.e., there exist diffeomorphisms $\phi: N \subseteq G \rightarrow N^{\prime} \subseteq \mathrm{G}^{\prime}$ and $\varphi: \mathbb{R}^{\ell} \rightarrow \mathbb{R}^{\ell^{\prime}}$ such that $T_{g} \phi \cdot \Xi(g, u)=$ $\Xi^{\prime}(\phi(g), \varphi(u))$. Then $T_{\mathbf{1}} \phi \cdot \Xi(\mathbf{1}, u)=\Xi^{\prime}\left(\mathbf{1}^{\prime}, \varphi(u)\right)$ and so $T_{\mathbf{1}} \phi \cdot \Gamma=\Gamma^{\prime}$. It remains to be shown that $T_{\mathbf{1}} \phi$ preserves the Lie bracket. We have that $\phi_{*}\left[\Xi_{u}, \Xi_{v}\right]=$ $\left[\phi_{*} \Xi_{u}, \phi_{*} \Xi_{v}\right]$ for left-invariant vector fields $\Xi_{u}=\Xi(\cdot, u)$ and $\Xi_{v}=\Xi(\cdot, v)$. Hence, $T_{\mathbf{1}} \phi \cdot\left[\Xi_{u}(\mathbf{1}), \Xi_{v}(\mathbf{1})\right]=\left[\Xi_{\varphi(u)}^{\prime}\left(\mathbf{1}^{\prime}\right), \Xi_{\varphi(v)}^{\prime}\left(\mathbf{1}^{\prime}\right)\right]=\left[T_{\mathbf{1}} \phi \cdot \Xi_{u}(\mathbf{1}), T_{\mathbf{1}} \phi \cdot \Xi_{v}(\mathbf{1})\right]$. Likewise $T_{\mathbf{1}} \phi \cdot\left[\Xi_{u}(\mathbf{1}),\left[\Xi_{u}(\mathbf{1}), \Xi_{w}(\mathbf{1})\right]\right]=\left[T_{\mathbf{1}} \phi \cdot \Xi_{u}(\mathbf{1}), T_{\mathbf{1}} \phi \cdot\left[\Xi_{v}(\mathbf{1}), \Xi_{w}(\mathbf{1})\right]\right]$ and similarly for higher order commutators. As the elements $\Xi_{u}(\mathbf{1}), u \in \mathbb{R}^{\ell}$ generate $\mathfrak{g}$, it follows that $T_{1} \phi$ is a Lie algebra isomorphism.

Conversely, suppose $\psi$ is a Lie algebra isomorphism such that $\psi \cdot \Gamma=\Gamma^{\prime}$. Then there exist neighbourhoods $N$ and $N^{\prime}$ of $\mathbf{1}$ and $\mathbf{1}^{\prime}$, respectively, and a local group isomorphism $\phi: N \rightarrow N^{\prime}$ such that $T_{1} \phi=\psi$ (see, e.g., [29]). Also, there exists a unique affine isomorphism $\varphi: \mathbb{R}^{\ell} \rightarrow \mathbb{R}^{\ell^{\prime}}$ such that $\psi \cdot \Xi(\mathbf{1}, u)=\Xi^{\prime}(\mathbf{1}, \varphi(u))$. Therefore, (locally) we get $T_{g} \phi \cdot \Xi(g, u)=T_{\mathbf{1}} L_{\phi(g)} \cdot \psi \cdot \Xi(\mathbf{1}, u)=T_{1} L_{\phi(g)} \cdot \Xi^{\prime}\left(\mathbf{1}^{\prime}, \varphi(u)\right)=$ $\Xi^{\prime}(\phi(g), \varphi(u))$. Hence $\Sigma$ and $\Sigma^{\prime}$ are $D F_{l o c}$-equivalent.

The global analogue of the characterization for detached feedback equivalence follows similarly (and so the proof is omitted).

Theorem 4. Two full-rank systems $\Sigma$ and $\Sigma^{\prime}$ are $D F$-equivalent if and only if there exists a Lie group isomorphism $\phi: \mathrm{G} \rightarrow \mathrm{G}^{\prime}$ such that $T_{\mathbf{1}} \phi \cdot \Gamma=\Gamma^{\prime}$.

We conclude the section with some specific examples on the classification, under local detached feedback equivalence, of systems on some low-dimensional Lie groups.
Example 3 ( see [12]). Any two-input inhomogeneous full-rank system on the Euclidean group $\mathrm{SE}(2)$ is $D F_{\text {loc }}$-equivalent to exactly one of the following full-rank systems:

$$
\begin{aligned}
\Sigma_{1} & : E_{1}+u_{1} E_{2}+u_{2} E_{3}, \\
\Sigma_{2, \alpha} & : \alpha E_{3}+u_{1} E_{1}+u_{2} E_{2} .
\end{aligned}
$$

Here $\alpha>0$ parametrizes a family of class representatives, each different value corresponding to a distinct non-equivalent representative.

Let $\Sigma$ be an inhomogeneous system with trace $\Gamma=\sum a_{i} E_{i}+\left\langle\sum b_{i} E_{i}, \sum c_{i} E_{i}\right\rangle$. If $E^{3}\left(\Gamma^{0}\right) \neq\{0\}$, then $\Gamma=a_{1}^{\prime} E_{1}+a_{2}^{\prime} E_{2}+\left\langle b_{1}^{\prime} E_{1}+b_{2}^{\prime} E_{2}, c_{1}^{\prime} E_{1}+c_{2}^{\prime} E_{2}+E_{3}\right\rangle$. (Here $E^{3}$ denotes the corresponding element of the dual basis.) As $\left(b_{1}^{\prime}\right)^{2}+\left(b_{2}^{\prime}\right)^{2} \neq 0$, the equation

$$
\left[\begin{array}{cc}
b_{2}^{\prime} & -b_{1}^{\prime} \\
b_{1}^{\prime} & b_{2}^{\prime}
\end{array}\right]\left[\begin{array}{l}
v_{1} \\
v_{2}
\end{array}\right]=\left[\begin{array}{l}
a_{2}^{\prime} \\
a_{1}^{\prime}
\end{array}\right]
$$

has a unique solution (with $v_{2} \neq 0$ ). Therefore

$$
\psi=\left[\begin{array}{ccc}
v_{2} b_{2}^{\prime} & v_{2} b_{1}^{\prime} & c_{1}^{\prime} \\
-v_{2} b_{1}^{\prime} & v_{2} b_{2}^{\prime} & c_{2}^{\prime} \\
0 & 0 & 1
\end{array}\right]
$$

is an automorphism such that $\psi \cdot \Gamma_{1}=\psi \cdot\left(E_{1}+\left\langle E_{2}, E_{3}\right\rangle\right)=\Gamma$. Thus $\Sigma$ is $D F_{l o c}$-equivalent to $\Sigma_{1}$. On the other hand, suppose $E^{3}\left(\Gamma^{0}\right)=\{0\}$. Then $\Gamma=a_{3} E_{3}+\left\langle E_{1}, E_{2}\right\rangle$. Hence $\psi=\operatorname{diag}\left(1,1, \operatorname{sgn}\left(a_{3}\right)\right)$ is an automorphism such that $\psi \cdot \Gamma=\alpha E_{3}+\left\langle E_{1}, E_{2}\right\rangle$ with $\alpha>0$. Thus $\Sigma$ is $D F_{\text {loc }}$-equivalent to $\Sigma_{2, \alpha}$. As the subspace $\left\langle E_{1}, E_{2}\right\rangle$ is invariant (under automorphisms), $\Sigma_{1}$ and $\Sigma_{2, \alpha}$ cannot be $D F_{l o c}$-equivalent. It is easy to show that $\Sigma_{2, \alpha}$ and $\Sigma_{2, \alpha^{\prime}}$ are $D F_{l o c}$-equivalent only if $\alpha=\alpha^{\prime}$.

Example 4 (see [10]). Any two-input homogeneous full-rank system on the pseudoorthogonal group $\mathrm{SO}(2,1)$ is $D F_{\text {loc }}$-equivalent to exactly one of the following fullrank systems:

$$
\begin{aligned}
& \Sigma_{1}: u_{1} E_{1}+u_{2} E_{2}, \\
& \Sigma_{2}: u_{1} E_{2}+u_{2} E_{3} .
\end{aligned}
$$

Let $\Sigma$ be a two-input homogeneous full-rank system with trace $\Gamma=\langle A, B\rangle$. The sign $\sigma(\Gamma)$ of $\Gamma$ is given by

$$
\sigma(\Gamma)=\operatorname{sgn}\left(\left|\begin{array}{ll}
A \odot A & A \odot B \\
A \odot B & B \odot B
\end{array}\right|\right) .
$$

(It is easy to show that $\sigma(\Gamma)$ does not depend on the parametrization.) As $\odot$ is preserved by automorphisms, it follows that $\sigma(\psi \cdot \Gamma)=\sigma(\Gamma)$. A straightforward computation shows that if $\sigma(\Gamma)=0$, then $\Sigma$ does not have full rank.

Suppose $\sigma(\Gamma)=-1$. Then we may assume that $a_{3} \neq 0$. Hence $\Gamma=$ $\left\langle a_{1}^{\prime} E_{1}+a_{2}^{\prime} E_{2}+E_{3}, r \sin \theta E_{1}+r \cos \theta E_{2}\right\rangle$. Thus

$$
\psi=\left[\begin{array}{ccc}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{array}\right]
$$

is an automorphism such that $\psi \cdot \Gamma=\left\langle a_{1}^{\prime \prime} E_{1}+E_{3}, E_{2}\right\rangle$. Now, as $\sigma(\psi \cdot \Gamma)=-1$, we have $\left(a_{1}^{\prime \prime}\right)^{2}-1<0$ and so $\psi \cdot \Gamma=\left\langle\sinh \vartheta E_{1}+\cosh \vartheta E_{3}, E_{2}\right\rangle$. Therefore

$$
\psi^{\prime}=\left[\begin{array}{ccc}
\cosh \vartheta & 0 & -\sinh \vartheta \\
0 & 1 & 0 \\
-\sinh \vartheta & 0 & \cosh \vartheta
\end{array}\right]
$$

is an automorphism such that $\psi^{\prime} \cdot \psi \cdot \Gamma=\left\langle E_{3}, E_{2}\right\rangle$. Thus $\Sigma$ is $D F_{\text {loc }}$-equivalent to $\Sigma_{1}$. If $\sigma(\Gamma)=1$, then a similar argument shows that there exists an automorphism $\psi$ such that $\psi \cdot \Gamma=\left\langle E_{1}, E_{2}\right\rangle$ (and so $\Sigma$ is $D F_{l o c}$-equivalent to $\Sigma_{2}$ ). Lastly, as $\sigma\left(\Gamma_{1}\right)=1$ and $\sigma\left(\Gamma_{2}\right)=-1$, it follows that $\Sigma_{1}$ and $\Sigma_{2}$ are not equivalent.

Example 5 (see [15]). The (four-dimensional) oscillator Lie group has parametrization

$$
\text { Osc : }\left[\begin{array}{cccc}
1 & -y \cos \theta-z \sin \theta & z \cos \theta-y \sin \theta & -2 x \\
0 & \cos \theta & \sin \theta & z \\
0 & -\sin \theta & \cos \theta & y \\
0 & 0 & 0 & 1
\end{array}\right]
$$

where $x, y, z, \theta \in \mathbb{R}$. Its Lie algebra likewise has parametrization

$$
\mathfrak{o s c}:\left[\begin{array}{cccc}
0 & -y & z & -2 x \\
0 & 0 & \theta & z \\
0 & -\theta & 0 & y \\
0 & 0 & 0 & 0
\end{array}\right]=x E_{1}+y E_{2}+z E_{3}+\theta E_{4}
$$

where $x, y, z, \theta \in \mathbb{R}$. The nonzero commutator relations are $\left[E_{2}, E_{3}\right]=E_{1}$, $\left[E_{2}, E_{4}\right]=-E_{3}$, and $\left[E_{3}, E_{4}\right]=E_{2}$. Osc decomposes as a semidirect product $\mathrm{H}_{3} \rtimes \mathrm{SO}$ (2) of the Heisenberg group $\mathrm{H}_{3}$ and orthogonal group SO (2); furthermore, it is a nontrivial central extension of the Euclidean group SE (2) ([25]). The oscillator group was first studied by Streater [35]; it is associated with the harmonic oscillator problem, from whence it gets its name. This group (and its higher dimensional analogues) have been studied by several authors in both differential geometry and mathematical physics (see, e.g., $[21,23,24,31]$ ).

Any homogeneous full-rank system on Osc is $D F_{l o c}$-equivalent to exactly one of the following full rank systems:

$$
\begin{array}{ll}
\Sigma^{(2,0)} & : u_{1} E_{2}+u_{2} E_{4} \\
\Sigma_{1}^{(3,0)} & : u_{1} E_{1}+u_{2} E_{2}+u_{3} E_{4} \\
\Sigma_{2}^{(3,0)} & : u_{1} E_{2}+u_{2} E_{3}+u_{3} E_{4} \\
\Sigma^{(4,0)} & : u_{1} E_{1}+u_{2} E_{2}+u_{3} E_{3}+u_{4} E_{4} .
\end{array}
$$

The group of automorphisms takes the form

$$
\operatorname{Aut}(\mathfrak{o s c}):\left[\begin{array}{cccc}
\sigma\left(x^{2}+y^{2}\right) & w y-\sigma v x & -w x-\sigma v y & u \\
0 & x & y & v \\
0 & -\sigma y & \sigma x & w \\
0 & 0 & 0 & \sigma
\end{array}\right]
$$

where $x, y, u, v, w \in \mathbb{R}, x^{2}+y^{2} \neq 0$, and $\sigma= \pm 1$. Clearly no single-input homogeneous system has full rank. Suppose $\Sigma$ is a two-input full-rank system with trace $\Gamma=\left\langle\sum a_{i} E_{i}, \sum b_{i} E_{i}\right\rangle$. As $\Sigma$ has full rank, it follows that $E^{4}(\Gamma) \neq\{0\}$. Hence $\Gamma=\left\langle a_{1}^{\prime} E_{1}+a_{2}^{\prime} E_{2}+a_{3}^{\prime} E_{3}+E_{4}, b_{1}^{\prime} E_{1}+b_{2}^{\prime} E_{2}+b_{3}^{\prime} E_{3}\right\rangle$. Therefore,

$$
\psi=\left[\begin{array}{cccc}
1 & a_{2}^{\prime} & a_{3}^{\prime} & -a_{1}-\left(a_{2}^{\prime}\right)^{2}-\left(a_{3}^{\prime}\right)^{2} \\
0 & 1 & 0 & -a_{2}^{\prime} \\
0 & 0 & 1 & -a_{3}^{\prime} \\
0 & 0 & 0 & 1
\end{array}\right]
$$

is an automorphism such that $\psi \cdot \Gamma=\left\langle E_{4}, b_{1}^{\prime \prime} E_{1}+r \cos \theta E_{2}+r \sin \theta E_{3}\right\rangle$ with $r>0$. (We have that $r \neq 0$ as $\Sigma$ has full rank.) Accordingly,

$$
\psi^{\prime}=\left[\begin{array}{cccc}
\frac{1}{r^{2}} & -\frac{b_{1}^{\prime \prime} \cos \theta}{r^{3}} & -\frac{b_{1}^{\prime \prime} \sin \theta}{r^{3}} & 0 \\
0 & \frac{\cos \theta}{\sin } & \frac{\sin \theta}{r} & \frac{b_{1}^{\prime \prime}}{r^{2}} \\
0 & -\frac{\sin \theta}{r} & \frac{\cos \theta}{r} & 0 \\
0 & 0 & 0 & 1
\end{array}\right]
$$

is an automorphism such that $\left(\psi^{\prime} \circ \psi\right) \cdot \Gamma=\left\langle E_{2}, \frac{b_{1}^{\prime \prime}}{r^{2}} E_{2}+E_{4}\right\rangle=\left\langle E_{2}, E_{4}\right\rangle$. Consequently $\Sigma$ is $D F_{\text {loc }}$-equivalent to $\Sigma^{(2,0)}$. The three-input case is similar, although somewhat more involved. (The four-input case is trivial.)

Remark 4. The examples discussed in this note deal only with the local case. The approach for the global case is very similar; however, one needs to first determine the subgroup $d \operatorname{Aut}(\mathrm{G})$ of $\operatorname{Aut}(\mathfrak{g})$. For $\operatorname{SE}(2)$ and $\mathrm{SO}(2,1)_{0}$ it turns out that $d \operatorname{Aut}(\mathrm{G})=\operatorname{Aut}(\mathfrak{g})$ (see, e.g., $[16,19])$. For the oscillator group, this does not hold true.

## 5 Closing Remarks

State space equivalence is a very strong equivalence relation. Hence, any general classification leads to a large number of equivalence classes and so is of little use (except perhaps in low dimensions, e.g., $[1,19]$ ). On the other hand, detached feedback equivalence is noticeably weaker, and so leads to far fewer equivalence classes. On three-dimensional Lie groups, a full classification (both local and global) of systems under detached feedback equivalence has been achieved ( $[8,10-12]$, and [16]; see also $[13,17]$ ). In the same vein, on several other low-dimensional (matrix) Lie groups, important classification results have also been obtained (cf. [2, 5, 15]).

Detached feedback equivalence has a natural extension to invariant optimal control problems (cf. [9,14]). Two optimal control problems are cost equivalent if the underlying control systems are detached feedback equivalent and the change of controls $\varphi$ is compatible with the costs. (Such a perspective was used to classify the corresponding sub-Riemannian structures on the Heisenberg groups [6]; see also [4,5].)

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## Rory Biggs

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Department of Mathematics and Applied Mathematics
University of Pretoria
0083 Hartfield, South Africa
E-mail: rory.biggs@up.ac.za
Claudiu C. Remsing
Department of Mathematics
Rhodes University
6140 Grahamstown, South Africa
E-mail: c.c.remsing@ru.ac.za

# Viscous flow through a porous medium filled by liquid with varying viscosity 

Anatoly Filippov, Yulia Koroleva


#### Abstract

The paper deals with study of a Stokes-Brinkman system with varying viscosity that describes the fluid flow along an ensemble of partially porous cylindrical particles using the cell approach. We have proved the existence and uniqueness of the solutions as well as derived some uniform estimates.


Mathematics subject classification: 76D03, 76D07, 76D10, 76S99..
Keywords and phrases: Fluid flow, Stokes system, Brinkman equation, varying viscosity.

## 1 Introduction

Pressure driven membrane processes (reverse osmosis, nano-, ultra and microfiltration), sedimentation, flows of underground water and crude oil are important examples of flow through porous media. Usually porous medium was modeled by a dense set of rigid impermeable (colloid) particles [4]. For now to achieve effective use of a porous medium in the above-mentioned areas, the structure of a porous layer should be viewed from different points of view. For example, it is not necessary that the particles always have a smooth homogeneous surface but also have a rough surface or a surface covered by a porous shell. The hydrodynamic models of colloid particle changed considerably over last decades. The latter attracts itself in terminology too: soft particles [8], i.e. particles with porous hydrodynamically permeable surface layer, draw now more attention than hard impermeable particles [4]. There has been also considerable recent interest in the use of beds of porous particles for biological applications such as perfusion chromatography for purifying proteins and other biomolecules and cell or enzyme immobilization. Therefore a number of technologies require the development of modeling of porous media. The mentioned porous media are frequently modeled as aggregates of particles and/or fibers. The cell model [4] has been very effectively used for investigation of the mentioned above flows. The basic principle of the cell model is to replace a system of randomly oriented particles by a periodic array of spheres or cylinders embedded in a center of spherical or cylindrical liquid cells. Appropriate boundary conditions on the cell boundary are supposed to take into account the influence of surrounding particles on the flow inside the cell and the force applied to the particle in the center of the cell. The four variants of these conditions are known as the Happel (the absence of tangential stresses on the cell surface), Kuwabara (the absence of vortexes - the

[^1]flow potentiality), Kvashnin (the cell symmetry), and Cunningham (the flow on the surface of cell is assumed to be uniform) models [15]. In the course of filtration processes the structure of the membrane can change due to (i) dissolution of particles, (ii) adsorption of polymers on the surfaces of the particles usually referred to as a poisoning. Both the above mentioned processes result in a formation of a porous shell (in the form of a colloidal layer or a gel layer) on the solid particles surface, which are usually hard to remove. The presence of porous shell on solid particles has a clear impact on the drag force exerted by the flow on the particles. Another situation where the slip velocity is of interest is flow over polymer brushes. Polymer chains attached to the surface of a particle create a porous shell around the particle, effectively increasing its diameter. Penetration of the outer flow into the polymer brush determines the transport of ions and other chemical species between the outer flow and the surface of the particle. Hence, the knowledge of the flow field at the interface between a highly porous medium and a liquid is of a substantial importance. Flow through porous shells is frequently modeled by Brinkman's equation [2], which is a modified form of the Darcy's equation. However, it has been observed that the results obtained based on the Brinkman's equations do not agree with the experimental data for non-homogeneous porous media. A modification of the Brinkman's equation was suggested in [14] for the media having non-homogeneous porosity. To overcome this problem it is possible also to use "variable viscosity model" for the liquid/porous boundary region. We assume below that porous shells under consideration have a uniform porosity but variable liquid viscosity inside porous layer in accordance with power or exponential law. The membranes under investigation below are supposed to be built by either non-porous particles with a rough surface or particles covered by a porous shell. The latter shells also have a rough surface, and a scale of roughness is equal or even bigger than the average pore size inside the shell. The important problem is a correct selection of boundary conditions on surfaces of non-porous but rough surfaces of particles or porous shell of particles. We use bellow the condition of "tangential stresses slippage" which is a jump of tangential stresses at the porous-liquid interface $[6,7]$. The aim of this paper is to prove the existence and uniqueness of the solutions of boundary value problems as well as derive some uniform estimates which will be useful for numerical simulations.

## 2 Statement of the problem

Describe the viscous flow through a porous medium, modeled as a set of parallel composite cylindrical particles, and filled by liquid with varying viscosity by two systems: the Stokes one

$$
\left\{\begin{array}{l}
\widetilde{\nabla} \widetilde{p}^{o}=\widetilde{\mu}^{0} \widetilde{\Delta} \widetilde{\mathbf{v}}^{0},  \tag{1}\\
\widetilde{\operatorname{div}} \widetilde{\mathbf{v}}^{o}=0
\end{array}\right.
$$



Figure 1. The flow parallel to the cylinders
outside the porous layer $\widetilde{a} \leq \widetilde{r} \leq \widetilde{b}$ and in the porous layer $\widetilde{R} \leq \widetilde{r} \leq \widetilde{a}$ by the Brinkman's system

$$
\left\{\begin{array}{l}
\widetilde{\nabla} \widetilde{p}^{i}=\widetilde{\operatorname{div}}\left(\widetilde{\mu}^{i} \widetilde{D} \widetilde{\mathbf{v}}^{i}\right)-\frac{\widetilde{\mu}^{o}}{\widetilde{k}} \widetilde{\mathbf{v}}^{i},  \tag{2}\\
\widetilde{\operatorname{div}} \widetilde{\mathbf{v}}^{i}=0
\end{array}\right.
$$

Here the tilde denotes dimensional variables, indices $o$ and $i$ refer to the external and porous zones respectively; $\widetilde{\mu}^{i}$ and $\widetilde{\mu}^{o}$ are the viscosities of the liquids inside Brinkman's layer and in liquid shell, correspondingly. The variable $\widetilde{k}$ is the specific permeability of the porous layer. We suppose that viscosity of clear liquid $\widetilde{\mu}^{o}$ is constant over region $\widetilde{a}<\tilde{r}<\tilde{b}$ and viscosity of Brinkmans liquid $\tilde{\mu}^{i}=\tilde{\mu}^{o}\left(\frac{\tilde{a}}{\tilde{r}}\right)^{\alpha}$ increases according to power law from $\tilde{\mu}^{o}$ at porous media-clear liquid interface to

$$
\tilde{\mu}^{o}\left(\frac{\tilde{a}}{\tilde{R}}\right)^{\alpha}
$$

at the interface between solid core and porous layer. Parameter $\alpha$ is needed in order to get necessary viscosity of Brinkmans liquid in the vicinity of the solid core. The unknown functions are $\widetilde{\mathbf{v}}^{o}, \widetilde{\mathbf{v}}^{i}-$ the velocity field and the pressure $\widetilde{p}^{o}, \widetilde{p}^{i}$.

Also the boundary conditions as follows are set:

$$
\begin{equation*}
\widetilde{\mathbf{v}}^{i}=0, \text { as } \widetilde{r}=\widetilde{R} \tag{3}
\end{equation*}
$$

the continuity condition:

$$
\begin{equation*}
\widetilde{\mathbf{v}}^{i}=\widetilde{\mathbf{v}}^{o}, \quad \tilde{\sigma}_{r r}^{o}=\widetilde{\sigma}_{r r}^{i}, \text { as } \widetilde{r}=\widetilde{a} \tag{4}
\end{equation*}
$$

The condition for a jump of tangential stresses at the interface between porous layer and clear liquid reads,

$$
\begin{equation*}
\widetilde{\sigma}_{r z}^{i}-\widetilde{\sigma}_{r z}^{o}=\frac{\beta \widetilde{\mu^{o}}}{\sqrt{\hat{k}}} \widetilde{v}_{z}^{o}, \quad \text { as } \widetilde{r}=\widetilde{a} \tag{5}
\end{equation*}
$$

Here $-\infty<\beta<\sqrt{\frac{\tilde{\mu}^{i}}{\tilde{\mu}^{0}}}$ is the dimensionless parameter which should be found from a physical experiment [3]. In case of flow which is parallel to the cylinders all four known conditions at the outer cell boundary are reduced to the scalar one [12]:

$$
\begin{equation*}
\frac{d \widetilde{v}_{z}^{o}}{d \widetilde{r}}=0, \text { as } \widetilde{r}=\widetilde{b} \tag{6}
\end{equation*}
$$

For the convenience of the analysis we pass to the dimensionless operators and variables by the following substitutions:

$$
\begin{align*}
& \frac{\tilde{b}}{\tilde{a}}=\frac{1}{\gamma}, r=\frac{\tilde{r}}{\tilde{a}}, z=\frac{\tilde{z}}{\tilde{a}}, \nabla=\tilde{\nabla} \cdot \tilde{a}, \Delta=\tilde{\Delta} \cdot \tilde{a}^{2}, \delta=\frac{\tilde{\delta}}{\tilde{a}}, R=\frac{\tilde{R}}{\tilde{a}}=1-\delta, \\
& \mathbf{v}=\frac{\tilde{\mathbf{v}}}{\tilde{U}} \quad p=\frac{\tilde{p}}{\tilde{p}_{0}}, \tilde{p}_{0}=\frac{\tilde{U} \cdot \tilde{\mu}^{o}}{\tilde{a}}, k=\frac{\tilde{k}}{\tilde{a}^{2}}>0, \omega=\frac{d p}{d z}, \tag{7}
\end{align*}
$$

where $\widetilde{U}$ is the cell (filtration) velocity $\tilde{U}=-\tilde{L}_{11} \frac{d \tilde{\tilde{p}}}{d \tilde{z}}$, where $\tilde{L}_{11}$ is the hydrodynamic permeability of the membrane [13].

Denote by $B_{\gamma}$ the layer

$$
B_{\gamma}=\left\{1 \leq r \leq \frac{1}{\gamma}, \varphi \in[0,2 \pi], z \in[0, \infty)\right\}
$$

and by $B_{R}$ the set

$$
B_{R}=\{R \leq r \leq 1, \varphi \in[0,2 \pi], z \in[0, \infty)\}
$$

In the dimensionless notations the systems (1) and (2) read as

$$
\left\{\begin{array}{l}
\nabla p^{o}=\mu^{o} \Delta \mathbf{v}^{o} \text { in } B_{\gamma}  \tag{8}\\
\operatorname{div}^{o}=0 \text { in } B_{\gamma}, \\
\frac{d v_{z}^{o}}{d r}=0 \text { on } r=\frac{1}{\gamma}, v_{z}^{o}=v_{z}^{i} \text { on } r=1
\end{array}\right.
$$

where $\mathbf{v}^{i}$ is given by

$$
\left\{\begin{array}{l}
\nabla p^{i}=\operatorname{div}\left(r^{-\alpha} D \mathbf{v}^{i}\right)-\frac{\mathbf{v}^{\mathbf{i}}}{k} \text { in } B_{R}  \tag{9}\\
\operatorname{div} \mathbf{v}^{i}=0 \text { in } B_{R} \\
v_{z}^{i}=0 \text { on } r=R \\
\frac{d v_{z}^{i}}{d r}=\frac{d v_{z}^{o}}{d r}+\frac{\beta}{\sqrt{k}} v_{z}^{o} \text { as } r=1
\end{array}\right.
$$

The problems (8) and (9) are linked to each other via the boundary condition $\mathbf{v}^{i}=\mathbf{v}^{o}$ on the common boundary $r=1$ which physically means the continuous flow regime.

Our goal is to investigate the qualitative properties of the obtained systems: existence and uniqueness of the solution as well as to derive some apriori estimates.

### 2.1 The flow parallel to cylinders. The case $\mu^{i}=\mu^{o} r^{-\alpha}$

Rewrite the problems (8) and (9) in cylindric coordinates ( $r, \varphi, z$ ) with help of formulas

$$
\begin{align*}
\nabla p & =\left(\frac{\partial p}{\partial r}, \frac{1}{r} \frac{\partial p}{\partial \varphi}, \frac{\partial p}{\partial z}\right) \\
\operatorname{div} v & =\left(\frac{1}{r} \frac{\partial}{\partial r}\left(r v_{r}\right)+\frac{1}{r} \frac{\partial v_{\varphi}}{\partial \varphi}+\frac{\partial v_{z}}{\partial z}\right),  \tag{10}\\
\Delta v & =\frac{1}{r} \frac{\partial}{\partial r}(r v)+\frac{1}{r^{2}} \frac{\partial^{2} v}{\partial \varphi^{2}}+\frac{\partial^{2} v}{\partial z^{2}}
\end{align*}
$$

and consider the case when the flow is parallel to the cylinders, i.e. the components of the solution satisfy $v_{r}^{i}=v_{r}^{o}=v_{\varphi}^{i}=v_{\varphi}^{o}=0$, while nonzero are $v_{z}^{i}$ as well as $v_{z}^{o}$. We show now that in such case the divergence free property of the velocity implies independence of velocity and $\frac{\partial p^{j}}{\partial z}$ on $z$-variable. Here index $j$ is $o$ or $i$. Indeed, for $j=o$ or $j=i$ the equation

$$
\operatorname{divv}^{j}=0 \Leftrightarrow\left(\frac{1}{r} \frac{\partial}{\partial r}\left(r v_{r}^{j}\right)+\frac{1}{r} \frac{\partial v_{\varphi}^{j}}{\partial \varphi}+\frac{\partial v_{z}^{j}}{\partial z}\right)=0 \Leftrightarrow \frac{\partial v_{z}^{j}}{\partial z}=0
$$

implies independence $v_{z}^{j}$ on $z$-variable. For an arbitrary $\mu^{i}(r)$ when the flow is parallel to $z$-direction, the term $\operatorname{div}\left(\mu^{i} D \mathbf{v}\right)$ becomes

$$
\frac{1}{r} \frac{d}{d r}\left(\mu^{i}\left(r \frac{d v_{z}}{d r}\right)\right)=\frac{d v_{z}}{d r}\left(\frac{d \mu^{i}}{d r}+\frac{\mu^{i}}{r}\right)+\mu^{i} \frac{d^{2} v_{z}}{d r^{2}}
$$

in the polar coordinates. Having in mind that $\frac{d p^{j}}{d z}=$ const $=\omega, j=o, i$, we arrive at the following one-dimensional Stokes and Brinkman's equations, where for the simplicity we omit the sub-index $z$ (i.e. the notation $v^{j}$ should be understood as $v_{z}^{j}$ ):

$$
\begin{gather*}
\frac{d^{2} v^{o}}{d r^{2}}+\frac{1}{r} \frac{d v^{o}}{d r}=\omega, 1<r<\frac{1}{\gamma}  \tag{11}\\
\frac{d^{2} v^{i}}{d r^{2}}-\frac{\alpha-1}{r} \frac{d v^{i}}{d r}=r^{\alpha}\left(\frac{v^{i}}{k}+\omega\right), \quad R<r<1 \tag{12}
\end{gather*}
$$

with boundary conditions

$$
\begin{align*}
& v^{i}=0 \text { as } r=R, \\
& v^{o}=v^{i} \text { as } r=1, \quad \frac{d v^{i}}{d r}-\frac{d v^{o}}{d r}=\frac{\beta}{\sqrt{k}} v^{o}, \quad r=1 .  \tag{13}\\
& \frac{d v^{o}}{d r}=0 \text { as } r=\frac{1}{\gamma} .
\end{align*}
$$

## 3 On the existence and uniqueness of the solution

### 3.1 Preliminaries

We recall some basic definitions of Sobolev spaces. The Sobolev space $H^{1}(\Omega)$ is defined as the completion of the set of functions from the space $C^{\infty}(\bar{\Omega})$ by the norm $\|u\|_{H^{1}(\Omega)}=\sqrt{\int_{\Omega}\left(u^{2}+|\nabla u|^{2}\right) d x}$; the space $H^{-1}(\Omega)$ denotes the dual space to $H^{1}$, i. e. the set of functionals defined on the elements in $H^{1}(\Omega)$. Following the traditions, we denote by $H$ the set of functions $u$ from $H^{1}(\Omega)$ such that $\operatorname{div} u=0$. Finally, $\stackrel{\circ}{L}_{2}(\Omega)$ consists of functions $u \in L_{2}$ satisfying the condition $\int_{\Omega} u d x=0$. In our analysis the following classical theorem will be used (see [1] and [5]):
Theorem 1 (Lions-Lax-Milgram Lemma). Let $U$ and $V$ be two real Hilbert spaces and let $B: U \times V \rightarrow \mathbb{R}$ be a continuous bilinear functional, where $V$ is continuously embedded in $U\left(\|u\|_{U} \leq c\|u\|_{V}\right)$. Suppose also that $B$ is coercive in the following sense: for some constant $c>0$ and all $u \in U,|B[u, u]| \geq c\|u\|_{U}^{2}$. Then, for all $f \in V^{*}$, there exists a unique solution $u=u_{f} \in U$ to the weak problem $B\left[u_{f}, v\right]=$ $\langle f, v\rangle$ for all $v \in V$. Moreover, the solution depends continuously on the given datum: $\left\|u_{f}\right\|_{U} \leq \frac{1}{c}\|f\|_{V^{*}}$.

### 3.2 The weak solution

Multiplying equations (11), (12) by $v^{o}, v^{i}$ respectively and integrating the result over the corresponding domains, we can define the weak solutions $v^{o}$ and $v^{i}$.

Definition 1. The function $v^{o} \in H^{1}\left(B_{R}\right)$ is called the weak solution to (11) if the following integral identity holds:

$$
\begin{equation*}
-\int_{1}^{\frac{1}{\gamma}}\left(\frac{d v^{o}}{d r}\right)^{2} d r+\int_{1}^{\frac{1}{\gamma}} \frac{v^{o}}{r} \frac{d v^{o}}{d r} d r=\left.\frac{d v^{o}}{d r}\right|_{r=1} v^{o}(1)+\omega \int_{1}^{\frac{1}{\gamma}} v^{o} d r \tag{14}
\end{equation*}
$$

The function $v^{i} \in H^{1}\left(B_{R}\right)$ is called the weak solution to (12) if it satisfies

$$
\begin{equation*}
\int_{R}^{1}\left(\frac{d v^{i}}{d r}\right)^{2} d r+(\alpha-1) \int_{R}^{1} \frac{1}{r} v^{i} \frac{d v^{i}}{d r} d r+\frac{1}{k} \int_{R}^{1} r^{\alpha}\left(v^{i}\right)^{2} d r+\omega \int_{R}^{1} r^{\alpha} v^{i} d r=\left.\frac{1}{2} \frac{d\left(v^{i}\right)^{2}}{d r}\right|_{r=1} \tag{15}
\end{equation*}
$$

Here we used integration by parts, the boundary conditions for $v^{i}$ and observation that

$$
v^{i} \frac{d v^{i}}{d r}=\frac{1}{2} \frac{d\left(v^{i}\right)^{2}}{d r} .
$$

By using the boundary conditions

$$
v^{o}=v^{i}, \quad \frac{d v^{i}}{d r}-\frac{d v^{0}}{d r}=\frac{\beta}{\sqrt{k}} v^{o}, \quad r=1,
$$

one can rewrite the identity (15) to the form

$$
\begin{align*}
\int_{R}^{1}\left(\frac{d v^{i}}{d r}\right)^{2} d r+(\alpha-1) \int_{R}^{1} \frac{1}{r} v^{i} \frac{d v^{i}}{d r} d r & +\frac{1}{k} \int_{R}^{1} r^{\alpha}\left(v^{i}\right)^{2} d r+\omega \int_{R}^{1} r^{\alpha} v^{i} d r=  \tag{16}\\
& =\left.\frac{1}{2} \frac{d\left(v^{o}\right)^{2}}{d r}\right|_{r=1}+\frac{\beta}{\sqrt{k}}\left(v^{o}(1)\right)^{2}
\end{align*}
$$

Remark 1. Exactly in the same way one can define the weak solution to (8) and (9) for an arbitrary viscosity $\mu^{i}(r)=\mu^{o} \mu(r)$. The integral identities will replace $r^{-\alpha}$ by the function $\mu(r)$.

### 3.3 The main result

Let us prove the existence and uniqueness of the weak solution. The following theorem gives such result.

Theorem 2. The unique solution $v^{i} \in H^{1}\left(B_{R}\right)$ to (15) does exist and satisfies the estimates

$$
\begin{align*}
\left\|r^{\frac{\alpha}{2}} v^{i}\right\|_{L_{2}(R, 1)}^{2} & \leq\left|v^{o}(1)\right| \frac{k \omega}{2}\left(1-\frac{1}{\gamma^{2}}\right)+\frac{\beta}{\sqrt{k}}\left(v^{o}(1)\right)^{2} \\
\left\|\frac{d v^{i}}{d r}\right\|_{L_{2}(R, 1)}^{2} & \leq\left|v^{o}(1)\right| \frac{k \omega}{2}\left(1-\frac{1}{\gamma^{2}}\right)+\frac{\beta}{\sqrt{k}}\left(v^{o}(1)\right)^{2} \tag{17}
\end{align*}
$$

where $v^{o}$ is the unique solution satisfying (11).
Proof. Let us analyze first the solvability of equation (11). It is easy to find the analytical solution to (11), which evidently coincides with the solution in the weak sense. Indeed,

$$
\frac{d}{d r}\left(r \frac{d v^{o}}{d r}\right)=r \omega \Leftrightarrow \frac{d v^{o}}{d r}=\frac{r \omega}{2}+\frac{C}{r} .
$$

Boundary condition $\frac{d v^{o}}{d r}=0$ at $r=\frac{1}{\gamma}$ implies that $C=-\frac{\omega}{2 \gamma^{2}}$. Integrating the equation once more, one derives that

$$
\begin{equation*}
v^{o}=C_{1}-\frac{\omega}{2 \gamma^{2}} \ln r+\frac{\omega r^{2}}{4}, \quad \text { where } C_{1}=v^{i}(1)-\frac{\omega}{4} \tag{18}
\end{equation*}
$$

due to the condition $v^{o}=v^{i}$ at $r=1$. The uniqueness of $v^{o}$ follows directly from formula (18) or can be derived from equations (11), assuming the existence of two different functions $v_{1}^{o} \neq v_{2}^{o}$. This technique is quite standard so we skip the full details.

Denote by $B\left[v^{i}, v^{i}\right]: H^{1}\left(B_{R}\right) \times H^{1}\left(B_{R}\right) \rightarrow \mathbb{R}$ the bilinear form

$$
B\left[v^{i}, v^{i}\right]=\int_{R}^{1}\left(\frac{d v^{i}}{d r}\right)^{2} d r+(\alpha-1) \int_{R}^{1} \frac{1}{r} v^{\frac{d}{2}} \frac{d v^{i}}{d r} d r+\frac{1}{k} \int_{R}^{1} r^{\alpha}\left(v^{i}\right)^{2} d r-\left.\frac{1}{2} \frac{d\left(v^{i}\right)^{2}}{d r}\right|_{r=1} .
$$

Define the functional on the space $H^{1}\left(B_{R}\right)$ :

$$
\left\langle f, v^{i}\right\rangle=-\omega \int_{R}^{1} r^{\alpha} v^{i} d r
$$

then the question on the existence and uniqueness of the solution (15) is reduced to solvability of

$$
B\left[v^{i}, v^{i}\right]=\left\langle f, v^{i}\right\rangle
$$

for any $f \in H^{-1}\left(B_{R}\right)$. Let us establish the coerciveness of $B\left[v^{i}, v^{i}\right]$ (see Theorem 1). Evaluating the boundary conditions

$$
\left.\frac{1}{2} \frac{d\left(v^{i}\right)^{2}}{d r}\right|_{r=1}=\left.\frac{d v^{o}}{d r}\right|_{r=1} v^{o}(1)+\frac{\beta}{\sqrt{k}}\left(v^{o}(1)\right)^{2}=v^{o}(1) \frac{\omega}{2}\left(1-\frac{1}{\gamma^{2}}\right)+\frac{\beta}{\sqrt{k}}\left(v^{o}(1)\right)^{2}
$$

and using the evident inequalities

$$
\begin{gathered}
\left(v^{o}\right)^{2}(1) \leq \int_{R}^{1} \frac{1}{r} v^{i} \frac{d v^{i}}{d r} d r \leq \frac{1}{2 R}\left(v^{o}\right)^{2}(1), \quad \frac{1}{k} \int_{R}^{1} r^{\alpha}\left(v^{i}\right)^{2} d r \geq \frac{R^{\alpha}}{k} \int_{R}^{1}\left(v^{i}\right)^{2} d r \\
\left(v^{o}(1)\right)^{2}=\left(v^{i}(1)\right)^{2}=\left(\int_{R}^{1} \frac{d v^{i}}{d r} d r\right)^{2} \leq \delta \int_{R}^{1}\left(\frac{d v^{i}}{d r}\right)^{2} d r
\end{gathered}
$$

we conclude that

$$
\begin{align*}
& \left|B\left[v^{i}, v^{i}\right]\right| \geq \left\lvert\, \int_{R}^{1}\left(\frac{d v^{i}}{d r}\right)^{2} d r+\frac{1}{k} \int_{R}^{1} r^{\alpha}\left(v^{i}\right)^{2} d r+\left(\frac{\alpha-1}{2}-\frac{\beta}{\sqrt{k}}\right)\left(v^{o}\right)^{2}(1)\right.  \tag{19}\\
& \left.-v^{o}(1) \frac{\omega}{2}\left(1-\frac{1}{\gamma^{2}}\right) \right\rvert\, \geq C\left(\alpha, \beta, \gamma, R, k, v^{o}(1)\right)\left\|v^{i}\right\|_{H_{1}\left(B_{R}\right)}^{2}
\end{align*}
$$

where

$$
C\left(\alpha, \beta, \gamma, R, k, v^{o}(1)\right) \text { is a constant and }\left\|v^{i}\right\|_{H_{1}\left(B_{R}\right)}^{2}=\int_{R}^{1}\left(\frac{d v^{i}}{d r}\right)^{2} d r+\int_{R}^{1}\left(v^{i}\right)^{2} d r
$$

Hence, the unique solution $v^{i}$ exists due to Lions-Lax-Milgram Lemma.
Observe also that the identity (16) imply the estimates

$$
\begin{align*}
& \left.\left\|r^{\frac{\alpha}{2}} v^{i}\right\|_{L_{2}(R, 1)}^{2} \leq \frac{k}{2}\left|\frac{d\left(v^{o}\right)^{2}}{d r}\right|_{r=1} \right\rvert\,+\frac{\beta}{\sqrt{k}}\left(v^{o}(1)\right)^{2}, \\
& \left.\left\|\frac{d v^{i}}{d r}\right\|_{L_{2}(R, 1)}^{2} \leq \frac{k}{2}\left|\frac{d\left(v^{o}\right)^{2}}{d r}\right|_{r=1} \right\rvert\,+\frac{\beta}{\sqrt{k}}\left(v^{o}(1)\right)^{2} . \tag{20}
\end{align*}
$$

Coming back to estimates (20) and evaluating

$$
\left.\frac{1}{2} \frac{d\left(v^{o}\right)^{2}}{d r}\right|_{r=1}=v^{o}(1) \frac{\omega}{2}\left(1-\frac{1}{\gamma^{2}}\right),
$$

we derive the asymptotics

$$
\begin{aligned}
& \left\|r^{\frac{\alpha}{2}} v^{i}\right\|_{L_{2}(R, 1)}^{2} \leq\left|v^{o}(1)\right| \frac{k \omega}{2}\left(1-\frac{1}{\gamma^{2}}\right)+\frac{\beta}{\sqrt{k}}\left(v^{o}(1)\right)^{2}, \\
& \left\|\frac{d v^{i}}{d r}\right\|_{L_{2}(R, 1)}^{2} \leq\left|v^{o}(1)\right| \frac{k \omega}{2}\left(1-\frac{1}{\gamma^{2}}\right)+\frac{\beta}{\sqrt{k}}\left(v^{o}(1)\right)^{2}
\end{aligned}
$$

## 4 Exponential viscosity

Assume now that

$$
\begin{equation*}
\widetilde{\mu}^{i}=\widetilde{\mu}^{o} e^{-\alpha\left(\frac{\tilde{r}}{a}-1\right)}, \quad \alpha>0 \tag{21}
\end{equation*}
$$

and again the flow is parallel to $z$-axis. Making an analogous steps to come to dimensionless form of the Brinkman's equation, one gets the equation

$$
\begin{equation*}
\frac{d^{2} v^{i}}{d r^{2}}+\left(-\alpha+\frac{1}{r}\right) \frac{d v^{i}}{d r}=e^{\alpha\left(\frac{r}{a}-1\right)}\left(\frac{v^{i}}{k}+\omega\right), R<r<1 \tag{22}
\end{equation*}
$$

with boundary conditions

$$
\begin{align*}
& v^{i}=0 \text { as } r=R, \\
& v^{i}=v^{o} \text { as } r=1,  \tag{23}\\
& \frac{d v^{i}}{d r}-\frac{d v^{o}}{d r}=\frac{\beta}{\sqrt{k}} v^{o}, \quad r=1 .
\end{align*}
$$

The weak solution to (22), (23) satisfies

$$
\begin{align*}
& \int_{R}^{1}\left(\frac{d v^{i}}{d r}\right)^{2} d r+\frac{1}{k} \int_{R}^{1} e^{\alpha\left(\frac{r}{a}-1\right)}\left(v^{i}\right)^{2} d r+\frac{\alpha}{2} \int_{R}^{1} \frac{d}{d r}\left(v^{i}\right)^{2} d r+\omega \int_{R}^{1} e^{\alpha\left(\frac{r}{a}-1\right)} v^{i} d r=  \tag{24}\\
& =\left.\frac{1}{2} \frac{d\left(v^{i}\right)^{2}}{d r}\right|_{r=1}+\frac{1}{2} \int_{R}^{1} \frac{1}{r} \frac{d}{d r}\left(v^{i}\right)^{2} d r .
\end{align*}
$$

Applying the Newton- Leibnitz formula and taking into account the boundary conditions for $v^{i}$, the identity (24) can be rewritten as follows:

$$
\begin{align*}
& \int_{R}^{1}\left(\frac{d v^{i}}{d r}\right)^{2} d r+\frac{1}{k} \int_{R}^{1} e^{\alpha\left(\frac{r}{a}-1\right)}\left(v^{i}\right)^{2} d r+\frac{\alpha+1}{2}\left(v^{o}(1)\right)^{2}-  \tag{25}\\
& -\frac{1}{2} \int_{R}^{1} \frac{\left(v^{i}\right)^{2}}{r^{2}} d r+\omega \int_{R}^{1} e^{\alpha\left(\frac{r}{a}-1\right)} v^{i} d r=\left.\frac{1}{2} \frac{d\left(v^{o}\right)^{2}}{d r}\right|_{r=1}+\frac{\beta}{\sqrt{k}}\left(v^{o}(1)\right)^{2} .
\end{align*}
$$

Similarly, one can prove the following theorem on the existence and uniqueness of the solution to (24).

Theorem 3. The solution to (24) does exist, is unique and satisfies estimates

$$
\begin{align*}
& \left\|e^{\frac{\alpha}{2}\left(\frac{r}{a}-1\right)} v^{i}\right\|_{L_{2}(R, 1)}^{2} \leq\left(1+\frac{2 R^{2}\left(1+\delta \beta k^{-\frac{1}{2}}\right)}{2 R^{2}-\delta}\right)\left|v^{o}(1) \frac{\omega k}{2}\left(1-\frac{1}{\gamma^{2}}\right)\right|  \tag{26}\\
& \left\|\frac{d v^{i}}{d r}\right\|_{L_{2}(R, 1)}^{2} \leq \frac{2 R^{2}}{2 R^{2}-\delta}\left|v^{o}(1) \frac{\omega k}{2}\left(1-\frac{1}{\gamma^{2}}\right)\right|
\end{align*}
$$

Proof. All steps of the proof are identical to the previously considered case in Lemma 2. We introduce the bilinear form $B\left[v^{i}, v^{i}\right]: H^{1}\left(B_{R}\right) \times H^{1}\left(B_{R}\right) \rightarrow \mathbb{R}$ :

$$
\begin{gathered}
B\left[v^{i}, v^{i}\right]=\int_{R}^{1}\left(\frac{d v^{i}}{d r}\right)^{2} d r-\frac{1}{2} \int_{R}^{1} \frac{\left(v^{i}\right)^{2}}{r^{2}} d r+\frac{1}{k} \int_{R}^{1} e^{\frac{\alpha}{2}\left(\frac{r}{a}-1\right)}\left(v^{i}\right)^{2} d r- \\
-\left.\frac{1}{2} \frac{d\left(v^{i}\right)^{2}}{d r}\right|_{r=1}+\frac{1}{2} \int_{R}^{1} \frac{1}{r} \frac{d}{d r}\left(v^{i}\right)^{2} d r
\end{gathered}
$$

and functional

$$
\left\langle f, v^{i}\right\rangle=\left.\frac{d v^{o}}{d r}\right|_{r=1} v^{o}(1)-\left(\frac{\alpha+1}{2}-\frac{\beta}{\sqrt{k}}\right)\left(v^{o}(1)\right)^{2}-\omega \int_{R}^{1} e^{\frac{\alpha}{2}\left(\frac{r}{a}-1\right)} v^{i} d r .
$$

In order to use the Lions-Lax-Milgram Lemma on the existence and uniqueness of the solution, it is required to get the estimate $\left|B\left[v^{i}, v^{i}\right]\right| \geq C\left\|v^{i}\right\|_{H_{1}\left(B_{R}\right)}^{2}$. In view of the inequality

$$
-\frac{1}{2} \int_{R}^{1} \frac{\left(v^{i}\right)^{2}}{r^{2}} d r \geq-\frac{1}{2} \int_{R}^{1}\left(v^{i}\right)^{2} d r,
$$

we get the desired bound

$$
\begin{equation*}
\left|B\left[v^{i}, v^{i}\right]\right| \geq \int_{R}^{1}\left(\frac{d v^{i}}{d r}\right)^{2} d r+\frac{1}{k} \int_{R}^{1}\left(e^{\alpha\left(\frac{r}{a}-1\right)}-\frac{k}{2}\right)\left(v^{i}\right)^{2} d r \geq C(\alpha, R, k)\left\|v^{i}\right\|_{H_{1}\left(B_{R}\right)}^{2} \tag{27}
\end{equation*}
$$

Here the constant $C(\alpha, R, k)=\min \left\{1,\left|\frac{e^{\alpha\left(\frac{R}{a}-1\right)}}{k}-\frac{1}{2}\right|\right\}$. We note that the classical Friedrich's inequality

$$
\begin{equation*}
\int_{R}^{1}\left(v^{i}\right)^{2} d r \leq \delta \int_{R}^{1}\left(\frac{d v^{i}}{d r}\right)^{2} d r \tag{28}
\end{equation*}
$$

is valid for function $v^{i}$ since it vanishes on the boundary $r=R$. Moreover, the constant is equal to the square of the strip $\{R \leq r \leq 1\} \times 1=\delta$. To obtain the estimates (26) we use the integral identity in the form

$$
\begin{align*}
& \int_{R}^{1}\left(\frac{d v^{i}}{d r}\right)^{2} d r+\frac{1}{k} \int_{R}^{1} e^{\alpha\left(\frac{r}{a}-1\right)}\left(v^{i}\right)^{2} d r+\frac{\alpha+1}{2}\left(v^{o}(1)\right)^{2}+\omega \int_{R}^{1} e^{\alpha\left(\frac{r}{a}-1\right)} v^{i} d r=  \tag{29}\\
& \left.\frac{1}{2} \frac{d\left(v^{o}\right)^{2}}{d r}\right|_{r=1}+\frac{\beta}{\sqrt{k}}\left(v^{o}(1)\right)^{2}+\frac{1}{2} \int_{R}^{1} \frac{\left(v^{i}\right)^{2}}{r^{2}} d r
\end{align*}
$$

It implies the following estimates:

$$
\begin{align*}
& \int_{R}^{1}\left(\frac{d v^{i}}{d r}\right)^{2} d r \leq\left.\frac{1}{2} \frac{d\left(v^{o}\right)^{2}}{d r}\right|_{r=1}+\frac{\beta}{\sqrt{k}}\left(v^{o}(1)\right)^{2}+\frac{1}{2} \int_{R}^{1} \frac{\left(v^{i}\right)^{2}}{r^{2}} d r \\
& \int_{R}^{1} e^{\alpha\left(\frac{r}{a}-1\right)}\left(v^{i}\right)^{2} d r \leq\left.\frac{1}{2} \frac{d\left(v^{o}\right)^{2}}{d r}\right|_{r=1}+\frac{\beta}{\sqrt{k}}\left(v^{o}(1)\right)^{2}+\frac{1}{2} \int_{R}^{1} \frac{\left(v^{i}\right)^{2}}{r^{2}} d r . \tag{30}
\end{align*}
$$

The first term in the right-hand side is bounded as in (17):

$$
\begin{equation*}
\left|\frac{1}{2} \frac{d\left(v^{o}\right)^{2}}{d r}\right|_{r=1}\left|\leq\left|v^{o}(1) \frac{\omega k}{2}\left(1-\frac{1}{\gamma^{2}}\right)\right|\right. \tag{31}
\end{equation*}
$$

Applying the inequality (28), we consider the third term:

$$
\begin{equation*}
\frac{1}{2} \int_{R}^{1} \frac{\left(v^{i}\right)^{2}}{r^{2}} d r \leq \frac{1}{2 R^{2}} \int_{R}^{1}\left(v^{i}\right)^{2} d r \leq \frac{\delta}{2 R^{2}} \int_{R}^{1}\left(\frac{d v^{i}}{d r}\right)^{2} d r \tag{32}
\end{equation*}
$$

Now we can use this result in the first inequality of (30):

$$
\begin{equation*}
\int_{R}^{1}\left(\frac{d v^{i}}{d r}\right)^{2} d r \leq\left|v^{o}(1) \frac{\omega k}{2}\left(1-\frac{1}{\gamma^{2}}\right)\right|+\frac{\delta}{2 R^{2}} \int_{R}^{1}\left(\frac{d v^{i}}{d r}\right)^{2} d r \tag{33}
\end{equation*}
$$

Consequently,

$$
\begin{equation*}
\int_{R}^{1}\left(\frac{d v^{i}}{d r}\right)^{2} d r \leq \frac{2 R^{2}}{2 R^{2}-\delta}\left|v^{o}(1) \frac{\omega k}{2}\left(1-\frac{1}{\gamma^{2}}\right)\right| \tag{34}
\end{equation*}
$$

Finally, the term $\frac{\beta}{\sqrt{k}}\left(v^{o}(1)\right)^{2}$ can also be estimated by $\int_{R}^{1}\left(\frac{d v^{i}}{d r}\right)^{2} d r$ with help of Friedrichs inequality (28) and Hölder inequality

$$
\int_{\Omega}|f(x) g(x)| d x \leq\left(\int_{\Omega}|f(x)|^{p} d x\right)^{\frac{1}{p}}\left(\int_{\Omega}|g(x)|^{q} d x\right)^{\frac{1}{q}}, \frac{1}{p}+\frac{1}{q}=1
$$

if one apply it for $\Omega=(R, 1), f=\frac{d v^{i}}{d r}, g=1$ and $p=q=2$ :

$$
\begin{align*}
\frac{\beta}{\sqrt{k}}\left(v^{o}(1)\right)^{2}=\frac{\beta}{\sqrt{k}}\left(v^{i}(1)\right)^{2}= & \frac{\beta}{\sqrt{k}}\left(\int_{R}^{1} \frac{d v^{i}}{d r} d r\right)^{2} \leq \frac{\delta \beta}{\sqrt{k}} \int_{R}^{1}\left(\frac{d v^{i}}{d r}\right)^{2} d r \leq  \tag{35}\\
& \leq \frac{2 R^{2} \delta \beta k^{-\frac{1}{2}}}{2 R^{2}-\delta}\left|v^{o}(1) \frac{\omega k}{2}\left(1-\frac{1}{\gamma^{2}}\right)\right|
\end{align*}
$$

The results in (33), (34) and (35) can be directly used to estimate the second line in (30):

$$
\begin{array}{r}
\int_{R}^{1} e^{\alpha\left(\frac{r}{a}-1\right)}\left(v^{i}\right)^{2} d r \leq\left|v^{o}(1) \frac{\omega k}{2}\left(1-\frac{1}{\gamma^{2}}\right)\right|+\frac{\delta}{2 R^{2}} \int_{R}^{1}\left(\frac{d v^{i}}{d r}\right)^{2} d r \leq  \tag{36}\\
\\
\left|v^{o}(1) \frac{\omega k}{2}\left(1-\frac{1}{\gamma^{2}}\right)\right|\left(1+\frac{2 R^{2}\left(1+\delta \beta k^{-\frac{1}{2}}\right)}{2 R^{2}-\delta}\right)
\end{array}
$$

## 5 Concluding remarks

Let us observe that estimates (17) and (26) show the continuous dependence of the solution $v^{i}$ on initial data $k, \omega, \beta, \gamma, \delta, R$ as well as on the solution $v^{o}$ at the common boundary $r=1$. Note also that factor $r^{\frac{\alpha}{2}}$ in the estimate (17) and similarly $e^{\alpha\left(\frac{r}{a}-1\right)}$ in (26) means the following asymptotical behaviour of $v^{i}$ : $v^{i} \sim C\left(v^{o}(1), k, \omega, \beta, \gamma\right) r^{-\frac{\alpha}{2}}$, where the constant $C\left(v^{o}(1), k, \omega, \beta, \gamma\right)$ depends on $v^{o}(1), k, \omega, \beta, \gamma$. Analogously, $v^{i} \sim C\left(v^{o}, k, \omega, \beta, \gamma, \delta, R\right) e^{-\frac{\alpha}{2}\left(\frac{r}{a}-1\right)}$ in the second case. Roughly speaking, the solution $v^{i}$ is proportional to square root of the viscosity. If
one apply the estimate (35) to the second term in right-hand side of (17), we get the upper bounds in the form which involves $R$ as well:

$$
\begin{align*}
\left\|r^{\frac{\alpha}{2}} v^{i}\right\|_{L_{2}(R, 1)}^{2} & \leq\left|v^{o}(1)\right| \frac{k \omega}{2}\left(1-\frac{1}{\gamma^{2}}\right)\left(1+\frac{2 R^{2} \delta \beta k^{-\frac{1}{2}}}{2 R^{2}-\delta}\right), \\
\left\|\frac{d v^{i}}{d r}\right\|_{L_{2}(R, 1)}^{2} & \leq\left|v^{o}(1)\right| \frac{k \omega}{2}\left(1-\frac{1}{\gamma^{2}}\right)\left(1+\frac{2 R^{2} \delta \beta k^{-\frac{1}{2}}}{2 R^{2}-\delta}\right) . \tag{37}
\end{align*}
$$

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Anatoly Filippov
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Yulia Koroleva
Department of Higher Mathematics
Gubkin State University of Oil and Gas
(National Research University)
Leninskij prospect 65-1
Moscow, 119991, Russia
E-mail: filippov.a@gubkin.ru
E-mail: koroleva.y@gubkin.ru

# Invariant conditions of stability of unperturbed motion governed by some differential systems in the plane 

Natalia Neagu, Victor Orlov, Mihail Popa


#### Abstract

Center-affine invariant conditions of the stability of unperturbed motion were determined for differential systems in the plane with polynomial nonlinearities in non-critical cases and for differential systems in the plane with polynomial nonlinearities up to the fourth degree inclusive in critical cases.


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## Introduction

Problems which require a general formulation of stability not only of equilibrium but also of motion arose in science and technics in the middle of XIX-th century.

Lyapunov (1857-1918) published his PhD thesis concerning the stability of motion in 1892, and it was translated into French and published in France in 1907. According to the French version, this work was reprinted in Russian, with some additions, in his collection of works [1] in 1956. The mentioned work contains many fruitful ideas and results of great importance. All the history related to the theory on stability of motion is considered to be divided into periods before and after Lyapunov.

First of all, A. M. Lyapunov gave a strict definition of the stability of motion, which was so successful that all scientists took it as fundamental one for their researches.

A lot of papers were written in the field of stability of motion. The universal scientific literature concerning the stability of motion contains thousands of papers, including hundreds of monographs and textbooks of many authors. This literature is rich in the development of this theory, as well as in its applications in practice.

Note that many problems on stability treated in these works are governed by two-dimensional (or multidimensional) autonomous polynomial differential systems. Methods of the theory of invariants for such systems were elaborated in the school of differential equations from Chişinău. Moreover, the theory of Lie algebras and Sibirsky graded algebras with applications in the qualitative theory of these equations [2-7] there were developed.

The stability of unperturbed motions using the theory of algebras, of invariants and of Lie algebras was studied for the first time in [8]. In this paper, the similar investigations are done for two-dimensional differential systems with polynomial nonlinearities.
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## 1 Definition of stability of unperturbed motion and of critical system

We consider the two-dimensional differential system with polynomial nonlinearities of perturbed motion (see, for example, [1] or [9]) of the form

$$
\begin{equation*}
\frac{d x^{j}}{d t}=a_{\alpha}^{j} x^{\alpha}+\sum_{i=1}^{l} a_{\alpha_{1} \alpha_{2} \ldots \alpha_{m_{i}}}^{j} x^{\alpha_{1}} x^{\alpha_{2}} \ldots x^{\alpha_{m_{i}}}\left(j, \alpha, \alpha_{1}, \ldots, \alpha_{m_{i}}=1,2 ; l<\infty\right) \tag{1}
\end{equation*}
$$

where $a_{\alpha_{1} \alpha_{2} \ldots \alpha_{m_{i}}}^{j}$ is a symmetric tensor in lower indices in which the total convolution is done and $\Gamma=\left\{m_{1}, m_{2}, \ldots, m_{l}\right\}\left(m_{i} \geq 2\right)$ is a finite set of distinct natural numbers. Coefficients and variables in (1) are given over the field of real numbers $\mathbb{R}$.

The system of the first approximation ([1], [9])

$$
\begin{equation*}
\frac{d x^{j}}{d t}=a_{\alpha}^{j} x^{\alpha} \quad(j, \alpha=1,2) \tag{2}
\end{equation*}
$$

plays an important role in studying differential systems (1). As it follows from [1] (or [9]), to unperturbed motion of system (1) the zero values of variables $x^{j}(t)$ $(j=1,2)$ correspond. Taking into account this fact, we have the following definition of stability by Lyapunov [9]:

If for any small positive value $\varepsilon$, however small, one can find a positive number $\delta$ such that for all perturbations $x^{j}\left(t_{0}\right)$ satisfying the condition

$$
\begin{equation*}
\sum_{j=1}^{2}\left(x^{j}\left(t_{0}\right)\right)^{2} \leq \delta \tag{3}
\end{equation*}
$$

the inequality

$$
\sum_{j=1}^{2}\left(x^{j}(t)\right)^{2}<\varepsilon
$$

is valid for any $t \geq t_{0}$, then the unperturbed motion $x^{j}=0(j=\overline{1,2})$ is called stable, otherwise it is called unstable.

If the unperturbed motion is stable and the number $\delta$ can be found however small such that for any perturbed motions satisfying (3) the condition

$$
\lim _{t \rightarrow \infty} \sum_{j=1}^{2}\left(x^{j}(t)\right)^{2}=0
$$

is valid, then the unperturbed motion is called asymptotically stable.
Inspired by the work [1] we have
Definition 1. The differential system (1) with polynomial nonlinearities will be called a critical system of Lyapunov type if the characteristic equation of the system of the first approximation (2) has one zero root and all other roots have negative real parts. When the real parts of the roots of the characteristic equation are different from zero, the system (1) will be called non-critical.

First, we will examine the non-critical case.
Lemma 1. The characteristic equation of system (1) and (2) is

$$
\begin{equation*}
\varrho^{2}+L_{1,2} \varrho+L_{2,2}=0, \tag{4}
\end{equation*}
$$

where the coefficients in (4) are center-affine invariants [2] and have the form

$$
\begin{equation*}
L_{1,2}=-I_{1}, \quad L_{2,2}=\frac{1}{2}\left(I_{1}^{2}-I_{2}\right) \tag{5}
\end{equation*}
$$

with

$$
\begin{equation*}
I_{1}=a_{\alpha}^{\alpha}, \quad I_{2}=a_{\beta}^{\alpha} a_{\alpha}^{\beta} . \tag{6}
\end{equation*}
$$

By means of the Lyapunov theorems on stability of unperturbed motion in the first approximation (2), the Hurwitz theorem on the signs of the roots of an algebraic equation (see, for example, [9]) and using Lemma 1 we have

Theorem 1. Assume that the center-affine invariants (5) of system (1) satisfy the inequalities $L_{1,2}>0, L_{2,2}>0$. Then the unperturbed motion $x^{1}=x^{2}=0$ of this system is asymptotically stable.

Theorem 2. If at least one of the center-affine invariant expressions (5) of system (1) is negative, then the unperturbed motion $x^{1}=x^{2}=0$ of this system is unstable.

## 2 Canonical form of a critical system of Lyapunov type

Remark 1. In the following, we will study critical systems of Lyapunov type in the first case, and such systems will be called critical systems or critical systems of Lyapunov type.

Lemma 2. The characteristic equation of system (2) (and therefore of system (1)) has one zero root and the other ones real and negative if and only if the following invariant conditions

$$
\begin{equation*}
I_{1}^{2}-I_{2}=0, \quad I_{1}<0 \tag{7}
\end{equation*}
$$

hold, where $I_{1}$ and $I_{2}$ are from (6).
The proof of Lemma 2 follows from the fact that the characteristic equation of system (2) and therefore of (1) has the form (4)-(5).

From [1] it follows
Lemma 3. Let for system (2) (for (1)) the invariant conditions (7) hold. Then the system (2) by a center-affine transformation can be brought to the form

$$
\begin{equation*}
\frac{d x^{1}}{d t}=0, \quad \frac{d x^{2}}{d t}=a_{\alpha}^{2} x^{\alpha} \quad(\alpha=\overline{1,2}) \tag{8}
\end{equation*}
$$

and, therefore, the system (1) can be written in the form

$$
\begin{gather*}
\frac{d x^{1}}{d t}=\sum_{i=1}^{l} a_{\alpha_{1} \alpha_{2} \ldots \alpha_{m_{i}}}^{1} x^{\alpha_{1}} x^{\alpha_{2}} \ldots x^{\alpha_{m_{i}}}, \\
\frac{d x^{2}}{d t}=a_{\alpha}^{2} x^{\alpha}+\sum_{i=1}^{l} a_{\alpha_{1} \alpha_{2} \ldots \alpha_{m_{i}}}^{2} x^{\alpha_{1}} x^{\alpha_{2}} \ldots x^{\alpha_{m_{i}}}\left(\alpha, \alpha_{1}, \ldots, \alpha_{m_{i}}=\overline{1,2} ; l<\infty\right) . \tag{9}
\end{gather*}
$$

Remark 2. The system (9) is called the canonical form of a critical system of Lyapunov type (1), where the first equation from (9) is called the critical equation and the second one - the non-critical equation.

For the case examined in this paper the Lyapunov's Theorem [1, §32] can be written in the following form:

Theorem 3. Let the characteristic equation of the matrix of linear part of differential system with polynomial nonlinearities have one zero root and other roots have negative real parts. Assume that the differential system of the perturbed motion (1) was brought to the form (9) and consider the equation

$$
\begin{equation*}
a_{\alpha}^{2} x^{\alpha}+\sum_{i=1}^{l} a_{\alpha_{1} \alpha_{2} \ldots \alpha_{m_{i}}}^{2} x^{\alpha_{1}} x^{\alpha_{2}} \ldots x^{\alpha_{m_{i}}}=0 \quad\left(\alpha, \alpha_{1}, \alpha_{2}, \ldots, \alpha_{m_{i}}=\overline{1,2} ; l<\infty\right) \tag{10}
\end{equation*}
$$

from which we determine the variable $x^{2}$ as a holomorphic function of the variable $x^{1}$, vanishing for $x^{1}=0$ (such determination of $x^{2}$ is always possible and is unique). Substitute the determined values into the polynomial

$$
\sum_{i=1}^{l} a_{\alpha_{1} \alpha_{2} \ldots \alpha_{m_{i}}}^{1} x^{\alpha_{1}} x^{\alpha_{2}} \ldots x^{\alpha_{m_{i}}} \quad\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m_{i}}=1,2 ; l<\infty\right) .
$$

If the obtained result is not identically zero, then we can develop it in an increasing powers series of $x^{1}$. When the lowest power of $x^{1}$ in this development is even, then the unperturbed motion is unstable. When the lowest power of $x^{1}$ is odd, then the unperturbed motion depends on the sign of the coefficient of $x^{1}$. The unperturbed motion will be unstable when this coefficient is positive and will be stable when the coefficient is negative. In the last case, any perturbed motion that corresponds to small enough perturbation will approach asymptotically the unperturbed motion.

If the obtained result is identically zero, then there exists a continuous series of stabilized motions to which the examined unperturbed motion belongs. All the motions of this series, close enough to the unperturbed motions, including the last one, will be stable. In this case, for small enough perturbations, any perturbed motion will tend asymptotically to one of the stabilized motions of the mention series.

## 3 Center-affine invariant conditions of stability of unperturbed motion for the system with quadratic nonlinearities

We examine the differential system with quadratic nonlinearities

$$
\begin{equation*}
\frac{d x^{j}}{d t}=a_{\alpha}^{j} x^{\alpha}+a_{\alpha \beta}^{j} x^{\alpha} x^{\beta} \quad(j, \alpha, \beta=1,2), \tag{11}
\end{equation*}
$$

where $a_{\alpha \beta}^{j}$ is a symmetric tensor in lower indices in which the total convolution is done.

It was shown in [4] that the set of unimodular comitants and invariants of the system (1) consists of some graded algebras, which in [7] were called the Sibirsky algebras. For system (11) these algebras were denoted in [4] by $S_{1,2}$ - the Sibirsky algebras of comitants and $S I_{1,2}$ - the Sibirsky algebras of invariants.

It was shown in [4] that the set of generators of these algebras (which is finite) consists of polynomial bases of the homogeneous center-affine comitants and invariants.

Based on this and on the polynomial bases of the center-affine comitants and invariants of system (11) given in [2], we can write the Sibirsky algebra in the form

$$
S_{1,2}=<I_{1}, I_{2}, \ldots, I_{16}, K_{1}, K_{2}, \ldots, K_{20} \mid f_{1}, f_{2}, \ldots, f_{27}>
$$

and

$$
S I_{1,2}=<I_{1}, I_{2}, \ldots, I_{16} \mid f_{1}, f_{2}, \ldots, f_{9}>,
$$

where $I_{r}$ and $K_{s}$ are the invariants and the comitants of these algebras, and $f_{j}$ are their syzygies.

Later on, we will use the following generators of the Sibirsky algebras of system (11), for which their tensorial forms from [2] are written as follows:

$$
\begin{gather*}
I_{1}=a_{\alpha}^{\alpha}, \quad I_{2}=a_{\beta}^{\alpha} a_{\alpha}^{\beta}, \quad I_{5}=a_{p}^{\alpha} a_{\gamma q}^{\beta} a_{\alpha \beta}^{\gamma} \varepsilon^{p q}, \quad K_{1}=a_{\alpha \beta}^{\alpha} x^{\beta}, \quad K_{2}=a_{\alpha}^{p} x^{\alpha} x^{q} \varepsilon_{p q}, \\
K_{3}=a_{\beta}^{\alpha} a_{\alpha \gamma}^{\beta} x^{\gamma}, K_{4}=a_{\gamma}^{\alpha} a_{\alpha \beta}^{\beta} x^{\gamma}, K_{5}=a_{\alpha \beta}^{p} x^{\alpha} x^{\beta} x^{q} \varepsilon_{p q}, \quad K_{7}=a_{\beta \gamma}^{\alpha} a_{\alpha \delta}^{\beta} x^{\gamma} x^{\delta},  \tag{12}\\
K_{8}=a_{\gamma}^{\alpha} a_{\delta}^{\beta} a_{\alpha \beta}^{\gamma} x^{\delta}, \quad K_{11}=a_{\alpha}^{p} a_{\beta \gamma}^{\alpha} x^{\beta} x^{\gamma} x^{q} \varepsilon_{p q}, \quad K_{12}=a_{\beta}^{\alpha} a_{\alpha \gamma}^{\beta} a_{\delta \mu}^{\gamma} x^{\delta} x^{\mu}, \\
K_{13}=a_{\gamma}^{\alpha} a_{\alpha \beta}^{\beta} a_{\delta \mu}^{\gamma} x^{\delta} x^{\mu},
\end{gather*}
$$

where $\varepsilon^{p q}\left(\varepsilon_{p q}\right)$ is the unit bivector with coordinates $\varepsilon^{11}=\varepsilon^{22}=0, \varepsilon^{12}=-\varepsilon^{21}=$ $1\left(\varepsilon_{11}=\varepsilon_{22}=0, \varepsilon_{12}=-\varepsilon_{21}=1\right)$.

Suppose the system (11) is critical of Lyapunov type. Then by Lemma 3 it can be brought to the canonical form (9)

$$
\begin{equation*}
\frac{d x^{1}}{d t}=a_{\alpha \beta}^{1} x^{\alpha} x^{\beta}, \quad \frac{d x^{2}}{d t}=a_{\alpha}^{2} x^{\alpha}+a_{\alpha \beta}^{2} x^{\alpha} x^{\beta} \quad(\alpha, \beta=1,2) . \tag{13}
\end{equation*}
$$

According to Theorem 3, we examine the equation (10) provided by non-critical equation of (13), which in the expanded form looks as

$$
\begin{equation*}
a_{1}^{2} x^{1}+a_{2}^{2} x^{2}+a_{11}^{2}\left(x^{1}\right)^{2}+2 a_{12}^{2} x^{1} x^{2}+a_{22}^{2}\left(x^{2}\right)^{2}=0 . \tag{14}
\end{equation*}
$$

In this case, under the conditions (5)-(6) and the inequality from (7) we have

$$
\begin{equation*}
I_{1}=a_{2}^{2}<0 \tag{15}
\end{equation*}
$$

Then from (14) we can write

$$
\begin{equation*}
x^{2}=-\frac{a_{1}^{2}}{a_{2}^{2}} x^{1}-\frac{a_{11}^{2}}{a_{2}^{2}}\left(x^{1}\right)^{2}-\frac{2 a_{12}^{2}}{a_{2}^{2}} x^{1} x^{2}-\frac{a_{22}^{2}}{a_{2}^{2}}\left(x^{2}\right)^{2} \tag{16}
\end{equation*}
$$

By Theorem 3, we seek $x^{2}$ as a holomorphic function of $x^{1}$. Then we can write

$$
\begin{equation*}
x^{2}=-\frac{a_{1}^{2}}{a_{2}^{2}} x^{1}+B_{2}\left(x^{1}\right)^{2}+B_{3}\left(x^{1}\right)^{3}+B_{4}\left(x^{1}\right)^{4}+\cdots \tag{17}
\end{equation*}
$$

Substituting (17) into (16) we get

$$
\begin{aligned}
& -\frac{a_{1}^{2}}{a_{2}^{2}} x^{1}+B_{2}\left(x^{1}\right)^{2}+B_{3}\left(x^{1}\right)^{3}+\cdots=-\frac{a_{1}^{2}}{a_{2}^{2}} x^{1}-\frac{a_{11}^{2}}{a_{2}^{2}}\left(x^{1}\right)^{2}-\frac{2 a_{12}^{2}}{a_{2}^{2}} x^{1}\left[-\frac{a_{1}^{2}}{a_{2}^{2}} x^{1}+\right. \\
& \left.\quad+B_{2}\left(x^{1}\right)^{2}+B_{3}\left(x^{1}\right)^{3}+\cdots\right]-\frac{a_{22}^{2}}{a_{2}^{2}}\left[-\frac{a_{1}^{2}}{a_{2}^{2}} x^{1}+B_{2}\left(x^{1}\right)^{2}+B_{3}\left(x^{1}\right)^{3}+\cdots\right]^{2}
\end{aligned}
$$

This implies that

$$
\begin{aligned}
& B_{2}\left(x^{1}\right)^{2}+B_{3}\left(x^{1}\right)^{3}+B_{4}\left(x^{1}\right)^{4}+\cdots=\left[-\frac{a_{11}^{2}}{a_{2}^{2}}+\frac{2 a_{1}^{2} a_{12}^{2}}{\left(a_{2}^{2}\right)^{2}}-\frac{\left(a_{1}^{2}\right)^{2} a_{22}^{2}}{\left(a_{2}^{2}\right)^{3}}\right]\left(x^{1}\right)^{2}+ \\
+ & {\left[-\frac{2 a_{12}^{2}}{a_{2}^{2}} B_{2}+\frac{2 a_{1}^{2} a_{22}^{2}}{\left(a_{2}^{2}\right)^{2}} B_{2}\right]\left(x^{1}\right)^{3}+\left[-\frac{2 a_{12}^{2}}{a_{2}^{2}} B_{3}-\frac{a_{22}^{2}}{a_{2}^{2}} B_{2}^{2}+2 \frac{a_{1}^{2} a_{22}^{2}}{\left(a_{2}^{2}\right)^{2}} B_{3}\right]\left(x^{1}\right)^{4}+\cdots }
\end{aligned}
$$

and we obtain

$$
\begin{gather*}
B_{2}=\frac{1}{\left(a_{2}^{2}\right)^{3}}\left[-\left(a_{2}^{2}\right)^{2} a_{11}^{2}+2 a_{1}^{2} a_{2}^{2} a_{12}^{2}-\left(a_{1}^{2}\right)^{2} a_{22}^{2}\right], B_{3}=\frac{2}{\left(a_{2}^{2}\right)^{2}}\left(-a_{2}^{2} a_{12}^{2}+a_{1}^{2} a_{22}^{2}\right) B_{2}, \\
B_{4}=\frac{1}{\left(a_{2}^{2}\right)^{2}}\left[-a_{2}^{2} a_{22}^{2} B_{2}^{2}+2\left(a_{1}^{2} a_{22}^{2}-a_{2}^{2} a_{12}^{2}\right) B_{3}\right], \ldots \tag{18}
\end{gather*}
$$

Substituting (17) into the right-hand side of the critical differential equation (13) we have

$$
a_{11}^{1}\left(x^{1}\right)^{2}+2 a_{12}^{1} x^{1} x^{2}+a_{22}^{1}\left(x^{2}\right)^{2}=A_{2}\left(x^{1}\right)^{2}+A_{3}\left(x^{1}\right)^{3}+A_{4}\left(x^{1}\right)^{4}+\cdots
$$

or in the expanded form we get

$$
\begin{gathered}
a_{11}^{1}\left(x^{1}\right)^{2}+2 a_{12}^{1} x^{1}\left[-\frac{a_{1}^{2}}{a_{2}^{2}} x^{1}+B_{2}\left(x^{1}\right)^{2}+B_{3}\left(x^{1}\right)^{3}+\cdots\right]+ \\
+a_{22}^{1}\left[-\frac{a_{1}^{2}}{a_{2}^{2}} x^{1}+B_{2}\left(x^{1}\right)^{2}+B_{3}\left(x^{1}\right)^{3}+\cdots\right]^{2}=A_{2}\left(x^{1}\right)^{2}+A_{3}\left(x^{1}\right)^{3}+A_{4}\left(x^{1}\right)^{4}+\cdots
\end{gathered}
$$

This implies that

$$
\begin{gather*}
A_{2}=\frac{1}{\left(a_{2}^{2}\right)^{2}}\left[\left(a_{2}^{2}\right)^{2} a_{11}^{1}-2 a_{1}^{2} a_{2}^{2} a_{12}^{1}+\left(a_{1}^{2}\right)^{2} a_{22}^{1}\right], \\
A_{3}=\frac{2}{a_{2}^{2}}\left(a_{2}^{2} a_{12}^{1}-a_{1}^{2} a_{22}^{1}\right) B_{2}, A_{4}=\frac{2}{a_{2}^{2}}\left(a_{2}^{2} a_{12}^{1}-a_{1}^{2} a_{22}^{1}\right) B_{3}+a_{22}^{1} B_{2}^{2}, \ldots \tag{19}
\end{gather*}
$$

By Theorem 3, to determine the stability of the unperturbed motion described by system (13), it is necessary to study the expressions (19).

Let us introduce the following notations

$$
\begin{gather*}
P=\left(a_{2}^{2}\right)^{2} a_{11}^{1}-2 a_{1}^{2} a_{2}^{2} a_{12}^{1}+\left(a_{1}^{2}\right)^{2} a_{22}^{1}, \quad Q=\left(a_{2}^{2}\right)^{2} a_{11}^{2}-2 a_{1}^{2} a_{2}^{2} a_{12}^{2}+\left(a_{1}^{2}\right)^{2} a_{22}^{2}, \\
R=\left(a_{2}^{2}\right)^{2} a_{11}^{1}-\left(a_{1}^{2}\right)^{2} a_{22}^{1}, \quad S=a_{1}^{2} a_{22}^{1}-a_{2}^{2} a_{12}^{1} \tag{20}
\end{gather*}
$$

and take into account that according to (15) we have $a_{2}^{2}<0$.
Next, we observe that the stability of the unperturbed motion can occur when $A_{2}=0$ from (19), i.e. when $P=0$ from (20).

Assume in (18) that $B_{2}=0$, then (20) yields $Q=0$. This implies that all $B_{3}, B_{4}, \ldots$ are equal to zero. From this it follows that all the coefficients $A_{3}, A_{4}, \ldots$ vanish and therefore the stability of the unperturbed motion holds.

Suppose $B_{2} \neq 0$. If $S \neq 0$, then the stability of the unperturbed motion is determined by the sign of $A_{3}$ from (19). If in (20) $S=0$, then $A_{3}=0$ and the coefficient $A_{4}$ from (19) is non-zero if $a_{22}^{1} \neq 0$. Therefore, the stability is possible only if $a_{22}^{1}=0$. Observe that when $S=P=0$ in (20), then $R=0$. Hence, when $a_{22}^{1}=0$ the last two equations in (20) yield $a_{11}^{1}=a_{12}^{1}=0$.

Taking into account the inequality (15) and Theorem 3, we obtain the following results for stability of the unperturbed motion determined by the system of perturbed motion (13).

Lemma 4. The stability of the unperturbed motion described by system (13) under conditions (7) is characterized by one of the following six possible cases:
I. $P \neq 0$, then the unperturbed motion is unstable;
II. $P=0, Q S>0$, then the unperturbed motion is unstable;
III. $P=0, Q S<0$, then the unperturbed motion is stable;
$I V . R=S=0, a_{22}^{1} Q \neq 0$, then the unperturbed motion is unstable;
$V . P=Q=0$, then the unperturbed motion is stable;
VI. $a_{11}^{1}=a_{12}^{1}=a_{22}^{1}=0$, then the unperturbed motion is stable.

In the last two cases the unperturbed motion belongs to some continuous series of stabilized motions, moreover, it is also asymptotically stable [10] in Case III. The expressions $P, Q, R, S$ are given in (20).

Later on, we make use of the following expressions of the invariants and comitants
of system (11) given in (12):

$$
\begin{gather*}
E_{1}=I_{1}^{2} K_{1}-I_{1}\left(K_{3}+K_{4}\right)+K_{8}, \\
E_{2}=I_{1}^{3}\left(K_{1}^{2}-K_{7}\right)+2 I_{1}^{2}\left(K_{1} K_{4}-2 K_{1} K_{3}-K_{13}\right)+2 I_{1}\left(I_{5} K_{2}+2 K_{3}^{2}-K_{4}^{2}\right)+  \tag{21}\\
+4 K_{8}\left(K_{4}-K_{3}\right)+2 I_{2} K_{12}, \quad E_{3}=I_{2} K_{1}+I_{1}\left(K_{4}-K_{3}\right)-K_{8}, \\
E_{4}=I_{1}\left(K_{11}-K_{1} K_{2}\right)+K_{2}\left(K_{4}-K_{3}\right), \quad E_{5}=K_{11}-I_{1} K_{5} .
\end{gather*}
$$

Lemma 5. Suppose the first equality from (7) holds. Then the system (11) by a center-affine transformation can be brought to the form

$$
\begin{equation*}
\frac{d x^{1}}{d t}=0, \quad \frac{d x^{2}}{d t}=a_{\alpha}^{2} x^{\alpha}+a_{\alpha \beta}^{2} x^{\alpha} x^{\beta} \quad(\alpha, \beta=1,2) \tag{22}
\end{equation*}
$$

if and only if the following condition

$$
\begin{equation*}
E_{5} \equiv 0 \tag{23}
\end{equation*}
$$

holds, where $E_{5}$ is from (21).
Proof. Suppose the first relation from (7) holds. This allows us to write for (11)

$$
\begin{equation*}
a_{1}^{1}=r a_{1}^{2}, \quad a_{2}^{1}=r a_{2}^{2} \tag{24}
\end{equation*}
$$

Denote by $\Delta_{i j}$ the minors of matrix of the coefficients from the right-hand sides of system (11), where $i$ and $j$ represent the number of columns of this matrix on which the minors are built. Then

$$
E_{5}=\Delta_{13}\left(x^{1}\right)^{3}+\left(\Delta_{23}+2 \Delta_{14}\right)\left(x^{1}\right)^{2} x^{2}+\left(\Delta_{15}+2 \Delta_{24}\right) x^{1}\left(x^{2}\right)^{2}+\Delta_{25}\left(x^{2}\right)^{3}
$$

By means of this expressions and of conditions (23)-(24) we have

$$
\begin{equation*}
a_{11}^{1}=r a_{11}^{2}, \quad a_{12}^{1}=r a_{12}^{2}, \quad a_{22}^{1}=r a_{22}^{2} \tag{25}
\end{equation*}
$$

Taking into account (24) and (25), the center-affine transformation $\bar{x}^{1}=x^{1}-$ $r x^{2}, \bar{x}^{2}=x^{2}$ brings the system (11) to the form (15). Lemma 5 is proved.

Theorem 4. Let for differential system of the perturbed motion (11) the invariant conditions (7) be satisfied. Then the stability of the unperturbed motion in system (11) is described by one of the following six possible cases:
I. $E_{1} \not \equiv 0$, then the unperturbed motion is unstable;
II. $E_{1} \equiv 0, \quad E_{2}>0$, then the unperturbed motion is unstable;
III. $E_{1} \equiv 0, \quad E_{2}<0$, then the unperturbed motion is stable;
$I V . E_{3} \equiv 0, E_{4} E_{5} \not \equiv 0$, then the unperturbed motion is unstable;
$V . E_{4} \equiv 0$, then the unperturbed motion is stable;
$V I . E_{5} \equiv 0$, then the unperturbed motion is stable.
In the last two cases the unperturbed motion belongs to some continuous series of stabilized motions, and moreover, it is also asymptotically stable in Case III. The expressions $E_{i}(i=\overline{1,5})$ are given in (21).

Proof. Observe that expressions (21), for system (13) under condition (15), are expressed by (20) as follows:

$$
\begin{gather*}
E_{1}=P x^{1}, \quad E_{2}=4 S\left[Q\left(x^{1}\right)^{2}-P x^{1} x^{2}\right] \\
E_{3}=R x^{1}-2 a_{2}^{2} S x^{2}, \quad E_{4}=-Q\left(x^{1}\right)^{3}+P\left(x^{1}\right)^{2} x^{2} \tag{26}
\end{gather*}
$$

Setting $E_{3} \equiv 0$, then by means of the polynomials $R$ and $S$ from (20), we get for $E_{5}$ from (21) the expression $E_{5}=-a_{2}^{2} a_{22}^{1}\left(\frac{a_{1}^{2}}{a_{2}^{2}} x^{1}+x^{2}\right)^{3}$.

Using the last assertion, the expressions (22) and Lemmas 4 and 5, we get the Cases I-VI. We mention that the comitant $E_{2}$ from (21) is even with respect to $x^{1}$ and $x^{2}$ and has the weight equal to zero [2] in the Cases II and III. This ensures that any center-affine transformation cannot change the sign of $E_{2}$. Theorem 4 is proved.

Remark 3. From Theorem 4, the conditions for Lyapunov's Example 2 [1, §32] are obtained setting $a_{1}^{1}=a_{2}^{1}=0, a_{1}^{2}=k, a_{2}^{2}=-1, a_{11}^{1}=a, a_{12}^{1}=\frac{1}{2} b, a_{22}^{1}=c$, $a_{11}^{2}=l, a_{12}^{2}=\frac{1}{2} m, a_{22}^{2}=n$ and $x^{1}=x, x^{2}=y$.

## 4 Critical system of Lyapunov type with cubic nonlinearities

Let the differential system of perturbed motion with polynomial nonlinearities of the form

$$
\begin{align*}
& \frac{d x}{d t}=c x+d y+p x^{3}+3 q x^{2} y+3 r x y^{2}+s y^{3}, \\
& \frac{d y}{d t}=e x+f y+t x^{3}+3 u x^{2} y+3 v x y^{2}+w y^{3} \tag{27}
\end{align*}
$$

where $c, d, e, f, p, q, r, s, t, u, v, w$ are arbitrary real coefficients.
Similar to the previous case, when the characteristic equation of (27) has one zero root and the other one is negative, i.e. the conditions (7) are satisfied, then system (27) by a center-affine transformation can be brought to its critical form

$$
\begin{gather*}
\frac{d x}{d t}=p x^{3}+3 q x^{2} y+3 r x y^{2}+s y^{3}  \tag{28}\\
\frac{d y}{d t}=e x+f y+t x^{3}+3 u x^{2} y+3 v x y^{2}+w y^{3} .
\end{gather*}
$$

According to (10) we write the equation

$$
\begin{equation*}
e x+f y+t x^{3}+3 u x^{2} y+3 v x y^{2}+w y^{3}=0 \tag{29}
\end{equation*}
$$

By (6)-(7) we have for system (28) that $I_{1}=f<0$. Then from the last relation we express $y$ and obtain

$$
\begin{equation*}
y=-\frac{e}{f} x-\frac{t}{f} x^{3}-3 \frac{u}{f} x^{2} y-3 \frac{v}{f} x y^{2}-\frac{w}{f} y^{3} . \tag{30}
\end{equation*}
$$

We seek $y$ as a holomorphic function of $x$. Then we can write

$$
\begin{equation*}
y=-\frac{e}{f} x+B_{2} x^{2}+B_{3} x^{3}+B_{4} x^{4}+B_{5} x^{5}+B_{6} x^{6}+B_{7} x^{7}+B_{8} x^{8}+B_{9} x^{9}+\cdots \tag{31}
\end{equation*}
$$

Substituting (31) into (30) and identifying the coefficients of the same powers of $x$ in the obtained relation we have

$$
\begin{gather*}
B_{2 n}=0, \forall n \in \mathbb{N}, B_{3}=-\frac{t}{f}+3 \frac{e u}{f^{2}}-3 \frac{e^{2} v}{f^{3}}+\frac{e^{3} w}{f^{4}}, \\
B_{5}=-3\left(\frac{u}{f}-2 \frac{e v}{f^{2}}+\frac{e^{2} w}{f^{3}}\right) B_{3}, \\
B_{7}=-3\left[\left(\frac{v}{f}-\frac{e w}{f^{2}}\right) B_{3}-3\left(\frac{u}{f}-2 \frac{e v}{f^{2}}+\frac{e^{2} w}{f^{3}}\right)^{2}\right] B_{3},  \tag{32}\\
B_{9}=-\left[\frac{w}{f} B_{3}^{3}+6\left(\frac{v}{f}-\frac{e w}{f^{2}}\right) B_{3} B_{5}+3\left(\frac{u}{f}-2 \frac{e v}{f^{2}}+\frac{e^{2} w}{f^{3}}\right) B_{7}\right], \ldots
\end{gather*}
$$

Substituting (31) into the right-hand side of the critical differential equation (28) we obtain

$$
\begin{gathered}
p x^{3}+3 q x^{2} y+3 r x y^{2}+s y^{3}= \\
=A_{2} x^{2}+A_{3} x^{3}+A_{4} x^{4}+A_{5} x^{5}+A_{6} x^{6}+A_{7} x^{7}+A_{8} x^{8}+A_{9} x^{9}+A_{10} x^{10}+A_{11} x^{11}+\cdots
\end{gathered}
$$

From this, taking into account (31) and (32) we have

$$
\begin{gather*}
A_{2 n}=0, \forall n \in \mathbb{N}, A_{3}=p-3 \frac{e q}{f}+3 \frac{e^{2} r}{f^{2}}-\frac{e^{3} s}{f^{3}}, \\
A_{5}=3\left(q-2 \frac{e r}{f}+\frac{e^{2} s}{f^{2}}\right) B_{3}, A_{7}=3\left[\left(r-\frac{e s}{f}\right) B_{3}^{2}+\left(q-2 \frac{e r}{f}+\frac{e^{2} s}{f^{2}}\right) B_{5}\right],  \tag{33}\\
A_{9}=s B_{3}^{3}+6\left(r-\frac{e s}{f}\right) B_{3} B_{5}+3\left(q-2 \frac{e r}{f}+\frac{e^{2} s}{f^{2}}\right) B_{7}, \\
A_{11}=3\left[s B_{3}^{2} B_{5}+2\left(r-\frac{e s}{f}\right) B_{3} B_{7}+\left(r-\frac{e s}{f}\right) B_{5}^{2}+\left(q-2 \frac{e r}{f}+\frac{e^{2} s}{f^{2}}\right) B_{9}\right], \ldots
\end{gather*}
$$

We introduce the following notations:

$$
\begin{gather*}
T=f^{3} p-3 e f^{2} q+3 e^{2} f r-e^{3} s, \quad U=-f^{3} t+3 e f^{2} u-3 e^{2} f v+e^{3} w \\
V=f^{2} q-2 e f r+e^{2} s, \quad W=f r-e s \tag{34}
\end{gather*}
$$

Then, from (32) and (33), we get

$$
\begin{gather*}
A_{3}=\frac{1}{f^{3}} T, \quad B_{3}=\frac{1}{f^{4}} U, \quad A_{5}=\frac{3}{f^{2}} V B_{3}, \quad A_{7}=3\left(\frac{1}{f} W B_{3}^{2}+\frac{1}{f^{2}} V B_{5}\right),  \tag{35}\\
A_{9}=s B_{3}^{3}+\frac{6}{f} W B_{3} B_{5}+\frac{3}{f^{2}} V B_{7}, \ldots
\end{gather*}
$$

Using Theorem 3, the expressions (34) and (35) ( $I_{1}=f<0$ ), we come to the following statement.

Lemma 6. The stability of unperturbed motion in the system of perturbed motion (28) is described by one of the following ten possible cases:
I. $T<0$, then the unperturbed motion is unstable;
II. $\quad T>0$, then the unperturbed motion is stable;
III. $T=0, U V>0$, then the unperturbed motion is unstable;
$I V . T=0, U V<0$, then the unperturbed motion is stable;
$V . \quad T=V=0, U \neq 0, W<0$, then the unperturbed motion is unstable;
VI. $\quad T=V=0, U \neq 0, W>0$, then the unperturbed motion is stable;
VII. $\quad T=V=W=0, s U>0$, then the unperturbed motion is unstable;
VIII. $T=V=W=0, s U<0$, then the unperturbed motion is stable;
$I X . \quad T=U=0$, then the unperturbed motion is stable;
X. $p=q=r=s=0$, then the unperturbed motion is stable.

In the last two cases, the unperturbed motion belongs to some continuous series of stabilized motions, moreover, in Cases II, IV, VI and VIII this motion is also asymptotically stable [10]. The expressions $T, U, V, W$ are given in (34).

Proof. Assume $A_{3}>0$, then from (35) we get $\frac{T}{f^{3}}>0$. Taking into account that $f<0$, it follows $T<0$. By Theorem 3 we have proved the Case I. Similarly the Case II is analyzed.

Suppose in (34) that $U \neq 0$. Then from (35) we have $B_{3} \neq 0$.
If $A_{3}=0$, i.e. $T=0$, then by (35) the stability or the instability of unperturbed motion is determined by the sign of expression $U V$. Then using Theorem 3 we proved the Cases III and IV.

If $T=A_{5}=0$, i.e. $V=0$, then by (35) the stability or the instability of unperturbed motion is determined according to the sign of expression $\frac{U^{2} W}{f^{9}}$. Taking into account that $f<0$, by Theorem 3 we get the Cases V and VI.

If $A_{3}=A_{5}=A_{7}=0(T=V=W=0)$, then the stability or the instability of unperturbed motion is determined by the sign of expression $A_{9}$, i.e. $\frac{s U}{f^{12}}$. From this, according to Theorem 3, we obtain the Cases VII and VIII. If $T=U=0$, then all $A_{k}(k \geq 3)$ are equal to zero. By Theorem 3 we have the Case IX. If $U \neq 0$ and $T=V=W=s=0$, then from (34) we obtain the Case X. Lemma 6 is proved.

Proceeding from the polynomial bases of center-affine comitants and invariants of the system (27) given in [11], we can write the Sibirsky algebras with generators
$S_{1,3}=\left\{J_{1}, J_{2}, \ldots, J_{20}, K_{1}, K_{2}, \ldots, K_{13}, Q_{1}, Q_{2}, \ldots, Q_{14}\right\}, S I_{1,3}=\left\{J_{1}, J_{2}, \ldots, J_{20}\right\}$,
where $J_{i}, K_{j}$ and $Q_{k}$ are invariants and comitants of these algebras.
For the system (27) we have the notations

$$
\begin{align*}
& x^{1}=x, a_{1}^{1}=c, a_{2}^{1}=d, a_{111}^{1}=p, a_{112}^{1}=q, a_{122}^{1}=r, a_{222}^{1}=s, \\
& x^{2}=y, a_{1}^{2}=e, a_{2}^{2}=f, a_{111}^{2}=t, a_{112}^{2}=u, a_{122}^{2}=v, a_{222}^{2}=w . \tag{36}
\end{align*}
$$

Further we will need the following generators of Sibirsky algebras $S_{1,3}$ and $S I_{1,3}$, which in tensorial form are written

$$
\begin{gather*}
J_{1} \equiv I_{1}=a_{\alpha}^{\alpha}, \quad J_{2} \equiv I_{2}=a_{\beta}^{\alpha} a_{\alpha}^{\beta}, \quad J_{3}=a_{\pi}^{\alpha} a_{k \alpha \beta}^{\beta} \varepsilon^{\pi k}, \quad J_{6}=a_{\pi}^{\alpha} a_{\gamma}^{\beta} a_{k \alpha \beta}^{\gamma} \varepsilon^{\pi k}, \\
K_{1}=a_{\beta}^{\alpha} x^{\beta} x^{\gamma} \varepsilon_{\alpha \gamma}, K_{2}=a_{\alpha \beta \gamma}^{\alpha} x^{\beta} x^{\gamma}, \quad K_{3}=a_{\alpha \beta \gamma}^{\pi} x^{\alpha} x^{\beta} x^{\gamma} x^{k} \varepsilon_{\pi k}, \\
Q_{1}=a_{\alpha}^{\pi} a_{\beta \gamma \delta}^{k} x^{\alpha} x^{\beta} x^{\gamma} x^{\delta} \varepsilon_{\pi k}, \quad Q_{2}=a_{\beta}^{\alpha} a_{\alpha \gamma \delta}^{\beta} x^{\gamma} x^{\delta},  \tag{37}\\
Q_{3}=a_{\gamma}^{\alpha} a_{\alpha \beta \delta}^{\beta} x^{\gamma} x^{\delta}, \quad Q_{4}=a_{\gamma}^{\alpha} a_{\delta}^{\beta} a_{\alpha \beta \eta}^{\gamma} x^{\delta} x^{\eta} .
\end{gather*}
$$

By means of these generators, we compose the following invariant expressions:

$$
\begin{gather*}
F_{1}=K_{1}\left(J_{6}-J_{1} J_{3}\right)+J_{1}\left[J_{1}^{2} K_{2}-J_{1}\left(Q_{2}+Q_{3}\right)+Q_{4}\right], \quad F_{2}=J_{6}-J_{1} J_{3}, \\
F_{3}=K_{1}\left[J_{3} K_{1}-J_{1}\left(J_{1} K_{2}+2 Q_{2}-Q_{3}\right)+Q_{4}\right]+J_{1}^{2}\left(J_{1} K_{3}+Q_{1}\right),  \tag{38}\\
F_{4}=J_{1} K_{2}-Q_{2}, \quad F_{5}=Q_{1} .
\end{gather*}
$$

Lemma 7. Suppose that the first relation from (7) is satisfied. Then the system (27) by a center-affine transformation can be brought to the form

$$
\frac{d x}{d t}=0, \quad \frac{d y}{d t}=e x+f y+t x^{3}+3 u x^{2} y+3 v x y^{2}+w y^{3}
$$

if and only if $F_{5} \equiv 0$, where $F_{5}$ is from (38).
The proof is similar to Lemma 6. We make use of $F_{5}$ which for system (27) has the form

$$
\begin{aligned}
F_{5}=\Delta_{13}\left(x^{1}\right)^{4} & +\left(\Delta_{23}+3 \Delta_{14}\right)\left(x^{1}\right)^{3} x^{2}+3\left(\Delta_{15}+\Delta_{24}\right)\left(x^{1}\right)^{2}\left(x^{2}\right)^{2}+ \\
& +\left(\Delta_{16}+3 \Delta_{25}\right) x^{1}\left(x^{2}\right)^{3}+\Delta_{26}\left(x^{2}\right)^{4},
\end{aligned}
$$

where $\Delta_{i j}$ are the minors of matrix of the coefficients from the right-hand sides of system (27) built on columns $i$ and $j$ of this matrix.

Theorem 5. Let for differential system of the perturbed motion

$$
\frac{d x^{j}}{d t}=a_{\alpha}^{j} x^{\alpha}+a_{\alpha \beta \gamma}^{j} x^{\alpha} x^{\beta} x^{\gamma} \quad(j, \alpha, \beta, \gamma=1,2)
$$

the invariant conditions $J_{1}^{2}-J_{2}=0, J_{1}<0$ be satisfied. Then the stability of the unperturbed motion is described by one of the following ten possible cases:
I. $F_{1}<0$, then the unperturbed motion is unstable;
II. $F_{1}>0$, then the unperturbed motion is stable;
III. $F_{1} \equiv 0, F_{2} F_{3}>0$, then the unperturbed motion is unstable;
IV. $F_{1} \equiv 0, F_{2} F_{3}<0$, then the unperturbed motion is stable;
V. $F_{1} \equiv 0, F_{2}=0, F_{3} \not \equiv 0, F_{4}<0$, then the unperturbed motion is unstable;
VI. $F_{1} \equiv 0, F_{2}=0, F_{3} \not \equiv 0, F_{4}>0$, then the unperturbed motion is stable;
VII. $F_{1} \equiv 0, F_{2}=0, F_{4} \equiv 0, F_{3} F_{5}>0$, then the unperturbed motion is unstable;
VIII. $F_{1} \equiv 0, F_{2}=0, F_{4} \equiv 0, F_{3} F_{5}<0$, then the unperturbed motion is stable;
$I X . F_{3} \equiv 0$, then the unperturbed motion is stable;
X. $F_{5} \equiv 0$, then the unperturbed motion is stable.

In the last two cases the unperturbed motion belongs to some continuous series of stabilized motions, and moreover, it is also asymptotically stable in Cases II, IV, VI, VIII. The expressions $F_{i}(i=\overline{1,5})$ are given in (38).

Proof. Observe that the first three expressions from (38), for critical system (28) with notations (36), look as follows:

$$
\begin{equation*}
F_{1}=T x^{2}, \quad F_{2}=V, \quad F_{3}=U x^{4}+T x^{3} y . \tag{39}
\end{equation*}
$$

Suppose that $F_{1} \equiv 0, F_{2}=0$. Then by means of the polynomials $T, V, W$ from (34), we get for expression $F_{4}$ from (38) that $F_{4}=W\left(\frac{e}{f} x+y\right)^{2}$. Using the expressions (39), the last assertion together with Lemmas 6 and 7, we obtain the Cases I-X. We note that the comitants $F_{1}, F_{2} F_{3}, F_{4}, F_{3} F_{5}$ from (38), used in the Cases I-VIII of Theorem 5, are even-degree comitants with respect to $x$ and $y$ and have the weights [2] equal to $0,0,0,-2$, respectively. Moreover, each one of these comitants (in the case when it is applied) is a binary form with a well defined sing. This ensures that any center-affine transformation cannot change their sign. Theorem 5 is proved.

## 5 Critical system of Lyapunov type with nonlinearities of degree four

We consider the differential system of perturbed motion with polynomial nonlinearities

$$
\begin{align*}
& \frac{d x}{d t}=c x+d y+g x^{4}+4 h x^{3} y+6 k x^{2} y^{2}+4 l x y^{3}+m y^{4}  \tag{40}\\
& \frac{d y}{d t}=e x+f y+n x^{4}+4 p x^{3} y+6 q x^{2} y^{2}+4 r x y^{3}+s y^{4}
\end{align*}
$$

where $c, d, e, f, g, h, k, l, m, n, p, q, r, s$ are real arbitrary coefficients.
Similar to the previous cases, when the characteristic equation of (40) has one zero root and the other one is negative, i.e. the conditions (7) are satisfied, then this system by a center-affine transformation can be brought to its critical form

$$
\begin{gather*}
\frac{d x}{d t}=g x^{4}+4 h x^{3} y+6 k x^{2} y^{2}+4 l x y^{3}+m y^{4},  \tag{41}\\
\frac{d y}{d t}=e x+f y+n x^{4}+4 p x^{3} y+6 q x^{2} y^{2}+4 r x y^{3}+s y^{4} .
\end{gather*}
$$

According to Theorem 3, we analyze the equation

$$
\begin{equation*}
e x+f y+n x^{4}+4 p x^{3} y+6 q x^{2} y^{2}+4 r x y^{3}+s y^{4}=0 \tag{42}
\end{equation*}
$$

As for system (40) we have $I_{1}=f<0$, then from (42) we express $y$ :

$$
\begin{equation*}
y=-\frac{e}{f} x-\frac{n}{f} x^{4}-4 \frac{p}{f} x^{3} y-6 \frac{q}{f} x^{2} y^{2}-4 \frac{r}{f} x y^{3}-\frac{s}{f} y^{4} . \tag{43}
\end{equation*}
$$

We seek $y$ as a holomorphic function of $x$. Then we can write

$$
\begin{gather*}
y=-\frac{e}{f} x+B_{2} x^{2}+B_{3} x^{3}+B_{4} x^{4}+B_{5} x^{5}+B_{6} x^{6}+B_{7} x^{7}+B_{8} x^{8}+B_{9} x^{9}+  \tag{44}\\
+B_{10} x^{10}+B_{11} x^{11}+B_{12} x^{12}+B_{13} x^{13}+B_{14} x^{14}+B_{15} x^{15}+B_{16} x^{16}+\cdots
\end{gather*}
$$

Substituting (44) into (43) and equating the coefficients of monomials in $x$, we find that

$$
\begin{gather*}
B_{i}=0(i=2,3,5,6,8,9,11,12,14,15, \ldots), B_{4}=-\left(\frac{n}{f}-4 \frac{e p}{f^{2}}+6 \frac{e^{2} q}{f^{3}}-4 \frac{e^{3} r}{f^{4}}+\frac{e^{4} s}{f^{5}}\right), \\
B_{7}=-4\left(\frac{p}{f}-3 \frac{e q}{f^{2}}+3 \frac{e^{2} r}{f^{3}}-\frac{e^{3} s}{f^{4}}\right) B_{4}, \\
B_{10}=-2\left[3\left(\frac{q}{f}-2 \frac{e r}{f^{2}}+\frac{e^{2} s}{f^{3}}\right) B_{4}^{2}+2\left(\frac{p}{f}-3 \frac{e q}{f^{2}}+3 \frac{e^{2} r}{f^{3}}-\frac{e^{3} s}{f^{4}}\right) B_{7}\right], \\
B_{13}=-4\left[\left(\frac{r}{f}-\frac{e s}{f^{2}}\right) B_{4}^{3}+3\left(\frac{q}{f}-2 \frac{e r}{f^{2}}+\frac{e^{2} s}{f^{3}}\right) B_{4} B_{7}+\right. \\
\left.+\left(\frac{p}{f}-3 \frac{e q}{f^{2}}+3 \frac{e^{2} r}{f^{3}}-\frac{e^{3} s}{f^{4}}\right) B_{10}\right], \\
B_{16}=-\left[\frac{s}{f} B_{4}^{4}+12\left(\frac{r}{f}-\frac{e s}{f^{2}}\right) B_{4}^{2} B_{7}+6\left(\frac{q}{f}-2 \frac{e r}{f^{2}}+\frac{e^{2} s}{f^{3}}\right)\left(2 B_{4} B_{10}+B_{7}^{2}\right)+\right. \\
\left.+4\left(\frac{p}{f}-3 \frac{e q}{f^{2}}+3 \frac{e^{2} r}{f^{3}}-\frac{e^{3} s}{f^{4}}\right) B_{13}\right], \ldots \tag{45}
\end{gather*}
$$

Substituting (44) into the right-hand side of the critical differential equation, then from (41) we get

$$
g x^{4}+4 h x^{3} y+6 k x^{2} y^{2}+4 l x y^{3}+m y^{4}=A_{2} x^{2}+A_{3} x^{3}+\cdots+A_{16} x^{16}+\cdots
$$

Hence, taking into account (44) and (45) we have

$$
\begin{gather*}
A_{i}=0(i=2,3,5,6,8,9,, 11,12,14,15, \ldots), A_{4}=g-4 \frac{e h}{f}+6 \frac{e^{2} k}{f^{2}}-4 \frac{e^{3} l}{f^{3}}+\frac{e^{4} m}{f^{4}}, \\
A_{7}=4\left(h-3 \frac{e k}{f}+3 \frac{e^{2} l}{f^{2}}-\frac{e^{3} m}{f^{3}}\right) B_{4} \\
A_{10}=2\left[3\left(k-2 \frac{e l}{f}+\frac{e^{2} m}{f^{2}}\right) B_{4}^{2}+2\left(h-3 \frac{e k}{f}+3 \frac{e^{2} l}{f^{2}}-\frac{e^{3} m}{f^{3}}\right) B_{7}\right] \\
A_{13}=4\left[\left(l-\frac{e m}{f}\right) B_{4}^{3}+3\left(k-2 \frac{e l}{f}+\frac{e^{2} m}{f^{2}}\right) B_{4} B_{7}+\right. \\
\left.+\left(h-3 \frac{e k}{f}+3 \frac{e^{2} l}{f^{2}}-\frac{e^{3} m}{f^{3}}\right) B_{10}\right] \\
A_{16}=m B_{4}^{4}+12\left(l-\frac{e m}{f}\right) B_{4}^{2} B_{7}+6\left(k-2 \frac{e l}{f}+\frac{e^{2} m}{f^{2}}\right)\left(2 B_{4} B_{10}+B_{7}^{2}\right)+  \tag{46}\\
+4\left(h-3 \frac{e k}{f}+3 \frac{e^{2} l}{f^{2}}-\frac{e^{3} m}{f^{3}}\right) B_{13}, \ldots
\end{gather*}
$$

Let us introduce the following notation:

$$
\begin{gather*}
A=f^{4} g-4 e f^{3} h+6 e^{2} f^{2} k-4 e^{3} f l+e^{4} m, \\
B=-f^{4} n+4 e f^{3} p-6 e^{2} f^{2} q+4 e^{3} f r-e^{4} s,  \tag{47}\\
C=f^{3} h-3 e f^{2} k+3 e^{2} f l-e^{3} m, \quad D=f^{2} k-2 e f l+e^{2} m, \quad E=f l-e m .
\end{gather*}
$$

Then taking into account (47), we obtain from (45)-(46) that

$$
\begin{gather*}
A_{4}=\frac{1}{f^{4}} A, \quad B_{4}=\frac{1}{f^{5}} B, \quad A_{7}=\frac{4}{f^{8}} B C, \quad A_{10}=2\left(\frac{3}{f^{12}} B^{2} D+\frac{2}{f^{3}} C B_{7}\right), \\
A_{13}=4\left(\frac{1}{f^{16}} B^{3} E+\frac{3}{f^{2}} D B_{4} B_{7}+\frac{1}{f^{3}} C B_{10}\right)  \tag{48}\\
A_{16}=m B_{4}^{4}+\frac{12}{f} E B_{4}^{2} B_{7}+\frac{6}{f^{2}} D\left(2 B_{4} B_{10}+B_{7}^{2}\right)+\frac{4}{f^{3}} C B_{13}, \ldots
\end{gather*}
$$

Lemma 8. The stability of unperturbed motion in the system of perturbed motion (41) is described by nine possible cases, if for expressions (47) ( $I_{1}=f<0$ ) the following conditions are satisfied:
I. $A \neq 0$, then the unperturbed motion is unstable;
II. $A=0, B C>0$, then the unperturbed motion is unstable;
III. $A=0, B C<0$, then the unperturbed motion is stable;
IV. $A=C=0, B D \neq 0$, then the unperturbed motion is unstable;
V. $A=C=D=0, B E>0$, then the unperturbed motion is unstable;
VI. $A=C=D=0, B E<0$, then the unperturbed motion is stable;
VII. $A=C=D=E=0, m B \neq 0$, then the unperturbed motion is unstable;
VIII. $A=B=0$, then the unperturbed motion is stable;
$I X . g=h=k=l=m=0$, then the unperturbed motion is stable.
In the last two cases, the unperturbed motion belongs to some continuous series of stabilized motions. Moreover, this motion is also asymptotically stable [10] in Cases III and VI. The expressions $A, B, C, D, E$ are given in (47).

Proof. If $A_{4} \neq 0$, then from (48) we have $A \neq 0$. By Theorem 3, we get the Case I.
Suppose in (47) that $B \neq 0$. Then (48) implies that $B_{4} \neq 0$. If $A_{4}=0$, i.e. $A=0$, then according to (48) the stability or the instability of unperturbed motion is determined by the sign of the expression $A_{7}$ (the sign of the product $B C$ ). Using Theorem 3 we obtain the Cases II and III.

When $A=A_{7}=0$, i.e. $C=0$, then from (48) we have $A_{10}=\frac{6}{f^{12}} B^{2} D$. If $D \neq 0$, then we obtain the Case IV (see Theorem 3).

Suppose $A=C=D=0$. Then from (48) it results that $A_{13} \neq 0$, when $B E \neq 0$. So the stability or the instability of the unperturbed motion is determined by the sign of expression $B E$. Using Theorem 3 we get the Cases V and VI.

When $A_{4}=A_{7}=A_{10}=A_{13}=0(B \neq 0)$, then we have $A=C=D=E=0$. If $A_{16} \neq 0$, then from (48) we obtain the Case VII. If $A=B=0$, then all $A_{k}(k \geq 4)$ vanish. By Theorem 3 we get the Case VIII. If $A=C=D=E=0$ and $m=0$, then (47) with $f<0$ implies the Case IX. Lemma 8 is proved.

Let $\varphi$ and $\psi$ be homogeneous comitants of degree $\rho_{1}$ and $\rho_{2}$ respectively of the phase variables $x$ and $y$ of a two-dimensional polynomial differential system. Then by [3] the transvectant

$$
\begin{equation*}
(\varphi, \psi)^{(j)}=\frac{\left(\rho_{1}-j\right)\left(\rho_{2}-j\right)}{\rho_{1}!\rho_{2}!} \sum_{i=0}^{j}(-1)^{j}\binom{j}{i} \frac{\partial^{j} \varphi}{\partial x^{j-i} \partial y^{i}} \frac{\partial^{j} \psi}{\partial x^{i} \partial y^{j-i}} \tag{49}
\end{equation*}
$$

is also a comitant for this system.
In the Iu. Calin's works, see for example [12], it is shown that by means of the transvectant (49) all generators of the Sibirsky algebras of comitants and invariants for any system of type (1) can be constructed.

We denote the homogeneities from the right-hand sides of system (40) as follows:

$$
\begin{align*}
& P_{1}(x, y)=c x+d y, \quad P_{4}(x, y)=g x^{4}+4 h x^{3} y+6 k x^{2} y^{2}+4 l x y^{3}+m y^{4} \\
& Q_{1}(x, y)=e x+f y, \quad Q_{4}(x, y)=n x^{4}+4 p x^{3} y+6 q x^{2} y^{2}+4 r x y^{3}+s y^{4} . \tag{50}
\end{align*}
$$

According to [13], we write the following comitants of the system (40)

$$
\begin{equation*}
R_{i}=P_{i}(x, y) y-Q_{i}(x, y) x, \quad S_{i}=\frac{1}{i}\left(\frac{\partial P_{i}(x, y)}{\partial x}+\frac{\partial Q_{i}(x, y)}{\partial y}\right), \quad(i=1,4) . \tag{51}
\end{equation*}
$$

Later on, we will need the following comitants and invariants from [13] of system (40) built by operations (49) and (51):

$$
\begin{gather*}
I_{1}=S_{1}, \quad I_{2}=\left(R_{1}, R_{1}\right)^{(2)}, \quad K_{1}=R_{4}, \quad K_{2}=S_{4}, \quad Q_{1}=R_{1}, \quad Q_{2}=S_{1}, \\
Q_{3}=\left(R_{4}, R_{1}\right)^{(2)}, Q_{4}=\left(R_{4}, R_{1}\right)^{(1)}, \quad Q_{5}=\left(S_{4}, S_{1}\right)^{(2)}, \quad Q_{6}=\left(S_{4}, R_{1}\right)^{(1)}, \\
\left.\left.\left.\left.Q_{19}=\llbracket R_{4}, R_{1}\right)^{(2)}, R_{1}\right)^{(2)}, \quad Q_{20}=\llbracket R_{4}, R_{1}\right)^{(2)}, R_{1}\right)^{(1)},  \tag{52}\\
\left.\left.\left.\left.\left.Q_{21}=\llbracket S_{4}, R_{1}\right)^{(2)}, R_{1}\right)^{(1)}, \quad Q_{43}=\llbracket R_{4}, R_{1}\right)^{(2)}, R_{1}\right)^{(2)}, R_{1}\right)^{(1)},
\end{gather*}
$$

where the sign " $[$ " denotes all the parentheses of the transvectant that have to be written in the left.

We consider for system (40) the following expressions composed of comitants and invariants from (52) that can be written in the form:

$$
\begin{gather*}
H_{1}=Q_{1}\left[Q_{2}\left(15 Q_{19}-8 Q_{21}\right)-10 Q_{43}+12 I_{1}^{2} Q_{5}\right]+Q_{2}^{2}\left[Q_{2}\left(4 K_{2} Q_{2}+5 Q_{3}-8 Q_{6}\right)-10 Q_{20}\right], \\
H_{2}=5 Q_{2}^{3}\left(K_{1} Q_{2}-2 Q_{4}\right)+2 Q_{1}^{2}\left(5 Q_{19}+4 Q_{21}-6 Q_{2} Q_{5}\right)-4 Q_{1} Q_{2}\left[Q_{2}\left(K_{2} Q_{2}-5 Q_{3}-2 Q_{6}\right)+\right. \\
\left.+5 Q_{20}\right], \quad H_{3}=Q_{2}\left(5 Q_{19}-6 Q_{21}+3 Q_{2} Q_{5}\right)-10 Q_{43}, \quad H_{4}=5 I_{1} Q_{5}+10 Q_{19}-2 Q_{21}, \\
H_{5}=Q_{1}, \quad H_{6}=5 I_{1} K_{2}+10 Q_{3}-6 Q_{6}, \quad H_{7}=8 K_{2} Q_{1}-5 K_{1} Q_{2}-10 Q_{4} . \tag{53}
\end{gather*}
$$

Lemma 9. Suppose that the first equality holds in (7). Then by a center-affine transformation the system (40) can be brought to the form

$$
\frac{d x}{d t}=0, \frac{d y}{d t}=e x+f y+n x^{4}+4 p x^{3} y+6 q x^{2} y^{2}+4 r x y^{3}+s y^{4}
$$

if and only if the condition $H_{7} \equiv 0$ is satisfied, where $H_{7}$ is from (53).

The proof of this Lemma is similar to Lemma 6. Here, the fact is used that $H_{7}$ from (53), for the system (40), is of the form

$$
\begin{aligned}
H_{7} & =10\left[\Delta_{13}\left(x^{1}\right)^{5}+\left(\Delta_{23}+4 \Delta_{14}\right)\left(x^{1}\right)^{4} x^{2}+2\left(2 \Delta_{24}+3 \Delta_{15}\right)\left(x^{1}\right)^{3}\left(x^{2}\right)^{2}+\right. \\
& \left.+2\left(2 \Delta_{16}+3 \Delta_{25}\right)\left(x^{1}\right)^{2}\left(x^{2}\right)^{3}+\left(\Delta_{17}+4 \Delta_{26}\right) x^{1}\left(x^{2}\right)^{4}+\Delta_{27}\left(x^{2}\right)^{5}\right]
\end{aligned}
$$

where $\Delta_{i j}$ are the minors of the matrix of coefficients from the right-hand sides of system (40), built on the columns $i$ and $j$ of this matrix.

Theorem 6. Let for system of perturbed motion (40) the invariant conditions (7) be satisfied. Then the stability of the unperturbed motion is described by one of the following nine possible cases:
I. $H_{1} \not \equiv 0$, then the unperturbed motion is unstable;
II. $H_{1} \equiv 0, H_{2} H_{3}>0$, then the unperturbed motion is unstable;
III. $H_{1} \equiv 0, H_{2} H_{3}<0$, then the unperturbed motion is stable;
IV. $H_{1} \equiv H_{3} \equiv 0, H_{2} H_{4} \not \equiv 0$, then the unperturbed motion is unstable;
V. $H_{1} \equiv H_{3} \equiv H_{4} \equiv 0, H_{2} H_{5} H_{6}>0$, then the unperturbed motion is unstable;
VI. $H_{1} \equiv H_{3} \equiv H_{4} \equiv 0, H_{2} H_{5} H_{6}<0$, then the unperturbed motion is stable;
VII. $H_{1} \equiv H_{3} \equiv H_{4} \equiv H_{6} \equiv 0, H_{2} H_{7} \not \equiv 0$, then the unperturbed motion is unstable;
VIII. $H_{2} \equiv 0$, then the unperturbed motion is stable;
$I X . H_{7} \equiv 0$, then the unperturbed motion is stable.
In the last two cases, the unperturbed motion belongs to some continuous series of stabilized motions, and moreover in Cases III, and VI this motion is also asymptotically stable [2]. The expressions $H_{i}(i=\overline{1,7})$ are given in (53).

Proof. The first three expressions from (53), for the critical system (41), give

$$
\begin{equation*}
H_{1}=10 A x^{3}, H_{2}=10 B x^{5}+10 A x^{4} y, H_{3}=10 C x \tag{54}
\end{equation*}
$$

Next the proof is based on Lemma 8. The Case I is obvious if we use (54). Put $H_{1}=0$, then by Lemma 8, from (54) we obtain the Cases II and III.

The product $\mathrm{H}_{2} \mathrm{H}_{3}$ is of even degree with respect to $x$ and has the weight equal to 0 [2]. Therefore, the expression $H_{2} H_{3}$ under any center-affine transformation does not change its sign. Using (54), the Case IV of Lemma 8 implies the Case IV of Theorem 6 and we have $\left(f=I_{1}<0\right)$

$$
\begin{equation*}
H_{2}=10 B x^{5}, H_{4}=10 D\left(\frac{e}{f} x+y\right) \tag{55}
\end{equation*}
$$

For the Cases V and VI of the Theorem, Lemma 8 yields $A=C=D=0$. Then from (54) and (55), we obtain the invariant equations for the examined cases. By means of these equations and the expressions from (53), we obtain

$$
\begin{equation*}
H_{2}=10 B x^{5}, \quad H_{5}=-f x\left(\frac{e}{f} x+y\right), \quad H_{6}=10 E\left(\frac{e}{f}+y\right)^{3} \tag{56}
\end{equation*}
$$

From this, we get $H_{2} H_{5} H_{6}=-100 f B E x^{6}\left(\frac{e}{f} x+y\right)^{4}$. This product is of even degree with respect to $x$ and $y$ and have the weight -2 and has a well defined sign. Hence, we have the Cases V-VI.

The Case VII of Theorem 6 is obtained by using the Case VII of Lemma 8 and the expressions (54)-(56). Indeed, for this case we have

$$
H_{7}=-10 f m\left(\frac{e}{f} x+y\right)^{4} .
$$

By means of $H_{7}$ and $H_{2}$ from (54), we get the Case VII with inequality $H_{2} H_{7}$.
The Case VIII of the Theorem results from (54) using the Case VIII of Lemma 8 and expressions (47)-(48). The Case IX results from the Case IX of Lemma 8 and the assertion of Lemma 9. Theorem 6 is proved.

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Natalia Neagu
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Tiraspol State University,
Ion Creangă State Pedagogical University, Chişinău, Republic of Moldova
E-mail: neagu_natusik@mail.ru
Victor Orlov
Technical University of Moldova,
Institute of Mathematics and Computer Science, Chişinău, Republic of Moldova
E-mail: orlovictor@gmail.com
Mihail Popa
Institute of Mathematics and Computer Science, Chişinǎu, Republic of Moldova
E-mail: mihailpomd@gmail.com

# A Note on 2-Hypersurfaces of the Nearly Kählerian Six-Sphere 

Ahmad Abu-Saleem, Mihail B. Banaru, Galina A. Banaru


#### Abstract

It is proved that hypersurfaces with type number two in a nearly Kählerian sphere $S^{6}$ admit almost contact metric structures of cosymplectic type that are non-cosymplectic.


Mathematics subject classification: 53B35, 53B50.
Keywords and phrases: Almost contact metric structure, structure of cosymplectic type, nearly Kählerian manifold, six-dimensional sphere, almost Hermitian manifold, type number.

## 1 Introduction

The six-dimensional sphere $S^{6}$ with a canonical nearly Kählerian structure was the first example of non-Kählerian almost Hermitian manifold. That is why it presents a special interest for researchers in the area of Hermitian geometry. Such outstanding geometers as A. Gray, V. F. Kirichenko, K. Sekigawa and N. Ejiri have studied diverse aspects of the geometry of nearly Kählerian six-dimensional sphere. Of course, the geometry of nearly Kählerian manifolds (or $W_{1}$-manifolds, after GrayHervella classification [14]) is a spacious and important part of Hermitian geometry.

It is known that almost contact metric structures are induced on oriented hypersurfaces of almost Hermitian manifolds. Many specialists observe that this fact determines the profound connection between the contact and Hermitian geometries. Almost contact metric structures on hypersurfaces of almost Hermitian manifolds were studied by some remarkable geometers. The work of D. E. Blair, S. Goldberg, S. Ishihara, S. Sasaki, H. Yanamoto and K. Yano are assumed classical. In the present note, almost contact metric structures on 2-hypersurfaces (i.e. on hypersurfaces with type number 2) of nearly Kählerian six-dimensional sphere are considered.

In [3] and [11], it was proved that if $t$ is the type number of an oriented hypersurface of the nearly Kählerian six-sphere $S^{6}$, then the condition $t \leq 1$ holds if and only if the induced almost contact metric structure on this hypersurface is nearly cosymplectic.

In this paper, an additional result on almost contact metric hypersurfaces of nearly Kählerian six-dimensional sphere is given. Namely, we shall show that 2type hypersurfaces of a nearly Kählerian six-sphere $S^{6}$ admit almost contact metric structures of cosymplectic type.
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## 2 Preliminaries

The almost Hermitian manifold is an even-dimensional manifold $M^{2 n}$ with a Riemannian metric $g=\langle\cdot, \cdot\rangle$ and an almost complex structure $J$. These objects must satisfy the following condition

$$
\langle J X, J Y\rangle=\langle X, Y\rangle, \quad X, Y \in \aleph\left(M^{2 n}\right),
$$

where $\aleph\left(M^{2 n}\right)$ is the module of smooth vector fields on $M^{2 n}$. All considered manifolds, tensor fields and similar objects are assumed to be of the class $C^{\infty}$.

The specification of an almost Hermitian structure on a manifold is equivalent to the setting of a $G$-structure, where $G$ is the unitary group $U(n)$ [5], [15]. Its elements are the frames adapted to the structure ( $A$-frames). They look as follows:

$$
\left(p, \varepsilon_{1}, \ldots, \varepsilon_{n}, \varepsilon_{\hat{1}}, \ldots, \varepsilon_{\hat{n}}\right)
$$

where $\varepsilon_{a}$ are the eigenvectors corresponded to the eigenvalue $i=\sqrt{-1}$, and $\varepsilon_{\hat{a}}$ are the eigenvectors corresponded to the eigenvalue $-i$. Here the index $a$ ranges from 1 to $n$, and we state $\hat{a}=a+n$.

The matrixes of the operator of the almost complex structure and of the Riemannian metric written in an $A$-frame look as follows, respectively:

$$
\left(J_{j}^{k}\right)=\left(\begin{array}{c|c}
i I_{n} & 0 \\
\hline 0 & -i I_{n}
\end{array}\right), \quad\left(g_{k j}\right)=\left(\begin{array}{c|c}
0 & I_{n} \\
\hline I_{n} & 0
\end{array}\right),
$$

where $I_{n}$ is the identity matrix; $k, j=1, \ldots, 2 n$.
We recall [16] that the fundamental form of an almost Hermitian manifold is determined by the relation

$$
F(X, Y)=\langle X, J Y\rangle, \quad X, Y \in \aleph\left(M^{2 n}\right)
$$

By direct computing it is easy to obtain that in an $A$-frame the fundamental form matrix looks as follows:

$$
\left(F_{k j}\right)=\left(\begin{array}{c|c}
0 & i I_{n} \\
\hline-i I_{n} & 0
\end{array}\right) .
$$

The first group of the Cartan structural equations of an almost Hermitian manifold written in an $A$-frame looks as follows [9], [16]:

$$
\begin{align*}
& d \omega^{a}=\omega_{b}^{a} \wedge \omega^{b}+B^{a b}{ }_{c} \omega^{c} \wedge \omega_{b}+B^{a b c} \omega_{b} \wedge \omega_{c} ;  \tag{1}\\
& d \omega_{a}=-\omega_{a}^{b} \wedge \omega_{b}+B_{a b}{ }^{c} \omega_{c} \wedge \omega^{b}+B_{a b c} \omega^{b} \wedge \omega^{c},
\end{align*}
$$

where

$$
\begin{aligned}
B^{a b}{ }_{c} & =-\frac{i}{2} J_{\hat{b}, c}^{a} ; B_{a b}{ }^{c}=\frac{i}{2} J_{b, \hat{c}}^{\hat{a}} ; \\
B^{a b c} & =\frac{i}{2} J_{[\hat{b}, \hat{c}]}^{a} ; B_{a b c}=-\frac{i}{2} J_{[b, c]}^{\hat{a}} .
\end{aligned}
$$

The systems of functions $\left\{B^{a b}{ }_{c}\right\},\left\{B_{a b}{ }^{c}\right\},\left\{B^{a b c}\right\},\left\{B_{a b c}\right\}$, are the components of the Kirichenko tensors of the almost Hermitian manifold [2], [9], $a, b, c=1, \ldots, n ;, \hat{a}=a+n$.

An almost Hermitian manifold is called nearly Kählerian [14], [16] if

$$
\nabla_{X}(F)(X, Y)=0, \quad X, Y \in \aleph\left(M^{2 n}\right)
$$

The almost contact metric structure on an odd-dimensional manifold $N$ is defined by the system of tensor fields $\{\Phi, \xi, \eta, g\}$ on this manifold, where $\xi$ is a vector field, $\eta$ is a covector field, $\Phi$ is a tensor of the type $(1,1)$ and $g=\langle\cdot, \cdot\rangle$ is the Riemannian metric [12],[16]. Moreover, the following conditions are fulfilled:

$$
\begin{gathered}
\eta(\xi)=1, \Phi(\xi)=0, \eta \circ \Phi=0, \Phi^{2}=-i d+\xi \otimes \eta \\
\langle\Phi X, \Phi Y\rangle=\langle\Phi X, \Phi Y\rangle-\eta(X) \eta(Y), X, Y \in \aleph(N)
\end{gathered}
$$

where $\mathcal{N}(N)$ is the module of smooth vector fields on $N$. As examples of almost contact metric structures we can consider the cosymplectic structure, the nearly cosymplectic structure, the Sasakian structure and the Kenmotsu structure.

The cosymplectic structure that is characterized by the following condition:

$$
\nabla \eta=0, \quad \nabla \Phi=0
$$

where $\nabla$ is the Levi-Civita connection of the metric. It has been proved that the manifold, admitting the cosymplectic structure, is locally equivalent to the product $M \times R$, where $M$ is a Kählerian manifold [15].

An almost contact metric structure $\{\Phi, \xi, \eta, g\}$ is called nearly cosymplectic if the following condition is fulfilled [16], [19]:

$$
\nabla_{X}(\Phi) Y+\nabla_{Y}(\Phi) X=0, X, Y \in \aleph(N)
$$

We note that the nearly cosymplectic structures have many remarkable properties and play an important role in contact geometry. We mark out a number of articles by H . Endo on the geometry of nearly cosymplectic manifolds as well as the fundamental research by E. V. Kusova on this subject [19].

Is it known if $(N,\{\Phi, \xi, \eta, g\})$ is an almost contact metric manifold, then an almost Hermitian structure is induced on the product $N \times R$ [12], [21]. If this almost Hermitian structure is integrable, then the input almost contact metric structure is called normal. A normal contact metric structure is called Sasakian [16]. On the other hand, we can characterize the Sasakian structure by the following condition:

$$
\nabla_{X}(\Phi) Y=\langle X, Y\rangle \xi-\eta(Y) X, \quad X, Y \in \aleph(N)
$$

For example, Sasakian structures are induced on totally umbilical hypersurfaces in a Kählerian manifold [21]. As it is well known, the Sasakian structures have many remarkable properties and play a fundamental role in contact geometry.

In 1972 Katsuei Kenmotsu introduced a new class of almost contact metric structures, defined by the condition:

$$
\nabla_{X}(\Phi) Y=\langle\Phi X, Y\rangle \xi-\eta(Y) \Phi X, X, Y \in \aleph(N) .
$$

The Kenmotsu manifolds are normal and integrable, but they are not contact manifolds. We mark out that the fundamental monograph by Gh. Pitiş [20] contains a detailed description of Kenmotsu manifolds and their generalizations and a set of important results on this subject.

At the end of this section, note that when we give a Riemannian manifold and its submanifold (in particular, its hypersurface), the rank of determined second fundamental form is called the type number [18].

## 3 The main result

Let us use the first group of Cartan structural equations of an almost contact metric structure on an oriented hypersurface $N^{2 n-1}$ of an almost Hermitian manifold $M^{2 n}[6],[21]:$

$$
\begin{gather*}
d \omega^{a}=\omega_{b}^{a} \wedge \omega^{b}+B_{c}^{a b} \omega^{c} \wedge \omega_{b}+B^{a b c} \omega_{b} \wedge \omega_{c}+ \\
+\left(\sqrt{2} B^{a n}{ }_{b}+i \sigma_{b}^{a}\right) \omega^{b} \wedge \omega+\left(-\sqrt{2} \tilde{B}^{n a b}-\frac{1}{\sqrt{2}} B^{a b}{ }_{n}-\frac{1}{\sqrt{2}} \tilde{B}^{a b n}+i \sigma^{a b}\right) \omega_{b} \wedge \omega ; \\
d \omega_{a}=-\omega_{a}^{b} \wedge \omega_{b}+B_{a b}{ }^{c} \omega_{c} \wedge \omega^{b}+B_{a b c} \omega^{b} \wedge \omega^{c}+ \\
+\left(\sqrt{2} B_{a n}{ }^{b}-i \sigma_{a}^{b}\right) \omega_{b} \wedge \omega+\left(-\sqrt{2} \tilde{B}_{n a b}-\frac{1}{\sqrt{2}} \tilde{B}_{a b n}-\frac{1}{\sqrt{2}} B_{a b}{ }^{n}-i \sigma_{a b}\right) \omega^{b} \wedge \omega ; \\
d \omega=\sqrt{2} B_{n a b} \omega^{a} \wedge \omega^{b}+\sqrt{2} B^{n a b} \omega_{a} \wedge \omega_{b}+  \tag{2}\\
+\left(\sqrt{2} B^{n a}{ }_{b}-\sqrt{2} B_{n b}{ }^{a}-2 i \sigma_{b}^{a}\right) \omega^{b} \wedge \omega_{a}+ \\
+\left(\tilde{B}_{n b n}+B_{n b}{ }^{n}+i \sigma_{n b}\right) \omega \wedge \omega^{b}+\left(\tilde{B}^{n b n}+B^{n b}{ }_{n}-i \sigma_{n}^{b}\right) \omega \wedge \omega_{b},
\end{gather*}
$$

where

$$
\tilde{B}^{a b c}=\frac{i}{2} J_{\hat{b}, \hat{c}}^{a} ; \quad \tilde{B}_{a b c}=-\frac{i}{2} J_{b, c}^{\hat{a}}
$$

and $\sigma$ is the second fundamental form of the immersion of $N$ into $M^{2 n}$. We also use the detailed structural equations (1) of a six-dimensional almost Hermitian submanifold of Cayley algebra [5], [6], [7]:

$$
\begin{align*}
& d \omega^{a}=\omega_{b}^{a} \wedge \omega^{b}+\frac{1}{\sqrt{2}} \varepsilon^{a b h} D_{h c} \omega^{c} \wedge \omega_{b}+\frac{1}{\sqrt{2}} \varepsilon^{a h[b} D_{h}^{c]} \omega_{b} \wedge \omega_{c} ; \\
& d \omega_{a}=-\omega_{a}^{b} \wedge \omega_{b}+\frac{1}{\sqrt{2}} \varepsilon_{a b h} D^{h c} \omega_{c} \wedge \omega^{b}+\frac{1}{\sqrt{2}} \varepsilon_{a h[b} D^{h}{ }_{c]} \omega^{b} \wedge \omega^{c} ; \tag{3}
\end{align*}
$$

Here $\varepsilon_{a b c}=\varepsilon_{a b c}^{123}, \varepsilon^{a b c}=\varepsilon_{123}^{a b c}$ are the components of the third-order Kronecher tensor;

$$
\begin{aligned}
& D^{h c}=D_{\hat{h} \hat{c}}, \quad D_{h}{ }^{c}=D_{h \hat{c}}, \quad D^{h}{ }_{c}=D_{\hat{h} c} ; \\
& D_{c j}=\mp T_{c j}^{8}+i T_{c j}^{7}, \quad D_{\hat{c} j}=\mp T_{\hat{c} j}^{8}-i T_{\hat{c} j}^{7},
\end{aligned}
$$

where $\left\{T_{k j}^{\varphi}\right\}$ are the components of the configuration tensor (in Gray-Kirichenko notation [13], [15]); $\varphi=7,8 a, b, c, d, g, h=1,2,3 ; \hat{a}=a+3 ; k, j=1,2,3,4,5,6$. Comparing these equations with (1), we get the expressions for the Kirichenko tensors of six-dimensional almost Hermitian submanifolds of Cayley algebra (in particular, for the nearly Kählerian six-dimensional sphere $S^{6}$ ):

$$
\begin{aligned}
& B^{a b}{ }_{c}=\frac{1}{\sqrt{2}} \varepsilon^{a b h} D_{h c} ; \quad B_{a b}{ }^{c}=\frac{1}{\sqrt{2}} \varepsilon_{a b h} D^{h c} ; \\
& B^{a b c}=\frac{1}{\sqrt{2}} \varepsilon^{a h[b} D_{h}^{c]} ; B_{a b c}=\frac{1}{\sqrt{2}} \varepsilon_{a h[b} D_{c]}^{h} .
\end{aligned}
$$

Knowing that the Kirichenko tensors $B^{a b}{ }_{c}$ and $B_{a b}{ }^{c}$ of the nearly Kählerian sixsphere vanish [8], we rewrite these structural equations as follows:

$$
\begin{gather*}
d \omega^{\alpha}=\omega_{\beta}^{\alpha} \wedge \omega^{\beta}+B^{\alpha \beta \gamma} \omega_{\beta} \wedge \omega_{\gamma}+ \\
+i \sigma_{\beta}^{\alpha} \omega^{\beta} \wedge \omega+\left(-\sqrt{2} \tilde{B}^{n \alpha \beta}-\frac{1}{\sqrt{2}} \tilde{B}^{\alpha \beta n}+i \sigma^{\alpha \beta}\right) \omega_{\beta} \wedge \omega \\
d \omega_{\alpha}=-\omega_{\alpha}^{\beta} \wedge \omega_{\beta}+B_{\alpha \beta \gamma} \omega^{\beta} \wedge \omega^{\gamma}- \\
-i \sigma_{\alpha}^{\beta} \omega_{\beta} \wedge \omega+\left(-\sqrt{2} \tilde{B}_{n \alpha \beta}-\frac{1}{\sqrt{2}} \tilde{B}_{\alpha \beta n}-i \sigma_{a \beta}\right) \omega^{\beta} \wedge \omega  \tag{4}\\
d \omega=\sqrt{2} B_{n \alpha \beta} \omega^{\alpha} \wedge \omega^{\beta}+\sqrt{2} B^{n \alpha \beta} \omega_{\alpha} \wedge \omega_{\beta}- \\
-2 i \sigma_{\beta}^{\alpha} \omega^{\beta} \wedge \omega_{\alpha}+\left(\tilde{B}_{n \beta n}+i \sigma_{n \beta}\right) \omega \wedge \omega^{\beta}+\left(\tilde{B}^{n \beta n}-i \sigma_{n}^{\beta}\right) \omega \wedge \omega_{\beta}
\end{gather*}
$$

On the other hand, we obtain the more precise structural equations of the nearly Kählerian structure on the six-sphere [9]:

$$
\begin{aligned}
d \omega^{a} & =\omega_{b}^{a} \wedge \omega^{b}+\mu \varepsilon^{a c b} \omega_{b} \wedge \omega_{c} \\
d \omega_{a} & =-\omega_{a}^{b} \wedge \omega_{b}+\bar{\mu} \varepsilon_{a c b} \omega^{b} \wedge \omega^{c} .
\end{aligned}
$$

If an almost contact metric hypersurface of a nearly Kählerian manifold is of type number two, then we get the simplest matrix of its second fundamental form:

$$
\left(\sigma_{p s}\right)=\left(\begin{array}{c|c|c} 
& 0 & \\
\left(\sigma_{\alpha \beta}\right) & \begin{array}{c}
\ldots \\
0
\end{array} & 0 \\
& 0 & 0 \ldots 0 \\
\hline 0 \ldots 0 & 0 & 0 \\
\hline 0 & \begin{array}{c}
0 \\
\ldots
\end{array} & \left(\sigma_{\bar{\alpha} \hat{\beta}}\right)
\end{array}\right), \quad p, s=1,2,3,4,5,
$$

and what is more

$$
\operatorname{rank}\left(\sigma_{\alpha \beta}\right)=\operatorname{rank}\left(\sigma_{\hat{\alpha} \hat{\beta}}\right)=1 .
$$

That is why from (4) and (5) we obtain the following Cartan structural equations of an almost contact metric structure on an oriented 2-hypersurface of nearly Kählerian six-sphere:

$$
\begin{gathered}
d \omega^{\alpha}=\omega_{\beta}^{\alpha} \wedge \omega^{\beta}+B^{\alpha \beta \gamma} \omega_{\beta} \wedge \omega_{\gamma}+\left(-\sqrt{2} \tilde{B}^{n \alpha \beta}-\frac{1}{\sqrt{2}} \tilde{B}^{\alpha \beta n}+i \sigma^{\alpha \beta}\right) \omega_{\beta} \wedge \omega \\
d \omega_{\alpha}=-\omega_{\alpha}^{\beta} \wedge \omega_{\beta}+B_{\alpha \beta \gamma} \omega^{\beta} \wedge \omega^{\gamma}+\left(-\sqrt{2} \tilde{B}_{n \alpha \beta}-\frac{1}{\sqrt{2}} \tilde{B}_{\alpha \beta n}-i \sigma_{a \beta}\right) \omega^{\beta} \wedge \omega \\
d \omega=0
\end{gathered}
$$

In [17], V. F. Kirichenko and I. V. Uskorev have introduced a new class of almost contact metric structure. Namely, they have defined the almost contact metric structure with the close contact form as the structures of cosymplectic type. As they have established, the condition

$$
d \omega=0
$$

is necessary and sufficient for an almost contact metric structure to be of cosymplectic type. V. F. Kirichenko and I. V. Uskorev have also proved that the structure of cosymplectic type is invariant under canonical conformal transformations [17]. We recall also that a conformal transformation of an almost contact metric structure $\{\Phi, \xi, \eta, g\}$ on the manifold $N$ is a transition to the almost contact metric structure $\{\tilde{\Phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g}\}$, where $\tilde{\Phi}=\Phi, \tilde{\xi}=e^{f} \xi, \tilde{\eta}=e^{-f} \eta$ and $\tilde{g}=e^{-2 f} g$. Here $f$ is a smooth function on the manifold $N$ [16], [21].

Evidently, a trivial example of structure of cosymplectic type is the cosymplectic structure with well-known Cartan structural equations [7], [16]:

$$
\begin{gathered}
d \omega^{\alpha}=\omega_{\beta}^{\alpha} \wedge \omega^{\beta} \\
d \omega_{\alpha}=-\omega_{\alpha}^{\beta} \wedge \omega_{\beta} \\
d \omega=0
\end{gathered}
$$

Another important example of the almost contact metric structure of cosymplectic type is the Kenmotsu structure with following Cartan structural equations [1], [16]:

$$
\begin{gathered}
d \omega^{\alpha}=\omega_{\beta}^{\alpha} \wedge \omega^{\beta}+\omega \wedge \omega^{\alpha} \\
d \omega_{\alpha}=-\omega_{\alpha}^{\beta} \wedge \omega_{\beta}+\omega \wedge \omega_{\alpha} \\
d \omega=0
\end{gathered}
$$

It is easy to see that the structural equations (6) perfectly correspond to the structure of cosymplectic type, but this almost contact metric structure is not cosymplectic or Kenmotsu. So, we have proved the following result.
Theorem 1. Hypersurfaces with type number two in a nearly Kählerian six-sphere admit non-cosymplectic and non-Kenmotsu almost contact metric structures of cosymplectic type.

## 4 Some comments

As we have mentioned, in [3] and [11] it was proved that the almost contact metric structure on a totally geodesic or on 1-type hypersurface in a nearly Kählerian six-sphere must be nearly cosymplectic. We remark that the nearly cosymplectic structure is not of cosymplectic type because its Cartan structural equations look as follows [4],[19]:

$$
\begin{gathered}
d \omega^{\alpha}=\omega_{\beta}^{\alpha} \wedge \omega^{\beta}+H^{\alpha \beta \gamma} \omega_{\beta} \wedge \omega_{\gamma}+H^{\alpha \beta} \omega_{\beta} \wedge \omega \\
d \omega_{\alpha}=-\omega_{\alpha}^{\beta} \wedge \omega_{\beta}+H_{\alpha \beta \gamma} \omega^{\beta} \wedge \omega^{\gamma}+H_{\alpha \beta} \omega^{\beta} \wedge \omega \\
d \omega=-\frac{2}{3} G_{\alpha \beta} \omega^{\alpha} \wedge \omega^{\beta}-\frac{2}{3} G^{\alpha \beta} \omega_{\alpha} \wedge \omega_{\beta}
\end{gathered}
$$

On the other hand, in [10], it has been proved that 2-hypersurfaces in an arbitrary Kählerian manifold also admit non-cosymplectic and non-Kenmotsu almost contact metric structures of cosymplectic type. Taking into account that the class of nearly Kählerian manifolds is situated "between" the classes of Kählerian and quasi-Kählerian manifolds [14], we can pose an open problem.

Problem. Find a characterization of the almost contact metric structure on a 2-type hypersurface in a quasi-Kählerian manifold. In particular, can the almost contact metric structure on such a hypersurface be of cosymplectic type?

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Ahmad Abu-Saleem
Received June 16, 2017
Al al-Bayt University Marfaq Jordan
E-mail:dr_ahmad57@yahoo.com
Mihail B. Banaru, Galina A. Banaru
Smolensk State University
4, Przhevalsky Street
Smolensk - 214000
Russia
E-mail: mihail.banaru@yahoo.com

# Post-quantum No-key Protocol 

N. A. Moldovyan, A. A. Moldovyan, V.A.Shcherbacov


#### Abstract

There is proposed three-pass no-key protocol that is secure to hypothetic attacks based on computations with using quantum computers. The main operations are multiplication and exponentiation in finite ground field $G F(p)$. Sender and receiver of secret message also use representation of some value $c \in G F(p)$ as product of two other values $R_{1} \in G F(p)$ and $R_{2} \in G F(p)$ one of which is selected at random. Then the values $R_{1}$ and $R_{2}$ are encrypted using different local keys.


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## 1 Introduction

An open problem of cryptography is design of post-quantum cryptographic algorithms and protocols [1,2]. The most practical public-key cryptoschemes are based on difficulty of discrete logarithm [3-5] and of factoring integers containing two large prime factors [6,7]. The three-pass no-key encryption protocol [3] based on the first problem represents significant practical interest, for example, to perform secure encryption with short shared keys [8].

Quantum computations are in progress and it is expected that in observable future it will be possible to implement polynomial algorithms solving the discrete logarithm and factoring problems [9]. Therefore researchers are looking for new cryptographic primitives and designs of cryptoschemes, for example, the hidden conjugacy search problem in finite non-commutative groups was proposed as primitive for designing post-quantum cryptoschemes [10-12].

In the present communication we propose post-quantum implementation of the three-pass no-key encryption protocol. In the proposed protocol there is used exponentiation in the finite ground field $\operatorname{GF}(p)$, where $p$ is a sufficiently large prime, like in the known no-key encryption protocol. However it is additionally used representation of some element of the field $G F(p)$ as product of two other elements one of which is selected at random and serves as an additional local key. Due to such representation performed independently on the side of the message sender and on the side of the receiver, solving the discrete logarithm problem (DLP) cannot be used to break the proposed protocol. No key encryption protocol [3] exploits commutative ciphers.

[^2]Encryption function $E$ is called commutative if it satisfies the following condition

$$
E_{K}\left[E_{Q}(M)\right]=E_{Q}\left[E_{K}(M)\right],
$$

where $K$ and $Q$ are encryption keys and $M$ is some plaintext, for arbitrary keys $K$ and $Q \neq K$.

The appropriate commutative encryption function is provided by the exponentiation encryption method by Pohlig and Hellman [13] that is described as follows.

Suppose $p$ is a 2048 -bit prime such that number $p-1$ contains a large prime divisor $q$ the size of which is $|q| \geq 256$ bits, for example, $p=2 q+1$.

To select an encryption/decryption key $(e, d)$ one needs to generate a random number $e$ that is mutually prime with $p-1$ and has size $|e| \geq 256$ bits and then to compute $d=e^{-1} \bmod p-1$.

The encryption procedure is described with the formula

$$
C=M^{e} \bmod p .
$$

Decryption of the ciphertext $C$ is performed as computing the value

$$
M=C^{d} \bmod p
$$

Suppose Alice wishes to send the secret message $M$ to Bob, using a public channel and no shared key. For this purpose they can use the following no key protocol:
(i) Alice chooses a random key $\left(e_{A}, d_{A}\right)$ and encrypts the message $M$ using the formula $C_{1}=M^{e_{A}} \bmod p$. Then she sends the ciphertext $C_{1}$ to Bob;
(ii) Bob chooses a random key $\left(e_{B}, d_{B}\right)$ and encrypts the ciphertext $C_{1}$ as follows: $C_{2}=C_{1}^{e_{B}} \bmod p$ and sends the ciphertext $C_{2}$ to Alice;
(iii) Alice decrypts the ciphertext $C_{2}$ obtaining the ciphertext $C_{3}: C_{3}=$ $C_{2}^{d_{A}} \bmod p$. Then she sends the ciphertext $C_{3}$ to Bob;
(iv) Bob computes the message $M=C_{3}^{d_{B}} \bmod p$.

This three-pass protocol provides security to passive attacks (potential adversary only intercepts the values sent via public channel, but does not masquerade as sender or receiver of secret message), since the used exponentiation cipher is as secure as discrete logarithm problem is hard.

However, the described protocol is not secure against attacks using hypothetic quantum computers.

We propose the following post-quantum implementation of the no-key protocol.

1. Alice generates two local keys in the form of two pairs of numbers $\left(e_{A 1}, d_{A 1}\right)$ and $\left(e_{A 2}, d_{A 2}\right)$ such that $d_{A 1}=e_{A 1}^{-1} \bmod p-1$ and $d_{A 2}=e_{A 2}^{-1} \bmod p-1$, and forms the pair of random numbers $R_{1}<p$ and $R_{2}<p$ such that $M=R_{1} R_{2} \bmod p$, where $M$ is some secret message. Then she encrypts the numbers $R_{1}$ and $R_{2}$, using formulas $C_{1}^{\prime}=R_{1}^{e_{A 1}} \bmod p$ and $C_{1}^{\prime \prime}=R_{2}^{e_{A 2}} \bmod p$, and sends the ciphertexts $C_{1}^{\prime}$ and $C_{1}^{\prime \prime}$ to Bob.
2. Bob generates his two local keys $\left(e_{B 1}, d_{B 1}\right)$ and $\left(e_{B 2}, d_{B 2}\right)$ and represents each of the numbers $C_{1}^{\prime}$ and $C_{1}^{\prime \prime}$ as product of the pair of random numbers $\left(R_{11}, R_{12}\right)$, where $R_{11}<p$ and $R_{12}<p$, and ( $R_{21}, R_{22}$ ), where $R_{21}<p$ and $R_{22}<p$, respectively: $R_{1}=R_{11} R_{12} \bmod p ; R_{2}=R_{21} R_{22} \bmod p$.

Then he generates two random values $L_{1}<p$ and $L_{2}<p$ and encrypts the numbers $R_{11}, R_{12}, R_{21}$, and $R_{22}$ as follows:

$$
\begin{array}{ll}
C_{2}^{\prime}=R_{11}^{e_{B 1}} L_{1}^{d_{B 2}} \bmod p ; & C_{2}^{\prime \prime \prime}=R_{21}^{e_{B 1}} L_{2}^{d_{B 2}} \bmod p ; \\
C_{2}^{\prime \prime}=R_{12}^{e_{22}} L_{1}^{-d_{B 1}} \bmod p ; \quad \bar{C}_{2}=R_{22}^{e_{B 2}} L_{2}^{-d_{B 1}} \bmod p,
\end{array}
$$

and sends the ciphertexts $C_{2}^{\prime}, C_{2}^{\prime \prime}, C_{2}^{\prime \prime \prime}$, and $\bar{C}_{2}$ to Alice.
3. Alice generates random numbers $N_{1}<p$ and $N_{2}<p$ and decrypts the ciphertexts $C_{2}^{\prime}, C_{2}^{\prime \prime}, C_{2}^{\prime \prime \prime}$, and $\bar{C}_{2}$ as follows:

$$
\begin{array}{ll}
C_{3}^{\prime}=\left(C_{2}^{\prime}\right)^{d_{A 1}} N_{1} \bmod p ; & C_{3}^{\prime \prime \prime}=\left(C_{2}^{\prime \prime \prime}\right)^{d_{A 2}} N_{1}^{-1} \bmod p ; \\
C_{3}^{\prime \prime}=\left(C_{2}^{\prime \prime}\right)^{d_{A 1}} N_{2} \bmod p ; & \bar{C}_{3}=\left(\overline{C_{2}}\right)^{d_{A 2}} N_{2}^{-1} \bmod p,
\end{array}
$$

and sends the ciphertexts $C_{3}^{\prime}, C_{3}^{\prime \prime}, C_{3}^{\prime \prime \prime}$, and $\bar{C}_{3}$ to Bob.
4. Bob recovers the secret message $M$ from the values $C_{3}^{\prime}, C_{3}^{\prime \prime}, C_{3}^{\prime \prime \prime}$, and $\bar{C}_{3}$ multiplying the numbers $S^{\prime}, S^{\prime \prime}, S^{\prime \prime \prime}$ and $\bar{S}$ that are computed as follows: $S^{\prime}=$ $\left(C_{3}^{\prime}\right)^{d_{B 1}} \bmod p ; S^{\prime \prime}=\left(C_{3}^{\prime \prime}\right)^{d_{B 2}} \bmod p ; S^{\prime \prime \prime}=\left(C_{3}^{\prime \prime \prime}\right)^{d_{B 1}} \bmod p ; \bar{S}=\left(\bar{C}_{3}\right)^{d_{B 2}} \bmod p ;$ $M=S^{\prime} S^{\prime \prime} S^{\prime \prime \prime} \bar{S} \bmod p$.
A correctness proof of the protocol is as follows:

$$
\begin{aligned}
& S^{\prime} \equiv\left(C_{3}^{\prime}\right)^{d_{B 1}} \equiv\left(C_{2}^{\prime}\right)^{d_{B 1} d_{A 1}} N_{1}^{d_{B 1}} \equiv R_{11}^{d_{B 1} d_{A 1} e_{B 1}} L_{1}^{d_{B 1} d_{A 1} d_{B 2}} N_{1}^{d_{B 1}} \equiv \\
& R_{11}^{d_{A 1}} L_{1}^{d_{B 1} d_{A 1} d_{B 2}} N_{1}^{d_{B 1}} \bmod p ; \\
& S^{\prime \prime} \equiv\left(C_{3}^{\prime \prime}\right)^{d_{B 2}} \equiv\left(C_{2}^{\prime \prime}\right)^{d_{B 2} d_{A 1}} N_{2}^{d_{B 2}} \equiv R_{12}^{d_{B 2} d_{A 1} e_{B 2}} L_{1}^{-d_{B 2} d_{A 1} d_{B 1}} N_{2}^{d_{B 2}} \equiv \\
& R_{12}^{d_{A 1}} L_{1}^{-d_{B 2} d_{A 1} d_{B 1}} N_{2}^{d_{B 2}} \bmod p ; \\
& S^{\prime \prime \prime} \equiv\left(C_{3}^{\prime \prime \prime}\right)^{d_{B 1}} \equiv\left(C_{2}^{\prime \prime \prime}\right)^{d_{B 1} d_{A 2}} N_{1}^{-d_{B 1}} \equiv R_{21}^{d_{B 1} d_{A 2} e_{B 1}} L_{2}^{d_{B 1} d_{A 1} d_{B 2}} N_{1}^{-d_{B 1}} \equiv \\
& R_{21}^{d_{A 2}} L_{2}^{d_{B 1} d_{A 2} d_{B 2}} N_{1}^{-d_{B 1}} \bmod p ; \\
& \bar{S} \equiv\left(\bar{C}_{3}\right)^{d_{B 2}} \equiv\left(\bar{C}_{2}\right)^{d_{B 2} d_{A 2}} N_{2}^{-d_{B 2}} \equiv R_{22}^{d_{B 2} d_{A 2} e_{B 2}} L_{2}^{-d_{B 2} d_{A 2} d_{B 1}} N_{2}^{-d_{B 2}} \equiv \\
& R_{22}^{d_{A 2}} L_{2}^{-d_{B 2} d_{A 2} d_{B 1}} N_{2}^{-d_{B 2}} \bmod p .
\end{aligned}
$$

Multiplying the numbers $S^{\prime}$ and $S^{\prime \prime}$ one gets

$$
\begin{aligned}
& S^{\prime} S^{\prime \prime} \equiv R_{11}^{d_{A 1}} L_{1}^{d_{B 1} d_{A 1} d_{B 2}} N_{1}^{d_{B 1}} R_{12}^{d_{A 1}} L_{1}^{-d_{B 2} d_{A 1} d_{B 1}} N_{2}^{d_{B 2}} \equiv \\
& \left(R_{11} R_{12}\right)^{d_{A 1}} N_{1}^{d_{B 1}} N_{2}^{d_{B 2}} \equiv\left(C_{1}^{\prime}\right)^{d_{A 1}} N_{1}^{d_{B 1}} N_{2}^{d_{B 2}} \equiv \\
& \left(R_{1}\right)^{d_{A 1} e_{A 1}} N_{1}^{d_{B 1}} N_{2}^{d_{B 2}} \equiv R_{1} N_{1}^{d_{B 1}} N_{2}^{d_{B 2}} \bmod p .
\end{aligned}
$$

Multiplying the numbers $S^{\prime \prime \prime}$ and $\bar{S}$ one gets

$$
\begin{aligned}
& S^{\prime \prime \prime} \bar{S} \equiv R_{21}^{d_{A 2}} L_{2}^{d_{B 1} d_{A 2} d_{B 2}} N_{1}^{-d_{B 1}} R_{22}^{d_{A 2}} L_{2}^{-d_{B 2} d_{A 2} d_{B 1}} N_{2}^{-d_{B 2}} \equiv \\
& \left(R_{21} R_{22}\right)^{d_{A 2}} N_{1}^{-d_{B 1}} N_{2}^{-d_{B 2}} \equiv\left(C_{2}^{\prime}\right)^{d_{A 2}} N_{1}^{-d_{B 1}} N_{2}^{-d_{B 2}} \\
& \equiv\left(R_{2}\right)^{d_{A 2} e_{A 2}} N_{1}^{-d_{B 1}} N_{2}^{-d_{B 2}} \equiv R_{2} N_{1}^{-d_{B 1}} N_{2}^{-d_{B 2}} \bmod p .
\end{aligned}
$$

Thus, we have

$$
S^{\prime} S^{\prime \prime} S^{\prime \prime \prime} \bar{S} \equiv R_{1} N_{1}^{d_{B 1}} N_{2}^{d_{B 2}} R_{2} N_{1}^{-d_{B 1}} N_{2}^{-d_{B 2}} \equiv R_{1} R_{2} \bmod p
$$

Therefore, $M=S^{\prime} S^{\prime \prime} S^{\prime \prime \prime} \bar{S} \bmod p$.
We invite the reader to participate in security analysis of the proposed protocol.

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N. A. Moldovyan

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St. Petersburg Institute for Informatics and Automation
of Russian Academy of Sciences
14 Liniya, 39, St.Petersburg, 199178

## Russia

E-mail: nmold@mail.ru
A. A. Moldovyan

ITMO University
Kronverksky pr., 10, St.Petersburg, 197101

## Russia

E-mail: maa1305@yandex.ru
V. A. Shcherbacov

Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
Academiei str. 5, MD-2028 Chişinău
Moldova
E-mail: scerb@math.md

# Some estimates for angular derivative at the boundary 

Bülent Nafi Örnek


#### Abstract

In this paper, we establish lower estimates for the modulus of the values of $f(z)$ on boundary of unit disc. For the function $f(z)=1+c_{1} z+c_{2} z^{2}+\ldots$ defined in the unit disc such that $f(z) \in \mathcal{N}(\beta)$ assuming the existence of angular limit at the boundary point $b$, the estimations below of the modulus of angular derivative have been obtained at the boundary point $b$ with $f(b)=\beta$. Moreover, Schwarz lemma for class $\mathcal{N}(\beta)$ is given. The sharpness of these inequalities has been proved.


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## 1 Introduction

Let $f$ be a holomorphic function in the unit disc $D=\{z:|z|<1\}, f(0)=0$ and $|f(z)|<1$ for $|z|<1$. In accordance with the classical Schwarz lemma, for any point $z$ in the disc $D$, we have $|f(z)| \leq|z|$ and $\left|f^{\prime}(0)\right| \leq 1$. Equality in these inequalities (in the first one, for $z \neq 0$ ) occurs only if $f(z)=z e^{i \theta}$, where $\theta$ is a real number ([6], p.329). For historical background about the Schwarz lemma and its applications on the boundary of the unit disc, we refer to $[2,19]$.

The basic tool in proving our results is the following lemma due to Jack.
Lemma 1 (Jack's lemma). Let $f(z)$ be holomorphic function in the unit disc $D$ with $f(0)=0$. Then if $|f(z)|$ attains its maximum value on the circle $|z|=r$ at a point $z_{0} \in D$, then there exists a real number $k \geq 1$ such that

$$
\frac{z_{0} f^{\prime}\left(z_{0}\right)}{f\left(z_{0}\right)}=k .
$$

Let $\mathcal{A}$ denote the class of functions

$$
f(z)=1+c_{1} z+c_{2} z^{2}+\ldots
$$

that are holomorphic in the unit disc $D$. Also, $\mathcal{N}(\beta)$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy

$$
\begin{equation*}
\Re\left(f(z)-\frac{z f^{\prime}(z)}{f(z)}\right)>\frac{\beta(3-2 \beta)}{2(1-\beta)}, \tag{1.1}
\end{equation*}
$$

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where $\beta<0$.
Let $f(z) \in \mathcal{N}(\beta)$ and define $\varphi(z)$ in $D$ by

$$
\varphi(z)=\frac{f(z)-1}{f(z)-(2 \beta-1)} .
$$

Obviously, $\varphi(z)$ is holomorphic function in the unit disc $D$ and $\varphi(0)=0$. We want to prove $|\varphi(z)|<1$ for $|z|<1$.

If there exists a point $z_{0} \in D$ such that

$$
\max _{|z| \leq\left|z_{0}\right|}|\varphi(z)|=\left|\varphi\left(z_{0}\right)\right|=1,
$$

then Jack's lemma gives us that $\varphi\left(z_{0}\right)=e^{i \theta}$ and $z_{0} \varphi^{\prime}\left(z_{0}\right)=k \varphi\left(z_{0}\right)(k \geq 1)$.
Thus we have

$$
\frac{z_{0} f^{\prime}\left(z_{0}\right)}{f\left(z_{0}\right)}=\frac{\frac{2(1-\beta) z_{0} \varphi^{\prime}\left(z_{0}\right)}{\left(1-\varphi\left(z_{0}\right)\right)^{2}}}{(1-\beta) \frac{1+\varphi\left(z_{0}\right)}{1-\varphi\left(z_{0}\right)}+\beta}=\frac{\frac{2(1-\beta) k e^{i \theta}}{\left(1-e^{i \theta}\right)^{2}}}{(1-\beta) \frac{1+e^{i \theta}}{1-e^{i \theta}}+\beta}
$$

Since

$$
\frac{e^{i \theta}}{\left(1-e^{i \theta}\right)^{2}}=\frac{e^{i \theta}}{1-2 e^{i \theta}+e^{2 i \theta}}=\frac{1}{e^{-i \theta}-2+e^{i \theta}}=\frac{1}{2(\cos \theta-1)}
$$

and

$$
\begin{aligned}
\frac{1+e^{i \theta}}{1-e^{i \theta}} & =\frac{1+\cos \theta+i \sin \theta}{1-\cos \theta-i \sin \theta}=\frac{(1+\cos \theta+i \sin \theta)(1-\cos \theta+i \sin \theta)}{(1-\cos \theta)^{2}+\sin ^{2} \theta} \\
& =\frac{i \sin \theta}{1-\cos \theta},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\frac{z_{0} f^{\prime}\left(z_{0}\right)}{f\left(z_{0}\right)} & =-\frac{\frac{(1-\beta) k}{1-\cos \theta}}{(1-\beta) \frac{\sin \theta}{1-\cos \theta}+\beta}=\frac{-(1-\beta) k}{(1-\beta) i \sin \theta+\beta(1-\cos \theta)} \\
& =\frac{-(1-\beta) k[-(1-\beta) i \sin \theta+\beta(1-\cos \theta)]}{\beta^{2}(1-\cos \theta)^{2}+(1-\beta)^{2} \sin ^{2} \theta}
\end{aligned}
$$

and

$$
\Re\left(\frac{z_{0} f^{\prime}\left(z_{0}\right)}{f\left(z_{0}\right)}\right)=\frac{-\beta(1-\beta) k(1-\cos \theta)}{\beta^{2}(1-\cos \theta)^{2}+(1-\beta)^{2} \sin ^{2} \theta} .
$$

If we write $1-\cos \theta=s$ and

$$
h(s)=\frac{s}{\beta^{2} s^{2}+(1-\beta)^{2}\left(2 s-s^{2}\right)}
$$

then we have

$$
\Re\left(\frac{z_{0} f^{\prime}\left(z_{0}\right)}{f\left(z_{0}\right)}\right)=-\beta(1-\beta) k h(s) .
$$

Since $h(s)$ takes its minimum value for $s=0$, we have that

$$
\Re\left(\frac{z_{0} f^{\prime}\left(z_{0}\right)}{f\left(z_{0}\right)}\right)=-\beta(1-\beta) k \frac{1}{2(1-\beta)^{2}} \geq \frac{-\beta}{2(1-\beta)} .
$$

Thus, we obtain

$$
\Re\left(f\left(z_{0}\right)-\frac{z_{0} f^{\prime}\left(z_{0}\right)}{f\left(z_{0}\right)}\right) \leq \beta+\frac{\beta}{2(1-\beta)}=\frac{\beta(3-2 \beta)}{2(1-\beta)} .
$$

This contradict (1.1). So, there is no point $z_{0} \in D$ such that $\varphi\left(z_{0}\right)=1$. This means that $|\varphi(z)|<1$ for $|z|<1$. Thus, from the Schwarz lemma, we obtain

$$
\left|f^{\prime}(0)\right| \leq 2(1-\beta) .
$$

Moreover, the equality $\left|f^{\prime}(0)\right|=2(1-\beta)$ occurs for the function

$$
f(z)=\frac{1-(2 \beta-1) z}{1-z} .
$$

That proves
Lemma 2. If $f(z) \in \mathcal{N}(\beta)$, then we have

$$
\begin{equation*}
\left|f^{\prime}(0)\right| \leq 2(1-\beta) \tag{1.3}
\end{equation*}
$$

The equality in (1.3) occurs for the function

$$
f(z)=\frac{1-(2 \beta-1) z}{1-z}
$$

This lemma yields a " $\mathcal{N}(\beta)$ version" of the classical Schwarz lemma for holomorphic function of one complex variable.

The following boundary version of the Schwarz lemma was proved in 1938 by Unkelbach in [20] and then rediscovered and partially improved by Osserman in 2000 [16].

Lemma 3. Let $f(z)$ be a holomorphic function self-mapping of $D$, that is $|f(z)|<1$ for all $z \in D$. Assume that there is $a b \in \partial D$ so that $f$ extends continuously to $b$, $|f(b)|=1$ and $f^{\prime}(b)$ exists. Then

$$
\begin{equation*}
\left|f^{\prime}(b)\right| \geq \frac{2}{1+\left|f^{\prime}(0)\right|} \tag{1.4}
\end{equation*}
$$

The equality in (1.4) holds if and only if $f$ is of the form

$$
f(z)=-z \frac{d-z}{1-d z}, \quad \forall z \in D
$$

for some constant $d \in(-1,0]$.

Corollary 1. Under the hypotheses lemma, we have

$$
\begin{equation*}
\left|f^{\prime}(b)\right| \geq 1 \tag{1.5}
\end{equation*}
$$

with equality only if $f$ is of the form

$$
f(z)=z e^{i \theta}
$$

where $\theta$ is a real number.
The following lemma, known as the Julia-Wolff lemma, is needed in the sequel [18].

Lemma 4 (Julia-Wolff lemma). Let $f$ be a holomorphic function in $E$, $f(0)=0$ and $f(D) \subset D$. If, in addition, the function $f$ has an angular limit $f(b)$ at $b \in \partial D$, $|f(b)|=1$, then the angular derivative $f^{\prime}(b)$ exists and $1 \leq\left|f^{\prime}(b)\right| \leq \infty$.

Inequality (1.4) and its generalizations have important applications in geometric theory of functions (see, e.g., $[6,18]$ ). Therefore, the interest to such type results is not vanished recently (see, e.g., $[1,2,4,5,10,11,16,17,19]$ and references therein).

Vladimir N. Dubinin has continued this line and has made a refinement on the boundary Schwarz lemma under the assumption that $f(z)=c_{p} z^{p}+c_{p+1} z^{p+1}+\ldots$, with a zero set $\left\{z_{k}\right\}$ (see [4]).
S. G. Krantz and D. M. Burns [9] and D. Chelst [3] studied the uniqueness part of the Schwarz lemma. In M. Mateljević's papers, for more general results and related estimates, see also ([12-15]).

Also, M. Jeong [8] showed some inequalities at a boundary point for different form of holomorphic functions and found the condition for equality and in [7] a holomorphic self map defined on the closed unit disc with fixed points only on the boundary of the unit disc.

## 2 Main Results

In this section, for holomorphic function $f(z)=1+c_{1} z+c_{2} z^{2}+\ldots$ belonging to the class of $\mathcal{N}(\beta)$, the modulus of the angular derivative of the function at the boundary point of the unit disc has been estimated.

Theorem 1. Let $f(z) \in \mathcal{N}(\beta)$. Assume that, for some $b \in \partial D, f$ has angular limit $f(b)$ at $b$ and $f(b)=\beta$. Then we have the inequality

$$
\begin{equation*}
\left|f^{\prime}(b)\right| \geq \frac{1-\beta}{2} \tag{2.1}
\end{equation*}
$$

The equality in (2.1) occurs for the function

$$
f(z)=\frac{1-(2 \beta-1) z}{1-z} .
$$

Proof. Consider the function

$$
\varphi(z)=\frac{f(z)-1}{f(z)-(2 \beta-1)} .
$$

$\varphi(z)$ is a holomorphic function in the unit disc $D$ and $\varphi(0)=0$. From the Jack's lemma and since $f(z) \in \mathcal{N}(\beta)$, we obtain $|\varphi(z)|<1$ for $|z|<1$. Also, we have $|\varphi(b)|=1$ for $b \in \partial D$.

From (1.5), we obtain

$$
1 \leq\left|\varphi^{\prime}(b)\right|=\frac{2(1-\beta)\left|f^{\prime}(b)\right|}{|f(b)-(2 \beta-1)|^{2}}=\frac{2(1-\beta)\left|f^{\prime}(b)\right|}{(\beta-(2 \beta-1))^{2}}=\frac{2(1-\beta)\left|f^{\prime}(b)\right|}{(1-\beta)^{2}}
$$

and

$$
1 \leq \frac{2\left|f^{\prime}(b)\right|}{1-\beta}
$$

So, we take the inequality (2.1).
Now, we shall show that the inequality (2.1) is sharp. Let

$$
f(z)=\frac{1-(2 \beta-1) z}{1-z}
$$

Then, we have

$$
f^{\prime}(z)=2 \frac{1-\beta}{(z-1)^{2}},
$$

and

$$
\left|f^{\prime}(-1)\right|=\frac{1-\beta}{2}
$$

Theorem 2. Under the same assumptions as in Theorem 1, we have

$$
\begin{equation*}
\left|f^{\prime}(b)\right| \geq \frac{(1-\beta)^{2}}{1-\beta+\left|f^{\prime}(0)\right|} \tag{2.2}
\end{equation*}
$$

The inequality (2.2) is sharp with equality for the function

$$
f(z)=\frac{1+a z+(2 \beta-1)\left(z^{2}+a z\right)}{1+2 a z+z^{2}},
$$

where $a=\frac{\left|f^{\prime}(0)\right|}{2(1-\beta)}$ is an arbitrary number from $[0,1]$ (see (1.3)).
Proof. Let $\varphi(z)$ be as in the proof of Theorem 1. Using the inequality (1.4) for the function $\varphi(z)$, we obtain

$$
\frac{2}{1+\left|\varphi^{\prime}(0)\right|} \leq\left|\varphi^{\prime}(b)\right|=\frac{2\left|f^{\prime}(b)\right|}{1-\beta} .
$$

Since

$$
\varphi^{\prime}(z)=\frac{2(1-\beta) f^{\prime}(z)}{(f(z)-(2 \beta-1))^{2}}
$$

and

$$
\left|\varphi^{\prime}(0)\right|=\frac{2(1-\beta)\left|f^{\prime}(0)\right|}{(f(0)-(2 \beta-1))^{2}}=\frac{2(1-\beta)\left|f^{\prime}(0)\right|}{(1-(2 \beta-1))^{2}}=\frac{\left|f^{\prime}(0)\right|}{2(1-\beta)},
$$

we have

$$
\frac{2}{1+\frac{\left|f^{\prime}(0)\right|}{2(1-\beta)}} \leq \frac{2\left|f^{\prime}(b)\right|}{1-\beta}
$$

and

$$
\left|f^{\prime}(b)\right| \geq \frac{2(1-\beta)^{2}}{2(1-\beta)+\left|f^{\prime}(0)\right|}
$$

To show that the inequality (2.2) is sharp, take the holomorphic function

$$
f(z)=\frac{1+a z+(2 \beta-1)\left(z^{2}+a z\right)}{1+2 a z+z^{2}} .
$$

Then

$$
f^{\prime}(1)=-\frac{1-\beta}{1+a}
$$

and

$$
\left|f^{\prime}(1)\right|=\frac{1-\beta}{1+a}
$$

Since $a=\frac{\left|f^{\prime}(0)\right|}{2(1-\beta)}$, we have

$$
\left|f^{\prime}(1)\right|=\frac{1-\beta}{1+\frac{\left|f^{\prime}(0)\right|}{2(1-\beta)}}=\frac{2(1-\beta)^{2}}{2(1-\beta)+\left|f^{\prime}(0)\right|}
$$

The inequality (2.2) can be strengthened as below by taking into account $c_{2}$ which is second coefficient in the expansion of the function $f(z)$. That is, taking into account two consecutive coeffients, the inequality (2.2) has been strengthened. This is given by the following Theorem.

Theorem 3. Let $f(z) \in \mathcal{N}(\beta)$. Assume that, for some $b \in \partial D$, $f$ has angular limit $f(b)$ at $b$ and $f(b)=\beta$. Then we have the inequality

$$
\begin{equation*}
\left|f^{\prime}(b)\right| \geq \frac{1-\beta}{2}\left(1+\frac{2\left(2(1-\beta)-\left|c_{1}\right|\right)^{2}}{4(1-\beta)^{2}-\left|c_{1}\right|^{2}+\left|2(1-\beta) c_{2}-c_{1}^{2}\right|}\right) . \tag{2.3}
\end{equation*}
$$

The inequality (2.3) is sharp with extremal function

$$
f(z)=\frac{1-(2 \beta-1) z}{1-z}
$$

Proof. Let $\varphi(z)$ be as in the proof of Theorem 1. By the maximum principle for each $z \in D$, we have $|\varphi(z)| \leq|z|$. So,

$$
\Theta(z)=\frac{\varphi(z)}{z}
$$

is a holomorphic function in $D$ and $|\Theta(z)|<1$ for $|z|<1$.
From equality of $\Theta(z)$, we have

$$
\begin{aligned}
\Theta(z) & =\frac{\varphi(z)}{z}=\frac{1}{z} \frac{f(z)-1}{f(z)-(2 \beta-1)} \\
& =\frac{1+c_{1} z+c_{2} z^{2}+c_{3} z^{3}+\ldots-1}{z\left(1+c_{1} z+c_{2} z^{2}+c_{3} z^{3}+\ldots-(2 \beta-1)\right)} \\
& =\frac{c_{1}+c_{2} z+c_{3} z^{2}+\ldots}{2(1-\beta)+c_{1} z+c_{2} z^{2}+c_{3} z^{3}+\ldots} .
\end{aligned}
$$

Thus, we take

$$
\begin{equation*}
|\Theta(0)|=\frac{\left|c_{1}\right|}{2(1-\beta)} \leq 1 \tag{2.4}
\end{equation*}
$$

and

$$
\left|\Theta^{\prime}(0)\right|=\frac{\left|2(1-\beta) c_{2}-c_{1}^{2}\right|}{4(1-\beta)^{2}}
$$

Moreover, it can be seen that

$$
\frac{b \varphi^{\prime}(b)}{\varphi(b)}=\left|\varphi^{\prime}(b)\right| \geq\left|(b)^{\prime}\right|=\frac{b(b)^{\prime}}{b}
$$

The function

$$
\Phi(z)=\frac{\Theta(z)-\Theta(0)}{1-\overline{\Theta(0)} \Theta(z)}
$$

is a holomorphic in the unit disc $D,|\Phi(z)|<1$ for $|z|<1, \Phi(0)=0$ and $|\Phi(b)|=1$ for $b \in \partial D$.

From (1.4), we obtain

$$
\begin{aligned}
\frac{2}{1+\left|\Phi^{\prime}(0)\right|} & \leq\left|\Phi^{\prime}(b)\right|=\frac{1-|\Theta(0)|^{2}}{|1-\overline{\Theta(0)} \Theta(b)|^{2}}\left|\Theta^{\prime}(b)\right| \leq \frac{1+|\Theta(0)|}{1-|\Theta(0)|}\left|\Theta^{\prime}(b)\right| \\
& =\frac{1+|\Theta(0)|}{1-|\Theta(0)|}\left\{\left|\varphi^{\prime}(b)\right|-1\right\}
\end{aligned}
$$

Since

$$
\Phi^{\prime}(z)=\frac{1-|\Theta(0)|^{2}}{(1-\overline{\Theta(0)} \Theta(z))^{2}} \Theta^{\prime}(z)
$$

$$
\left|\Phi^{\prime}(0)\right|=\frac{\left|\Theta^{\prime}(0)\right|}{1-|\Theta(0)|^{2}}=\frac{\frac{\left|2(1-\beta) c_{2}-c_{1}^{2}\right|}{4(1-\beta)^{2}}}{1-\left(\frac{\left|c_{1}\right|}{2(1-\beta)}\right)^{2}}=\frac{\left|2(1-\beta) c_{2}-c_{1}^{2}\right|}{4(1-\beta)^{2}-\left|c_{1}\right|^{2}},
$$

we take

$$
\begin{aligned}
\frac{2}{1+\frac{\left|2(1-\beta) c_{2}-c_{1}^{2}\right|}{4(1-\beta)^{2}-\left|c_{1}\right|^{2}}} & \leq \frac{1+\frac{\left|c_{1}\right|}{2(1-\beta)}}{1-\frac{\left|c_{1}\right|}{2(1-\beta)}}\left\{\frac{2\left|f^{\prime}(b)\right|}{1-\beta}-1\right\} \\
& =\frac{2(1-\beta)+\left|c_{1}\right|}{2(1-\beta)-\left|c_{1}\right|}\left\{\frac{2\left|f^{\prime}(b)\right|}{1-\beta}-1\right\} .
\end{aligned}
$$

Therefore, we obtain

$$
1+\frac{2\left(4(1-\beta)^{2}-\left|c_{1}\right|^{2}\right)}{4(1-\beta)^{2}-\left|c_{1}\right|^{2}+\left|2(1-\beta) c_{2}-c_{1}^{2}\right|} \frac{2(1-\beta)-\left|c_{1}\right|}{2(1-\beta)+\left|c_{1}\right|} \leq \frac{2\left|f^{\prime}(b)\right|}{1-\beta}
$$

and

$$
\left|f^{\prime}(b)\right| \geq \frac{1-\beta}{2}\left(1+\frac{2\left(2(1-\beta)-\left|c_{1}\right|\right)^{2}}{4(1-\beta)^{2}-\left|c_{1}\right|^{2}+\left|2(1-\beta) c_{2}-c_{1}^{2}\right|}\right) .
$$

So, we obtain the inequality (2.3).
To show that the inequality (2.3) is sharp, take the holomorphic function

$$
f(z)=\frac{1-(2 \beta-1) z}{1-z}
$$

Then

$$
\left|f^{\prime}(-1)\right|=\frac{1-\beta}{2}
$$

Since $\left|c_{1}\right|=2(1-\beta),(2.3)$ is satisfied with equality.
If $f(z)-1$ has no zeros different from $z=0$ in Theorem 3, the inequality (2.3) can be further strengthened. It has been investigated in the case of having only one point $b$ in the unit disc $D$ of the function $f(z)$. That inequality is stronger than the inequalities which have been expressed above. This is given by the following Theorem.

Theorem 4. Let $f(z) \in \mathcal{N}(\beta)$ and $f(z)-1$ has no zeros in $D$ except $z=0$ and $c_{1}>0$. Assume that, for some $b \in \partial D$, $f$ has angular limit $f(b)$ at $b$ and $f(b)=\beta$. Then we have the inequality

$$
\begin{equation*}
\left|f^{\prime}(b)\right| \geq \frac{1-\beta}{2}\left(1-\frac{2(1-\beta)\left|c_{1}\right| \ln ^{2}\left(\frac{\left|c_{1}\right|}{2(1-\beta)}\right)}{2(1-\beta)\left|c_{1}\right| \ln \left(\frac{\left|c_{1}\right|}{2(1-\beta)}\right)-\left|2(1-\beta) c_{2}-c_{1}^{2}\right|}\right) . \tag{2.5}
\end{equation*}
$$

In addition, the equality in (2.5) occurs for the function

$$
f(z)=\frac{1-(2 \beta-1) z}{1-z} .
$$

Proof. Let $c_{1}>0$ in the expression of the function $f(z)$. Having in mind the inequality (2.4) and the function $f(z)-1$ has no zeros in $D$ except $D-\{0\}$, we denote by $\ln \Theta(z)$ the holomorphic branch of the logarithm normed by the condition

$$
\ln \Theta(0)=\ln \left(\frac{\left|c_{1}\right|}{2(1-\beta)}\right)<0
$$

The auxiliary function

$$
\Gamma(z)=\frac{\ln \Theta(z)-\ln \Theta(0)}{\ln \Theta(z)+\ln \Theta(0)}
$$

is holomorphic in the unit disc $D,|\Gamma(z)|<1, \Gamma(0)=0$ and $|\Gamma(b)|=1$ for $b \in \partial D$.
From (1.4), we obtain

$$
\begin{aligned}
\frac{2}{1+\left|\Gamma^{\prime}(0)\right|} & \leq\left|\Gamma^{\prime}(b)\right|=\frac{|2 \ln \Theta(0)|}{|\ln \Theta(b)+\ln \Theta(0)|^{2}}\left|\frac{\Theta^{\prime}(b)}{\Theta(b)}\right| \\
& =\frac{-2 \ln \Theta(0)}{\ln ^{2} \Theta(0)+\arg ^{2} \Theta(b)}\left\{\left|\varphi^{\prime}(b)\right|-1\right\}
\end{aligned}
$$

Since

$$
\begin{aligned}
\left|\Gamma^{\prime}(0)\right| & =\frac{-1}{\ln \left(\frac{\left|c_{1}\right|}{2(1-\beta)}\right)} \frac{\frac{\left|2(1-\beta) c_{2}-c_{1}^{2}\right|}{4(1-\beta)^{2}}}{\frac{c_{1} \mid}{2(1-\beta)}} \\
& =\frac{-1}{\ln \left(\frac{\left|c_{1}\right|}{2(1-\beta)}\right)} \frac{\left|2(1-\beta) c_{2}-c_{1}^{2}\right|}{2(1-\beta)\left|c_{1}\right|}
\end{aligned}
$$

and replacing $\arg ^{2} \Theta(b)$ by zero, then we have

$$
\frac{1}{1-\frac{1}{\ln \left(\frac{\left|c_{1}\right|}{2(1-\beta)}\right)} \frac{\left|2(1-\beta) c_{2}-c_{1}^{2}\right|}{2(1-\beta)\left|c_{1}\right|}} \leq \frac{-1}{\ln \left(\frac{\left|c_{1}\right|}{2(1-\beta)}\right)}\left\{\frac{2\left|f^{\prime}(b)\right|}{1-\beta}-1\right\}
$$

and

$$
1-\frac{2(1-\beta)\left|c_{1}\right| \ln ^{2}\left(\frac{\left|c_{1}\right|}{2(1-\beta)}\right)}{2(1-\beta)\left|c_{1}\right| \ln \left(\frac{\left|c_{1}\right|}{2(1-\beta)}\right)-\left|2(1-\beta) c_{2}-c_{1}^{2}\right|} \leq \frac{2\left|f^{\prime}(b)\right|}{1-\beta} .
$$

Thus, we obtain the inequality (2.5) with an obvious equality case.
In the following Theorem, we shall give an estimate below $\left|f^{\prime}(b)\right|$ according to the first nonzero Taylor coeffcient of about two zeros, namely $z=0$ and $z_{1} \neq 0$.

Theorem 5. Let $f(z) \in \mathcal{N}(\beta)$ and $f\left(z_{1}\right)=1$ for $0<\left|z_{1}\right|<1$. Suppose that, for some $b \in \partial D, f$ has an angular limit $f(b)$ at $b, f(b)=\beta$. Then we have the inequality

$$
\begin{align*}
& \left|f^{\prime}(b)\right| \geq \frac{1-\beta}{2}\left(1+\frac{1-\left|z_{1}\right|^{2}}{\left|b-z_{1}\right|^{2}}+\frac{2(1-\beta)\left|z_{1}\right|-\left|f^{\prime}(0)\right|}{2(1-\beta)\left|z_{1}+\right| f^{\prime}(0)}\right.  \tag{2.6}\\
& \times\left[\left.1+\frac{4(1-\beta)^{2}\left|z_{1}\right|^{2}+\left|f^{\prime}\left(z_{1}\right)\right| \mid\left(1-\left.z_{1}\right|^{2}\left|f^{\prime}(0)\right|-2(1-\beta)\left|f^{\prime}\left(z_{1}\right)\right|\left(1-\left|z_{1}\right|^{2}\right)-2(1-\beta)\left|f^{\prime}(0)\right|\right.}{4(1-\beta)^{2}\left|z_{1}\right|^{2}+\left|f^{\prime}\left(z_{1}\right)\right|\left(1-\left|z_{1}\right|^{2}\right)\left|f^{\prime}(0)\right|+2(1-\beta)\left|f^{\prime}\left(z_{1}\right)\right|\left(1-\left|z_{1}\right|^{2}\right)+2(1-\beta)\left|f^{2}(0)\right|} \right\rvert\, \frac{1 b-\left.z_{1}\right|^{2}}{\mid b-2(1)} .\right.
\end{align*}
$$

The inequality (2.6) is sharp, with equality for each possible values $\left|f^{\prime}(0)\right|=$ $2 e(1-\beta)$ and $\left|f^{\prime}\left(z_{1}\right)\right|=2 f(1-\beta)\left(0 \leq e \leq 2(1-\beta)\left|z_{1}\right|, 0 \leq f \leq 2(1-\beta) \frac{\left|z_{1}\right|}{1-\left|z_{1}\right|^{2}}\right)$.
Proof. Let

$$
q(z)=\frac{z-z_{1}}{1-\overline{z_{1}} z}
$$

Also, let $h: D \rightarrow D$ be a holomorphic function and a point $z_{1} \in D$ in order to satisfy

$$
\left|\frac{h(z)-h\left(z_{1}\right)}{1-\overline{h\left(z_{1}\right)} h(z)}\right| \leq\left|\frac{z-z_{1}}{1-\overline{z_{1} z}}\right|=|q(z)|
$$

and

$$
\begin{equation*}
|h(z)| \leq \frac{\left|h\left(z_{1}\right)\right|+|q(z)|}{1+\left|h\left(z_{1}\right)\right||q(z)|} \tag{2.7}
\end{equation*}
$$

by Schwarz-Pick lemma [8]. If $p: D \rightarrow D$ is holomorphic function and $0<\left|z_{1}\right|<1$, letting

$$
h(z)=\frac{p(z)-p(0)}{z(1-\overline{p(0)} p(z))}
$$

in (2.7), we obtain

$$
\left|\frac{p(z)-p(0)}{z(1-\overline{p(0)} p(z))}\right| \leq \frac{\left|\frac{p\left(z_{1}\right)-p(0)}{z_{1}\left(1-\overline{p(0) p} p\left(z_{1}\right)\right)}\right|+|q(z)|}{1+\left|\frac{p\left(z_{1}\right)-p(0)}{z_{1}\left(1-\overline{p(0)} p\left(z_{1}\right)\right)}\right||q(z)|}
$$

and

$$
\begin{equation*}
|p(z)| \leq \frac{|p(0)|+|z| \frac{|C|+|q(z)|}{1+|C||(z)|}}{1+|p(0)||z| \frac{|C|+|q(z)|}{1+|C| q(z) \mid}}, \tag{2.8}
\end{equation*}
$$

where

$$
C=\frac{p\left(z_{1}\right)-p(0)}{z_{1}\left(1-\overline{p(0)} p\left(z_{1}\right)\right)} .
$$

Without loss of generality, we will assume that $b=1$. If we take

$$
p(z)=\frac{\varphi(z)}{z \frac{z-z_{1}}{1-z_{1} z}},
$$

then

$$
p\left(z_{1}\right)=\frac{\varphi^{\prime}\left(z_{1}\right)\left(1-\left|z_{1}\right|^{2}\right)}{z_{1}}, \quad p(0)=\frac{\varphi^{\prime}(0)}{-z_{1}}
$$

and

$$
C=\frac{\frac{\varphi^{\prime}\left(z_{1}\right)\left(1-\left|z_{1}\right|^{2}\right)}{z_{1}}+\frac{\varphi^{\prime}(0)}{z_{1}}}{z_{1}\left(1+\frac{\varphi^{\prime}\left(z_{1}\right)\left(1-\left|z_{1}\right|^{2}\right)}{z_{1}} \frac{\varphi^{\prime}(0)}{z_{1}}\right)},
$$

where $|C| \leq 1$. Let $|p(0)|=\alpha$ and

$$
\mathrm{T}=\frac{\left|\frac{\varphi^{\prime}\left(z_{1}\right)\left(1-\left|z_{1}\right|^{2}\right)}{z_{1}}\right|+\left|\frac{\varphi^{\prime}(0)}{z_{1}}\right|}{\left|z_{1}\right|\left(1+\left|\frac{\varphi^{\prime}\left(z_{1}\right)\left(1-\left|z_{1}\right|^{2}\right)}{z_{1}}\right|\left|\frac{\varphi^{\prime}(0)}{z_{1}}\right|\right)}
$$

From (2.8), we get

$$
|\varphi(z)| \leq|z||q(z)| \frac{\alpha+|z| \frac{\mathrm{T}+|q(z)|}{1+\mathrm{T}|q(z)|}}{1+\alpha|z| \frac{\mathrm{T}+|q(z)|}{1+\mathrm{T}|q(z)|}}
$$

and

$$
\begin{equation*}
\frac{1-|\varphi(z)|}{1-|z|} \geq \frac{1+\alpha|z| \frac{\mathrm{T}+|q(z)|}{1+\mathrm{T}|q(z)|}-\alpha|z||q(z)|-|q(z)||z|^{2} \frac{\mathrm{~T}+|q(z)|}{1+\mathrm{T}|q(z)|}}{(1-|z|)\left(1+\alpha|z| \frac{\mathrm{T}+|q(z)|}{1+\mathrm{T}|q(z)|}\right)}=s(z) \tag{2.9}
\end{equation*}
$$

Let $\kappa(z)=1+\alpha|z| \frac{\mathrm{T}+|q(z)|}{1+\mathrm{T}|q(z)|}$ and $\tau(z)=1+\mathrm{T}|q(z)|$. Then

$$
s(z)=\frac{1-|z|^{2}|q(z)|^{2}}{(1-|z|) \kappa(z) \tau(z)}+\mathrm{T}|q(z)| \frac{1-|z|^{2}}{(1-|z|) \kappa(z) \tau(z)}+|z| \mathrm{T} \alpha \frac{1-|q(z)|^{2}}{(1-|z|) \kappa(z) \tau(z)} .
$$

Since

$$
\begin{gathered}
\lim _{z \rightarrow 1} \kappa(z)=\lim _{z \rightarrow 1} 1+\alpha|z| \frac{\mathrm{T}+|q(z)|}{1+\mathrm{T}|q(z)|}=1+\alpha \\
\lim _{z \rightarrow 1} \tau(z)=\lim _{z \rightarrow 1} 1+\mathrm{T}|q(z)|=1+\mathrm{T}
\end{gathered}
$$

and

$$
1-|q(z)|^{2}=1-\left|\frac{z-z_{1}}{1-\overline{z_{1}} z}\right|^{2}=\frac{\left(1-\left|z_{1}\right|^{2}\right)\left(1-|z|^{2}\right)}{\left|1-\overline{z_{1}} z\right|^{2}}
$$

passing to the angular limit in (2.9) gives

$$
\begin{aligned}
\left|\varphi^{\prime}(z)\right| & \geq \frac{2}{(1+\alpha)(1+\mathrm{T})}\left(1+\frac{1-\left|z_{1}\right|^{2}}{\left|1-z_{1}\right|^{2}}+\mathrm{T}+\alpha \mathrm{T} \frac{1-\left|z_{1}\right|^{2}}{\left|1-z_{1}\right|^{2}}\right) \\
& =1+\frac{1-\left|z_{1}\right|^{2}}{\left|1-z_{1}\right|^{2}}+\frac{1-\alpha}{1+\alpha}\left(1+\frac{1-\mathrm{T}}{1+\mathrm{T}} \frac{1-\left|z_{1}\right|^{2}}{\left|1-z_{1}\right|^{2}}\right) .
\end{aligned}
$$

Moreover, since

$$
\frac{1-\alpha}{1+\alpha}=\frac{1-|p(0)|}{1+|p(0)|}=\frac{1-\left|\frac{\varphi^{\prime}(0)}{z_{1}}\right|}{1+\left|\frac{\varphi^{\prime}(0)}{z_{1}}\right|}=\frac{\left|z_{1}\right|-\left|\varphi^{\prime}(0)\right|}{\left|z_{1}\right|+\left|\varphi^{\prime}(0)\right|}
$$

$$
\begin{aligned}
& =\frac{2(1-\beta)\left|z_{1}\right|-\left|f^{\prime}(0)\right|}{2(1-\beta)\left|z_{1}\right|+\left|f^{\prime}(0)\right|}, \\
& \frac{1-\mathrm{T}}{1+\mathrm{T}}=\frac{1-\frac{\left|\frac{\varphi^{\prime}\left(z_{1}\right)\left(1-\left|z_{1}\right|^{2}\right)}{z_{1}}\right|+\left|\frac{\varphi^{\prime}(0)}{z_{1}}\right|}{\left|z_{1}\right|\left(1+\left|\frac{\varphi^{\prime}\left(z_{1}\right)\left(1-\left|z_{1}\right|^{2}\right)}{z_{1}}\right|\left|\frac{\varphi^{\prime}(0)}{z_{1}}\right|\right)}}{1+\frac{\left|\frac{\varphi^{\prime}\left(z_{1}\right)\left(1-\left|z_{1}\right|^{2}\right)}{z_{1}}\right|+\left|\frac{\varphi^{\prime}(0)}{z_{1}}\right|}{\left|z_{1}\right|\left(1+\left|\frac{\varphi^{\prime}\left(z_{1}\right)\left(1-\left|z_{1}\right|^{2}\right)}{z_{1}}\right|\left|\frac{\varphi^{\prime}(0)}{z_{1}}\right|\right)}} \\
& =\frac{1-\frac{\frac{f^{\prime}\left(z_{1}\right)}{2(1-\beta)}\left(1-\left|z_{1}\right|^{2}\right)}{z_{1}}\left|+\left|\frac{\frac{f^{\prime}(0)}{2(1-\beta)}}{z_{1}}\right|\right.}{\left|z_{1}\right|\left(1+\left|\frac{\frac{f^{\prime}\left(z_{1}\right)}{2(1-\beta)}\left(1-\left|z_{1}\right|^{2}\right)}{z_{1}}\right|\left|\frac{\frac{f^{\prime}(0)}{2(1-\beta)}}{z_{1}}\right|\right)}
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{1-\mathrm{T}}{1+\mathrm{T}}=\frac{\left|z_{1}\right|\left(1+\left|\frac{\frac{f^{\prime}\left(z_{1}\right)}{2(1-\beta)}\left(1-\left|z_{1}\right|^{2}\right)}{z_{1}}\right|\left|\frac{\frac{f^{\prime}(0)}{2(1-\beta)}}{z_{1}}\right|\right.}{\left|z_{1}\right|\left(\left.1+\left|\frac{\frac{f^{\prime}\left(z_{1}\right)}{2(1-\beta)}\left(1-\left|z_{1}\right|^{2}\right)}{z_{1}}\right| \right\rvert\, \frac{f^{\prime}\left(z_{1}\right)}{2(1-\beta)}\left(1-\left|z_{1}\right|^{2}\right)\right.} z_{1}\left|-\left|\frac{\frac{f^{\prime}(0)}{2(1-\beta)}}{z_{1}}\right|\right)+\left|\frac{\frac{f^{\prime}\left(z_{1}\right)}{2(1-\beta)}\left(1-\left|z_{1}\right|^{2}\right)}{z_{1}}\right|+\left|\frac{f^{\prime}(0)}{2(1-\beta)} z_{1}\right| \\
& =\frac{4(1-\beta)^{2}\left|z_{1}\right|^{2}+\left|f^{\prime}\left(z_{1}\right)\right|\left(1-\left|z_{1}\right|^{2}\right)\left|f^{\prime}(0)\right|-2(1-\beta)\left|f^{\prime}\left(z_{1}\right)\right|\left(1-\left|z_{1}\right|^{2}\right)-2(1-\beta)\left|f^{\prime}(0)\right|}{4(1-\beta)^{2}\left|z_{1}\right|^{2}+\left|f^{\prime}\left(z_{1}\right)\right|\left(1-\left|z_{1}\right|^{2}\right)\left|f^{\prime}(0)\right|+2(1-\beta)\left|f^{\prime}\left(z_{1}\right)\right|\left(1-\left|z_{1}\right|^{2}\right)+2(1-\beta)\left|f^{\prime}(0)\right|},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
& \left|\varphi^{\prime}(1)\right| \geq 1+\frac{1-\left|z_{1}\right|^{2}}{\left|1-z_{1}\right|^{2}}+\frac{2(1-\beta)\left|z_{1}\right|-\left|f^{\prime}(0)\right|}{2(1-\beta)\left|z_{1}\right|+\left|f^{\prime}(0)\right|} \\
& \times\left[1+\frac{4(1-\beta)^{2}\left|z_{1}\right|^{2}+\left|f^{\prime}\left(z_{1}\right)\right|\left(1-\left|z_{1}\right|^{2}\right)\left|f^{\prime}(0)\right|-2(1-\beta)\left|f^{\prime}\left(z_{1}\right)\right|\left(1-\left|z_{1}\right|^{2}\right)-2(1-\beta)\left|f^{\prime}(0)\right|}{4(1-\beta)^{2}\left|z_{1}\right|^{2}+\left|f^{\prime}\left(z_{1}\right)\right|\left(1-\left|z_{1}\right|^{2}\right)\left|f^{\prime}(0)\right|+2(1-\beta)\left|f^{\prime}\left(z_{1}\right)\right|\left(1-\left|z_{1}\right|^{2}\right)+2(1-\beta)\left|f^{\prime}(0)\right|} \frac{1-\left.z_{1}\right|^{2}}{11-\left.z_{1}\right|^{2}}\right] .
\end{aligned}
$$

From definition of $\varphi(z)$, we have

$$
\varphi^{\prime}(z)=\frac{2(1-\beta) f^{\prime}(z)}{(f(z)-(2 \beta-1))^{2}}
$$

and

$$
\left|\varphi^{\prime}(1)\right|=\left|\frac{2(1-\beta) f^{\prime}(1)}{(f(1)-(2 \beta-1))^{2}}\right|=\frac{2\left|f^{\prime}(1)\right|}{1-\beta}
$$

Thus, we obtain the inequality (2.6).
Now, we shall show that the inequality (2.6) is sharp.
Since

$$
p(z)=\frac{\varphi(z)}{z \frac{z-z_{1}}{1-z_{1} z}}
$$

is holomorphic function in the unit disc and $|p(z)| \leq 1$ for $z \in D$, we obtain

$$
\left|\varphi^{\prime}(0)\right| \leq\left|z_{1}\right|
$$

and

$$
\left|\varphi^{\prime}\left(z_{1}\right)\right| \leq \frac{\left|z_{1}\right|}{1-\left|z_{1}\right|^{2}} .
$$

We take $z_{1} \in(-1,0)$ and arbitrary two numbers $e$ and $f$, such that $0 \leq e \leq$ $2(1-\beta)\left|z_{1}\right|, 0 \leq f \leq 2(1-\beta) \frac{\left|z_{1}\right|}{1-\left|z_{1}\right|^{2}}$.

Let

$$
\mathrm{K}=\frac{\frac{f\left(1-\left|z_{1}\right|^{2}\right)}{z_{1}}+\frac{e}{z_{1}}}{z_{1}\left(1+e f \frac{1-\left|z_{1}\right|^{2}}{z_{1}^{2}}\right)}=\frac{1}{z_{1}^{2}} \frac{f\left(1-\left|z_{1}\right|^{2}\right)+e}{1+e f \frac{1-\left|z_{1}\right|^{2}}{z_{1}^{2}}} .
$$

The auxiliary function

$$
t(z)=z \frac{z-z_{1}}{1-\overline{z_{1}} z} \frac{\frac{-e}{z_{1}}+z \frac{\mathrm{~K}+\frac{z-z_{1}}{1-\bar{z}_{1} z}}{1+\mathrm{K} \frac{z-z_{1}}{1-z_{1} z}}}{1-\frac{e}{z_{1}} z \frac{\mathrm{~K}+\frac{z-z_{1}}{1-z_{1} z}}{1+\mathrm{K} \frac{z-z_{1}}{1-z_{1} z}}}
$$

is holomorphic in $D$ and $|t(z)|<1$ for $z \in D$. Let

$$
\begin{equation*}
\frac{f(z)-1}{f(z)-(2 \beta-1)}=z \frac{z-z_{1}}{1-\overline{z_{1}} z} \frac{\frac{-e}{z_{1}}+z \frac{\mathrm{~K}+\frac{z-z_{1}}{1-\overline{\overline{1}_{2}}}}{1+\mathrm{K} \frac{z-z_{1}}{1-z_{1}}}}{1-\frac{e}{z_{1}} z \frac{\mathrm{~K}+\frac{z-z_{1}}{1-\overline{12} z}}{1+\mathrm{K} \frac{z-z_{1}}{1-\overline{\overline{1}_{1} z}}}} . \tag{2.10}
\end{equation*}
$$

So, we have

$$
f(z)=\frac{1-(2 \beta-1) z \frac{z-z_{1}}{1-\overline{z_{1}} z} \frac{\frac{-e}{z_{1}}+z \frac{\mathrm{~K}+\frac{z-z_{1}}{1-\overline{z_{1}} z}}{1+\mathrm{K} \frac{z-z_{1}}{1-\overline{z_{1}} z}}}{1-\frac{e}{z_{1}} z \frac{\mathrm{~K}+\frac{z-z_{1}}{1-\overline{z_{1}} z}}{1+\mathrm{K} \frac{z-z_{1}}{1-\overline{z_{1}} z}}}}{1-z \frac{z-z_{1}}{1-z_{1} z} \frac{\frac{-e}{z_{1}}+z \frac{\mathrm{~K}+\frac{z-z_{1}}{1-\overline{z_{1} z}}}{1+\mathrm{K} \frac{z-z_{1}}{1-\overline{z_{1} z}}}}{1-\frac{e}{z_{1}} z \frac{\mathrm{~K}+\frac{z-z_{1}}{1-\overline{z_{1}} z}}{1+\mathrm{K} \frac{z-z_{1}}{1-z_{1} z}}}}
$$

Therefore, we take $\left|f^{\prime}(0)\right|=2 e(1-\beta)$ and $\left|f^{\prime}\left(z_{1}\right)\right|=2 f(1-\beta)$.
From (2.10), with the simple calculations, we obtain

$$
\begin{aligned}
\frac{2(1-\beta) f^{\prime}(1)}{(f(1)-(2 \beta-1))^{2}} & =1+\frac{1-z_{1}^{2}}{\left(1-z_{1}\right)^{2}}+\frac{\left(1+\frac{1-z_{1}^{2}}{\left(1-z_{1}\right)^{2}} \frac{1-\mathrm{K}^{2}}{(1+)^{2}}\right)\left(1-\frac{e}{z_{1}}\right)+\frac{e}{z_{1}}\left(1+\frac{1-z_{1}^{2}}{\left(1-z_{1}\right)^{2}} \frac{1-\mathrm{K}^{2}}{(1+\mathrm{k})^{2}}\right)\left(1-\frac{e}{z_{1}}\right)}{\left(1-\frac{e}{z_{1}}\right)^{2}} \\
& =1+\frac{1-z_{1}^{2}}{\left(1-z_{1}\right)^{2}}+\frac{e+z_{1}}{-e+z_{1}}\left(1+\frac{1-z_{1}^{2}}{\left(1-z_{1}\right)^{2}} \frac{z_{1}^{2}+e f\left(1-z_{1}^{2}\right)-f\left(1-z_{1}^{2}\right)-e}{z_{1}^{2}+e f\left(1-z_{1}^{2}\right)+f\left(1-z_{1}^{2}\right)+e}\right)
\end{aligned}
$$

and
$f^{\prime}(1)=\frac{1-\beta}{2}\left(1+\frac{1-z_{1}^{2}}{\left(1-z_{1}\right)^{2}}+\frac{e+z_{1}}{-e+z_{1}}\left(1+\frac{1-z_{1}^{2}}{\left(1-z_{1}\right)^{2}} \frac{z_{1}^{2}+e f\left(1-z_{1}^{2}\right)-f\left(1-z_{1}^{2}\right)-e}{z_{1}^{2}+e f\left(1-z_{1}^{2}\right)+f\left(1-z_{1}^{2}\right)+e}\right)\right)$.
Since $z_{1} \in(-1,0)$, the last equality shows that (2.6) is sharp.

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Bülent Nafi Ornek Received August 7, 2017<br>Department of Computer Engineering<br>Amasya University<br>Merkez-Amasya 05100, Turkey<br>E-mail: nafiornek@gmail.com,<br>nafi.ornek@amasya.edu.tr


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