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The Cotton tensor and Chern-Simons invariants in dimension 3: an introduction

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Abstract. We review, with complete proofs, the theory of Chern-Simons invariants for oriented Riemannian 3-manifolds. The Cotton tensor is the first-order variation of the Chern-Simons invariant. We deduce that it is conformally invariant, and trace- and divergence-free, from the corresponding properties of the Chern-Simons invariant. Moreover, the Cotton tensor vanishes if and only if the metric is locally conformally flat. In the last part of the paper we survey the link of Chern-Simons invariants with the eta invariant and with the central value of the Selberg zeta function of odd type.

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1 Motivation

Let (M, g) be an oriented Riemannian 3-manifold. It is natural to ask if, like it happens in dimension 2, the metric g is locally conformally flat. There exists an obstruction to local conformal flatness in dimension 3, discovered by Émile Cotton [5] in 1899. This obstruction is a symmetric, traceless, conformally covariant, and divergence-free 2-tensor on M. In 1974 it was further unveiled by Chern and Simons [4] that the Cotton tensor is *variational*: if M is compact, there exists a $\mathbb{R}/\mathbb{Z}$-valued function on the space of Riemannian metrics, the Chern-Simons invariant, whose gradient at a given metric g is the Cotton tensor $\mathfrak{Cott}(g)$.

In this survey paper we give a short and self-contained introduction to the theory of Chern-Simons invariants for three-dimensional Riemannian manifolds. As consequences, we will prove in a conceptual way some of the important properties of $\mathfrak{Cott}$. Our approach relies on the paper by Chern and Simons [4], with modern notation and focusing on the dimension 3. We use Besse's book [3] as reference for standard formulas in Riemannian geometry.

Chern-Simons invariants are defined here with respect to some Riemannian metric, but they actually depend on just a connection. Moreover they can be defined in arbitrary dimensions, while we have limited ourselves to dimension 3. The main focus of interest regarding Chern-Simons invariants shifted in recent years towards invariants of smooth manifolds, independent of any background metric, obtained by averaging Chern-Simons invariants with respect to a (mathematically non-rigorous) measure over an infinite-dimensional space of connections. Such an invariant is called a “topological quantum field theory” and currently plays a prominent role in mathematical physics. Witten’s foundational paper [11] highlighted a link between this Chern-Simons TQFT and the Jones polynomial of knots. We do not attempt here any foray into these fancy fields.

Since on one hand a comprehensive list of references would dwarf the size of the paper, and since on the other hand our text is self-contained except for the last section, we have kept the references to a minimum, hoping that the numerous authors having significant contributions to the subject will not feel slighted. We have limited our ambitions to understanding Riemannian Chern-Simons invariants of 3-manifolds, in the goal of offering the reader a firm, albeit tiny, first foothold into Chern-Simons realm. The Cotton tensor comes as a reward for our self-limitation. Evidently, no claim of originality is made below other than the presentation, itself strongly influenced by [6].

These notes are accessible to readers familiar with the basic objects of Riemannian geometry: differential forms, Levi-Civita connection, curvature, Ricci tensor, scalar curvature. They formed the topic of a concentrated doctoral course at the University of Bucharest in March 2014. I am grateful to Sorin Dăscălescu for the invitation.

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2 Definitions and background

Throughout the paper, M is a Riemannian manifold of dimension 3, and g its metric. Let

$$\nabla : C^\infty(M, TM) \to C^\infty(M, \Lambda^1 M \otimes TM)$$

denote the Levi-Civita connection, and

$$d^\nabla : C^\infty(M, \Lambda^k M \otimes TM) \longrightarrow C^\infty(M, \Lambda^{k+1} M \otimes TM)$$

its extension to k-forms twisted by vector fields. The square of this operator equals, in every degree k, the curvature operator R, viewed as an endomorphism-valued 2-form:

$$(d^\nabla)^2 = R \in C^\infty(M, \Lambda^2 M \otimes \mathrm{End}(TM)).$$

The second Bianchi identity is an immediate consequence:

$$d^\nabla(R) = [d^\nabla, (d^\nabla)^2] = 0. \tag{1}$$

The *Ricci tensor* is the contraction on positions $\{1, 4\}$ of R viewed as a $(3, 1)$ tensor:

$$\mathfrak{Ric}(U, V) = \mathrm{tr}\left[X \mapsto R_{XU} V\right].$$

The *scalar curvature* $\mathfrak{scal}$ is the trace of $\mathfrak{Ric}$ with respect to g.

Let $(d^\nabla)^*$ denote the formal adjoint of d^∇. This operator restricted to symmetric 2-tensors is sometimes called the *divergence* operator. By twice contracting the $(4,1)$ tensor from (1) on positions $\{3,4\}$ and then $\{1,5\}$, we obtain

$$(d^\nabla)^* \left(\mathfrak{Ric} - \tfrac{\mathfrak{scal}}{2} g\right) = 0. \tag{2}$$

Definition 1. The *Schouten tensor* of the metric g is essentially the Ricci tensor, namely

$$\mathfrak{Sch} := \mathfrak{Ric} - \tfrac{\mathfrak{scal}}{4} g.$$

Notice that since the dimension of M is 3, $\mathrm{tr}(\mathfrak{Sch}) = \frac{\mathfrak{scal}}{4}$. Also from (2), $\mathfrak{Sch}$ is not divergence-free unless $\mathfrak{scal}$ is constant.

Definition 2. The *Cotton form* of g is $\mathfrak{cott} := d^\nabla \mathfrak{Sch} \in C^\infty(M, \Lambda^2 M \otimes TM)$. Assuming M to be oriented, the *Cotton tensor* of g is the bilinear form

$$\mathfrak{Cott} := * d^\nabla \mathfrak{Sch} \in C^\infty(M, \Lambda^1 M \otimes TM),$$

where $*$ is the Hodge operator transforming 2-forms into 1-forms on M.

Recall that the Hodge operator in dimension 3 is defined in terms of an oriented orthonormal frame S_1, S_2, S_3 as follows:

$$*1 = S^1 \wedge S^2 \wedge S^3, \quad *S^1 = S^2 \wedge S^3, \quad *S^2 = S^3 \wedge S^1, \quad *S^3 = S^1 \wedge S^2 \tag{3}$$

and satisfies $*^2 = \mathrm{Id}_{\Lambda^* M}$.

3 Chern-Simons forms of degree 3

Let M be a smooth manifold of any dimension, $m \geq 1$ a natural number, and θ a 1-form on M with values in $\mathrm{gl}(m, \mathbb{R})$, i.e., θ is a $m \times m$ matrix with entries θ_{ij}, $i, j \in \{1, \ldots, m\}$, which are 1-forms on M. The Chern-Simons form associated to θ is defined by

$$\mathfrak{cs}(\theta) := \mathrm{tr}\left(\theta \wedge d\theta + \tfrac{2}{3} \theta \wedge \theta \wedge \theta\right) \in \Lambda^3(M).$$

In the above trace, the wedge product of forms-valued matrices is the usual product of matrices, where the entries are multiplied using the wedge product on $\Lambda^*(M)$. The exterior differential $d\theta$ is obtained by applying the de Rham differential d to each entry θ_{ij}, yielding a $\mathrm{gl}(m, \mathbb{R})$-valued 2-form.

Lemma 1. *Let $\{\theta_t\}_{t \in \mathbb{R}}$ be a smooth family of forms and denote by $\dot\theta := \frac{d\theta_t}{dt}|_{t=0}$ its variation in $t = 0$. Then*

$$\tfrac{d}{dt} \mathfrak{cs}(\theta_t)_{|t=0} = d\mathrm{tr}(\dot\theta \wedge \theta) + 2\mathrm{tr}(\dot\theta \wedge (d\theta + \theta \wedge \theta)).$$

Proof. For matrix-valued forms $\alpha, \beta \in \Lambda^*(M) \otimes \mathrm{gl}(m, \mathbb{R})$ we have the trace identity

$$\mathrm{tr}(\alpha \wedge \beta) = (-1)^{\deg \alpha \cdot \deg \beta} \mathrm{tr}(\beta \wedge \alpha)$$

although of course the matrix-valued forms α and β need not commute. We also use the obvious rule for taking the exterior differential of a trace, namely

$$d\mathrm{tr}(\alpha \wedge \beta) = \mathrm{tr}(d\alpha \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge d\beta).$$

We compute by differentiating under the trace by the Leibniz rule and using the above trace identity:

$$\begin{aligned} \tfrac{d}{dt}\mathfrak{cs}(\theta_t)_{|t=0} &= \mathrm{tr}(\dot\theta \wedge d\theta + \theta \wedge d\dot\theta + 2\dot\theta \wedge \theta \wedge \theta) \\ &= \mathrm{tr}(d\dot\theta \wedge \theta - \dot\theta \wedge d\theta + 2\dot\theta \wedge d\theta + 2\dot\theta \wedge \theta \wedge \theta) \end{aligned}$$

which gives the desired formula. □

Set $\Omega := d\theta + \theta \wedge \theta \in \Lambda^2(M) \otimes \mathrm{gl}(m, \mathbb{R})$. Lemma 1 can be rewritten

$$\tfrac{d}{dt}\mathfrak{cs}(\theta_t)_{|t=0} = d\mathrm{tr}(\dot\theta \wedge \theta) + 2\mathrm{tr}(\dot\theta \wedge \Omega).$$

Lemma 2. *The exterior derivative of* $\mathfrak{cs}(\omega)$ *is*

$$d\mathfrak{cs}(\theta) = \mathrm{tr}(\Omega \wedge \Omega).$$

Proof. We have by the trace identity

$$d\mathfrak{cs}(\theta) = \mathrm{tr}(d\theta \wedge d\theta + 2d\theta \wedge \theta \wedge \theta) = \mathrm{tr}\left((d\theta + \theta \wedge \theta)^2\right)$$

where in the last equality we used $\mathrm{tr}(\theta^4) = 0$ by anti-symmetry. □

4 The Chern-Simons invariant of a Riemannian metric

From now on M is an oriented compact 3-manifold without boundary. The tangent bundle to such a manifold is trivial by a classical result of Stiefel [9, Satz 21] (in fact, for this result one does not even need M to be compact, but we do not use here this more general statement). Choose a global orthonormal frame $S = (S_1, S_2, S_3)$, obtained for instance by applying the Gram-Schmidt procedure to some arbitrary smooth frame. In such a frame, the Levi-Civita connection can be expressed as

$$\nabla = d + \omega$$

where the so(3)-valued connection 1-form $\omega = (\omega_{ij})_{i,j=1,2,3}$ is given by

$$\omega_{ij} = \langle \nabla S_j, S_i \rangle$$

(since the metric is parallel, we have $\omega_{ij} = -\omega_{ji}$).

Definition 3. The Chern-Simons 3-form $\mathfrak{cs}(M, g, S)$ is defined by

$$\mathfrak{cs}(M, g, S) := \mathfrak{cs}(\omega) = \mathrm{tr}(\omega \wedge d\omega + \tfrac{2}{3}\omega \wedge \omega \wedge \omega) \in \Lambda^3(M),$$

where the trace is the usual trace on 3×3 matrices. The Chern-Simons invariant of (M, g) with respect to the frame S is

$$\mathfrak{CS}(M, g, S) := -\tfrac{1}{16\pi^2} \int_M \mathfrak{cs}(M, g, S) \in \mathbb{R}.$$

The triviality of the vector bundle TM entails the vanishing of its "primary" characteristic classes (Stiefel-Whitney, Pontriagin and Euler). At the same time, this triviality implies the existence of a global connection 1-form, the essential ingredient in the construction of the Chern-Simons invariant. Since the invariant exists under the condition that all primary characteristic classes vanish, it is viewed as a kind of "secondary" characteristic class, dependent on the metric, and *a priori* also on the choice of the frame.

Proposition 1. *Let S, S' be two orthonormal frames on M linked by some* $\mathrm{SO}(3)$*-valued function $\mathfrak{a} : M \to \mathrm{SO}(3)$, i.e., $S' = S\mathfrak{a}$. Then*

$$\mathfrak{CS}(M, g, S) - \mathfrak{CS}(M, g, S') \in \mathbb{Z}.$$

Proof. The connection form changes by

$$\omega' = \mathfrak{a}^{-1}\omega\mathfrak{a} + \mathfrak{a}^{-1}d\mathfrak{a}.$$

By a simple computation, the Chern-Simons form of ω' is given by

$$\mathfrak{cs}(\omega') = \mathfrak{cs}(\omega) + d\mathrm{tr}(\mathfrak{a}^{-1}\omega \wedge d\mathfrak{a}) - \tfrac{1}{3}\mathrm{tr}((\mathfrak{a}^{-1}d\mathfrak{a})^3).$$

The form $\mathfrak{a}^{-1}d\mathfrak{a}$ equals the pull-back via the map $\mathfrak{a} : M \to \mathrm{SO}(3)$ of the Maurer-Cartan 1-form ω_{MC} that we recall in Lemma 3 below. By that lemma, the integral on $\mathrm{SO}(3)$ of the 3-form $\mathrm{tr}((\omega_{MC})^3)$ equals $48\pi^2$, thus $\frac{1}{48\pi^2}\mathrm{tr}((\omega_{\mathrm{MC}})^3)$ is a generator of $H^3(\mathrm{SO}(3), \mathbb{Z})$. It follows that the integral on M of $\frac{1}{48\pi^2}\mathrm{tr}((\mathfrak{a}^{-1}d\mathfrak{a})^3)$ is an integer, equal to the degree of the map $\mathfrak{a}$. ☐

We see that the constant of normalization was chosen so that for different choices of S, the Chern-Simons integral changes by some integer. In other words, the invariant is well-defined modulo $\mathbb{Z}$ independently of S, and will be denoted $\mathfrak{CS}(M, g) \in \mathbb{R}/\mathbb{Z}$.

5 An example

Let us compute the Chern-Simons invariant of the Lie group $\mathrm{SO}(3)$. Let $\mathbb{H}$ be the quaternion algebra. The group of quaternions of length 1 (which is just the sphere $S^3 \subset \mathbb{H}$) acts on $\mathbb{H}$ via *right* multiplication, preserving at the same time the standard Hermitian metric and the structure of complex vector space given by left multiplication with complex numbers. We get in this way a unitary representation of S^3. By compactness, the representation lies in SU(2), and since it is clearly faithful, it provides a Lie group isomorphism $\mathrm{SU}(2) \to S^3$.

Conjugation by quaternions of length 1 is a real representation of S^3 (which we henceforth identify with SU(2)). It acts orthogonally on $\mathbb{H} = \mathbb{R}^4$ and preserves the real line, thus it also preserves its orthogonal complement $\mathbb{H}' = \mathbb{R}^3$, the 3-dimensional space of purely imaginary quaternions. By connectedness, it must take values in SO(3). The kernel of this representation is the intersection of the center of $\mathbb{H}$ with S^3, thus it consists of $\{\pm 1\}$. Moreover the representation is surjective since it contains every reflection around an axis in $\mathbb{H}'$.

We have obtained a $2:1$ covering $\mathrm{SU}(2) \to \mathrm{SO}(3)$ of Lie groups, with deck group $\{\pm 1\}$ acting isometrically. Endow SO(3) with the metric g induced from this covering.

Let

$$I := \mathrm{ad}_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix}, \quad J := \mathrm{ad}_j = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}, \quad K := \mathrm{ad}_k = \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4}$$

be the image in the Lie algebra so(3) of the standard orthonormal basis $\{i, j, k\}$ in $T_1 S^3$ (recall that the tangent space to SO(3) at the identity can be identified with the Lie algebra of anti-symmetric 3×3 matrices). We transport these vectors on SO(3) by left translations, thus obtaining left-invariant vector fields denoted by the same letters I, J, K. Their Lie bracket is given by:

$$[I, J] = 2K, \qquad [J, K] = 2I, \qquad [K, I] = 2J.$$

From the Koszul formula (6) and left-invariance, we deduce

$$\nabla_I J = K, \qquad \nabla_J K = I, \qquad \nabla_K I = J.$$

The connection 1-form ω is thus given by

$$\omega(I) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \omega(J) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \qquad \omega(K) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

implying that $\mathrm{tr}(\omega^3) = -6 d\mathrm{vol}_g$. Since the coefficients of ω are constant, it follows from the Cartan formula

$$d\omega(I, J) = -2\omega(K), \qquad d\omega(J, K) = -2\omega(I), \qquad d\omega(K, I) = -2\omega(J).$$

Since $\mathrm{tr}\left(\omega(I)^2\right) = \mathrm{tr}\left(\omega(I)^2\right) = \mathrm{tr}\left(\omega(I)^2\right) = -2$, it follows $\mathrm{tr}(\omega \wedge d\omega) = 12 d\mathrm{vol}_g$. In conclusion,

$$\mathfrak{cs}(\omega) = 8 d\mathrm{vol}_g.$$

The volume of the sphere S^3 with its standard metric is obtained as follows:

$$\mathrm{vol}(S^3)=\int_{-\pi/2}^{\pi/2}\cos^2(t)\mathrm{vol}(S^2)dt=4\pi\cdot\tfrac{\pi}{2}=2\pi^2,$$

so $\mathrm{vol}(\mathrm{SO}(3),g)=\pi^2$. We deduce

$$\mathfrak{CS}(\mathrm{SO}(3),g)=-\tfrac{1}{2}. \tag{5}$$

Corollary 1. *The Chern-Simons invariant of S^3 with its standard metric vanishes in $\mathbb{R}/\mathbb{Z}$.*

Proof. The $2:1$ covering $S^3\to\mathrm{SO}(3)$ is an isometry, hence $\int_{S^3}\mathfrak{cs}(S^3)=2\int_{\mathrm{SO}(3)}\mathfrak{cs}(\mathrm{SO}(3))$. The result follows from (5). □

Lemma 3. *Let ω_{MC} denote the Maurer-Cartan 1-form on $\mathrm{SO}(3)$, namely $\omega_{\mathrm{MC}}(X)=X\in\mathrm{so}(3)$ for every left-invariant vector field X. Then*

$$\mathrm{tr}(\omega_{\mathrm{MC}}^3)=-48d\mathrm{vol}_g.$$

Proof. It follows from (4) that

$$\begin{aligned}\mathrm{tr}(\omega_{\mathrm{MC}}(I)\omega_{\mathrm{MC}}(J)\omega_{\mathrm{MC}}(K))&=\mathrm{tr}(IJK)=-8,\\ \mathrm{tr}(\omega_{\mathrm{MC}}(I)\omega_{\mathrm{MC}}(K)\omega_{\mathrm{MC}}(J))&=\mathrm{tr}(IKJ)=8.\end{aligned}$$

From the trace identity, the lemma follows. □

6 Conformal invariance

One of the striking properties of the Riemannian Chern-Simons invariant is its conformal invariance: the invariant does not change (modulo $\mathbb{Z}$) when the metric varies in a fixed conformal class.

Theorem 1. *Let (M,g) be a closed oriented Riemannian 3-fold, and $f\in C^\infty(M,\mathbb{R})$ an arbitrary conformal factor. Then*

$$\mathfrak{CS}(M,g)=\mathfrak{CS}(M,e^{2f}g).$$

Proof. For $t\in\mathbb{R}$ set $g^t:=e^{2tf}g$, thus in particular $g^0=g$. Fix an orthonormal frame $S=(S_1,S_2,S_3)$ on (M,g), and define $S_j^t:=e^{-tf}S_j$, $j=1,2,3$. Then (S_1^t,S_2^t,S_3^t) form an orthonormal frame on (M,g^t). Starting from the Koszul formula

$$\begin{aligned}2\langle\nabla_XY,Z\rangle=&X\langle Y,Z\rangle+Y\langle X,Z\rangle-Z\langle X,Y\rangle\\ &+\langle[X,Y],Z\rangle+\langle[Z,X],Y\rangle+\langle[Z,Y],X\rangle,\end{aligned}\tag{6}$$

we deduce that the Levi-Civita connections of $g^1=e^{2f}g$ and that of g differ by

$$\nabla^1_XY-\nabla_XY=X(f)Y+Y(f)X-\langle X,Y\rangle_g\mathrm{grad}^g(f).$$

for every vector fields X, Y. We apply this identity to deduce

$$\langle \nabla^1_X S^1_j, S^1_i \rangle_{g^1} = \langle \nabla_X S_j, S_i \rangle + \langle S_j(f) S_i, X \rangle - \langle S_i(f) S_j, X \rangle.$$

In other words, if we identify vectors and 1-forms using the metric g, we express the connection 1-form of ∇^1 in the frame S^1 as

$$\omega^1 = \omega + \alpha$$

where

$$\alpha_{ij}(X) = \langle S_j(f) S_i - S_i(f) S_j, X \rangle.$$

Applying this identity to g^t, we get $\omega^t = \omega + t\alpha$. From Lemma 1 with $\dot{\omega} = \alpha$, we obtain

$$\tfrac{d}{dt}\mathfrak{cs}(\omega^t)_{|t=0} = d\mathrm{tr}(\alpha \wedge \omega) + 2\mathrm{tr}(\alpha \wedge R) \tag{7}$$

where $R \in \Lambda^2(M, \mathrm{so}(3))$ is the Riemannian curvature tensor written in the frame S.

Lemma 4. *The trace* $\mathrm{tr}(\alpha \wedge R)$ *vanishes.*

Proof. Let $(k, h, l) \in \Sigma(3)^+$ denote even permutations of $\{1, 2, 3\}$. We write

$$\mathrm{tr}(\alpha \wedge R) = d\mathrm{vol}_g \sum_{(k,h,l)\in\Sigma(3)^+} \sum_{i,j=1}^{3} \alpha_{ji}(S_k) R_{hlij}.$$

But $\alpha_{ji}(S_k) = S_i(f)\delta_{kj} - S_j(f)\delta_{ki}$, hence

$$\mathrm{tr}(\alpha \wedge R) = d\mathrm{vol}_g \sum_{(k,h,l)\in\Sigma(3)^+} \left(\sum_{i=1}^{3} S_i(f) R_{hlik} - \sum_{j=1}^{3} S_j(f) R_{hlkj} \right).$$

Both terms vanish by the first Bianchi identity (see [3, Prop. 1.85d]). □

By this lemma, (7) can be rewritten

$$\tfrac{d}{dt}\mathfrak{cs}(\omega^t)_{|t=0} = d\mathrm{tr}(\alpha \wedge \omega).$$

Now apply this to ω_t instead of ω. We have seen that $\frac{d\omega^t}{dt} = \alpha$ is independent of t. Therefore the identity

$$\tfrac{d}{dt}\mathfrak{cs}(\omega^t) = d\mathrm{tr}(\alpha \wedge \omega^t)$$

is valid for all t. Moreover, $\mathrm{tr}(\alpha\wedge\alpha) = 0$ and so finally we deduce $\frac{d\mathfrak{cs}(\omega^t)}{dt} = d\mathrm{tr}(\alpha\wedge\omega)$. This is independent of t, and by integrating from 0 to 1 we get $\mathfrak{cs}(\omega^1) = \mathfrak{cs}(\omega) + d\mathrm{tr}(\alpha \wedge \omega)$. The theorem is a direct consequence of Stokes' formula. □

7 First-order variation of the Chern-Simons invariant

Recall the definitions of the Cotton form, respectively tensor, on a Riemannian 3-manifold (M, g):

$$\mathfrak{cott} = d^{\nabla}\mathfrak{Sch}, \qquad \mathfrak{Cott} = *\mathfrak{cott}.$$

Let L^2 denote the Hilbert space of square-integrable symmetric 2-tensors on M with respect to the scalar product and the volume form induced by g.

Theorem 2. *Let $\Omega := d\omega + \omega \wedge \omega \in \Lambda^2(M, \mathrm{so}(3))$ denote the curvature form of g in the frame S. Let $(g^t)_{t\in\mathbb{R}}$ be a 1-parameter family of Riemannian metrics, and denote by $\dot{g} := \frac{d}{dt}g^t{}_{|t=0}$ its first-order variation. Then*

$$\dot{\mathfrak{CS}} := \tfrac{d}{dt}\mathfrak{CS}(M, g^t)_{|t=0} = -\tfrac{1}{8\pi^2}\langle \dot{g}, \mathfrak{Cott}\rangle_{L^2} = -\tfrac{1}{8\pi^2}\int_M \langle \dot{g}, \mathfrak{Cott}(g)\rangle_g d\mathrm{vol}_g.$$

Proof. We first construct a smooth family $(S^t)_{t\in\mathbb{R}}$ of global frames on M such that S^t is orthonormal for g^t. For this, consider the metric $g^{\mathcal{Z}} := dt^2 + g^t$ on $\mathcal{Z} := \mathbb{R} \times M$, and set S^t to be the parallel transport along the segment $[0, t]$ of the frame $S^0 := S$. A direct computation shows that the lines $\mathbb{R} \times \{p\}$ are geodesics for $g^{\mathcal{Z}}$ for all $p \in M$, thus S^t is indeed an orthonormal frame in the slice $\{t\} \times M$.

Let ω^t be the connection 1-form of g^t in the frame S^t, and set $\dot{\omega} := \frac{d\omega^t}{dt}|_{t=0}$. Applying Lemma 1 and interchanging the integral on M with the t-differential, we find

$$-16\pi^2\dot{\mathfrak{CS}} = 2\int_M \mathrm{tr}(\dot{\omega} \wedge \Omega).$$

Let $X \in \mathcal{V}(M)$ be a vector field on M, extended on $\mathcal{Z}$ to be constant in the t direction, in other words $L_T X = 0$ or equivalently $[T, X] = 0$ where we denote $T := \frac{\partial}{\partial t}$. Then

$$\dot{\omega}_{ij}(X) = \partial_t g^t(\nabla^t_X S^t_j, S^t_i) = \partial_t g^{\mathcal{Z}}(\nabla^{\mathcal{Z}}_X S^t_j, S^t_i) = \langle \nabla^{\mathcal{Z}}_T \nabla^{\mathcal{Z}}_X S^t_j, S^t_i\rangle = \langle R^{\mathcal{Z}}_{TX} S^t_j, S^t_i\rangle.$$

We used above the fact that $\nabla^{\mathcal{Z}}_T S^t_j = 0$ (by construction of S^t_j) and the commutation of T and X. From the symmetries of the Riemannian curvature, we get

$$\dot{\omega}_{ij} = R^{\mathcal{Z}}_{S_j, S_i} T$$

(we identify vectors and 1-forms using the metric g). Let W be the Weingarten operator of $\{0\} \times M \hookrightarrow \mathcal{Z}$. By the Codazzi-Mainardi equation [3, 1.72d],

$$R^{\mathcal{Z}}_{S_j, S_i} T = d^{\nabla} W(S_j, S_i)$$

or equivalently $R^{\mathcal{Z}} T = d^{\nabla} W$ as vector-valued 2-forms. The operator W, essentially the second fundamental form, can be computed in terms of the first variation h of g^t, namely

$$W = g^{-1}\mathrm{I\!I} = \tfrac{1}{2} g^{-1}\dot{g}.$$

It follows that $\dot\omega = \frac{1}{2}d^\nabla \dot g$, where $d^\nabla \dot g$ is viewed as a so(3)-valued 1-form using the basis S. Since the trace is independent of the basis,

$$-16\pi^2\dot{\mathfrak{CS}} = \int_M \mathrm{tr}(d^\nabla \dot g \wedge R)$$

where R is the Riemannian curvature tensor viewed as a section in $\Lambda^2(M)\otimes \mathrm{End}(TM)$, and $d^\nabla \dot g$ is viewed as a section in $\mathrm{End}(TM)\otimes\Lambda^1(M)$. Using the symmetry of R, this becomes

$$-2\int_M \langle d^\nabla \dot g, *_{34}R\rangle d\mathrm{vol}_g,$$

where the Hodge star operator $*_{34}$ acts on the last two positions in R. By definition of the adjoint operator, this equals

$$-2\langle \dot g, d^{\nabla^*} *_{34} R\rangle_{L^2}.$$

Now the adjoint of d^∇ on 2-forms with values in T^*M is just $-*_{12}\, d^\nabla *_{12}$, thus

$$-16\pi^2\dot{\mathfrak{CS}} = 2\langle \dot g, *_{12}d^\nabla *_{12} *_{34}R\rangle_{L^2}.$$

The 2-tensor $*_{12} *_{34} R$ is easily computed:

$$*_{12} *_{34} R = \mathfrak{Ric} - \tfrac{\mathfrak{scal}}{2}g =: Q = \mathfrak{Sch} - \tfrac{\mathfrak{scal}}{4}g.$$

We remark that h is symmetric but $d^{\nabla^*} *_{34} R$ is not necessarily so. Of course, the skew-symmetric component of $d^{\nabla^*} *_{34} R$ will not contribute towards $\dot{\mathfrak{CS}}$ since h is symmetric. Thus,

$$-8\pi^2\dot{\mathfrak{CS}} = \langle \dot g, (*_{12}d^\nabla *_{12} *_{34}R)_{\mathrm{sym}}\rangle_{L^2} = \langle \dot g, (*_{12}d^\nabla Q)_{\mathrm{sym}}\rangle_{L^2}.$$

We now claim that $(*_{12}d^\nabla Q)_{\mathrm{sym}} = *_{12}d^\nabla\mathfrak{Sch} = \mathfrak{Cott}$. This means two things:

- $*_{12}d^\nabla\mathfrak{Sch}$ is symmetric;
- $*_{12}d^\nabla(\mathfrak{scal}\cdot g)$ is skew-symmetric.

For any function f, $*_{12}d^\nabla(fg) = *df$ is a 2-form, so the second fact is evident. As for the first, we use the identity [3, 1.94]

$$\mathrm{tr}_{13}d^\nabla\mathfrak{Sch} = d^{\nabla^*}(\mathfrak{Ric} - \tfrac{\mathfrak{scal}}{2}g) = 0,$$

where tr_{13} denotes trace with respect to g on positions $\{1,3\}$. Take (i,j,k) to be a cyclic permutation of $(1,2,3)$. Then

$$0 = (\mathrm{tr}_{13}d^\nabla\mathfrak{Sch})(S_k) = d^\nabla\mathfrak{Sch}(S_i,S_k,S_i) + d^\nabla\mathfrak{Sch}(S_j,S_k,S_j),$$

which is equivalent (by immediate algebraic considerations) to the desired symmetry

$$*_{12}d^\nabla\mathfrak{Sch}(S_i,S_j) = *_{12}d^\nabla\mathfrak{Sch}(S_j,S_i).$$

□

8 Properties of the Cotton tensor

Let (M°, g°) be a Riemannian 3-manifold, not necessarily compact. Then every point $x \in M^\circ$ has a neighborhood which can be isometrically embedded in some compact manifold (M, g). We will prove below some local properties of the Cotton tensor of (M, g), which are therefore shared by $\mathfrak{Cott}(M^\circ, g^\circ)$.

Proposition 2. *The Cotton tensor* $*d^\nabla \mathfrak{Sch}$ *is symmetric.*

Proof. This was shown in the last part of the proof of Theorem 2. □

Proposition 3. *The Cotton tensor is trace-free.*

Proof. Let $f \in C^\infty(M, \mathbb{R})$ be arbitrary, and set $g^t := e^{2tf} g$. Its first-order variation at $t = 0$ is given by $2fg$. By conformal invariance, $\mathfrak{CS}(M, g^t)$ is constant in time. On the other hand, by Theorem 2, we get

$$0 = 16\pi^2 \frac{d\mathfrak{CS}(M, g^t)}{dt}|_{t=0} = \langle 2fg, \mathfrak{Cott}(g)\rangle = 2\int_M f \operatorname{tr}(\mathfrak{Cott}(g)) d\mathrm{vol}_g.$$

This means that $\operatorname{tr}(\mathfrak{Cott}(g))$ is L^2-orthogonal on every smooth function on M, hence it must vanish identically. □

Proposition 4. *The Cotton tensor is divergence-free, i.e.,* $d^{\nabla^*}\mathfrak{Cott}(g) = 0$.

Proof. Let $X \in \mathcal{V}(M)$ be a vector field, and ϕ_t the 1-parameter group of diffeomorphisms obtained by integrating X on M. Set $g^t := \phi_t^* g$, so in particular $\dot{g} = L_X g$. Since g and g^t are isometric, it is rather evident that $\mathfrak{CS}(M, g^t) = \mathfrak{CS}(M, g)$. By Theorem 2 we deduce that $\langle L_X g, \mathfrak{Cott}(g)\rangle_{L^2} = 0$. But $L_X g = \frac{1}{2}(d^\nabla X)_{\mathrm{sym}}$, so by Proposition 2 and the definition of the adjoint, $\langle L_X g, \mathfrak{Cott}(g)\rangle = \frac{1}{2}\langle X, d^{\nabla^*}\mathfrak{Cott}\rangle$. Hence $d^{\nabla^*}\mathfrak{Cott}$ is L^2-orthogonal to every vector field X, so it must vanish identically. □

Proposition 5. *The Cotton tensor is conformally covariant, in the sense that for every* $f \in C^\infty(M)$*, we have*

$$\mathfrak{Cott}(e^{2f} g) = e^{-f}\mathfrak{Cott}(g).$$

The Cotton form $\mathfrak{cott} = d^\nabla \mathfrak{Sch}$ *is conformally invariant:*

$$\mathfrak{cott}(e^{2f} g) = \mathfrak{cott}(g).$$

Proof. Let h be any symmetric 2-tensor, and choose a family g^t of metrics with $\dot{g} = h$, for instance $g^t = g + th$ for small t. By conformal invariance of the Chern-Simons invariant, $\mathfrak{CS}(M, e^{2f} g^t) = \mathfrak{CS}(M, g^t)$ so their first variations in $t = 0$ must be equal. Therefore by Theorem 2, $\langle e^{2f} h, \mathfrak{Cott}(e^{2f} g)\rangle_{L^2(e^{2f}g)} = \langle h, \mathfrak{Cott}(g)\rangle_{L^2(g)}$, or equivalently

$$\langle e^f h, \mathfrak{Cott}(e^{2f} g)\rangle_{L^2(g)} = \langle h, \mathfrak{Cott}(g)\rangle_{L^2(g)}.$$

Since h was arbitrary, we deduce that $e^f \mathfrak{Cott}(e^{2f}g) = \mathfrak{Cott}(g)$. Using the obvious rescaling properties of the Hodge star operator under conformal transformations in dimension 3 acting on 1-forms, we deduce that the Cotton forms of g and $e^{2f}g$ are equal. □

These four propositions can of course be proved directly from the definitions, by local computations. We hope nevertheless that the reader will admit the qualitative advantage of proving properties of the Cotton tensor via Chern-Simons invariants. The difficulty of the proof was hidden in the definition and properties of the latter, but in exchange we gain a superior insight for the properties of the former.

9 Conformal immersions in $\mathbb{R}^4$

Theorem 3. *Let (M, g) be a closed oriented Riemannian 3-manifold and assume there exists a conformal immersion $\imath : M \to \mathbb{R}^4$. Then the Chern-Simons invariant $\mathfrak{CS}(M, g)$ vanishes.*

Proof. Since the Chern-Simons invariant is conformally invariant, by replacing g with $\imath^* g^{\mathbb{R}^4}$ we can assume that $\imath$ is an isometric immersion. Let $N : M \to S^3$ be the Gauss map of the immersion, i.e., $N(x)$ is the unit normal to $\imath_*(T_xM)$ chosen such that if (S_1, S_2, S_3) is an oriented frame in T_xM, then $(N(x), S_1, S_2, S_3)$ is positively oriented in $\mathbb{R}^4$. We identify $\mathbb{R}^4$ with the quaternion algebra as in Section 5. For every point $\eta \in S^3$, consider the orthonormal frame in $T_\eta S^3$

$$U_1 = i\eta, \qquad U_2 = j\eta, \qquad U_3 = k\eta.$$

Similarly, for every $x \in M$ consider the orthonormal frame in T_xM

$$S_1 = \imath_*^{-1}(iN(x)), \qquad S_2 = \imath_*^{-1}(jN(x)), \qquad S_3 = \imath_*^{-1}(kN(x)).$$

We claim that in these frames, the connection 1-forms on S^3 and on M are related by

$$N^*\omega^{S^3} = \omega^M. \tag{8}$$

This is a local statement so we can assume that M is a hypersurface in $\mathbb{R}^4$. Notice that for every $x \in M$, we have $S_i(x) = U_i(N(x))$, $i = 1, 2, 3$. Let $X \in T_xM$ be a vector tangent to a curve γ in M with $x = \gamma(0)$. Then using the definition of the Levi-Civita connection for hypersurfaces, we get

$$\langle \nabla^M_X S_1, S_2 \rangle = \langle \partial_t S_1(\gamma(t)), S_2 \rangle = \langle i\partial_t N(\gamma(t)), jN_x \rangle,$$
$$\langle \nabla^{S^3}_{N_*X} U_1, U_2 \rangle = \langle \partial_t U_1(N(\gamma(t))), U_2 \rangle = \langle i\partial_t N(\gamma(t)), jN_x \rangle$$

so $\omega^M_{21}(X) = \omega^{S^3}_{21}(N_*X)$. The same arguments for the other pairs of indices end the proof of (8). Then by definition, the Chern-Simons forms on M and S^3 are related by

$$N^*\mathfrak{cs}(S^3, U) = \mathfrak{cs}(M, S).$$

It follows that $\mathfrak{CS}(M, g) = \deg(N)\mathfrak{CS}(S^3)$ is an integer, by Corollary 1 (where $\deg(N) \in \mathbb{Z}$ is the topological degree of the Gauss map $N : M \to S^3$). □

In particular, it follows from this theorem and (5) that SO(3) cannot be conformally immersed in $\mathbb{R}^4$, although it is locally isometric to $S^3 \subset \mathbb{R}^4$.

10 Locally conformally flat metrics

We end our excursion into Cotton territory by solving the lcf (locally conformally flat) problem in dimension 3:

> *When does a Riemannian metric g on a 3-manifold M admit, for every $x \in M$, a conformal factor $f \in C^\infty(V)$ defined on some neighborhood $V \ni x$ such that $e^{2f}g$ is flat?*

From Proposition 5, one necessary condition for a positive answer is the vanishing of the Cotton tensor $\mathfrak{Cott} = *d^\nabla \mathfrak{Sch}$, where $\mathfrak{Sch}$ is the Schouten tensor. The following result is due to Émile Cotton [5].

Theorem 4. *A Riemannian metric g on a 3-manifold M is locally conformally flat if and only if its Cotton tensor $\mathfrak{Cott}(g)$ vanishes.*

The Cotton tensor and the Cotton form are obtained from one another through the Hodge star, so they vanish simultaneously. In dimension 3 the Schouten tensor determines completely the curvature tensor since for every vectors U, V, X we have (see [3, 1.119b])

$$R_{U,V}X = \langle X, V\rangle \mathfrak{Sch}(U) + \langle \mathfrak{Sch}(X), V\rangle U - \langle \mathfrak{Sch}(X), U\rangle V - \langle U, X\rangle \mathfrak{Sch}(V), \quad (9)$$

or in other words

$$RX = -X \wedge \mathfrak{Sch} - \mathfrak{Sch}(X) \wedge g. \quad (10)$$

Moreover, $\mathfrak{Sch}$ and $\mathfrak{Ric}$ determine one another, since $\mathrm{tr}(\mathfrak{Sch}) = \mathfrak{scal}(g)/4$. Hence g is conformally flat if and only if it is conformally Schouten-flat.

Proof. Take $g_1 = e^{2f}g$ for some $f \in C^\infty(M)$. We know by Proposition 5 that $\mathfrak{cott} = \mathfrak{cott}_1$. Thus, if $\mathfrak{cott}_1 = 0$ it follows that $\mathfrak{cott} = 0$.

Conversely, recall from [3, 1.159] the formula for the conformal change of the Schouten tensor: if we set $X := df$,

$$\mathfrak{Sch} - \mathfrak{Sch}_1 = \nabla X - X \otimes X + \tfrac{1}{2}|X|^2 g. \quad (11)$$

Suppose that we can solve locally the equation in the unknown X

$$\mathfrak{Sch} = \nabla X - X \otimes X + \tfrac{1}{2}|X|^2 g \quad (12)$$

for some vector field X (identified to a 1-form using g). Then the term ∇X is symmetric, hence X must be closed. From the Poincaré lemma, X is locally exact, thus there exists (locally) a function f with $X = df$. By combining (11) and (12), the metric $g_1 := e^{2f}g$ will be Schouten-flat. Thus, in order to finish the proof it is enough to show that the equation (12) is locally solvable under the assumption that $\mathfrak{cott} = 0$.

Rewrite (12) as the overdetermined system

$$\nabla X = \mathfrak{Sch} + X \otimes X - \tfrac{1}{2}|X|^2 g \tag{13}$$

and apply the twisted exterior differential d^∇ in both sides. We get

$$RX = \mathfrak{cott} + d^\nabla(X \otimes X - \tfrac{1}{2}|X|^2 g) \tag{14}$$

where R is the curvature tensor. Using (13), the fact that X is closed and g is parallel, we compute

$$\begin{aligned} d^\nabla(X \otimes X - \tfrac{1}{2}|X|^2 g) = & -X \wedge \nabla X - \langle \nabla X, X\rangle \wedge g \\ = & -X \wedge \mathfrak{Sch} + \tfrac{1}{2}|X|^2 X \wedge g \\ & - \mathfrak{Sch}(X) \wedge g - |X|^2 X \wedge g + \tfrac{1}{2}|X|^2 X \wedge g \end{aligned}$$

which, substituting in (14) and using the assumption $\mathfrak{cott} = 0$, reduces to the constraint

$$RX = -X \wedge \mathfrak{Sch} - \mathfrak{Sch}(X) \wedge g$$

already noted above (10). Thus the system (12) is *involutive*, so by the Frobenius theorem it is locally integrable. For completeness, let us prove this integrability by hand, without invoking the Frobenius theorem.

Choose local coordinates x_1, x_2, x_3 in M, and for $j = 1, 2, 3$ denote by $\partial_j = \frac{\partial}{\partial x_j}$ the coordinate vector fields. We fix $X(0)$, and extend X along the axis $\{x_2 = x_3 = 0\}$ in a neighborhood of the origin by solving the equation (13) in the direction of ∂_1:

$$\nabla_{\partial_1} X = \mathfrak{Sch}(\partial_1) + \langle X, \partial_1\rangle X - \tfrac{1}{2}|X|^2 \partial_1. \tag{15}$$

This is an ODE with smooth coefficients, hence the solution $X(t, 0, 0)$ exists for small time t and is uniquely determined by the initial value $X(0)$. Now for every t, extend X along the lines $\{x_1 = t, x_3 = 0\}$ using the ODE obtained from (13) in the direction of ∂_2:

$$\nabla_{\partial_2} X = \mathfrak{Sch}(\partial_2) + \langle X, \partial_2\rangle X - \tfrac{1}{2}|X|^2 \partial_2. \tag{16}$$

Again, the solution $X(t, s, 0)$ exists for small time s, depends smoothly on the parameter t and on the variable s, and is uniquely determined by the values of X in $(t, 0, 0)$. Finally, we extend X along the lines $\{x_1 = t, x_2 = s\}$ using (13) in the direction of ∂_3. This defines a smooth vector field X in a neighborhood of the origin, but we must still prove that (13) is satisfied.

By construction we know (15) only at points of the form $(t, 0, 0)$, and (16) only on the plane $x_3 = 0$. Let us prove that (15) holds on the plane $\{x_3 = 0\}$. For this, set

$$A := \nabla_{\partial_1} X - (\mathfrak{Sch}(\partial_1) + \langle X, \partial_1\rangle X - \tfrac{1}{2}|X|^2 \partial_1). \tag{17}$$

Since (15) is valid at $(t,0,0)$ we know that $A(t,0,0)=0$ for every t.

The fact that the system (12) is involutive should imply that A satisfies a linear system of ODE's in the direction of x_2. Explicitly we compute using (16) repeatedly:

$$\begin{aligned}\nabla_{\partial_2} A = {} & \nabla_{\partial_2}\nabla_{\partial_1} X - \nabla_{\partial_2}\mathfrak{Sch}(\partial_1) - \partial_2\langle X,\partial_1\rangle X - \langle X,\partial_1\rangle\nabla_{\partial_2} X \\ & + \langle \nabla_{\partial_2} X, X\rangle\partial_1 + \tfrac{1}{2}|X|^2\nabla_{\partial_2}\partial_1.\end{aligned}$$

Now use $\nabla_{\partial_2}\nabla_{\partial_1} X = R_{\partial_2\partial_1} X + \nabla_{\partial_1}\nabla_{\partial_2} X$ and $d^\nabla\mathfrak{Sch} = \mathfrak{cott} = 0$. We get from (16)

$$\begin{aligned}\nabla_{\partial_2} A &= R_{\partial_2\partial_1} X + \nabla_{\partial_1}\left(\mathfrak{Sch}(\partial_2) + \langle X,\partial_2\rangle X - \tfrac{1}{2}|X|^2\partial_2\right) - \nabla_{\partial_2}\mathfrak{Sch}(\partial_1) \\ &\quad - \partial_2\langle X,\partial_1\rangle X - \langle X,\partial_1\rangle\nabla_{\partial_2} X + \langle\nabla_{\partial_2} X, X\rangle\partial_1 + \tfrac{1}{2}|X|^2\nabla_{\partial_2}\partial_1 \\ &= R_{\partial_2\partial_1} X + dX(\partial_1,\partial_2)X + \langle X,\partial_2\rangle\nabla_{\partial_1} X - \langle X,\partial_1\rangle\nabla_{\partial_2} X \\ &\quad - \langle\nabla_{\partial_1} X, X\rangle\partial_2 + \langle\nabla_{\partial_2} X, X\rangle\partial_1.\end{aligned}$$

Substitute $\nabla_{\partial_2} X$ and $\nabla_{\partial_1} X$ in the equation above, using (16) and (17):

$$\begin{aligned}\nabla_{\partial_2} A &= R_{\partial_2\partial_1} X + dX(\partial_1,\partial_2)X + \langle X,\partial_2\rangle A - \langle A, X\rangle\partial_2 \\ &\quad + \langle X,\partial_2\rangle\mathfrak{Sch}(\partial_1) - \langle X,\partial_1\rangle\mathfrak{Sch}(\partial_2) \\ &\quad - \langle\mathfrak{Sch}(\partial_1), X\rangle\partial_2 + \langle\mathfrak{Sch}(\partial_2), X\rangle\partial_1 \\ &= dX(\partial_1,\partial_2)X + \langle X,\partial_2\rangle A - \langle A, X\rangle\partial_2\end{aligned}$$

where in the last equality we have used (9). From (17), (16) and the symmetry of the Schouten tensor,

$$dX(\partial_1,\partial_2) = \langle\nabla_{\partial_1} X,\partial_2\rangle - \langle\nabla_{\partial_2} X,\partial_1\rangle = \langle A,\partial_2\rangle.$$

Hence we get

$$\nabla_{\partial_2} A = \langle A,\partial_2\rangle + \langle X,\partial_2\rangle A - \langle A, X\rangle\partial_2 = L(A) \tag{18}$$

where L is an endomorphism of TM. Therefore the vector field A is a solution of the linear ODE (18) in the variable x_2, with zero initial values, hence it vanishes identically.

By precisely the same argument applied to the pairs of variables $\{x_1,x_3\}$ and $\{x_2,x_3\}$, we see that (15) and (16) continue to hold at every point (t,s,x_3). This means that X is a solution to (13) as claimed. □

From the proof, it appears that X is closed, and uniquely determined by the initial value $X(0)$. Therefore the conformal factor f, the primitive of X, is uniquely determined by four parameters, its value and its differential at 0. These 4 degrees of freedom arise from the fact that locally on $\mathbb{R}^3$, the Lie algebra of conformal Killing vector fields has dimension 10, while the subalgebra of Killing vector fields has dimension 6.

11 Links with other invariants

We survey below two beautiful mathematical objects related to the Chern-Simons invariant. This section is of course no longer self-contained.

11.1 The eta invariant

The Chern-Simons invariant is strongly related to the *eta invariant* of the odd signature operator on M. The eta invariant is a real-valued invariant of closed oriented 3-manifolds, initially introduced by Atiyah, Patodi and Singer [2] as a correction term in Hirzebrich's signature formula on a 4-manifold with boundary. It can be defined for every elliptic self-adjoint differential operator A acting on the sections of some vector bundle E, but its construction is non-elementary: one needs to understand the *spectrum* of A, which is the discrete subset of $\mathbb{R}$ of eigenvalues (with multiplicity) of A viewed as a self-adjoint unbounded operator in $L^2(M,E)$. The non-zero part of the spectrum is denoted $\mathrm{Spec}(A)^*$.

The eta function, a meromorphic function in the variable $z \in \mathbb{C}$, is defined for $\Re(z) > 3$ by the absolutely convergent series

$$\eta(A;z) = \dim\ker(A) + \sum_{\lambda\in\mathrm{Spec}(A)^*} \mathrm{sign}(\lambda)|\lambda|^{-z}.$$

The eigenvalues of A grow sufficiently fast to ensure absolute convergence in a half-plane, for instance if A is of order 1, then the series defining $\eta(A,z)$ is absolutely convergent for $\Re(z) > 3$. The function thus obtained extends to $\mathbb{C}$ with possible simple poles in $z \in \{2,-2,-4,-6,\dots\}$, in particular $z = 0$ is a regular point, and $\eta(A)$ is by definition that regular value.

When A is the self-adjoint odd signature operator acting on $\Lambda^1(M) \oplus \Lambda^3(M)$,

$$A := *d - d*$$

(here $*$ is the Hodge star defined in (3)), the resulting eta invariant is denoted $\eta(M,g)$ to highlight its dependence on the metric.

Theorem 5 ([2])**.** *Modulo $\mathbb{Z}$, the following equality holds:*

$$3\eta(M,g) \equiv 2\mathfrak{CS}(M,g) \mod \mathbb{Z}.$$

The proof relies on the signature formula of Atiyah, Patodi and Singer [1] on an oriented 4-manifold X with boundary M:

$$\mathrm{signature}(X) = -\tfrac{1}{24\pi^2}\int_X \mathrm{tr}((R^X)^2) - \eta(M,g).$$

Here $\mathrm{signature}(X) \in \mathbb{Z}$ is the signature of the intersection form on the relative cohomology $H^2(X,M;\mathbb{R})$, defined as the difference of the dimensions of maximal subspaces in $H^2(X,M;\mathbb{R})$ along which the intersection form is positive, respectively negative definite. The metric g^X on X is of product type near the boundary, in the sense that $L_\nu g^X = 0$ for ν the geodesic vector field with respect to g^X orthogonal to M (here L_ν denotes Lie derivative). Moreover, g^X restricts to g on M.

We only care about the right-hand side of the signature formula modulo integers:

$$\eta(M,g) + \tfrac{1}{24\pi^2}\int_X \mathrm{tr}((R^X)^2) \in \mathbb{Z}. \tag{19}$$

Recall that for every orthonormal frame S on M, we have by definition

$$\mathfrak{CS}(M,g) + \tfrac{1}{16\pi^2}\int_M \mathfrak{cs}(M,g,S) \in \mathbb{Z}. \tag{20}$$

To give the idea of the proof of Theorem 5, suppose that we can find X a compact oriented four-manifold bounded by M (this is always possible by a result of Thom [10] about the oriented cobordism ring). Extend g to a metric g^X on X, and suppose that S completed with the inner unit normal vector field can be extended to a frame S^X on X (this is *not* always possible, for topological reasons). Nevertheless, whenever these assumptions are fulfilled, write using Stokes' formula and Lemma 2

$$\int_M \mathfrak{cs}(M,g,S) = \int_X \mathrm{tr}((R^X)^2).$$

This equality holds for instance when X is a cylinder, with diffeomorphic boundary components (M,g), (M',g') with opposite orientations. Keeping g' fixed and using (19), (20), we see that $\frac{3}{2}\eta(M,g) - \mathfrak{CS}(M,g)$ is constant (modulo $\mathbb{Z}$) on the space of Riemannian metrics on M. That constant is shown in [2] to be either 0 or $\frac{1}{2}$, according to the parity of signature(X) if we choose the filling manifold X to be Spin (using the vanishing of the Spin cobordism group in dimension 3).

11.2 The Selberg zeta function

Assume (M,g) is hyperbolic, i.e., its sectional curvatures are constant and equal to -1. As above, M is supposed to be compact and orientable. Then M is isometric to a quotient $\Gamma\backslash\mathbb{H}^3$, where $\mathbb{H}^3$ is the hyperbolic 3-space, and $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$ is a discrete subgroup of oriented isometries (i.e., a Kleinian group) consisting only of loxodromic elements: every non-trivial element of Γ is conjugated in $\mathrm{PSL}(2,\mathbb{C})$ to a matrix of the form $\begin{bmatrix} v_\gamma & 0 \\ 0 & v_\gamma^{-1}\end{bmatrix}$ with $|v_\gamma| < 1$, where of course v_γ and its inverse are the eigenvalues of the matrix γ. The complex number $q_\gamma := v_\gamma^2$ is called the *multiplier* of γ, corresponding to the fact that the action of γ on the Riemann sphere (the ideal boundary of $\mathbb{H}^3$) is conjugated to the multiplication by q_γ.

Closed geodesics in M are in one-to-one correspondence with (non-trivial) conjugacy classes in $\Gamma \simeq \pi_1(M)$. A geodesic c_γ coresponding to a conjugacy class $[\gamma]$ determines thus the multiplier $q_\gamma = e^{-(l_\gamma + i\theta_\gamma)}$. Viewed geometrically, l_γ is the length of c_γ, while $e^{i\theta_\gamma}$ is the holonomy along $c_{[\gamma]}$. Both quantities are expressible in terms of the trace $\mathrm{tr}(\gamma) = v_\gamma + v_\gamma^{-1} \in \mathbb{C}$.

The *Selberg zeta function of odd type* was defined by Millson [7] as an infinite product over the set $\mathcal{P}$ of primitive conjugacy classes of Γ (an element in Γ is said to be primitive if it is not a nontrivial power of another element). The definition is a particular case of a construction from Selberg's foundational paper [8]:

$$\mathcal{Z}_\Gamma(\lambda) = \prod_{[\gamma]\in\mathcal{P}} \prod_{m,n=0}^{\infty} \frac{1 - q_\gamma^m (\overline{q}_\gamma)^{n+1} e^{-\lambda l_\gamma}}{1 - q_\gamma^{m+1}(\overline{q}_\gamma)^n e^{-\lambda l_\gamma}}.$$

The product is absolutely convergent in the half-plane $\{\Re(\lambda) > 0\}$, and has a meromorphic extension to the whole complex plane. Like the Riemann zeta function, the Selberg zeta function displays a symmetry around 0 (moreover, it is known to have zeros only on the imaginary axis). The central value $\mathcal{Z}_\Gamma(0)$ can be interpreted heuristically as the divergent product

$$\mathcal{Z}_\Gamma(0) = \prod_{[\gamma]\in\mathcal{P}} \prod_{n\geq 1} \frac{1-(\overline{q}_\gamma)^n}{1-q_\gamma^n}.$$

Theorem 6 (Millson)**.** *The central value of the Selberg zeta function of odd type on a 3-dimensional hyperbolic manifold $M = \Gamma\backslash\mathbb{H}^3$ is related to the eta invariant by the identity*

$$\exp(i\pi\eta(M)) = \mathcal{Z}_\Gamma(0).$$

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Lower bound on product of binomial coefficients

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Abstract. We give a lower bound on a product of binomial coefficients, connected with primality proving or construction of high multiplicative order elements in finite fields.

Mathematics subject classification: 05E40, 11T30.
Keywords and phrases: Product of binomial coefficients, primality proving, certificate, finite field, multiplicative order, lower bound.

1 Introduction and Preliminaries

Let m be a positive integer. The problem of finding such integers d_-, d $(0 \le d_- \le d < m)$ that the product of binomial coefficients

$$C(d_-, d) = \binom{m}{d_-}\binom{d}{d_-}\binom{2m - d_- - d - 1}{m - d - 1} \qquad (1)$$

is large, appears in the following two cases.

1. AKS primality proving algorithm optimization.

Efficient primality tests (determining whether a given number is prime or composite) are needed in applications: a number of cryptographic protocols use big prime numbers.

In 2002 M. Agrawal, N. Kayal and N. Saxena [1] presented a deterministic polynomial time algorithm AKS that determines whether an input number n is prime or composite. It was proved [3] that AKS algorithm runs in $(\log n)^{7.5+o(1)}$ time. Significantly modified versions of AKS [3, 4] are also known with $(\log n)^{4+o(1)}$ running time. The algorithm in [3] uses a notion of certificate for an integer n. It is proved that if we have found the certificate for an integer, then this integer is a power of a prime. Then it is easy to decide if the integer is prime. During the certificate finding an essential point is to verify an inequality, for which it is necessary to calculate an expression of the form (1). We choose numbers d_-, d to construct the certificate.

2. Construction of high multiplicative order elements for finite field extensions.

The problem of constructing efficiently a primitive element for a given finite field is notoriously difficult in the computational theory of finite fields. That is why one considers less restrictive question: to find an element with high multiplicative order [8, 9]. It is sufficient in this case to obtain a lower bound on the order. High order elements are needed in several applications. Such applications include but are not limited to cryptography, coding theory, pseudorandom number generation and combinatorics. The problem is considered both for general and special finite fields.

General extensions are considered in [7, 13]. For special finite fields, it is possible to construct elements which can be proved to have much higher orders. Extensions connected with a notion of Gauss period are considered in [2, 10]. Extensions based on Kummer polynomials are considered in [5, 6, 11].

F_q denotes finite field with q elements. According to [6, Lemma 2.1] we have the following lemma for extensions on a base of Kummer polynomials.

Lemma 1. *Let q be a prime power. Let m be a positive divisor of $q-1$. Let $x^m - a$ ($a \in F_q^*$) be an irreducible polynomial in F_q and θ be one of its roots in the extension $F_{q^m} = F_q[x]/(x^m - a)$. Then for any $b \in F_q$ the element $\theta + b$ has the multiplicative order at least $D = \max\limits_{0 \le d_- \le d < m} C(d_-, d)$.*

One can see that the product of the form (1) is present in Lemma 1. Therefore, the problem of product (1) maximization is important. It is shown in [3] that approximately for $d \approx m/2$ and $d_- \approx 0,2928m$ we have $C(d_-, d) \approx 5,8284^m$. Note that the value $5,8284^m$ is not a lower bound on D, but only some approximate value. Indeed, consider the following numerical examples.

For $m = 37$, maximum D is achieved at $d_- = 10, d = 17$ and is equal to $D = C(10, 17) \approx 2,81 \cdot 10^{25}$. We do not compute precise integer value of D, because we only need to compare it with $(5,8284)^{37} = 2,12 \cdot 10^{28}$. For $m = 511$, we have $D = C(149, 254) \approx 4,17 \cdot 10^{386}$. At the same time, $(5,8284)^{511} = 1,57 \cdot 10^{391}$.

From the point of view of applications (in particular, cryptography) an exact theoretical lower bound on D is desired. We give in the paper such explicit lower bound on maximum of the product (1) of binomial coefficients. In particular, bounds on binomial coefficients from [12] are used. The following inequality for binomial coefficients has been obtained in [12, Theorem 2.8, inequality (2.12)].

Lemma 2. *If r, s, t are integers with the conditions $s > r \ge 1$ and $t \ge 2$, then*

$$\binom{st}{rt} > (1/\sqrt{2\pi}) \cdot e^{r-1/(8t)} \cdot t^{-1/2} \frac{s^{s(t-1)+1}}{(s-r)^{(s-r)(t-1)-r+1} \cdot r^{rt+1/2}}. \tag{2}$$

For $r = 1$ we have the following corollary from inequality (2) [12, Corollary 2.9, inequality (2.13)].

Corollary 1. *For $s > 1$ and $t \ge 2$ the following inequality holds:*

$$\binom{st}{t} > (1/\sqrt{2\pi}) \cdot e^{1-1/(8t)} \cdot t^{-1/2} \frac{s^{s(t-1)+1}}{(s-1)^{(s-1)(t-1)}}. \tag{3}$$

Lemma 3. *The following equalities are true for binomial coefficients:*

$$\binom{u}{v} = \frac{u}{u-v} \binom{u-1}{v}, \tag{4}$$

$$\binom{u}{v} = \frac{u}{v} \binom{u-1}{v-1}. \tag{5}$$

Proof. To prove (4) note that $\binom{u}{v} = \frac{(u-v+1)\cdot\ldots\cdot u}{1\cdot 2\cdot\ldots\cdot v}$ and $\binom{u-1}{v} = \frac{(u-v)\cdot\ldots\cdot(u-1)}{1\cdot 2\cdot\ldots\cdot v}$. The observation that $\binom{u-1}{v-1} = \frac{(u-v+1)\cdot\ldots\cdot(u-1)}{1\cdot 2\cdot\ldots\cdot(v-1)}$ allows to prove (5). □

2 Main result

We give below in Theorem a lower bound on the maximum D of the product (1) of binomial coefficients. The proof of the theorem uses inequalities (2) and (3) from respectively Lemma 2 and Corollary 1.

Theorem 1. *Put $h = 4 \cdot 5^{5/4}/3^{3/2}$. For $m \geq 8$ the following lower bound holds:*

$$D > \frac{h^m}{30m^{3/2}}. \tag{6}$$

Proof. Take $k = m \ mod \ 4$, $d_- = (m-k)/4$, $d = (m-k)/2$. Show first that for $k \in \{0, 1, 2, 3\}$

$$\binom{m}{d_-} = \binom{m}{(m-k)/4} > \beta(k)\binom{m-k}{(m-k)/4}, \tag{7}$$

where $\beta(0) = 1$, $\beta(1) = \frac{32}{25}$, $\beta(2) = \frac{16\cdot 14}{13\cdot 11}$, $\beta(3) = \frac{32\cdot\ 28\cdot 24}{27\cdot 23\cdot 19}$.

$\beta(0) = 1$ is clear. For $k = 1$, apply (4) to the left side of (7):

$$\binom{m}{(m-1)/4} = \frac{4m}{3m+1}\binom{m-1}{(m-1)/4}.$$

Since, for $m \geq 8$, the inequality $\frac{4m}{3m+1} \geq \frac{32}{25}$ holds, we have the above-mentioned $\beta(1)$. For $k = 2$, apply (4) subsequently 2 times:

$$\binom{m}{(m-2)/4} = \frac{4m}{3m+2}\binom{m-1}{(m-2)/4}, \binom{m-1}{(m-2)/4} = \frac{4(m-1)}{3(m-1)+1}\binom{m-2}{(m-2)/4}.$$

As, for $m \geq 8$, the conditions $\frac{4m}{3m+2} \geq \frac{16}{13}$, $\frac{4(m-1)}{3(m-1)+1} \geq 14/11$ are true, we have the foresaid $\beta(2)$. For $k = 3$, apply (4) subsequently 3 times:

$$\binom{m}{(m-3)/4} = \frac{4m}{3m+3}\binom{m-1}{(m-3)/4}, \binom{m-1}{(m-3)/4} = \frac{4(m-1)}{3(m-1)+2}\binom{m-2}{(m-3)/4},$$

$$\binom{m-2}{(m-3)/4} = \frac{4(m-2)}{3(m-2)+1}\binom{m-3}{(m-3)/4}.$$

Since, for $m \geq 8$, the inequalities $\frac{4m}{3m+3} \geq \frac{32}{27}$, $\frac{4(m-1)}{3(m-1)+2} \geq \frac{28}{23}$ and $\frac{4(m-2)}{3(m-2)+1} \geq \frac{24}{19}$ hold, we obtain the aforementioned $\beta(3)$.

Show now that

$$\binom{2m-d_- -d-1}{m-d-1} = \binom{2m-3(m-k)/4-1}{m-2(m-k)/4-1} > \delta(k)\binom{5(m-k)/4}{2(m-k)/4}, \tag{8}$$

where $\delta(0) = \frac{1}{3}$, $\delta(1) = \frac{39}{25}$, $\delta(2) = \frac{21\cdot 19\cdot 17}{8\cdot 13\cdot 11}$, $\delta(3) = \frac{5\cdot 41\cdot 37\cdot 33\cdot 29}{2\cdot 14\cdot 27\cdot 23\cdot 19}$.

For $k = 0$, apply (5) to the left side of (8):

$$\binom{5m/4-1}{2m/4-1} = \frac{2m/4-1}{5m/4-1}\binom{5m/4}{2m/4}.$$

As, for $m \geq 8$, the condition $\frac{2m/4-1}{5m/4-1} \geq \frac{3}{9}$ is true, we have $\delta(0) = \frac{1}{3}$. For $k = 1$, apply the equality (4):

$$\binom{5(m-1)/4+1}{2(m-1)/4} = \frac{5(m-1)/4+1}{3(m-1)/4+1}\binom{5(m-1)/4}{2(m-1)/4}.$$

Since, for $m \geq 8$, the inequality $\frac{5(m-1)/4+1}{3(m-1)/4+1} \geq \frac{39}{25}$ holds, we have the aforesaid $\delta(1)$. For $k = 2$, first apply (5):

$$\binom{5(m-2)/4+3}{2(m-2)/4+1} = \frac{5(m-2)/4+3}{2(m-2)/4+1}\binom{5(m-2)/4+2}{2(m-2)/4}.$$

Then apply (4) subsequently 2 times:

$$\binom{5(m-2)/4+2}{2(m-2)/4} = \frac{5(m-2)/4+2}{3(m-2)/4+2}\cdot\frac{5(m-2)/4+1}{3(m-2)/4+1}\binom{5(m-2)/4}{2(m-2)/4}.$$

For $m \geq 8$, since $\frac{5(m-2)/4+3}{2(m-2)/4+1} \geq \frac{21}{8}$, $\frac{5(m-2)/4+2}{3(m-2)/4+2} \geq \frac{19}{13}$ and $\frac{5(m-2)/4+1}{3(m-2)/4+1} \geq \frac{17}{11}$ are true, we obtain the foregoing $\delta(2)$. For $k = 3$, first apply (5) 2 times:

$$\binom{5(m-3)/4+5}{2(m-3)/4+2} = \frac{5(m-3)/4+5}{2(m-3)/4+2}\cdot\frac{5(m-3)/4+4}{2(m-3)/4+1}\binom{5(m-3)/4+3}{2(m-3)/4}.$$

Then apply (4) subsequently 3 times:

$$\binom{5(m-3)/4+3}{2(m-3)/4} = \frac{5(m-3)/4+3}{3(m-3)/4+3}\cdot\frac{5(m-3)/4+2}{3(m-3)/4+2}\cdot\frac{5(m-3)/4+1}{3(m-3)/4+1}\binom{5(m-3)/4}{2(m-3)/4}.$$

For $m \geq 8$, since $\frac{5(m-3)/4+5}{2(m-3)/4+2} \geq \frac{5}{2}$, $\frac{5(m-3)/4+4}{2(m-3)/4+1} \geq \frac{41}{14}$, $\frac{5(m-3)/4+3}{3(m-3)/4+3} \geq \frac{37}{27}$, $\frac{5(m-3)/4+2}{3(m-3)/4+2} \geq \frac{33}{23}$ and $\frac{5(m-3)/4+1}{3(m-3)/4+1} \geq \frac{29}{19}$ hold, we have the forementioned $\delta(3)$.

Combining (7) and (8), we obtain that for $k \in \{0, 1, 2, 3\}$ and $n = m - k$ the following inequality holds

$$D > \beta(k)\delta(k)\binom{n}{n/4}\binom{n/2}{n/4}\binom{5n/4}{2n/4}. \tag{9}$$

Now we give, using inequalities (2) or (3), lower bounds on each binomial coefficient on the right side of (9). Applying the inequality (2) to the first coefficient on the right side of (9) (in this case $t = n/4$, $s = 4$), we have:

$$\binom{n}{n/4} > (1/\sqrt{2\pi}) \cdot e^{1-1/(2n)}(n/4)^{-1/2}\frac{4^{n-3}}{3^{3n/4-3}}. \tag{10}$$

Note, that $t \geq 2$ must hold, that is $m-k \geq 8$, and if $k=0$, then $m \geq 8$. Applying the inequality (2) to the second coefficient (in this case $t = n/4$, $s = 2$), we have:

$$\binom{n/2}{n/4} > (1/\sqrt{2\pi}) \cdot e^{1-1/(2n)}(n/4)^{-1/2}2^{n/2-1}. \tag{11}$$

Applying the inequality (3) to the third coefficient (in this case $t = n/4$, $s = 5$, $r = 2$), we have:

$$\binom{5n/4}{2n/4} > \frac{1}{\sqrt{2\pi}} \cdot e^{2-1/(2n)}(n/4)^{-1/2}\frac{5^{5n/4-4}}{2^{n/2-4} \cdot 3^{3n/4+1/2}}. \tag{12}$$

Substituting the inequalities (10), (11), (12) in the inequality (9), and taking into account that $\frac{1}{e^{3/(2(m-k))}} \geq \frac{1}{e^{3/(2(m-3))}}$, $1 < e^{3/(2(m-3))} < 1,35$ for $m \geq 8$, $\frac{1}{(m-k)^{3/2}} \geq \frac{1}{m^{3/2}}$, we obtain the bound

$$D > \frac{3^7 \cdot e^4}{10^5 \cdot \pi^{3/2} \cdot 1,35} \cdot \frac{\beta(k)\delta(k)}{h^k} \cdot \frac{h^m}{m^{3/2}}. \tag{13}$$

Since $\frac{3^7 \cdot e^4}{10^5 \cdot \pi^{3/2} \cdot 1,35} > 0,1588$, minimal value for $\frac{\beta(k)\delta(k)}{h^k}$ is at $k = 3$ and equals to 0,21, the last inequality is transformed into the bound (6). □

Obtained lower bound (6) on the product (1) of binomial coefficients is exact theoretical bound and comparable with the corresponding value from [3]. Taking into account in (6) that $5,7556 < 4 \cdot 5^{5/4}/3^{3/2} < 5,7557$, we have the following corollary.

Corollary 2. *For $m \geq 8$ the following inequality holds: $D > \frac{5,7556^m}{30m^{3/2}}$.*

Clearly for big enough m the main contribution on the right side of the last inequality is given by the term $(5,7556)^m$.

Remark that our result is a lower bound on D for $m \geq 8$ with constant 5,7556. If allow m to be bigger, say $m \geq 32$, then one obtains similar lower bound with constant 5,8230. To achieve this, choose in the proof of the theorem $d_- = m/4 + m/32$, $d = m/2$. For $m \geq 1024$, taking $d_- = m/4 + m/32 + m/128 + m/512 + m/1024$, $d = m/2$, one can obtain a bound with constant 5,8284.

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On 2-absorbing Primary Subsemimodules over Commutative Semirings

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Abstract. In this paper, we define 2-absorbing primary subsemimodules of a semimodule M over a commutative semiring S with $1 \neq 0$ which is a generalization of primary subsemimodules of semimodules. A proper subsemimodule N of a semimodule M is said to be a 2-absorbing primary subsemimodule of M if $abm \in N$ implies $ab \in \sqrt{(N : M)}$ or $am \in N$ or $bm \in N$ for some $a, b \in S$ and $m \in M$. It is proved that if K is a subtractive subsemimodule of M and $\sqrt{(K : M)}$ is a subtractive ideal of S, then K is a 2-absorbing primary subsemimodule of M if and only if whenever $IJN \subseteq K$ for some ideals I, J of S and a subsemimodule N of M, then $IJ \subseteq \sqrt{(K : M)}$ or $IN \subseteq K$ or $JN \subseteq K$. In this paper, we prove a number of results concerning 2-absorbing primary subsemimodules.

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1 Introduction

The notion of a semiring was first introduced by H. S. Vandiver in 1934 [16]. After that various research have been done in this area and several applications have been found in various branches of mathematics and computer science. The concepts of prime and primary ideals are essential ingredients in ideal theory and algebraic geometry. Prime subsemimodule and primary subsemimodules have been used in soft mathematics and studied by many authors (for example see [3], [4], [7], [8] and [10]) during the last decade. The concept of 2-absorbing subsemimodule which is a generalization of a prime subsemimodule was studied in [13]. In this paper, we introduce the concept of 2-absorbing primary subsemimodule which is a generalization of the primary subsemimodule. Throughout the paper, a semiring S will be considered as commutative with identity $1 \neq 0$ and a left S-semimodule means a unitary semimodule.

A commutative semiring is a commutative semigroup $(S, \cdot)$ and a commutative monoid $(S, +, 0_S)$ in which 0_S is the additive identity and $0_S \cdot x = x \cdot 0_S = 0_S$ for all $x \in S$, both are connected by ring like distributivity. A non-empty subset I of a semiring S is called an ideal of S if whenever $a, b \in I$ and $s \in S$, then $a + b \in I$ and $sa, as \in I$. An ideal I of S is said to be proper if $I \neq S$. A left S-semimodule M is a commutative monoid $(M, +)$ which has a zero element 0_M, together with an operation $S \times M \to M$, denoted by $(a, x) \to ax$ such that for all $a, b \in S$ and

$x, y \in M$,
(i) $a(x+y) = ax + ay$,
(ii) $(a+b)x = ax + bx$,
(iii) $(ab)x = a(bx)$,
(iv) $0_S \cdot x = 0_M = a \cdot 0_M$.

A non-empty subset N of an S-semimodule M is a subsemimodule of M if N is closed under addition and scalar multiplication. A proper subsemimodule N of an S-semimodule M is called subtractive if whenever $a, a+b \in N$, $b \in M$ then $b \in N$. Let N be a subsemimodule of M. Then, an associated ideal of N is defined as $(N :_S M)$ or simply $(N : M)$ denote the ideal $\{s \in S : sM \subseteq N\}$ and $(N : m) = \{a \in S : am \in N \text{and} \, m \in M\}$. Recall ([3], [4], [9], [10], [13], [15]) the following: A non-zero proper ideal I of S is said to be a 2-absorbing ideal of S if whenever $abc \in I$ for any $a, b, c \in S$, then $ab \in I$ or $ac \in I$ or $bc \in I$. A proper ideal I of S is said to be a 2-absorbing primary ideal of S if whenever $a, b, c \in S$ with $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$, where $\sqrt{I} = \{s \in S$: there exists $n \in N$ with $s^n \in I\}$ denotes the radical of I. A proper subsemimodule N of an S-semimodule M is said to be a prime subsemimodule if for $a \in S$, $m \in M$, and $am \in N$, either $m \in N$ or $a \in (N :_S M)$. A proper subsemimodule N of M is said to be a 2-absorbing subsemimodule of M if whenever $a, b \in S$, $m \in M$ with $abm \in N$, then $ab \in (N :_S M)$ or $am \in N$ or $bm \in N$. A proper subsemimodule N of M is said to be a primary subsemimodule of M if whenever $a \in S$, $m \in M$ and $am \in N$, then $a \in \sqrt{(N : M)}$ or $m \in N$, where $\sqrt{(N : M)} = \{a \in S : a^n M \subseteq N \, \text{for some} \, n \geq 1\}$. A proper subsemimodule N of an S-semimodule M is said to be a strong subsemimodule if for each $x \in N$ there exists $y \in N$ such that $x + y = 0$.

2 2-absorbing primary subsemimodules

In this section, we introduce the concept of 2-absorbing primary subsemimodule of a semimodule M over a commutative semiring S and investigate some properties and results.

Definition 1. Let M be a semimodule over a commutative semiring S and N be a proper subsemimodules of M. Then N is said to be a 2-absorbing primary subsemimodule of M if whenever $abm \in N$ where $a, b \in S$ and $m \in M$, then $ab \in \sqrt{(N : M)}$ or $am \in N$ or $bm \in N$.

It is easy to see that every 2-absorbing subsemimodule of a semimodule M over a commutative semiring S is a 2-absorbing primary subsemimodule of M but converse need not be true. For instance, consider a Z_0^+-semimodule $M = Z_{16} = \{0, 1, 2, ..., 15\}$. Take a subsemimodule $N = \{0, 8\}$, generated by 8. Then $(N : M) = \{a \in S : aM \subseteq N\} = \{0, 8, 16, ...\}$ and $\sqrt{(N : M)} = \{a \in S : a^n \in (N : M)\} = \{0, 2, 4, 8....\}$. Now, $2.2.2 \in N$ but $2.2 \notin N$ and $2.2 \notin (N : M)$. Therefore, N is not a 2-absorbing subsemimodule of M but it is a 2-absorbing primary subsemimodule of

M, as $2.2 \in \sqrt{(N:M)}$. Also, every primary subsemimodule of M is a 2-absorbing primary subsemimodule but converse need not be true. For example, let S be $Z^* = Z_0^+$ and let $M = Z^* \times Z^*$ be a semimodule over S. If we take the subsemimodule $N = \{0\} \times 4Z^*$ of M, then $(N : M)$=$\{0\}$ and $\sqrt{(N:M)} = \{0\}$. Here, N is a 2-absorbing primary subsemimodule of M but N is not a primary subsemimodule of M. Because $2 \cdot (0, 2) \in N$ but $2 \notin \sqrt{(N:M)}$ and $(0, 2) \notin N$.

Result 1. *Let M be a semimodule and N be a proper subtractive subsemimodule of M and let $m \in M$. Then the following holds:*

(i) $(N : M)$ *is a subtractive ideal of S.*

(ii) $(0 : M)$ *and* $(N : m)$ *are subtractive ideals of S, where* $(0 : M)$=$\{a \in S : aM \subseteq \{0\}\}$.

Proof. Proof is straightforward. □

Theorem 1. *Let N be a subtractive 2-absorbing primary subsemimodule of a semimodule M. Then, $(N : M)$ is a 2-absorbing primary ideal of S.*

Proof. Let $abc \in (N : M)$ for some $a, b, c \in S$. Let $ab \notin (N : M)$ and $bc \notin \sqrt{(N:M)}$. This implies $ab \notin (N : M)$ and $bc \notin (N : M)$. Therefore, there exists $x, y \in M$ such that $abx \notin N$ and $bcy \notin N$ but $ac(bx + by) \in N$. Since N is a 2-absorbing primary subsemimodule of M, we have either $ac \in \sqrt{(N:M)}$ or $a(bx + by) \in N$ or $c(bx + by) \in N$. If $ac \in \sqrt{(N:M)}$, then there is nothing to prove. If $a(bx+by) \in N$, then $aby \notin N$ (as N is a subtractive). Consider $abcy \in N$. Since N is a 2-absorbing primary subsemimodule and $aby \notin N$, $bcy \notin N$, we have $ac \in \sqrt{(N:M)}$. Similarly, if $c(bx + by) \in N$, then we have $cbx \notin N$. Consider $abcx \in N$. Since N is a 2-absorbing primary subsemimodule and $abx \notin N$, $bcx \notin N$, we have $ac \in \sqrt{(N:M)}$. This implies that $(N : M)$ is a 2-absorbing primary ideal of S. □

Theorem 2. *Let N be a 2-absorbing primary subsemimodule of an S-semimodule M. Then $\sqrt{(N:M)}$ is a 2-absorbing ideal of S.*

Proof. Let N be a 2-absorbing primary subsemimodule of an S-semimodule M. Then by Theorem 1, we have $(N : M)$ is a 2-absorbing primary ideal of S. By [Theorem 2, [15]], we have $\sqrt{(N:M)}$ is a 2-absorbing ideal of S. □

Theorem 3. *Let N be a 2-absorbing primary subsemimodule of a semimodule M such that $\sqrt{(N:M)} = P$ for some prime ideal P of S. Then for some $m \in M \setminus N$, $\sqrt{(N:m)}$ is a prime ideal of S.*

Proof. Let N be a 2-absorbing primary subsemimodule of M. Then by Theorem 2, $\sqrt{(N:M)}$ is a 2-absorbing ideal of S. Let $a, b \in S$ be such that $ab \in \sqrt{(N:m)}$, where $m \in M \setminus N$. Therefore, $(ab)^n \in (N : m)$, that is, $a^n b^n m \in N$ for some positive integer n. This gives, either $a^n m \in N$ or $b^n m \in N$ or $a^n b^n \in \sqrt{(N:M)}$ since N

is a 2-absorbing primary subsemimodule of M. If $a^n m \in N$ or $b^n m \in N$, that is, $a^n \in (N : m)$ or $b^n \in (N : m)$, then $\sqrt{(N : m)}$ is prime. If $a^n b^n \in \sqrt{(N : M)}$, we have $(a^n b^n)^m \in (N : M)$ for some positive integer m. Thus, $ab \in \sqrt{(N : M)} = P$. Therefore, either $a \in P$ or $b \in P$ since P is prime. Hence $a \in \sqrt{(N : M)} \subseteq \sqrt{(N : m)}$ or $b \in \sqrt{(N : M)} \subseteq \sqrt{(N : m)}$. Consequently, $\sqrt{(N : m)}$ is a prime ideal of S. □

Theorem 4. *Let $f : M \mapsto M'$ be a homomorphism of a S-semimodules M and M'. If N is a 2-absorbing primary subsemimodule of M', then $f^{-1}(N)$ is also a 2-absorbing primary subsemimodule of M.*

Proof. Let $abm \in f^{-1}(N)$ for some $a, b \in S$ and $m \in M$. Then $f(abm) \in N$, that is, $abf(m) \in N$. Since N is a 2-absorbing primary subsemimodule of M', therefore $ab \in \sqrt{(N : M')}$ or $af(m) \in N$ or $bf(m) \in N$. Hence, $ab \in f^{-1}(\sqrt{(N : M')})$ or $am \in f^{-1}(N)$ or $bm \in f^{-1}(N)$. Since $f^{-1}(\sqrt{(N : M')}) \subseteq \sqrt{f^{-1}(N : M')}$, we have $f^{-1}(N)$ is a 2-absorbing primary subsemimodule of M. □

Theorem 5. *Let M be an S-semimodule, N be a 2-absorbing primary subsemimodule of M and K be a subsemimodule of M such that $K \nsubseteq N$. Then $N \cap K$ is a 2-absorbing primary subsemimodule of K.*

Proof. Clearly, $N \cap K$ is a proper subsemimodule of K. Let $abx \in N \cap K$ where $a, b \in S$ and $x \in K$. Since $abx \in N$ and N is a 2-absorbing primary subsemimodule of M, therefore either $ax \in N$ or $bx \in N$ or $ab \in \sqrt{(N : M)}$. If $ax \in N$ or $bx \in N$, then $ax \in N \cap K$ or $bx \in N \cap K$. If $ab \in \sqrt{(N : M)}$, then $(ab)^n M \subseteq N$ for some positive integer n. In particular, $(ab)^n K \subseteq N$ which implies $(ab)^n K \subseteq N \cap K$ for some positive integer n. Thus, $ab \in \sqrt{(N \cap K : K)}$. Hence $N \cap K$ is a 2-absorbing primary subsemimodule of K. □

Theorem 6. *Let M and M' be S-semimodules, $f : M \mapsto M'$ be an epimorphism such that $f(0) = 0$ and N be a subtractive strong subsemimodule of M. If N is a 2-absorbing primary subsemimodule of M with $ker f \subseteq N$, then $f(N)$ is a 2-absorbing primary subsemimodule of M'*

Proof. Let N be a 2-absorbing primary subsemimodule of M and $abx \in f(N)$ for some $a, b \in S$ and $x \in M'$. Since $abx \in f(N)$, there exists an element $x' \in N$ such that $abx = f(x')$. Since f is an epimorphism and $x \in M'$, then there exists $y \in M$ such that $f(y) = x$. As $x' \in N$ and N is a strong subsemimodule of M, therefore there exists $x'' \in N$ such that $x' + x'' = 0$, which gives $f(x' + x'') = 0$. Therefore, $abx + f(x'') = 0$ or $f(aby) + f(x'') = 0$ implies $aby + x'' \in ker f \subseteq N$. Thus, we have $aby \in N$, since N is a subtractive subsemimodule of M. Since N is a 2-absorbing primary, we conclude that $ab \in \sqrt{(N : M)}$ or $ay \in N$ or $by \in N$. Thus, $ab \in f(\sqrt{(N : M)}) \subseteq \sqrt{f(N : M)}$ or $f(ay) \in f(N)$ or $f(by) \in f(N)$ and hence $ab \in \sqrt{(f(N) : M')}$ or $ax \in f(N)$ or $bx \in f(N)$. Thus, $f(N)$ is a 2-absorbing primary subsemimodule of M'. □

A subsemimodule N of an S-semimodule M is said to be irreducible if $N = N_1 \cap N_2$, where N_1 and N_2 are subsemimodules of M, then either $N = N_1$ or $N = N_2$. We now give a characterization of 2-absorbing primary subsemimodule of an S-semimodule of M, when N is irreducible.

Theorem 7. *Let N be a proper subtractive subsemimodule of an S-semimodule M. Let $\sqrt{(N : M)}$ be a 2-absorbing ideal of S. If N is an irreducible subsemimodule of M, then N is a 2-absorbing primary subsemimodule of M if and only if $(N : r) = (N : r^2)$ for all $r \in S \setminus \sqrt{(N : M)}$.*

Proof. Let N be a 2-absorbing primary subsemimodule of M and let $r \in S \setminus \sqrt{(N : M)}$. We will show that $(N : r) = (N : r^2)$. Clearly, $(N : r) \subseteq (N : r^2)$. Let $x \in (N : r^2)$, so $r^2x \in N$. Therefore, either $rx \in N$ or $r^2 \in \sqrt{(N : M)}$, since N is a 2-absorbing primary subsemimodule. If $rx \in N$, then $x \in (N : r)$. Otherwise, $r^2 \in \sqrt{(N : M)}$ gives $r \in \sqrt{(N : M)}$, a contradiction. Thus $(N : r) = (N : r^2)$.
Conversely, let $r_1r_2x \in N$ for some $r_1, r_2 \in S$ and $x \in M$. Let $r_1r_2 \notin \sqrt{(N : M)}$. Then we will show that $r_1x \in N$ or $r_2x \in N$. We claim that $r_1 \notin \sqrt{(N : M)}$ and $r_2 \notin \sqrt{(N : M)}$ because if $r_1 \in \sqrt{(N : M)}$ and $r_2 \in \sqrt{(N : M)}$, then $r_1r_2 \in (\sqrt{(N : M)})^2 \subseteq \sqrt{(N : M)}$, which is a contradiction. Therefore we may assume that either $(N : r_1) = (N : r_1^2)$ or $(N : r_2) = (N : r_2^2)$. Suppose $(N : r_1) = (N : r_1^2)$. Let $r_1x \notin N$ and $r_2x \notin N$, then $N \subseteq (N + Sr_1x) \cap (N + Sr_2x)$. Let $y \in (N+Sr_1x)\cap(N+Sr_2x)$. Then $y = n_1+s_1r_1x = n_2+s_2r_2x$ where $n_1, n_2 \in N$ and $s_1, s_2 \in S$. Now, $r_1y = r_1n_1 + s_1r_1^2x = r_1n_2 + r_1r_2s_2x$ and $s_2r_1r_2x, rn_1, rn_2 \in N$, so $s_1r_1^2x \in N$, as N is subtractive. This implies $s_1x \in (N : r_1^2)$ but $(N : r_1) = (N : r_1^2)$. Therefore $s_1xr_1 \in N$ and so $y \in N$. Hence $(N + Sr_1x) \cap (N + Sr_2x) \subseteq N$. Consequently, $(N + Sr_1x) \cap (N + Sr_2x) = N$, a contradiction (since N is an irreducible subsemimodule). Hence N is a 2-absorbing primary subsemimodule of M. $\square$

Lemma 1. *Let N be a subtractive 2-absorbing primary subsemimodule of an S-semimodule M. Suppose that $abJ \subseteq N$ for some subsemimodule J of M. If $ab \notin \sqrt{(N : M)}$, then $aJ \subseteq N$ or $bJ \subseteq N$.*

Proof. Let $abJ \subseteq N$ for some $a, b \in S$ and for some subsemimodule J of M. Suppose $aJ \nsubseteq N$ and $bJ \nsubseteq N$, then $aj_1 \notin N$ and $bj_2 \notin N$ for some $j_1, j_2 \in J$. Since $abj_1 \in N$ and $ab \notin \sqrt{(N : M)}$ and $aj_1 \notin N$, we have $bj_1 \in N$. Again, since $abj_2 \in N$ and $ab \notin \sqrt{(N : M)}$ and $bj_2 \notin N$, we have $aj_2 \in N$. Now, $ab(j_1 + j_2) \in N$ and $ab \notin \sqrt{(N : M)}$, we have either $a(j_1 + j_2) \in N$ or $b(j_1 + j_2) \in N$. If $a(j_1 + j_2) \in N$ and $aj_2 \in N$, we get $aj_1 \in N$, a contradiction. Similarly, if $b(j_1 + j_2) \in N$ and $bj_1 \in N$, then $bj_2 \in N$ (since N is subtractive), a contradiction. Thus, $aJ \subseteq N$ or $bJ \subseteq N$. $\square$

We know that, if K is a subtractive subsemimodule, then $(K : M)$ is also subtractive. In the next theorem, we will assume that K and $\sqrt{(K : M)}$ are subtractive subsemimodule of M and subtractive ideal of S respectively.

Theorem 8. *Let K be a subtractive subsemimodule of M and $\sqrt{(K:M)}$ be a subtractive ideal of S. If K is a 2-absorbing primary subsemimodule of M, then whenever $IJN \subseteq K$ for some ideals I, J of S and a subsemimodule N of M, then $IJ \subseteq \sqrt{(K:M)}$ or $IN \subseteq K$ or $JN \subseteq K$.*

Proof. Let K be a 2-absorbing primary subsemimodule of M and let $IJN \subseteq K$ for some ideals I, J of S and a subsemimodule N of M, such that $IJ \nsubseteq \sqrt{(K:M)}$. We show that $IN \subseteq K$ or $JN \subseteq K$. If possible, suppose that $IN \nsubseteq K$ and $JN \nsubseteq K$. Then there exist $a_1 \in I$ and $b_1 \in J$ such that $a_1N \nsubseteq K$ and $b_1N \nsubseteq K$. Since $a_1b_1N \subseteq K$ and $a_1N \nsubseteq K$ and $b_1N \nsubseteq K$, we have $a_1b_1 \in \sqrt{(K:M)}$ by Lemma 1. Next, we have $IJ \nsubseteq \sqrt{(K:M)}$, therefore for some $a \in I$ and $b \in J$, $ab \notin \sqrt{(K:M)}$. Since $abN \subseteq K$ and $ab \notin \sqrt{(K:M)}$, we have $aN \subseteq K$ or $bN \subseteq K$ by Lemma 1. Here three cases arise.

Case I: $aN \subseteq K$ but $bN \nsubseteq K$. Since $a_1bN \subseteq K$ and $bN \nsubseteq K$ and $a_1N \nsubseteq K$, by Lemma 1 we have $a_1b \in \sqrt{(K:M)}$. Now, $aN \subseteq K$ but $a_1N \nsubseteq K$, therefore $(a+a_1)N \nsubseteq K$. Since $(a+a_1)bN \subseteq K$ and $bN \nsubseteq K$ and $(a+a_1)N \nsubseteq K$ implies $(a+a_1)b \in \sqrt{(K:M)}$ by Lemma 1. Since $(a+a_1)b \in \sqrt{(K:M)}$ and $a_1b \in \sqrt{(K:M)}$, we have $ab \in \sqrt{(K:M)}$, as $\sqrt{(K:M)}$ is subtractive, a contradiction.

Case II: When $bN \subseteq K$ but $aN \nsubseteq K$. Since $ab_1N \subseteq K$ and $aN \nsubseteq K$ and $b_1N \nsubseteq K$, then by Lemma 1, $ab_1 \in \sqrt{(K:M)}$. Since $bN \subseteq K$ and $b_1N \nsubseteq K$, we have $(b+b_1)N \nsubseteq K$. Since $a(b+b_1)N \subseteq K$ and $aN \nsubseteq K$ and $(b+b_1)N \nsubseteq K$, we have $a(b+b_1) \in \sqrt{(K:M)}$ by Lemma 1. Since $a(b+b_1) \in \sqrt{(K:M)}$ and $ab_1 \in \sqrt{(K:M)}$, we have $ab \in \sqrt{(K:M)}$ (since $\sqrt{(K:M)}$ is subtractive), a contradiction.

Case III: When $aN \subseteq K$ and $bN \subseteq K$. Since $bN \subseteq K$ and $b_1N \nsubseteq K$ it implies $(b+b_1)N \nsubseteq K$. Since $a_1(b+b_1)N \subseteq K$ and $(b+b_1)N \nsubseteq K$ and $a_1N \nsubseteq K$, we conclude that $a_1(b+b_1) \in \sqrt{(K:M)}$, by Lemma 1. Since $a_1b_1 \in \sqrt{(K:M)}$ and $a_1(b+b_1) \in \sqrt{(K:M)}$, we have $a_1b \in \sqrt{(K:M)}$, as $\sqrt{(K:M)}$ is subtractive. Again, $aN \subseteq K$ and $a_1N \nsubseteq K$ implies $(a+a_1)N \nsubseteq K$. Since $(a+a_1)b_1N \subseteq K$ and $(a+a_1)N \nsubseteq K$ and $b_1N \nsubseteq K$, then we have $(a+a_1)b_1 \in \sqrt{(K:M)}$ by Lemma 1. Since $a_1b_1 \in \sqrt{(K:M)}$ and $(a+a_1)b_1 \in \sqrt{(K:M)}$, then $ab_1 \in \sqrt{(K:M)}$. Since $(a+a_1)(b+b_1)N \subseteq K$ and $(a+a_1)N \nsubseteq K$ and $(b+b_1)N \nsubseteq K$, then by Lemma 1 $(a+a_1)(b+b_1) \in \sqrt{(K:M)}$. Since $ab_1, a_1b, a_1b_1 \in \sqrt{(K:M)}$, we have $ab \in \sqrt{(K:M)}$ (since $\sqrt{(K:M)}$ is subtractive), a contradiction. Hence $IN \subseteq K$ or $JN \subseteq K$.

□

Definition 2. ([3], Definition 1) A subsemimodule N of an S-semimodule M is called a partitioning subsemimodule (=Q-subsemimodule) if there exists a nonempty subset Q of M such that

(i) $SQ \subseteq Q$, where $SQ = \{rq : r \in S, q \in Q\}$;
(ii) $M = \cup\{q + N : q \in Q\}$;
(iii) If $q_1, q_2 \in Q$, then $(q_1 + N) \cap (q_2 + N) \neq \emptyset$ if and only if $q_1 = q_2$.

Let M be an S-semimodule, and let N be a Q-subsemimodule of M. Define $M/N_{(Q)} = \{q + N : q \in Q\}$. Then $M/N_{(Q)}$ forms an S-semimodule under the operations $\oplus$ and $\odot$ defined as follows: $(q_1+N) \oplus (q_2+N) = q_3+N$ where $q_3 \in Q$ is the unique element such that $q_1+q_2+N \subseteq (q_3+N)$ and $r \odot (q_1+N) = q_4+N$, where $r \in S$ and $q_4 \in Q$ is the unique element such that $rq_1 + N \subseteq q_4 + N$. Then, this S-semimodule $M/N_{(Q)}$ is called the quotient semimodule of M by N. By the definition of Q-subsemimodule, there exists a unique $q_0 \in Q$ such that $0_M + N \subseteq q_0 + N$. Then $q_0 + N$ is a zero element of M/N. But, for every $q \in Q$ from (i) one obtains $0_M = 0_s q \in Q$; hence $q_0 = 0_M$.
For deeper understandings of Q-subsemimodules of semimodule, we refer ([3],[4], [8],[14]).

Theorem 9. *Let M be an S-semimodule, N be a Q-subsemimodule of M and P be a subtractive subsemimodule of M such that $N \subseteq P$. Then P is a 2-absorbing primary subsemimodule of M if and only if $P/N_{(Q\cap P)}$ is a 2-absorbing primary subsemimodule of $M/N_{(Q)}$.*

Proof. Let P be a 2-absorbing primary subsemimodule of M. Let $a, b \in S$ and $q + N \in M/N_{(Q)}$ be such that $ab \odot q + N = q_1 + N \in P/N_{(Q\cap P)}$ where $q_1 \in Q \cap P$ is a unique element such that $abq + N \subseteq q_1 + N$. So $abq = q_1 + x_1$, for some $x_1 \in N \subseteq P$. Since P is a 2-absorbing primary subsemimodule of M, either $(ab)^n \in (P : M)$ or $aq \in P$ or $bq \in P$ for some positive integer n. First, let $a^n b^n \in (P : M)$. Consider, $a^n b^n \odot q_2 + N = q_3 + N$ where $q_2 + N \in M/N_{(Q)}$ and $q_3 \in Q$ is a unique element such that $a^n b^n q_2 + N \subseteq q_3 + N$. So, $a^n b^n q_2 = q_3 + x_2$ for some $x_2 \in N \subseteq P$. Since $a^n b^n \in (P : M)$, we have $a^n b^n q_2 \in P$, which gives $q_3 \in P$, as P is subtractive. Thus, we have $q_3 \in Q \cap P$ which gives $a^n b^n \odot q_2 + N = q_3 + N \in P/N_{(Q\cap P)}$ and hence $ab \in \sqrt{(P/N_{(Q\cap P)} : M/N_{(Q)})}$. If $aq \in P$, consider $a \odot q + N = q_4 + N$ where $q_4 \in Q$ is a unique element such that $aq + N \subseteq q_4 + N$. This gives, $aq = q_4 + x_3$ for some $x_3 \in N \subseteq P$. Since P is subtractive, we have $q_4 \in P$ Hence $a \odot (q + N) = q_4 + N \in P/N_{(Q\cap P)}$. Similarly, we can prove that $b \odot (q + N) \in P/N_{(Q\cap P)}$. Consequently, $P/N_{(Q\cap P)}$ is a 2-absorbing primary subsemimodule of $M/N_{(Q)}$.

Conversely, let $P/N_{(Q\cap P)}$ be a 2-absorbing primary subsemimodule of $M/N_{(Q)}$. Let $abx \in P$ for some $a, b \in S$ and $x \in M$. Since, N is a Q-subsemimodule of M and $x \in M$, we have $x \in q + N$ where $q \in Q$. So $abx \in abq + N$. Now, let $ab \odot (q+N) = q_5+N$ where $q_5 \in Q$ is a unique element such that $abq+N \subseteq q_5+N$. This gives, $abx = q_5 + x_4$ for some $x_4 \in N \subseteq P$. Therefore, we have $q_5 \in P$, since P is subtractive. Thus, $q_5 \in Q \cap P$ and hence $ab \odot (q + N) = q_5 + N \in P/N_{(Q\cap P)}$. Thus, we have $a^m b^m \in (P/N_{(Q\cap P)} : M/N_{(Q)})$ or $a \odot (q+N) \in P/N_{(Q\cap P)}$

or $b \odot (q + N) \in P/N_{(Q\cap P)}$ for some positive integer m, since $P/N_{(Q\cap P)}$ is a 2-absorbing primary subsemimodule. Let $a^m b^m \in (P/N_{(Q\cap P)} : M/N_{(Q)})$ for some positive integer m. Let $y \in M$. Then there exists a unique element $q_6 \in Q$ such that $y \in q_6 + N \in M/N_{(Q)}$. Now, $a^m b^m \odot (q_6 + N) \in P/N_{(Q\cap P)}$. Therefore, there exists a unique element $q_7 \in Q \cap P$ such that $a^m b^m q_6 + N \subseteq q_7 + N$ which gives $a^m b^m y \in a^m b^m q_6 + N \subseteq q_7 + N$. Thus, $a^m b^m y = q_7 + x_5 \in P$ for some $x_5 \in N \subseteq P$. Hence $a^m b^m y \in P$ and hence $a^m b^m \in (P : M)$. Let $a \odot (q + N) \in P/N_{(Q\cap P)}$. Then, there exists unique $q_8 \in Q \cap P$ such that $aq + N \subseteq q_8 + N$. We have, $x \in q + N$ implies $ax \in aq + N \subseteq q_8 + N$. Therefore, $ax = q_8 + x_6$ for some $x_6 \in N \subseteq P$. Hence $ax \in P$. Similarly, $bx \in P$. $\square$

Theorem 10. *Let M be an S-semimodule, N be a Q-subsemimodule of M and P be a subtractive subsemimodule of M such that $N \subseteq P$. If N and $P/N_{(Q\cap P)}$ are 2-absorbing primary subsemimodules of M and $M/N_{(Q)}$ respectively, then P is a 2-absorbing primary subsemimodule of M.*

Proof. Let N and $P/N_{(Q\cap P)}$ be 2-absorbing primary subsemimodules of M and $M/N_{(Q)}$ respectively. Let $abx \in P$ for some $a, b \in S$ and $x \in M$. If $abx \in N$, then we are done (since N is a 2-absorbing primary subsemimodule of M). So, let $abx \notin N$. Since $x \in M$, there exists a unique element $q_1 \in Q$ such that $x \in q_1 + N$ gives $abx \in ab \odot (q_1 + N)$. This gives, $abx \in abq_1 + N \subseteq q_2 + N$ where q_2 is a unique element of Q. Since $abx \in P$ and $N \subseteq P$, we have $q_2 \in P$. Therefore, $ab \odot (q_1 + N) \in P/N_{(Q\cap P)}$. Thus, either $ab \in \sqrt{(P/N_{(Q\cap P)} : M/N_{(Q)})}$ or $a \odot (q_1 + N) \in P/N_{(Q\cap P)}$ or $b \odot (q_1 + N) \in P/N_{(Q\cap P)}$. Now, it is similar to the proof of the converse part of the last theorem. $\square$

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Solvability of a nonlinear integral equation arising in kinetic theory

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Abstract. In the paper the question of solvability of an Urysohn type nonlinear integral equation arising in kinetic theory of gases has been studied. We prove the existence of a positive and bounded solution and also suggest an approach for the construction of a solution. We also show that there is a qualitative difference between solutions in the linear and nonlinear cases. In the nonlinear case the solution is a positive and bounded function, while the corresponding linear equation has an alternating solution, which possesses linear growth at infinity.

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1 Introduction

The paper is devoted to the study and solution of the following Urysohn nonlinear integral equation

$$F(x) = g(x) + \int_0^\infty W(x,t,F(t))dt, \tag{1.1}$$

with respect to the unknown function $F(x)$, where

$$g(x) = \frac{2\varepsilon c}{3\sqrt{\pi}} \int_0^\infty e^{-xs} e^{-\frac{1}{s^2}} (s^2+1) \frac{ds}{s^4}, \tag{1.2}$$

$$\begin{aligned} W(x,t,F(t)) &= \frac{2}{3\sqrt{\pi}} \sqrt{F(t)} \times \\ \times \int_0^\infty \left[e^{-|x-t|s} + (1-\varepsilon) e^{-(x+t)s}\right] &e^{-\frac{1}{s^2 F(t)}} \left[\frac{1}{s^2 F(t)} + 1\right] \frac{ds}{s}. \end{aligned} \tag{1.3}$$

Equation (1.1), as well as its intrinsic mathematical interest, has important applications in kinetic theory of gases (see [1–3]). Equation (1.1) may be derived from the Boltzmann model equation. By equation (1.1) the flow of a rarefied gas in the half-space $x > 0$ bounded by flat plate $x = 0$ is described. The function $F(x)$ represents temperature distribution near the wall. Here x is the distance from the wall, $c = \dfrac{\beta}{\alpha}$

$(0 < c \leq 1)$, where α is the mean value of density in the boundary layer and β is the density of particles reflected from the wall. We will assume that c is previously known. ε is the accomodation coefficient $(0 < \varepsilon \leq 1)$.

In the present note we prove the existence theorem of a positive and bounded solution of equation (1.1) and also suggest the approach for the construction of a solution. We also show that there is a qualitative difference between solutions in the nonlinear and linear cases. In the nonlinear case the solution is a positive and bounded function, while the corresponding linear equation has an alternating solution, which possesses linear growth at infinity.

2 The existence of a bounded solution for an Urysohn type nonlinear integral equation

Below we formulate the theorem of global solvability of equation (1.1) in the space of bounded functions for arbitrary values of $c > 0$ and $\alpha > 0$.

We consider the following function

$$\xi(t) = t^4 - ct^3 - 1, \quad t \in \mathbb{R}^+ \equiv [0, +\infty). \tag{2.1}$$

We note that $\xi(0) = -1$, $\xi'(t) = 4t^3 - 3ct^2 \geq 0$ if $t \in \left[\frac{3c}{4}, +\infty\right)$ and $\xi'(t) \leq 0$ if $t \in \left[0, \frac{3c}{4}\right]$, $\xi(c) < 0$, $\lim\limits_{t\to\infty} \xi(t) = +\infty$, then there exists a unique point $t_0 > c$ such that $\xi(t_0) = 0$, moreover, for $t > t_0$, $\xi(t) > 0$.

We introduce the following iterations for equation (1.1):

$$F_{n+1}(x) = g(x) + \int\limits_0^\infty W(x, t, F_n(t))dt, \tag{2.2}$$

$$F_0(x) = t_0^2 = c_0. \tag{2.3}$$

It is easy to verify that the function W defined by (1.3) is monotone increasing in the third argument, i. e.

$$W(x, t, z) \ \uparrow \text{w.r.t.} z. \tag{2.4}$$

Indeed, since $\rho(z) = \left(\frac{s^2}{z} + 1\right)\sqrt{z}e^{-\frac{s^2}{z}} \uparrow w.r.t. \ z, z \geq 0$, then from the representation of W it follows that $W \uparrow$ w.r.t. z.

Below we prove by induction that $F_n(x)$ is monotone decreasing in n

$$1) \quad F^{(n)} \downarrow \text{w.r.t. } n \quad \text{and} \quad 2) \quad F^{(n)}(x) \geq g(x). \tag{2.5}$$

Let $n = 0$. We have

$$\begin{gathered} F_1(x) = g(x) + \int\limits_0^\infty W(x, t, F_0(t))dt = \\ = J_1(x) + c_0 - J_2(x) = F_0(x) + J_1(x) - J_2(x), \end{gathered} \tag{2.6}$$

where

$$J_1(x) = \frac{2\varepsilon c}{3\sqrt{\pi}} \int\limits_0^\infty e^{-xs} e^{-\frac{1}{s^2}} \left(\frac{1}{s^2} + 1\right) \frac{ds}{s^2}, \tag{2.7}$$

$$J_2(x) = \frac{2\varepsilon}{3\sqrt{\pi}} \sqrt{c_0} \int\limits_0^\infty e^{-xs} e^{-\frac{1}{s^2 c_0}} \left(\frac{1}{s^2 c_0} + 1\right) \frac{ds}{s^2}. \tag{2.8}$$

We must prove that $J_2(x) \geq J_1(x)$ for each $x \in \mathbb{R}^+$. It is sufficient to prove that for each $x \in \mathbb{R}^+$ the inequality holds

$$c e^{-\frac{1}{s^2}} \left(\frac{1}{s^2} + 1\right) \leq \sqrt{\frac{1}{c_0}} e^{-\frac{1}{s^2 c_0}} \left(c_0 + \frac{1}{s^2}\right). \tag{2.9}$$

Let us consider the following function

$$\varphi(s^2) = c\sqrt{c_0} e^{\frac{1}{s^2}\left(\frac{1}{c_0} - 1\right)} \left(\frac{1}{s^2} + 1\right), \;\; s^2 \in \mathbb{R}^+. \tag{2.10}$$

Note that $s_0^2 = c_0 - 1$ is the unique maximum point for φ. Therefore

$$\varphi(s^2) \leq \varphi(s_0^2) = c\sqrt{c_0} \left(\frac{1}{c_0 - 1} + 1\right) e^{-\frac{1}{c_0}}. \tag{2.11}$$

Using the well-known inequality

$$e^{-x} \leq \frac{1}{1+x}, \quad x \geq 0, \tag{2.12}$$

from (2.11) we obtain

$$\varphi(s^2) \leq \frac{c c_0^2 \sqrt{c_0}}{(c_0^2 - 1)}. \tag{2.13}$$

First we prove that

$$\frac{c\sqrt{c_0} c_0}{(c_0^2 - 1)} \leq 1. \tag{2.14}$$

Since $c_0 = t_0^2 > 1$ (because $t_0^4 = c t_0^3 + 1 > 1 \Rightarrow t_0^2 > 1$), then inequality (2.14) is equivalent to the following inequality:

$$c\sqrt{c_0} c_0 \leq (c_0^2 - 1). \tag{2.15}$$

As $\xi(t) \uparrow$ in t on $[t_0, +\infty)$, then $\xi(\sqrt{c_0}) \geq \xi(t_0) = 0$ or $\xi(\sqrt{c_0}) = c_0^2 - c\sqrt{c_0} c_0 - 1 \geq 0$, i. e. (2.14) is proved. Taking into consideration (2.14), from (2.13), we obtain

$$\varphi(s^2) = \frac{c c_0^2 \sqrt{c_0}}{(c_0^2 - 1)} \leq c_0 \leq c_0 + \frac{1}{s^2}. \tag{2.16}$$

From (2.16) follows (2.9). Therefore we have $J_2(x) \geq J_1(x)$. Considering the last inequality and relation (2.6) we come to the inequality $F_1(x) \leq F_0(x)$. We assume that $F_n(x) \le F_{n-1}(x)$ for some $n \in \mathbb{N}$. Since $W(x,t,z)$ monotonically increases in the third argument z then from (2.2) it follows that

$$F_{n+1}(x) \le F_n(x). \tag{2.17}$$

Now we prove that the sequence of functions $\{F_n(x)\}_{n=0}^{\infty}$ is bounded by $g(x)$.

First, we show that $t_0^2 > \frac{c}{2}$. Assume the contrary: $t_0^2 \le \frac{c}{2}$. Since $t_0 > c$ then we have $c < \sqrt{\frac{c}{2}}$ or

$$c < \frac{1}{2}. \tag{2.18}$$

On the other hand,

$$0 = t_0^4 - ct_0^3 - 1 < t_0^4 - 1.$$

Hence, we obtain $t_0^2 > 1$. But since $t_0^2 < \frac{c}{2}$ then we obtain inequality $c > 2$.

Taking into consideration (2.18), from the last inequality we come to contradiction. Therefore,

$$t_0^2 > \frac{c}{2}. \tag{2.19}$$

Now, due to (2.19) from (2.3), we have

$$F_0(x) = t_0^2 > \frac{c}{2} \ge g(x),$$

because

$$g(x) \le \frac{2}{3\sqrt{\pi}} c \int_0^\infty e^{-\frac{1}{s^2}} \left(\frac{s^2+1}{s^4} \right) ds = \frac{c}{2}.$$

Let $F_n(x) \ge g(x)$ for some $n \in \mathbb{N}$. Then taking into consideration monotonicity and nonnegativity of the function W, we obtain

$$F_{n+1}(x) \ge g(x) + \int_0^\infty W(x,t,g(t))dt \ge g(x). \tag{2.20}$$

Therefore the sequence of functions $\{F_n(x)\}_{n=0}^{\infty}$ has a pointwise limit as $n \to \infty$. In accordance with B.Levi's theorem the function F satisfies equation (1.1) and the double inequalities

$$g(x) \le F(x) \le c_0 \equiv t_0^2. \tag{2.21}$$

Thus the following theorem holds

Theorem 1. *Let* $0 < c \le 1$ *is a given number. Then nonlinear integral equation* (1.1) *has a positive measurable and bounded solution* $F(x)$. *The following inequalities hold*

$$g(x) \le F(x) \le c_0 \equiv t_0^2, \tag{2.22}$$

where t_0 *is the unique positive root of the following algebraic equation* $t^4 - ct^3 - 1 = 0$.

3 Linearization of a Urysohn nonlinear integral equation (1.1). Qualitative difference between solutions in the linear and nonlinear cases

Usually in kinetic theory in linear approximation the function $F(x)$ is represented as:

$$F(x) = 1 + \Delta f(x), \tag{3.1}$$

where $\Delta f(x)$ is the temperature perturbation $(\Delta f(x) \ll 1)$. Taking into account (3.1), expanding the function $W(x,t,F(t))$ by the third argument in a power series about zero and holding the first expansion term, we obtain the following Wiener-Hopf-Hankell type linear integral equation with respect to $\Delta f(x)$:

$$\Delta f(x) = g_1(x) + \int_0^\infty [K(x-t) + (1-\varepsilon)K(x+t)]\Delta f(t)dt. \tag{3.2}$$

Here

$$\begin{gathered} K(x) = \int_0^\infty e^{-|x|s} G(s)ds, \\ G(s) = \frac{2}{3}\frac{1}{\sqrt{\pi}}\frac{1}{s}e^{-\frac{1}{s^2}}\left(\frac{1}{s^4} + \frac{1}{2s^2} + \frac{1}{2}\right), \end{gathered} \tag{3.3}$$

$$\begin{gathered} g_1(x) = \int_0^\infty e^{-xs} G_1(s)ds, \\ G_1(s) = \frac{2\varepsilon}{3\sqrt{\pi}s^4}(c-1)(s^2+1)e^{-\frac{1}{s^2}}. \end{gathered} \tag{3.4}$$

It is easy to check that kernel (3.3) satisfies the conservative condition

$$K \geq 0, \quad \int_{-\infty}^{+\infty} K(x)dx = 1. \tag{3.5}$$

Due to linearity the solution of equation (3.2) can be written as:

$$\triangle f(x) = -\triangle f_1(x) + \triangle f_2(x), \tag{3.6}$$

where $\triangle f_1(x)$ and $\triangle f_2(x)$ are the solutions of inhomogeneous and homogeneous equations, respectively

$$\Delta f_1(x) = -g_1(x) + \int_0^\infty [K(x-t) + (1-\varepsilon)K(x+t)]\Delta f_1(t)dt \tag{3.7}$$

($-g_1(x) \geq 0$ because of $c \in (0,1]$),

$$\Delta f_2(x) = \int_0^\infty [K(x-t) + (1-\varepsilon)K(x+t)]\Delta f_2(t)dt. \tag{3.8}$$

There are numerous works devoted to study and solutions of equations (3.7) and (3.8) (see [4, 5] and references therein). Without going into details we note that equation (3.7) has positive bounded solution, which possesses finite limit at infinity (see [4, 6]).

The solution of corresponding homogeneous equation (3.8) has the form (see [5])

$$\triangle f_2(x) = \frac{1}{\sqrt{\nu_2}} x + q(x), \tag{3.9}$$

here $q(x)$ is the well-known Hopf function, and ν_2 is the second moment of the kernel $K(x)$. Thus we have

$$\begin{gathered} \triangle f(x) = \frac{1}{\sqrt{\nu_2}} x + q(x) - \triangle f_1(x) \quad \text{and} \\ \triangle f(x) \sim \frac{1}{\sqrt{\nu_2}} x, \quad \text{as} \quad x \to +\infty. \end{gathered} \tag{3.10}$$

Conclusion. Note that the linear equation (3.2) possesses an alternating solution with the asymptotic $O(x)$ as x tends to $+\infty$, while the solution of initial nonlinear equation (1.1) is a positive bounded function $F(x)$. Moreover, $g(x) \leq F(x) \leq c_0,\ x \in \mathbb{R}^+$. The qualitative difference between the solutions is conditioned by linearization of equation (1.1). In fact the linearization can distort the problem and the corresponding linear equation can not adequately describe the problem from a physical point of view.

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Relation between Levinson center, chain recurrent set and center of Birkhoff for compact dissipative dynamical systems

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Abstract. In this paper we prove the analogues of Birkhoff's theorem for one-sided dynamical systems (both with continuous and discrete times) with noncompact space having a compact global attractor. The relation between Levinson center, chain recurrent set and center of Birkhoff is established for compact dissipative dynamical systems.

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1 Introduction

Let X be a compact metric space, $(X, \mathbb{R}, \pi)$ be a flow on X, $M \subseteq X$ be a nonempty compact and invariant subset of X. Denote $\Omega(M) := \{x \in M :$ there exist $\{x_n\} \subset M$ and $\{t_n\} \subset \mathbb{R}$ such that $x_n \to x,\ t_n \to +\infty$ as $n \to \infty$ and $\pi(t_n, x_n) \to x\}$. Recall that the point $x \in X$ is called Poisson stable if $x \in \omega_x \bigcap \alpha_x$, where by ω_x (respectively, α_x) the ω (respectively, α)-limits set of x is denoted. The following result is well known (see, for example, [1, 14]).

Theorem 1 (Birkhoff's theorem). *The following statements hold:*

1. *there exists a nonempty, compact and invariant subset $\mathfrak{B}(\pi) \subseteq X$ with the properties:*
   (i) $\Omega(\mathfrak{B}(\pi)) = \mathfrak{B}(\pi)$;
   (ii) $\mathfrak{B}(\pi)$ *is the maximal compact invariant subset of J with the property (i).*
2. $\mathfrak{B}(\pi) = \overline{\mathcal{P}(\pi)}$, *i. e., the set of all Poisson stable points $\mathcal{P}(\pi)$ of the dynamical system $(X, \mathbb{R}, \pi)$ is dense in $\mathfrak{B}(\pi)$.*

Remark 1. 1. The set $\mathfrak{B}(\pi)$ is called the Bikkhoff center of dynamical system $(X, \mathbb{R}, \pi)$.

2. Note that Birkhoff theorem remains true also for the discrete dynamical systems $(X, \mathbb{Z}, \pi)$. This fact was established in the work of V. S. Bondarchuk and V. A. Dobrynsky [1].

3. The second statement of Theorem 1 remains true if we replace the center of Birkhoff $\mathfrak{B}(\pi)$ by arbitrary compact invariant set $M \subseteq J$ with the property $\Omega(M) = M$. Namely the following equality takes place: $M = \overline{\mathcal{P}(\pi) \bigcap M}$.

The main result of this paper is the proof of the analogues of Birkhoff theorem for the one-sided dynamical systems (both with continuous and discrete times) with noncompact phase space having a compact global attractor.

2 Birkhoff center

Definition 1. A dynamical system $(X, \mathbb{T}, \pi)$ is said to be:

1. pointwise dissipative if there exists a nonempty compact subset $K \subseteq X$ such that
$$\lim_{t \to +\infty} \rho(\pi(t, x), K) = 0 \tag{1}$$
for all $x \in X$;

2. compactly dissipative if there exists a nonempty compact subset $K \subseteq X$ such that (1) holds uniformly with respect to x on every compact subset from X.

Remark 2. Every compact dissipative dynamical system is pointwise dissipative. The converse, generally speaking, is not true (see, for example, [4, Ch.I]).

Theorem 2 (see [4, Ch.I])**.** *Suppose that $(X, \mathbb{T}, \pi)$ is a compact dissipative dynamical system, then there exists a nonempty, compact, invariant subset $J \subseteq X$ possessing the following properties:*

1. *J attracts every compact subset A from X, i. e.,*
$$\lim_{t \to +\infty} \rho(\pi(t, x), J) = 0$$
uniformly with respect to $x \in A$;

2. *J is orbitally stable, i.e., for all $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $\rho(x, J) < \delta$ implies $\rho(\pi(t, x), J) < \varepsilon$ for all $t \geq 0$;*

3. *J is the maximal compact invariant subset of X.*

Let M be a positively invariant and closed subset of X. Denote by $J_x^+(M) := \{p \in X :$ there exist $\{x_n\} \subseteq M$ and $t_n \to +\infty$ such that $x_n \to x$ and $\pi(t_n, x_n) \to p$ as $n \to +\infty$ $\}$.

Lemma 1. *Let M be a positively invariant and closed subset of X. If $p_n \to p$, $x_n \to x$ as $n \to \infty$ and $p_n \in J_{x_n}^+(M)$, then $p \in J_x^+(M)$.*

Proof. Let ε be an arbitrary positive number, $p_n \to p$ and $x_n \to x$ as $n \to \infty$. Then there exists a number $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$\rho(p_n, p) < \varepsilon/3 \quad \text{and} \quad \rho(x_n, x) < \varepsilon/3$$

for all $n \geq n_0$. Since $p_n \in J^+_{x_n}(M)$ for all $n \in \mathbb{N}$, then there exist $\{x_n^m\} \subseteq M$ and $\{t_n^m\}$ (for all $m \in \mathbb{N}$) such that $x_n^m \to x_n$, $t_n^m \to +\infty$ and $\pi(t_n^m, x_n^m) \to p_n$ as $m \to \infty$. In particular, for given ε there exists $n < m_n = m_n(\varepsilon) \in \mathbb{N}$ such that

$$\rho(x_n^m, x_n) < \varepsilon/3 \quad \text{and} \quad \rho(\pi(t_n^m, x_n^m), p_n) < \varepsilon/3$$

for all $m \geq m_n$. Denote by $\bar{x}_n := x_n^{m_n}$ and $\bar{t}_n := t_n^{m_n} > n$. Note that $\{\bar{x}_n\} \subseteq M$, $\bar{t}_n \to +\infty$ as $n \to \infty$ and

$$\rho(\bar{x}_n, x) = \rho(x_n^{m_n}, x) \leq \rho(x_n^{m_n}, x_n) + \rho(x_n, x) < \varepsilon/3 + \varepsilon/3 < \varepsilon$$

for all $n \geq n_0(\varepsilon)$, i.e., $\bar{x}_n \to x$ as $n \to \infty$. In addition we have

$$\rho(\pi(\bar{t}_n, \bar{x}_n), p) = \rho(\pi(t_n^{m_n}, x_n^{m_n}), p) \leq \rho(\pi(t_n^{m_n}, x_n^{m_n}), p_n) + \rho(p_n, p) < \varepsilon/3 + \varepsilon/3 < \varepsilon$$

for all $n \geq n_0$. Thus for the point p we find the sequence $\{\bar{x}_n\} \subseteq M$ and $\bar{x}_n \to +\infty$ as $n \to \infty$ such that $\bar{x}_n \to x$ and $\pi(\bar{t}_n, \bar{x}_n) \to p$ as $n \to \infty$, i. e., $p \in J^+_x(M)$. Lemma is proved. $\square$

Lemma 2. *Let M be a positively invariant and closed subset of X and $x \in X$. The following statements hold:*

1. *$J^+_x(M) \subseteq M$ for all $x \in M$;*
2. *the set $J^+_x(M)$ is closed and positively invariant;*
3. *if M is compact, then $J^+_x(M)$ is invariant.*

Proof. Let $p \in J^+_x(M)$ and $t \in \mathbb{T}$, then there are $\{x_n\}$ and $t_n \to +\infty$ such that $x_n \to x$ and $\pi(t_n, x_n) \to p$ as $n \to \infty$. Then we have $\pi(t, p) = \lim\limits_{n\to\infty} \pi(t, \pi(t_n, x_n)) = \lim\limits_{n\to\infty} \pi(t + t_n, x_n)$ and, consequently, $\pi(t, p) \in J^+_x(M)$ because $x_n \in M$ and M is closed and positively invariant. Finally, it is evident that $J^+_x(M) \subseteq M$ for all $x \in M$.

Now we will establish the second statement of Lemma. Let $\{p_n\}$ be a sequence from $J^+_x(M)$ such that $p_n \to p$ as $n \to \infty$, then $p_n \in J^+_{x_n}(M)$ where $x_n := x$ for all $n \in \mathbb{N}$. By Lemma 1 $p \in J^+_x(M)$ because $p_n \to p$ and $x_n \to x$ as $n \to \infty$. Let us show now that the set $J^+_x(M)$ is positively invariant. Indeed, let $t \in \mathbb{T}$ and $p \in J^+_x(M)$, then there are $\{x_n\} \subseteq M$ and $t_n \to +\infty$ as $n \to \infty$ such that $\pi(t_n, x_n) \to p$ as $n \to \infty$. Note that $\pi(t, p) = \lim\limits_{n\to\infty} \pi(t + t_n, x_n)$ and, consequently, $\pi(t, p) \in J^+_x(M)$.

Suppose that the set M is compact and $p \in J^+_x(M)$, then there are $\{x_n\} \subseteq M$ and $t_n \to +\infty$ as $n \to \infty$ such that $\pi(t_n, x_n) \to p$ as $n \to \infty$. Let $t \in \mathbb{T}$ be an arbitrary number, then for sufficiently large $n \in \mathbb{N}$ we have $t_n - t \in \mathbb{T}$ because $t_n \to +\infty$ as $n \to \infty$. Since the set M is positively invariant and compact, then without loss of generality we can suppose that the sequence $\{\pi(t_n - t, x_n)\}$ is convergent. Denote by p_t its limit, then we obtain $p = \lim_{n\to\infty} \pi(t_n - t + t, x_n) = \lim_{n\to\infty} \pi(t, \pi(t_n - t, x_n)) = \pi(t, p_t)$ and, consequently, $p \in \pi(t, J_x^+(M))$, i. e., $J_x^+(M) \subseteq \pi(t, J_x^+(M))$ for all $t \in \mathbb{T}$. Thus $J_x^+(M)$ is positively and negatively invariant, i.e., it is invariant. □

Definition 2. Let M be a subset of X. A point $x \in X$ is said to be non-wandering with respect to M if $x \in J_x^+(M)$.

Denote by $\Omega(M) := \{x \in M : \ x \in J_x^+(M)\}$ the set of all non-wandering points of M with respect to M.

Remark 3. Let A and B be two closed and positively invariant subsets of X, then $\Omega(A) \subseteq \Omega(B)$.

Definition 3. A point $p \in X$ is said to be:

- Poisson stable in the positive direction if $x \in \omega_x$;
- Poisson stable in the negative direction if there exists an entire trajectory $\gamma_x \in \Phi_x$ such that $x \in \alpha_{\gamma_x}$, where $\alpha_{\gamma_x} := \{q \in X : \text{ there exists } t_n \to -\infty \text{ such that } \gamma_x(t_n) \to q \text{ as } n \to \infty\}$;
- Poisson stable if it is Poisson stable in the both directions.

Lemma 3. *Let M be a nonempty, closed and positively invariant set, then the following statements hold:*

1. *the set $\Omega(M)$ is closed;*
2. *if $p \in M$ is Poisson stable in the positive direction, then $p \in \Omega(M)$;*
3. *if the point $p \in M$ and $\gamma \in \Phi_p$ is an entire trajectory such that $\gamma(\mathbb{S}) \subset M$ and $p \in \alpha_\gamma$, then $p \in \Omega(M)$.*

Proof. The first statement directly follows from Lemma 1 and definition of $\Omega(M)$.

Let $p \in M$ and $p \in \omega_p$, then there exists a sequence $t_n \to +\infty$ such that $\pi(t_n, p) \to p$ as $n \to \infty$. Let $p_n := p$ for all $n \in \mathbb{N}$, then $p_n \to p$ and $\pi(t_n, p_n) \to p$ as $n \to \infty$. This means that $p \in J_p^+(M)$, i.e., $p \in \Omega(M)$.

Let $p \in M$, $\gamma \in \Phi_p$, $\gamma(\mathbb{S}) \subset M$ and $p \in \alpha_\gamma$. Then there exists a sequence $t_n \to +\infty$ such that $\gamma(-t_n) \to p$ as $n \to \infty$. Denote by $p_n := \gamma(-t_n)$, then $p_n \to p$ and $p = \pi(t_n, p_n) \to p$ as $n \to \infty$. Thus $p \in J_p^+(M)$ and, consequently, $p \in \Omega(M)$. □

Lemma 4. *Suppose that M is a nonempty, compact positively invariant set and $\mathcal{M}$ is a nonempty, compact minimal subset of M, then $\mathcal{M} \subseteq \Omega(M)$.*

Proof. Let $p \in \mathcal{M}$ and $\gamma \in \Phi_p$ be an entire trajectory of $(X, \mathbb{T}, \pi)$ passing through p at the initial moment such that $\gamma(\mathbb{S}) \subseteq M$. Since $\mathcal{M}$ is minimal, ω_p and α_γ are nonempty, compact and invariant we have $\alpha_\gamma = \omega_p = \mathcal{M}$. In particular there exists a sequence $\tau_n \to +\infty$ such that $p_n := \gamma(-\tau_n) \to p$ as $n \to \infty$. Note that $\pi(\tau_n, p_n) = p$ for all $n \in \mathbb{N}$ and, consequently, $p \in \Omega(\mathcal{M}) \subseteq \Omega(M)$. □

Corollary 1. *If M is a nonempty, compact positively invariant set, then $\Omega(M) \neq \emptyset$.*

Proof. Let M be a nonempty, compact and positively invariant set of $(X, \mathbb{T}, \pi)$. By Birkhoff theorem there exists a nonempty minimal subset $\mathcal{M} \subseteq M$ and by Lemma 4 we have $\mathcal{M} \subseteq \Omega(M)$. □

Denote by Φ_x the set of all entire trajectories γ_x of $(X, \mathbb{T}, \pi)$ passing through the point x at the initial moment $t = 0$.

Lemma 5. *Suppose that M is a nonempty, compact and positively invariant set. Then $\Omega(M)$ is a nonempty, compact and positively invariant subset of M.*

Proof. By Corollary 1 the set $\Omega(M)$ is a nonempty subset. By Lemma 1 the set $\Omega(M)$ is closed. Since $\Omega(M) \subseteq M$ and M is compact, then $\Omega(M)$ is so. Let now $p \in \Omega(M)$ and $t \in \mathbb{T}$, then there are $p_n \to p$ ($p_n \in M$) and $t_n \to +\infty$ as $n \to \infty$ such that $p = \lim_{n\to\infty} \pi(t_n, p_n)$. Note that $\pi(t, p) = \lim_{n\to\infty} \pi(t, \pi(t_n, p_n)) = \lim_{n\to\infty} \pi(t_n, \pi(t, p_n))$ and, consequently, $\pi(t, p) \in J^+_{\pi(t,p)}(M)$ because $\lim_{n\to\infty} \pi(t, p_n) = \pi(t, p)$ and $\{\pi(t, p_n)\} \subseteq M$. This means that $\pi(t, p) \in \Omega(M)$, i.e., $\Omega(M)$ is positively invariant. □

Lemma 6. *Let M be a nonempty positively invariant subset of X, then the following statements hold:*

1. *if $(X, \mathbb{T}, \pi)$ is pointwise dissipative, then $\Omega(M)$ is nonempty, closed and positively invariant;*
2. *if the dynamical system $(X, \mathbb{T}, \pi)$ is compactly dissipative and J is its Levinson center, then the set $\Omega(M)$ is nonempty, compact, positively invariant and $\Omega(M) \subseteq J$;*
3. *if the dynamically system $(X, \mathbb{T}, \pi)$ is point dissipative (but not compactly dissipative), then the set $\Omega(X)$, generally speaking, is not compact.*

Proof. Since $(X, \mathbb{T}, \pi)$ is pointwise dissipative, then $\Omega_M := \overline{\bigcup\{\omega_x : x \in M\}} \subseteq X$ is a nonempty compact invariant subset of $(X, \mathbb{T}, \pi)$ and by Birkhoff's theorem in Ω_M there exists at least one compact minimal subset $\mathcal{M} \subseteq \Omega \subseteq X$. By Corollary 1 $\Omega(M) \neq \emptyset$. Let us show that $\Omega(M)$ is closed. If $p = \lim_{n\to\infty} p_n$ and $p_n \in \Omega(M)$, then $p_n \in J^+_{p_n}(M)$. By Lemma 1 we have $p \in J^+_p(M)$, i.e., $p \in \Omega(M)$. If $p \in \Omega(M)$ and $t \in \mathbb{T}$, then there are $p_n \in M$ and $t_n \to +\infty$ such that $p = \lim_{n\to\infty} \pi(t_n, p_n)$ and, consequently, $\pi(t, p) = \lim_{n\to\infty} \pi(t, \pi(t_n, p_n)) = \lim_{n\to\infty} \pi(t_n, \pi(t, p_n))$, i.e., $\pi(t, p) \in J^+_{\pi(t,p)}(M)$ because $\lim_{n\to\infty} \pi(t, p_n) = \pi(t, p)$. This means that $\pi(t, p) \in \Omega(M)$, i.e., $\Omega(M)$ is positively invariant.

Let $(X, \mathbb{T}, \pi)$ be compactly dissipative and $x \in \Omega(M)$, then there exist $\{x_n\} \subseteq M$ and $t_n \to +\infty$ such that $x_n \to x$ and $\pi(t_n, x_n) \to x$ as $n \to \infty$. Denote $K_0 := \overline{\{x_n\}}$, where by bar the closure in X is denoted. Then we have

$$\rho(\pi(t_n, x_n), J) \leq \sup_{p \in K_0} \rho(\pi(t_n, p), J), \tag{2}$$

where J is Levinson center of $(X, \mathbb{T}, \pi)$. Passing to limit in (2) we obtain $x \in J$. By the first item the set $\Omega(X)$ is nonempty, compact and positively invariant.

To prove the third item it is sufficient to construct an example with the corresponding properties. To this end we note that in the works [5] and [8] a dynamical system $(X, \mathbb{T}, \pi)$ with the following properties was constructed:

1. $(X, \mathbb{T}, \pi)$ is point dissipative, but it is not compactly dissipative;

2. $\Omega(X)$ is an unbounded set and, consequently, it is not compact.

Lemma is proved. □

Let $(X, \mathbb{T}, \pi)$ be a compact dissipative dynamical system and J be its Levinson center and $M \subseteq X$ be a nonempty, closed and positively invariant subset from X. Denote by $M_1 := \Omega(M)$ the set of all non-wandering (with respect to M) points of $(X, \mathbb{T}, \pi)$. By Lemma 6 the set M_1 is a nonempty, compact and positively invariant subset of J. We denote by $M_2 := \Omega(M_1) \subseteq M_1$ the set of all non-wandering (with respect to M_1) points. By Corollary 1 and Lemma 5 the set M_2 is nonempty, compact and positively invariant. Analogously we define the set $M_3 := \Omega(M_2) \subseteq M_2$ which is also a nonempty, compact and positively invariant set. We can continue this process and we will obtain $M_n := \Omega(M_{n-1})$ for all $n \in \mathbb{N}$. Thus we have a sequence $\{M_n\}_{n\in\mathbb{N}}$ possessing the following properties:

1. for all $n \in \mathbb{N}$ the set M_n is nonempty, compact and positively invariant;

2. $J \supseteq M_1 \supseteq M_2 \supseteq M_3 \supseteq \ldots \supseteq M_n \supseteq M_{n+1} \supseteq \ldots$.

Denote by $M_\lambda := \bigcap\limits_{n=1}^{\infty} M_n$, then M_λ is a nonempty, compact (since the set J is compact) and invariant subset of J. Now we define the set $M_{\lambda+1} := \Omega(M_\lambda)$ and we can continue this process to obtain the following sequence

$$J \supseteq M_1 \supseteq M_2 \supseteq M_3 \supseteq \ldots \supseteq M_n \supseteq$$
$$M_{n+1} \supseteq \ldots \supseteq M_\lambda \supseteq M_{\lambda+1} \supseteq \ldots \supseteq M_{\lambda+k} \supseteq \ldots.$$

Now construct the set $M_\mu := \bigcap\limits_{k=1}^{\infty} M_{\mu+k}$ and we denote by $M_{\mu+1} := \Omega(M_\mu)$ and so on. Thus we will obtain a transfinite sequence of nonempty, compact and positively invariant subsets

$$\begin{gathered} J \supseteq M_1 \supseteq M_2 \supseteq M_3 \supseteq \ldots \supseteq M_n \supseteq \\ M_{n+1} \supseteq \ldots \supseteq M_\lambda \supseteq \ldots \supseteq M_\lambda \supseteq \ldots \supseteq M_\mu \supseteq \ldots. \end{gathered} \tag{3}$$

Since J is a nonempty compact set, then in the sequence (3) there is at most a countable family of different elements, i.e., there exists a γ such that $M_{\nu+1} = M_\nu$.

Definition 4. The set $\mathfrak{B}(M) := M_\nu$ is said to be the center of Birkhoff for the closed and positively invariant set M. If $M = X$, then the set $\mathfrak{B}(\pi) := \mathfrak{B}(X)$ is said to be the Birkhoff center of compact dissipative dynamical system $(X, \mathbb{T}, \pi)$.

Let $(X,\mathbb{T},\pi)$ be a compact dissipative dynamical system and J be its Levinson center. Denote $P(\pi) := \{p \in X : \ p \in \omega_x\}$, then by Lemma 3 we have $P(\pi) \subseteq \mathfrak{B}(\pi) \subseteq J$.

Let K be a nonempty subset of X. Denote by $C(\mathbb{T},K)$ the set of all continuous mappings $f : \mathbb{T} \mapsto K$ equipped with the compact-open topology.

Lemma 7. *Let $(X,\mathbb{T},\pi)$ be a compact dissipative dynamical system and $\mathfrak{B}(\pi)$ be its Birkhoff center. Then the following statements hold:*

1. *$\mathfrak{B}(\pi)$ is a nonempty, compact and invariant set;*
2. *$\mathfrak{B}(\pi)$ is a maximal compact invariant subset M of X such that $\Omega(M) = M$.*

Proof. By Lemma 6 $\mathfrak{B}(\pi)$ is a nonempty, compact and positively invariant set. To finish the proof of the first statement it is sufficient to establish that the set $\mathfrak{B}(\pi)$ is negatively invariant, i.e., $\mathfrak{B}(\pi) \subset \pi(t,\mathfrak{B}(\pi))$ for all $t \in \mathbb{T}$. To this end it is sufficient to show that for all $x \in \mathfrak{B}(\pi)$ the set of all entire trajectories γ_x of $(X,\mathbb{T},\pi)$ passing through the point x at the initial moment with the condition $\gamma_x(\mathbb{S}) \subseteq \mathfrak{B}(\pi)$ is nonempty. Let $x \in \mathfrak{B}(\pi)$. Since $\Omega(\mathfrak{B}(\pi)) = \mathfrak{B}(\pi)$, then there are $\{x_n\} \subseteq \mathfrak{B}(\pi)$ and $\{\tau_n\} \subseteq \mathbb{T}$ such that $x_n \to x$, $\tau_n \to +\infty$ and $\pi(\tau_n, x_n) \to x$. Denote by γ_n the function from $C(\mathbb{S},\mathfrak{B}(\pi))$ defined by the equality $\gamma_n(t) = \pi(t+\tau_n, x_n)$ for all $t \geq -\tau_n$ and $\gamma_n(t) = x_n$ for all $t \leq -\tau_n$. We will show that the sequence $\{\gamma_n\}$ is relatively compact in $C(\mathbb{S},\mathfrak{B}(\pi))$. Let $l > 0$. Since the set $\mathfrak{B}(\pi)$ is compact, then it is sufficient to check that the sequence $\{\gamma_n\}$ is equi-continuous on the interval $[-l,l]$. If we suppose that it is not true then there exist $\varepsilon_0 > 0$, $\delta_n \to$ and $t_n^1, t_n^2 \in [-l,l]$ such that

$$|t_n^1 - t_n^2| < \delta_n \ \text{ and } \ \rho(\gamma_n(t_n^1), \gamma_n(t_n^2)) \geq \varepsilon_0 \tag{4}$$

for all $n \in \mathbb{N}$. Without loss of generality we may consider that the sequence $\{\gamma_n(-l)\}$ is convergent and denote its limit by $\bar{x}$. From inequality (4) we have

$$\varepsilon_0 \leq \rho(\gamma_n(t_n^1), \gamma_n(t_n^1)) = \rho(\pi(l+t_n^1, \gamma_n(-l)), \pi(l+t_n^2, \gamma_n(-l))). \tag{5}$$

Passing to limit in inequality (5) as $n \to \infty$ and taking into consideration (4), we obtain $\varepsilon_0 \leq \rho(\pi(l+\bar{t},\bar{x}), \pi(l+\bar{t},\bar{x})) = 0$, where $\bar{t} := \lim\limits_{n\to\infty} t_n^1 = \lim\limits_{n\to\infty} t_n^2$. The obtained contradiction proves our statement. Thus the sequence $\{\gamma_n\}$ is equi-continuous on $[-l,l]$ and the set $\cup_{n=1}^{\infty}\gamma_n([-l,l]) \subseteq \mathfrak{B}(\pi)$ is relatively compact. Taking into account that l is an arbitrary positive number we conclude that the sequence $\{\gamma_n\}$ is relatively compact in $C(\mathbb{S},\mathfrak{B}(\pi))$. We may suppose that the sequence $\{\gamma_n\}$ is convergent. Denote by $\gamma := \lim\limits_{n\to\infty} \gamma_n$, then $\gamma(0) = x := \lim\limits_{n\to\infty} \pi(\tau_n, x_n)$ and $\gamma \in \Phi_x$ such that $\gamma(\mathbb{S}) \subseteq \mathfrak{B}(\pi) = \Omega(\mathfrak{B}(\pi))$, because by construction $\gamma_n(\mathbb{S}) \subseteq \mathfrak{B}(\pi)$ for all $n \in \mathbb{N}$.

Let now $M \subseteq X$ be an arbitrary nonempty, compact and invariant subset of X with the property $\Omega(M) = M$. Then by construction of $\mathfrak{B}(M)$ we have $\mathfrak{B}(M) = M$. On the other hand $M \subseteq J$, where J is the Levinson center of the compact dissipative dynamical system $(X,\mathbb{T},\pi)$ and, consequently, $\mathfrak{B}(M) \subseteq \mathfrak{B}(X) = \mathfrak{B}(\pi)$. Lemma is completely proved. $\square$

Definition 5. Recall that the mapping $f : X \mapsto X$ is said to be open if for all $p \in X$ and $\delta > 0$ the set $f(B(p, \delta))$ is open.

Let $p \in \mathfrak{B}(\pi)$ and $\varepsilon > 0$. Denote by $\tilde{B}(p, \varepsilon) := B(p, \varepsilon) \bigcap \mathfrak{B}(\pi)$.

Lemma 8. *Let $(X, \mathbb{T}, \pi)$ be a compact dissipative dynamical system and $\mathfrak{B}(\pi)$ be its Birkhoff center. Then the following statements hold:*

1. *for all $p \in \mathfrak{B}(\pi)$, $\varepsilon > 0$ and $t_0 \in \mathbb{T}$ there exists a number $t = t(p, \varepsilon, t_0) > t_0$ such that $\pi(t, \tilde{B}(p, \varepsilon)) \bigcap \tilde{B}(p, \varepsilon) \neq \emptyset$;*

2. *for all $\varepsilon > 0$, $L > 0$ and $p \in \mathfrak{B}(\pi)$ there are $q \in \tilde{B}(p, \varepsilon)$, $\delta = \delta(L, \varepsilon) > 0$ and $t > L$ such that*
$$\tilde{B}(q, \delta) \bigcup \pi(t, \tilde{B}(q, \delta)) \subset \tilde{B}(p, \varepsilon).$$

Proof. Suppose that under the conditions of Lemma the first statement is not true. Then there exist $p_0 \in \mathfrak{B}(\pi)$, $\varepsilon_0 > 0$ and $t_0 \in \mathbb{T}$ such that

$$\pi(t, \tilde{B}(p_0, \varepsilon_0)) \bigcap \tilde{B}(p_0, \varepsilon_0) = \emptyset \tag{6}$$

for all $t \geq t_0$. On the other hand since $p_0 \in \mathfrak{B}(\pi)$, then there exist $\{p_n\} \subseteq \mathfrak{B}(\pi)$ and $t_n \to +\infty$ such that $\pi(t_n, p_n) \to p$ as $n \to \infty$ and, consequently,

$$\pi(t_n, \tilde{B}(p, \varepsilon_0)) \bigcap \tilde{B}(p, \varepsilon_0) \neq \emptyset \tag{7}$$

for all $n \in \mathbb{N}$. Conditions (6) and (7) are contradictory. The obtained contradiction proves our statement.

Now we will establish the second statement. Let $\varepsilon > 0$, $L > 0$ and $p \in \mathfrak{B}(\pi)$. Since $p \in J_p^+(\mathfrak{B}(\pi))$, then there are $q \in \tilde{B}(p, \varepsilon)$ and $t > L$ such that $\pi(t, q) \in \tilde{B}(p, \varepsilon)$. Let μ be a positive number such that $\tilde{B}(\pi(t, q), \mu) \subset \tilde{B}(p, \varepsilon)$. By continuity of the map $\pi(t, \cdot) : \mathfrak{B}(\pi) \mapsto \mathfrak{B}(\pi)$ there exists a positive number $\delta = \delta(t, q, \varepsilon)$ such that $\tilde{B}(q, \delta) \subset \tilde{B}(p, \varepsilon)$ and $\pi(t, \tilde{B}(q, \delta)) \subset \tilde{B}(\pi(t, q), \mu) \subset \tilde{B}(p, \varepsilon)$. $\square$

Lemma 9. *Suppose that $(X, \mathbb{T}, \pi)$ is a dynamical system and the following conditions hold:*

1. *the space X is compact;*

2. *X is an invariant set, i. e., $\pi(t, X) = X$ for all $t \in \mathbb{T}$;*

3. *$\Omega(X) = X$.*

Then for all $x \in X$, $\varepsilon > 0$ and $l > 0$ there exists a number $t > l$ such that

$$\pi^{-t} B(x, \varepsilon) \bigcap B(x, \varepsilon) \neq \emptyset.$$

Proof. Let $x \in X$ and l, ε be two arbitrary positive numbers. Since $x \in J_x^+$, then there are sequences $\{x_n\} \subseteq X$ and $\{t_n\} \subseteq \mathbb{T}$ such that

$$x_n \to x, \ t_n \to +\infty \ \text{ and } \ \pi(t_n, x_n) \to x \tag{8}$$

as $n \to \infty$. For the sufficiently large $n \in \mathbb{N}$ we have

$$t_n > l \ \text{ and } \ x_n, \pi(t_n, x_n) \in B(x, \varepsilon). \tag{9}$$

Let $\gamma_n \in \Phi_{\pi(t_n,x_n)}$ be a full trajectory of $(X, \mathbb{T}, \pi)$ passing through $\pi(t_n, x_n)$ at the initial moment $t = 0$ such that $\gamma_n(s) = \pi(s + t_n, x_n)$ for all $s \geq -t_n$. Then $\gamma_n(-t_n) = x_n \in B(x, \varepsilon)$ and $x_n = \gamma_n(-t_n) \in \pi^{-t_n}(x_n) \subseteq \pi^{-t_n} B(x, \varepsilon)$. Thus we will have

$$x_n \in \pi^{-t_n} B(x, \varepsilon) \bigcap B(x, \varepsilon) \neq \emptyset \tag{10}$$

for all sufficiently large $n \in \mathbb{N}$. □

Corollary 2. *Under the conditions of Lemma 9 for all $x \in X$, $\varepsilon > 0$ and $l > 0$ there exists $t > l$ such that $B(x, \varepsilon) \bigcap \pi^t B(x, \varepsilon) \neq \emptyset$.*

Proof. By Lemma 9 for all $x \in X$, $\varepsilon > 0$ and $l > 0$ there exists $t > l$ such that $\pi^{-t} B(x, \varepsilon) \bigcap B(x, \varepsilon) \neq \emptyset$ and, consequently,

$$\pi^t(\pi^{-t} B(x, \varepsilon) \bigcap B(x, \varepsilon)) \subseteq B(x, \varepsilon) \bigcap \pi^t B(x, \varepsilon) \neq \emptyset.$$

□

Corollary 3. *Suppose that the dynamical system $(X, \mathbb{T}, \pi)$ is compact dissipative and $\mathfrak{B}(\pi)$ is its Birkhoff's center, then for all $x \in \mathfrak{B}(\pi)$, $\varepsilon > 0$ and $l > 0$ there exists a number $t > l$ such that $\pi^{-t} \tilde{B}(x, \varepsilon) \bigcap \tilde{B}(x, \varepsilon) \neq \emptyset$.*

Proof. This statement directly follows from Lemmas 7 and 9. □

Theorem 3. *Suppose that $(X, \mathbb{T}, \pi)$ is a compact dissipative dynamical system, for all $t > 0$ the mapping $\tilde{\pi}(t, \cdot) := \pi(t, \cdot)\big|_{\mathfrak{B}(\pi)}$ is open, then the set of all Poisson stable in the positive direction points of $(X, \mathbb{T}, \pi)$ is dense in $\mathfrak{B}(\pi)$, i. e., $\mathfrak{B}(\pi) = \overline{P(\pi)}$.*

Proof. By Lemma 3 we have $P(\pi) \subseteq \mathfrak{B}(\pi)$ and, consequently, $\overline{P(\pi)} \subseteq \mathfrak{B}(\pi)$. To finish the proof of Theorem it is sufficiently to show that $\overline{P(\pi)} \supseteq \mathfrak{B}(\pi)$.

Let $p \in \mathfrak{B}(\pi)$ and ε be an arbitrary (sufficient small) positive number. Let $\{t_n\}$ be an increasing sequence such that $\tau_n \to +\infty$. By Lemma 8 (item 2) there exists $t_1 > \tau_1$ such that

$$\tilde{B}[x_1, \varepsilon_1] \subseteq \tilde{B}[p, \varepsilon] \ \text{ and } \ \pi(t_1, \tilde{B}[x_1, \varepsilon_1]) \subseteq \tilde{B}[p, \varepsilon].$$

Since the mapping $\pi(t_1, \cdot)$ is open, then we can choose $x_1 \in \mathfrak{B}(\pi)$ and $\varepsilon_1 > 0$ such that

$$\tilde{B}[x_1, \varepsilon_1] \subset \pi(t_1, \tilde{B}[p, \varepsilon]) \subseteq \tilde{B}[p, \varepsilon].$$

By Lemma 8 there is $t_2 > \tau_2$ such that we will have

$$\tilde{B}[x_2, \varepsilon_2] \subseteq \tilde{B}[x_1, \varepsilon_1] \quad \text{and} \quad \pi(t_2, \tilde{B}[x_2, \varepsilon_2]) \subseteq \tilde{B}[x_1, \varepsilon_1].$$

Since the mapping $\pi(t_2, \cdot)$ is open we can again choose $x_2 \in \mathfrak{B}(\pi)$ and $0 < \varepsilon_2 < \varepsilon_1/2$ such that

$$\tilde{B}[x_3, \varepsilon_3] \subseteq \tilde{B}[x_2, \varepsilon_2] \quad \text{and} \quad \pi(t_3, \tilde{B}[x_3, \varepsilon_3]) \subseteq \tilde{B}[x_2, \varepsilon_2].$$

Reasoning analogously we can construct sequences $\{x_n\} \subseteq \mathfrak{B}(\pi)$ and $\{\varepsilon_n\}$ such that $\varepsilon_n < \varepsilon_{n-1}/2$, $\tilde{B}[x_n, \varepsilon_n] \subset \tilde{B}[x_{n-1}, \varepsilon_{n-1}]$ and $\pi(t_n, \tilde{B}[x_n, \varepsilon_n]) \subseteq \tilde{B}[x_{n-1}, \varepsilon_{n-1}]$ for all $n \in \mathbb{N}$, where $\varepsilon_0 := \varepsilon$ and $x_0 := p$. Since $\mathfrak{B}(\pi)$ is a nonempty compact set, then $\Lambda := \bigcap\limits_{n=0}^{\infty} \tilde{B}(x_n, \varepsilon_n) \neq \emptyset$ and it consists of a unique point. Let $\{x\} = \Lambda$. We will show that the point x is Poisson stable in the positive direction. In fact, if $L > 0$ is a sufficiently large number and $\delta > 0$, respectively, sufficiently small number, then we choose a natural number $m \in \mathbb{N}$ with the condition that $t_m > L$ and $\varepsilon_m < \delta$, then $\pi(t_n, \tilde{B}[x_n, \varepsilon_n]) \subseteq \tilde{B}[x_m, \varepsilon_m] \subseteq \tilde{B}[x, \delta]$ for all $n > m$. In particular $\pi(t_n, x) \in \tilde{B}[x, \delta]$ for all $n > m$, i.e., $x \in \omega_x$. Thus $x \in \tilde{B}(p, \varepsilon)$ and, consequently, $\mathfrak{B}(\pi) \subseteq \overline{P(\pi)}$. Theorem is proved. □

Remark 4. 1. Note that the mappings $\tilde{\pi}(t, \cdot)$ $(t \in \mathbb{T})$ are open, if on $\mathfrak{B}(\pi)$ the dynamical system $(X, \mathbb{T}, \pi)$ is invertible, i.e., for all $t \in \mathbb{T}$ the mapping $\tilde{\pi}(t, \cdot) : \mathfrak{B}(\pi) \mapsto \mathfrak{B}(\pi)$ is a homeomorphism.

2. If the dynamical system $(X, \mathbb{T}, \pi)$ is invertible on $\mathfrak{B}(\pi)$, then by Theorem 1.14 [14, Ch.III] (see also Proposal 1.1 from [1], where the analogue of Theorem 1.4 for the discrete dynamical systems was proved) in the set $\mathfrak{B}(\pi)$ the set of all Poisson stable (both in the positive and negative directions) points from X is dense.

Let $(X, \mathbb{T}, \pi)$ be a compact dissipative dynamical system. Recall that a compact set $M \subseteq X$ is called a weak attractor of the dynamical system $(X, \mathbb{T}, \pi)$ if $\omega_x \cap M \neq \emptyset$ for all $x \in X$. In this section we establish the relationship between weak attractors of the dynamical system $(X, \mathbb{T}, \pi)$ and its Levinson center.

Theorem 4 (see [4, Ch.I])**.** *Let $(X, \mathbb{T}, \pi)$ be compactly dissipative, J be its Levinson center and M be a compact weak attractor of the dynamical system $(X, \mathbb{T}, \pi)$. Then $J = J^+(M)$.*

Denote by $J_x^+ := \{p \in X :$ there exist the sequences $x_n \to x$ and $t_n \to +\infty$ such that $\pi(t_n, x_n) \to p$ as $n \to \infty\}$ and $J^+(M) := \bigcup\{J_x^+ : \ x \in M\}$.

Lemma 10. *Let $M \subseteq X$ be a nonempty, compact, positively invariant and minimal subset of X. Then the following statements hold:*

1. *the set M is invariant, i.e., $\pi(t, M) = M$ for all $t \in \mathbb{T}$;*

2. *for every $x \in M$ each full trajectory $\gamma \in \Phi_x$ is Poisson stable, i.e., $x \in \omega_x = \alpha_\gamma$.*

Proof. Let $t_0 \in \mathbb{T}$ and $M' := \pi(t_0, M)$, then $M' \subseteq M$ and $\pi(t, M') = \pi(t + t_0, M) \subseteq M$. Since M is a nonempty, compact and positively invariant set, then the set M' is so. Taking into consideration that M is a minimal set we conclude that $M = \pi(t_0, M)$ for all $t_0 \in \mathbb{T}$ and, consequently, it is invariant.

Let now $x \in M$ be an arbitrary point from M, then ω_x is a nonempty, compact and positively invariant subset of M. Since the set M is minimal, then we have $\omega_x = M$. Let now $\gamma \in \Phi_x$ be an arbitrary full trajectory of $(X, \mathbb{T}, \pi)$ with the properties: $\gamma(0) = x$ and $\gamma(\mathbb{S}) \subseteq M$, then its α-limit set $\alpha_\gamma \subseteq M$ is a nonempty and compact subset of $\omega_x = M$. If $p \in \alpha_\gamma$, then there exists a sequence $s_n \to -\infty$ such that $p = \lim\limits_{n\to\infty} \gamma(s_n)$. For all $t \in \mathbb{T}$ the sequence $\{\gamma(t+s_n)\} \subseteq M$ is relatively compact and, consequently, without loss of generality, we may suppose that $\{\gamma(t + s_n)\}$ converges. Denote by p_t its limit, i.e., $p_t := \lim\limits_{n\to\infty} \gamma(t + s_n)$. Note that

$$\pi(t, p) = \lim_{n\to\infty} \pi(t, \gamma(s_n)) = \lim_{n\to\infty} \gamma(t + s_n) \in \alpha_\gamma \subseteq M$$

for all $t \in \mathbb{T}$ and, consequently, ω_p is a nonempty, compact, positively invariant subset of M. On the other hand we have $\omega_p \subseteq \alpha_\gamma \subseteq M$. Since the set M is minimal, then we obtain $M = \omega_p \subseteq \alpha_\gamma \subseteq M$ and, consequently, $\alpha_\gamma = M$. Thus we have $x \in \omega_x = \alpha_\gamma = M$. Lemma is completely proved. □

Theorem 5. *Let $(X, \mathbb{T}, \pi)$ be a compact dissipative dynamical system, J be its Levinson center and $\mathfrak{B}(\pi)$ be the Birkhoff center of $(X, \mathbb{T}, \pi)$. Then the following equality takes place: $J = J^+(\mathfrak{B}(\pi))$.*

Proof. By Lemmas 3 and 6 we have $\overline{\mathcal{P}(\pi)} \subseteq \mathfrak{B}(\pi) \subseteq J$ and $\overline{\mathcal{P}(\pi)}$ is a nonempty and compact subset of J. It is not difficult to show that the set $\mathcal{P}(\pi)$ is a weak attractor for $(X, \mathbb{T}, \pi)$. In fact, let $x \in X$ be an arbitrary point of X. Since the dynamical system $(X, \mathbb{T}, \pi)$ is compact dissipative, then the ω-limit set ω_x of the point x is a nonempty, compact and positively invariant subset of X. By theorem of Birkhoff in ω_x there exists a nonempty, compact, positively invariant and minimal subset $M \subseteq \omega_x$. By Lemma 10 every point p from M is Poisson stable and, consequently, $M \subseteq \mathcal{P}(\pi) \subseteq \overline{\mathcal{P}(\pi)} \subseteq \mathfrak{B}(\pi)$. Thus we have $M \subseteq \omega_x \bigcap \mathfrak{B}(\pi)$ for each $x \in X$, i. e., $\mathfrak{B}(\pi)$ is a weak attractor of $(X, \mathbb{T}, \pi)$. Now to finish the proof of Theorem it is sufficient to apply Theorem 4. □

3 Chain recurrent motions

Let $\Sigma \subseteq X$ be a compact positively invariant set, $\varepsilon > 0$ and $t > 0$.

Definition 6. The collection $\{x = x_0, x_1, x_2, \dots, x_k = y; t_0, t_1, \dots, t_k\}$ of the points $x_i \in \Sigma$ and the numbers $t_i \in \mathbb{T}$ such that $t_i \geq t$ and $\rho(x_i t_i, x_{i+1}) < \varepsilon$ ($i = 0, 1, \dots, k-1$) is called (see, for example, [2,3,6,7,12] and the bibliography therein) a (ε, t, π)-chain joining the points x and y.

Remark 5. Without loss of generality we can suppose always that $t_i \leq 2t$, where t_i and t the numbers figuring in Definition 6 (see, for example, [2, Ch.I]).

We denote by $P(\Sigma)$ the set $\{(x,y) : x,y \in \Sigma, \forall\ \varepsilon > 0\ \forall\ t > 0\ \exists\ (\varepsilon,t,\pi)$-chain joining x and $y\}$. The relation $P(\Sigma)$ is closed, invariant and transitive [2, 6, 10–12].

Definition 7. The point $x \in \Sigma$ is called chain recurrent (in Σ) if $(x,x) \in P(\Sigma)$.

We denote by $\mathfrak{R}(\Sigma)$ the set of all chain recurrent (in Σ) points from Σ.

Remark 6. Note that if Σ_1 and Σ_2 are two positively invariant subsets of $(X,\mathbb{T},\pi)$ with condition $\Sigma_1 \subseteq \Sigma_2$, then $\mathfrak{R}(\Sigma_1) \subseteq \mathfrak{R}(\Sigma_2)$.

Definition 8. Let $A \subseteq X$ be a nonempty positively invariant set. The set A is called (see, for example, [9]) internally chain recurrent if $\mathfrak{R}(A) = A$, and internally chain transitive if the following stronger condition holds: for any $a, b \in A$ and any $\varepsilon > 0$ and $t > 0$, there is an (ε,t,π)-chain in A connecting a and b.

The set of all chain recurrent points for $(X,\mathbb{T},\pi)$ is denoted by $\mathfrak{R}(\Sigma)$, i. e., $\mathfrak{R}(\Sigma) := \{x \in \Sigma : (x,x) \in P(\Sigma)\}$. On $\mathfrak{R}(\Sigma)$ we will introduce a relation $\sim$ as follows: $x \sim y$ if and only if $(x,y) \in P(\Sigma)$ and $(y,x) \in P(\Sigma)$. It is easy to check that the introduced relation $\sim$ on $\mathfrak{R}(\Sigma)$ is a relation of equivalence and, consequently, it is easy to decompose it into the classes of equivalence $\{\mathfrak{R}_\lambda : \lambda \in \mathcal{L}\}$ (internally chain transitive subsets), i. e., $\mathfrak{R}(\Sigma) = \sqcup\{\mathfrak{R}_\lambda :\ \lambda \in \mathcal{L}\}$. By Proposal 2.6 from [2] (see also [6] and [10–12] for the semi-group dynamical systems) the defined above components of the decomposition of the set $\mathfrak{R}(\Sigma)$ are closed and positively invariant.

Lemma 11 (see [9])**.** *Let $x \in X$ and $\gamma \in \Phi_x$. The ω-limit (respectively, α-limit) set of positive (respectively, negative) pre-compact orbit of the point x is internally chain transitive, i. e., $\mathfrak{R}(\omega_x) = \omega_x$ (respectively, $\mathfrak{R}(\alpha_\gamma) = \alpha_\gamma$).*

Let $(X,\mathbb{T},\pi)$ be a compact dissipative dynamical system and J be its Levinson center. Denote by $\mathfrak{R}(\pi) := \mathfrak{R}(J)$.

Problem. Suppose that $(X,\mathbb{T},\pi)$ is a compact dissipative dynamical system and J is its Levinson center. To prove that $\mathfrak{R}(\pi) = \mathfrak{R}(X)$ or to construct a corresponding counterexample.

Remark 7. In the connection with the Problem formulated above it is interesting to note that in the works [5, 8] an example of dynamical system $(X,\mathbb{T},\pi)$ is constructed which posses the following properties:

1. $(X,\mathbb{T},\pi)$ is point dissipative;
2. $(X,\mathbb{T},\pi)$ is asymptotically compact;
3. $(X,\mathbb{T},\pi)$ is not compact dissipative;
4. $\mathfrak{R}(X)$ is an unbounded subset of X.

Denote by $C(\mathbb{T}\times X, X)$ the set of all continuous functions $\pi : \mathbb{T}\times X \mapsto X$ equipped with the compact-open topology. If $K \subset X$ is a compact subset from X, then we denote by

$$d_K(f,g) := \sup_{L>0} \min\{\sup_{0\le t\le L,\ x\in K} \rho(f(t,x),g(t,x)), L^{-1}\} \tag{11}$$

and $\mathcal{D} := \{d_K : K \in C(X)\}$ a family of pseudo-metrics which generates the compact-open topology on $C(\mathbb{T} \times X, X)$, where $C(X)$ is the family of all compact subsets from X.

Remark 8. Note that for all $\varepsilon > 0$ the inequality $d_K(f,g) < \varepsilon$ is equivalent to $\sup\limits_{0\le t\le \varepsilon^{-1},\ x\in K} \rho(f(t,x), g(t,x)) < \varepsilon$ (see, for example,[13, Ch.I] or [14, Ch.IV]).

Definition 9. Recall [2, Ch.I] that the collection $[x_1, x_2, \dots, x_k := y; t_1, t_2, \dots, t_{k-1}]$ is called a generalized chain joining x and y if the following conditions are fulfilled:

1. $t_i \ge t$;
2. $\rho(x, x_1) < \varepsilon$;
3. $\rho(\pi(t_i, x_i), x_{i+1}) < \varepsilon$ $(1 = 1, \dots, k-1)$.

Remark 9. In the book [2, Ch.I] it is shown that in the definition of chain recurrence the (ε, t, f)-chains can be replaced by generalized (ε, t, f)-chains.

Theorem 6. *Suppose that the following conditions hold:*

1. *$\mathcal{M} \subset C(\mathbb{T} \times X, X)$ is a compact subset from $C(\mathbb{T} \times X, X)$;*
2. *for all $\pi \in \mathcal{M}$ the dynamical system $(X, \mathbb{T}, \pi)$ is compact dissipative and J_π is its Levinson center;*
3. *the set $J := \overline{\bigcup\{J_\pi : \pi \in \mathcal{M}\}}$ is compact.*

Then the mapping $F : \mathcal{M} \mapsto 2^J$ defined by equality $F(\pi) := \mathfrak{R}(\pi)$ is upper semi-continuous, where by 2^J the space of all compact subsets from J equipped with the Hausdorff metric is denoted.

Proof. Let $\pi_n, \pi \in \mathcal{M}$ and $d_J(\pi_n, \pi) \to 0$, $a_n \in \mathfrak{R}(\pi_n)$ and $a_n \to a$ as $n \to \infty$. We need to show that $a \in \mathfrak{R}(\pi)$. Let ε be an arbitrary positive number and $0 < \delta < \varepsilon/4$. There exists a number $n_0 \in \mathbb{N}$ such that $\rho(a_n, a) < \delta$ and $d_J(\pi_n, \pi) < \delta$ for all $n \ge n_0$. Since $a_n \in \mathfrak{R}(\pi_n)$, then there is a $(\delta, \varepsilon^{-1}, \pi_n)$-chain from a_n to a_n, i.e., there exists a collection $\{x_0 = a_n, x_1, \dots, x_{k-1}, x_k = a_n; t_0, \dots, t_{k-1}\}$ such that

$$\rho(\pi_n(t_i, x_i), x_{i+1}) < \delta,\ \varepsilon^{-1} \le t_i \le 2\varepsilon^{-1}\ (i = 0, 1, \dots, k-1).$$

Thus the collection $[x_0, x_1, \dots, x_{k-1}, a; t_0, t_1, \dots, t_{k-1}]$ is a generalized $(2\delta, \varepsilon^{-1}, \pi_n)$-chain joining a with a. From the inequality $d_J(\pi_n, \pi) < \delta$ it follows that

$$\rho(\pi_n(t, x), \pi(t, x)) < \delta\ (x \in J,\ 0 \le t \le \delta^{-1} < 4\varepsilon^{-1})$$

and, consequently, the above indicated generalized $(2\delta, \varepsilon^{-1}, \pi_n)$-chain is also a generalized $(\varepsilon, \varepsilon^{-1}, \pi)$ chain from a to a. Since ε is an arbitrary positive number, then $a \in \mathfrak{R}(\pi)$. $\square$

Lemma 12. *Suppose that $(X, \mathbb{T}, \pi)$ is compact dissipative and J if its Levinson center, then $\omega_x \subseteq \mathfrak{R}(J) = \mathfrak{R}(\pi)$ for all $x \in X$.*

Proof. Let $x \in X$ be an arbitrary point. Since $(X, \mathbb{T}, \pi)$ is compact dissipative, then ω_x is a nonempty, compact, and invariant subset of J, then $\mathfrak{R}(\omega_x) \subseteq \mathfrak{R}(J) = \mathfrak{R}(\pi)$. By Lemma 11 we have $\omega_x = \mathfrak{R}(\omega_x)$ and, consequently, $\omega_x \subseteq \mathfrak{R}(\pi)$. ☐

Lemma 13 (see [4, Ch.IV])**.** *If the compact invariant set Σ from X contains only a finite number of minimal sets, then the relation $\sim$ decomposes the set $\mathfrak{R}(\Sigma)$ into the finite number of different classes of equivalence (internally chain transitive sets).*

Remark 10. 1. Lemma 13 was established in [4, Ch.IV] for the two-sided (group) dynamical systems.

2. For the one-sided (semi-group) dynamical systems this statement may be proved by slight modifications of the arguments from [4, Ch.IV].

3. For two-sided dynamical systems ($\mathbb{T} = \mathbb{S}$) with infinite number of compact minimal subsets Lemma 13 remains true if in addition the dynamical system $(X, \mathbb{S}, \pi)$ satisfies some condition of hyperbolicity (see Theorem 4.1 [4, Ch.IV]).

Lemma 14 (see [9])**.** *Let M be an isolated (local maximal) invariant set and $\mathfrak{R}$ be a compact internally chain transitive set for $(X, \mathbb{T}, \pi)$. Assume that $M \bigcap \mathfrak{R} \neq \emptyset$ and $M \subseteq \mathfrak{R}$.*

Then

a. there exists a point $u \in \mathfrak{R} \setminus M$ such that $\omega_u \subseteq M$;

b. there exists a point $w \in \mathfrak{R} \setminus M$ and an entire trajectory $\gamma \in \Phi_w$ such that $\alpha_\gamma \subseteq M$.

Theorem 7. *Assume that the following conditions hold:*

1. the dynamical system $(X, \mathbb{T}, \pi)$ is compactly dissipative and J is its Levinson center;

2. there exists a finite number n of compact minimal subsets $M_i \subseteq J$ $(i = 1, 2, \dots, k)$ of $(X, \mathbb{T}, \pi)$;

3. the collection of subsets $\{M_1, M_2, \dots, n\}$ does not admit k-cycles;

4. for all $x \in X$ there exists a number $i \in \{1, 2, \dots, n\}$ such that $\omega_x = M_i$.

Then any compact internally chain transitive set $\mathfrak{R}_\lambda(\pi)$ is a minimal set of $(X, \mathbb{T}, \pi)$, i. e., there exists $i \in \{1, 2, \dots, n\}$ such that $\mathfrak{R}_\lambda = M_i$.

Proof. Let $\mathfrak{R}_\lambda(\pi)$ be a compact internally chain transitive set for $(X, \mathbb{T}, \pi)$. Since $\mathfrak{R}_\lambda(\pi)$ is a compact positively invariant set, then by Birkhoff's theorem in $\mathfrak{R}_\lambda(\pi)$ there exists a nonempty compact minimal set $M_i \subseteq \mathfrak{R}_\lambda(\pi)$ $(i_1 \in \{1, 2, \dots, n\})$. We will show that $\mathfrak{R}_\lambda(\pi) = M_{i_1}$. If we suppose that it is not true, then by Lemma 14 there exists a point $x_1 \in \mathfrak{R}_\lambda(\pi) \setminus M_{i_1}$ and an entire trajectory $\gamma_1 \in \Phi_{x_1}$ such that

$\alpha_{\gamma_1} \subseteq M_{i_1}$. By conditions of Theorem there exists a number $i_2 \in \{1, 2, \dots, n\}$ such that $\omega_{x_1} = M_{i_2}$. Since $M_{i_2} \subseteq \mathfrak{R}_\lambda(\pi)$ and $\mathfrak{R}_\lambda(\pi) \neq M_{i_2}$ then by Lemma 14 there exists a point $x_2 \in \mathfrak{R}_\lambda(\pi) \setminus M_{i_2}$ and an entire trajectory $\gamma_2 \in \Phi_{x_2}$ such that $\alpha_{\gamma_2} = M_{i_2}$ and there exists a number $i_3 \in \{1, 2, \dots, n\}$ such that $\omega_{x_2} = M_{i_3}$. Since there is only a finite number of M_i's, we will eventually arrive at a cyclic chain of some minimal sets of $(X, \mathbb{T}, \pi)$, which contradicts our assumption. $\square$

Corollary 4. *Under the conditions of Theorem 7 we have* $\mathfrak{R}(\pi) = \coprod_{i=1}^{n} M_i$.

Theorem 8. *Suppose that* $(X, \mathbb{T}, \pi)$ *is a bounded dissipative dynamical system and* J *is its Levinson center. Then for every* $\delta > 0$ *and* $B \in \mathcal{B}(X)$ *there exists* $L = L(\delta, B) > 0$ *(*$L \in \mathbb{T}$*) such that*

$$\pi([0, L], x) \bigcap B(\mathfrak{R}(J), \delta) \neq \emptyset \textit{ for all } x \in B,$$

i. e., for all $x \in B$ *there exists* $l = l(x) \in [0, L]$ *such that*

$$\pi(l, x) \in B(\mathfrak{R}(J), \delta).$$

Proof. If we suppose that the statement of Theorem is not true, then there are $\delta_0 > 0$, $B_0 \in \mathcal{B}(X)$, $L_n \geq n$ and $x_n \in B_0$ such that

$$\rho(\pi(t, x_n), \mathfrak{R}(J)) \geq \delta_0 \tag{12}$$

for all $t \in [0, L_n]$. Let $s_n := [L_n/2]$ and $\tilde{x}_n := \pi(s_n, x_n)$. Note that

$$\rho(\tilde{x}_n, J) = \rho(\pi(s_n, x_n), J) \leq \beta(\pi(s_n, B_0), J) \to 0 \tag{13}$$

as $n \to \infty$, because $s_n \to \infty$ and J attracts the bounded subset B_0 as $t \to +\infty$. From (13) it follows that the sequence $\{\tilde{x}_n\}$ is relatively compact. Thus, without loss of generality we can suppose that the sequence $\{\tilde{x}_n\}$ is convergent. Denote $\tilde{x} = \lim\limits_{n\to\infty} \tilde{x}_n$, then by (13) we obtain $\tilde{x} \in J$. On the other hand by (12) we obtain

$$\rho(\pi(t, \tilde{x}_n), \mathfrak{R}(J)) = \rho(\pi(t + s_n, x_n), \mathfrak{R}(J)) \geq \delta_0 \tag{14}$$

for all $t \in [-s_n, s_n]$. Let $\gamma \in \mathcal{F}_{\tilde{x}}$ be the full trajectory of dynamical system $(X, \mathbb{T}, \pi)$ passing through $\{x\}$ at the initial moment $t = 0$ and defined by equality $\gamma(t) = \lim\limits_{n\to\infty} \pi(t + s_n, x_n)$ for all $t \in \mathbb{S}$. Note that $\gamma(\mathbb{S}) \subseteq J$ because for every $t \in \mathbb{S}$ we have

$$\rho(\pi(t + s_n, x_n), J) \leq \rho(\pi(t + s_n, B_0), J) \tag{15}$$

for sufficiently large n, and passing to limit in (15) as $n \to \infty$ we obtain $\gamma(t) \in J$ for all $t \in \mathbb{S}$. By Lemma 12 we have $\omega_{\tilde{x}} \subseteq \mathfrak{R}(J)$. But from (14) it follows that $\gamma(t) \notin \mathfrak{R}(J)$ for all $t \in \mathbb{S}$ and, consequently, $\omega_{\tilde{x}} \bigcap \mathfrak{R}(J) = \emptyset$. The obtained contradiction proves our statement. Theorem is proved. $\square$

Corollary 5. *Suppose that the following conditions hold:*

1. *$(X, \mathbb{T}, \pi)$ is a bounded dissipative dynamical system and J its Levinson center;*

2. *$(X, \mathbb{T}, \pi)$ is a gradient system;*

3. *$Fix(\pi) = \{p_1, p_2, \ldots, p_m\}$;*

4. *$Fix(\pi)$ does not contain any k-cycle ($k \ge 1$).*

Then for every $\delta > 0$ and $B \in \mathcal{B}(X)$ there exists $L = L(\delta, B) > 0$ ($L \in \mathbb{T}$) such that

$$\pi([0, L], B) \bigcap B(Fix(\pi), \delta) \ne \emptyset,$$

i. e., for all $x \in B$ there exists $l = l(x) \in [0, L]$ such that

$$\pi(l, x) \in B(Fix(\pi), \delta).$$

Proof. This statement follows from Theorems 7 and 8. □

Theorem 9. *Suppose that the following conditions are fulfilled:*

1. *the dynamical system $(X, \mathbb{T}, \pi)$ admits a compact global attractor J which attracts every bounded subset $B \in \mathcal{B}(X)$;*

2. *$\mathfrak{R}(J)$ consists of finite number of different classes of equivalence $\mathfrak{R}_1$ $\mathfrak{R}_2$, ..., $\mathfrak{R}_k$.*

Then for every $\tilde{\delta} > 0$ there exists $\delta \in (0, \tilde{\delta})$ such that for every $x \in B(\mathfrak{R}_i, \delta)$ ($i = \overline{1, k}$) with $\pi(t, x) \in B(\mathfrak{R}_i, \delta)$ for all $t \in [0, T)$ and $\pi(T, x) \notin B(\mathfrak{R}_i, \delta)$ we have $\pi(t, x) \notin B(\mathfrak{R}_i, \delta)$ for each $t \ge T$ (i. e., never returns again in $B(\mathfrak{R}_i, \delta)$ for all $t \ge T$).

Proof. By Lemma 4.3 [4, Ch.IV] in the collection $\{\mathfrak{R}_1, \mathfrak{R}_2, \ldots, \mathfrak{R}_k\}$ there is no r-cycles ($r \ge 1$). We will show that if we suppose that the statement of Theorem is not true, then we will have a contradiction this the fact formulated above. In fact. Suppose that Theorem is wrong, then there are $\mathfrak{R}_{i_0}$, $B(\mathfrak{R}_{i_0}, \delta_0)$ ($\delta_0 > 0$), $T_n \in \mathbb{T}$, $T'_n > T_n$ and a sequence $\{x_n\} \subset B(\mathfrak{R}_{i_0}, \delta_0)$ such that

$$\pi(T_n, x_n) \notin B(\mathfrak{R}_{i_0}, \delta_0) \text{ and } \pi(T'_n, x_n) \in B(\mathfrak{R}_{i_0}, 1/n).$$

Without loss of generality we can suppose that $\pi(t, x_n) \in B(\mathfrak{R}_{i_0}, \delta_0)$ for all $t \in [0, T_n)$.

Note that $T_n \to \infty$ as $n \to \infty$. If we suppose that it is not so, then we can consider that $\{T_n\}$ is bounded (otherwise we can extract a subsequence $\{T_{k_n}\}$ which converges to $+\infty$ as n goes to ∞), i. e., there exists a number $L > 0$ such that

$$\pi(t, x_n) \notin B(\mathfrak{R}_{i_0}, \delta_0) \tag{16}$$

for all $t \geq L$ and $n \in \mathbb{N}$. Since $x_n \in B(\mathfrak{R}_{i_0}, 1/n)$, then without loss of generality we can suppose that $\{x_n\}$ is convergent. Denote by $p := \lim\limits_{n\to\infty} x_n$, then $p \in \mathfrak{R}_{i_0}$ and passing into limit in (16) as $n \to \infty$ we obtain

$$\pi(t,p) \notin B(\mathfrak{R}_{i_0}, \delta_0) \tag{17}$$

for all $t \geq L$. On the other hand

$$\pi(t,p) \in \mathfrak{R}_{i_0} \tag{18}$$

for all $t \geq 0$ because the set $\mathfrak{R}_{i_0}$ is invariant. Relations (17) and (18) are contradictory. The obtained contradiction proves our statement.

Denote by $\tilde{x}_n := \pi(T_n, x_n)$, then we have

1. $\tilde{x}_n \notin B(\mathfrak{R}_{i_0}, \delta_0)$ for all $n \in \mathbb{N}$;
2. $\pi(t, \tilde{x}_n) = \pi(t + T_n, x_n) \in B(\mathfrak{R}_{i_0}, \delta_0)$ for all $-T_n \leq t < 0$;
3. $\pi(\tilde{T}'_n, \tilde{x}_n) \in B(\mathfrak{R}_{i_0}, 1/n)$ for all $n \in \mathbb{N}$, where $\tilde{T}'_n := T'_n - T_n > 0$.

Since $x_n \in B(\mathfrak{R}_{i_0}, 1/n)$, $T_n \to +\infty$ and $(X, \mathbb{T}, \pi)$ is compactly dissipative, then the sequence $\{\tilde{x}_n\}$ is relatively compact and without loss of generality we can suppose that it is convergent. Denote by $\tilde{x} := \lim\limits_{n\to\infty} \tilde{x}_n$ and consider $\gamma \in \Phi_{\tilde{x}}$, where $\gamma(t) := \lim\limits_{n\to\infty} \pi(t + T_n, x_n)$ for all $t \in \mathbb{S}$.

Note that $\tilde{T}'_n \to +\infty$ as $n \to \infty$. In fact, if we suppose that it is not true, then without loss of generality we can consider that $\{\tilde{T}'_n\}$ is bounded, for example, $\tilde{T}'_n \in [0, L]$ for all $n \in \mathbb{N}$, where L is some positive number. Let $l := \lim\limits_{n\to\infty} \tilde{T}'_n$, then $l \in [0, L]$ (if it is necessary we can extract a convergent subsequence from $\{\tilde{T}'_n\}$). Then from (iii) we obtain $\pi(l, \tilde{x}) \in \mathfrak{R}_{i_0}$ and, consequently, $\tilde{x} \in \mathfrak{R}_{i_0}$ because $\mathfrak{R}_{i_0}$ is invariant. The obtained contradiction proves our statement.

We will show that $\gamma(t) \in J$ for all $t \in \mathbb{S}$. In fact

$$\rho(\pi(t + T_n, x_n), J) \leq \beta(\pi(t + T_n, K), J) \to 0$$

as $n \to \infty$, where $K := \overline{\{x_n\}}$ and by bar the closure in the space X is denoted. Now we note that $\gamma(t) \in B(\mathfrak{R}_{i_0}, \delta_0)$ for all $t < 0$. Since the set $\mathfrak{R}_{i_0}$ is local maximal, then without loss of generality we can suppose that in $B(\mathfrak{R}_{i_0}, \delta_0)$ the invariant set $\mathfrak{R}_{i_0}$ is maximal and, consequently, $\alpha_\gamma \subseteq \mathfrak{R}_{i_0}$. On the other hand $\omega_{\tilde{x}} \subseteq \mathfrak{R}(J)$ and, consequently, there exists a number $i_1 \in \{1, 2, \ldots, k\}$ such that $\omega_{\tilde{x}} \subseteq \mathfrak{R}_{i_1}$. Since the collection $\{\mathfrak{R}_1, \mathfrak{R}_2, \ldots, \mathfrak{R}_k\}$ has not r-cycles ($r \geq 1$), then $i_1 \neq i_0$.

Since $\tilde{x}_n \to \tilde{x}$ as $n \to \infty$ and $\omega_{\tilde{x}} \subseteq \mathfrak{R}_{i_1}$, then by integral continuity for all $n \in \mathbb{N}$ there exists a number $T^1_n > 0$ such that $\pi(T^1_n, \tilde{x}_n) \in B(\mathfrak{R}_{i_1}, 1/n)$. Taking into account that $\tilde{T}'_n \to +\infty$ as $n \to \infty$ and Theorem 8 we can consider that $T^1_n \leq \tilde{T}'_n$. On the other hand by Theorem 8 for all $n \in \mathbb{N}$ there exists $T^2_n \in (T^1_n, \tilde{T}'_n)$ such that $\pi(T^2_n, \tilde{x}_n) \notin B(\mathfrak{R}_{i_1}, \delta_0)$. Repeating the reasoning above for the set $\mathfrak{R}_{i_1}$ and the sequence $\{\tilde{x}_n\}$ we can find a full trajectory γ_1 so that $\alpha_{\gamma_1} \subseteq \mathfrak{R}_{i_1}$ and $\omega_{\tilde{x}_1} \subseteq \mathfrak{R}_{i_2}$, where $i_2 \neq i_0, i_1$ and $\tilde{x}_1 := \gamma_1(0)$.

Reasoning analogously we will construct a sequence $\{\gamma, \gamma_1, \dots, \gamma_p\}$ $(p \leq k-1)$ so that $\alpha_{\gamma_p} \subseteq \mathfrak{R}_{i_p}$ and $\omega_{\tilde{x}_p} \subseteq \mathfrak{R}_{i_{p+1}}$ $(\gamma_0 := \gamma)$. Since the family $\{\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_k\}$ contains a finite number of sets $\mathfrak{R}_p$, then after the finite number $q \leq k$ of steps we will have $\mathfrak{R}_{i_p} = R_{i_0}$, i.e., we will obtain a q-cycle. The obtained contradiction proves our Theorem. □

Corollary 6. *Suppose that the following conditions hold:*

1. $(X, \mathbb{T}, \pi)$ *is a bounded dissipative dynamical system and* J *its Levinson center;*
2. $(X, \mathbb{T}, \pi)$ *is a gradient system;*
3. $Fix(\pi) = \{p_1, p_2, \dots, p_m\}$*;*
4. $Fix(\pi)$ *does not contain any* k*-cycle* $(k \geq 1)$*.*

Then for every $\tilde{\delta} > 0$ *there exists* $\delta \in (0, \tilde{\delta})$ *such that for every* $x \in B(\mathfrak{R}_i, \delta)$ $(i = \overline{1,k})$ *with* $\pi(t,x) \in B(\mathfrak{R}_i, \delta)$ *for all* $t \in [0, T)$ *and* $\pi(T,x) \notin B(\mathfrak{R}_i, \delta)$ *we have* $\pi(t,x) \notin B(\mathfrak{R}_i, \delta)$ *for each* $t \geq T$ *(i. e., never returns again in* $B(\mathfrak{R}_i, \delta)$ *for all* $t \geq T$*).*

Proof. This statement follows from Theorems 8 and 9. □

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Free rectangular tribands

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Abstract. We introduce the notion of a rectangular triband, construct a free rectangular triband and describe its structure.

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Keywords and phrases: Trioid, rectangular triband, free rectangular triband, triband of subtrioids, dimonoid, semigroup.

1 Introduction

Recall that a vector space (set) T equipped with three binary associative operations $\dashv$, $\vdash$ and $\perp$ that satisfy the following axioms: $(x \dashv y) \dashv z = x \dashv (y \vdash z)\,(T1)$, $(x \vdash y) \dashv z = x \vdash (y \dashv z)\,(T2)$, $(x \dashv y) \vdash z = x \vdash (y \vdash z)\,(T3)$, $(x \dashv y) \dashv z = x \dashv (y \perp z)\,(T4)$, $(x \perp y) \dashv z = x \perp (y \dashv z)\,(T5)$, $(x \dashv y) \perp z = x \perp (y \vdash z)\,(T6)$, $(x \vdash y) \perp z = x \vdash (y \perp z)\,(T7)$, $(x \perp y) \vdash z = x \vdash (y \vdash z)\,(T8)$ for all $x, y, z \in T$, is called a trialgebra (trioid) [1]. So, the notion of a trialgebra is based on the notion of a trioid and all results obtained for trioids can be applied to trialgebras. This connection between trioids and trialgebras gives a motivation for studying trioids. Another reason for our interest in trioids is their connection with dimonoids [2, 3]. For a general introduction and basic theory see [1, 4].

The first step in the study of idempotent semigroups has been made by David McLean [5] who used rectangular bands for the description of the structure of an arbitrary band. Rectangular dimonoids (rectangular dibands) first appeared in the researches of the structure of dibands of subdimonoids (see [6]). Using rectangular dibands, a structure theorem on idempotent dimonoids was given in [7]. The free rectangular diband was constructed in [8].

In this paper we introduce the notion of a rectangular triband and give examples of rectangular tribands (Lemmas 1–4). We also construct a free rectangular triband (Theorem 1), describe its structure (Theorems 3–4) and the automorphism group (Lemma 5). As a consequence of Theorem 2, some least congruences on free rectangular tribands are described (Corollary 2).

2 Preliminaries

A nonempty subset A of a trioid $(T, \dashv, \vdash, \perp)$ is called a subtrioid if for any $a, b \in T$, a, $b \in A$ it follows that $a \dashv b$, $a \vdash b, a \perp b \in A$. An idempotent semigroup

S is called a rectangular band if

$$xyx = x \tag{1}$$

for all $x, y \in S$. It is clear that in any rectangular band the identity

$$xyz = xz \tag{2}$$

holds.

A trioid $(T, \dashv, \vdash, \perp)$ is called an idempotent trioid or a triband [9] if semigroups $(T, \dashv)$, $(T, \vdash)$ and $(T, \perp)$ are idempotent semigroups. A trioid $(T, \dashv, \vdash, \perp)$ will be called a rectangular trioid or a rectangular triband, if semigroups $(T, \dashv)$, $(T, \vdash)$ and $(T, \perp)$ are rectangular bands.

Note that the class of all rectangular tribands is a subvariety of the variety of all trioids. A trioid which is free in the variety of rectangular tribands will be called a free rectangular triband.

Recall the definition of a triband of subtrioids which was introduced in [9].

If $f : T_1 \to T_2$ is a homomorphism of trioids, then the corresponding congruence on T_1 will be denoted by Δ_f.

Let S be an arbitrary trioid, J be some idempotent trioid and let $\alpha : S \to J : x \mapsto x\alpha$ be a homomorphism. Then every class of the congruence Δ_α is a subtrioid of the trioid S, and the trioid S itself is a union of such trioids S_ξ, $\xi \in J$, that

$$x\alpha = \xi \Leftrightarrow x \in S_\xi = \Delta_\alpha^x = \{t \in S \,|\, (x, t) \in \Delta_\alpha\},$$
$$S_\xi \dashv S_\varepsilon \subseteq S_{\xi \dashv \varepsilon}, \quad S_\xi \vdash S_\varepsilon \subseteq S_{\xi \vdash \varepsilon}, \quad S_\xi \perp S_\varepsilon \subseteq S_{\xi \perp \varepsilon},$$
$$\xi \neq \varepsilon \Rightarrow S_\xi \cap S_\varepsilon = \varnothing.$$

In this case we say that S is decomposable into a triband of subtrioids (or S is a triband J of subtrioids S_ξ $(\xi \in J)$). If J is an idempotent semigroup (band), then we say that S is a band J of subtrioids S_ξ $(\xi \in J)$. If J is a commutative band, then we say that S is a semilattice J of subtrioids S_ξ $(\xi \in J)$. If J is a left (right) zero semigroup, then we say that S is a left (right) band J of subtrioids S_ξ $(\xi \in J)$.

Observe that the notion of a triband of subtrioids generalizes the notion of a diband of subdimonoids [6] (see also [7]) and the notion of a band of semigroups [10].

Recall that a nonempty set D equipped with two binary associative operations $\dashv$ and $\vdash$ satisfying the axioms $(T1) - (T3)$ is called a dimonoid [2, 3]. If $D = (D, \dashv, \vdash)$ is a dimonoid, then the trioid $(D, \dashv, \vdash, \dashv)$ (respectively, $(D, \dashv, \vdash, \vdash)$) will be denoted by $(D)^{\dashv}$ (respectively, $(D)^{\vdash}$). It is clear that $(D)^{\dashv}$ and $(D)^{\vdash}$ are different as trioids but they coincide as dimonoids.

Consider the following dimonoids from [8] which will be used in Section 4.

Let X be an arbitrary nonempty set. Let $X_{\ell z} = (X, \dashv)$, $X_{rz} = (X, \vdash)$, $X_{rb} = X_{\ell z} \times X_{rz}$ be a left zero semigroup, a right zero semigroup and a rectangular band, respectively. By [8] $X_{\ell z, rz} = (X, \dashv, \vdash)$ is the free left zero and right zero dimonoid (or the free left and right diband).

Define operations $\dashv$ and $\vdash$ on X^2 by

$$(x,y)\dashv(a,b)=(x,b),\quad (x,y)\vdash(a,b)=(a,b)$$

for all $(x,y),(a,b)\in X^2$. By [8] $(X^2,\dashv,\vdash)$ is a free (rb,rz)-dimonoid. It is denoted by $X_{rb,rz}$.

Define operations $\dashv$ and $\vdash$ on X^2 by

$$(x,y)\dashv(a,b)=(x,y),\quad (x,y)\vdash(a,b)=(x,b)$$

for all $(x,y),(a,b)\in X^2$. By [8] $(X^2,\dashv,\vdash)$ is a free $(\ell z,rb)$-dimonoid. It is denoted by $X_{\ell z,rb}$.

Define operations $\dashv$ and $\vdash$ on X^3 by

$$(x_1,x_2,x_3)\dashv(y_1,y_2,y_3)=(x_1,x_2,y_3),$$

$$(x_1,x_2,x_3)\vdash(y_1,y_2,y_3)=(x_1,y_2,y_3)$$

for all $(x_1,x_2,x_3),(y_1,y_2,y_3)\in X^3$. The algebra $(X^3,\dashv,\vdash)$ is denoted by $FRct(X)$. According to Theorem 1 from [8] $FRct(X)$ is a free rectangular diband.

As usual, $\mathbb{N}$ denotes the set of all positive integers.

3 Rectangular tribands

In this section we give new examples of rectangular tribands and construct a free rectangular triband of an arbitrary rank.

We first give examples of rectangular tribands.

It is immediate to prove the following three lemmas.

Let $I_n=\{1,2,...,n\}$, $n>1$, and let $\{X_i\}_{i\in I_n}$ be a family of arbitrary nonempty sets X_i, $i\in I_n$. Define operations $\dashv$, $\vdash$ and $\perp$ on $\prod_{i\in I_3}X_i$ by

$$(a_1,b_1,c_1)\dashv(a_2,b_2,c_2)=(a_1,b_1,c_1),$$

$$(a_1,b_1,c_1)\vdash(a_2,b_2,c_2)=(a_1,b_2,c_2),$$

$$(a_1,b_1,c_1,)\perp(a_2,b_2,c_2)=(a_1,b_1,c_2)$$

for all $(a_1,b_1,c_1),(a_2,b_2,c_2)\in\prod_{i\in I_3}X_i$. It is clear that $(\prod_{i\in I_3}X_i,\perp,\vdash)$ is a rectangular diband [8] and $(\prod_{i\in I_3}X_i,\dashv)$ is a left zero semigroup.

Lemma 1. $(\prod_{i\in I_3}X_i,\dashv,\vdash,\perp)$ *is a rectangular triband.*

If $X_i=X$ for all $i\in I_3$, then denote the algebra $(\prod_{i\in I_3}X_i,\dashv,\vdash,\perp)$ by $X_{lz,rd}$.

Define operations $\dashv$, $\vdash$ and $\perp$ on $\prod_{i\in I_3}X_i$ by

$$(a_1,b_1,c_1)\dashv(a_2,b_2,c_2)=(a_1,b_1,c_2),$$

$$(a_1,b_1,c_1)\vdash(a_2,b_2,c_2)=(a_2,b_2,c_2),$$

$$(a_1,b_1,c_1,)\perp(a_2,b_2,c_2)=(a_1,b_2,c_2)$$

for all $(a_1,b_1,c_1),(a_2,b_2,c_2)\in\prod_{i\in I_3}X_i$. It is clear that $(\prod_{i\in I_3}X_i,\dashv,\perp)$ is a rectangular diband [8] and $(\prod_{i\in I_3}X_i,\vdash)$ is a right zero semigroup.

Lemma 2. $(\prod_{i\in I_3} X_i, \dashv, \vdash, \perp)$ *is a rectangular triband.*

If $X_i = X$ for all $i \in I_3$, then denote the algebra $(\prod_{i\in I_3} X_i, \dashv, \vdash, \perp)$ by $X_{rd,rz}$.

Define operations $\dashv$, $\vdash$ and $\perp$ on $\prod_{i\in I_2} X_i$ by

$$(a_1, b_1)\dashv(a_2, b_2) = (a_1, b_1), \quad (a_1, b_1)\vdash(a_2, b_2) = (a_2, b_2),$$

$$(a_1, b_1)\perp(a_2, b_2) = (a_1, b_2)$$

for all $(a_1, b_1), (a_2, b_2) \in \prod_{i\in I_2} X_i$. It is clear that $(\prod_{i\in I_2} X_i, \dashv, \vdash)$ is a left zero and right zero dimonoid [8] and $(\prod_{i\in I_2} X_i, \perp)$ is a rectangular band.

Lemma 3. $(\prod_{i\in I_2} X_i, \dashv, \vdash, \perp)$ *is a rectangular triband.*

If $X_i = X$ for all $i \in I_2$, then denote $(\prod_{i\in I_2} X_i, \dashv, \vdash, \perp)$ by $X^{rb}_{lz,rz}$. Note that the trioid $X^{rb}_{lz,rz}$ was first constructed in [9].

Define operations $\dashv$, $\vdash$ and $\perp$ on $\prod_{i\in I_{2k}} X_i$, where $k \in \mathbb{N}$, by

$$(x_1, x_2, ..., x_{2k})\dashv(y_1, y_2, ..., y_{2k}) = (x_1, x_2, ..., x_{2k-1}, y_{2k}),$$

$$(x_1, x_2, ..., x_{2k})\vdash(y_1, y_2, ..., y_{2k}) = (x_1, y_2, ..., y_{2k}),$$

$$(x_1, x_2, ..., x_{2k})\perp(y_1, y_2, ..., y_{2k}) = (x_1, x_2, ..., x_k, y_{k+1}, ..., y_{2k})$$

for all $(x_1, x_2, ..., x_{2k}), (y_1, y_2, ..., y_{2k}) \in \prod_{i\in I_{2k}} X_i$.

Lemma 4. *For any $k > 1$, $\left(\prod_{i\in I_{2k}} X_i, \dashv, \vdash, \perp\right)$ is a rectangular triband.*

Proof. By Lemma 4 from [8] $(\prod_{i\in I_{2k}} X_i, \dashv, \vdash, \perp)$ satisfies the axioms $(T1) - (T3)$ of a trioid and the associativity of operations $\dashv, \vdash$. For all $(x_1, x_2, ..., x_{2k})$, $(y_1, y_2, ..., y_{2k}), (z_1, z_2, ..., z_{2k}) \in \prod_{i\in I_{2k}} X_i$ obtain

$$((x_1, x_2, ..., x_{2k})\perp(y_1, y_2, ..., y_{2k}))\perp(z_1, z_2, ..., z_{2k}) =$$

$$= (x_1, x_2, ..., x_k, y_{k+1}, ..., y_{2k})\perp(z_1, z_2, ..., z_{2k}) =$$

$$= (x_1, x_2, ..., x_k, z_{k+1}, ..., z_{2k}) = (x_1, x_2, ..., x_{2k})\perp(y_1, y_2, ..., y_k, z_{k+1}, ..., z_{2k}) =$$

$$= (x_1, x_2, ..., x_{2k})\perp((y_1, y_2, ..., y_{2k})\perp(z_1, z_2, ..., z_{2k})),$$

$$((x_1, x_2, ..., x_{2k})\dashv(y_1, y_2, ..., y_{2k}))\dashv(z_1, z_2, ..., z_{2k}) =$$

$$= (x_1, x_2, ..., x_{2k-1}, y_{2k})\dashv(z_1, z_2, ..., z_{2k}) =$$

$$= (x_1, x_2, ..., x_{2k-1}, z_{2k}) = (x_1, x_2, ..., x_{2k})\dashv(y_1, y_2, ..., y_k, z_{k+1}, ..., z_{2k}) =$$

$$= (x_1, x_2, ..., x_{2k})\dashv((y_1, y_2, ..., y_{2k})\perp(z_1, z_2, ..., z_{2k})),$$

$$((x_1, x_2, ..., x_{2k})\perp(y_1, y_2, ..., y_{2k}))\dashv(z_1, z_2, ..., z_{2k}) =$$

$$= (x_1, x_2, ..., x_k, y_{k+1}, ..., y_{2k})\dashv(z_1, z_2, ..., z_{2k}) =$$

$$= (x_1, x_2, ..., x_k, y_{k+1}, ..., y_{2k-1}, z_{2k}) = (x_1, x_2, ..., x_{2k})\perp(y_1, y_2, ..., y_{2k-1}, z_{2k}) =$$

$$= (x_1, x_2, ..., x_{2k})\perp((y_1, y_2, ..., y_{2k})\dashv(z_1, z_2, ..., z_{2k})),$$

$$((x_1, x_2, ..., x_{2k})\dashv(y_1, y_2, ..., y_{2k}))\perp(z_1, z_2, ..., z_{2k}) =$$

$$= (x_1, x_2, ..., x_{2k-1}, y_{2k})\perp(z_1, z_2, ..., z_{2k}) =$$

$$= (x_1, x_2, ..., x_k, z_{k+1}, ..., z_{2k}) = (x_1, x_2, ..., x_{2k})\perp(y_1, z_2, ..., z_{2k}) =$$

$$= (x_1, x_2, ..., x_{2k})\perp((y_1, y_2, ..., y_{2k})\vdash(z_1, z_2, ..., z_{2k})),$$

$$((x_1, x_2, ..., x_{2k})\vdash(y_1, y_2, ..., y_{2k}))\perp(z_1, z_2, ..., z_{2k}) = (x_1, y_2, ..., y_{2k})\perp(z_1, z_2, ..., z_{2k}) =$$

$$= (x_1, y_2, ..., y_k, z_{k+1}, ..., z_{2k}) = (x_1, x_2, ..., x_{2k})\vdash(y_1, y_2, ..., y_k, z_{k+1}, ..., z_{2k}) =$$

$$= (x_1, x_2, ..., x_{2k})\vdash((y_1, y_2, ..., y_{2k})\perp(z_1, z_2, ..., z_{2k})),$$

$$((x_1, x_2, ..., x_{2k})\perp(y_1, y_2, ..., y_{2k}))\vdash(z_1, z_2, ..., z_{2k}) =$$

$$= (x_1, x_2, ..., x_k, y_{k+1}, ..., y_{2k})\vdash(z_1, z_2, ..., z_{2k}) =$$

$$= (x_1, z_2, ..., z_{2k}) = (x_1, x_2, ..., x_{2k})\vdash(y_1, z_2, ..., z_{2k}) =$$

$$= (x_1, x_2, ..., x_{2k})\vdash((y_1, y_2, ..., y_{2k})\vdash(z_1, z_2, ..., z_{2k})).$$

Thus, $(\prod_{i\in I_{2k}} X_i, \dashv, \vdash, \perp)$ satisfies the axioms $(T4) - (T8)$ of a trioid and the associativity of $\perp$ and so, it is a trioid. Obviously, $(\prod_{i\in I_{2k}} X_i, \dashv, \vdash, \perp)$ is idempotent. Show that it is a rectangular triband. We have

$$(x_1, x_2, ..., x_{2k})\dashv(y_1, y_2, ..., y_{2k})\dashv(x_1, x_2, ..., x_{2k}) =$$

$$= (x_1, x_2, ..., x_{2k-1}, y_{2k})\dashv(x_1, x_2, ..., x_{2k}) = (x_1, x_2, ..., x_{2k}),$$

$$(x_1, x_2, ..., x_{2k})\vdash(y_1, y_2, ..., y_{2k})\vdash(x_1, x_2, ..., x_{2k}) =$$

$$= (x_1, y_2, ..., y_{2k})\vdash(x_1, x_2, ..., x_{2k}) = (x_1, x_2, ..., x_{2k}),$$

$$(x_1, x_2, ..., x_{2k})\perp(y_1, y_2, ..., y_{2k})\perp(x_1, x_2, ..., x_{2k}) =$$

$$= (x_1, x_2, ..., x_k, y_{k+1}, ..., y_{2k})\perp(x_1, x_2, ..., x_{2k}) = (x_1, x_2, ..., x_{2k}).$$

Thus, $(\prod_{i\in I_{2k}} X_i, \dashv, \vdash, \perp)$ is a rectangular triband. □

Obviously, operations of $(\prod_{i\in I_2} X_i, \dashv, \vdash, \perp)$ coincide and it is a rectangular band.

Let X be an arbitrary nonempty set. We denote the trioid $(X^4, \dashv, \vdash, \perp)$ by $FRT(X)$.

The main result of this section is the following

Theorem 1. *$FRT(X)$ is a free rectangular triband.*

Proof. By Lemma 4 $FRT(X)$ is a rectangular triband. Let $(T, \dashv', \vdash', \perp')$ be an arbitrary rectangular trioid and $\sigma : X \to T$ be an arbitrary map. Define the map

$$\tau : \ FRT(X) \to (T, \dashv', \vdash', \perp') :$$

$$(a, b, c, d) \mapsto (a, b, c, d)\tau = (a\sigma \vdash' b\sigma) \perp' (c\sigma \dashv' d\sigma).$$

In order to prove that τ is a homomorphism we will use axioms of a trioid and the identities (1), (2). One can get

$$((a_1, b_1, c_1, d_1) \dashv (a_2, b_2, c_2, d_2))\tau = (a_1, b_1, c_1, d_2)\tau = (a_1\sigma \vdash' b_1\sigma) \perp' (c_1\sigma \dashv' d_2\sigma) =$$
$$= (a_1\sigma \vdash' b_1\sigma) \perp' ((c_1\sigma \dashv' d_1\sigma) \dashv' (c_2\sigma \dashv' d_2\sigma)) =$$
$$= ((a_1\sigma \vdash' b_1\sigma) \perp' (c_1\sigma \dashv' d_1\sigma)) \dashv' (c_2\sigma \dashv' d_2\sigma) =$$
$$= ((a_1\sigma \vdash' b_1\sigma) \perp' (c_1\sigma \dashv' d_1\sigma)) \dashv' (a_2\sigma \vdash' b_2\sigma) \dashv' (c_2\sigma \dashv' d_2\sigma) =$$
$$= ((a_1\sigma \vdash' b_1\sigma) \perp' (c_1\sigma \dashv' d_1\sigma)) \dashv'$$
$$\dashv' ((a_2\sigma \vdash' b_2\sigma) \perp' (c_2\sigma \dashv' d_2\sigma)) = (a_1, b_1, c_1, d_1)\tau \dashv' (a_2, b_2, c_2, d_2)\tau,$$
$$((a_1, b_1, c_1, d_1) \vdash (a_2, b_2, c_2, d_2))\tau = (a_1, b_2, c_2, d_2)\tau = (a_1\sigma \vdash' b_2\sigma) \perp' (c_2\sigma \dashv' d_2\sigma) =$$
$$= a_1\sigma \vdash' (b_2\sigma \perp' (c_2\sigma \dashv' d_2\sigma)) = a_1\sigma \vdash' ((b_2\sigma \vdash' a_2\sigma \vdash' b_2\sigma) \perp' (c_2\sigma \dashv' d_2\sigma)) =$$
$$= a_1\sigma \vdash' ((b_2\sigma \vdash' (a_2\sigma \vdash' b_2\sigma)) \perp' (c_2\sigma \dashv' d_2\sigma)) =$$
$$= a_1\sigma \vdash' (b_2\sigma \vdash' ((a_2\sigma \vdash' b_2\sigma) \perp' (c_2\sigma \dashv' d_2\sigma))) =$$
$$= a_1\sigma \vdash' ((a_2\sigma \vdash' b_2\sigma) \perp' (c_2\sigma \dashv' d_2\sigma)) =$$
$$= a_1\sigma \vdash' b_1\sigma \vdash' (c_1\sigma \dashv' d_1\sigma) \vdash' ((a_2\sigma \vdash' b_2\sigma) \perp' (c_2\sigma \dashv' d_2\sigma)) =$$
$$= (a_1\sigma \vdash' b_1\sigma) \vdash' ((c_1\sigma \dashv' d_1\sigma) \vdash' ((a_2\sigma \vdash' b_2\sigma) \perp' (c_2\sigma \dashv' d_2\sigma))) =$$
$$= ((a_1\sigma \vdash' b_1\sigma) \perp' (c_1\sigma \dashv' d_1\sigma)) \vdash' ((a_2\sigma \vdash' b_2\sigma) \perp' (c_2\sigma \dashv' d_2\sigma)) =$$
$$= (a_1, b_1, c_1, d_1)\tau \vdash' (a_2, b_2, c_2, d_2)\tau,$$
$$((a_1, b_1, c_1, d_1) \perp (a_2, b_2, c_2, d_2))\tau = (a_1, b_1, c_2, d_2)\tau = (a_1\sigma \vdash' b_1\sigma) \perp' (c_2\sigma \dashv' d_2\sigma) =$$
$$= ((a_1\sigma \vdash' b_1\sigma) \perp' (c_1\sigma \dashv' d_1\sigma)) \perp' ((a_2\sigma \vdash' b_2\sigma) \perp' (c_2\sigma \dashv' d_2\sigma)) =$$
$$= (a_1, b_1, c_1, d_1)\tau \perp' (a_2, b_2, c_2, d_2)\tau.$$

Thus, τ is a homomorphism and $FRT(X)$ is free. □

Corollary 1. *The free rectangular triband $FRT(X)$ generated by a finite set X is finite. Specifically, if $|X| = n$, then $|FRT(X)| = n^4$.*

Denote the symmetric group on X by $\Im[X]$ and the automorphism group of a trioid M by $Aut\,M$. It is not difficult to see that the set $\{(a, a, a, a) \,|\, a \in X\}$ is generating for $FRT(X)$. From here obtain the following description of the automorphism group of the free rectangular triband.

Lemma 5. $Aut\,FRT(X) \cong \Im[X]$.

4 Decompositions of $\boldsymbol{FRT(X)}$

In this section we describe the structure of free rectangular tribands and characterize some least congruences on free rectangular tribands.

For all $i, j \in X$ put

$$\Lambda_{(i)} = \{(a,b,c,d) \in FRT(X) \mid a = i\} \ ,$$

$$\Lambda_{[i]} = \{(a,b,c,d) \in FRT(X) \mid d = i\} \ ,$$

$$\Lambda_{(i,j)} = \{(a,b,c,d) \in FRT(X) \mid (a,d) = (i,j)\} \ .$$

The following theorem gives decompositions of $FRT(X)$ into bands of subtrioids.

Theorem 2. *Let $FRT(X)$ be a free rectangular triband. Then*

(i) $FRT(X)$ is a left band X_{lz} of subtrioids $\Lambda_{(i)}, i \in X_{lz}$, such that $\Lambda_{(i)} \cong X_{rd,rz}$ for every $i \in X_{lz}$;

(ii) $FRT(X)$ is a right band X_{rz} of subtrioids $\Lambda_{[i]}, i \in X_{rz}$, such that $\Lambda_{[i]} \cong X_{lz,rd}$ for every $i \in X_{rz}$;

(iii) $FRT(X)$ is a rectangular band X_{rb} of subtrioids $\Lambda_{(i,j)}, (i,j) \in X_{rb}$, such that $\Lambda_{(i,j)} \cong X^{rb}_{lz,rz}$ for every $(i,j) \in X_{rb}$.

Proof. (i) By Theorem 1 the map $\pi_{lz} : FRT(X) \to X_{lz} : (a,b,c,d) \mapsto a$ is a homomorphism. Then $\Lambda_{(i)}, i \in X_{lz}$, is a class of $\Delta_{\pi_{lz}}$ which is a subtrioid of $FRT(X)$. It is immediate to check that for every $i \in X_{lz}$ the map

$$\Lambda_{(i)} \to X_{rd,rz} : (i,b,c,d) \mapsto (b,c,d)$$

is an isomorphism.

(ii) By Theorem 1 the map $\pi_{rz} : FRT(X) \to X_{rz} : (a,b,c,d) \to d$ is a homomorphism. Then $\Lambda_{[i]}$, $i \in X_{rz}$, is a class of $\Delta_{\pi_{rz}}$ which is a subtrioid of $FRT(X)$. It is clear that for every $i \in X_{rz}$ the map

$$\Lambda_{[i]} \to X_{lz,rd} : (a,b,c,i) \mapsto (a,b,c)$$

is an isomorphism.

(iii) By Theorem 1 the map $\pi_{rb} : FRT(X) \to X_{rb} : (a,b,c,d) \to (a,d)$ is a homomorphism. Then $\Lambda_{(i,j)}, (i,j) \in X_{rb}$, is a class of $\Delta_{\pi_{rb}}$ which is a subtrioid of $FRT(X)$. It can be shown that for every $(i,j) \in X_{rb}$ the map

$$\Lambda_{(i,j)} \to X^{rb}_{lz,rz} : (i,b,c,j) \mapsto (b,c)$$

is an isomorphism. $\square$

If ρ is a congruence on a trioid $(T, \dashv, \vdash, \perp)$ such that operations of $(T, \dashv, \vdash, \perp)/\rho$ coincide and it is a left zero semigroup (respectively, right zero semigroup, rectangular band, semilattice), then we say that ρ is a left zero congruence (respectively, right zero congruence, rectangular band congruence, semilattice congruence).

From Theorem 2 we obtain

Corollary 2. *Let $FRT(X)$ be a free rectangular triband. Then*
(i) $\Delta_{\pi_{lz}}$ is the least left zero congruence on $FRT(X)$;
(ii) $\Delta_{\pi_{rz}}$ is the least right zero congruence on $FRT(X)$;
(iii) $\Delta_{\pi_{rb}}$ is the least rectangular band congruence on $FRT(X)$.

Proof. (i) It is well-known that every left zero semigroup is a free left zero semigroup. By Theorem 2 (i) we obtain (i).

The proofs of (ii) and (iii) are similar. □

From Theorem 5 [11] it follows that any rectangular triband is semilattice indecomposable, i.e. the least semilattice congruence on a rectangular triband coincides with the universal relation on this trioid.

For all $i, j, k \in X$ put

$$\Lambda_{(i,j,k)} = \{(a,b,c,d) \in FRT(X) \mid (a,b,c) = (i,j,k)\} ,$$

$$\Lambda_{[i,j,k]} = \{(a,b,c,d) \in FRT(X) \mid (b,c,d) = (i,j,k)\} ,$$

$$\Lambda_{[i,j]} = \{(a,b,c,d) \in FRT(X) \mid (b,c) = (i,j)\} .$$

The following theorem gives decompositions of $FRT(X)$ into tribands of subsemigroups.

Theorem 3. *Let $FRT(X)$ be a free rectangular triband. Then*
(i) $FRT(X)$ is a triband $X_{lz,rd}$ of subsemigroups $\Lambda_{(i,j,k)}, (i,j,k) \in X_{lz,rd}$, such that $\Lambda_{(i,j,k)} \cong X_{rz}$ for every $(i,j,k) \in X_{lz,rd}$;
(ii) $FRT(X)$ is a triband $X_{rd,rz}$ of subsemigroups $\Lambda_{[i,j,k]}, (i,j,k) \in X_{rd,rz}$, such that $\Lambda_{[i,j,k]} \cong X_{lz}$ for every $(i,j,k) \in X_{rd,rz}$;
(iii) $FRT(X)$ is a triband $X^{rb}_{lz,rz}$ of subsemigroups $\Lambda_{[i,j]}, (i,j) \in X^{rb}_{lz,rz}$, such that $\Lambda_{[i,j]} \cong X_{rb}$ for every $(i,j) \in X^{rb}_{lz,rz}$.

Proof. (i) By Theorem 1 the map

$$\pi_{lz,rd} : FRT(X) \to X_{lz,rd} : (a,b,c,d) \mapsto (a,b,c)$$

is a homomorphism. Then $\Lambda_{(i,j,k)}, (i,j,k) \in X_{lz,rd}$, is a class of $\Delta_{\pi_{lz,rd}}$ which is a subtrioid of $FRT(X)$. If $(a_1,b_1,c_1,d_1), (a_2,b_2,c_2,d_2) \in \Lambda_{(i,j,k)}$, then $(a_1,b_1,c_1) = (a_2,b_2,c_2) = (i,j,k)$ and

$$(a_1,b_1,c_1,d_1) \dashv (a_2,b_2,c_2,d_2) = (a_1,b_1,c_1,d_2) = (i,j,k,d_2),$$

$$(a_1,b_1,c_1,d_1) \vdash (a_2,b_2,c_2,d_2) = (a_1,b_2,c_2,d_2) = (i,j,k,d_2),$$

$$(a_1,b_1,c_1,d_1) \perp (a_2,b_2,c_2,d_2) = (a_1,b_1,c_2,d_2) = (i,j,k,d_2).$$

Hence operations of $\Lambda_{(i,j,k)}$ coincide and so, it is a semigroup. It is easy to cheek that for every $(i,j,k) \in X_{lz,rd}$ the map $\Lambda_{(i,j,k)} \to X_{rz} : (i,j,k,d) \mapsto d$ is an isomorphism.

(ii) By Theorem 1 the map

$$\pi_{rd,rz} : FRT(X) \to X_{rd,rz} : (a,b,c,d) \mapsto (b,c,d)$$

is a homomorphism. Then $\Lambda_{[i,j,k]}, (i,j,k) \in X_{rd,rz}$, is a class of $\Delta_{\pi_{rd,rz}}$which is a subtrioid of $FRT(X)$. If $(a_1,b_1,c_1,d_1),(a_2,b_2,c_2,d_2) \in \Lambda_{[i,j,k]}$, then $(b_1,c_1,d_1) = (b_2,c_2,d_2) = (i,j,k)$ and

$$(a_1,b_1,c_1,d_1)\dashv(a_2,b_2,c_2,d_2) = (a_1,b_1,c_1,d_2) = (a_1,i,j,k),$$

$$(a_1,b_1,c_1,d_1)\vdash(a_2,b_2,c_2,d_2) = (a_1,b_2,c_2,d_2) = (a_1,i,j,k),$$

$$(a_1,b_1,c_1,d_1)\perp(a_2,b_2,c_2,d_2) = (a_1,b_1,c_2,d_2) = (a_1,i,j,k).$$

Hence operations of $\Lambda_{[i,j,k]}$ coincide and so, it is a semigroup. One can check that for every $(i,j,k) \in X_{rd,rz}$ the map $\Lambda_{[i,j,k]} \to X_{lz} : (a,i,j,k) \mapsto a$ is an isomorphism.

(iii) By Theorem 1 the map

$$\pi^{rb}_{lz,rz} : FRT(X) \to X^{rb}_{lz,rz} : (a,b,c,d) \mapsto (b,c)$$

is a homomorphism. Then $\Lambda_{[i,j]}, (i,j) \in X^{rb}_{lz,rz}$, is a class of $\Delta_{\pi^{rb}_{lz,rz}}$ which is a subtrioid of $FRT(X)$. If $(a_1,b_1,c_1,d_1),(a_2,b_2,c_2,d_2) \in \Lambda_{[i,j]}$, then $(b_1,c_1) = (b_2,c_2) = (i,j)$ and

$$(a_1,b_1,c_1,d_1)\dashv(a_2,b_2,c_2,d_2) = (a_1,b_1,c_1,d_2) = (a_1,i,j,d_2),$$

$$(a_1,b_1,c_1,d_1)\vdash(a_2,b_2,c_2,d_2) = (a_1,b_2,c_2,d_2) = (a_1,i,j,d_2),$$

$$(a_1,b_1,c_1,d_1)\perp(a_2,b_2,c_2,d_2) = (a_1,b_1,c_2,d_2) = (a_1,i,j,d_2).$$

Hence operations of $\Lambda_{[i,j]}$ coincide and so, it is a semigroup. An immediate verification shows that for every $(i,j) \in X^{rb}_{lz,rz}$ the map $\Lambda_{[i,j]} \to X_{rb} : (a,i,j,d) \mapsto (a,d)$ is an isomorphism. □

For all $i,j,k \in X$ put

$$V_{(i)} = \{(a,b,c,d) \in FRT(X) \mid b = i\}\ ,$$

$$V_{[i]} = \{(a,b,c,d) \in FRT(X) \mid c = i\}\ ,$$

$$V_{(i,j,k)} = \{(a,b,c,d) \in FRT(X) \mid (a,b,d) = (i,j,k)\}\ ,$$

$$V_{[i,j,k]} = \{(a,b,c,d) \in FRT(X) \mid (a,c,d) = (i,j,k)\}\ ,$$

$$V_{(i,j)} = \{(a,b,c,d) \in FRT(X) \mid (a,b) = (i,j)\}\ ,$$

$$V_{[i,j]} = \{(a,b,c,d) \in FRT(X) \mid (a,c) = (i,j)\}\ ,$$

$$V_{(i,j]} = \{(a,b,c,d) \in FRT(X) \mid (b,d) = (i,j)\}\ ,$$

$$V_{[i,j)} = \{(a,b,c,d) \in FRT(X) \mid (c,d) = (i,j)\}.$$

The following theorem gives decompositions of $FRT(X)$ into tribands of subtrioids.

Theorem 4. *Let $FRT(X)$ be a free rectangular triband. Then*

(i) $FRT(X)$ is a triband $(X_{lz,rz})^{\dashv}$ of subtrioids $V_{(i)}, i \in (X_{lz,rz})^{\dashv}$, such that $V_{(i)} \cong (FRct(X))^{\vdash}$ for every $i \in X_{lz,rz}$;

(ii) $FRT(X)$ is a triband $(X_{lz,rz})^{\vdash}$ of subtrioids $V_{[i]}, i \in (X_{lz,rz})^{\vdash}$, such that $V_{[i]} \cong (FRct(X))^{\dashv}$ for every $i \in X_{lz,rz}$;

(iii) $FRT(X)$ is a triband $(FRct(X))^{\dashv}$ of subtrioids $V_{(i,j,k)}, (i,j,k) \in (FRct(X))^{\dashv}$, such that $V_{(i,j,k)} \cong (X_{lz,rz})^{\vdash}$ for every $(i,j,k) \in FRct(X)$.

(iv) $FRT(X)$ is a triband $(FRct(X))^{\vdash}$ of subtrioids $V_{[i,j,k]}, (i,j,k) \in (FRct(X))^{\vdash}$, such that $V_{[i,j,k]} \cong (X_{lz,rz})^{\dashv}$ for every $(i,j,k) \in FRct(X)$;

(v) $FRT(X)$ is a triband $(X_{lz,rb})^{\dashv}$ of subtrioids $V_{(i,j)}, (i,j) \in (X_{lz,rb})^{\dashv}$, such that $V_{(i,j)} \cong (X_{rb,rz})^{\vdash}$ for every $(i,j) \in X_{lz,rb}$;

(vi) $FRT(X)$ is a triband $(X_{lz,rb})^{\vdash}$ of subtrioids $V_{[i,j]}, (i,j) \in (X_{lz,rb})^{\vdash}$, such that $V_{[i,j]} \cong (X_{rb,rz})^{\dashv}$ for every $(i,j) \in X_{lz,rb}$;

(vii) $FRT(X)$ is a triband $(X_{rb,rz})^{\dashv}$ of subtrioids $V_{(i,j]}, (i,j) \in (X_{rb,rz})^{\dashv}$, such that $V_{(i,j]} \cong (X_{lz,rb})^{\vdash}$ for every $(i,j) \in X_{rb,rz}$;

(viii) $FRT(X)$ is a triband $(X_{rb,rz})^{\vdash}$ of subtrioids $V_{[i,j)}, (i,j) \in (X_{rb,rz})^{\vdash}$, such that $V_{[i,j)} \cong (X_{lz,rb})^{\dashv}$ for every $(i,j) \in X_{rb,rz}$.

Proof. (i) By Theorem 1 the map

$$\pi^{\dashv}_{lz,rz} : FRT(X) \to (X_{lz,rz})^{\dashv} : (a,b,c,d) \mapsto b$$

is a homomorphism. Then $V_{(i)}, i \in X_{lz,rz}$, is a class of $\Delta_{\pi^{\dashv}_{lz,rz}}$ which is a subtrioid of $FRT(X)$. If $(a_1,b_1,c_1,d_1),(a_2,b_2,c_2,d_2) \in V_{(i)}$, then $b_1 = b_2 = i$ and

$$(a_1,b_1,c_1,d_1)\dashv(a_2,b_2,c_2,d_2) = (a_1,b_1,c_1,d_2) = (a_1,i,c_1,d_2),$$

$$(a_1,b_1,c_1,d_1)\vdash(a_2,b_2,c_2,d_2) = (a_1,b_2,c_2,d_2) = (a_1,i,c_2,d_2),$$

$$(a_1,b_1,c_1,d_1)\perp(a_2,b_2,c_2,d_2) = (a_1,b_1,c_2,d_2) = (a_1,i,c_2,d_2).$$

Hence operations $\vdash$ and $\perp$ of $V_{(i)}$ coincide. It is easy to cheek that for every $i \in X_{lz,rz}$ the map

$$V_{(i)} \to (FRct(X))^{\vdash} : (a,i,c,d) \mapsto (a,c,d)$$

is an isomorphism.

(ii) By Theorem 1 the map

$$\pi^{\vdash}_{lz,rz} : FRT(X) \to (X_{lz,rz})^{\vdash} : (a,b,c,d) \mapsto c$$

is a homomorphism. Then $V_{[i]}, i \in X_{lz,rz}$, is a class of $\Delta_{\pi^{\vdash}_{lz,rz}}$ which is a subtrioid of $FRT(X)$. If $(a_1,b_1,c_1,d_1),(a_2,b_2,c_2,d_2) \in V_{[i]}$, then $c_1 = c_2 = i$ and

$$(a_1,b_1,c_1,d_1)\dashv(a_2,b_2,c_2,d_2) = (a_1,b_1,c_1,d_2) = (a_1,b_1,i,d_2),$$

$$(a_1,b_1,c_1,d_1)\vdash(a_2,b_2,c_2,d_2) = (a_1,b_2,c_2,d_2) = (a_1,b_2,i,d_2),$$

$$(a_1, b_1, c_1, d_1) \bot (a_2, b_2, c_2, d_2) = (a_1, b_1, c_2, d_2) = (a_1, b_1, i, d_2).$$

Hence operations $\dashv$ and $\bot$ of $V_{[i]}$ coincide. It is easy to cheek that for every $i \in X_{lz,rz}$ the map

$$V_{[i]} \to (FRct(X))^{\dashv} : (a, b, i, d) \mapsto (a, b, d)$$

is an isomorphism.

(iii) By Theorem 1 the map

$$\pi^{\dashv}_{FRct} : FRT(X) \to (FRct(X))^{\dashv} : (a, b, c, d) \mapsto (a, b, d)$$

is a homomorphism. Then $V_{(i,j,k)}, (i, j, k) \in FRct(X)$, is a class of $\Delta_{\pi^{\dashv}_{FRct}}$ which is a subtrioid of $FRT(X)$. If $(a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2) \in V_{(i,j,k)}$, then $(a_1, b_1, d_1) = (a_2, b_2, d_2) = (i, j, k)$ and

$$(a_1, b_1, c_1, d_1) \dashv (a_2, b_2, c_2, d_2) = (a_1, b_1, c_1, d_2) = (i, j, c_1, k),$$

$$(a_1, b_1, c_1, d_1) \vdash (a_2, b_2, c_2, d_2) = (a_1, b_2, c_2, d_2) = (i, j, c_2, k),$$

$$(a_1, b_1, c_1, d_1) \bot (a_2, b_2, c_2, d_2) = (a_1, b_1, c_2, d_2) = (i, j, c_2, k).$$

Hence operations $\vdash$ and $\bot$ of $V_{(i,j,k)}$ coincide. It is clear that for every $(i, j, k) \in FRct(X)$ the map

$$V_{(i,j,k)} \to (X_{lz,rz})^{\vdash} : (i, j, c, k) \mapsto c$$

is an isomorphism.

(iv) By Theorem 1 the map

$$\pi^{\vdash}_{FRct} : FRT(X) \to (FRct(X))^{\vdash} : (a, b, c, d) \mapsto (a, c, d)$$

is a homomorphism. Then $V_{[i,j,k]}, (i, j, k) \in FRct(X)$, is a class of $\Delta_{\pi^{\vdash}_{FRct}}$ which is a subtrioid of $FRT(X)$. If $(a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2) \in V_{[i,j,k]}$, then $(a_1, c_1, d_1) = (a_2, c_2, d_2) = (i, j, k)$ and

$$(a_1, b_1, c_1, d_1) \dashv (a_2, b_2, c_2, d_2) = (a_1, b_1, c_1, d_2) = (i, b_1, j, k),$$

$$(a_1, b_1, c_1, d_1) \vdash (a_2, b_2, c_2, d_2) = (a_1, b_2, c_2, d_2) = (i, b_2, j, k),$$

$$(a_1, b_1, c_1, d_1) \bot (a_2, b_2, c_2, d_2) = (a_1, b_1, c_2, d_2) = (i, b_1, j, k).$$

Hence operations $\dashv$ and $\bot$ of $V_{[i,j,k]}$ coincide. One can verify that for every $(i, j, k) \in FRct(X)$ the map

$$V_{[i,j,k]} \to (X_{lz,rz})^{\dashv} : (i, b, j, k) \mapsto b$$

is an isomorphism.

(v) By Theorem 1 the map

$$\pi^{\dashv}_{lz,rb} : FRT(X) \to (X_{lz,rb})^{\dashv} : (a, b, c, d) \mapsto (a, b)$$

is a homomorphism. Then $V_{(i,j)}, (i,j) \in X_{lz,rb}$, is a class of $\Delta_{\pi^{\dashv}_{lz,rb}}$ which is a subtrioid of $FRT(X)$. If $(a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2) \in V_{(i,j)}$, then $(a_1, b_1) = (a_2, b_2) = (i, j)$ and

$$(a_1, b_1, c_1, d_1) \dashv (a_2, b_2, c_2, d_2) = (a_1, b_1, c_1, d_2) = (i, j, c_1, d_2),$$

$$(a_1, b_1, c_1, d_1) \vdash (a_2, b_2, c_2, d_2) = (a_1, b_2, c_2, d_2) = (i, j, c_2, d_2),$$

$$(a_1, b_1, c_1, d_1) \perp (a_2, b_2, c_2, d_2) = (a_1, b_1, c_2, d_2) = (i, j, c_2, d_2).$$

Hence operations $\vdash$ and $\perp$ of $V_{(i,j)}$ coincide. One can cheek that for every $(i, j) \in X_{lz,rb}$ the map

$$V_{(i,j)} \to (X_{rb,rz})^{\vdash} : (i, j, c, d) \mapsto (c, d)$$

is an isomorphism.

(vi) By Theorem 1 the map

$$\pi^{\vdash}_{lz,rb} : FRT\ (X) \to (X_{lz,rb})^{\vdash} : (a, b, c, d) \mapsto (a, c)$$

is a homomorphism. Then $V_{[i,j]}, (i, j) \in X_{lz,rb}$, is a class of $\Delta_{\pi^{\vdash}_{lz,rb}}$ which is a subtrioid of $FRT(X)$. If $(a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2) \in V_{[i,j]}$, then $(a_1, c_1) = (a_2, c_2) = (i, j)$ and

$$(a_1, b_1, c_1, d_1) \dashv (a_2, b_2, c_2, d_2) = (a_1, b_1, c_1, d_2) = (i, b_1, j, d_2),$$

$$(a_1, b_1, c_1, d_1) \vdash (a_2, b_2, c_2, d_2) = (a_1, b_2, c_2, d_2) = (i, b_2, j, d_2),$$

$$(a_1, b_1, c_1, d_1) \perp (a_2, b_2, c_2, d_2) = (a_1, b_1, c_2, d_2) = (i, b_1, j, d_2).$$

Hence operations $\dashv$ and $\perp$ of $V_{[i,j]}$ coincide. It can be shown that for every $(i, j) \in X_{lz,rb}$ the map

$$V_{[i,j]} \to (X_{rb,rz})^{\dashv} : (i, b, j, d) \mapsto (b, d)$$

is an isomorphism.

(vii) By Theorem 1 the map

$$\pi^{\dashv}_{rb,rz} : FRT\ (X) \to (X_{rb,rz})^{\dashv} : (a, b, c, d) \mapsto (b, d)$$

is a homomorphism. Then $V_{(i,j]}, (i, j) \in X_{rb,rz}$, is a class of $\Delta_{\pi^{\dashv}_{rb,rz}}$ which is a subtrioid of $FRT(X)$. If $(a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2) \in V_{(i,j]}$, then $(b_1, d_1) = (b_2, d_2) = (i, j)$ and

$$(a_1, b_1, c_1, d_1) \dashv (a_2, b_2, c_2, d_2) = (a_1, b_1, c_1, d_2) = (a_1, i, c_1, j),$$

$$(a_1, b_1, c_1, d_1) \vdash (a_2, b_2, c_2, d_2) = (a_1, b_2, c_2, d_2) = (a_1, i, c_2, j),$$

$$(a_1, b_1, c_1, d_1) \perp (a_2, b_2, c_2, d_2) = (a_1, b_1, c_2, d_2) = (a_1, i, c_2, j).$$

Hence operations $\vdash$ and $\perp$ of $V_{(i,j]}$ coincide. Clearly, for every $(i, j) \in X_{rb,rz}$ the map

$$V_{(i,j]} \to (X_{lz,rb})^{\vdash} : (a, i, c, j) \mapsto (a, c)$$

is an isomorphism.

(viii) By Theorem 1 the map

$$\pi^{\vdash}_{rb,rz} : FRT(X) \to (X_{rb,rz})^{\vdash} : (a,b,c,d) \mapsto (c,d)$$

is a homomorphism. Then $V_{[i,j)}, (i,j) \in X_{rb,rz}$, is a class of $\Delta_{\pi^{\vdash}_{rb,rz}}$ which is a subtrioid of $FRT(X)$. If $(a_1,b_1,c_1,d_1),(a_2,b_2,c_2,d_2) \in V_{[i,j)}$, then $(c_1,d_1) = (c_2,d_2) = (i,j)$ and

$$(a_1,b_1,c_1,d_1) \dashv (a_2,b_2,c_2,d_2) = (a_1,b_1,c_1,d_2) = (a_1,b_1,i,j),$$

$$(a_1,b_1,c_1,d_1) \vdash (a_2,b_2,c_2,d_2) = (a_1,b_2,c_2,d_2) = (a_1,b_2,i,j),$$

$$(a_1,b_1,c_1,d_1) \perp (a_2,b_2,c_2,d_2) = (a_1,b_1,c_2,d_2) = (a_1,b_1,i,j).$$

Hence operations $\dashv$ and $\perp$ of $V_{[i,j)}$ coincide. Evidently, for every $(i,j) \in X_{rb,rz}$ the map

$$V_{[i,j)} \to (X_{lz,rb})^{\dashv} : (a,b,i,j) \mapsto (a,b)$$

is an isomorphism. □

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Estimates of stability radius of multicriteria Boolean problem with Hölder metrics in parameter spaces

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Abstract. We consider multiple objective combinatorial linear problem in the situation where parameters of objective functions are exposed to perturbations. We study quantitative characteristic of stability (stability radius) of the problem assuming that there are Hölder metrics in the space of solutions and the criteria space.

Mathematics subject classification: 90C09, 90C27, 90C29, 90C31.
Keywords and phrases: Boolean programming, multicriteria optimization, stability radius, Pareto set, Hölder metric.

1 Introduction

The main difficulty while studying stability of discrete optimization problems is their combinatorial complexity. Small changes of initial data make a model behave in an unpredictable manner. In addition, in the case of several conflicting objectives the problem complexity may only be increased (see e.g. [1, 2]).

There are a lot of papers devoted to different approaches dealing with uncertainty in discrete models, both in single and multicriteria cases (see e.g. [3–5]). One of such approaches is known as quantitative approach. This approach aims to derive quantitative bounds for feasible initial data changes preserving a given property of the solution set (or of a single solution) or/and create algorithms for the bounds calculation. The limit level of perturbations of problem parameters which preserve the property of invariance is called stability radius. The present work continues a line of investigations initiated in [6–9] that focuses on studying the stability radius of multicriteria Boolean optimization problems with various types of metrics in the parameter space. We have obtained the lower and upper bounds for the stability radius of the multicriteria combinatorial linear problem on the assumption that Hölder norms are specified in the space of solutions and in the space of criteria.

2 Problem statement and basic definitions

Let $\mathbf{R}^m$ be the space of criteria, $\mathbf{R}^n$ be the space of solutions, C be an $m \times n$ matrix with the rows $C_i = (c_{i1}, c_{i2}, \dots, c_{in}) \in \mathbf{R}^n$, $i \in N_m = \{1, 2, \dots, m\}$, $x = (x_1, x_2, \dots, x_n)^T \in X \subseteq \mathbf{E}^n$, $n \geq 2$, $\mathbf{E} = \{0, 1\}$, $|X| \geq 2$. Let a linear vector criterion

$$Cx = (C_1x, C_2x, \ldots, C_mx)^T \to \min_{x \in X}$$

be specified on the set of Boolean vectors (solutions) X.

Under a m-criterion problem Boolean problem $Z^m(C)$, $C \in \mathbf{R}^{m\times n}$, we understand the problem of finding the Pareto set, i.e. the set of efficient (Pareto optimal) solutions

$$P^m(C) = \{x \in X : X(x, C) = \emptyset\},$$

where

$$X(x, C) = \{x' \in X : Cx' \leq Cx \ \&\ Cx' \neq Cx\}.$$

Since X is finite, the set $P^m(C)$ is not empty for any matrix $C \in \mathbf{R}^{m\times n}$.

We will perturb elements of the matrix C by adding matrices C' from $\mathbf{R}^{m\times n}$ to it. Thus, the perturbed problem $Z^m(C + C')$ has the form

$$(C + C')x \to \min_{x \in X}$$

and the Pareto set of such a problem has the form $P^m(C + C')$.

For any natural number d in the real space $\mathbf{R}^d$, we specify the Hölder norm l_p, $p \in [1, \infty]$, i.e., the norm of a vector $y = (y_1, y_2, \ldots, y_d)$ is understood to be the number

$$\|y\|_p = \begin{cases} \Big(\sum\limits_{i \in N_d} |y_i|^p\Big)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\limits_{i \in N_d} |y_i| & \text{if } p = \infty. \end{cases}$$

For any $p, q \in [1, \infty]$, let us define the Hölder norm l_p in the space of solutions $\mathbf{R}^n$ and the Hölder norm l_q in the criteria space $\mathbf{R}^m$. Thereby, the norm $\|C\|_{pq}$ of the matrix $C \in \mathbf{R}^{m\times n}$ is defined as the norm of the vector whose components are the norms of the matrix rows $C_1, C_2, \ldots, C_m$, i.e.

$$\|C\|_{pq} = \big\|\big(\|C_1\|_p, \|C_2\|_p, \ldots, \|C_m\|_p\big)^T\big\|_q.$$

It is easy to see that for any $p, q \in [1, \infty]$ the following inequalities hold

$$\|C_i\|_p \leq \|C\|_{pq}, \quad i \in N_m. \tag{1}$$

Obviously, for each $\alpha \geq 0$, $p \in [1, \infty]$ and vector $a = (a_1, a_2, \ldots, a_n) \in \mathbf{R}^n$ with components $|a_i| = \alpha$, $i \in N_n$, the following equality holds

$$\|a\|_p = \alpha n^{1/p}. \tag{2}$$

Let $l_{p'}$ be a conjugate norm in the space of solutions $\mathbf{R}^n$ and, as is well known, the numbers p and p' are related by the condition

$$1/p + 1/p' = 1.$$

As usual, we assume that $p' = 1$ if we have $p = \infty$ and that $p' = \infty$ if we have $p = 1$. Thus, henceforth, we assume that the domain of varying the numbers p and p' is the interval $[1, \infty]$ and that the numbers p and p' themselves are related by the above-mentioned condition. In addition, we impose that $1/p = 1$ if $p = \infty$.

We will use the well-known Hölder inequality

$$ab \leq \|a\|_p \|b\|_{p'}, \tag{3}$$

where $a = (a_1, a_2, \ldots, a_n) \in \mathbf{R}^n$ and $b = (b_1, b_2, \ldots, b_n)^T \in \mathbf{R}^n$.

As usually (see e. g. [6–9]), by the radius of stability of the problem $Z^m(C)$ we mean the quantity

$$\rho^m(p, q) = \begin{cases} \sup \Xi & \text{if } \Xi \neq \emptyset, \\ 0 & \text{if } \Xi = \emptyset, \end{cases}$$

where

$$\Xi = \big\{\varepsilon > 0 : \forall C' \in \Omega_{pq}(\varepsilon) \ \big(P^m(C + C') \subseteq P^m(C)\big)\big\},$$

$$\Omega_{pq}(\varepsilon) = \{C \in \mathbf{R}^{m \times n} : \|C\|_{pq} < \varepsilon\}.$$

Thus, the stability radius of the problem $Z^m(C)$ is the limiting perturbation of elements of the matrix C in the space $\mathbf{R}^{m \times n}$ that does not produce new efficient solutions. The set $\Omega_{pq}(\varepsilon)$ is called the set of perturbing matrices.

In the trivial case, where $P^m(C) = X$, the inclusion $P^m(C + C') \subseteq P^m(C)$ holds for any perturbing matrix $C' \in \Omega_{pq}(\varepsilon)$, $\varepsilon > 0$. Therefore, no one perturbation of the problem parameters can cause appearance of new efficient solutions, i. e. stability radius of such problem is unbounded above. The problem $Z^m(C)$ for which $P^m(C) \neq X$ will be called non-trivial.

3 Estimates of the stability radius

For a non-trivial problem $Z^m(C)$ and any $p, q \in [1, \infty]$ we assume

$$\varphi^m(p) = \min_{x \in X \setminus P^m(C)} \ \max_{x' \in P^m(x, C)} \ \min_{i \in N_m} \frac{C_i(x - x')}{\|x - x'\|_{p'}},$$

$$\psi^m(p, q) = \min\big\{\sigma^m(p), \ n^{1/p} m^{1/q} \varphi^m(\infty)\big\},$$

where

$$P^m(x, C) = P^m(C) \cap X(x, C),$$

$$\sigma^m(p) = \min\{\|C_i\|_p : i \in N_m\}.$$

Theorem 1. *For any $p, q \in [1, \infty]$ and $m \in \mathbf{N}$, the stability radius $\rho^m(p, q)$ of a non-trivial problem $Z^m(C)$ has the following bounds:*

$$\varphi^m(p) \leq \rho^m(p, q) \leq \psi^m(p, q).$$

Proof. First, let us prove the inequality $\rho^m(p,q) \geq \varphi^m(p)$ which is trivial in the case $\varphi^m(p) = 0$. Assume $\varphi^m(p) > 0$. Let $C' \in \Omega_{pq}(\varphi^m(p))$ be a perturbing matrix with rows C'_i, $i \in N_m$. By the definition of the number $\varphi^m(p)$ and according to (1), for any solution $x \in X \setminus P^m(C)$, there exists an effective solution $x^0 \in P^m(x, C)$ such that

$$\frac{C_i(x - x^0)}{\|x - x^0\|_{p'}} \geq \varphi^m(p) > \|C'\|_{pq} \geq \|C'_i\|_p, \quad i \in N_m.$$

Whence, using the Hölder inequality (3) we find

$$(C_i + C'_i)(x - x^0) \geq C_i(x - x^0) - \|C'_i\|_p \|x - x^0\|_{p'} > 0, \quad i \in N_m.$$

Thus, $x \notin P^m(C + C')$. Therefore, every ineffective solution of the problem $Z^m(C)$ retains its ineffectiveness in perturbed problem $Z^m(C + C')$. Hence $P^m(C + C') \subseteq P^m(C)$ for every perturbing matrix $C' \in \Omega_{pq}(\varphi^m(p))$, i. e. $\rho^m(p,q) \geq \varphi^m(p)$.

Now, let us prove the inequality $\rho^m(p,q) \leq n^{1/p} m^{1/q} \varphi^m(\infty)$. According to the definition of the number $\varphi^m(\infty)$ there exists a solution $x^0 \in X \setminus P^m(C)$ such that for each solution $x \in P^m(x^0, C)$ there exists an index $k = k(x) \in N_m$ satisfying

$$C_k(x^0 - x) \leq \varphi^m(\infty) \|x^0 - x\|_1. \tag{4}$$

Choose an arbitrary number ε that obeys the condition $\varepsilon > n^{1/p} m^{1/q} \varphi^m(\infty)$ and specify elements of the perturbing matrix $C^0 = [c^0_{ij}] \in \mathbf{R}^{m \times n}$ with rows C^0_i, $i \in N_m$, as follows

$$c^0_{ij} = \begin{cases} -\delta & \text{if } i \in N_m,\ x^0_j = 1, \\ \delta & \text{if } i \in N_m,\ x^0_j = 0, \end{cases}$$

where $\varphi^m(\infty) < \delta < \varepsilon / n^{1/p} m^{1/q}$. Using (2) we derive

$$\|C^0_i\|_p = \delta n^{1/p}, \quad i \in N_m,$$

$$\|C^0\|_{pq} = \delta n^{1/p} m^{1/q}, \quad C^0 \in \Omega_{pq}(\varepsilon),$$

$$C^0_i(x^0 - x) = -\delta \|x^0 - x\|_1 < 0, \quad i \in N_m. \tag{5}$$

Therefore, taking into account inequality (4) we obtain

$$(C_k + C^0_k)(x^0 - x) = C_k(x^0 - x) + C^0_k(x^0 - x) \leq (\varphi^m(\infty) - \delta)\|x^0 - x\|_1 < 0.$$

As a result we have

$$\forall x \in P^m(x^0, C) \quad \big(x \notin X(x^0, C + C^0)\big). \tag{6}$$

If $X(x^0, C + C^0) = \emptyset$, then $x^0 \in P^m(C + C^0)$. Assume $X(x^0, C + C^0) \neq \emptyset$. In this case, due to external stability of the Pareto set $P^m(C + C^0)$ (see e. g. p. 34 in [10]) there exists a solution $x^* \in P^m(x^0, C + C^0)$. Let us prove that $x^* \notin P^m(C)$. Assume to the contrary that $x^* \in P^m(C)$. According to (6) this yields $x^* \in P^m(C) \setminus P^m(x^0, C)$. Therefore, there are only two cases: the equality

$Cx^* = Cx^0$ holds or the inequality $Cx^* \leq Cx^0$ does not hold. In the first case, taking into account (5) we have

$$(C_i + C_i^0)(x^0 - x^*) < 0, \quad i \in N_m.$$

In the second case, there exists an index $l \in N_m$ such that $C_l x^* > C_l x^0$. Taking into account (5) again we obtain

$$(C_l + C_l^0)(x^0 - x^*) < 0.$$

In both cases we obtained contradictions with the inclusion $x^* \in P^m(x^0, C + C^0)$.

Summarizing the above we state that for any number $\varepsilon > n^{1/p} m^{1/q} \varphi^m(\infty)$ there exist perturbing matrix $C^0 \in \Omega_{pq}(\varepsilon)$ and an inefficient solution (x^0 or x^*) of the problem $Z^m(C)$ such that it becomes efficient in perturbed problem $Z^m(C + C^0)$. Hence

$$\forall \varepsilon > n^{1/p} m^{1/q} \varphi^m(\infty) \; \exists C^0 \in \Omega_{pq}(\varepsilon) \;\; \big(P^m(C + C^0) \not\subseteq P^m(C)\big),$$

i. e.

$$\rho^m(p, q) \leq n^{1/p} m^{1/q} \varphi^m(\infty).$$

It remains to prove that $\rho^m(p, q) \leq \sigma^m(p)$. Let x^0 be an inefficient solution of the problem $Z^m(C)$ and the index $k \in N_m$ be such that

$$\sigma^m(p) = \|C_k\|_p. \tag{7}$$

Assuming that $\varepsilon > \sigma^m(p)$ we denote a number δ with the conditions

$$0 < \delta n^{1/p} < \varepsilon - \sigma^m(p). \tag{8}$$

We define the vector $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$ by

$$\eta_j = \begin{cases} -\delta & \text{if } x_j^0 = 1, \\ \delta & \text{if } x_j^0 = 0. \end{cases}$$

Then we have

$$\|\eta\|_p = \delta n^{1/p} \tag{9}$$

and for each solution $x \in X \setminus \{x^0\}$ we obtain

$$\eta(x^0 - x) = -\delta \|x^0 - x\|_1 < 0. \tag{10}$$

We specify the rows $C_i^0 \in \mathbf{R}^n$, $i \in N_m$, of the perturbing matrix $C^0 \in \mathbf{R}^{m \times n}$ by the rule

$$C_i^0 = \begin{cases} \eta - C_i & \text{if } i = k, \\ \mathbf{0} & \text{if } i \in N_m \setminus \{k\}. \end{cases}$$

Hence, taking into account (10) we derive

$$C_k^0(x^0 - x) = (\eta - C_k)(x^0 - x) = -\delta \|x^0 - x\|_1 - C_k(x^0 - x).$$

It follows from equalities (7) and (9) and inequality (8) that

$$\|C^0\|_{pq} = \|C^0_k\|_p = \|\eta - C_k\|_p \leq \|\eta\|_p + \|C_k\|_p = \delta n^{1/p} + \sigma^m(p) < \varepsilon.$$

Consequently, for each solution $x \in X \setminus \{x^0\}$ we deduce

$$(C_k + C^0_k)(x^0 - x) = -\delta\|x^0 - x\|_1 < 0,$$

i.e. $x \notin X(x^0, C + C^0)$, where $C^0 \in \Omega_{pq}(\varepsilon)$. Using $x^0 \notin X(x^0, C + C^0)$ we get $X(x^0, C + C^0) = \emptyset$, which implies $x^0 \in P^m(C + C^0)$. Due to $x^0 \notin P^m(C)$ the inequality $\rho^m(p, q) \leq \varepsilon$ is true for any number $\varepsilon > \sigma^m(p)$. Thus we have proved that $\rho^m(p, q) \leq \sigma^m(p)$. This with proved inequality $\rho^m(p, q) \leq n^{1/p}m^{1/q}\varphi^m(\infty)$ implies $\rho^m(p, q) \leq \psi^m(p, q)$. □

4 Corollaries

As corollaries of Theorem 1 we obtain the following results.

Corollary 1 [11]. $\varphi^m(p) \leq \rho^m(p, p) \leq (nm)^{1/p}\varphi^m(\infty)$.

Corollary 2 [6]. $\rho^m(\infty, \infty) = \varphi^m(\infty) = \min\limits_{x \in X \setminus P^m(C)} \max\limits_{x' \in P^m(x,C)} \min\limits_{i \in N_m} \dfrac{C_i(x - x')}{\|x - x'\|_1}$.

Corollary 3 [12]. $\varphi^m(p) \leq \rho^m(p, \infty) \leq n^{1/p}\varphi^m(\infty)$.

Corollary 4 [13]. $\varphi^m(\infty) \leq \rho^m(\infty, q) \leq m^{1/q}\varphi^m(\infty)$.

Note that the paper [13] describes a class of problems $Z^m(C)$ for which the following formula holds

$$\rho^m(\infty, q) = m^{1/q}\varphi^m(\infty), \quad q \in [1, \infty].$$

This means that the upper-bound of Corollary 4 is achievable.

The following known result proves that the lower-bound estimate of the stability radius is achievable.

Theorem 2 [9]. *If* $|P^m(C)| = 1$, *then for any numbers* $p, q \in [1, \infty]$ *the stability radius is expressed by the formula*

$$\rho^m(p, q) = \varphi^m(p).$$

We denote the stability radius of scalar problem $Z^1(C)$

$$Cx \to \min_{x \in X}, \quad C \in \mathbf{R}^{1 \times n}, \quad X \subseteq \mathbf{E}^n,$$

by $\rho^1(p)$, $p \in [1, \infty]$.

Corollary 5. $\varphi^1(p) \leq \rho^1(p) \leq n^{1/p}\varphi^1(\infty)$.

The paper [12] describes a class of scalar linear problems $Z^1(C)$ for which the following formula holds

$$\rho^1(p) = n^{1/p}\varphi^1(\infty), \quad p \in [1, \infty].$$

Therefore, the upper-bound of Corollary 5 is achievable.

Corollaries 2 and 5 imply the following known result.

Corollary 6 [14, 15]. $\rho^1(\infty) = \varphi^1(\infty)$.

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On Soft Trees

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Abstract. In this paper, we introduce the notions of soft trees, soft cycles, soft bridges, soft cutnodes, and describe various methods of construction of soft trees. We investigate some of their fundamental properties.

Mathematics subject classification: 05C99, 03E72.
Keywords and phrases: Soft sets, soft cycle, soft bridge, soft cutnode, soft trees.

1 Introduction and Preliminaries

In 1975, Rosenfeld [11] first discussed the concept of fuzzy graphs whose basic idea was introduced by Kauffmann [8] in 1973. Rosenfeld also proposed the fuzzy relations between fuzzy sets and developed the structure of fuzzy graphs, obtaining analogs of several graph theoretical concepts. Moreover, Bhattacharya [5] gave some remarks on fuzzy graphs. Bhutani and Rosenfeld [6] introduced the concept of $M-$ strong fuzzy graphs and described some of their properties. Recently, Thumbakara and George [13] discussed the concept of soft graphs in the specific way. On the other hand, Akram and Nawaz [4] have introduced the concepts of soft graphs and vertex-induced soft graphs in broad spectrum. In this paper, we introduce the concepts of soft trees, soft cycles, soft bridges, soft cutnodes and investigate some of their properties. We discuss some interesting properties of soft trees as a generalization of crisp trees. We also introduce some operations including union, intersection, AND operation and OR operation on soft trees.

Soft sets were proposed by Molodtsov in 1999, which provides a general mechanism for uncertainty modelling in a wide variety of applications [8, 12]. Let U be an initial universe of objects and P be the set of all parameters associated with objects in U, called a *parameter space*. In most cases parameters are considered to be attributes, characteristics or properties of objects in U. The power set of U is denoted by $\mathcal{P}(U)$.

Definition 1 (see [10])**.** A pair $\mathfrak{S} = (F, A)$ is called a *soft set* over U, where $A \subseteq P$ is a parameter set and $F\colon A \to \mathcal{P}(U)$ is a set-valued mapping, called the *approximate function* of the soft set $\mathfrak{S}$.

Let $G^* = (V, E)$ be a crisp graph and A be any nonempty set. Let subset R of $A \times V$ be an arbitrary relation from A to V. A mapping (or set-valued function) from A to $\mathcal{P}(V)$ written as $F : A \to \mathcal{P}(V)$ can be defined as $F(x) = \{y \in V : xRy\}$. The pair (F, A) is a soft set over V.

Definition 2 (see [3])**.** A *soft graph* $G = (G^*, F, K, A)$ is a 4-tuple such that

(a) $G^* = (V, E)$ is a simple graph,

(b) A is a non-empty set of parameters,

(c) (F, A) is a soft set over V,

(d) (K, A) is a soft set over E,

(e) $H(x) = (F(x), K(x))$ is a subgraph of G^* for all $x \in A$.

In what follows, we will use G^* for a simple graph, G for a soft graph and $H(x)$ for subgraph.

Definition 3 (see [3])**.** Let G be a soft graph of G^*. Then G is said to be a *complete soft graph* if every $H(x)$ is a complete graph for all $x \in A$.

2 Soft Trees

Definition 4. Let G be a soft graph of G^*. Then G is said to be a *soft tree* if every $H(x)$ is a tree for all $x \in A$.

Example 1. Consider a simple graph $G^* = (V, E)$, where $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E = \{v_1v_2, v_2v_3, v_3v_5, v_3v_4, v_4v_5, v_5v_6, v_6v_1\}$. Let $A = \{v_2, v_6\} \subseteq V$. We define an approximate function

$$F : A \to \mathcal{P}(V) \text{ by } F(x) = \{y \in V : xRy \Leftrightarrow d(x, y) \leq 1\}.$$

That is, $F(v_2) = \{v_1, v_2, v_3\}$, $F(v_6) = \{v_1, v_5, v_6\}$. Thus, $(F, A) = \{F(v_2), F(v_6)\}$ is a soft set over V. We now define an approximate function $K : A \to \mathcal{P}(E)$ by

$$K(x) = \{xy \in E : xRxy \Leftrightarrow xy \subseteq F(x)\}.$$

That is, $K(v_2) = \{v_1v_2, v_2v_3\}$, $K(v_6) = \{v_5v_6, v_6v_1\}$. Thus, $(K, A) = \{K(v_2), K(v_6)\}$ is a soft set over E. By routine calculations, it is easy to see that $H(v_2) = (F(v_2), K(v_2))$, $H(v_6) = (F(v_6), K(v_6))$ are connected subgraphs of G^* and also trees as shown in Fig. 1.

Hence, $G = \{H(v_2) = (F(v_2), K(v_2)), H(v_6) = (F(v_6), K(v_6))\}$ is a soft tree of G^*.

Theorem 1. *Let $H(x)$ be subgraph with $n \geq 3$ vertices of G^* and G a soft tree of G^*. Then G is not a complete soft graph.*

Proof. Suppose on contrary that G is a complete soft graph, then every subgraph $H(x)$, for all $x \in A$ is complete. Suppose u, v be arbitrary nodes of $H(x)$ and they are connected by an edge uv. Since $H(x)$ is a graph with $n \geq 3$ vertices, then we can always find at least one vertex w which is connected to v by an edge wv and to u by an edge wu, because $H(x)$ is a complete graph. Then there exists a cycle uvw. Therefore, $H(x)$ cannot be a tree which contradicts the fact that $H(x)$ is a connected subgraph of soft tree G. Hence, G can not be a complete soft graph. $\square$

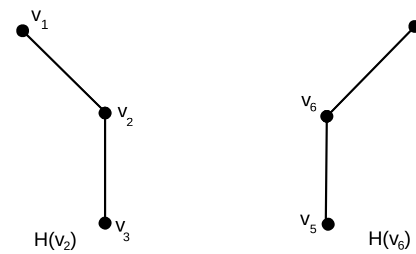

Figure 1. Subtrees

Definition 5. Let G be a soft graph of G^*. Then G is said to be a *soft cycle* if $H(x)$ is a cycle for all $x \in A$.

Example 2. Consider a simple graph $G^* = (V, E)$, where $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{v_1v_2, v_1v_4, v_2v_4, v_2v_3, v_3v_4, v_4v_5, v_5v_1\}$. Let $A = \{v_3, v_5\} \subseteq V$. We define an approximate function $F : A \to \mathcal{P}(V)$ by

$$F(x) = \{y \in V : xRy \Leftrightarrow d(x, y) \leq 1\}.$$

That is, $F(v_3) = \{v_2, v_3, v_4\}$, $F(v_5) = \{v_1, v_4, v_5\}$. Thus, $(F, A) = \{F(v_3), F(v_5)\}$ is a soft set over V. We now define an approximate function $K : A \to \mathcal{P}(E)$ by

$$K(x) = \{xy \in E : xRxy \Leftrightarrow xy \subseteq F(x)\}.$$

That is, $K(v_3) = \{v_2v_3, v_3v_4, v_4v_2\}$, $K(v_5) = \{v_1v_4, v_4v_5, v_5v_1\}$. Thus, $(K, A) = \{K(v_3), K(v_5)\}$ is a soft set over E. By routine calculations, it is easy to see that $H(v_3) = (F(v_3), K(v_3))$, $H(v_5) = (F(v_5), K(v_5))$ are connected subgraphs of G^* and also cycles as shown in the Fig. 2. Hence, $G = \{H(v_3) = (F(v_3), K(v_3)), H(v_5) =$

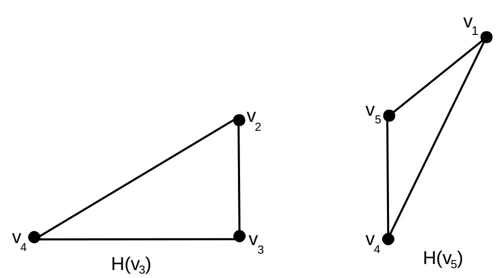

Figure 2. Subcycles

$(F(v_5), K(v_5))\}$ is a soft cycle of G^*.

Definition 6. Let G be a soft graph of G^*. Let u, v be two nodes and $H(x)$ a subgraph of G^*, then an edge $uv \in H(x)$ is called a *soft bridge* of G if removal of uv disconnects the $H(x)$.

Definition 7. Let G be a soft graph of G^*. Let u be a node of G^*, then u is called a *soft cutnode* of G if deletion of it from some $H(x)$, a subgraph of G, disconnects the $H(x)$.

In other words, we can say that u is a *soft cutnode* if it is a cutnode of some $H(x)$, $x \in A$.

Example 3. Consider a simple graph $G^* = (V, E)$, where $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E = \{v_1v_2, v_2v_6, v_2v_3, v_3v_4, v_5v_6\}$. Let $A = \{v_1, v_4\} \subseteq V$. We define an approximate function $F : A \to \mathcal{P}(V)$ by

$$F(x) = \{y \in V : xRy \Leftrightarrow e(y) \leq e(x)\},$$

where $e(v_1) = e(v_3) = e(v_6) = 3$, $e(v_2) = 2$, $e(v_4) = e(v_5) = 4$,., That is, $F(v_1) = \{v_1, v_2, v_3, v_6\}$, $F(v_4) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. Thus, $(F, A) = \{F(v_1), F(v_4)\}$ is a soft set over V. We now define an approximate function $K : A \to \mathcal{P}(E)$ by

$$K(x) = \{xy \in E : xRxy \Leftrightarrow xy \subseteq F(x)\}.$$

That is, $K(v_1) = \{v_1v_2, v_2v_6, v_2v_3\}$, $K(v_4) = \{v_1v_2, v_2v_6, v_2v_3, v_3v_4, v_5v_6\}$. Thus, $(K, A) = \{K(v_1), K(v_4)\}$ is a soft set over E. By routine calculations, it is easy to see that $H(v_1) = (F(v_1), K(v_1))$ and $H(v_4) = (F(v_4), K(v_4))$ are connected subgraphs of G^* as shown in Fig. 3. Therefore, $G = \{H(v_1) =$

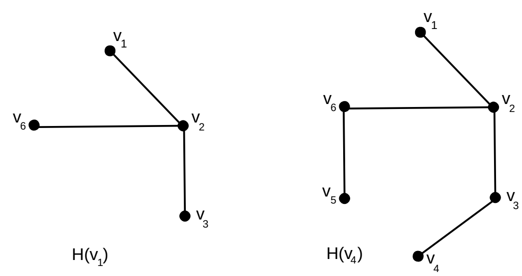

Figure 3. Connected subgraphs

$(F(v_1), K(v_1)), H(v_4) = (F(v_4), K(v_4))\}$ is a soft graph. In $H(v_1)$, all edges v_2v_6, v_1v_2, v_2v_3 are bridges because removal of any edge from $H(v_1)$ disconnects it as shown in Fig. 4. In $H(v_4)$, $v_1v_2, v_2v_6, v_2v_3, v_3v_4, v_5v_6$ are bridges, because removal of any edge from $H(v_4)$ disconnects it as shown in Fig. 5. Therefore, $v_1v_2, v_2v_6, v_2v_3, v_3v_4, v_5v_6$ are soft bridges of G. v_2 is a cutnode of $H(v_1)$ because deletion of it from $H(v_1)$ disconnects the $H(v_1)$ as shown in Fig. 6. Here v_2, v_3, v_6 are cutnodes of $H(v_4)$ because deletion of each of them from $H(v_4)$ disconnects the $H(v_4)$ as shown in Fig. 7. Therefore, v_2, v_3, v_6 are soft cutnodes of G.

Theorem 2. *If w is a common node of at least two soft bridges, then w is a soft cutnode.*

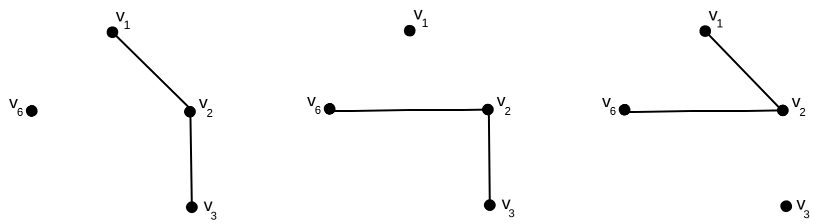

Figure 4. Disconnected subgraphs of $H(v_1)$

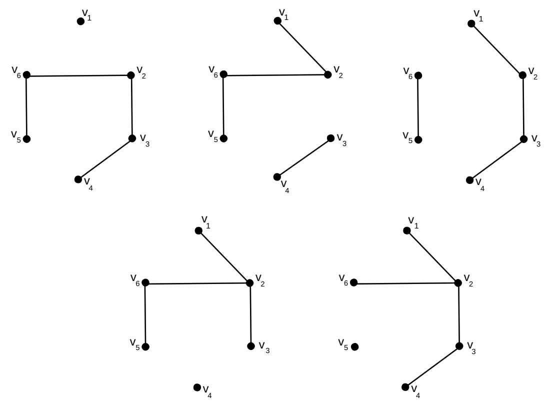

Figure 5. Disconnected subgraph of $H(v_4)$

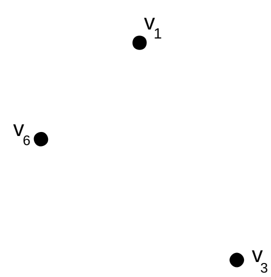

Figure 6. Disconnected subgraphs of $H(v_1)$

Proof. Let v_1w and wv_2 be two soft bridges of G. Then v_1w and wv_2 are bridges of some $H(x)$, that is, there exist some u, v such that v_1w is on every $u - v$ path. Clearly, if we delete w, then all the edges associated with it get removed. Then every $u - v$ path is disconnected. Thus, $H(x)$ is disconnected and w is a cutnode. Hence, w is a soft cutnode. □

Remark 1. The converse statement of Theorem 2 is not true as it can be seen in the following example.

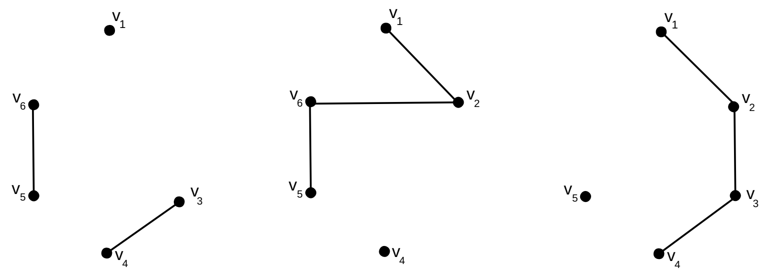

Figure 7. Disconnected subgraphs of $H(v_4)$

Example 4. Consider a simple graph $G^* = (V, E)$, where $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{v_1v_2, v_2v_3, v_3v_1, v_3v_4, v_4v_5\}$. Let $A = \{v_1, v_2\} \subseteq V$. We define an approximate function $F : A \to \mathcal{P}(V)$ by

$$F(x) = \{y \in V : xRy \Leftrightarrow d(x, y) \geq 0\}.$$

That is, $F(v_1) = \{v_1, v_2, v_3, v_4, v_5\} = F(v_2)$. Thus, $(F, A) = \{F(v_1) = F(v_2)\}$ is a soft set over V. We now define an approximate function $K : A \to \mathcal{P}(E)$ by

$$K(x) = \{xy \in E : xRxy \Leftrightarrow xy \subseteq F(x)\}.$$

That is, $K(v_1) = \{v_1v_2, v_2v_3, v_1v_3, v_3v_4, v_4v_5\} = K(v_2)$. Thus, $(K, A) = \{K(v_1) = K(v_2)\}$ is a soft set over E.

Thus, $H(v_1) = (F(v_1), K(v_1))$ and $H(v_2) = (F(v_2), K(v_2))$ are connected subgraphs of G^* as shown in Fig. 8. In both $H(v_1)$ and $H(v_2)$, v_3v_4, v_4v_5 are bridges as

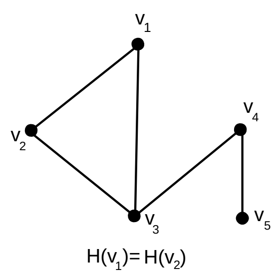

Figure 8. Connected subgraphs

shown in Fig. 9. Therefore, v_3v_4, v_4v_5 are soft bridges of G. v_3, v_4 are cutnodes as shown in Fig. 10. Therefore, v_3, v_4 are soft cutnodes of G. Here v_3 is a soft cutnode but it is not a common node of two soft bridges.

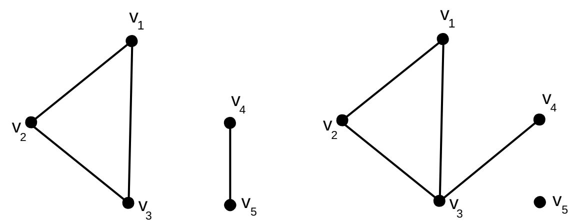

Figure 9. Disconnected subgraphs of $H(v_1) = H(v_2)$

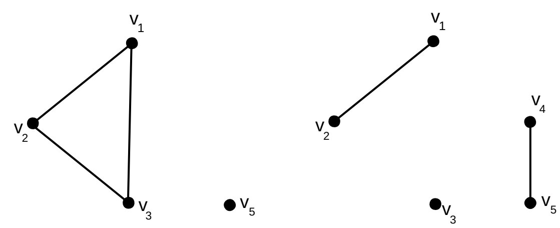

Figure 10. Disconnected subgraphs of $H(v_1) = H(v_2)$

We state the following theorems without their proofs.

Theorem 3. *A complete soft graph has no soft cutnodes.*

Theorem 4. *If G is a soft tree, then all edges of G are the soft bridges of G.*

Theorem 5. *If G is a soft tree, then internal nodes of G are the soft cutnodes of G.*

2.1 Operations on soft trees

Definition 8. Let $G_1 = (G^*, F_1, K_1, A)$ and $G_2 = (G^*, F_2, K_2, B)$ be two soft trees of G^*. The *union of two soft trees* G_1 and G_2 is a soft graph and defined as $G = G_1 \cup G_2 = (G^*, F, K, C)$ if $H(x) = (F(x), K(x))$ for all $x \in C$ is a subgraph, where $C = A \cup B$ and for all $x \in C$,

$$F(x) = \begin{cases} F_1(x) & \text{if } x \in A - B, \\ F_2(x) & \text{if } x \in B - A, \\ F_1(x) \cup F_2(x) & \text{if } x \in A \cap B. \end{cases}$$

$$K(x) = \begin{cases} K_1(x) & \text{if } x \in A - B, \\ K_2(x) & \text{if } x \in B - A, \\ K_1(x) \cup K_2(x) & \text{if } x \in A \cap B. \end{cases}$$

Theorem 6. *Let $G_1 = (G^*, F_1, K_1, A)$ and $G_2 = (G^*, F_2, K_2, B)$ be two soft trees of G^* such that $A \cap B = \emptyset$, then $G_1 \cup G_2$ is a soft tree.*

Proof. The union of $G_1 = (G^*, F_1, K_1, A)$ and $G_2 = (G^*, F_2, K_2, B)$ is defined as

$$G = G_1 \cup G_2 = (G^*, F, K, C),$$

where $C = A \cup B$ and for all $x \in C$,

$$F(x) = \begin{cases} F_1(x) & \text{if } x \in A - B, \\ F_2(x) & \text{if } x \in B - A, \\ F_1(x) \cup F_2(x) & \text{if } x \in A \cap B. \end{cases}$$

$$K(x) = \begin{cases} K_1(x) & \text{if } x \in A - B, \\ K_2(x) & \text{if } x \in B - A, \\ K_1(x) \cup K_2(x) & \text{if } x \in A \cap B. \end{cases}$$

Since $A \cap B = \emptyset$, then $A - B = A$ and $B - A = B$. Thus,

$$F(x) = \begin{cases} F_1(x) & \text{if } x \in A, \\ F_2(x) & \text{if } x \in B. \end{cases}$$

$$K(x) = \begin{cases} K_1(x) & \text{if } x \in A, \\ K_2(x) & \text{if } x \in B. \end{cases}$$

$H_1(x) = (F_1(x), K_1(x))$ and $H_2(x) = (F_2(x), K_2(x))$ are trees, since G_1 and G_2 are soft trees. Therefore, $H = (F(x), K(x))$ is a tree and $G = (G^*, F, K, C)$ is a soft tree. □

Remark 2. If $A \cap B \neq \emptyset$, then union of two soft trees may not be a soft tree as it can be seen in the following example.

Example 5. Consider a simple graph $G^* = (V, E)$, where $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1\}$. Let $G_1 = (G^*, F_1, K_1, A)$ and $G_2 = (G^*, F_2, K_2, B)$ be two soft trees of G^*.

Let $A = \{v_3, v_4\} \subseteq V$ and $B = \{v_4\} \subseteq V$. We define approximate functions $F_1 : A \to \mathcal{P}(V)$ and $F_2 : B \to \mathcal{P}(V)$ by

$$F_1(x) = \{y \in V : xRy \Leftrightarrow d(x, y) \leq 1\} \ \forall x \in A,$$

i. e., $F_1(v_3) = \{v_2, v_3, v_4\}$, $F_1(v_4) = \{v_3, v_4, v_5\}$, and

$$F_2(x) = \{y \in V : xRy \Leftrightarrow d(x, y) \geq 1\} \ \forall x \in B,$$

i. e., $F_2(v_4) = \{v_1, v_2, v_3, v_5\}$, respectively. Thus, $F(v_3) = F_1(v_3) = \{v_2, v_3, v_4\}$, $F(v_4) = F_1(v_4) \cup F_2(v_4) = \{v_1, v_2, v_3, v_4, v_5\}$. Thus, $(F, A) = \{F(v_3), F(v_4)\}$ is a soft set over V.

We now define approximate functions $K_1 : A \to \mathcal{P}(E)$ and $K_2 : B \to \mathcal{P}(E)$ by

$$K_1(x) = \{xy \in E : xRxy \Leftrightarrow xy \subseteq F_1(x)\} \ \forall x \in A,$$

i. e., $K_1(v_3) = \{v_2v_3, v_3v_4\}$, $K_1(v_4) = \{v_3v_4, v_4v_5\}$, and

$$K_2(x) = \{xy \in E : xRxy \Leftrightarrow xy \subseteq F_2(x)\}\ \forall x \in B,$$

i. e., $K_2(v_4) = \{v_1v_2, v_2v_3, v_5v_1\}$, respectively. Thus, $K(v_3) = K_1(v_3) = \{v_2v_3, v_3v_4\}$, $K(v_4) = K_1(v_4) \cup K_2(v_4) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1\}$. Thus, $(K, A) = \{K(v_3), K(v_4)\}$ is a soft set over E. By routine calculations, it is easy to see that $H(v_3) = (F(v_3), K(v_3))$ and $H(v_4) = (F(v_4), K(v_4))$ are connected subgraphs of G^*. $H(v_3) = (F(v_3), K(v_3))$ is a tree but $H(v_4) = (F(v_4), K(v_4))$ is a cycle as shown in Fig. 11. Hence, $G = \{H(v_3) = (F(v_3), K(v_3)), H(v_4) = (F(v_4), K(v_4))\}$

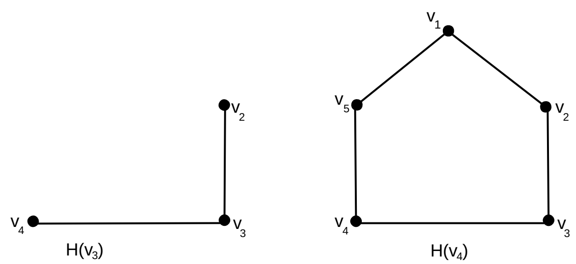

Figure 11. Connected subgraphs

is not a soft tree of G^*.

Definition 9. Let $G_1 = (G^*, F_1, K_1, A)$ and $G_2 = (G^*, F_2, K_2, B)$ be two soft trees of G^*. The *intersection of two soft trees* G_1 and G_2 is a soft graph and defined as $G = G_1 \cap G_2 = (G^*, F, K, C)$ if $H(x) = (F(x), K(x))$ for all $x \in C$ is a subgraph, where $C = A \cup B$ and for all $x \in C$,

$$F(x) = \begin{cases} F_1(x) & \text{if } x \in A - B, \\ F_2(x) & \text{if } x \in B - A, \\ F_1(x) \cap F_2(x) & \text{if } x \in A \cap B. \end{cases}$$

$$K(x) = \begin{cases} K_1(x), & \text{if } x \in A - B, \\ K_2(x), & \text{if } x \in B - A, \\ K_1(x) \cap K_2(x), & \text{if } x \in A \cap B. \end{cases}$$

Theorem 7. *Let $G_1 = (G^*, F_1, K_1, A)$ and $G_2 = (G^*, F_2, K_2, B)$ be two soft trees of G^* such that $A \cap B = \emptyset$, then $G_1 \cap G_2$ is a soft tree.*

Proof. The intersection of $G_1 = (G^*, F_1, K_1, A)$ and $G_2 = (G^*, F_2, K_2, B)$ is defined as

$$G = G_1 \cap G_2 = (G^*, F, K, C),$$

where $C = A \cup B$ and for all $x \in C$,

$$F(x) = \begin{cases} F_1(x) & \text{if } x \in A - B, \\ F_2(x) & \text{if } x \in B - A, \\ F_1(x) \cap F_2(x) & \text{if } x \in A \cap B. \end{cases}$$

$$K(x) = \begin{cases} K_1(x) & \text{if } x \in A - B, \\ K_2(x) & \text{if } x \in B - A, \\ K_1(x) \cap K_2(x) & \text{if } x \in A \cap B. \end{cases}$$

Since $A \cap B = \emptyset$, then $A - B = A$ and $B - A = B$. Thus,

$$F(x) = \begin{cases} F_1(x) & \text{if } x \in A, \\ F_2(x) & \text{if } x \in B. \end{cases}$$

$$K(x) = \begin{cases} K_1(x) & \text{if } x \in A, \\ K_2(x) & \text{if } x \in B. \end{cases}$$

$H_1(x) = (F_1(x), K_1(x))$ and $H_2(x) = (F_2(x), K_2(x))$ are trees, since G_1 and G_2 are soft trees. Therefore, $H = (F(x), K(x))$ is a tree and $G = (G^*, F, K, C)$ is a soft tree. $\square$

Remark 3. If $A \cap B \neq \emptyset$, then intersection of two soft trees may not be a soft tree as it can be seen in the following example.

Example 6. Consider a simple graph $G^* = (V, E)$, where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$. Let $G_1 = (G^*, F_1, K_1, A)$ and $G_2 = (G^*, F_2, K_2, B)$ be two soft trees of G^*. Let $A = \{v_2, v_4\} \subseteq V$ and $B = \{v_3, v_4\} \subseteq V$. We define approximate functions $F_1 : A \to \mathcal{P}(V)$ and $F_2 : B \to \mathcal{P}(V)$ by

$$F_1(x) = \{y \in V : xRy \Leftrightarrow d(x, y) \geq 1\} \ \forall x \in A,$$

i. e., $F_1(v_2) = \{v_1, v_3, v_4\}$, $F_1(v_4) = \{v_1, v_2, v_3\}$, and

$$F_2(x) = \{y \in V : xRy \Leftrightarrow d(x, y) \leq 1\} \ \forall x \in B,$$

i. e., $F_2(v_3) = \{v_2, v_3, v_4\}$, $F_2(v_4) = \{v_1, v_3, v_4\}$, respectively. Thus, $F(v_2) = F_1(v_2) = \{v_1, v_3, v_4\}$, $F(v_3) = F_2(v_3) = \{v_2, v_3, v_4\}$, $F(v_4) = F_1(v_4) \cap F_2(v_4) = \{v_1, v_3\}$. Thus, $(F, A) = \{F(v_2), F(v_3), F(v_4)\}$ is a soft set over V. We now define approximate functions $K_1 : A \to \mathcal{P}(E)$ and $K_2 : B \to \mathcal{P}(E)$ by

$$K_1(x) = \{xy \in E : xRxy \Leftrightarrow xy \subseteq F_1(x)\} \ \forall x \in A,$$

i. e., $K_1(v_2) = \{v_3v_4, v_4v_1\}$, $K_1(v_4) = \{v_1v_2, v_2v_3\}$, and

$$K_2(x) = \{xy \in E : xRxy \Leftrightarrow xy \subseteq F_2(x)\} \ \forall x \in B,$$

i. e., $K_2(v_3) = \{v_2v_3, v_3v_4\}$, $K_2(v_4) = \{v_3v_4, v_4v_1\}$, respectively. Thus, $K(v_2) = K_1(v_2) = \{v_3v_4, v_4v_1\}$, $K(v_3) = K_2(v_3) = \{v_2v_3, v_3v_4\}$, $K(v_4) = K_1(v_4) \cap K_2(v_4) = \{\}$. Thus, $(K, A) = \{K(v_2), K(v_3), K(v_4)\}$ is a soft set over E. By routine calculations, it is easy to see that $H(v_2) = (F(v_2), K(v_2))$ and $H(v_3) = (F(v_3), K(v_3))$ are connected subgraphs of G^* as well as trees as shown in Fig. 12. But $H(v_4) = (F(v_4), K(v_4))$ is not a connected subgraph and hence, not a tree as shown in Fig. 13. Hence, $G = \{H(v_2) = (F(v_2), K(v_2)), H(v_3) = (F(v_3), K(v_3)), H(v_4) = (F(v_4), K(v_4))\}$ is not a soft tree of G^*.

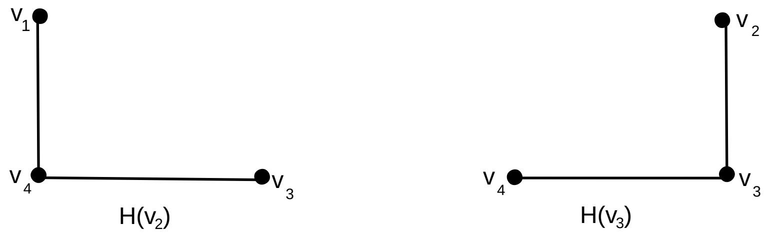

Figure 12. Subtrees

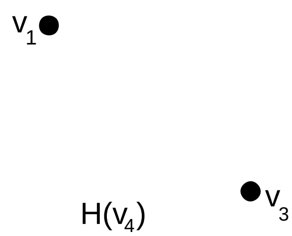

Figure 13. Disconnected subgraph

Definition 10. Let $G_1 = (G^*, F_1, K_1, A)$ and $G_2 = (G^*, F_2, K_2, B)$ be two soft trees of G^*. The *OR operation on two soft trees* G_1 and G_2 is a soft tree and defined as

$$G = G_1 \vee G_2 = (G^*, F, K, C),$$

if the subgraph $H(a, b) = (F(a, b), K(a, b))$, for all $(a, b) \in C$ is a tree, where $C = A \times B$ and for all $(a, b) \in C$, $F(a, b) = F_1(a) \cup F_2(b)$ and $K(a, b) = K_1(a) \cup K_2(b)$.

Example 7. Consider a simple graph $G^* = (V, E)$, where $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1\}$. Let $G_1 = (G^*, F_1, K_1, A)$ and $G_2 = (G^*, F_2, K_2, B)$ be two soft trees of G^*.

Let $A = \{v_2, v_5\} \subseteq V$ and $B = \{v_1, v_3\} \subseteq V$. Then $A \times B = \{(v_2, v_1), (v_2, v_3), (v_5, v_1), (v_5, v_3)\}$.

We define approximate functions $F_1 : A \to \mathcal{P}(V)$ and $F_2 : B \to \mathcal{P}(V)$ by

$$F_1(x) = \{y \in V : xRy \Leftrightarrow d(x, y) \leq 1\} \ \forall x \in A,$$

i.e., $F_1(v_2) = \{v_1, v_2, v_3\}$, $F_1(v_5) = \{v_1, v_4, v_5\}$, and

$$F_2(x) = \{y \in V : xRy \Leftrightarrow d(x, y) > 1\} \ \forall x \in B,$$

i.e., $F_2(v_1) = \{v_3, v_4\}$, $F_2(v_3) = \{v_1, v_5\}$, respectively. Thus, $F(v_2, v_1) = F_1(v_2) \cup F_2(v_1) = \{v_1, v_2, v_3, v_4\}$, $F(v_2, v_3) = F_1(v_2) \cup F_2(v_3) = \{v_1, v_2, v_3, v_5\}$, $F(v_5, v_1) = F_1(v_5) \cup F_2(v_1) = \{v_1, v_3, v_4, v_5\}$, $F(v_5, v_3) = F_1(v_5) \cup F_2(v_3) = \{v_1, v_4, v_5\}$. Thus, $(F, A) = \{F(v_2, v_1), F(v_2, v_3), F(v_5, v_1), F(v_5, v_3)\}$ is a soft set over V.

We now define approximate functions $K_1 : A \to \mathcal{P}(E)$ and $K_2 : B \to \mathcal{P}(E)$ by

$$K_1(x) = \{xy \in E : xRxy \Leftrightarrow xy \subseteq F_1(x)\}\ \forall x \in A,$$

i. e., $K_1(v_2) = \{v_1v_2, v_2v_3\}$, $K_1(v_5) = \{v_4v_5, v_5v_1\}$, and

$$K_2(x) = \{xy \in E : xRxy \Leftrightarrow xy \subseteq F_2(x)\}\ \forall x \in B,$$

i. e., $K_2(v_1) = \{v_3v_4\}$, $K_2(v_3) = \{v_5v_1\}$, respectively. Thus, $K(v_2, v_1) = K_1(v_2) \cup K_2(v_1) = \{v_1v_2, v_2v_3, v_3v_4\}$, $K(v_2, v_3) = K_1(v_2) \cup K_2(v_3) = \{v_1v_2, v_2v_3, v_5v_1\}$, $K(v_5, v_1) = K_1(v_5) \cup K_2(v_1) = \{v_3v_4, v_4v_5, v_5v_1\}$, $K(v_5, v_3) = K_1(v_5) \cup K_2(v_3) = \{v_4v_5, v_5v_1\}$. Thus, $(K, A) = \{K(v_2, v_1), K(v_2, v_3), K(v_5, v_1), K(v_5, v_3)\}$ is a soft set over E. Hence, $H(v_2, v_1) = (F(v_2, v_1), K(v_2, v_1))$, $H(v_2, v_3) = (F(v_2, v_3), K(v_2, v_3))$, $H(v_5, v_1) = (F(v_5, v_1), K(v_5, v_1))$ and $H(v_5, v_3) = (F(v_5, v_3), K(v_5, v_3))$ are connected subgraphs of G^* and are also trees as shown in Fig. 14. Hence, $G =$

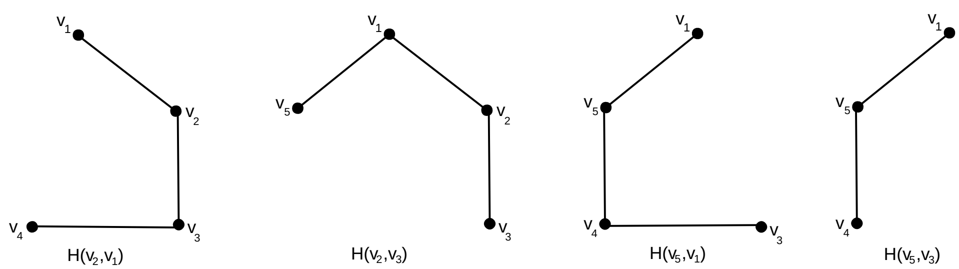

Figure 14. Subtrees

$\{H(v_2, v_1) = (F(v_2, v_1), K(v_2, v_1)), H(v_2, v_3) = (F(v_2, v_3), K(v_2, v_3)), H(v_5, v_1) = (F(v_5, v_1), K(v_5, v_1)), H(v_5, v_3) = (F(v_5, v_3), K(v_5, v_3))\}$ is a soft tree.

Definition 11. Let $G_1 = (G^*, F_1, K_1, A)$ and $G_2 = (G^*, F_2, K_2, B)$ be two soft trees of G^*. The *AND operation on two soft trees* G_1 and G_2 is a soft tree and defined as

$$G = G_1 \wedge G_2 = (G^*, F, K, C)$$

if the subgraph $H(a, b) = (F(a, b), K(a, b))$, for all $(a, b) \in C$, is a tree, where $C = A \times B$ and for all $(a, b) \in C$, $F(a, b) = F_1(a) \cap F_2(b)$ and $K(a, b) = K_1(a) \cap K_2(b)$.

Example 8. Consider a simple graph $G^* = (V, E)$, where $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1\}$. Let $G_1 = (G^*, F_1, K_1, A)$ and $G_2 = (G^*, F_2, K_2, B)$ be two soft trees of G^*. Let $A = \{v_1, v_3\} \subseteq V$ and $B = \{v_2, v_4\} \subseteq V$. Then $A \times B = \{(v_1, v_2), (v_1, v_4), (v_3, v_2), (v_3, v_4)\}$. We define approximate functions $F_1 : A \to \mathcal{P}(V)$ and $F_2 : B \to \mathcal{P}(V)$ by

$$F_1(x) = \{y \in V : xRy \Leftrightarrow d(x, y) \leq 1\}\ \forall x \in A,$$

i. e., $F_1(v_1) = \{v_1, v_2, v_5\}$, $F_1(v_3) = \{v_2, v_3, v_4\}$, and

$$F_2(x) = \{y \in V : xRy \Leftrightarrow d(x, y) \geq 1\}\ \forall x \in B,$$

i. e., $F_2(v_2) = \{v_1, v_3, v_4, v_5\}$, $F_2(v_4) = \{v_1, v_2, v_3, v_5\}$, respectively. Thus, $F(v_1, v_2) = F_1(v_1) \cap F_2(v_2) = \{v_1, v_5\}$, $F(v_1, v_4) = F_1(v_1) \cap F_2(v_4) = \{v_1, v_2, v_5\}$, $F(v_3, v_2) = F_1(v_3) \cap F_2(v_2) = \{v_3, v_4\}$, $F(v_3, v_4) = F_1(v_3) \cap F_2(v_4) = \{v_2, v_3\}$. Thus, $(F, A) = \{F(v_1, v_2), F(v_1, v_4), F(v_3, v_2), F(v_3, v_4)\}$ is a soft set over V.

We now define approximate functions $K_1 : A \to \mathcal{P}(E)$ and $K_2 : B \to \mathcal{P}(E)$ by

$$K_1(x) = \{xy \in E : xRxy \Leftrightarrow xy \subseteq F_1(x)\} \ \forall x \in A,$$

i. e., $K_1(v_1) = \{v_1v_2, v_5v_1\}$, $K_1(v_3) = \{v_2v_3, v_3v_4\}$, and

$$K_2(x) = \{xy \in E : xRxy \Leftrightarrow xy \subseteq F_2(x)\} \ \forall x \in B,$$

i. e., $K_2(v_2) = \{v_3v_4, v_4v_5, v_5v_1\}$, $K_2(v_4) = \{v_1v_2, v_2v_3, v_5v_1\}$. Thus, $K(v_1, v_2) = K_1(v_1) \cap K_2(v_2) = \{v_5v_1\}$, $K(v_1, v_4) = K_1(v_1) \cap K_2(v_4) = \{v_1v_2, v_5v_1\}$, $K(v_3, v_2) = K_1(v_3) \cap K_2(v_2) = \{v_3v_4\}$, $K(v_3, v_4) = K_1(v_3) \cap K_2(v_4) = \{v_2v_3\}$. Thus, $(K, A) = \{K(v_1, v_2), K(v_1, v_4), K(v_3, v_2), K(v_3, v_4)\}$ is a soft set over E. Hence, $H(v_1, v_2) = (F(v_1, v_2), K(v_1, v_2))$, $H(v_1, v_4) = (F(v_1, v_4), K(v_1, v_4))$, $H(v_3, v_2) = (F(v_3, v_2), K(v_3, v_2))$ and $H(v_3, v_4) = (F(v_3, v_4), K(v_3, v_4))$ are connected subgraphs of G^* and are also trees as shown in Fig. 15. Hence, $G = \{H(v_1, v_2) =$

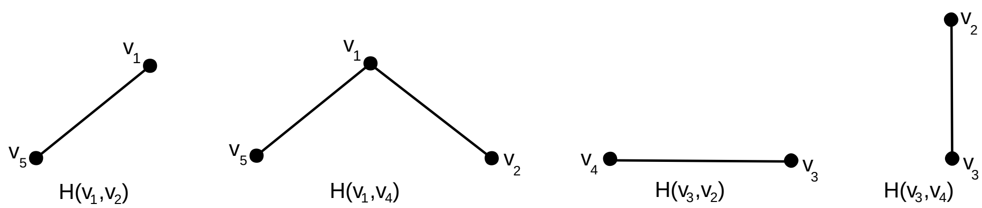

Figure 15. Subtrees

$(F(v_1, v_2),\ K(v_1, v_2))$, $H(v_1, v_4) = (F(v_1, v_4),\ K(v_1, v_4))$, $H(v_3, v_2) = (F(v_3, v_2), K(v_3, v_2))$, $H(v_3, v_4) = (F(v_3, v_4), K(v_3, v_4))\}$ is a soft tree.

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On LCA groups with locally compact rings of continuous endomorphisms. II

Valeriu Popa

Abstract. For certain classes $\mathcal{S}$ of locally compact abelian groups, we determine the groups $X \in \mathcal{S}$ with the property that the ring $E(X)$ of continuous endomorphisms of X is locally compact in the compact-open topology.

Mathematics subject classification: 16W80.
Keywords and phrases: LCA groups, rings of continuous endomorphisms, compact-open topology.

1 Introduction

Let $\mathcal{L}$ be the class of locally compact abelian groups. For $X \in \mathcal{L}$, let $E(X)$ denote the ring of continuous endomorphisms of X, taken with the compact-open topology.

In the present paper, we continue our work begun in [17] concerning the problem of characterizing the groups $X \in \mathcal{L}$ for which $E(X)$ is locally compact. Our main results are as follows. We establish some necessary conditions and, respectively, some sufficient conditions on X in order for $E(X)$ be locally compact. For groups in $\mathcal{L}$ containing a lattice and for densely divisible torsion-free groups in $\mathcal{L}$, we give a complete solution to the considered problem. We also determine the topological torsion groups $X \in \mathcal{L}$ with the property that $E(A/B)$ is locally compact for all closed subgroups A, B of X such that $A \supset B$.

2 Notation

We will follow the notation used in [17]. In addition, for $X, Y \in \mathcal{L}$ and $f \in H(X, Y)$, we denote by f^* the transpose of f, i.e. the homomorphism $f^* \in H(Y^*, X^*)$ defined by the rule $f^*(\gamma) = \gamma \circ f$ for all $\gamma \in Y^*$. If C is a closed subgroup of X and $n \in \mathbb{N}_0$, we set $\frac{1}{n}C = \{x \in X \mid nx \in C\}$. We will also make use of the discrete group $\mathbb{Z}$ of integers, and of the groups of reals $\mathbb{R}$ and of p-adic numbers $\mathbb{Q}_p$, where $p \in \mathbb{P}$, all taken with their usual topologies. Finally, if $(X_i)_{i \in I}$ is a family of topological groups (rings) such that, for each $i \in I$, X_i admits an open subgroup (subring) U_i, then $\prod_{i \in I}(X_i; U_i)$ stands for the local direct product of $(X_i)_{i \in I}$ with respect to $(U_i)_{i \in I}$. Recall that $\prod_{i \in I}(X_i; U_i)$ is the subgroup (subring) of $\prod_{i \in I} X_i$ consisting of all families $(x_i)_{i \in I}$ such that $x_i \in U_i$ for all but finitely many $i \in I$, topologized by declaring all neighborhoods of zero in the topological group (ring) $\prod_{i\in I} U_i$ to be a fundamental system of neighborhoods of zero in $\prod_{i\in I}(X_i; U_i)$.

3 Local compactness of some homomorphism groups

In this preparatory section, we determine the groups $X \in \mathcal{L}$ with the property that the topological groups $H(X,\mathbb{R})$, $H(\mathbb{R},X)$, $H(X,\mathbb{Q})$, $H(\mathbb{Q},X)$, $H(X,\mathbb{Q}^*)$, and $H(\mathbb{Q}^*,X)$ are locally compact.

We first recall the following definition, due to V. Charin [4].

Definition 1. A topological group X is said to be a group of finite (special) rank if there exists a natural number r such that every finite subset F of X topologically generates a subgroup with no more than r topological generators, i.e. $\overline{\langle F\rangle} = \overline{\langle x_1,\dots,x_k\rangle}$ for some $x_1,\dots,x_k \in X$ and $k \le r$. The smallest r with this property is called the special rank of X. In case no such r exists, X is said to have infinite special rank.

As is well known, a discrete torsion-free group $X \in \mathcal{L}$ has finite special rank r if and only if its torsion-free rank is equal to r. It is also known that if $X \in \mathcal{L}$ is a topologically p-primary group for some $p \in \mathbb{P}$, then X has finite special rank r if and only if

$$X \cong G_1 \times \cdots \times G_r,$$

where every $G_i, 1 \le i \le r$, is topologically isomorphic with one of the groups $\mathbb{Q}_p$, $\mathbb{Z}_p$, $\mathbb{Z}(p^\infty)$, or $\mathbb{Z}(p^n)$ for some $n \in \mathbb{N}_0$ [5, Theorem 5].

We now begin the study of local compactness of the mentioned homomorphism groups. For $H(X,\mathbb{R})$ and $H(X,\mathbb{Q})$, we have

Theorem 1. *Let X be a group in $\mathcal{L}$ containing a compact open subgroup. The following conditions are equivalent:*

(i) *$H(X,\mathbb{R})$ is locally compact.*

(ii) *$H(X,\mathbb{Q})$ is locally compact.*

(iii) *$X/k(X)$ has finite rank.*

Proof. The fact that (i) and (iii) are equivalent follows from [15, Lemma 3.2]. Let us establish the equivalence of (ii) and (iii). Assume (ii), and let Ω be a compact neighborhood of zero in $H(X,\mathbb{Q})$. By the definition of the compact-open topology, there is a compact subset K of X such that $\Omega_{X,\mathbb{Q}}(K,\{0\}) \subset \Omega$. Let $\pi : X \to X/k(X)$ be the canonical projection. Since X has a compact open subgroup, $X/k(X)$ is discrete, and hence $\pi(K)$ is finite. Let $G = \langle \pi(K)\rangle_*$. It is clear that G has finite rank [12, p. 41]. We shall show that $G = X/k(X)$. Assume the contrary, and pick an arbitrary non-zero $b \in \big(X/k(X)\big)/G$. Since G is pure in $X/k(X)$, the quotient group $\big(X/k(X)\big)/G$ is torsion-free, so $o(b) = \infty$. Letting $\varphi : X/k(X) \to \big(X/k(X)\big)/G$ denote the canonical projection, write $b = \varphi(b')$ for some $b' \in X/k(X)$. Now, given any

$r \in \mathbb{Q}$, let $\xi_r : \big(X/k(X)\big)/G \to \mathbb{Q}$ be the extension of the group homomorphism from $\langle b\rangle$ to $\mathbb{Q}$ which carries b to r [8, Theorem 21.1]. Then $\xi_r \circ \varphi \circ \pi \in \Omega_{X,\mathbb{Q}}(K,\{0\})$, so $r \in \Omega b'$. Since $r \in \mathbb{Q}$ was chosen arbitrarily, we get $\mathbb{Q} \subset \Omega b'$, which is a contradiction because $\Omega b'$ is finite and $\mathbb{Q}$ is infinite. This proves that $G = X/k(X)$, so (i) implies (iii).

To see the converse, assume (iii), and pick any elements $a_1, \dots, a_m \in X$ such that $a_1 + k(X), \dots, a_m + k(X)$ form a basis in $X/k(X)$. We claim that

$$\Omega_{X,\mathbb{Q}}\big(\{a_1, \dots, a_m\}, \{0\}\big) = \{0\},$$

which means that $H(X, \mathbb{Q})$ is discrete. To see this, fix any $a \in X \setminus k(X)$. Then there exist $n \in \mathbb{N}_0$ and $l_1, \dots, l_m \in \mathbb{Z}$ such that

$$n(a + k(X)) = \sum_{i=1}^{m} l_i(a_i + k(X)),$$

and hence

$$na - \sum_{i=1}^{m} l_i a_i \in k(X).$$

Pick any $f \in \Omega_{X,\mathbb{Q}}\big(\{a_1, \dots, a_m\}, \{0\}\big)$. Since $k(\mathbb{Q}) = \{0\}$, we have $k(X) \subset \ker(f)$. It follows that

$$nf(a) = \sum_{i=1}^{m} l_i f(a_i) = 0,$$

so $f(a) = 0$. Since $a \in X \setminus k(X)$ was chosen arbitrarily, it follows that $f = 0$, and hence (iii) implies (ii). $\square$

As a direct consequence, we derive the following:

Corollary 1. *Let X be a group in $\mathcal{L}$ containing a compact open subgroup. The following conditions are equivalent:*

(i) *$H(\mathbb{R}, X)$ is locally compact.*

(ii) *$H(\mathbb{Q}^*, X)$ is locally compact.*

(iii) *$c(X)$ has finite dimension.*

Proof. Since $H(\mathbb{R}, X) \cong H(X^*, \mathbb{R})$ and $H(\mathbb{Q}^*, X) \cong H(X^*, \mathbb{Q})$ [11, Ch. IV, Theorem 4.2, Corollary 2], the assertion follows from Theorem 1 and duality. $\square$

For $H(\mathbb{Q}, X)$, we have:

Theorem 2. *For a group $X \in \mathcal{L}$, the following statements are equivalent:*

(i) *$H(\mathbb{Q}, X)$ is locally compact.*

(ii) *There is a symmetric open neighborhood V of zero in X such that $(\frac{1}{n}V)\cap d(X)$ is relatively compact for all $n \in \mathbb{N}_0$.*

(iii) *There is an open subgroup F of $\overline{d(X)}$ such that $(\frac{1}{n}F) \cap \overline{d(X)}$ is compactly generated for all $n \in \mathbb{N}_0$.*

Proof. Assume (i), and let Ω be a compact neighborhood of zero in $H(\mathbb{Q}, X)$. Then there is a finite subset $K = \{a_1, \ldots, a_k\}$ of $\mathbb{Q}$ and an open neighborhood U of zero in X such that $\Omega_{\mathbb{Q},X}(K, U) \subset \Omega$. As is well known, the finitely generated subgroups of $\mathbb{Q}$ are cyclic [8, p. 17], so $\langle K\rangle = \langle a\rangle$ for some $a \in \mathbb{Q}$. Write $a_i = m_i a$ with $m_i \in \mathbb{Z}$ for all $i = 1, \ldots, k$. Further, set $m = \max_{1\le i\le k} |m_i|$, and choose a symmetric open neighborhood V of zero in X such that

$$\underbrace{V + \cdots + V}_{m} \subset U.$$

We claim that $(\frac{1}{n}V) \cap d(X)$ is relatively compact for all $n \in \mathbb{N}_0$. Indeed, given any $f \in \Omega_{\mathbb{Q},X}(\{a\}, V)$, we have

$$f(a_i) = m_i f(a) \in \underbrace{V + \cdots + V}_{m} \subset U$$

for all $i = 1, \ldots, k$. Consequently,

$$\Omega_{\mathbb{Q},X}(\{a\}, V) \subset \Omega_{\mathbb{Q},X}(K, U) \subset \Omega,$$

proving that $\Omega_{\mathbb{Q},X}(\{a\}, V)$ has compact closure in $H(\mathbb{Q}, X)$. It follows from the Ascoli's theorem that for each $q \in \mathbb{Q}$, the orbit $\Omega_{\mathbb{Q},X}(\{a\}, V)q$ is relatively compact in X. Now, fix any $n \in \mathbb{N}_0$ and any $x \in (\frac{1}{n}V) \cap d(X)$. Then $nx \in V$. Define $h \in H(\langle \frac{a}{n}\rangle, d(X))$ by setting $h(\frac{a}{n}) = x$. Since $d(X)$ is divisible, h extends to a homomorphism $\widehat{h} \in H(\mathbb{Q}, d(X))$ [8, Theorem 21.1]. Let j be the canonical injection of $d(X)$ into X. We have $\widehat{h}(a) = n\widehat{h}(\frac{1}{n}a) = nx \in V$, so $j \circ \widehat{h} \in \Omega_{\mathbb{Q},X}(\{a\}, V)$, and hence

$$x \in \Omega_{\mathbb{Q},X}(\{a\}, V)\frac{a}{n}.$$

Since $x \in \left(\frac{1}{n}V\right) \cap d(X)$ was chosen arbitrarily, we get

$$\left(\frac{1}{n}V\right) \cap d(X) \subset \Omega_{\mathbb{Q},X}(\{a\}, V)\frac{a}{n},$$

proving that $\left(\frac{1}{n}V\right) \cap d(X)$ is relatively compact in X. So (i) implies (ii).

Now assume (ii), and fix an arbitrary $n \in \mathbb{N}_0$. It follows from [7, Exercise 1.3.D(a)] that

$$\overline{\left(\frac{1}{n}V\right) \cap \overline{d(X)}} = \overline{\left(\frac{1}{n}V\right) \cap d(X)},$$

so $\overline{\left(\frac{1}{n}V\right) \cap \overline{d(X)}}$ is compact, and hence the subgroup $\left\langle\left(\frac{1}{n}V\right) \cap \overline{d(X)}\right\rangle$ is compactly generated in $\overline{d(X)}$ [9, (5.13)]. Since $\left(\frac{1}{n}V\right) \cap \overline{d(X)}$ is open in $\overline{d(X)}$, it also follows that

$\langle(\frac{1}{n}V) \cap \overline{d(X)}\rangle$ is closed in X [9, (5.5)]. In a similar manner, $\langle V \cap \overline{d(X)}\rangle$ is open in $\overline{d(X)}$, so closed in X, and hence $\frac{1}{n}\langle V \cap \overline{d(X)}\rangle$ is closed in X because multiplication by n is continuous. We assert that

$$\Big\langle\Big(\frac{1}{n}V\Big) \cap \overline{d(X)}\Big\rangle = \frac{1}{n}\Big\langle V \cap \overline{d(X)}\Big\rangle \cap \overline{d(X)}. \tag{1}$$

Indeed, if $x \in (\frac{1}{n}V) \cap \overline{d(X)}$, then $nx \in V \cap \overline{d(X)}$, so $x \in \langle(\frac{1}{n}V) \cap \overline{d(X)}\rangle \cap \overline{d(X)}$, proving that

$$\Big\langle\Big(\frac{1}{n}V\Big) \cap \overline{d(X)}\Big\rangle \subset \frac{1}{n}\Big\langle V \cap \overline{d(X)}\Big\rangle \cap \overline{d(X)}.$$

To see the inverse inclusion, pick an arbitrary $x \in \frac{1}{n}\langle V \cap \overline{d(X)}\rangle \cap \overline{d(X)}$. Since

$$\begin{aligned}\langle V \cap \overline{d(X)}\rangle &= \langle \overline{V \cap \overline{d(X)}}\rangle \\ &= \langle \overline{V \cap d(X)}\rangle = \overline{\langle V \cap d(X)\rangle},\end{aligned}$$

we conclude that there exist $m \in \mathbb{N}_0$, $l_1, \ldots, l_m \in \mathbb{Z}$, and $a_1, \ldots, a_m \in V \cap d(X)$ such that

$$nx - \sum_{i=1}^{m} l_i a_i \in V.$$

Further, since $d(X)$ is divisible, we can write $a_i = nb_i$ with $b_i \in d(X)$ for all $i = 1, \ldots, m$. It follows that

$$n(x - \sum_{i=1}^{m} l_i b_i) \in V,$$

so

$$x - \sum_{i=1}^{m} l_i b_i \in \frac{1}{n}V,$$

and hence

$$x - \sum_{i=1}^{m} l_i b_i \in \Big(\frac{1}{n}V\Big) \cap \overline{d(X)}.$$

As $b_1, \ldots, b_m \in (\frac{1}{n}V) \cap \overline{d(X)}$, this proves that $x \in \Big\langle\Big(\frac{1}{n}V\Big) \cap \overline{d(X)}\Big\rangle$, so

$$\frac{1}{n}\Big\langle V \cap \overline{d(X)}\Big\rangle \cap \overline{d(X)} \subset \Big\langle\Big(\frac{1}{n}V\Big) \cap \overline{d(X)}\Big\rangle,$$

proving (1). Finally, taking $F = \langle V \cap \overline{d(X)}\rangle$, we conclude that (ii) implies (iii).

Next assume (iii), and let U be a symmetric open neighborhood of zero in X such that $\overline{U}$ is compact and $F = \langle U \cap \overline{d(X)}\rangle$ [9, (5.13)]. We shall show that $\Omega_{\mathbb{Q},X}(\{1\}, U)$ is relatively compact in $H(\mathbb{Q}, X)$. Since $\mathbb{Q}$ is discrete, it is clear that $\Omega_{\mathbb{Q},X}(\{1\}, U)$

is equicontinuous. Fix any $l, n \in \mathbb{N}_0$. To show that $\Omega_{\mathbb{Q},X}(\{1\},U)\frac{l}{n}$ is relatively compact in X, observe first that

$$\Omega_{\mathbb{Q},X}(\{1\},U)\frac{l}{n} \subset l\Big(\big(\frac{1}{n}U\big) \cap \overline{d(X)}\Big). \tag{2}$$

Indeed, for any $f \in \Omega_{\mathbb{Q},X}(\{1\},U)$, we have $nf(\frac{1}{n}) = f(1) \in U$, so $f(\frac{1}{n}) \in \big(\frac{1}{n}U\big) \cap \overline{d(X)}$ because $\mathbb{Q}$ is divisible, and hence

$$f(\frac{l}{n}) = lf(\frac{1}{n}) \in l\Big(\big(\frac{1}{n}U\big) \cap \overline{d(X)}\Big).$$

Since

$$\big(\frac{1}{n}U\big) \cap \overline{d(X)} = \frac{1}{n}\big(U \cap \overline{d(X)}\big) \cap \overline{d(X)}$$

it is clear that the inclusion (2) will assure the compactness of $\overline{\Omega_{\mathbb{Q},X}(\{1\},U)\frac{l}{n}}$ if we show that $\frac{1}{n}\big(U \cap \overline{d(X)}\big)$ has compact closure. Now, since $G = \big(\frac{1}{n}F\big) \cap \overline{d(X)}$ is compactly generated, we can write $G = A \oplus B \oplus C$, where $A \cong \mathbb{R}^d$ and $B \cong \mathbb{Z}^s$ for some $d, s \in \mathbb{N}$, and C is a compact subgroup of G [9, (9.8)]. Let $\pi_A, \pi_B, \pi_C \in E(G)$ be the canonical projections of G onto A, B, and C, respectively. Since $\big(\frac{1}{n}V\big) \cap \overline{d(X)} \subset G$ and $1_G = \pi_A + \pi_B + \pi_C$, where 1_G is the identity mapping on G, we have

$$\frac{1}{n}\big(U \cap \overline{d(X)}\big) \subset \pi_A\big(\frac{1}{n}\big(U \cap \overline{d(X)}\big)\big) + \pi_B\big(\frac{1}{n}\big(U \cap \overline{d(X)}\big)\big) + \pi_C\big(\frac{1}{n}\big(U \cap \overline{d(X)}\big)\big).$$

But

$$\pi_A(\frac{1}{n}\big(U \cap \overline{d(X)}\big)) \subset \frac{1}{n}\pi_A(U \cap \overline{d(X)}) \cap A,$$

$$\pi_B(\frac{1}{n}\big(U \cap \overline{d(X)}\big)) \subset \frac{1}{n}\pi_B(U \cap \overline{d(X)}) \cap B$$

and

$$\pi_C(\frac{1}{n}\big(U \cap \overline{d(X)}\big)) \subset \frac{1}{n}\pi_C(U \cap \overline{d(X)}) \cap C,$$

so

$$\frac{1}{n}\big(U \cap \overline{d(X)}\big) \subset \frac{1}{n}\pi_A(U \cap \overline{d(X)}) \cap A + \frac{1}{n}\pi_B(U \cap \overline{d(X)}) \cap B + \frac{1}{n}\pi_C(U \cap \overline{d(X)}) \cap C,$$

proving that $\frac{1}{n}\big(U \cap \overline{d(X)}\big)$ has compact closure in X. It follows by the Ascoli's theorem that $\Omega_{\mathbb{Q},X}(\{1\},U)$ is relatively compact in $H(\mathbb{Q},X)$, and hence (iii) implies (i). □

In order to dualize the preceding theorem, we will need the following lemma.

Lemma 1. *Let $X \in \mathcal{L}$. For every closed subgroup C of X and every $n \in \mathbb{N}_0$, $A(X^*, nC) = \frac{1}{n}A(X^*, C)$.*

Proof. We have

$$\begin{aligned} A(X^*, nC) &= \{\gamma \in X^* \mid \gamma(nx) = 0 \text{ for all } x \in C\} \\ &= \{\gamma \in X^* \mid n\gamma(x) = 0 \text{ for all } x \in C\} \\ &= \{\gamma \in X^* \mid n\gamma \in A(X, C)\} \\ &= \frac{1}{n} A(X^*, C). \end{aligned}$$

□

Corollary 2. *For a group $X \in \mathcal{L}$, the following statements are equivalent:*

(i) *$H(X, \mathbb{Q}^*)$ is locally compact.*

(ii) *There is a closed subgroup C of X such that $m(X) \subset C$, $C/m(X)$ is compact, and $X/\overline{\overline{nC} + m(X)}$ has no small subgroups for all $n \in \mathbb{N}_0$.*

Proof. Assume (i). Since $H(\mathbb{Q}, X^*) \cong H(X, \mathbb{Q}^*)$ [11, Ch. IV, Theorem 4.2, Corollary 2], it follows from Theorem 2 that there is an open subgroup F of $\overline{d(X^*)}$ such that $\left(\frac{1}{n}F\right) \cap \overline{d(X^*)}$ is compactly generated for all $n \in \mathbb{N}_0$. Set $C = A(X, F)$. Clearly, $m(X^*) \subset C$ and $C/m(X^*) \cong \left(\overline{d(X^*)}/F\right)^*$ [6, Exercise 3.8.7], so $C/m(X^*)$ is compact [9, (5.21) and (23,17)]. By Lemma 1, we have

$$A(X^*, nC) = \frac{1}{n} A(X^*, C) = \frac{1}{n} F,$$

so $\overline{nC} = A(X, \frac{1}{n}F)$, and hence

$$\begin{aligned} A\big(X, (\frac{1}{n}F) \cap \overline{d(X^*)}\big) &= \overline{A(X, \frac{1}{n}F) + A(X, \overline{d(X^*)})} \\ &= \overline{\overline{nC} + m(X)} \end{aligned}$$

for all $n \in \mathbb{N}_0$. It follows from [9, (23.25)] that

$$\left(X/\overline{\overline{nC} + m(X)}\right)^* \cong \left(\frac{1}{n}F\right) \cap \overline{d(X^*)},$$

so $X/\overline{\overline{nC} + m(X)}$ has no small subgroups [1, Proposition 7.9] for all $n \in \mathbb{N}_0$. Consequently, (i) implies (ii).

Now assume (ii), and set $F = A(X^*, C)$. Since $m(X) \subset C$, we clearly have $F \subset \overline{d(X^*)}$. Further, since $\overline{d(X^*)}/F \cong \big(C/m(X)\big)^*$, it is also clear that F is open in $\overline{d(X^*)}$. Finally, given any $n \in \mathbb{N}_0$. we have

$$\left(\left(\frac{1}{n}F\right) \cap \overline{d(X^*)}\right)^* \cong X/\overline{\overline{nC} + m(X)},$$

so $\left(\frac{1}{n}F\right) \cap \overline{d(X^*)}$ is compactly generated [1, Proposition 7.9]. It follows from Theorem 2 that $H(\mathbb{Q}, X^*)$, and hence $H(X, \mathbb{Q}^*)$, is locally compact, proving that (ii) implies (i). □

4 Some necessary and some sufficient conditions

In this section, we reduce the study of local compactness of the ring $E(X)$ for general groups $X \in \mathcal{L}$ to some more special groups. We also establish some sufficient conditions for local compactness of $E(X)$.

Definition 2. A group $X \in \mathcal{L}$ is caled residual if $d(X) \subset k(X)$ and $c(X) \subset m(X)$.

Theorem 3. *Let $X \in \mathcal{L}$. If $E(X)$ is locally compact, then*

$$X \cong \mathbb{R}^d \times \mathbb{Q}^r \times (\mathbb{Q}^*)^s \times T,$$

where $d, r, s \in \mathbb{N}$ and T is a residual group in $\mathcal{L}$ such that $E(T)$ is locally compact.

In addition, if $d \neq 0$, then $T/k(T)$ is of finite rank and $c(T)$ is of finite dimension.

If $r \neq 0$, then $T/k(T)$ is of finite rank and $\overline{d(T)}$ admits an open subgroup F such that $(\frac{1}{n}F) \cap \overline{d(T)}$ is compactly generated for all $n \in \mathbb{N}_0$.

If $s \neq 0$, then $c(T)$ is of finite dimension and T admits a compact subgroup C such that $m(T) \subset C$, $C/m(T)$ is compact, and $T/\overline{\overline{nC} + m(T)}$ has no small subgroups for all $n \in \mathbb{N}_0$.

Proof. By [1, Theorem 9.3], we can write $X = C \oplus D \oplus S \oplus T$, where $C \cong \mathbb{R}^d$ for some $d \in \mathbb{N}$, $D \cong \mathbb{Q}^{(r)}$ and $S \cong (\mathbb{Q}^*)^s$ for some cardinal numbers r and s, and T is a residual group in $\mathcal{L}$. Since D, S, and T are topological direct summands of X, we conclude from [17, Lemma 2] that $E(D)$, $E(S)$, and $E(T)$ are locally compact. Further, r and s must be finite by virtue of [17, Corollary 2 and Corollary 4]. Taking account of [9, (23,34)(c) and (23,34)(d)], the remaining assertions follow from the results of Section 3. □

We also have

Theorem 4. *Let X be a residual group in $\mathcal{L}$. If $E(X)$ is locally compact, then X satisfies one of the following conditions:*

(i) *$X/k(X)$ is of finite rank and $c(X)$ is of finite dimension.*

(ii) *$X/k(X)$ is of finite rank, $c(X)$ is of infinite dimension, and $m(x) = k(X)$.*

(iii) *$X/k(X)$ is of infinite rank, $c(X)$ is of finite dimension, and $d(X) = c(X)$.*

(iv) *$X/k(X)$ is of infinite rank, $c(X)$ is of infinite dimension, $d(X) = c(X)$, and $m(x) = k(X)$.*

Proof. Let $E(X)$ be locally compact. We show first that if $X/k(X)$ is of infinite rank, then $d(X) = c(X)$. Indeed, assume $X/k(X)$ is of infinite rank. By the local compactness of $E(X)$, there exist a compact subset K of X and an open neighborhood U of zero in X such that $U \subset K$ and $\Omega_X(K, U)$ is relatively compact in $E(X)$. Since X is residual and $\langle K \rangle$ is compactly generated, we can write $\langle K \rangle = A \oplus B$, where A is compact and $B \cong \mathbb{Z}^n$ for some $n \in \mathbb{N}_0$. Clearly, $A \subset k(X)$ and $k(X) \cap B = \{0\}$.

Let $\pi : X \to X/k(X)$ be the canonical projection. Since $B \cong \pi(B)$, the pure subgroup $\pi(B)_*$ has finite rank in $X/k(X)$ [12, p. 41], so $X/k(X) \neq \pi(B)_*$, and hence $\big(X/k(X)\big)/\pi(B)_*$ is a non-zero torsion-free group. Fix an arbitrary $c \in X$ such that $\pi(c) \notin \pi(B)_*$. It follows by the Ascoli's theorem that $\Omega_X(K,U)c$ is relatively compact in X. Our goal is to show that $d(X) \subset \Omega_X(K,U)c$. To this end, pick any $z \in d(X)$, and define $\xi_z \in H(\langle\varphi(\pi(c))\rangle, d(X))$ by setting $\xi(\varphi(\pi(c))) = z$, where $\varphi : X/k(X) \to \big(X/k(X)\big)/\pi(B)_*$ is the canonical projection. Let us denote by $\widehat{\xi_z} \in H\big(\big(X/k(X)\big)/\pi(B)_*, d(X)\big)$ the extension of ξ_z to $\big(X/k(X)\big)/\pi(B)_*$ and by j the canonical injection of $d(X)$ into X. We have $j \circ \widehat{\xi_z} \circ \varphi \circ \pi \in \Omega_X(K,U)$ and $z = (j \circ \widehat{\xi_z} \circ \varphi \circ \pi)(c)$, so $z \in \Omega_X(K,U)c$. Since $z \in d(X)$ was picked arbitrarily, we deduce that $d(X) \subset \Omega_X(K,U)c$, so $\overline{d(X)}$ is compact, and hence $\overline{d(X)} = c(X)$ by [9, (24.24)]. Consequently, if $X/k(X)$ is of infinite rank, then $d(X) = c(X)$ [9, (24.25)]. Now, since $E(X^*)$ is locally compact too [17, Lemma 1], we conclude as above for X that if $X^*/k(X^*)$ is of infinite rank, then $d(X^*) = c(X^*)$. It follows by duality that if $c(X)$ is of infinite dimension, then $m(X) = k(X)$.

We further combine these facts, to get the conclusion. First suppose that $X/k(X)$ is of finite rank. If $X^*/k(X^*)$ is of finite rank too, then $c(X)$ is of finite dimension, and hence we have (i). On the other hand, if $X^*/k(X^*)$ is of infinite rank, then $c(X)$ is of infinite dimension and, as we know from the above, also $m(X) = k(X)$, so in this case we have (ii). Next suppose that $X/k(X)$ is of infinite rank. Then we know from the above that $d(X) = c(X)$. Thus, if $X^*/k(X^*)$ is of finite rank, then $c(X)$ is of finite dimension, and in this case we are led to (iii). Finally, if $X^*/k(X^*)$ is of infinite rank, we are led to (iv). □

We will need the following lemma, which is an adaption of Lemma 3 from [10].

Lemma 2. *For any groups $X, Y \in \mathcal{L}$, the following statements are equivalent:*

(i) *There is a neighborhood Ω of zero in $H(X,Y)$ such that Ωx is compact in Y for all $x \in X$.*

(ii) *There is a neighborhood Ω of zero in $H(Y^*, X^*)$ which operates equicontinuously on Y^*.*

Proof. Assume (i). By the definition of the compact-open topology, there exist a compact subset K of X and an open neighborhood U of zero in Y such that $\Omega_{X,Y}(K,U) \subset \Omega$. Since X and Y are locally compact, we can choose an open neighborhood V of zero in X and an open neighborhood W of zero in Y such that $\overline{V}$ and $\overline{W}$ are compact. Let $K_0 = K \cup \overline{V}$ and $U_0 = U \cap W$. It is clear that $\Omega_{X,Y}(K_0,U_0) \subset \Omega_{X,Y}(K,U)$, so $\Omega_{X,Y}(K_0,U_0)$ has compact closure in $H(X,Y)$. Moreover, for any compact subset C of X, the set

$$\Omega_{X,Y}(K_0,U_0)C = \{f(x) \mid f \in \Omega_{X,Y}(K_0,U_0) \text{ and } x \in C\}$$

has compact closure in Y. Indeed, by the compactness of C, there exist elements $x_1, \ldots, x_m \in C$ such that $C \subset \cup_{i=1}^m (x_i + V)$. Given any $x \in C$, we then have

$x - x_{i_0} \in V$ for some $i_0 \in \{1, \ldots, m\}$, whence

$$f(x) \in f(x_{i_0}) + f(V) \subset \Omega_{X,Y}(K_0, U_0)x_i + U_0$$

for all $f \in \Omega_{X,Y}(K_0, U_0)$. Consequently,

$$\Omega_{X,Y}(K_0, U_0)C \subset \bigcup_{i=1}^{m} \overline{\Omega_{X,Y}(K_0, U_0)x_i} + \overline{U_0},$$

proving that $\Omega_{X,Y}(K_0, U_0)C$ has compact closure in Y. We shall show that the set

$$\Omega_{X,Y}(K_0, U_0)^* = \{f^* \in H(Y^*, X^*) \mid f \in \Omega_{X,Y}(K_0, U_0)\}$$

is equicontinuous in $H(Y^*, X^*)$. Let O be an arbitrary neighborhood of zero in X^*. We may assume that $O = \Omega_{X,\mathbb{T}}(C, D)$, where C is a compact subset of X and D is an open neighborhood of zero in $\mathbb{T}$. For this C, let $C' = \overline{\Omega_{X,Y}(K_0, U_0)C}$. Then C' is a compact subset of Y, so $O' = \Omega_{Y,\mathbb{T}}(C', D)$ is a neighborhood of zero in Y^*. Now, it is easily seen that $f^*(O') \subset O$ for all $f \in \Omega_{X,Y}(K_0, U_0)$, so $\Omega_{X,Y}(K_0, U_0)^*$ is equicontinuous at zero, and hence on Y^*. This proves that (i) implies (ii).

Now assume (ii), and let Φ be the neighborhood of zero in $H(X, Y)$ such that $\Omega = \{f^* \mid f \in \Phi\}$ [11, Ch. IV, Theorem 4.2, Corollary 2]. We claim that Φ operates with relatively compact orbits. Pick any $a \in X$. It suffices to show that $\xi_Y(\Phi a)$ is relatively compact in Y^{**}, where $\xi_Y : Y \to Y^{**}$ is the canonical topological isomorphism of Y, i.e. $\xi_Y(y)(\gamma) = \gamma(y)$ for all $y \in Y$ and $\gamma \in Y^*$. Observe that

$$\xi_Y(\Phi a) = \big\{\xi_X(a) \circ f^* \mid f \in \Phi\big\},$$

where $\xi_X : X \to X^{**}$ is the canonical topological isomorphism of X. To see that $\xi_Y(\Phi a)$ is equicontinuous, pick an arbitrary neighborhood D of zero in $\mathbb{T}$, and set $O = \{\gamma \in X^* \mid \xi(a)(\gamma) \in D\}$. Since $\xi(a)$ is continuous, O is a neighborhood of zero in X^*. Further, since $\Phi^* = \Omega$ is equicontinuous, there is a neighborhood W of zero in Y^* such that $f^*(W) \subset O$ for all $f \in \Phi^*$. It follows that $(\xi(a) \circ f^*)(W) \subset D$ for all $f \in \Phi^*$, proving that $\xi_Y(\Phi a)$ is equicontinuous. Finally, since $\mathbb{T}$ is compact, it is also clear that $\xi_Y(\Phi a)$ operates with relatively compact orbits. Consequently, $\xi_Y(\Phi a)$ is relatively compact in Y^{**} by the Ascoli's theorem. □

We now establish some sufficient conditions for the local compactness of $E(X)$.

Theorem 5. *Let X be a group in $\mathcal{L}$ satisfying the following conditions:*

i) *$c(X) \cap k(X)$ has finite dimension.*

ii) *For each $p \in S(X)$, $\Big(k(X)/\big(c(X) \cap k(X)\big)\Big)_p$ has finite rank.*

iii) *$X/\big(c(X) + k(X)\big)$ has finite rank.*

Then $E(X)$ is locally compact.

Proof. We can write $X = C \oplus Y$, where $C \cong \mathbb{R}^d$ for some $d \in \mathbb{N}$ and Y contains a compact open subgroup. Then

$$E(X) \cong \begin{pmatrix} E(\mathbb{R}^d) & H(Y, \mathbb{R}^d) \\ H(\mathbb{R}^d, Y) & E(Y) \end{pmatrix}.$$

Now, since $H(Y, \mathbb{R}^d) \cong H(Y, \mathbb{R})^d$ and $H(\mathbb{R}^d, Y) \cong H(\mathbb{R}, Y)^d$ [9, (23.34)(c) and (23.34)(d)], we conclude from Theorem 1 and Theorem 2 that $H(Y, \mathbb{R}^d)$ and $H(\mathbb{R}^d, Y)$ are locally compact. As $E(\mathbb{R}^d)$ is locally compact too, it suffices to show that $E(Y)$ is locally compact. To this purpose, pick any elements $a_1, \ldots, a_m$ of Y such that $a_1 + k(Y), \ldots, a_m + k(Y)$ form a basis in $Y/k(Y)$, and a compact open subgroup U of Y. We claim that

$$\Omega = \Omega_Y(\{a_1, \ldots, a_m\} \cup U, U)$$

is relatively compact in $E(Y)$. Let a be an arbitrary element in Y. Then there exist $n \in \mathbb{N}_0$, $l_1, \ldots, l_m \in \mathbb{Z}$, and $b \in k(Y)$ such that $na = b + \sum_{i=1}^m l_i a_i$. Moreover, by multiplying the above equation through by the order of $b + U$ in $k(Y)/U$, if necessary, we may assume that $b \in U$. Now, given any $f \in \Omega$, we have

$$nf(a) = f(b) + \sum_{i=1}^m l_i f(a_i) \subset U,$$

so $f(a) \in \frac{1}{n}U$. Consequently, to conclude that Ω operates with relatively compact orbits, it suffices to show that $\frac{1}{n}U$ is compact. It is clear that $\big(\frac{1}{n}U\big)/U$ is a torsion group of bounded order, so $\frac{1}{n}U \subset k(Y)$, and hence $\big(\frac{1}{n}U\big)/U$ is a subgroup of bounded order of $k(Y)/U$. Since

$$k(Y)/U \cong \big(k(Y)/c(Y)\big)/\big(U/c(Y)\big),$$

we deduce from condition (ii) that the primary components of $k(Y)/U$ have finite rank. Further, since $\big(\frac{1}{n}U\big)/U$ is a subgroup of bounded order of $k(Y)/U$, we conclude that $\big(\frac{1}{n}U\big)/U$ is finite, so $\frac{1}{n}U$ is compact. Consequently, Ω operates with relatively compact orbits.

Further, observe that X^* too satisfies the hypotheses of the theorem. Indeed, by [9, (24,17)],[6, Proposition 3.3.3], and [9, (23,25)], we have

$$\begin{aligned} c(X^*) \cap k(X^*) &= A(X^*, c(X) + k(X)) \\ &\cong \Big(X/\big(c(X) + k(X)\big)\Big)^*, \end{aligned}$$

so $c(X^*) \cap k(X^*)$ has finite dimension by (iii) and [9, (24.28)]. Similarly, since

$$\Big(X^*/\big(c(X^*) + k(X^*)\big)\Big)^* \cong c(X) \cap k(X),$$

we deduce from (i) that $X^*/\big(c(X^*) + k(X^*)\big)$ has finite rank. Finally, we see from [6, Exercise 3.8.7] and [9, (6.9)] that

$$\begin{aligned}\Big(k(X^*)/\big(c(X^*)\cap k(X^*)\big)\Big)^* &\cong \big(c(X)+k(X)\big)/c(X)\\ &= \big(C\oplus k(X)\big)/\big(C\oplus \big(c(X)\cap k(X)\big)\big)\\ &\cong k(X)/\big(c(X)\cap k(X)\big).\end{aligned}$$

Given any $p\in S(X)$, we then have

$$\Big(k(X^*)/\big(c(X^*)\cap k(X^*)\big)\Big)_p \cong \Big(k(X)/\big(c(X)\cap k(X)\big)\Big)_p^*,$$

so $\Big(k(X^*)/\big(c(X^*)\cap k(X^*)\big)\Big)_p$ has finite rank by (ii) and [5, Theorem 4]. It follows that X^* too satisfies the hypotheses of the theorem. Consequently, we can conclude by using the same argument as with X that $E(X^*)$ admits a neighborhood of zero, which operates with relatively compact orbits. It follows from Lemma 2 that $E(X)$ admits a neighborhood of zero, which operates equicontinuously on X. It remains to apply the Ascoli's theorem. ☐

Remark 1. In [10, n° 9], M. Levin has shown that $A\Big(\prod_{n\in\mathbb{N}_0}\big(\mathbb{Z}(p^{2n});p^n\mathbb{Z}(p^{2n})\big)\Big)$ is locally compact although $\prod_{n\in\mathbb{N}_0}\big(\mathbb{Z}(p^{2n});p^n\mathbb{Z}(p^{2n})\big)$ has infinite rank. With similar arguments, it is easy to see that $E\Big(\prod_{n\in\mathbb{N}_0}\big(\mathbb{Z}(p^{2n});p^n\mathbb{Z}(p^{2n})\big)\Big)$ is locally compact as well, so the inverse of Theorem 5 is not valid.

5 Groups containing a lattice

Let X be a group in $\mathcal{L}$. A subgroup L of X is called a lattice in X if L is discrete and X/L is compact. If there exists such a subgroup L in X, then X is said to contain a lattice. If X decomposes as a topological direct sum of a discrete subgroup and a compact one, then it is said to contain a lattice trivially. If X contains a lattice but cannot be decomposed as a topological direct sum of a discrete group and a compact one, it is said to contain a lattice non-trivially.

In the present section, we answer the question of the local compactness of $E(X)$ in the case when X contains a lattice. In preparation for this we first establish a lemma, which introduces a topology, called the Birkhoff topology, on the group of units of a topological ring and shows how this topology is related to the topology of that ring.

Lemma 3. *Let E be a topological ring with identity* 1, *and let $E^\times$ be the group of invertible elements of E.*

(i) *If $\mathcal{B}$ is a filter base of neighborhoods of zero in E, then the set*

$$\mathcal{B}^\times = \big\{\big[(1+B)\cap E^\times\big]\cap\big[(1+B)\cap E^\times\big]^{-1} \mid B\in\mathcal{B}\big\}$$

is a filter base of neighborhoods of 1 *for a group topology on $E^\times$, which we call the Birkhoff topology of $E^\times$.*

(ii) *If E^{op} is the opposite topological ring of E and $E \times E^{op}$ is the topological direct product of topological rings E and E^{op}, then $E^{\times}$ with the Birkhoff topology is topologically isomorphic to a closed subgroup of the multiplicative monoid of $E \times E^{op}$. In particular, if E is locally compact, then $E^{\times}$ with its Birkhoff topology is locally compact too.*

Proof. (i) Since $\mathcal{B} \neq \varnothing$, it is clear that $\mathcal{B}^{\times} \neq \varnothing$ as well. Also, since every $B \in \mathcal{B}$ contains 0, we see that every element of $\mathcal{B}^{\times}$ contains 1, so $\varnothing \notin \mathcal{B}^{\times}$. Further, given any $B_1, B_2 \in \mathcal{B}$, there is $B_3 \in \mathcal{B}$ such that $B_3 \subset B_1 \cap B_2$. It follows that

$$[(1 + B_3) \cap E^{\times}] \subset [(1 + B_1) \cap E^{\times}] \cap [(1 + B_2) \cap E^{\times}],$$

so

$$[(1 + B_3) \cap E^{\times}]^{-1} \subset [(1 + B_1) \cap E^{\times}]^{-1} \cap [(1 + B_2) \cap E^{\times}]^{-1},$$

and hence $[(1 + B_3) \cap E^{\times}] \cap [(1 + B_3) \cap E^{\times}]^{-1}$ is contained in the set

$$\Big([(1 + B_1) \cap E^{\times}] \cap [(1 + B_1) \cap E^{\times}]^{-1}\Big) \cap \Big([(1 + B_2) \cap E^{\times}] \cap [(1 + B_2) \cap E^{\times}]^{-1}\Big).$$

Consequently, $\mathcal{B}^{\times}$ is a filter base on $E^{\times}$.

Next we show that $\mathcal{B}^{\times}$ satisfies the conditions (GV'_I), (GV'_{II}), and (GV'_{III}) of [2, Ch. III, §1, n° 2]. Let U be a neighborhood of zero in E. We can choose neighborhoods O and V of zero in E such that $O + O \subset U$, $V + V \subset O$, and $VV \subset O$. Then

$$[(1 + V) \cap E^{\times}][(1 + V) \cap E^{\times}] \subset [(1 + U) \cap E^{\times}],$$

so

$$\begin{aligned}\big([(1 + V) \cap E^{\times}] \cap [(1 + V) \cap E^{\times}]^{-1}\big)\big([(1 + V) \cap E^{\times}] \cap [(1 + V) \cap E^{\times}]^{-1}\big)& \\ \subset [(1 + U) \cap E^{\times}] \cap [(1 + U) \cap E^{\times}]^{-1}&,\end{aligned}$$

and hence (GV'_I) holds. Further, since

$$\Big([(1 + U) \cap E^{\times}] \cap [(1 + U) \cap E^{\times}]^{-1}\Big)^{-1} = [(1 + U) \cap E^{\times}]^{-1} \cap [(1 + U) \cap E^{\times}],$$

it is clear that (GV'_{II}) holds too. Finally, given any $a \in E^{\times}$, we can choose neighborhoods Φ and W of zero in E such that $\Phi a \subset U$ and $a^{-1}W \subset \Phi$, whence $a^{-1}Wa \subset U$. But then

$$a^{-1}[(1 + W) \cap E^{\times}]a \subset [(1 + U) \cap E^{\times}],$$

so

$$a^{-1}[(1 + W) \cap E^{\times}]^{-1}a \subset [(1 + U) \cap E^{\times}]^{-1},$$

and hence

$$a^{-1}\Big([(1 + W) \cap E^{\times}] \cap [(1 + W) \cap E^{\times}]^{-1}\Big)a \subset [(1 + U) \cap E^{\times}] \cap [(1 + U) \cap E^{\times}]^{-1}.$$

This proves (GV'_{III}). It follows that there is a unique group topology on $E^\times$, admitting $\mathcal{B}^\times$ as a filter base of neighborhoods of 1.

(ii) Recall that E^{op} is the topological ring in which the underlying set, the additive structure, and the topology are those of E, and whose multiplication is obtained by multiplying in E with reverse order. Consider the topological direct product $E \times E^{op}$. Since the mappings $(u, v) \to u \circ v$ and $(u, v) \to v \circ u$ from $E \times E^{op}$ to E are continuous, the sets

$$S = \{(u, v) \in E \times E^{op} \mid u \circ v = 1\}$$

and

$$T = \{(u, v) \in E \times E^{op} \mid v \circ u = 1\}$$

are closed in $E \times E^{op}$. It follows that $S \cap T$ is closed in $E \times E^{op}$. Clearly,

$$S \cap T = \{(u, u^{-1}) \in E \times E^{op} \mid u \in E^\times\}.$$

Moreover, $S \cap T$ has a group structure with respect to component-wise multiplication. Further, if we endow $S \cap T$ with the induced topology, then $S \cap T$ becomes a topological group. Indeed, the multiplication in $S \cap T$ is the restriction to $S \cap T$ of the multiplication in $E \times E^{op}$, and hence is continuous. Similarly, taking of inverses in $S \cap T$ is the restriction to $S \cap T$ of the mapping $(u, v) \to (v, u)$ from $E \times E^{op}$ onto $E \times E^{op}$, and hence is continuous too. It remains to observe that the mapping $\xi : u \to (u, u^{-1})$ is an isomorphism of topological groups from $E^\times$ onto $S \cap T$. Indeed, ξ is, clearly, an isomorphism of groups. Now, if U is a neighborhood of zero in E, then

$$\xi\Big([(1 + U) \cap E^\times] \cap [(1 + U) \cap E^\times]^{-1}\Big) = \Big((1 + U) \times (1 + U)\Big) \cap (S \cap T),$$

so ξ is bicontinuous. □

Specializing to the case $E = E(X)$, we have the following

Corollary 3. *Let $X \in \mathcal{L}$. Then $A(X)$ coincides with $E(X)^\times$ taken with its Birkhoff topology, and hence $A(X)$ is topologically isomorphic to a closed subgroup of the multiplicative monoid of $E(X) \times E(X)^{op}$.*

We are now prepared to describe all the groups $X \in \mathcal{L}$ containing a lattice for which $E(X)$ is locally compact. First, we consider the case when X contains a lattice non-trivially.

Theorem 6. *Let X be a group in $\mathcal{L}$ containing a lattice non-trivially. The following statements are equivalent:*

(i) *$E(X)$ is locally compact.*

(ii) *$A(X)$ is locally compact.*

(iii) *X satisfies the following conditions:*

1) *$c(X) \cap k(X)$ has finite dimension.*
2) *For each $p \in S(X)$, $\Big(k(X)/\big(c(X) \cap k(X)\big)\Big)_p$ has finite rank.*
3) *$X/\big(c(X) + k(X)\big)$ has finite rank.*

Proof. The fact that (i) implies (ii) follows from Corollary 3, the fact that (ii) implies (iii) follows from [10, Theorem 5], and the fact that (iii) implies (i) follows from Theorem 5. □

For the case of groups containing a lattice trivially, we have:

Theorem 7. *Let X be a group in $\mathcal{L}$ containing a lattice trivially, say $X = L \oplus C$ with L discrete and C compact. Then $E(X)$ is locally compact if and only if $E(L)$ and $E(C)$ are both locally compact.*

Proof. We have

$$E(X) \cong \begin{pmatrix} E(L) & H(C, L) \\ H(L, C) & E(C) \end{pmatrix}.$$

Since L is discrete, $H(L, C)$ is equicontinuous. Since C is compact, $H(L, C)$ operates with relatively compact orbits. Consequently, $H(L, C)$ is compact by the Ascoli's theorem. On the other hand, $H(C, L)$ is discrete because $\Omega_{C,L}(C, \{0\}) = \{0\}$. It follows that $E(X)$ is locally compact if and only if $E(L)$ and $E(C)$ are both locally compact. □

Remark 2. Taking account of the results in [17], the problem of determining the groups $X \in \mathcal{L}$ containing a lattice for which the ring $E(X)$ is locally compact is completely solved. In a similar way, the results of [17] and those of Section 3 can be used to describe the structure of any group $X \in \mathcal{L}$ with locally compact ring $E(X)$, which decomposes as a topological direct product of a finite number of copies of $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Q}^*$, and a group containing a lattice trivially. For example, this can be done for compactly generated groups [9, (9.8)], for groups with no small subgroups [1, Proposition 7.9], for groups with open connected component [1, Corollary 6.8], and for groups with compact subgroup of compact elements [1, Corollary 6.10], respectively.

We close this section by transferring to $E(X)$ a result of P. Plaumann for $A(X)$. We need the following definition from [13].

Definition 3. Let $X \in \mathcal{L}$. A factor of X is a quotient of the form A/B, where A and B are closed subgroups of X such that $A \supset B$.

Theorem 8. *For a topological torsion group $X \in \mathcal{L}$, the following statements are equivalent:*

(i) *$E(F)$ is locally compact for every factor F of X.*

(ii) *$A(F)$ is locally compact for every factor F of X.*

(iii) *For each $p \in S(X)$, X_p has finite rank.*

Proof. The fact that (i) implies (ii) follows from Corollary 3, the fact that (ii) implies (iii) follows from [13, Theorem 3.6 and Lemma 3.1], and the fact that (iii) implies (i) follows from Theorem 5 because every factor of X has primary components of finite rank [4, 1)]. □

6 Densely divisible torsion-free groups

In this final section, we answer the question of local compactness of the ring $E(X)$ for densely divisible torsion-free groups $X \in \mathcal{L}$. We begin with a special case.

Theorem 9. *Let $p \in \mathbb{P}$, and let X be a densely divisible, torsion-free, topological p-primary group in $\mathcal{L}$. The ring $E(X)$ is locally compact if and only if $X \cong \mathbb{Q}_p^r$ for some $r \in \mathbb{N}$.*

Proof. Let $E(X)$ be locally compact. Then $E(X^*)$ is locally compact as well. It is also clear that X^* is densely divisible and torsion-free. Let Ω be a compact neighborhood of zero in $E(X^*)$. By the definition of the compact-open topology, there exist a compact subset K of X^* and an open neighborhood U of zero in X^* such that $\Omega_{X^*}(K,U) \subset \Omega$. Since X^* is totally disconnected [1, Theorem 3.5], there is a compact open subgroup V of X^* such that $V \subset U$ [9, (7.5)], whence $\Omega_{X^*}(K,V) \subset \Omega_{X^*}(K,U)$, and hence $\Omega_{X^*}(K,V)$ is compact in $E(X^*)$.

We claim that $\frac{1}{p^n}V$ is compact for all $n \in \mathbb{N}$. To see this, fix any non-zero character $\alpha \in d(X^*)$, and let D_α be the minimal divisible subgroup of X^* containing α. Then $\overline{D_\alpha} \cong \mathbb{Q}_p$ [14, Lemma 2.4], so $X^* = \overline{D_\alpha} \oplus \Gamma$ for some closed subgroup Γ of X^* [1, Proposition 6.23]. Let $\pi_\alpha, \pi_\Gamma \in E(X^*)$ be the canonical projections of X^* onto $\overline{D_\alpha}$ and Γ, respectively. As $\pi_\alpha(K)$ is compact in $\overline{D_\alpha}$, we have $\pi_\alpha(K) \subset \frac{1}{p^{n_K}}\overline{\langle\alpha\rangle}$ for some $n_K \in \mathbb{N}_0$. Pick any $n \in \mathbb{N}_0$ and any $\beta \in d(X^*) \cap \frac{1}{p^n}V$, and let $\alpha' \in \overline{D_a}$ be the unique element satisfying $p^{n+n_K}\alpha' = \alpha$. Further, define $f \in H(\overline{\langle\alpha'\rangle} \oplus \Gamma, \overline{D_\alpha})$ by setting $f(\alpha') = \beta$ and $f(\gamma) = 0$ for all $\gamma \in \Gamma$. Since $\overline{\langle\alpha'\rangle} \oplus \Gamma$ is open in X^*, f extends to continuous group homomorphism $\hat{f} : X^* \to \overline{D_\alpha}$, so $j \circ \hat{f} \in E(X^*)$, where $j : \overline{D_\alpha} \to X^*$ is the canonical injection. Now, given any $\chi \in K$, we have

$$\begin{aligned}\hat{f}(\chi) = \hat{f}(\pi_\alpha(\chi)) \in \hat{f}\big(\frac{1}{p^{n_K}}\overline{\langle\alpha\rangle}\big) &= \hat{f}\big(\overline{\langle\frac{1}{p^{n_K}}\alpha\rangle}\big) \\ &= \hat{f}\big(\overline{\langle p^n\alpha'\rangle}\big) \subset \overline{\langle p^n\beta\rangle} \subset V,\end{aligned}$$

so $j \circ \hat{f} \in \Omega_{X^*}(K,V)$. Since $\beta \in d(X^*) \cap \frac{1}{p^n}V$ was chosen arbitrarily, it follows from [7, Theorem 1.3.6] that

$$\frac{1}{p^n}V = \overline{d(X^*) \cap \frac{1}{p^n}V} \subset \Omega_{X^*}(K,V)\alpha',$$

so $\frac{1}{p^n}V$ is compact.

Next, let $W = A(X, V)$. Clearly, W is compact and open in X [1, P. 22(e)]. Given any $n \in \mathbb{N}_0$, we deduce from Lemma 1 that

$$A(X^*, p^n W) = \frac{1}{p^n} A(X^*, W),$$

so $p^n W = A(X, \frac{1}{p^n}V)$. It follows that $p^n W$ is open in X, and hence in W. But $W \cong \mathbb{Z}_p^\nu$ for some cardinal number ν [3, Ch. III, §1, Proposition 3]. Consequently, ν must be finite, i.e. $\nu = r$ for some $r \in \mathbb{N}$.

The converse is clear, because $E(\mathbb{Q}_p^r)$ is topologically isomorphic to the matrix ring $M_r(\mathbb{Q}_p)$ over the field of p-adic numbers $\mathbb{Q}_p$, taken with its usual product topology. □

With this preparation, we can prove:

Theorem 10. *Let X be a densely divisible, torsion-free group in $\mathcal{L}$. The ring $E(X)$ is locally compact if and only if*

$$X \cong \mathbb{R}^d \times \mathbb{Q}^r \times (\mathbb{Q}^*)^s \times \prod_{p \in S(X)} (\mathbb{Q}_p^{r_p}; \mathbb{Z}_p^{r_p}),$$

where $d, r, s,$ and the r_p's are natural numbers.

Proof. Assume that $E(X)$ is locally compact. It follows from Theorem 3 that

$$X \cong \mathbb{R}^d \times \mathbb{Q}^r \times (\mathbb{Q}^*)^s \times T,$$

where $d, r, s \in \mathbb{N}$ and T is a residual in $\mathcal{L}$ such that $E(T)$ is locally compact. Now, in view of our hypotheses, $\overline{d(T)} = T$ and $m(T) = \{0\}$, whence $k(T) = T$ and $c(T) = \{0\}$. Consequently, T is a topological torsion group in $\mathcal{L}$, and hence

$$E(T) \cong \prod_{p \in S(X)} (E(T_p); \Omega_{T_p}(U_p, U_p)),$$

where, for each $p \in S(X)$, U_p is a compact open subgroup of T_p [16, (2.2)]. It follows that, for every $p \in S(X)$, $E(T_p)$ is locally compact ([3, p. 9] or [9, (6.16)(c)]), so $T_p \cong \mathbb{Q}_p^{r_p}$ for some $r_p \in \mathbb{N}_0$ by virtue of Theorem 9, and hence $T \cong \prod_{p \in S(X)} (\mathbb{Q}_p^{r_p}; \mathbb{Z}_p^{r_p})$ by [3, Ch. III, Proposition 4].

To show the converse, we write

$$X = A \oplus B \oplus C \oplus D,$$

where $A \cong \mathbb{R}^d$, $B \cong \mathbb{Q}^r$, $C \cong (\mathbb{Q}^*)^s$, and $D \cong \prod_{p \in S(X)} (\mathbb{Q}_p^{r_p}; \mathbb{Z}_p^{r_p})$. It is clear that $c(X) = A \oplus C$ and $k(X) = C \oplus D$, so $c(X) \cap k(X) = C$. We also have

$$c(X) + k(X) = A \oplus C \oplus D,$$

so $X/\big(c(X) + k(X)\big) \cong B$. Finally, given any $p \in S(X)$, we have

$$\Big(k(X)/\big(c(X) \cap k(X)\big)\Big)_p \cong \mathbb{Q}_p^{r_p}.$$

It remains to apply Theorem 5. □

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A note on weak structures due to Császár

A. K. Das

Abstract. Weak structures has been introduced by Á. Császár and it has been shown that every generalized topology and every minimal structure is a weak structure. Recently E. Ekici introduced and studied the structure $r(w)$ in a weak structure w on X. In general the structure $r(w)$ need not be a topology on X. In this paper we have shown that under some conditions $r(w)$ is a topology on X. Further, comparision of two weak structures has been studied.

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In [2], Császár introduced and studied generalized stuctures and in [1, 3] introduced generalized operators. Recently in [4], Császár introduced a new notion called weak structures. Let X be a non-empty set and $\mathcal{P}$ be its power set. A structure on X is a subset of $\mathcal{P}$ and an operation on X is a function from $\mathcal{P}$ to $\mathcal{P}$. A structure w on X is called a weak structure on X if and only if $\emptyset \in w$ [4]. Weak structures are briefly notrd as WS. If w is a WS on X, then every member of w is known as w-open and complement of a w-open set is known as w-closed. Let w be a WS on X and $A \subset X$ then the union of all w-open subsets of A is denoted as $i_w A$ and the intersection of all w-closed sets containing A is denoted as $c_w A$. Further with the help of i_w and c_w, several other structures such as $\alpha(w)$, $\beta(w)$, $\sigma(w)$, $\pi(w)$ and $\rho(w)$ have been introduced and studied in [4]. E. Ekici in [5], studied properties of the structures $\alpha(w)$, $\beta(w)$, $\sigma(w)$, $\pi(w)$ and $\rho(w)$ and introduced $r(w)$ and $rc(w)$. It is also shown that if w is a WS on X then each of the structures $\alpha(w)$, $\beta(w)$, $\sigma(w)$, $\pi(w)$ and $\rho(w)$ is a generalized topology. So it is natural to ask which structure under which condition becomes topology. In this paper, we have shown that under some conditions $r(w)$ is a topology.

Definition 1. [5] Let w be a WS on X and $A \subset X$. Then
(i) $A \in r(w)$ if $A = i_w(c_w(A))$,
(ii) $A \in rc(w)$ if $A = c_w(i_w(A))$.

Lemma 1. *Let w be a WS on X, then $\emptyset \in r(w)$ if any one of the followings holds:*
(i) there exist $U, V \in w$ such that $(X - U) \cap (X - V) = \emptyset$.
(ii) $\bigcap_{X-U \in w} U = \emptyset$.
(iii) for $\bigcap_{X-U \in w} U = V \neq \emptyset$ *there does not exist any $W \in w$ such that $W \subset V$.*

Proof. (i) Let w be a WS on X and $U, V \in w$ be such that $(X - U) \cap (X - V) = \emptyset$. Since $(X-U)$ and $(X-V)$ are two disjoint w-closed sets, $c_w(\emptyset) = \emptyset$. So $i_w(c_w(\emptyset)) = \emptyset$. Hence $\emptyset \in r(w)$.

(ii) If $\bigcap\limits_{X-U \in w} U = \emptyset$, then $c_w(\emptyset) = \emptyset$. Thus $i_w(c_w(\emptyset)) = \emptyset$. Hence $\emptyset \in r(w)$.

(iii) If $\bigcap\limits_{X-U \in w} U = V \neq \emptyset$, then $c_w(\emptyset) = V$. Since there does not exist any $W \in w$ such that $W \subset V$, $i_w(V) = \emptyset$. Thus $i_w(c_w(\emptyset)) = i_w(V) = \emptyset = \emptyset$. Hence $\emptyset \in r(w)$. □

Lemma 2. *Let w be a WS on X, then $X \in r(w)$ if either $X \in w$ or $\bigcup\limits_{U \in w} U = X$.*

Lemma 3. *If w is a WS on X and $U \in w$ is such that for every $V \in w$, $V \subset (X-U)$, then $X \in rc(w)$.*

Lemma 4. *Let w be a WS on X in which every pair of members of w is disjoint and $\bigcup\limits_{U \in w} U = X$. Then every member of w belongs to $r(w)$.*

Proof. Let w be a WS on X. Let every pair of members of w be disjoint and $\bigcup\limits_{U \in w} U = X$. Then for every $A \in w$, $c_w A = \cap\{B : B \in w, A \subset (X - B)\} = A$. Since $A \in w$, $i_w c_w A = i_w A = A$. Thus $A \in r(w)$. □

Lemma 5. *Let w be a WS on X in which every pair of members of w is disjoint and $\bigcup\limits_{U \in w} U = X$. Then arbitrary union of members of w belongs to $r(w)$.*

Proof. Let w be a WS on X and let A_α be a collection of members of w. Since $\bigcup\limits_{U \in w} U = X$ and every pair of members of w, $c_w(\cup A_\alpha) = \cap\{B : (X - B) \in w, \cap A_\alpha \subset B\} = \cup A_\alpha$. So $i_w c_w(\cup A_\alpha) = i_w(\cup A_\alpha) = \cup A_\alpha$. Thus $\cup A_\alpha \in w$. □

Theorem 1. *Let w be a WS on X in which every pair of members of w is disjoint and $\bigcup\limits_{U \in w} U = X$. Then $r(w)$ is a topology on X.*

Proof. Since every pair of members of w is disjoint and $\bigcup\limits_{U \in w} U = X$, either (ii) or (iii) of Lemma 1 holds. Thus $\emptyset \in r(w)$. Since $\bigcup\limits_{U \in w} U = X$, by Lemma 2, $X \in r(w)$. By Lemma 4, every member of w belongs to $r(w)$ and arbitrary union of members of w also belogs to w by Lemma 5. Since the intersection of members of w is empty, finnite intersection of members of w belongs to $r(w)$. Hence $r(w)$ is a topology on X. □

Remark 1. Let w be a WS on X in which every pair of members of w is disjoint and $\bigcup\limits_{U \in w} U = X$. Then it can also be shown that $rc(w)$ is a topology on X.

Let w and ν be two structures on X. The structure ν is said to be finer than w if for every member of w is a member of ν. The power set $\mathcal{P}$ of X is the finest structure on X and $\{\emptyset\}$ is the weakest structure on X. Two structures w and ν are said to be non-comparable if neither w is finer than ν nor ν is finer than w.

Observation 1. Let w and ν be two WSs on X and ν is finer than w. Then $r(w)$ and $r(\nu)$ are non-comparable.

Observation 2. Let w and ν be two WSs on X. Then
(i) $r(w) \cap r(\nu) \neq r(w \cap \nu)$.
(ii) $r(w) \cup r(\nu) \neq r(w \cup \nu)$.

The above observations are established by the following example.

Example 1. Let $X = \{a, b, c\}$, $w = \{\emptyset, \{a\}, \{b\}\}$ and $\nu = \{\emptyset, \{a\}, \{b\}, \{b, c\}$.
(i) Then ν is finer than w but $r(w) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $r(\nu) = \{\emptyset, \{a\}, \{b, c\}, X\}$ are non-comparable.
(ii) $r(w \cap \nu) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $r(w) \cap r(\nu) = \{\emptyset, \{a\}\}$.
(iii) $r(w \cup \nu) = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $r(w) \cup r(\nu) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$

Lemma 6. *Let w and ν be two WSs on X and ν is finer than w. Then $r(w) \cap r(\nu) \subset r(w \cap \nu)$ and $r(w \cup \nu) \subset r(w) \cup r(\nu)$.*

If WSs w and ν are non-comparable then the above result need not hold can be seen from the following example.

Example 2. Let $X = \{a, b, c\}$, $w = \{\emptyset, \{a\}, \{b\}\}$ and $\nu = \{\emptyset, \{a\}, \{c\}$.
$r(w) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $r(\nu) = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$.
$r(w \cup \nu) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, X\}$ and
$r(w) \cup r(\nu) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$.

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