BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(51), 2006, Pages 3–16
ISSN 1024–7696

Quotient rings of pseudonormed rings

S.A. Aleschenko, V.I. Arnautov

Abstract. The present article is devoted to the study of the connection between the restriction of a pseudonorm of a pseudonormed ring on various subrings and the pseudonorm of quotient ring. The basic results of this article were announced in [2].

Mathematics subject classification: 16W60, 13A18.
Keywords and phrases: Pseudonorm, pseudonormed ring, isometric homomorphism, semi-isometric isomorphism, subring, quotient ring.

1 Introduction

1.1 Definition. A real function ξ on a ring R is called a pseudonorm if the following conditions are satisfied:

1. $\xi(x) \geqslant 0$ for all $x \in R$;
2. $\xi(x) = 0$ iff $x = 0$;
3. $\xi(x - y) \leqslant \xi(x) + \xi(y)$ for all $x, y \in R$;
4. $\xi(x \cdot y) \leqslant \xi(x) \cdot \xi(y)$ for all $x, y \in R$.

1.2 Remark. The condition 3 is equivalent to the following conditions: $\xi(x + y) \leqslant \leqslant \xi(x) + \xi(y)$ and $\xi(-x) = \xi(x)$ for all $x, y \in R$.

1.3 Definition. The pseudonorm ξ is called a norm if the condition $\xi(x \cdot y) = = \xi(x) \cdot \xi(y)$ is satisfied for all $x, y \in R$.

1.4 Remark. It is clear that any pseudonorm ξ defines some separated topology on a ring R. However, the same topology can be defined by various pseudonorms.

1.5 Definition. Let (R, ξ) and $\left(\bar{R}, \bar{\xi}\right)$ be pseudonormed rings. A homomorphism $\varphi : R \to \bar{R}$ is called an isometric homomorphism if $\bar{\xi}(\varphi(x)) = \inf\{\, \xi(x + a) \mid a \in \in Ker\varphi\}$ for all $x \in R$.

If φ is also an isomorphism then the concept of isometric homomorphism coincides with the concept of isometric isomorphism in usual sense.

The following isomorphism theorem is frequently applied in algebra.

1.6 Theorem. *Let R be a ring and B be a subring of the ring R. If N is an ideal of the ring R then the quotient rings $B/(B \cap N)$ and $(B + N)/N$ are isomorphic.*

1.7 Remark. In particular, if the condition $B \cap N = \{\, 0\}$ is satisfied in the theorem 1.6 then the rings B and $(B + N)/N$ are isomorphic.

1.8 Remark. Let (R, ξ) be a topological or pseudonormed ring. In order to formulate analogues of this theorem it is natural to demand that the isomorphism preserves the topology or the pseudonorm, respectively. So:

– if ξ is a topology then the isomorphism should be a homeomorphism;

– if ξ is a norm or a pseudonorm then the isomorphism should be an isometric isomorphism.

Therefore situation is more difficult in this case.

First, it is necessary to define the corresponding structure $\bar{\xi}$ (the topology or the pseudonorm, respectively) on the quotient ring R/A.

We shall consider one of the most natural definitions of $\bar{\xi}$ for the topology or the pseudonorm ξ.

A. If ξ is a topology then the topology $\bar{\xi}$ is defined by $\bar{\xi} = \sup\{\tau \mid \tau$ is a ring topology on R/A and the canonical homomorphism $f_A : (R, \xi) \to (R/A, \tau)$ is a continuous homomorphism $\}$ in topological algebra.

In this case $f_A : (R, \xi) \to \left(R/A, \bar{\xi}\right)$ is a surjective, continuous and open homomorphism. Such homomorphisms are called topological homomorphisms.

B. If ξ is a pseudonorm then the pseudonorm $\bar{\xi}$ is defined by the equality $\bar{\xi}\,(x + A) = \inf\{\ \xi\,(x + a)\ \mid a \in A\ \}$ in the theory of the normed rings, i.e. the canonical homomorphism $f_A : (R, \xi) \to \left(R/A, \bar{\xi}\right)$ is an isometric homomorphism (see Definition 1.5).

If ξ is a topology or a pseudonorm then the ring $\left(R/A, \bar{\xi}\right)$ is designated by $(R, \xi)/A$ hereinafter.

Second, theorem 1.6 is not always true for the above mentioned topology or the pseudonorm on the quotient rings R/A.

This article is devoted to the study of analogues of Theorem 1.6 for pseudonormed rings. (Analogues of Theorem 1.6 for topological rings have been investigated in [1]).

1.9 Remark. If (R, ξ) and $\left(\bar{R}, \bar{\xi}\right)$ are the pseudonormed rings, $f : (R, \xi) \to \left(\bar{R}, \bar{\xi}\right)$ is a surjection and an isometric homomorphism then the mapping $\tilde{f} : (R, \xi)/(Kerf) \to$ $\to \left(\bar{R}, \bar{\xi}\right)$ defined by the equality $\tilde{f}\,(r + Kerf) = f\,(r)$ is an isometric isomorphism.

1.10 Remark. The topology or the pseudonorm $\bar{\xi}$ defined above on the quotient ring R/A is a separated topology or a separated pseudonorm if and only if A is a closed ideal in the topological ring (R, ξ) or (R, τ_ξ), respectively.

If ξ is a pseudonorm then the topology $\tau_{\bar{\xi}}$ coincides with the topology on the topological ring $(R, \tau_\xi)/A$.

2 Basic results

2.1 Theorem. *Let (R, ξ) and $\left(\tilde{R}, \tilde{\xi}\right)$ be pseudonormed rings, $\varphi : R \mapsto \tilde{R}$ be a ring isomorphism. The inequality $\tilde{\xi}\,(\varphi\,(x)) \leqslant \xi\,(x)$ is satisfied for all $x \in R$ iff there exists:*

– A pseudonormed ring $\left(\hat{R},\hat{\xi}\right)$ such that the pseudonormed ring (R,ξ) is a subring of the pseudonormed ring $\left(\hat{R},\hat{\xi}\right)$;

– An isometric homomorphism $\hat{\varphi}:\left(\hat{R},\hat{\xi}\right)\to\left(\tilde{R},\tilde{\xi}\right)$ such that $\hat{\varphi}$ is an extension of the isomorphism φ, i.e.

$\hat{\varphi}\left(x\right)=\varphi\left(x\right)$ and $\tilde{\xi}\left(\varphi\left(x\right)\right)=\inf\left\{\,\hat{\xi}\left(x+a\right)\,|\,a\in Ker\hat{\varphi}\right\}$ for all $x\in R$.

Proof. Necessity. Let the inequality $\tilde{\xi}\left(\varphi\left(x\right)\right)\leqslant\xi\left(x\right)$ be valid for all $x\in R$. We shall consider the ring $\hat{R}$ which is the direct product of rings R and $\tilde{R}$, i.e. $\hat{R}=\left\{\,\hat{r}=\left(a,\tilde{b}\right)\,\middle|\,a\in R,\ \tilde{b}\in\tilde{R}\right\}$ is a ring with operations of addition $\hat{r}_1+\hat{r}_2=$ $=\left(a_1+a_2,\tilde{b}_1+\tilde{b}_2\right)$ and multiplication $\hat{r}_1\cdot\hat{r}_2=\left(a_1\cdot a_2,\ \tilde{b}_1\cdot\tilde{b}_2\right)$, where $\hat{r}_1=\left(a_1,\tilde{b}_1\right)$ and $\hat{r}_2=\left(a_2,\tilde{b}_2\right)$.

Let's define the pseudonorm $\hat{\xi}$ on the ring $\hat{R}$ as follows: $\hat{\xi}\left(\hat{r}\right)=\max\left\{\,\xi\left(a\right),\tilde{\xi}\left(\tilde{b}\right)\right\}$, where $\hat{r}=\left(a,\tilde{b}\right)$. It is clear that the function $\hat{\xi}$ satisfies the axioms of pseudonorm.

Let's consider the subring $R'=\{\,a'=(a,\varphi\left(a\right))\mid a\in R\}$ of the ring $\hat{R}$. It follows from the inequality $\tilde{\xi}\left(\varphi\left(a\right)\right)\leqslant\xi\left(a\right)$ that

$$\xi'\left(a'\right)=\hat{\xi}\left(\left(a,\varphi\left(a\right)\right)\right)=\max\left\{\,\xi\left(a\right),\tilde{\xi}\left(\varphi\left(a\right)\right)\right\}=\xi\left(a\right).$$

If we put in correspondence to an element $a\in R$ the element $(a,\varphi\left(a\right))\in R'$ then the mapping defined by this rule is an isometric isomorphism of the pseudonormed rings (R,ξ) and (R',ξ'). Therefore we shall identify any element $a\in R$ with the element $(a,\varphi(a))\in R'$. Hence, we shall not distinguish the pseudonormed rings (R,ξ) and (R',ξ'), i.e. we can assume that the pseudonormed ring (R,ξ) is a subring of the pseudonormed ring $\left(\hat{R},\hat{\xi}\right)$.

We shall consider as mapping $\hat{\varphi}:\left(\hat{R},\hat{\xi}\right)\to\left(\tilde{R},\tilde{\xi}\right)$ the mapping defined by the equality $\hat{\varphi}\left(\left(a,\tilde{b}\right)\right)=\tilde{b}$. Then $\hat{\varphi}\left(a\right)=\hat{\varphi}\left(\left(a,\varphi\left(a\right)\right)\right)=\varphi\left(a\right)$ for any $a\in R$, i.e. the mapping $\hat{\varphi}$ is an extension of the isomorphism φ.

Then $Ker\hat{\varphi}=\left\{\,\hat{r}\in\hat{R}\,|\,\hat{\varphi}\left(\hat{r}\right)=0\right\}=\left\{\,\left(a,\tilde{b}\right)\in\hat{R}\,\middle|\,\tilde{b}=0\right\}=\{\,(a,0)\,|\,a\in R\}$ is an ideal of the ring $\hat{R}$ and

$$\inf\left\{\,\hat{\xi}\left(\hat{r}+\hat{a}\right)\,|\,\hat{a}\in Ker\hat{\varphi}\right\}=\inf\left\{\,\hat{\xi}\left(\left(r,\tilde{r}\right)+\left(a,0\right)\right)\,|\,a\in R\right\}=$$
$$=\inf\left\{\,\hat{\xi}\left(\left(r+a,\tilde{r}\right)\right)\,|\,a\in R\right\}=\inf_{a\in R}\left\{\,\max\left\{\,\xi\left(r+a\right),\tilde{\xi}\left(\tilde{r}\right)\right\}\right\}\leqslant$$
$$\leqslant\max\left\{\,\xi\left(0\right),\tilde{\xi}\left(\tilde{r}\right)\right\}=\tilde{\xi}\left(\tilde{r}\right)=\tilde{\xi}\left(\hat{\varphi}\left(\left(r,\tilde{r}\right)\right)\right)=\tilde{\xi}\left(\hat{\varphi}\left(\hat{r}\right)\right).$$

Thus,

$$\tilde{\xi}\left(\hat{\varphi}\left(\hat{r}\right)\right)\geqslant\inf\left\{\,\hat{\xi}\left(\hat{r}+\hat{a}\right)\,|\,\hat{a}\in Ker\hat{\varphi}\right\}. \qquad (1)$$

On the other hand, for any $a \in Ker\varphi$ and $\hat{r} = (r, \tilde{r}) \in \hat{R}$ the inequality

$$\max\Big\{\xi\left(r+a\right), \tilde{\xi}\left(\tilde{r}\right)\Big\} \geqslant \tilde{\xi}\left(\tilde{r}\right)$$

also takes place.

The set $\Big\{\max\Big\{\xi\left(r+a\right), \tilde{\xi}\left(\tilde{r}\right)\Big\} \mid a \in Ker\hat{\varphi}\Big\}$ is bounded below by the number $\tilde{\xi}\left(\tilde{r}\right)$, therefore $\inf\Big\{\max\Big\{\xi\left(r+a\right), \tilde{\xi}\left(\tilde{r}\right)\Big\} \mid a \in Ker\hat{\varphi}\Big\} \geqslant \tilde{\xi}\left(\tilde{r}\right)$. We have

$$\inf\Big\{\hat{\xi}\left(\hat{r}+\hat{a}\right) \mid \hat{a} \in Ker\hat{\varphi}\Big\} = \inf\Big\{\hat{\xi}\left((r+a, \tilde{r})\right) \mid a \in R\Big\} =$$

$$= \inf_{a \in Ker\hat{\varphi}}\Big\{\max\Big\{\xi\left(r+a\right), \tilde{\xi}\left(\tilde{r}\right)\Big\}\Big\} \geqslant \tilde{\xi}\left(\tilde{r}\right) = \tilde{\xi}\left(\hat{\varphi}\left(\hat{r}\right)\right).$$

Hence,

$$\inf\Big\{\hat{\xi}\left(\hat{r}+\hat{a}\right) \mid \hat{a} \in Ker\hat{\varphi}\Big\} \geqslant \tilde{\xi}\left(\hat{\varphi}\left(\hat{r}\right)\right). \tag{2}$$

From inequalities (1) and (2) we shall receive the required equality:

$$\tilde{\xi}\left(\hat{\varphi}\left(\hat{r}\right)\right) = \inf\Big\{\hat{\xi}\left(\hat{r}+\hat{a}\right) \mid \hat{a} \in Ker\hat{\varphi}\Big\}, \tag{3}$$

i.e. $\hat{\varphi} : \left(\hat{R}, \hat{\xi}\right) \to \left(\tilde{R}, \tilde{\xi}\right)$ is an isometric homomorphism.

Sufficiency. Let $(\hat{R}, \hat{\xi})$ be a pseudonormed ring and $\hat{\varphi} : \left(\hat{R}, \hat{\xi}\right) \to \left(\tilde{R}, \tilde{\xi}\right)$ be an isometric homomorphism such that the pseudonormed ring (R, ξ) is a subring of the pseudonormed ring $(\hat{R}, \hat{\xi})$ and the homomorphism $\hat{\varphi}$ is an extension of the isomorphism φ. Then

$$\xi\left(x\right) = \hat{\xi}\left(x\right) \geqslant \inf\Big\{\hat{\xi}\left(x+a\right) \mid a \in Ker\hat{\varphi}\Big\} = \tilde{\xi}\left(\varphi\left(x\right)\right),$$

i.e. the inequality $\tilde{\xi}\left(\varphi\left(x\right)\right) \leqslant \xi\left(x\right)$ is valid for any $x \in R$.

The theorem is proved.

2.2 Definition. Let (R, ξ) and $\left(\bar{R}, \bar{\xi}\right)$ be pseudonormed rings. An isomorphism $f : R \to \bar{R}$ is said to be a semi-isometric isomorphism if there exists a pseudonormed ring $\left(\hat{R}, \hat{\xi}\right)$ such that the following conditions are valid:

– the ring R is an ideal in the ring $\hat{R}$;

– $\hat{\xi}\Big|_R = \xi$;

– the isomorphism f can be extended up to an isometric homomorphism $\hat{f}$: $\left(\hat{R}, \hat{\xi}\right) \to \left(\bar{R}, \bar{\xi}\right)$ of the pseudonormed rings, i.e.

$$\bar{\xi}\left(\hat{f}\left(\hat{r}\right)\right) = \inf\Big\{\hat{\xi}\left(\hat{r}+i\right) \Big| i \in Ker\hat{f}\Big\} \quad \text{for all } \hat{r} \in \hat{R}.$$

2.3 Theorem. *Let* (R, ξ) *and* $\left(\bar{R}, \bar{\xi}\right)$ *be pseudonormed rings and* $f : R \to \bar{R}$ *be a ring isomorphism. Then the following statements are equivalent:*

I. The isomorphism $f : (R, \xi) \to \left(\bar{R}, \bar{\xi}\right)$ *is a semi-isometric isomorphism of the pseudonormed rings;*

II. $\dfrac{\xi(a \cdot b)}{\xi(b)} \leqslant \bar{\xi}(f(a)) \leqslant \xi(a)$ *and* $\dfrac{\xi(b \cdot a)}{\xi(b)} \leqslant \bar{\xi}(f(a)) \leqslant \xi(a)$ *for any* $a \in R$ *and* $b \in R \backslash \{0\}$*;*

III. There exist a pseudonormed ring $\left(\tilde{R}, \tilde{\xi}\right)$ *and a homomormism* $\tilde{f} : \tilde{R} \to \bar{R}$ *such that:*

a) R *is an ideal in the ring* $\tilde{R}$, $\tilde{\xi}\Big|_R = \xi$ *and* $\tilde{f}\Big|_R = f$*;*

b) $\bar{\xi}(f(r)) = \min\left\{\tilde{\xi}(r + a) \,\Big|\, a \in Ker\tilde{f}\right\}$ *for every* $r \in R$*, i.e. for every* $r \in R$ *there exists an element* $a_r \in Ker\tilde{f}$ *such that* $\bar{\xi}(f(r)) = \tilde{\xi}(r + a_r)$*.*

Proof $I \Rightarrow II$**.**

1. Let $f : (R, \xi) \to \left(\bar{R}, \bar{\xi}\right)$ be a semi-isometric isomorphism. Then it follows from Definition 2.2 that there exist a pseudonormed ring $\left(\hat{R}, \hat{\xi}\right)$ and an isometric homomormism $\hat{f} : \left(\hat{R}, \hat{\xi}\right) \to \left(\bar{R}, \bar{\xi}\right)$ such that R is an ideal of the ring $\hat{R}$, $\hat{\xi}\Big|_R = \xi$ and $\hat{f}\Big|_R = f$.

Since $\hat{f}$ is an isometric homomorphism then $\bar{\xi}(\hat{f}(\hat{r})) = \inf\{\hat{\xi}(\hat{r} + i) \mid i \in Ker\hat{f}\}$ for any $\hat{r} \in \hat{R}$. It means that this equality is valid also for $r \in R$, i.e.

$$\bar{\xi}\left(\hat{f}(r)\right) = \inf\left\{\hat{\xi}(r + i) \,\Big|\, i \in Ker\hat{f}\right\}.$$

Since $\hat{\xi}\Big|_R = \xi$ and $\hat{f}\Big|_R = f$ then we have

$$\bar{\xi}(f(r)) = \bar{\xi}\left(\hat{f}(r)\right) = \inf\left\{\hat{\xi}(r + i) \,\Big|\, i \in Ker\hat{f}\right\} \leqslant \hat{\xi}(r + 0) = \hat{\xi}(r) = \xi(r).$$

Thus the inequality $\bar{\xi}(f(r)) \leqslant \xi(r)$ is valid for any $r \in R$.

2. Let's show in the beginning that $R \cap Ker\hat{f} = \{0\}$.

Since $\hat{f}\Big|_R = f$ and $f : R \to \bar{R}$ is a ring isomorphism then $R \cap Ker\hat{f} = \left\{i \in R \,\Big|\, \hat{f}(i) = 0\right\} = \{i \in R \mid f(i) = 0\} = \{0\}$.

3. Let's verify the inequality $\dfrac{\xi(r \cdot a)}{\xi(a)} \leqslant \bar{\xi}(f(r))$ for any $r \in R$, $a \in R \backslash \{0\}$. Let $j \in Ker\hat{f}$ and $\hat{r} = r + j \in \hat{R}$. Then $\hat{r} \cdot a = (r + j) \cdot a = r \cdot a + j \cdot a$.

Since $R \cap Ker\hat{f} = \{0\}$ then $(a \cdot j) \in R \cap Ker\hat{f} = \{0\}$. It means that $\hat{r} \cdot a = r \cdot a + 0 = r \cdot a \in R$. Then

$$\xi(r \cdot a) = \hat{\xi}(r \cdot a) = \hat{\xi}(\hat{r} \cdot a) \leqslant \hat{\xi}(\hat{r}) \cdot \hat{\xi}(a) = \hat{\xi}(\hat{r}) \cdot \xi(a) = \hat{\xi}(r + j) \cdot \xi(a).$$

Hence

$$\frac{\xi(r\cdot a)}{\xi(a)} \leqslant \hat{\xi}(r+j) \quad \text{for any} \quad j\in Ker\hat{f}.$$

The set $\left\{\hat{\xi}(r+j)\,\middle|\, j\in Ker\hat{f}\right\}$ is bounded below by the number $\dfrac{\xi(r\cdot a)}{\xi(a)}$. It means that the number $\dfrac{\xi(r\cdot a)}{\xi(a)}$ is one of the lower bounds of that set. Therefore

$$\frac{\xi(r\cdot a)}{\xi(a)} \leqslant \inf\left\{\hat{\xi}(r+i)\,\middle|\, i\in Ker\hat{f}\right\} = \bar{\xi}\left(\hat{f}(r)\right) = \bar{\xi}(f(r)).$$

The inequality $\dfrac{\xi(a\cdot r)}{\xi(a)} \leqslant \bar{\xi}\left(\hat{f}(r)\right) = \bar{\xi}(f(r))$ is similarly proved.

Hence $I \Rightarrow II$ is proved.

Proof $II \Rightarrow III$**.**

Let the mapping $f:(R,\xi)\to\left(\bar{R},\bar{\xi}\right)$ possesses the following properties:

f is an isomorphism;

$\bar{\xi}(f(a)) \leqslant \xi(a)$ for any $a\in R$;

$\dfrac{\xi(a\cdot b)}{\xi(b)} \leqslant \bar{\xi}(f(a))$ and $\dfrac{\xi(b\cdot a)}{\xi(b)} \leqslant \bar{\xi}(f(a))$ for any $a\in R$ and $b\in R\backslash\{0\}$.

Let's prove that the statement III is valid.

Let's consider the ring $\tilde{R} = R\oplus\bar{R} = \left\{(r,\bar{r})\,\middle|\, r\in R, \bar{r}\in\bar{R}\right\}$ which is the direct sum of the rings R and $\bar{R}$. Let's define the real-valued function $\tilde{\xi}$ on $\tilde{R}$ as follows:

$$\tilde{\xi}((r,\bar{r})) = \xi\left(r - f^{-1}(\bar{r})\right) + \bar{\xi}(\bar{r}).$$

Let's define the mapping $\tilde{f}:\left(\tilde{R},\tilde{\xi}\right)\to\left(\bar{R},\bar{\xi}\right)$ by the equality $\tilde{f}((r,\bar{r})) = f(r)$.

1. Let's show that $\tilde{\xi}$ is a pseudonorm on the ring $\tilde{R}$.

1.1. It is obvious that $\tilde{\xi}((r,\bar{r})) \geqslant 0$ for all $r\in R$ and $\bar{r}\in\bar{R}$ because the pseudonorms ξ and $\bar{\xi}$ accept non-negative values, i.e. the condition 1 of the definition of a pseudonorm is valid.

1.2. Since $\xi(x) = 0 \Leftrightarrow x = 0$ and $\bar{\xi}(\bar{y}) = 0 \Leftrightarrow \bar{y} = 0$ then $\tilde{\xi}((r,\bar{r})) = 0 \Leftrightarrow \xi\left(r - f^{-1}(\bar{r})\right) + \bar{\xi}(\bar{r}) = 0 \Leftrightarrow \left\{\begin{array}{l}\xi\left(r - f^{-1}(\bar{r})\right) = 0\\ \bar{\xi}(\bar{r}) = 0\end{array}\right. \Leftrightarrow$

$$\Leftrightarrow \left\{\begin{array}{l} r - f^{-1}(\bar{r}) = 0\\ \bar{r} = 0\end{array}\right. \Leftrightarrow \left\{\begin{array}{l} r = 0\\ \bar{r} = 0\end{array}\right. \Leftrightarrow (r,\bar{r}) = 0.$$

Thus, the condition 2 of the definition of a pseudonorm is valid, i.e. $\tilde{\xi}((r,\bar{r})) = 0$ iff $(r,\bar{r}) = 0$.

1.3. Since the inequalities $\xi\left(x_1 - x_2\right) \leqslant \xi\left(x_1\right) + \xi\left(x_2\right)$ and $\bar{\xi}\left(\bar{y}_1 - \bar{y}_2\right) \leqslant \bar{\xi}\left(\bar{y}_1\right) + {} + \bar{\xi}\left(\bar{y}_2\right)$ are valid for any $x_1,\ x_2 \in R$ and $\bar{y}_1,\ \bar{y}_2 \in \bar{R}$ then

$$\tilde{\xi}\left(\left(r - q, \bar{r} - \bar{q}\right)\right) = \xi\left(r - q - f^{-1}\left(\bar{r} - \bar{q}\right)\right) + \bar{\xi}\left(\bar{r} - \bar{q}\right) =$$

$$= \xi\left(r - q - f^{-1}\left(\bar{r}\right) + f^{-1}\left(\bar{q}\right)\right) + \bar{\xi}\left(\bar{r} - \bar{q}\right) =$$
$$= \xi\left(\left(r - f^{-1}\left(\bar{r}\right)\right) - \left(q - f^{-1}\left(\bar{q}\right)\right)\right) + \bar{\xi}\left(\bar{r} - \bar{q}\right) \leqslant$$
$$\leqslant \xi\left(r - f^{-1}\left(\bar{r}\right)\right) + \xi\left(q - f^{-1}\left(\bar{q}\right)\right) + \bar{\xi}\left(\bar{r}\right) + \bar{\xi}\left(\bar{q}\right) =$$
$$= \left(\xi\left(r - f^{-1}\left(\bar{r}\right)\right) + \bar{\xi}\left(\bar{r}\right)\right) + \left(\xi\left(q - f^{-1}\left(\bar{q}\right)\right) + \bar{\xi}\left(\bar{q}\right)\right) = \tilde{\xi}\left(\left(r, \bar{r}\right)\right) + \tilde{\xi}\left(\left(q, \bar{q}\right)\right).$$

We have shown that the condition 3 of the definition of a pseudonorm is valid, i.e. $\tilde{\xi}\left(\left(r - q, \bar{r} - \bar{q}\right)\right) \leqslant \tilde{\xi}\left(\left(r, \bar{r}\right)\right) + \tilde{\xi}\left(\left(q, \bar{q}\right)\right)$ for all $r,\ q \in R$ and $\bar{r},\ \bar{q} \in \bar{R}$.

1.4. Let's verify the inequality $\tilde{\xi}\left(\left(r \cdot q, \bar{r} \cdot \bar{q}\right)\right) \leqslant \tilde{\xi}\left(\left(r, \bar{r}\right)\right) \cdot \tilde{\xi}\left(\left(q, \bar{q}\right)\right)$ for any $r,\ q \in R$ and $\bar{r},\ \bar{q} \in \bar{R}$.

Really,

$$\tilde{\xi}\left(\left(r \cdot q, \bar{r} \cdot \bar{q}\right)\right) = \xi\left(r \cdot q - f^{-1}\left(\bar{r} \cdot \bar{q}\right)\right) + \bar{\xi}\left(\bar{r} \cdot \bar{q}\right) =$$
$$= \xi\left(r \cdot q - f^{-1}\left(\bar{r}\right) \cdot f^{-1}\left(\bar{q}\right)\right) + \bar{\xi}\left(\bar{r} \cdot \bar{q}\right) =$$
$$= \xi\left(\left(r \cdot q - r \cdot f^{-1}\left(\bar{q}\right)\right) + \left(r \cdot f^{-1}\left(\bar{q}\right) - f^{-1}\left(\bar{r}\right) \cdot f^{-1}\left(\bar{q}\right)\right)\right) + \bar{\xi}\left(\bar{r} \cdot \bar{q}\right).$$

Since the inequality $\xi\left(x_1 + x_2\right) \leqslant \xi\left(x_1\right) + \xi\left(x_2\right)$ is valid for any $x_1,\ x_2 \in R$ then

$$\xi\left(\left(r \cdot q - r \cdot f^{-1}\left(\bar{q}\right)\right) + \left(r \cdot f^{-1}\left(\bar{q}\right) - f^{-1}\left(\bar{r}\right) \cdot f^{-1}\left(\bar{q}\right)\right)\right) + \bar{\xi}\left(\bar{r} \cdot \bar{q}\right) \leqslant$$
$$\leqslant \xi\left(r \cdot q - r \cdot f^{-1}\left(\bar{q}\right)\right) + \xi\left(r \cdot f^{-1}\left(\bar{q}\right) - f^{-1}\left(\bar{r}\right) \cdot f^{-1}\left(\bar{q}\right)\right) + \bar{\xi}\left(\bar{r} \cdot \bar{q}\right) =$$
$$= \xi\left(r \cdot \left(q - f^{-1}\left(\bar{q}\right)\right)\right) + \xi\left(\left(r - f^{-1}\left(\bar{r}\right)\right) \cdot f^{-1}\left(\bar{q}\right)\right) + \bar{\xi}\left(\bar{r} \cdot \bar{q}\right).$$

Since the inequalities $\xi\left(x_1 \cdot x_2\right) \leqslant \bar{\xi}\left(f\left(x_1\right)\right) \cdot \xi\left(x_2\right)$ and $\xi\left(x_1 \cdot x_2\right) \leqslant {} \leqslant \xi\left(x_1\right) \cdot \bar{\xi}\left(f\left(x_2\right)\right)$ are valid for any $x_1,\ x_2 \in R$ then

$$\xi\left(r \cdot \left(q - f^{-1}\left(\bar{q}\right)\right)\right) + \xi\left(\left(r - f^{-1}\left(\bar{r}\right)\right) \cdot f^{-1}\left(\bar{q}\right)\right) + \bar{\xi}\left(\bar{r} \cdot \bar{q}\right) \leqslant$$
$$\leqslant \bar{\xi}\left(f\left(r\right)\right) \cdot \xi\left(q - f^{-1}\left(\bar{q}\right)\right) + \xi\left(r - f^{-1}\left(\bar{r}\right)\right) \cdot \bar{\xi}\left(f\left(f^{-1}\left(\bar{q}\right)\right)\right) + \bar{\xi}\left(\bar{r} \cdot \bar{q}\right).$$

The inequality $\bar{\xi}\left(\bar{y}_1 \cdot \bar{y}_2\right) \leqslant \bar{\xi}\left(\bar{y}_1\right) \cdot \xi\left(\bar{y}_2\right)$ is valid for any $\bar{y}_1,\ \bar{y}_2 \in \bar{R}$. Therefore

$$\bar{\xi}\left(f\left(r\right)\right) \cdot \xi\left(q - f^{-1}\left(\bar{q}\right)\right) + \xi\left(r - f^{-1}\left(\bar{r}\right)\right) \cdot \bar{\xi}\left(\bar{q}\right) + \bar{\xi}\left(\bar{r} \cdot \bar{q}\right) \leqslant$$

$$\leqslant \bar{\xi}\left(f\left(r\right)\right) \cdot \xi\left(q - f^{-1}\left(\bar{q}\right)\right) + \xi\left(r - f^{-1}\left(\bar{r}\right)\right) \cdot \bar{\xi}\left(\bar{q}\right) + \bar{\xi}\left(\bar{r}\right) \cdot \bar{\xi}\left(\bar{q}\right) =$$
$$= \bar{\xi}\left(\left(f\left(r\right) - \bar{r}\right) + \bar{r}\right) \cdot \xi\left(q - f^{-1}\left(\bar{q}\right)\right) + \xi\left(r - f^{-1}\left(\bar{r}\right)\right) \cdot \bar{\xi}\left(\bar{q}\right) + \bar{\xi}\left(\bar{r}\right) \cdot \bar{\xi}\left(\bar{q}\right).$$

Since the inequality $\bar{\xi}\left(\bar{y}_1 + \bar{y}_2\right) \leqslant \bar{\xi}\left(\bar{y}_1\right) + \xi\left(\bar{y}_2\right)$ is valid for any $\bar{y}_1,\ \bar{y}_2 \in \bar{R}$ then

$$\bar{\xi}\left(\left(f\left(r\right) - \bar{r}\right) + \bar{r}\right) \cdot \xi\left(q - f^{-1}\left(\bar{q}\right)\right) + \xi\left(r - f^{-1}\left(\bar{r}\right)\right) \cdot \bar{\xi}\left(\bar{q}\right) + \bar{\xi}\left(\bar{r}\right) \cdot \bar{\xi}\left(\bar{q}\right) \leqslant$$

$$\leqslant \left(\bar{\xi}\left(f\left(r\right) - \bar{r}\right) + \bar{\xi}\left(\bar{r}\right)\right) \cdot \xi\left(q - f^{-1}\left(\bar{q}\right)\right) + \xi\left(r - f^{-1}\left(\bar{r}\right)\right) \cdot \bar{\xi}\left(\bar{q}\right) + \bar{\xi}\left(\bar{r}\right) \cdot \bar{\xi}\left(\bar{q}\right) =$$

$$= \left(\bar{\xi}\left(f\left(r\right)-\bar{r}\right)+\bar{\xi}\left(\bar{r}\right)\right)\cdot\xi\left(q-f^{-1}\left(\bar{q}\right)\right)+\left(\xi\left(r-f^{-1}\left(\bar{r}\right)\right)+\bar{\xi}\left(\bar{r}\right)\right)\cdot\bar{\xi}\left(\bar{q}\right).$$

Since the inequality $\bar{\xi}\left(f\left(x\right)\right)\leqslant\xi\left(x\right)$ is valid for any $x\in R$ then

$$\left(\bar{\xi}\left(f\left(r\right)-\bar{r}\right)+\bar{\xi}\left(\bar{r}\right)\right)\cdot\xi\left(q-f^{-1}\left(\bar{q}\right)\right)+\left(\xi\left(r-f^{-1}\left(\bar{r}\right)\right)+\bar{\xi}\left(\bar{r}\right)\right)\cdot\bar{\xi}\left(\bar{q}\right)\leqslant$$

$$\leqslant\left(\xi\left(f^{-1}\left(f\left(r\right)-\bar{r}\right)\right)+\bar{\xi}\left(\bar{r}\right)\right)\cdot\xi\left(q-f^{-1}\left(\bar{q}\right)\right)+\left(\xi\left(r-f^{-1}\left(\bar{r}\right)\right)+\bar{\xi}\left(\bar{r}\right)\right)\cdot\bar{\xi}\left(\bar{q}\right)=$$

$$=\left(\xi\left(r-f^{-1}\left(\bar{r}\right)\right)+\bar{\xi}\left(\bar{r}\right)\right)\cdot\xi\left(q-f^{-1}\left(\bar{q}\right)\right)+\left(\xi\left(r-f^{-1}\left(\bar{r}\right)\right)+\bar{\xi}\left(\bar{r}\right)\right)\cdot\bar{\xi}\left(\bar{q}\right)=$$

$$=\left(\xi\left(r-f^{-1}\left(\bar{r}\right)\right)+\bar{\xi}\left(\bar{r}\right)\right)\cdot\left(\xi\left(q-f^{-1}\left(\bar{q}\right)\right)+\bar{\xi}\left(\bar{q}\right)\right)=\tilde{\xi}\left(\left(r,\bar{r}\right)\right)\cdot\tilde{\xi}\left(\left(q,\bar{q}\right)\right).$$

Thus, the condition 4 of the definition of a pseudonorm is valid, i.e. $\tilde{\xi}\left(\left(r\cdot q,\bar{r}\cdot\bar{q}\right)\right)\leqslant\tilde{\xi}\left(\left(r,\bar{r}\right)\right)\cdot\tilde{\xi}\left(\left(q,\bar{q}\right)\right)$ for any $r,\, q\in R$ and $\bar{r},\,\bar{q}\in\bar{R}$.

We have shown that the function $\tilde{\xi}\left(\left(r,\bar{r}\right)\right)=\xi\left(r-f^{-1}\left(\bar{r}\right)\right)+\bar{\xi}\left(\bar{r}\right)$ defines a pseudonorm on the ring $\tilde{R}$.

2. Let's identify the ring R with the set of pairs $\{\,(r,0)\mid r\in R\}$. It is obvious that R is an ideal of the ring $\hat{R}$.

Let's consider the restrictions of the pseudonorm $\tilde{\xi}$ and the homomorphism $\tilde{f}$ on the ring $R=\{\,(r,0)\mid r\in R\}$, i.e. $\tilde{\xi}\left(\left(r,0\right)\right)=\xi\left(r-f^{-1}\left(0\right)\right)+\bar{\xi}\left(0\right)=\xi\left(r-0\right)+0=$ $=\xi\left(r\right)$ and $\tilde{f}\left(\left(r,0\right)\right)=f\left(r\right)$.

We have that $\tilde{\xi}\Big|_R=\xi$ and $\tilde{f}\Big|_R=f$.

3. Let's show that $\tilde{f}:\left(\tilde{R},\tilde{\xi}\right)\to\left(\bar{R},\bar{\xi}\right)$ is an isometric homomorphism.

3.1. Since f is an isomorphism and $\tilde{f}\Big|_R=f$ then

$$Ker\tilde{f}=\left\{\tilde{r}\in\tilde{R}\,\middle|\,\tilde{f}\left(\tilde{r}\right)=0\right\}=\left\{\left(r,\bar{r}\right)\in\tilde{R}\,\middle|\,\tilde{f}\left(\left(r,\bar{r}\right)\right)=0\right\}=$$

$$=\left\{\left(r,\bar{r}\right)\in\tilde{R}\,|\,f\left(r\right)=0\right\}=\left\{\left(r,\bar{r}\right)\in\tilde{R}\,|\,r=0\right\}=\{\,(0,\bar{r})\mid\bar{r}\in\bar{R}\}.$$

It means that the kernel of the homomorphism is $Ker\tilde{f}=\{\,(0,\bar{r})\mid\bar{r}\in\bar{R}\}$.

3.2. Let's take any $(r,\bar{r})\in\tilde{R}$ and $(0,\bar{j})\in Ker\tilde{f}$. Then

$$\tilde{\xi}\left(\left(r,\bar{r}\right)+\left(0,\bar{j}\right)\right)=\tilde{\xi}\left(\left(r,\bar{r}+\bar{j}\right)\right)=\xi\left(r-f^{-1}\left(\bar{r}+\bar{j}\right)\right)+\bar{\xi}\left(\bar{r}+\bar{j}\right)=$$

$$=\xi\left(r-f^{-1}\left(\bar{r}\right)-f^{-1}\left(\bar{j}\right)\right)+\bar{\xi}\left(\bar{r}+\bar{j}\right)\geqslant\bar{\xi}\left(f\left(r\right)-\bar{r}-\bar{j}\right)+\bar{\xi}\left(\bar{r}+\bar{j}\right)\geqslant\bar{\xi}\left(f\left(r\right)\right).$$

Thus, the inequality $\bar{\xi}\left(f\left(r\right)\right)\leqslant\tilde{\xi}\left(\left(r,\bar{r}\right)+\left(0,\bar{j}\right)\right)$ is valid for the element $(r,\bar{r})\in\tilde{R}$ and any element $(0,\bar{j})\in Ker\tilde{f}$. It means that $\bar{\xi}\left(f\left(r\right)\right)$ is one of the lower bounds of the set $\left\{\tilde{\xi}\left(\left(r,\bar{r}\right)+\left(0,\bar{j}\right)\right)\,\middle|\,(0,\bar{j})\in Ker\tilde{f}\right\}$. Therefore, the inequality

$$\bar{\xi}\left(f\left(r\right)\right)\leqslant\inf\left\{\tilde{\xi}\left(\left(r,\bar{r}\right)+\left(0,\bar{j}\right)\right)\,\middle|\,(0,\bar{j})\in Ker\tilde{f}\right\}\tag{4}$$

is valid for any $(r,\bar{r})\in\tilde{R}$.

3.3. Let's take any element $(r,\bar{r}) \in \tilde{R}$. Let $\bar{j}_0 = f(r) - \bar{r} \in \bar{R}$. Then $(r,\bar{r}) + (0,\bar{j}_0) = (r,\bar{r}) + (0, f(r) - \bar{r}) = (r, f(r))$, that is

$$\tilde{\xi}((r,\bar{r}) + (0,\bar{j}_0)) = \tilde{\xi}((r, f(r))) = \xi\left(r - f^{-1}(f(r))\right) + \bar{\xi}(f(r)) =$$

$$\xi(r-r) + \bar{\xi}(f(r)) = \xi(0) + \bar{\xi}(f(r)) = 0 + \bar{\xi}(f(r)) = \bar{\xi}(f(r)).$$

Thus, for any $(r,\bar{r}) \in \tilde{R}$ there exists $\bar{j}_0 = (0, f(r) - \bar{r}) \in Ker\tilde{f}$ such that $\tilde{\xi}((r,\bar{r}) + (0,\bar{j}_0)) = \bar{\xi}(f(r))$.

From here the inequality

$$\inf\left\{ \tilde{\xi}(\tilde{r} + j) \,\middle|\, j \in Ker\tilde{f} \right\} \leqslant \tilde{\xi}((r,\bar{r}) + (0,\bar{j}_0)) = \bar{\xi}(f(r)) \tag{5}$$

follows. From inequalities (4) and (5) the equality $\bar{\xi}(f(r)) = \inf\{\tilde{\xi}((r,\bar{r}) + +(0,\bar{j})) \mid (0,\bar{j}) \in Ker\tilde{f}\}$ follows. Besides it follows from the equality $\tilde{\xi}((r,\bar{r}) + +(0,\bar{j}_0)) = \bar{\xi}(f(r))$ that

$$\bar{\xi}(f(r)) = \min\left\{ \tilde{\xi}((r,\bar{r}) + (0,\bar{j})) \,\middle|\, (0,\bar{j}) \in Ker\tilde{f} \right\} =$$

$$= \min\{\tilde{\xi}((r,0) + (0,\bar{i})) \mid (0,\bar{i}) \in Ker\tilde{f}\}, \text{ where } (0,\bar{i}) = (0,\bar{r}) + (0,\bar{j}) \in Ker\tilde{f}.$$

We have shown that there exist a pseudonormed ring $\left(\tilde{R},\tilde{\xi}\right)$ and a homomorphism $\tilde{f} : \tilde{R} \to \bar{R}$ such that:

R is an ideal of the ring $\tilde{R}$, $\tilde{\xi}\Big|_R = \xi$ and $\tilde{f}\Big|_R = f$;

$\bar{\xi}(f(r)) = \min\{\tilde{\xi}((r,0) + (0,\bar{i})) \mid (0,\bar{i}) \in Ker\tilde{f}\}$, for every $r \in R$, i.e. for any $r \in R$ there exists an element $(0,\bar{a}_r) \in Ker\tilde{f}$ such that $\bar{\xi}(f(r)) = \tilde{\xi}((r,0) + (0,\bar{a}_r))$.

Hence $II \Rightarrow III$ is proved.

Proof $III \Rightarrow I$.

From the condition 3 of the theorem there exist a pseudonormed ring $\left(\tilde{R},\tilde{\xi}\right)$ and a homomorphism $\tilde{f} : \tilde{R} \to \bar{R}$ such that:

R is an ideal of the ring $\tilde{R}$;

$\tilde{\xi}\Big|_R = \xi$, $\tilde{f}\Big|_R = f$;

$\bar{\xi}(f(r)) = \min\left\{ \tilde{\xi}(r+a) \,\middle|\, a \in Ker\tilde{f} \right\}$ for every $r \in R$.

Let $\tilde{r} \in \tilde{R}$. As $f : R \to \bar{R}$ is an isomorphism then there exists a unique element $r \in R$ such that $f(r) = \tilde{f}(\tilde{r})$. Since the isomorphism f is the restriction of the homomorphism $\tilde{f}$ on the ring R then $f(r) = \tilde{f}(r)$. It means that $\tilde{f}(r) = \tilde{f}(\tilde{r})$. Then $\tilde{f}(r - \tilde{r}) = 0$. Hence, the element $r - \tilde{r}$ belongs to the kernel of the homomorphism $\tilde{f}$.

Then

$$\bar{\xi}\left(\tilde{f}(\tilde{r})\right) = \bar{\xi}(f(r)) = \min\left\{ \tilde{\xi}(r+a) \,\middle|\, a \in Ker\tilde{f} \right\} =$$

$$= \min\left\{ \tilde{\xi}(r + a + (\tilde{r} - \tilde{r})) \,\middle|\, a \in Ker\tilde{f} \right\} =$$

$$= \min\left\{ \tilde{\xi}\left(\tilde{r} + (a + (r - \tilde{r}))\right) \,\Big|\, a \in Ker\tilde{f} \right\} = \min\left\{ \tilde{\xi}\left(\tilde{r} + j\right) \,\Big|\, j \in Ker\tilde{f} \right\}.$$

Since for any set of real numbers S having the least element this element coincides with $\inf S$ then

$$\bar{\xi}\left(\tilde{f}\left(\tilde{r}\right)\right) = \inf\left\{ \tilde{\xi}\left(\tilde{r} + j\right) \,\Big|\, j \in Ker\tilde{f} \right\}.$$

Thus, f can be extended up to the isometric homomorphism $\tilde{f} : \left(\tilde{R}, \tilde{\xi}\right) \to \left(\bar{R}, \bar{\xi}\right)$, and $f : (R, \xi) \to \left(\bar{R}, \bar{\xi}\right)$ is a semi-isometric isomorphism by Definition 2.2.

The theorem is proved.

2.4 Corollary. *If (R, ξ) is a pseudonormed ring with the unit e and $\xi(e) = 1$ then any semi-isometric isomorphism of (R, ξ) is isometric.*

Let's consider the inequality $\dfrac{\xi(a \cdot b)}{\xi(b)} \leqslant \bar{\xi}(f(a)) \leqslant \xi(a)$ for $b = e$. We have $\xi(a) = \dfrac{\xi(a \cdot e)}{1} = \dfrac{\xi(a \cdot e)}{\xi(e)} \leqslant \bar{\xi}(f(a)) \leqslant \xi(a)$. Therefore $\bar{\xi}(f(a)) = \xi(a)$.

2.5 Corollary. *If (R, ξ) is a normed ring then any semi-isometric isomorphism of (R, ξ) is isometric.*

Really, in normed rings the equality $\xi(a \cdot b) = \xi(a) \cdot \xi(b)$ is valid. From this equality it follows that $\xi(a) = \dfrac{\xi(a) \cdot \xi(b)}{\xi(b)} = \dfrac{\xi(a \cdot b)}{\xi(b)} \leqslant \bar{\xi}(f(a)) \leqslant \xi(a)$. It means that $\bar{\xi}(f(a)) = \xi(a)$.

2.6 Corollary. *Let R and $\bar{R}$ be rings with zero multiplication (i.e. $a \cdot b = 0$ for all $a, b \in R$ and $\bar{a} \cdot \bar{b} = 0$ for all $\bar{a}, \bar{b} \in \bar{R}$). If ξ and $\bar{\xi}$ are pseudonorms on R and $\bar{R}$, accordingly, and $f : R \to \bar{R}$ is a ring isomorphism such that $\bar{\xi}(f(r)) \leqslant \xi(r)$ for every $r \in R$ then the isomorphism f is semi-isometric.*

Really, since $\dfrac{\xi(r \cdot q)}{\xi(q)} = 0 \leqslant \bar{\xi}(f(r)) \leqslant \xi(q)$ then from Theorem 2.3 it follows that $f : (R, \xi) \to \left(\bar{R}, \bar{\xi}\right)$ is a semi-isometric isomorphism.

2.7 Corollary. *Let (R, ξ) and $\left(\bar{R}, \bar{\xi}\right)$ be pseudonormed rings and $f : (R, \xi) \to \left(\bar{R}, \bar{\xi}\right)$ be a semi-isometric isomorphism. If $\tilde{\xi}$ is a pseudonorm on $\bar{R}$ such that $\bar{\xi}(f(r)) \leqslant$ $\leqslant \tilde{\xi}(f(r)) \leqslant \xi(r)$ for every $r \in R$ then $f : (R, \xi) \to \left(\bar{R}, \tilde{\xi}\right)$ is a semi-isometric isomorphism.*

Really, $\dfrac{\xi(r \cdot q)}{\xi(q)} \leqslant \bar{\xi}(f(r)) \leqslant \tilde{\xi}(f(r)) \leqslant \xi(q)$. It means that $f : (R, \xi) \to \left(\bar{R}, \bar{\xi}\right)$ is a semi-isometric isomorphism.

2.8 Theorem. *Let (R, ξ) and $\left(\bar{R}, \bar{\xi}\right)$ be pseudonormed rings and $f : R \to \bar{R}$ be a ring isomorphism. Then the following statements are equivalent:*

I. $\xi(a) \geqslant \bar{\xi}(f(a))$ and $\xi(a \cdot b) \leqslant \bar{\xi}(f(a)) \cdot \bar{\xi}(f(b))$ for any $a, b \in R$.

II. There exist a pseudonormed ring $\left(\hat{R}, \hat{\xi}\right)$ and a homomormism $\hat{f} : \hat{R} \to \bar{R}$ such that:

R is an ideal in the ring $\hat{R}$, $\hat{\xi}\Big|_R = \xi$ and $\hat{f}\Big|_R = f$;

$\bar{\xi}(f(r)) = \min\left\{ \hat{\xi}(r+a) \,\Big|\, a \in Ker\hat{f} \right\}$ for every $r \in R$, i.e. for every $r \in R$ there exists an element $a_r \in Ker\hat{f}$ such that $\bar{\xi}(f(r)) = \hat{\xi}(r+a_r)$;

The annihilator of the ring $\hat{R}$ contains $Ker\hat{f}$, i.e. $Ker\hat{f} \subseteq \left\{ a \in \hat{R} \,\Big|\, a \cdot \hat{R} = = \hat{R} \cdot a = 0 \right\}$ (in particular, $\left(Ker\hat{f}\right)^2 = 0$).

Proof ***I*** $\Rightarrow$ ***II*.** Let the mapping $f : (R, \xi) \to \left(\bar{R}, \bar{\xi}\right)$ possesses the following properties:

f is an isomorphism;

$\bar{\xi}(f(a)) \leqslant \xi(a)$ for any $a \in R$;

$\xi(a \cdot b) \leqslant \bar{\xi}(f(a)) \cdot \bar{\xi}(f(b))$ for any $a \in R$, $b \in R \backslash \{0\}$.

Let's prove that the statement II is valid.

Let $\left(\bar{R}', \bar{\xi}\right)$ be a ring with zero multiplication which elements belong to $\bar{R}$. We shall consider the ring $\hat{R} = R \oplus \bar{R}' = \left\{(r, \bar{r}) \,\middle|\, r \in R, \bar{r} \in \bar{R}'\right\}$ which is the direct sum of the rings R and $\bar{R}'$. Let's define the real-valued function $\tilde{\xi}$ on $\tilde{R}$ as in Theorem 2.3, i.e. $\hat{\xi}((r, \bar{r})) = \xi\left(r - f^{-1}(\bar{r})\right) + \bar{\xi}(\bar{r})$.

Let's define the mapping $\tilde{f} : \left(\tilde{R}, \tilde{\xi}\right) \to \left(\bar{R}, \bar{\xi}\right)$ by the equality $\tilde{f}((r, \bar{r})) = f(r)$.

Let's show that $\hat{\xi}$ is a pseudonorm on the ring $\hat{R}$.

The conditions 1 – 3 of Definition 1.1 can be verified by analogy with Theorem 2.3.

Let's verify the condition 4 of the definition of a pseudonorm. Since $\bar{R}'$ is a ring with zero multiplication then $\hat{\xi}((r, \bar{r}) \cdot (q, \bar{q})) = \hat{\xi}((r \cdot q, \bar{r} \cdot \bar{q})) = \hat{\xi}((r \cdot q, 0)) = = \xi(r \cdot q)$ for any $r, q \in R$ and $\bar{r}, \bar{q} \in \bar{R}'$.

It follows from the inequality $\xi(a \cdot b) \leqslant \bar{\xi}(f(a)) \cdot \bar{\xi}(f(b))$ that

$$\xi(r \cdot q) \leqslant \bar{\xi}(f(r)) \cdot \bar{\xi}(f(q)) = \bar{\xi}((f(r) - \bar{r}) + \bar{r}) \cdot \bar{\xi}((f(q) - \bar{q}) + \bar{q}).$$

Since the inequality $\bar{\xi}(\bar{y}_1 + \bar{y}_2) \leqslant \bar{\xi}(\bar{y}_1) + \xi(\bar{y}_2)$ is valid for any $\bar{y}_1$, $\bar{y}_2 \in \bar{R}'$ then

$$\bar{\xi}((f(r) - \bar{r}) + \bar{r}) \cdot \bar{\xi}((f(q) - \bar{q}) + \bar{q}) \leqslant \left(\bar{\xi}(f(r) - \bar{r}) + \bar{\xi}(\bar{r})\right) \cdot \left(\bar{\xi}(f(q) - \bar{q}) + \bar{\xi}(\bar{q})\right).$$

It follows from the inequality $\bar{\xi}(f(a)) \leqslant \xi(a)$ that

$$\left(\bar{\xi}(f(r) - \bar{r}) + \bar{\xi}(\bar{r})\right) \cdot \left(\bar{\xi}(f(q) - \bar{q}) + \bar{\xi}(\bar{q})\right) \leqslant$$

$$\leqslant \left(\xi\left(r - f^{-1}(\bar{r})\right) + \bar{\xi}(\bar{r})\right) \cdot \left(\xi\left(q - f^{-1}(\bar{q})\right) + \bar{\xi}(\bar{q})\right) = \hat{\xi}((r, \bar{r})) \cdot \hat{\xi}((q, \bar{q})).$$

Thus, the condition 4 of the definition of a pseudonorm is valid, i.e. $\hat{\xi}((r \cdot q, \bar{r} \cdot \bar{q})) \leqslant \leqslant \hat{\xi}((r, \bar{r})) \cdot \hat{\xi}((q, \bar{q}))$ for any r, $q \in R$ and $\bar{r}$, $\bar{q} \in \bar{R}'$.

We have shown that the function $\hat{\xi}((r, \bar{r})) = \xi\left(r - f^{-1}(\bar{r})\right) + \bar{\xi}(\bar{r})$ defines a pseudonorm on the ring $\hat{R}$.

Like in Theorem 2.3 let's identify the ring R with the set of pairs $\{\,(r,0)\mid r\in R\}$ which is an ideal in the ring $\hat{R}$. Let's consider the restrictions of the pseudonorm $\hat{\xi}$ and homomorphism $\hat{f}$ on the ring $R=\{\,(r,0)\mid r\in R\}$, i.e. $\hat{\xi}\left((r,0)\right)= =\xi\left(r-f^{-1}\left(0\right)\right)+\bar{\xi}\left(0\right)=\xi\left(r-0\right)+0=\xi\left(r\right)$ and $\hat{f}\left((r,0)\right)=f\left(r\right)$.

We have that $\hat{\xi}\Big|_R=\xi$ and $\hat{f}\Big|_R=f$.

Let's show that the annihilator of the ring $\hat{R}$ contains $Ker\hat{f}$.

Since $f:R\to\bar{R}$ is an isomorphism then $Ker\hat{f}=\left\{\,(r,\bar{r})\in\hat{R}|\ \hat{f}\left((r,\bar{r})\right)=0\right\}= =\left\{\,(r,\bar{r})\in\hat{R}|\ f\left(r\right)=0\right\}=\left\{\,(0,\bar{r})\in\hat{R}|\ \bar{r}\in\bar{R}'\right\}$.

Since $(0,\bar{r})\cdot(a,\bar{a})=(0\cdot a,\bar{r}\cdot\bar{a})=(0,0)$ and $(a,\bar{a})\cdot(0,\bar{r})=(a\cdot 0,\bar{a}\cdot\bar{r})=(0,0)$ for any $(0,\bar{r})\in Ker\hat{f}$ and $(a,\bar{a})\in\hat{R}$ then $Ker\hat{f}\subset Ann\hat{R}$.

Let's show that $\hat{f}:\left(\hat{R},\hat{\xi}\right)\to\left(\bar{R},\bar{\xi}\right)$ is an isometric homomorphism by analogy with Theorem 2.3. Let $(r,\bar{r})\in\hat{R}$ and $(0,\bar{j})\in Ker\hat{f}$. Then

$$\hat{\xi}\left((r,\bar{r})+(0,\bar{j})\right)=\hat{\xi}\left((r,\bar{r}+\bar{j})\right)=\xi\left(r-f^{-1}\left(\bar{r}+\bar{j}\right)\right)+\bar{\xi}\left(\bar{r}+\bar{j}\right)=$$

$$=\xi\left(r-f^{-1}\left(\bar{r}\right)-f^{-1}\left(\bar{j}\right)\right)+\bar{\xi}\left(\bar{r}+\bar{j}\right)\geqslant\bar{\xi}\left(f\left(r\right)-\bar{r}-\bar{j}\right)+\bar{\xi}\left(\bar{r}+\bar{j}\right)\geqslant\bar{\xi}\left(f\left(r\right)\right).$$

Thus, the inequality $\bar{\xi}\left(f\left(r\right)\right)\leqslant\hat{\xi}\left((r,\bar{r})+(0,\bar{j})\right)$ is valid for the element $(r,\bar{r})\in\hat{R}$ and any element $(0,\bar{j})\in Ker\hat{f}$. It means that $\bar{\xi}\left(f\left(r\right)\right)$ is one of the lower bounds of the set $\left\{\,\hat{\xi}\left((r,\bar{r})+(0,\bar{j})\right)\ \Big|\,(0,\bar{j})\in Ker\hat{f}\right\}$. Therefore, the inequality

$$\bar{\xi}\left(f\left(r\right)\right)\leqslant\inf\left\{\,\hat{\xi}\left((r,\bar{r})+(0,\bar{j})\right)\ \Big|\,(0,\bar{j})\in Ker\hat{f}\right\}\tag{6}$$

is valid for any $(r,\bar{r})\in\hat{R}$.

Let $(r,\bar{r})\in\hat{R}$ and $\bar{j}_0=f\left(r\right)-\bar{r}\in\bar{R}$. Then $(r,\bar{r})+(0,\bar{j}_0)=(r,\bar{r})+(0,f\left(r\right)-\bar{r})= =(r,f\left(r\right))$. It means that

$$\widehat{\xi}\left((r,\bar{r})+(0,j_0)\right)=\widehat{\xi}\left((r,f(r))\right)=\xi\left(r-f^{-1}(f(r))\right)+\bar{\xi}\left(f(r)\right)=$$

$$=\xi(r-r)+\bar{\xi}\left(f\left(r\right)\right)=\xi\left(0\right)+\bar{\xi}\left(f\left(r\right)\right)=0+\bar{\xi}\left(f\left(r\right)\right)=\bar{\xi}\left(f\left(r\right)\right).$$

Thus for any $(r,\bar{r})\in\hat{R}$ there exists $\bar{j}_0=(0,f(r-\bar{r}))\in Ker\hat{f}$ such that $\hat{\xi}\left((r,\bar{r})+(0,\bar{j}_0)\right)=\bar{\xi}\left(f\left(r\right)\right)$.

From here the inequality

$$\inf\left\{\,\hat{\xi}\left(\hat{r}+j\right)\ \Big|\,j\in Ker\hat{f}\right\}\leqslant\hat{\xi}\left((r,\bar{r})+(0,\bar{j}_0)\right)=\bar{\xi}\left(f\left(r\right)\right)\tag{7}$$

follows.

From inequalities (6) and (7) the equality

$$\bar{\xi}\left(f\left(r\right)\right)=\inf\left\{\,\hat{\xi}\left((r,\bar{r})+(0,\bar{j})\right)\ \Big|\,(0,\bar{j})\in Ker\hat{f}\right\}$$

follows.

Besides it follows from the equality $\hat{\xi}\left((r,\bar{r})+(0,\bar{j}_0)\right)=\bar{\xi}\left(f\left(r\right)\right)$ that

$$\bar{\xi}\left(f\left(r\right)\right)=\min\left\{\hat{\xi}\left((r,\bar{r})+(0,\bar{j})\right)\,\middle|\,(0,\bar{j})\in Ker\hat{f}\right\}=$$
$$=\min\left\{\hat{\xi}\left((r,0)+(0,\bar{i})\right)\,\middle|\,(0,\bar{i})\in Ker\hat{f}\right\},$$

where $(0,\bar{i})=((0,\bar{r})+(0,\bar{j}))\in Ker\hat{f}$.

We have shown that there exist a pseudonormed ring $\left(\hat{R},\hat{\xi}\right)$ and a homomorphism $\hat{f}:\hat{R}\to\bar{R}$ such that:

R is an ideal of the ring $\hat{R}$, $\hat{\xi}\Big|_R=\xi$ and $\hat{f}\Big|_R=f$;

$\bar{\xi}\left(f\left(r\right)\right)=\min\left\{\hat{\xi}\left((r,0)+(0,\bar{i})\right)\,\middle|\,(0,\bar{i})\in Ker\hat{f}\right\}$ for every $r\in R$, i.e. for any $r\in R$ there exist an element $(0,\bar{a}_r)\in Ker\hat{f}$ such that $\bar{\xi}\left(f\left(r\right)\right)=\hat{\xi}\left((r,0)+(0,\bar{a}_r)\right)$;

The annihilator of the ring $\hat{R}$ contains $Ker\hat{f}$.

Hence $I\Rightarrow II$ is proved.

Proof $II\Rightarrow I$. Let (R,ξ) and $\left(\bar{R},\bar{\xi}\right)$ be pseudonormed rings, $f:R\to\bar{R}$ be a ring isomorphism. Let $\left(\hat{R},\hat{\xi}\right)$ be a pseudonormed ring and $\hat{f}:\hat{R}\to\bar{R}$ be a homomorphism such that the following conditions are valid:

a) R is an ideal in the ring $\hat{R}$; $\hat{\xi}\Big|_R=\xi$ and $\hat{f}\Big|_R=f$;

b) $\bar{\xi}\left(f\left(r\right)\right)=\min\left\{\hat{\xi}\left(r+a\right)\,\middle|\,a\in Ker\hat{f}\right\}$ for every $r\in R$;

c) The annihilator of a ring $\hat{R}$ contains $Ker\hat{f}$.

It follows from Theorem 2.3 that the inequality $\xi\left(a\right)\geqslant\bar{\xi}\left(f\left(a\right)\right)$ is valid for any $a\in R$.

Let's show that the inequality $\xi\left(a\cdot b\right)\leqslant\bar{\xi}\left(f\left(a\right)\right)\cdot\bar{\xi}\left(f\left(b\right)\right)$ is valid for any $a,b\in R$ as well.

The equalities $\bar{\xi}\left(f\left(a\right)\right)=\min\left\{\hat{\xi}\left(a+i\right)\,\middle|\,i\in Ker\hat{f}\right\}=\hat{\xi}\left(a+i_a\right)$ and $\bar{\xi}\left(f\left(b\right)\right)=$ $=\min\left\{\hat{\xi}\left(b+j\right)\,\middle|\,j\in Ker\hat{f}\right\}=\hat{\xi}\left(b+j_b\right)$ are valid for any $a,b\in R$, where i_a, j_b are some elements from $Ker\hat{f}$. It means that

$$\bar{\xi}\left(f\left(a\right)\right)\cdot\bar{\xi}\left(f\left(b\right)\right)=\hat{\xi}\left(a+i_a\right)\cdot\hat{\xi}\left(b+j_b\right)\geqslant\hat{\xi}\left((a+i_a)\cdot(b+j_b)\right)=$$
$$=\hat{\xi}\left(a\cdot b+a\cdot j_b+i_a\cdot b+i_a\cdot j_b\right).$$

Sinse $i_a,j_b\in Ker\hat{f}\subset Ann\hat{R}$ then $a\cdot j_b=i_a\cdot b=i_a\cdot j_b=0$. Hence $\bar{\xi}\left(f\left(a\right)\right)\cdot\bar{\xi}\left(f\left(b\right)\right)\geqslant\hat{\xi}\left(a\cdot b+a\cdot j_b+i_a\cdot b+i_a\cdot j_b\right)=\hat{\xi}\left(a\cdot b\right)$ for any $a,b\in R$.

Thus, $\xi\left(a\right)\geqslant\bar{\xi}\left(f\left(a\right)\right)$ and $\xi\left(a\cdot b\right)\leqslant\bar{\xi}\left(f\left(a\right)\right)\cdot\bar{\xi}\left(f\left(b\right)\right)$ for any $a,b\in R$.

The theorem is proved.

References

[1] Arnautov V.I. *A semitopological isomorphism of topological rings.* Matematicheskie issledovaniya, vyp. 4:2(12), Kishinev, Shtiintsa, 1969, p. 3–16 (in Russian).

[2] Aleschenko S.A., Arnautov V.I. *Factor rings of pseudonormed rings.* Second Conference of the Mathematical Society of the Republic of Moldova, dedicated to the 40 anniversary of the foundation of the Institute of Mathematics and Computer Science of ASM, Chişinău, 2004.

Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
Academiei str. 5, MD-2028 Kishinev
Moldova
E-mail: *arnautov@math.md*

Received May 18, 2006

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(51), 2006, Pages 17–28
ISSN 1024–7696

Numerical treatment of the Kendall equation in the analysis of priority queueing systems

A.Iu. Bejan

Abstract. We investigate here how to treat numerically the Kendall functional equation occuring in the theory of branching processes and queueing theory. We discuss this question in the context of priority queueing systems with switchover times. In numerical analysis of such systems one deals with functional equations of the Kendall type and efficient numerical treatment of these is necessary in order to estimate important system performance characteristics.

Mathematics subject classification: Primary 65C50; Secondary 65B99.
Keywords and phrases: Kendall equation, priority queueing systems, switchover times.

1 Introduction

1.1 Preliminary notes

It is well known that the Kendall equation is one of fundamental functional equations which appear in the queueing theory and the theory of branching processes.

Consider a system $M|G|1$ with Poisson(λt) input flow of requests and random service time B with c.d.f. $B(t)$. The busy periods in such system are independent and identically distributed (i.i.d.) random variables (r.v.'s) with some cumulative distribution function (c.d.f.) $\Pi(t)$. How does $\Pi(t)$ depend on $B(t)$ and λ? Let $\beta(s)$ be the Laplace-Stieltjes transform of $B(t)$ and $\pi(s)$—the Laplace-Stieltjes transforms of $\Pi(t)$. It is well-known result that $\pi(s)$ satisfies the following functional equation:

$$\pi(s) = \beta(s + \lambda(1 - \pi(s))). \tag{1}$$

The equation (1) is known as *Kendall equation* due to Kendall (1951). It is not easy generally to obtain the analytical solution to this equation. Even in simple cases (choice of $B(t)$) the solution $\pi(s)$ may be analytically intractable. This, with the fact that $\pi(s)$ should be inverted in order to obtain full information on busy periods' distribution, leads to the necessity in numerical method for obtaining the solution of (1) and providing the absolute error of the evaluation. Such method is known and it is based on the fact that the right side of Kendall equation, being considered as a functional operator, has a fixed point, see Abate and Whitt (1992).

The Kendall equation makes part of many analytical results regarding distribution of busy periods in priority queueing systems $M_r|G_r|1$—queueing systems with

---

Poisson input flows of requests distinguished by importance and one server. Often such results are stated in the form of the systems of functional equations expressed in terms of Laplace-Stieltjes transforms.

For example, the LST $\pi(s)$ of the busy periods' c.d.f. $\Pi(t)$ in the priority queueing system $M_r|G_r|1$ with switchover times under the scheme "with losses" (for the description of such systems appeal to Mishkoy et al. (2006)) is determined by the following system of functional equations (see Klimov and Mishkoy (1979)):

$$\begin{cases} \pi_k(s) = \frac{\sigma_{k-1}}{\sigma_k}\{\pi_{k-1}(s+\lambda_k) + \Delta_{k-1}(s)\nu_k(s+\lambda_k[1-\overline{\pi_k}(s)])\} + \frac{\lambda_k}{\sigma_k}\pi_{kk}(s), \\ \Delta_{k-1}(s) = \pi_{k-1}(s+\lambda_k[1-\overline{\pi_k}(s)]) - \pi_{k-1}(s+\lambda_k), \\ \pi_{kk}(s) = \nu_k(s+\lambda_k[1-\overline{\pi_k}(s)])\overline{\pi_k}(s), \\ \overline{\pi_k}(s) = h_k(s+\lambda_k[1-\overline{\pi_k}(s)]), \\ \nu_k(s) = c_k(s+\sigma_{k-1})\{1-\frac{\sigma_{k-1}}{s+\sigma_{k-1}}[1-c_k(s+\sigma_{k-1})]\pi_{k-1}(s)\}^{-1}, \\ h_k(s) = \beta_k(s+\sigma_{k-1}) + \frac{\sigma_{k-1}}{s+\sigma_{k-1}}[1-\beta_k(s+\sigma_{k-1})]\pi_{k-1}(s)\nu_k(s),\ k=1,\ldots,r, \\ \pi_0(s) \equiv 0. \end{cases}$$

Here $\pi_r(s) \equiv \pi(s)$, λ_i is the parameter of the i^{th} Poisson input flow, and σ_i stands for $\sum\limits_{i=1}^{k} \lambda_i$.

For the same model with zero switchover times as an immediate corollary the following result follows:

$$\begin{cases} \sigma_k\pi_k(s) = \sigma_{k-1}\pi_{k-1}(s+\lambda_k[1-\pi_{kk}(s)]) + \lambda_k\pi_{kk}(s), \\ \pi_{kk}(s) = h_k(s+\lambda_k[1-\pi_{kk}(s)]), \\ h_k(s) = \beta_k(s+\sigma_{k-1}) + \frac{\sigma_{k-1}}{s+\sigma_{k-1}}[1-\beta_k(s+\sigma_{k-1})]\pi_{k-1}(s),\ k=1,\ldots,r, \\ \pi_0(s) \equiv 0. \end{cases}$$

One can notice the presence of the Kendall equations in such systems.

In the light of these facts, is suggested an acceleration of the scheme of obtaining a numerical solution to the Kendall equation which gives a gain in the number of operations (iterations) needed to solve one-dimensional problem (the case of $M|G|1$ system) as well as multidimensional problem (the case of $M_r|G_r|1$ system).

1.2 Notations and supporting results

1.2.1 Laplace and Laplace-Stieltjes transforms

Definition 1. Let $f(t)$ be a complex-valued function of real argument satisfying the following conditions: (i) $f(t) = 0\ \forall t < 0$, (ii) it is a function of bounded variation on any segment $[0,T]$, (iii) $\exists s_0, A \in \mathbb{R}$ s.t. $|f(t)| \leq Ae^{s_0 t}$. The Laplace transform of a function $f(t)$ is denoted by $\hat{f}(s)$ and is given by

$$\hat{f}(s) \overset{\partial ef}{=} \int\limits_0^\infty e^{-st}f(t)dt, \tag{2}$$

where $s \in \mathbb{C} : \Re s > s_0$.

This is a general definition. Conditions (i)-(iii) are required for correctness and existence of the transform. The Laplace transform is analytical in the right half-plane $\Re s > s_0$. The infimum of all such s for which the Laplace transform exists is called the *abscissa of convergence* and is denoted by σ_0. The Laplace transform for a real-valued function is defined automatically by Definition 1.

The Laplace transform is unique, in the sense that, given two functions $f_1(t)$ and $f_2(t)$ with the same transform, i.e. $\hat{f}_1(t) \equiv \hat{f}_2(t)$, the integral

$$\int\limits_0^T N(t)dt$$

of the null function $N(t) \overset{\partial ef}{=} f_1(t) - f_2(t)$ vanishes for all $T > 0$ (Lerch's theorem for integral transforms). The Laplace transform is linear:

$$a\hat{f}(t) + b\hat{g}(t) = a\hat{f}(t) + b\hat{g}(t) \tag{3}$$

The Laplace transform of a convolution $h(t) = f(t) * g(t) = \int\limits_0^\infty f(t-\tau)g(\tau)d\tau$ is given by

$$\hat{h}(t) = \hat{f}(t)\hat{g}(t) \tag{4}$$

Definition 2. The Laplace-Stieltjes transform of a real-valued function $F(t)$ of real argument is denoted by $\check{F}(s)$ and is given by

$$\check{F}(s) \overset{\partial ef}{=} \int\limits_0^\infty e^{-st} dF(t), \tag{5}$$

where $s \in \mathbb{C}$, whenever this integral exists. The integral here is the *Lebesgue-Stieltjes integral.*

Often s is a real variable, and then the LST is a real-valued function. If $F(t)$ is differentiable, i.e. $dF(t) = f(t)dt$, then the Laplace-Stieltjes transform of $F(t)$ is just a Laplace transform of its derivative.

Theorem 1. *(Uniqueness) Two different probability distributions have different Laplace-Stieltjes transforms.*

Theorem 2. *(Continuity) Let $F_n(t)$ be a cumulative distribution function with LST $\varphi_n(s)$, $n = 1, 2, \ldots$. If $F_n \to F$, where F is a possibly improper distribution with LST $\varphi(s)$, then $\varphi_n(s) \to \varphi(s)$ $\forall s > 0$. Conversely, if a sequence $\{\varphi_n(s)\}$ converges to $\varphi(s)$ for any $s > 0$, then φ is an LST of a possibly improper distribution F, s.t. $F_n \to F$. The limiting distribution F will be proper (or a probability distribution indeed) iff $\varphi(s) \to 1$ when $s \downarrow 0$.*

For the proofs of Theorem 1 and Theorem 2 see Feller (1971).

The n^{th} moment of the non-negative random variable X with p.d.f. $f_X(t)$ may be obtained via its Laplace transform in the following way:

$$\mathbb{E}[X^n] = (-1)^n \hat{f}_X^{(n)}(s)|_{s=0} \tag{6}$$

Abate and Whitt (1996) investigated functional operators that map one or more probability distributions on the positive real line into another using their Laplace-Stieltjes transforms.

1.2.2 Complete monotonicity and Bernstein theorem

We introduce now an important notion for our further considerations—a notion of *complete monotonicity.*

Definition 3. A real valued function φ on $[0, \infty)$ is said to be *completely monotone* (c.m.) if

$$(-1)^n \varphi^{(n)}(s) \geq 0 \ \forall n \in \mathbb{N} \cup \{0\} \ \forall s \in (0, \infty). \tag{7}$$

Example 1. Functions $s^{-\alpha}$, $e^{-\alpha s}$ $(\alpha \geq 0)$, $\frac{1}{1+s}$, $\frac{1}{s}$ are completely monotone functions. Function $\varphi(a+bs)$ is completely monotone when $\varphi(s)$ is completely monotone $(a \geq 0,\ b > 0)$.

We will make use of the following important properties of complete monotone functions in our further exposition. Functions φ and ψ are considered to be defined on $\mathbb{R}^+$.

Property 1. *If φ and ψ are two complete monotone functions, then their linear combination $\alpha\varphi + \beta\psi$ is a complete monotone function ($\alpha^2 + \beta^2 > 0$).*

Proof. The affirmation follows directly from the definition of complete monotone function. □

Property 2. *If φ and ψ are two complete monotone functions, then their product $\varphi\psi$ is a complete monotone function.*

Proof. We use the method of mathematical induction to show that

$$(-1)^n (\varphi\psi)^{(n)} \geq 0 \ \forall n = 0, 1, 2, ... \tag{8}$$

Assume that first n derivatives of $\varphi\psi$ alternate in sign. We first not that $\varphi\psi$ is nonnegative and $-\varphi'$ and $-\psi'$ are complete monotone functions. Therefore, (i) $-(\varphi\psi)' = -\varphi'\psi - \varphi\psi' \geq 0$ and this corresponds to the case $n = 1$ (the basis of induction), and (ii) we can apply the induction hypothesis for the products $-\varphi'\psi$ and $-\varphi\psi'$. But this immediately means that $\varphi\psi$ alternates in sign $n + 1$ times. By the principle of mathematical induction the property is proven. □

Property 3. *If φ is complete monotone and ψ is a non-negative function, s.t. ψ' is a complete monotone function, then $\varphi(\psi)$ is complete monotone.*

Proof. First, note that $\varphi(\psi)$ is a nonnegative function on $\mathbb{R}^+$. Then, notice that $-\varphi'(\psi)$ and ψ' are complete monotone functions. This makes $-\varphi'(\psi(s)) = -\varphi'(\psi)\psi'$ to be complete monotone (Property 2). We have proven that $-\left[\varphi(\psi)\right]'$ is a complete monotone function, i.e. $\varphi(\psi)$ is necessarily complete monotone. The property is proven. ☐

Laplace transforms of positive Borel measures on $\mathbb{R}^+$ are completely characterized by the Bernstein theorem in terms of complete monotonicity.

Theorem 6. (Bernstein (1928)) *Function φ is complete monotone iff there exists a unique nonnegative Borel measure μ on $[0, \infty]$, s.t. $\mu([0, \infty]) = \varphi(0^+)$ and $\forall s > 0$ $\varphi(s) = \int\limits_0^\infty e^{-sx}\mu(dt)$. Here $[0, \infty]$ is a one-point compactification of $[0, \infty)$.*

Remark 1. The theorem says that the class of complete monotone functions $\varphi(s)$ on half-line $\mathbb{R}^+$, such that $\varphi(0^+) \leq 1$ coincides with Laplace-Stieltjes transforms of cumulative distribution functions.

Example 2. It was mentioned above that the LST of the busy period in the system $M|G|1$ satisfies the Kendall equation. It can be shown (Feller (1971), Gnedenko et al (1971)), that Kendall equation determines a unique function $\pi(s)$ which is analytic in the right-half complex plane $\Re s > 0$. More of this, if the system workload $\rho = -\frac{\beta(0)}{\lambda} < 1$, then $\pi(0) = 1$, and the c.d.f. $\Pi(t)$ is a proper cumulative distribution function. The moments of the busy period Π in the system $M|G|1$ can be easily calculated using (6). For example, evaluating the first and second derivatives of $\pi(s)$ at zero in (1), one gets:

$$\mathbb{E}[\Pi] = -\pi'(0) = -\frac{\beta'(0)}{1-\rho} = \frac{\mathbb{E}[B]}{1-\rho},$$
$$\mathbb{E}[\Pi^2] = \pi''(0) = \frac{\mathbb{E}[B^2]}{(1-\rho)^3},$$

so that

$$\mathrm{var}[\Pi] = \mathbb{E}[\Pi^2] - \mathbb{E}[\Pi]^2 = \frac{\mathbb{E}[B^2] - \mathbb{E}[B]^2 + \rho\mathbb{E}[B]^2}{(1-\rho)^3}.$$

However, in order to obtain full information on the busy period Π one needs to invert $\pi(s)$. This should be found either analytically or evaluated numerically first.

In the case when $\rho > 1$, the following takes place: $\pi(0) < 1$, and $\Pi(t)$ is an improper c.d.f., i.e. $\lim\limits_{t\to\infty} \Pi(t) < 1$, that means that a busy period is of indefinite length with a positive probability. The case $\rho = 1$ is very special one.

A thorough discussion on Kendall equation treatment follows next.

2 Kendall equation: numerical treatment

2.1 Kendall fixed point operator

We discuss here in more detail the treatment of the Kendall equation

$$\hat{\varphi}(s) = \hat{g}(s + \rho - \rho\hat{\varphi}(s)), \tag{9}$$

where $\rho > 0$ and $\hat{g}(s)$ is a Laplace transform of some p.d.f. $g(s)$ associated with some proper c.d.f. $G(s)$. In other words, in virtue of Bernstein theorem (Theorem 6 and Remark 1), we suppose $\hat{g}(s)$ to be a complete monotone function, such that $\hat{g}(0^+) \leq 1$. The coefficient ρ is some non-negative real number.

Denote by $\mathcal{CM}$ the set of all complete monotone functions, by $\mathcal{CM}_1$ the set $\{\varphi(s) \in \mathcal{CM} \,|\, \varphi(0^+) \leq 1\}$, i.e., $\mathcal{CM}_1$ is the subset of complete monotone functions which correspond to proper or improper c.d.f.'s on $\mathbb{R}^+$.

To analyze (9) we introduce the following operator, which we will call *the Kendall operator*:

$$K_{\hat{g}}[\varphi](s) : \mathcal{CM}_1 \mapsto \mathcal{CM}_{\hat{g}} := \mathrm{Im}(K_{\hat{g}}) \subseteq \mathcal{CM}_1, \tag{10}$$

$$K_{\hat{g}}[\varphi](s) \overset{\partial ef}{=} \hat{g}(s + \rho - \rho\hat{\varphi}(s))), \text{ for some } \hat{g} \in \mathcal{CM}_1, \text{ and } \rho > 0. \tag{11}$$

First, we show that indeed $\mathrm{Im}(K_{\hat{g}}) \subseteq \mathcal{CM}_1$. To see this, note that

$$\psi_\varphi(s) = s + \rho(1 - \hat{\varphi}(s))$$

is a positive function with complete monotone derivative:

$$\begin{aligned} \psi'_\varphi(s) &= 1 - \rho\hat{\varphi'}(s) > 1 > 0, \\ \psi''_\varphi(s) &= -\rho\hat{\varphi}''(s) \leq 0, \\ &\ldots . \end{aligned}$$

We can apply Property 3 for $\hat{g}(\psi_\varphi(s))$ to see that $K_{\hat{g}}[\varphi](s)$ is complete monotone. Furthermore, this function is continuous as composition of continuous functions, therefore

$$K_{\hat{g}}[\varphi](0^+) = \hat{g}(\rho(1 - \hat{\varphi}(0^+))) \leq \hat{g}(0^+) \leq 1.$$

We conclude that $\mathrm{Im}(K_{\hat{g}}) \subseteq \mathcal{CM}_1$ for any given $\hat{g} \in \mathcal{CM}_1$.

The following theorem is well-known. Its proof is constructive and it is important for us.

Theorem 7. *The Kendall equation (9) has unique solution $\hat{\varphi}(s) \leq 1$, s.t. $\hat{\varphi}(s)$ is a Laplace-Stieltjes transform of some (probability) distribution Φ which is proper (and then it is a probability distribution) when $-\rho\hat{g}'(0) \leq 1$, and improper otherwise.*

Proof. Consider the following equation

$$Q_s(x) = \hat{g}(s + \rho - \rho x) - x = 0 \tag{12}$$

for a fixed $s > 0$ and for some $x \in [0, 1]$. The function $Q_s(x)$ is a convex function in respect to x (for any given $s > 0$), since its second derivative $\rho^2\hat{g}''(s + \rho(1 - x)) > 0$. Moreover, $Q_s(0) > 0$ and $Q_s(1) < 0$, therefore $Q_s(x)$ has a unique root x^* in $[0, 1]$. Allowing s to take any value from $\mathbb{R}^+$ one obtains the existence and uniqueness of the solution to the Kendall equation.

We want to show now that this solution is a c.m. function. Denote by $0(s)$ and $1(s)$ functions identical to zero and one on $\mathbb{R}^+$, corresp.

Consider two functional sequences: $\mathcal{U} = \{K^n_{\hat{g}}[0](s)\}_{n=0}^{\infty}$ with $K^0_{\hat{g}}[0](s) \equiv 0$ and $\mathcal{D} = \{K^n_{\hat{g}}[1](s)\}_{n=0}^{\infty}$ with $K^0_{\hat{g}}[1](s) \equiv 1 \ \forall s > 0$. We show that these sequences converge in a point-wise sense to the solution $\hat{\varphi}(s)$ to the Kendall equation (9).

It is obvious that $K^0_{\hat{g}}[0](s) \leq K^1_{\hat{g}}[0](s) \ \forall s > 0$, since $\hat{g}$ is a c.m. function. Suppose that $K^{n-1}_{\hat{g}}[0](s) \leq K^n_{\hat{g}}[0](s) \ \forall s > 0$. Then,

$$K^{n+1}_{\hat{g}}[0](s) = \hat{g}(s+\rho-\rho K^n_{\hat{g}}[0](s)) \geq \hat{g}(s+\rho-\rho K^{n-1}_{\hat{g}}[0](s)) = K^n_{\hat{g}}[0](s) \ \forall s > 0.$$

By induction

$$K^n_{\hat{g}}[0](s) \leq K^{n+1}_{\hat{g}}[0](s) \leq \hat{g}(0) \leq 1(s) \ \forall s > 0.$$

Thus, the sequence $\mathcal{U}$ is a monotone sequence of c.m. functions and is bounded from above—it has a limit which is a c.m. function (by Theorem 2) and which satisfies the Kendall equation, i.e. $\lim_{n\to\infty} K^n_{\hat{g}}[0](s) = \hat{\varphi}(s)$. This limit $\hat{\varphi}(s)$ is an LST of some probability distribution Φ with a total mass $\Phi([0,\infty]) = \varphi(0) \leq 1$.

One can show in a full analogy that $\mathcal{D}$ is a non-increasing sequence bounded from below by $0(s)$ and its limit is the unique solution to the Kendall equation:

$$\lim_{n\to\infty} K^n_{\hat{g}}[1](s) = \hat{\varphi}(s).$$

It remains to show that the probability distribution corresponding to the solution of the Kendall equation is proper iff $-\rho\hat{g}'(0) \leq 1$. To show this, consider

$$Q_0(x) = \hat{g}(\rho(1-x)) - x = 0, \tag{13}$$

as in (12), for $x = \varphi(0) = \lim_{n\to\infty} K^n_{\hat{g}}[0](s)$, and, thus, $x = \varphi(0)$ is the smallest root of (13) on $[0,1]$. Moreover, since $\hat{g}$ is an LST of a probability distribution, $Q_0(0) > 0$ and $Q_0(1) = 0$. So, the only possibility for (13) to have 2 roots on $[0,1]$ can be realized when $Q_0'(1) > 0$ (note that $Q_0''(x) > 0 \ \forall x \in (0,1)$), see Figure 1. The inequality $Q_0'(1) > 0$ is equivalent to the condition $-\rho\hat{g}'(0) > 1$. If $-\rho\hat{g}'(0) \leq 1$ then $\varphi(0) = 1$ and Φ is a proper probability distribution. ☐

We might also wish to consider operator $K_{\hat{g}}[\varphi](s)$ as well as the Kendall equation for complex argument s, s.t. $\Re s \geq 0$. Once obtained a c.m. function of real argument we can consider it as a function of complex arguments—this is justified by the principle of analytic continuation.

Theorem 8. *The sequence $\{K^n_{\hat{g}}[\hat{f}](s)\}_{n=0}^{\infty}$, where $\hat{f}$ is an LST of some possibly improper c.d.f. F, converges to the unique solution $\varphi(s)$ of the corresponding Kendall equation. The claim holds for complex s, s.t. $\Re s > 0$.*

Proof. Let first s be a real argument. It was proven in Theorem 7 that the Kendall equation has a unique solution. Moreover, the evidence of convergence of $\{\hat{\varphi}_n(s) := K^n_{\hat{g}}[\hat{f}](s)\}_{n=0}^{\infty}$ to the solution $\varphi(s)$ for any real $s > 0$ can be obtained in a constructive way exactly as it was made for the functions $\hat{f}(s) \equiv 0(s)$ and $\hat{f}(s) \equiv$

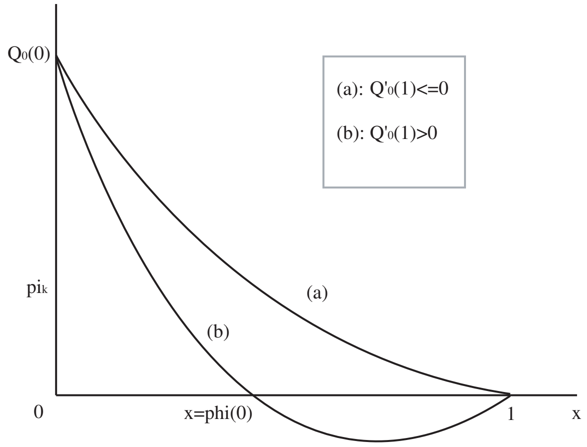

Figure 1. Alternative for $x = \varphi(0)$.

$1(s)$. Restricting attention to real s suffices to imply the point-wise convergence of corresponding c.d.f.'s Φ_n to Φ—the c.d.f. of the solution φ (Theorem 2):

$$\hat{\varphi}_n(s) \to \hat{\varphi}(s) \Rightarrow \Phi_n(t) \to \Phi(t).$$

However, the point-wise convergence for c.d.f.'s implies the transform convergence for all complex $s = x + iy$, s.t. $\Re s = x > 0$, since

$$\Re\hat{\phi}_n(s) = \int\limits_0^\infty \Re e^{-st} d\Phi_n(t) = \int\limits_0^\infty e^{-st} \cos yt \, d\Phi_n(t) \to \int\limits_0^\infty e^{-st} \cos yt \, d\Phi(t), \tag{14}$$

and

$$\Im\hat{\phi}_n(s) = \int\limits_0^\infty \Im e^{-st} d\Phi_n(t) = \int\limits_0^\infty e^{-st} \sin yt \, d\Phi_n(t) \to \int\limits_0^\infty e^{-st} \sin yt \, d\Phi(t), \tag{15}$$

so that

$$\begin{aligned} &\Re\hat{\phi}_n(s) \to \Re\hat{\phi}(s), \\ &\Im\hat{\phi}_n(s) \to \Im\hat{\phi}(s). \end{aligned} \tag{16}$$

We have the following result: $\hat{\phi}_n(s) \to \hat{\phi}(s)$ for all complex s with $\Re s > 0$.

The existence and uniqueness of the solution to the Kendall equation for complex s follows from the existence and uniqueness of the solution for non-negative real s and the principle of analytic continuity. □

The result and construction similar to that given in Theorem 7 was well-known for decades now. The fact that the iterations in the Kendall equation also work for complex values of the argument s was empirically found by a few authors (e.g. Doshi (1983)). The first proof of this result can be found in Abate and Whitt (1992). Our proof is similar, although some of its parts are different. It is heavily based on the notion of complete monotone function and the Bernstein theorem.

2.2 Adjustments in iterations of the Kendall operator

We know that if $\beta(s)$ is a complete monotone function and $\pi(s)$ satisfies the Kendall equation

$$\pi(s) = \beta(s + \rho(1 - \pi(s))),$$

then $\pi(s)$ is also a complete monotone function and can be numerically estimated at the point $s = s^* \geq 0$ using the following iteration procedure:

$$\begin{aligned} \pi(s^*) &= \lim_{n\to\infty} \pi^{(n)}(s^*), \text{ where} \\ \pi^{(n)}(s^*) &= \beta(s^* + \rho(1 - \pi^{(n-1)}(s^*))), \text{ with } \pi^{(0)}(s^*) \in [0, 1]. \end{aligned} \tag{17}$$

Here, the sequence $\{\pi^{(n)}(s^*)\}_{n=0}^{\infty}$ is non-increasing if $\pi^{(0)}(s^*) > \pi(s^*)$, and non-decreasing if $\pi^{(0)}(s^*) < \pi(s^*)$.

Remark 2. Note that the convergence of $\{\pi^{(n)}(s)\}_{n=0}^{\infty}$ to the solution $\pi(s)$ for complex s is assured by Theorem 8. However, there is no general result on the monotonicity of such convergence unless s is real. Abate and Whitt (1992) provided an example with no monotonicity (in any sense) of iterations in the complex case.

Using the property of complete monotonicity of the solution to the Kendall equation one can decrease the number of iterations to obtain the solution with a given precision. The idea is to set initial guess $\pi^{(0)}(s^*)$ of $\pi(s^*)$ closer to this value (s^* is a real non-negative number).

Suppose we are evaluating the solution to the Kendall equation on a regular grid $\{0, h, 2h, \ldots, (k-2)h, (k-1)h, kh, \ldots\}$. Suppose that π_{k-2} and π_{k-1} are the values which approximate $\pi(s)$ at the points $s = (k-2)h$ and $s = (k-1)h$. Then, let

$$\pi^{(0)}(s^* = kh) := \frac{\pi_{k-1} + \max(0, 2\pi_{k-1} - \pi_{k-2})}{2} \tag{18}$$

be the initial value for the process of iterations to the Kendall equation to evaluate $\pi(s)$ at $s = s^*$. The simple rationale behind this setting can easily be seen from Figure 2.

Alternatively, setting

$$\tilde{\pi}^{(0)}(s^*) := \pi_{k-1} \approx \pi(s^* - h),$$

and

$$\underset{\sim}{\pi}^{(0)}(s^*) := \max(0, 2\pi_{k-1} - \pi_{k-2}) \approx \max(0, 2\pi(s^* - h) - \pi(s^* - 2h)),$$

one may produce two sequences $\{\tilde{\pi}^{(n)}(s^*)\}_{n=0}^{\infty} \searrow \pi(s^*)$ and $\{\underset{\sim}{\pi}^{(n)}(s^*)\}_{n=0}^{\infty} \nearrow \pi(s^*)$ by iterating the Kendall equation:

$$\begin{aligned} \tilde{\pi}^{(n)}(s^*) &= \beta(s^* + \rho(1 - \tilde{\pi}^{(n-1)}(s^*))), \\ \underset{\sim}{\pi}^{(n)}(s^*) &= \beta(s^* + \rho(1 - \underset{\sim}{\pi}^{(n-1)}(s^*))), \end{aligned} \tag{19}$$

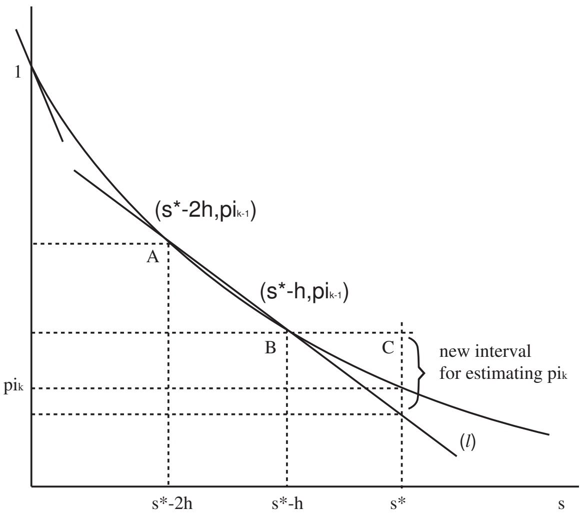

Figure 2. Improvement for iterations in Kendall equation

until

$$\epsilon_n = \frac{\tilde{\pi}^{(n)}(s^*) - \underset{\sim}{\pi}^{(n)}(s^*)}{2} < \epsilon,$$

where ϵ is a given precision. Finally, evaluate $\pi(s^*) \approx \frac{\tilde{\pi}^{(n)}(s^*) + \underset{\sim}{\pi}^{(n)}(s^*)}{2}$. The dependence of the number of iterations on the order of accuracy in iterations of the Kendall equation (for a particular value of s) is shown in Figure 3. The comparison between improved and not improved iteration process for solving the Kendall equation (for different types of service distribution) is shown in Figure 4.

3 Concluding remarks

It might seem the adjustments in iterations of the Kendall equation give non-essential gain in the number of operations needed to perform the iterations in order to achieve a solution with a certain level of precision in the case of $M|G|1$. However, in the context of the study of priority queueing systems with switchover times one needs to perform the iteration process quite many times. In the light of this, the adjustment process can be efficiently used for the acceleration of the numerical scheme's performance. Using this acceleration procedure the algorithms of busy period determination for priority queueing systems (e.g., the algorithm `BPLST` in Bejan (2004)) can be reviewed and improved.

Acknowledgement. This work was done under support of the SCOPES grant IB7320-110720.

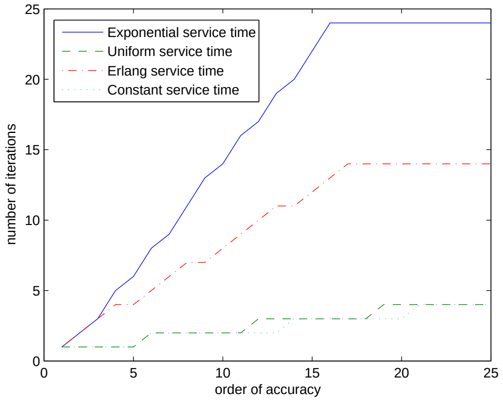

Figure 3. Number of iterations needed to solve the Kendall equation with corresponding accuracy (for different types of service distribution). The plot was obtained evaluating LST at $s = 2$—different values of s will produce different plots! This can be seen from Figure 4.

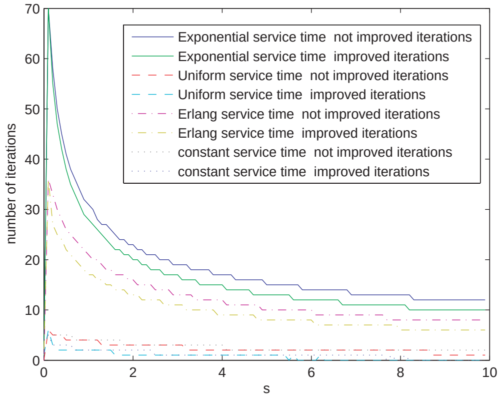

Figure 4. Comparison between improved and not improved iteration process for solving the Kendall equation; order of accuracy is 15.

References

[1] Abate J., Whitt W. *Solving probability transform functional equations for numerical inversion.* Oper. Res. Lett., 1992, **12**, p. 275–281.

[2] Abate J., Whitt W. *An operational calculus for probability distributions via Laplace transforms.* Adv. Appl. Probab., 1996, **28**, p. 75–113.

[3] Bejan A. *On algorithms of busy time period evaluation in priority queues with orientation time.* Communications of the Second Conference of the Mathematical Society of the Republic of Moldova, 2004, p. 32–36.

[4] Bernstein S.N. *Sur les fonctions absolument monotones.* Acta Math., 1928, **52**, p. 1–66.

[5] Doshi B.T. (1983)*An $M|G|1$ queue with a hybrid discipline.* Bell System Tech. J., 1983, **62**, p. 1251–1271.

[6] Feller W. *An Introduction to Probability Theory and its Application, Vol. 2.* Wiley, 1971.

[7] Gnedenko B.V. et al *Priority Queueing Systems.* Moscow State University Press, 1973 (in Russian).

[8] Kendall D.G. Some problems in the theory of queues. J. R. Stat. Soc. Ser. B Stat. Methodol, 1951, **13**, N 2, p. 151–185.

[9] Klimov G.P., Mishkoy G.K. *Priority queueing systems with switching.* Moscow State University Press, 1979 (in Russian).

[10] Mishkoy Gh., Giordano S., Andronati N., Bejan A. *Priority Queueing Systems with Switchover Times: Generalized Models for QoS and CoS Network Technologies and Analysis.* Technical report (`WEB: http://www.vitrum.md/andrew/PQSST.pdf`).

State University of Moldova
str. A. Mateevici, 60
Chişinău MD-2009, Moldova
and
Heriot-Watt University
School of Mathematical and Computer Sciences
Riccarton, EH14 4AS Edinburgh
Scotland, UK
E-mail:*A.I.Bejan@ma.hw.ac.uk*

Received March 3, 2006

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(51), 2006, Pages 29–34
ISSN 1024–7696

On the Division of Abstract Manifolds in Cubes

Mariana Bujac, Sergiu Cataranciuc, Petru Soltan

Abstract. We prove that in the class of abstract multidimensional manifolds without borders only torus V_1^n of dimension $n \geq 1$ can be divided in abstract cubes with the property: every face I^m from V_1^n is shared by 2^{n-m} cubes, $m = 0, 1, \ldots, n-1$. The abstract torus V_1^n is realized in E^d, $n+1 \leq d \leq 2n+1$, so it results that in the class of all n-dimensional combinatorial manifolds [1] *only* torus respects this propriety. Torus is autodual because of this propriety.

Mathematics subject classification: 18F15, 32Q60, 32C10.
Keywords and phrases: Abstract manifold, abstract cubic manifold, cubiliaj, Euler characteristic.

In paper [7, p.402] the scheme of the main types of n-dimensional manifolds it is presented, but the type of abstract manifolds which have been introduced recently in the papers [3–5] is missing. These abstract n-dimensional manifolds can be isomorphicly represented in E^d, $n+1 \leq d \leq 2n+1$. So we obtain combinatorial manifolds [1] which belong to the scheme mentioned above. We investigate abstract manifolds, which are defined by multi-ary relations and do not investigate directly combinatorial manifolds because we can obtain new results from abstract and more general point of view [6]. The base of an abstract manifold's definition is an abstract simplex S^n, which is defined on the set of $(n+1)$ elements from the $(n+1)$-ary relation of distinct elements.

First let's mention

Definition 1 [3]. *The complex of multi-ary relations,* $K^n = \{S_\lambda^m : \lambda \in \Lambda, card\Lambda < \infty, 0 \leq m \leq n\}$*, denoted* V_Δ^n*, is called an* ***abstract n-dimensional manifold without borders*** *if it satisfies the following postulates:*

A. *any abstract simplex* $S^{n-1} \in V_\Delta^n$ *is a common face exactly for two abstract* n*-dimensional simplexes;*

B. *for any simplexes* $S_i^n, S_j^n \in V_\Delta^n$*,* $i \neq j$*, there exists a sequence of* n*-dimensional simplexes* $S_1^n = S_i^n, S_2^n, \ldots, S_k^n = S_j^n$*,* $k \geq 2$*, where* $S_r^n \cap S_{r+1}^n = S_{r,r+1}^{n-1}$*,* $r \in \{1, 2, \ldots, k-1\}$*;*

C. *for* $\forall S^m \in V_\Delta^n$ *it holds that* $\exists S^n \in V_\Delta^n$*, such that* S^m *is a face of* S^n*,* $m \in \{0, 1, \ldots, n-1\}$*;*

D. *for any two disjoint simplexes* $\forall S_i^n, S_j^n \in V_\Delta^n$, *where* $S_i^n \cap S_j^n = S^m$, *it holds that* $\exists S_1^n = S_i^n, S_2^n, \ldots, S_k^n = S_j^n$, *such that* $\bigcap\limits_{l=1}^{k} S_l^n = S^m$.

We are interested only in the examination of oriented manifolds [3, 4]. Let's mention

Definition 2 [2, 6]**.** *The cubic complex* $K^n = \{I_\lambda^m : \lambda \in \Lambda, card\Lambda < \infty, 0 \le m \le n\}$, *denoted* $V_\square^n$, *is called an* ***abstract cubic n-dimensional manifold without borders*** *if the following properties are satisfied*:

A. *any* $(n-1)$*-dimensional cube is a common face exactly for two n-dimensional cubes from* K^n*;*

B. *for* $\forall I_i^n, I_j^n \in K^n, i \neq j$, *there exists a sequence of cubes from* K^n, $I_{i_1}^n = I_i^n, I_{i_2}^n \ldots, I_{i_q}^n = I_j^n$, *where* $I_r^n \cap I_{r+1}^n = I_{r,r+1}^{n-1}, r \in \{i_1, i_2, \ldots, i_{q-1}\}$*;*

C. *for* $\forall I^p \in K^n$, $0 \le p \le n-1$, *it holds* $\exists I^n \in K^n$, *where* I^p *is a face of* I^n*;*

D. *for any disjoint cubes* $\forall I_i^n, I_j^n \in K^n$, $I_i^n \cap I_j^n = I^p$, $2 \le p < n$, *there exists a sequence of abstract cubes from B.,* $I_{i_1}^n = I_i^n, I_{i_2}^n, \ldots, I_{i_q}^n = I_j^n$, *such that* $\bigcap\limits_{j=1}^{q} I_{i_j}^n = I^p$.

We are interested also in the examination of oriented cubic manifolds [6].

Definition 1 is based on a finite complex of multi-ary relations, but Definition 2 is formulated using a finite number of abstract cubes, which are defined by abstract simplexes. So Definition 1 and 2 are equivalent and in the following we use only the notation V^n.

Definition 3. *The property of n-dimensional abstract manifold without borders* V_p^n, *which is determined of a cubic complex* K^n, *such that* ***every*** *m-dimensional cube,* $0 \le m \le n$, *belongs to* 2^{n-m} *n-dimensional cubes, is called a* ***normal cubiliaj***[1] *of* V_p^n.

Let's define now a finite product of edges (abstract 1-dimensional cubes [4]) analogous with cartesian product.

Definition 4. *Let* $I_1^1, I_2^1, \ldots, I_r^1$ *be some 1-dimensional oriented abstract cubes. By induction*

1. $I_1^1 \otimes I_2^1 = I^2$, *where* I^2 *is a 2-dimensional abstract cube* [4] *and* $\overset{\circ}{I^2} = \overset{\circ}{I_1^1} \otimes \overset{\circ}{I_2^1}$ [4] *is his vacuum.*

(r-1). *Let's consider that* $I^{r-1} = I^{r-2} \otimes I_{r-1}^1$ *is defined, where* I^{r-1} *is an* $(r-1)$*-dimensional abstract cube* [4] *and* $\overset{\circ}{I^{r-1}} = \overset{\circ}{I_1^{r-2}} \otimes \overset{\circ}{I_{r-2}^1}$ *is its vacuum.*

---

[1] The notion of cubilaj was borrowed from the papers [10, 11].

r. *Inductively we define r-dimensional abstract cube* I^r *in the following way:* $I^r = I^{r-1} \otimes I^1_r$, *where* $\overset{\circ}{I^r} = \overset{\circ}{I^{r-1}} \otimes \overset{\circ}{I^1_r}$. *The cube* I^r *is called a* **cartesian product** *of cubes* $I^1_1, I^1_2, \dots, I^1_r$ *and will be denoted by*

$$I^r = \prod_{i=1}^{r} I^1_i \tag{1}$$

Let's consider n abstract oriented circumferences (1-dimensional manifolds): $V^1_1, V^1_2, \dots, V^1_n$ with the length (the number of 1-dimensional cubes) $d_1, d_2, \dots, d_n$. Using (1), we consider the cartesian product:[2]

$$K^n = \prod_{i=1}^{n} V^1_i \tag{2}$$

In accordance with Definition 2, it is obvious that (2) establishes an n-dimensional abstract manifold without borders which posseses a normal cubiliaj. Moreover, the Euler characteristic of V^1_i is $\chi(V^1_i) = 0, i \in \{1, 2, \dots, n\}$, so [7]:

$$\chi(K^n) = \prod_{i=1}^{n} \chi(V^1_i) = 0. \tag{3}$$

Consequently we have

Corollary 1. *The product* (2) *establishes an abstract torus* V^n_1 *(see Figure 1).*

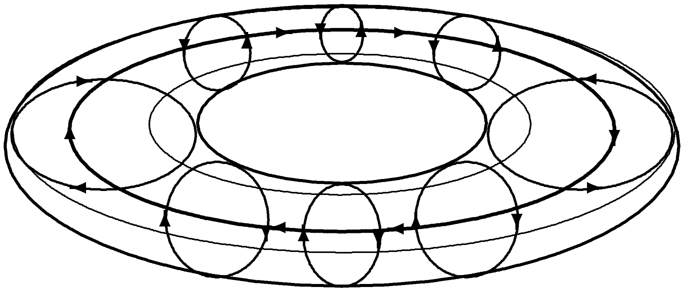

Figure 1

This corollary results from the fact that for every n (odd or even) (3) is true.

It holds

Theorem 1. *An abstract oriented manifold without borders which has a normal cubiliaj is a torus* V^n_1 *if and only if* V^n *is established by the cartesian product* (2).

---

[2] The formula (2) is realized in E^{2n+1} [1], so it define an n-dimensional torus as a cartesian product of n circumferences [13].

Proof. *The sufficiency* is obvious because of Corollary 1.

The necessity is simple. Let V^n be an abstract manifold which has a normal cubiliaj and I^n an abstract cube of V^n. $a_1, a_2, \dots, a_n$ are n oriented arcs with common origin which determine the manifold V^n. Let's consider the class of equivalence of "parallel" arcs A_1 ([$8-10, 12$]) and the class of $(n-1)$-dimensional cubes of V^n which are determined by elements from the class A_1.

Let's denote the last class by V_1^{n-1}. It is obvious that the last one is an abstract submanifold of V^n which possesses hereditarily a normal cubiliaj. Coherently let's move along the arc a_1. The end of the arc a_1 belongs to another abstract submanifold V_2^{n-1} of V^n which is "parallel" with V_1^{n-1}. Suppose that the manifold V_2^{n-1} is "perpendicular" to another arc b_1, coherent to a_1 (otherwise we give a new orientation to it). In the same reasoning we can obtain another manifold V_3^n which has a normal cubiliaj. By induction we can construct a 1-dimensional contour without cross points because of the finite number of cubes from V^n. If the intersections exists then V^n doesn't have a normal cubiliaj. So we obtain the first oriented abstract circumference V_1^1. By induction of the index i of a_i, considering the class of equivalence A_i of arcs "parallel" to a_i for V_i^{n-i}, $i \in \{1, 2, \dots, n-1\}$, we construct the $(n-1)$ oriented abstract circumference. So we have the abstract circumferences $V_1^1, V_2^1, \dots, V_{n-1}^1$. The submanifold V^1 of V^n which is perpendicular to $a_1, a_2, \dots, a_{n-1}$ (see figure 1, the thick meridian) possesses hereditarily a normal cubiliaj. So we have $V^1 = V_n^1$. Using the formula (2) we obtain the proof of Theorem 1.

It holds

Theorem 2. *Let V_p^n, $p \neq 1$, be a coherent oriented abstract manifold without borders* [1]*. This manifold does not possess a normal cubiliaj.*

Proof. By contradiction. We consider a submanifold V_p^{n-1} of V_p^n, $p \neq 1$, which can be obtain in the same way as in the proof of Theorem 2, using the arc $a_1 \in I^n$. Analogously to the proofs' procedure of Theorem 2, we can obtain n n-dimensional contours without autointersection, $V_1^1, V_2^1, \dots, V_n^1$, which cartesian product is

$$V_p^n = \prod_{i=1}^{n} V_i^i. \tag{4}$$

In accordance with Theorem 1, the product (4) represents a torus V_1^n which possesses a normal cubiliaj. This contradiction (for $p \neq 1$) results from a false assumption. Theorem 2 is proved.

Form Theorems 1 and 2 we obtain

Fundamental theorem. *A unique abstract n-dimensional manifold without borders V_p^n, where $n \geq 0$, which possesses a normal cubiliaj is the torus V_1^n.*

Remark 1. *In parer* [9] *was established that the sphere $S^2 \subset E^2$ does not possess a normal cubiliaj.*

It holds

Theorem 3. *Only the abstract torus, V_1^p, which possesses a normal cubiliaj, represents an autodual manifold corresponding to this cubiliaj.*

Proof. Let $\alpha_0, \alpha_1, \ldots, \alpha_n$ be the numbers of abstract cubes of V_1^n, with respective dimension $0, 1, 2, \ldots, n$. So we have:

$$\chi(V_1^n) = \sum_{i=1}^{n}(-1)^i \alpha_i = 0. \tag{5}$$

Considering the cubic complex K_d^n with the class of the cubes $\alpha_n, \alpha_{n-1}, \ldots, \alpha_0$ having the dimensions $0, 1, \ldots, n$ respectively and invariant incidences, we obtain that K_d^n is isomorphic to the complex $K^n = V_1^n$. So V_1^n is determined uniquely by K_d^n. From (5) it results:

$$\chi(V_1^n) = \chi(K_d^n) = \sum_{i=0}^{n}(-1)^{n-i} \alpha_{n-i} = 0. \tag{6}$$

So the initial normal cubiliaj of V_1^n is isomorphic to the normal cubiliaj which is established by K_d^n (see Figure 2.)

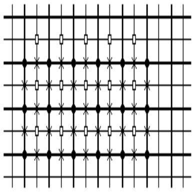

Figure 2

So this fact represents the *autoduality* of the torus V_1^n. Only this one is represented by a normal cubiliaj. In accordance with the Fundamental theorem such kind of autodualism has only the abstract torus V_1^n. Theorem 3 is proved.

Remark 2. *When the above results were obtained as something additional in the solving of application problems, we were informed about the papers* [11 − 13]. *This helped us to change the terms' names. The problems formulated by the famous mathematician Serghei Novikov inspired us to additional examinations. We do this with gratitude.*

In the following paper we will indicate the value of the Fundamental theorem in the transmission, receiving and picking up of information.

References

[1] Boltyanski V. *Gomotopiceskia teoria nepreryvnyh otobrajenii i vektornyh polei*. Moskva, Izd. Akademii Nauk SSSR, 1955 (in Russian).

[2] Bujac M. *Clasificarea varietăţilor abstracte multidimensionale orientabile şi fără borduri*. Analele Ştiinţifice ale USM, Seria "Ştiinţe fizico-matematice", Chişinău, 2003, p. 247–250.

[3] Bujac M., Soltan P. *On the Abstract and Nonoriented Varieties*. Annals of the Tiberiu Popoviciu Seminar of Functional Equations, Appriximation and Convexity, vol. 2, Mediamira Science Publisher, Cluj-Napoca, 2004, p. 11–16.

[4] Bujac M. *The Multidimensional Directed Euler Tour of Cubic Manifold*. Buletinul Academiei de Ştiinţe a Republicii Moldova, Matematica, 2006, N 1(50), p. 15–22.

[5] Bordei A., Martini H., Soltan P. *Diluate Homologies and Fixed Elements in a Complex of Multi-ary Relations*. Utilitas Matematica, 2004, **LXVI**, p. 267–286.

[6] Martini H., Soltan P. *On the Homologies of Multi-ary Relations*. JCMCC45, 2003, p. 219–243.

[7] *Matematiceskaia entynclopedya*, Tom 5, Moskva, 1985.

[8] Soltan P., Zambiţchii D, Prisăcaru C *Ekstremaliniye zadaci na grafah i algoritmy ih reshenya*. Chişinău, 1973 (in Russian).

[9] Soltan P., Prisăcaru C *Zadacha Shteinera na grafah*. DAN SSSR, 1971, **198**, N 1 (in Russian).

[10] Karalashvili O. *Ob otobrajeniah cubiceskih mnogobrazii v standartnuiu reshetcu Evklidova prostranstva*. Trudy Matematiceskogo instituta AN SSSR, Tom 196, 1991 (in Russian).

[11] Dolbilin N., Stanco M., Stogrin M. *Kvadriliaji i parametrizatzii resetkah Tiklov*. Trudy Matematiceskogo instituta AN SSSR, Tom 196, 1991 (in Russian).

[12] Dolbilin N., Stanco M., Stogrin M. *Kombinatornye voprosy dvuhmernoy modely Izinga*. Trudy Matematiceskogo instituta AN SSSR, Tom 196, 1991 (in Russian).

[13] Bishop R., Crittendeu R. *Geometry of Manifolds*. New York–London, 1964.

Moldova State University
Faculty of Mathematics and Computer Science
60 A. Mateevici street, Chişinău
MD−2009, Moldova
E-mail: *marianabujac@yahoo.com*
caseg@usm.md
psoltan@usm.md

Received April 21, 2006

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(51), 2006, Pages 35–44
ISSN 1024–7696

The numerical analysis of the tense condition of a solid body with the asymmetrical tensor of strains

Vasile Ceban, Ion Naval

Abstract. In this paper an approach permitting to make calculation of non-steady fields of elastic bodies with an asymmetrical stress tensor is proposed. On the basis of integral equations the explicit difference network, founded on S.K. Godunov method named "disintegrations of a gap" is constructed. The versions are considered, when the difference network approximates an initial set of equations with the first and second order of accuracy.

Mathematics subject classification: 74H15.
Keywords and phrases: Asymmetrical stress tensor .

The classical theory of elasticity is based on the model of a solid body in which interaction between particles is realized only by central forces.

However, it is impossible to explain satisfactorily the regularities of some phenomena, for instance, the spreading of the short acoustic waves in a crystalline solid body, polycrystalline metal and high polymer.

The theoretical results do not give satisfactory concordance with experimental data for a body with obviously expressed polycrystalline structure, in the complex tense condition with high gradient of the tension.

The model elaborated for the explanation of these phenomena differers from classical one by the fact that tense condition on elementary platform is characterized alongside with the vector of the power tension by the vector of moment tension, referred to the same unit platform as the vector of the usual tension.

The model of elastic moments medium was created in which infinitely small volume possesses six degrees of freedom, and the interaction between elements of the medium is realized by power and moments tensions [1–4].

Here we are concentrated on the consideration of the variant in which the motion of the medium point is completely described by the vector of the onward displacement , but the vector of the angular tumbling is equal to the local rotation of the medium in the sense of the usual theory of elasticity, i.e.

$$\overrightarrow{\omega} = \frac{1}{2} rot \overrightarrow{u}.$$

Such model is known under the name of continuum Kossera with straight rotation [5–7]. Each point of the medium has 3 degrees of freedom, and its motion is

---

completely defined by the vector $\overrightarrow{u}$. In this variant of the moment theory of elasticity four independent elasticity constants: λ, μ, l, k are considered ; λ, μ are the Lame parameters, l is the constant with dimension of the length, k is a dimensionless constant of the type of Poison coefficient.

The waves of the expansion in the Kossera medium with straight rotation do not differ from the expansion waves in the classical theory of elasticity, however, the tension moments influence essentially on the distortion waves consideration. We shall present basic system of the equations for the case of the asymmetrical theory of elasticity we are interested in.

The equations of the motion for the plane case in a rectangular coordinate system will be of the form:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = \rho \frac{\partial^2 u}{\partial t^2},$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = \rho \frac{\partial^2 v}{\partial t^2}, \tag{1}$$

where u, v are components of the displacement vector $\overrightarrow{u}$, and $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \sigma_{yx}$ are components of the tension tensor.

In this case $\sigma_{xy} \neq \sigma_{yx}$, and the relations between components of the tension and deformation taking in account inertness of the medium's particles rotation are of the form [8]

$$\sigma_{xx} = \lambda(\varepsilon_{xx} + \varepsilon_{yy}) + 2\mu\varepsilon_{xx}, \qquad \sigma_{yy} = \lambda(\varepsilon_{xx} + \varepsilon_{yy}) + 2\mu\varepsilon_{yy},$$

$$\sigma_{xy} = \mu\left[\varepsilon_{xy} - l^2\left(\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2} - \frac{k^2\rho}{\mu}\frac{\partial^2\omega}{\partial t^2}\right)\right] \tag{2}$$

$$\sigma_{xy} = \mu\left[\varepsilon_{xy} + l^2\left(\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2} - \frac{k^2\rho}{\mu}\frac{\partial^2\omega}{\partial t^2}\right)\right],$$

where ρ is the density, $\omega = \partial u/\partial y - \partial v/\partial x$.

As a consequence of the relations (2) the equation of the medium particles rotation is examined in the following form:

$$2k^2\rho l^2 \frac{\partial^2\omega}{\partial t^2} = \frac{\partial M_{zx}}{\partial x} + \frac{\partial M_{zy}}{\partial y} + (\sigma_{xy} - \sigma_{yx}),$$

where $\sigma_{xy} + \sigma_{yx} = 2\mu\varepsilon_{xy}$.

The internal moments M_{zx} and M_{zy} are expressed by ω as follows

$$M_{zx} = 2\mu l^2 \frac{\partial\omega}{\partial x}, \quad M_{zy} = 2\mu l^2 \frac{\partial\omega}{\partial y}. \tag{3}$$

The numerical integration of the hyperbolic system (4) is realized in the presence of the initial and border conditions by the method, founded on approximations of the equations by finite differences taking into account relations along characteristic directions.

For the numerical solution of the moment theory of elasticity equations, S. K. Godunov [9] explicit difference scheme will be considered. It requires to pass to the system of equations in first-order partial derivatives. Taking in consideration the velocity of the displacement, and differentiating (2) by time, after uncomplicated transformations (4) will be obtained.

Here with the bar the corresponding functions differentiated by time are marked. From now on for convenience, the bar will be omitted. It is known that the method requires the determination of the characteristic equations and relations for them. In accordance with S.K. Godunov method for the determination of the characteristics in x direction free terms and derivatives by y are excluded from the system (4).

It is easy to note that in the obtained system of equations we have:

a) Two equations form up closed group giving the unknown functions σ_{xx}, u;

b) Two equations form up closed group giving the unknown functions σ_{xy}, v;

c) Two equations form up closed group giving the unknown functions M_{zx}, ω

$$\begin{gathered}
\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}=\rho\frac{\partial u}{\partial t},\qquad \frac{\partial\sigma_{yx}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}=\rho\frac{\partial v}{\partial t},\\
\frac{\partial\sigma_{xx}}{\partial t}=\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+2\mu\frac{\partial u}{\partial x},\\
\frac{\partial\sigma_{yy}}{\partial t}=\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+2\mu\frac{\partial v}{\partial y},\\
\frac{\partial\sigma_{xy}}{\partial t}+\frac{\partial\sigma_{yx}}{\partial t}=2\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),\\
2k^2l^2\rho\frac{\partial\omega}{\partial t}=\frac{\partial M_{zx}}{\partial x}+\frac{\partial M_{zy}}{\partial y}+\sigma_{xy}-\sigma_{yx},\\
\frac{\partial M_{zx}}{\partial t}=2\mu l^2\frac{\partial\omega}{\partial x},\qquad \frac{\partial M_{zy}}{\partial t}=2\mu l^2\frac{\partial\omega}{\partial y}.
\end{gathered}\tag{4}$$

Father, following the known procedure, we shall multiply the first (the second) equation of the obtained first group of "one-dimensional" system on the α_1 (α_2) and add both equations.

As a result we get

$$\frac{\partial}{\partial t}\{\alpha_1\rho\, u+\alpha_2\sigma_{xx}\}=\frac{\partial}{\partial x}\{\alpha_2(\lambda+2\mu)\, u+\alpha_1\sigma_{xx}\}.\tag{5}$$

Our purpose consists in obtaining the characteristic equation for the Reman invariant F in the following form:

$$\frac{\partial F}{\partial t}=\eta\frac{\partial F}{\partial x}.$$

The linear combination of the velocity and tension (5) will be an invariant F if the following conditions are fulfilled

$$\alpha_1\rho = \alpha_2(\lambda + 2\mu)/\eta; \quad \alpha_1/\eta = \alpha_2.$$

In this homogeneous system one needs to determine such a value η for which its nonzero solution exists. Consequently,

$$(\rho - (\lambda + 2\mu)/\eta^2)\alpha_1 = 0; \quad (\rho - (\lambda + 2\mu)/\eta^2)\alpha_2 = 0. \tag{6}$$

Since for the existence of a nonzero solution either α_1, or α_2 must be different from zero, we find eigenvalues η_i:

$$\eta_{1,2} = \pm\sqrt{(\lambda + 2\mu)/\rho} = \pm c_1,$$

where c_1 is the velocity of the longitudinal wave spreading.

Proceeding similarly, for the second group of equations we shall get

$$\eta_{3,4} = \pm\sqrt{\mu/\rho} = \pm c_2,$$

where c_2 is the velocity of the transversal wave spreading.

The third group of equations gives

$$\frac{\partial}{\partial t}\{\alpha_1 M_{zx} + 2\alpha_2 k^2\rho l^2\omega\} = \frac{\partial}{\partial x}\{\alpha_2 M_{zx} + 2\alpha_1\mu l^2\omega\}. \tag{7}$$

The linear combination (7) will be an invariant F if the following conditions are fulfilled:

$$\alpha_1 = \alpha_2/\eta, \qquad \alpha_1\mu/\eta = \alpha_2 k^2\rho. \tag{8}$$

In homogeneous system (8) it is necessary to determine such value of η, for which a nonzero solution for this system exists. Consequently,

$$(k^2\rho - \mu/\eta^2)\alpha_1 = 0, \qquad (k^2\rho - \mu/\eta^2)\alpha_2 = 0. \tag{9}$$

Since the existence of a nonzero solution necessitates that either α_1, or α_2 must be different from zero, we find eigenvalues η_i:

$$\eta_{5,6} = \pm\sqrt{\mu/k^2/\rho} = \pm c_3.$$

Because in the medium only one transversal wave must be, we require $k = 1$ and $c_3 = c_2$. Really, if the volume deformation $\delta = \partial u/\partial x + \partial v/\partial y$ will be introduced, then it is possible to reduce the system of equations (4) to the following form:

$$\Delta\delta - \frac{1}{c_1^2}\delta_{tt} = 0; \quad \Delta\omega - \frac{1}{c_2^2}\omega_{tt} - l^2\left(\Delta\omega - \frac{k^2}{c_2^2}\omega_{tt}\right) = 0. \tag{10}$$

As can be seen from (10), this equation is a wave one relative to σ, as well as in the classical linear theory of elasticity while shift deformation ω satisfies the fourth order equation. By the study of the flat waves spreading in the elastic medium three-dimensional and shift deformation are expressed through four arbitrary functions depending on the flat waves [10]

$$\mathrm{Z}_{1,2} = t - \theta x \mp \sqrt{c_1^2 - \theta^2} y, \qquad \mathrm{Z}_{3,4} = t - \theta x \mp \sqrt{c_2^2 - \theta^2} y,$$

where θ is an arbitrary parameter.

The equation (10) relative to δ allows arbitrary solutions for flat waves through $\mathrm{Z}_{1,2}$.

We research the equation (10) for ω. Let $\omega = f\left(\gamma t - \alpha y - \beta x\right)$. Substituting in equation (10), we shall get

$$\left(\alpha^2 + \beta^2 - \gamma^2/c_2^2\right) f^{''} - l^2 \left(\alpha^2 + \beta^2\right)\left(\alpha^2 + \beta^2 - k^2\gamma^2/c_2^2\right) f^{IV} = 0 \tag{11}$$

From the given equation it follows that flat waves of the type $\omega = f\left(\gamma t - \alpha y - \beta x\right)$ for arbitrary type of the functions f are possible only by the condition $k = 1$, $\alpha^2 + \beta^2 - \gamma^2/c_2^2 = 0$.

So, further we shall everywhere consider $k = 1$.

We shall consider further the case of $\eta_{1,2} = \pm c_1$. Then from (6) it follows that α_1 is an arbitrary constant value, which we take equal to ± 1, and the factor $\alpha_2 = \pm 1/c_1$.

Then the invariant $F_1 = \rho u \pm \sigma_{xx}/c_1$.

Similarly it is possible to obtain $F_2 = \rho v \pm \sigma_{xy}/c_2$; $F_3 = M_{zx} \pm 2\alpha_2 \rho l^2/c_2 \omega$.

This means that along the characteristics $\frac{\partial x}{\partial t} = \pm \eta_k$ the relations $F_k = const$ are fulfilled, because

$$dF_k = \frac{\partial F_k}{\partial t} dt + \frac{\partial F_k}{\partial x} dx = \left(\frac{\partial F_k}{\partial t} \pm \eta_k \frac{\partial F_k}{\partial x}\right) dt = 0.$$

By analogy with "one-dimensional" system relative to the variable x, "one-dimensional" system relative to the variable y can be considered, obtained from system (4) when excluding derivatives by y.

The direct comparison of these systems shows that they become completely identical after establishing the correspondence:

$$x \leftrightarrow y, \quad u \leftrightarrow v, \quad \sigma_{xx} \leftrightarrow \sigma_{yy}, \quad \sigma_{xy} \leftrightarrow \sigma_{yx}, \quad M_{zx} \leftrightarrow M_{zy}, \quad F_k \leftrightarrow \Phi_k.$$

The characteristics and relations for them are automatically obtained with account of this correspondence.

The invariants F_k and $\Phi_k (k = 1, 2, 3)$ possess important features that along straight line $\pm c_k t + x = const$ they keep constant values.

These features are put in the base of the finite difference scheme construction.

In the domain of the arguments x and y variation we shall introduce uniform differences schemes as follows. The area will be limited by the contour, given by the restrictions $0 \leq x \leq a$, $0 \leq y \leq b$. We shall cover this square-wave area with the net as follows: for x

$$h_x = a/I, \quad x_i = i \cdot h_x, i = 1, 2, \cdots, I,$$

and for y

$$h_y = a/J, \quad y_j = j \cdot h_y, j = 1, 2, \cdots, J.$$

The nodes of the deference net, in which we shall define the unknown functions, choose in the cell center, formed by orthogonal net. All unknown functions are related to the center of accounting cell and are considered constant within a separate cell. Since the problem is dynamic then in the difference scheme the function values are present on two temporary stratums t_n and t_{n+1}, with step on time $\tau_n = t_{n+1} - t_n$.

We shall mark the functions defined on upper temporary layer t_{n+1}, by $\varphi^{i-1/2,j-1/2}$ or $\overline{\varphi}$, and on the under-stratums by $\varphi_{i-1/2,j-1/2}$ or φ. The approximation of the system of equations (4) is built on the base of integral identity, equivalent to the system (4).

The solution will be defined in the three-dimensional space with coordinate x, y, t.

We shall consider the elementary volume V, formed by the coordinate planes $x = x_i, x = x_{i-1}, y = y_j, y = y_{j-1}, t = t_n, t = t_{n-1}$. Let us take integral from the first equation of the system (4) on volume V:

$$\rho \int\limits_V \frac{\partial u}{\partial t} dV = \int\limits_V \left[\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} \right] dV$$

Considering that $dV = dxdydt$ and applying integrations formulas by parts, we get

$$\rho \int\limits_{S_1} u|_{t_n}^{t_{n+1}} dxdy = \int\limits_{S_2} \sigma_{xx}|_{x_i}^{x_{i+1}} dtdy + \int\limits_{S_3} \sigma_{xy}|_{y_j}^{y_{j+1}} dxdt.$$

Here through S_1, S_2, S_3 we shall mark the areas sides of the elementary volume. Approximating the obtained integrals on the areas S_1, S_2, S_3 by square formula of the central rectangle, get the following difference equation

$$\rho \left(\overline{u} - u \right) h_x h_y = \left(\Sigma_{xxi,j-1/2} - \Sigma_{xxi-1,j-1/2} \right) h_y \tau +$$

$$+ \left(\Sigma_{xyi-1/2,j} - \Sigma_{xyi-1/2,j-1} \right) h_x \tau.$$

The functions $u, \sigma_{xx}, \sigma_{xy}$ are determined on the lower temporary layer t_n, and $\overline{u}, \overline{\sigma}_{xx}, \overline{\sigma}_{xy}$ are determined on the temporary layer t_{n+1}. In the center of the lateral sides S_1, S_2 "greater" values are defined, their expressions through "small" values are given below.

The obtained equation will be divided by $h_x \cdot h_y \cdot \tau$

$$\rho \frac{\overline{u} - u}{\tau} = \frac{\Sigma_{xxi,j-1/2} - \Sigma_{xxi-1,j-1/2}}{h_x} + \frac{\Sigma_{xyi-1/2,j} - \Sigma_{xyi-1/2,j-1}}{h_y}.$$

The rest of the equations are approximated similarly, and that gives

$$\rho \frac{\overline{v} - v}{\tau} = \frac{\Sigma_{yxi,j-1/2} - \Sigma_{yxi-1,j-1/2}}{h_x} + \frac{\Sigma_{yyi-1/2,j} - \Sigma_{yyi-1/2,j-1}}{h_y},$$

$$\frac{\overline{\sigma}_{xx}-\sigma_{xx}}{\tau}=(\lambda+2\mu)\,\frac{U_{i,j-1/2}-U_{i-1,j-1/2}}{h_x}+\lambda\frac{V_{i-1/2,j}-V_{i-1/2,j-1}}{h_y},$$

$$\frac{\overline{\sigma}_{yy}-\sigma_{yy}}{\tau}=\lambda\frac{U_{i,j-1/2}-U_{i-1,j-1/2}}{h_x}+(\lambda+2\mu)\,\frac{V_{i-1/2,j}-V_{i-1/2,j-1}}{h_y},$$

$$\frac{\overline{\sigma}_{xy}-\sigma_{xy}}{\tau}+\frac{\overline{\sigma}_{yx}-\sigma_{yx}}{\tau}=2\mu\left(\frac{U_{i-1/2,j}-U_{i-1/2,j-1}}{h_y}+\frac{V_{i,j-1/2}-V_{i-1,j-1/2}}{h_x}\right),\quad(12)$$

$$2\rho l^2\frac{\overline{\omega}-\omega}{\tau}=\frac{M_{zxi,j-1/2}-M_{zxi-1,j-1/2}}{h_x}+\frac{M_{zyi-1/2,j}-M_{zyi-1/2,j-1}}{h_y}+\sigma_{xy}-\sigma_{yx},$$

$$\frac{\overline{M}_{zx}-M_{zx}}{\tau}=2\mu l^2\frac{\Omega_{i,j-1/2}-\Omega_{i-1,j-1/2}}{h_x},$$
$$\frac{\overline{M}_{zy}-M_{zy}}{\tau}=2\mu l^2\frac{\Omega_{i-1/2,j}-\Omega_{i-1/2,j-1}}{h_y}.$$

The built system of equations in finite differences (12) approximates the system of the differential equations (4). For finishing of the equations in finite differences building is required to indicate the way of the "greater" values calculation through "small". "Greater" values are defined in the center of the lateral sides of the volume V. Their expression through "small" values φ in internal nodes of the net are found from characteristic correlations, considered above.

"Greater" values Φ in the x direction are calculated as follows . We shall consider two nearby cells on temporary layer t_n, the centers of which are marked by $i-1/2$, $j-1/2$ and $i+1/2$, $j-1/2$. We shall conduct from these point characteristics with slopping $\pm c_1$, but from point of their intersection we shall lower characteristics with slopping $\pm c_2$. Since along these straight linear combinations of the unknown function maintain constant values, it is possible to write following correlations

$$\rho U_{i,j-1/2}+\Sigma_{xxi,j-1/2}/c_1=\rho u_{i+1/2,j-1/2}+\sigma_{xxi+1/2,j-1/2}/c_1,$$
$$\rho U_{i,j-1/2}-\Sigma_{xxi,j-1/2}/c_1=\rho u_{i-1/2,j-1/2}-\sigma_{xxi-1/2,j-1/2}/c_1.\quad(13)$$

Solving system (13) relative to "greater" values, we get

$$U_{i,j-1/2}=(u_{i+1/2,j-1/2}+u_{i-1/2,j-1/2})/2+$$
$$+(\sigma_{xxi+1/2,j-1/2}-\sigma_{xxi-1/2,j-1/2})/2/\rho/c_1,$$

$$\Sigma_{xxi,j-1/2}=\rho c_1(u_{i+1/2,j-1/2}-u_{i-1/2,j-1/2})/2+$$

$$+(\sigma_{xxi+1/2,j-1/2}+\sigma_{xxi-1/2,j-1/2})/2.$$

Doing similar discourses and transformations, we shall define the rest "greater" values, in direction y similarly as in direction x.

"Greater" values, on the border of the area are defined from three relations for the characteristics and three border conditions.

The initial conditions for the examined scheme are written by change continuous function by "small" values, determined on the net.

By decompositions of the discrete function, entering into difference scheme, in Taylor row in the neighborhood of the point (x_i, y_j, t_n) it is possible to show that built accounting scheme has a first approximation order. Stability condition can be received in the base of the Churant-Fridrix-Levi criterion, which confirms that velocity in differences of the wave spreading on each directions, must not be less, than velocity of the wave for differential approach.

Then we get

$$\tau \leq \left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)^{-1},$$

where $\tau_1 \leq \frac{h_x}{c_1}$, $\tau_1 \leq \frac{h_y}{c_1}$.

When execution of this condition is ensured, the solution of the built difference problem will be converging with the first order accuracy to exact solution. In order to study possibilities of the using at calculation of the second order accuracy schemes we shall consider certain modification of the scheme built above. Increasing of the scheme accuracy can be reached by account of centering differences function on temporary interval and taking into account of the change tangent to the verge of the sought function [11].

In obtained above grid-characteristic scheme, auxiliary "greater" values on the cell border were calculated at condition of constancy values of the vector solution components within each elementary cell. In this case, for determination of the values in cross point of the characteristics with lower time plane, in essence, is used zero order interpolation that provides the first order accuracy and monotonicity of the difference schemes. Increasing approximation till the second oder can be reached by considering more exact interpolation formulas.

For calculation of intermediate values of the sought function on previous temporary layer $n\tau$ we use modified formula for square interpolations.

$$u_{\pm}=u_{k\pm1/2}+g^{*}\left(u_{k\mp1/2}-u_{k\pm1/2}\right)+$$

$$+Ag^{*}\left(1-g^{*}\right)\left(u_{k\mp1/2}-2u_{k\pm1/2}+u_{k\pm3/2}\right)/2.$$

For the bounded nodes formula with unilateral finite differences is truthful

$$u_{\pm}=u_{k\pm1/2}+g^{*}\left(u_{k\mp1/2}-u_{k\pm3/2}\right)+$$

$$+Ag^{*}\left(1-g^{*}\right)\left(u_{k\pm1/2}-2u_{k\pm3/2}+u_{k\pm5/2}\right)/2.$$

Here signs "+", "–" correspond to positive and negative slopping of the characteristics $g^* = (1 - \tau c_1/h)/2$; A is the parameter, which value is defined from condition $\|u_n - [u]_h\| \to \min$, where u_n is obtained discrete solution; $[u]_h$ is projection of the sample problem exact solution on the grid; $\|\cdot\|$ is the norm in the space of grid function; h is the spatial step.

It is known that schemes of the raised accuracy order to do not possess monotonicity feature, but choice of the best value of the parameter A allows to minimize the dispersion of the finite differences solutions comparatively exact and vastly reduce drid viscosity, which show the first-order accuracy scheme.

So, the second order accuracy difference scheme differs from first-order accuracy scheme by way of expressing of the "greater" values through "small".

For nodes of the grid, located in internal area, we write following caracteristics correlations in x direction

$$\rho U_{i,j-1/2} + \Sigma_{xxi,j-1/2}/c_1 = s_{k+1/2} + g^* \left(s_{k-1/2} - s_{k+1/2}\right) +$$
$$+ A g^* \left(1 - g^*\right) \left(s_{k-1/2} - 2 s_{k+1/2} + s_{k+3/2}\right)/2,$$

$$\rho U_{i,j-1/2} - \Sigma_{xxi,j-1/2}/c_1 = s_{k-1/2} + g^* \left(s_{k+1/2} - s_{k-1/2}\right) +$$
$$+ A g^* \left(1 - g^*\right) \left(s_{k+1/2} - 2 s_{k-1/2} + s_{k-3/2}\right)/2,$$

were $s_\pm = \rho u_{i\pm1/2,j-1/2} + \sigma_{xxi\pm1/2,j-1/2}/c_1$.

We shall mark the right parts of the last correlations through $S_{i+1/2}$ and $S_{i-1/2}$. Then "greater" values are expressed through "small" by formula

$$U_{i,j-1/2} = 0.5 \left(S_{i+1/2} + S_{i+1/2}\right)/\rho,$$
$$\Sigma_{xxi,j-1/2} = 0.5 c_1 \left(S_{i+1/2} - S_{i+1/2}\right).$$

Similarly all rest "greater" values are got.

Study of the stability by Furie method gives condition

$$\tau \le (a_+/b_-)\tau_+\tau_- / \sqrt{\tau_-^2 + (a_+/b_-)^2 \tau_+^2},$$

were $\tau_+ = \max(\tau_x, \tau_y)$, $\tau_- = \min(\tau_x, \tau_y)$, $a_+ = \max(\lambda_i, \mu_i)$, $b_- = \min(\lambda_i, \mu_i)$, $(i = 1, 2)$, $\tau_x = h_x/a_+$, $\tau_y = h_y/a_+$ are the limiting steps, defined from condition of stability corresponding to one dimentional scheme.

For square greed $\tau_+ = \tau_- = \tau_0 = h/a_+$ and $\tau \le (a_+/b_-)\tau_0[1 + (a_+/b_-)^2]^{-1/2}$.

A more severe condition have in the case of the first-order accuracy scheme

$$\tau \le (a_+/b_-)\tau_+\tau_-[\tau_- + (a_+/b_-)\tau_+],$$

from which for square grid follows $\tau \le (a_+/b_-)\tau_0/[1 + (a_+/b_-)]$.

In order to obtain more high accuracy order results brings to expediency of the hybrid difference schemes using with flows correction, taking as support, described above scheme variants.

References

[1] KUVSHINSCHY E.V., AIARO E.L. *Continuum theory of assymetric tension.* SBPH, 1963, **5**, N 9 (in Russian).

[2] PALIMOV V.A. *The basic equations of assymetric tension.* AMM, 1964, **28**, N 3 (in Russian).

[3] COSSERAT E. *Theorie des corps deformables.* Paris, 1909.

[4] TOUPIN R. A. *Theories of elasticity with couple-stress.* Arch. R. Mech., 1914, N 17(85).

[5] AIARO E.L., KUVSHINSCHY E. V, *The basic equations of the elasticity theory for the medium with rotated interaction of particles.* SBF, 1960, **2**, N 7 (in Russian).

[6] SAVIN G.I. *Basic flate problems of the moment tensions theory.* Kiev, Ed. KSU, 1965 (in Russian).

[7] KOITER W. *Translation in russian in the coll.* Mechanics, 1965, N 3.

[8] MINDLIN D.D., TIRSTEN G. F. *Moment tentions effects in the linear theory of tensions. Collections of translations.* Mechanics, 1964, N 4.

[9] GODUNOV S.K., ZABRODIN A.V. AND OTHER. *Numerical solution of the multidimention problems of the gas dynamics.* Moskva, Ed. Science, 1976 (in Russian).

[10] FRANC F., MIZES R. *Integral and differential equations of the mathematical physics*, GTTI, 1936, ch. 12.

[11] M.N. ABUZAROV, V.G. BAGENOV, A.V. COCHETCOV. *About new effective approach to the precise effectivness of the Godunov scheem. Applied problems of the regidity and plasticity.* Union Interuniversity collection, 1987, p. 44–50 (in Russian).

Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
str. Academiei 5
Chişinău, MD−2028
Moldova
E-mail: *inaval@math.md*

Received July 18, 2005

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(51), 2006, Pages 45–52
ISSN 1024–7696

On the coproducts of cyclics in commutative modular and semisimple group rings

Peter Danchev

Abstract. We study certain properties of the coproducts (= direct sums) of cyclic groups in commutative modular and semisimple group rings. Our results strengthen a statement due to T. Zh. Mollov (Pliska, Stud. Math. Bulgar., 1981) and also they may be interpreted as a natural continuation of a recent investigation of ours (Serdica Math. J., 2003).

Mathematics subject classification: 16U60, 16S34, 20K10, 20K20, 20K21.
Keywords and phrases: Group rings, coproducts of cyclic groups, $p^{\omega+n}$-projective groups, criteria.

1 Introduction

Let G be an arbitrary multiplicative abelian group and let R be an arbitrary commutative unitary ring of any characteristic. By using standard abbreviations, we now formally introduce the basic concepts. For such a group G, G_p denotes its p-primary component of torsion which can be represented as $G_p = \cup_{n<\omega} G[p^n]$ where $G[p^n] = \{g \in G : g^{p^n} = 1\}$ is the p^n-socle of G, and G^1 denotes the first Ulm subgroup of G. For such a ring R, $char(R)$ denotes its characteristic which is a non-negative integer m with the property that either $m.1_R = 0$ and $m \neq 0$ is minimal with this property, whence we write $char(R) = m \neq 0$, or otherwise $char(R) = 0$ whenever $m \neq n$ implies $m.1_R \neq n.1_R$.

We recall that the field F is of the first kind with respect to the prime p if the degree $(F(\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots) : F) = \infty$, where ϵ_i are the primitive p^i-roots of unity for $i = 1, 2, \cdots, n, \cdots$. In the remaining variant when $(F(\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots) : F) < \infty$, the field F is called a field of the second kind with respect to p.

An example of a field of the first kind with respect to any prime number is the field $\mathbb{Q}$ of all rationals, while the fields $\mathbb{R}$ and $\mathbb{C}$ of all real and complex numbers, respectively, are examples of fields of the second kind with respect to all primes.

Our attention in the present exploration is concentrated on the Sylow p-group $S(RG)$ consisting of all normalized p-elements in the group ring RG. For $A \leq G$, we denote by $I(RG; A)$ the relative augmentation ideal of RG with respect to the subgroup A.

All other unexplained exclusively notions and notation are standard and are in agreement with the cited in the bibliography research papers.

The work is structured as follows: In the second paragraph, divided into two sections, we study the behavior of coproducts of cyclic groups in a modular and a semisimple aspect.

The first section treats the modular case by finding a criterion for $S(RG)/S(RA)$ to be a coproduct of cyclic groups provided that $char(R) = p$, p is a prime, under some additional restrictions on G and A; for instance such as G is torsion and $\coprod_{q \neq p} G_q \subseteq A$. As a consequence, we obtain a necessary and sufficient condition for $S(RG)$ to be a $C_{\omega+1}$-group in the sense of Megibben (= a *pillared* group in other terms due to P. D. Hill). These criteria expand the corresponding ones of T. Mollov in [12]. We also point out that a counterexample due to Mollov in his reviewer's report [16], concerning our assertion in [4], is an absurdity.

The second section deals with the semisimple case, where we establish criteria for $S(RG)$ to belong to major classes of abelian groups provided R is either a field of the first kind or of the second kind with respect to p, both with $char(R) \neq p$. To this aim, we deduce a new extension of a group-theoretic attainment of Dieudonné [11].

We also eliminate some minor misprints in [4].

We end the paper with concluding discussion and remarks where we list certain unanswered questions and conjectures.

2 Main Results

1. *Modular group rings*

The next affirmation was proved by Mollov [12, Proposition 6 b)] in a slightly modified form for the case when R is a field. In his proof such a limitation on R to be a field is essential. We now generalize this claim to an arbitrary commutative ring R with 1 and prime characteristic p by the usage of a different and more smooth approach.

We are now in a position to restate in an equivalent form the cited above Mollov's result.

Theorem 1. *Suppose G is an abelian group with a subgroup A, suppose M is a p-divisible group with $M_p = 1$ and suppose $char(R) = p$ is a prime. Then $S(R(G_p \times M))/S(R(A_p \times M))$ is a coproduct of cyclic groups if and only if G_p/A_p is a coproduct of cyclic groups.*

Proof. Evidently, $G_p/A_p \cong G_pS(R(A_p \times M))/S(R(A_p \times M)) \subseteq S(R(G_p \times M))/S(R(A_p \times M))$ since $G_p \cap S(R(A_p \times M)) = A_p$. Thus the necessity obviously holds.

In order to prove the sufficiency, we assume that G_p/A_p is a coproduct of cyclic groups. Appealing to the classical Kulikov's criterion, $G_p/A_p = \cup_{n<\omega}(G_n/A_p)$, $A_p \subseteq G_n \subseteq G_{n+1} \leq G_p$ and $G_n \cap G_p^{p^n} \subseteq A_p$ that is equivalent to $G_n \cap G^{p^n} \subseteq A_p$. Therefore, $G_p = \cup_{n<\omega}G_n$ and $G_p \times M = \cup_{n<\omega}(G_n \times M)$, whence $S(R(G_p \times M)) = \cup_{n<\omega}S(R(G_n \times M))$ and $S(R(G_p \times M))/S(R(A_p \times M)) = \cup_{n<\omega}[S(R(G_n \times M))/S(R(A_p \times M))]$. Finally, by exploiting the modular law, we compute that

$S(R(G_n\times M))\cap S^{p^n}(R(G_p\times M)) = S(R(G_n\times M))\cap S(R^{p^n}(G_p^{p^n}\times M)) = S(R^{p^n}[(G_n\times M)\cap(G_p^{p^n}\times M)]) \subseteq S(R(M\times A_p))$, where the last inclusion holds because of the relationships $M\subseteq G^{p^n}$ and $(G_n\times M)\cap(G_p^{p^n}\times M) = M((G_n\times M)\cap G_p^{p^n}) = M((G_n\times M)\cap G^{p^n}) = M(G_n\cap G^{p^n})\subseteq M\times A_p$ which are fulfilled over every natural $n\in\mathbb{N}$. This substantiates the claim that the quotient $S(R(G_p\times M))/S(R(A_p\times M))$ is a coproduct of cyclic groups. Hence the theorem.

Corollary 1. *Suppose G is an abelian group such that $G = G_p\times M$, where M is p-divisible, suppose $A\leq G$ such that $M\subseteq A$ and suppose char$(R) = p$ is a prime. Then $S(RG)/S(RA)$ is a coproduct of cyclic groups $\Longleftrightarrow$ G/A is a coproduct of cyclic groups $\Longleftrightarrow$ G_p/A_p is a coproduct of cyclic groups.*

Proof. It easily follows that $A = A_p\times M$. Therefore Theorem 1 applies to show that $S(RG)/S(RA)$ is a coproduct of cycles only when so is $G/A\cong G_p/A_p$.

Corollary 2. *Suppose that G is a torsion abelian group with $\coprod_{q\neq p}G_q\subseteq A\leq G$, and suppose that char$(R) = p$ is a rational prime. Then $S(RG)/S(RA)$ is a coproduct of cyclic groups $\Longleftrightarrow$ $G/A\cong G_p/A_p$ is a coproduct of cyclic groups.*

Proof. It follows immediately from Theorem 1 because $G = G_p\times\coprod_{q\neq p}G_q$, where the latter direct component is p-divisible.

We come now to the original formulation of the Mollov's statement from [12].

Corollary 3 (Mollov, 1981). *Assume that G is an abelian group, p a prime, $G = G_p\times M, A\leq G_p$, and that R is a field of char$(R) = p$. If M is a p-divisible group, then $S(RG)/S(R(A\times M))$ is a coproduct of cyclic groups $\Longleftrightarrow$ G_p/A is a coproduct of cyclic groups.*

Proof. The assertion follows at the substitutions $G = G_p\times M$ and $A = A_p$.

Remark. Proposition 6 a) in [12] was firstly extended in [2] to arbitrary abelian groups and commutative rings with identity of prime characteristic p without nilpotent elements. When G is a p-group, similar expansions of Propositon 6 b) were given by us in [3]. The real advantage here is that we have considered a more general coefficient ring.

For any ordinal number λ an abelian p-group H is termed by C. Megibben a C_λ-group if H/H^{p^α} is totally projective for all $\alpha<\lambda$. Apparently, every abelian p-group is a C_ω-group.

Our next statement concerns the finding of a criterion when $S(RG)$ is a $C_{\omega+1}$-group, that is, the first Ulm factor $S(RG)/S^1(RG)$ is a coproduct of cyclic groups (these groups are called also *pillared* by P.D. Hill).

Proposition 1. *Let G be a torsion abelian group and R a perfect commutative ring with 1 of prime char$(R) = p$. Then $S(RG)$ is a $C_{\omega+1}$-group $\Longleftrightarrow$ G_p is a $C_{\omega+1}$-group.*

Proof. Follows directly from Theorem 1 by putting $A_p = G_p^1$.

We close the modular case with some special critical commentaries (for a convenience, the symbols are as in [4]): The purported in [16] counterexample of Mollov on the Proof 2 of Proposition 1 from [4] is demonstrably false. In fact, Mollov constructed an abelian p-group A so that $A = H \times L$ where H is an infinite coproduct of cycles and L is finite cyclic of order p. Henceforth, $A^p = H^p \subseteq H$. Besides, the Mollov's choice $H_k = H$ is tendentious since in our proof in [4] $H = \cup_{k=1}^{\infty} H_k, H_k \subseteq H_{k+1}$ and $H_k \cap H^{p^{s_k}} = 1$, for each $k < \omega$ and some $s_k \in \mathbb{N}$, where H_k are **proper** subgroups of H (see, for instance, the well-known criterion of Kulikov for coproducts of torsion cyclic groups).

2. Semi-simple group rings

We start here with the specification that in [4] (e.g. the Abstract of [4]) the letter K denotes the first kind field with respect to p of $char(K) \neq p$ with the extra restriction that its spectrum $s_p(K)$ about p contains all naturals that is valid only in the situations when we consider the purity or the direct factor properties of G in $S(KG)$ (i.e. in Lemma on purity on p.38, Theorem 7, Corollary 8, Claim 13, and the comments after Problem 17).

Moreover, as it was truly noted in [16], the criteria (4) and (5), respectively (4') and (5'), of [4] are true only in the infinite case. The word "infinite" was omitted involuntarily. Also in (5) and (5') the condition that the abelian p-group A is a "direct sum of cyclics" follows at once by the relation $A^{p^i} = 1$, which means that A is bounded, so it is out of use and can be dropped.

As usual, (p^t) designates a cyclic group of order p^t.

Now, we shall prepare the finite case.

Claim 1. *Let G be an abelian p-group and let R be a field of the first kind with respect to p of $char(R) \neq p$. Then $S(RG)$ cannot be non-trivial finite elementary or finite reduced homogeneous no elementary.*

Proof. If G is a finite abelian p-group with exponent $exp(G) = p^j, j \geq 1$ and R is a field of the first kind with respect to p of $char(R) \neq p$, consulting with ([13], [15]), we write $S(RG) \cong \coprod_{\delta_{i_0}-1}(p^{i_0}) \times \coprod_{\delta_{i_1}}(p^{i_1}) \times \cdots \times \coprod_{\delta_{i_r}}(p^{i_r})$, where $i_0, i_1, \cdots, i_r \in s_p(R) = \{i_0, i_1, i_2, \cdots | i_0 < i_1 < i_2 < \cdots\}$, i_r is the minimal number so that $i_r \geq j$, $i' \in s_p(R)$ plus the condition that $i' < i$ is the minimal number with this property if it generally exists and where the numbers δ_i are described in the following manner: $\delta_i = (|G[p^i]| - |G[p^{i'}]|)/(R(\varepsilon_i) : R), i \neq i_0$; $\delta_i = |G[p^{i_0}]|, i = i_0$. Moreover, $\delta_i = 0$ whenever $i > i_r$ and $i \in s_p(R)$, since for any $t \in \mathbb{N} : G[p^t] = G[p^{t+1}] \iff G^{p^t} = 1 \iff G = G[p^t]$ as well as $\delta_{i_0} = 1 \iff G[p^{i_0}] = 1 \iff G = 1$ because $i_0 \geq 1$. This shows that our claim really holds true.

Claim 2. *Let G be a finite abelian p-group and R a field of the second kind with respect to p of $char(R) \neq p$. Then $S(RG)$ is elementary $\iff p = 2, R \neq R(\varepsilon_2)$ and $G^p = 1$.*

Proof. Referring to [13,14] it follows that $S(RG) \cong \coprod_{|G|-1}(p)$, where $p = 2, R \neq R(\varepsilon_2)$ and $|G| = |G[p]|$. Since G is finite and $G[p] \subseteq G$, we have $G = G[p]$, so $G^p = 1$.

Remark. In the criteria (5) and (5') from [4] we observe that $i = exp(G) \notin s_p(K)$, otherwise $S(KA)$ is not, never, reduced homogeneous no elementary. Hence we conclude that $s_p(K)$ does not contain all naturals in that situation.

We strongly emphasize that in assertions (7) and (7') of [4] there is a typos. Below we give the correct formulating.

In fact, "$S(KA)$, respectively $U_p(KA)$, is p^α-projective for some $\alpha \geq \omega \iff A$ is a direct sum of cyclics" should be written and read as "$S(KA)$, respectively $U_p(KA)$, is p^α-projective for some $\alpha \leq \omega \iff A$ is (a bounded or an unbounded) direct sum of cyclics".

For the case when $\alpha \geq \omega + 1$, the problem seems to be very difficult. In the next lines, we obtain some partial advantage in this way. In order to do this, we foremost note the classical fact that in [11] it was firstly constructed by Dieudonné the existence of a separable $p^{\omega+1}$-projective abelian p-group which is not a coproduct of cycles.

We formulate the following conjecture, namely:

Conjecture 1. Assume that G is an abelian p-group and that R is the first kind field with respect to p of $char(R) \neq p$. Then $S(RG)$ is p^α-projective for $\alpha \geq \omega + 1$ $\iff S(RG)$ is separable p^α-projective $\iff G$ is separable p^α-projective.

The first implication follows easily, because by [14] we have that $S^1(RG) = 1$ since $S(RG)$ reduced. The second one is difficult. This difficulty ensues via the following reason, thus contrasting with the modular case (see [2]). Utilizing the Nunke's criterion for $p^{\omega+n}$-projectivity where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ (see, for instance, [17]), if G is $p^{\omega+n}$-projective then there is $P \leq G[p^n]$ so that G/P is a coproduct of cyclic groups. On the other hand, $S(RG)/[(1 + I(RG; P)) \cap S(RG)] \cong S(R(G/P))$ is a coproduct of cyclic groups according to [14], whereas $(1+I(RG; P)) \cap S(RG)$ is not always bounded at p^n (i.e. equivalently is not always a subgroup of $S(RG)[p^n]$).

Now, we shall verify the second implication of the foregoing stated conjecture under the truthfulness of another conjecture (see [4, Problem 17]). Indeed, we assume that the *Generalized Direct Factor Conjecture* ([4, Problem 17]), which says that $S(RG)/G$ is a coproduct of cyclic groups whenever $G^1 = 1$ under the same circumstances on R and G as in Conjecture 1, holds in the affirmative and also additionally assume that the spectrum $s_p(R)$ of R about p, defined as in ([13-15]), contains all natural numbers, that is $\mathbb{N} \subseteq s_p(R)$. Invoking [7] or the Lemma from [4], G is pure in $S(RG)$. When there are gaps in the $s_p(R)$, we know that G is not always pure in $S(RG)$.

Furthermore, we distinguish two different approaches.

First Approach. Bearing in mind a classical theorem of L.Ya. Kulikov, G must be a direct factor of $S(RG)$, thus writing $S(RG) \cong G \times S(RG)/G$. Thereby $S(RG)$ is completely characterized by making use of [8].

Consequently, $S(RG)$ is separable p^α-projective for any ordinal $\alpha \iff G$ is separable p^α-projective for that α.

Second Approach. We shall apply successfully the following new version of the Generalized Dieudonné Criterion (for a strengthening in other ways see also [9] and [10]).

Criterion. *Suppose $A \leq G$ so that G/A is a coproduct of cyclic groups. Then, for some $m \geq 0$, G is $p^{\omega+m}$-projective* $\iff \exists\, C \leq A[p^m] : A = \cup_{n<\omega} A_n, C \subseteq A_n \subseteq A_{n+1}$

and for every $n \in \mathbb{N}$*:* $A_n \cap G^{p^n} \subseteq C \iff \exists\ C \leq A[p^m] : A[p] = \cup_{n<\omega} B_n, B_n \subseteq B_{n+1}$ *and for each* $n \in \mathbb{N}$*:* $B_n \cap G^{p^n} \subseteq C$.

Proof. "$\Rightarrow$". With the aid of the Nunke's criterion [17], there is a p-group T such that $T \leq G[p^m], G = \cup_{n<\omega} G_n, T \subseteq G_n \subseteq G_{n+1}$ and $G_n \cap G^{p^n} \subseteq T$. Therefore $A = \cup_{n<\omega} A_n$, where $A_n = G_n \cap A$, and we subsequently compute $A_n \cap G^{p^n} = G_n \cap G^{p^n} \cap A \subseteq A \cap T$. So, by the setting $A \cap T = C$, we are done. For the second equivalent form of the criterion, we observe that $A[p]C/C \subseteq (A/C)[p] = \cup_{n<\omega}(E_n/C)$ where $E_n \subseteq E_{n+1} \subseteq A$ and $E_n \cap G^{p^n} \subseteq C$, hence $A[p] = \cup_{n<\omega} E_n[p]$ where $B_n = E_n[p]$.

"$\Leftarrow$". Under the assumptions, there is $C \leq A[p^m] \leq G[p^m]$ so that A/C is a countable union of an ascending chain of subgroups with bounded in G/C heights. After this, because of the isomorphism $G/C/A/C \cong G/A$, the latter factor-group is a coproduct of cycles. Consequently, with the Dieudonné criterion [11] in hand (see [9] and [10] as well), we derive that G/C must be a coproduct of cyclic groups. Finally, Nunke's criterion in [17] is applicable to obtain the claim.

This terminates the proof of the criterion.

And so, since by hypotheses $S(RG)/G$ is a coproduct of cycles whenever $G^1 = 1$ and G is pure in $S(RG)$, the preceding criterion applies to show that $S(RG)$ is separable $p^{\omega+m}$-projective if and only if so does G; $m \in \mathbb{N}$.

Remark. If G/A is a coproduct of cycles and A is $p^{\omega+m}$-projective, it does not follow in general that G is $p^{\omega+m}$-projective as well.

Major consequences arise for various choices of the subgroup A, specifically when $A = G^p$; or $A = G[p]$ so $G/G[p] \cong G^p$; or $A = L$, a large subgroup of G (see [6]) - we notice that G/L is a coproduct of cyclic groups.

Finally, we remark that when R is a field of the second kind with respect to p of $char(R) \neq p$, $S(RG)$ is p^α-projective for some arbitrary ordinal $\alpha \iff S(RG)$ is p^ω-projective (= a coproduct of cycles). This is so since, by consulting with [14], $S(RG)$ is ever a coproduct of cyclic and quasi-cyclic groups (= a coproduct of cocyclic groups). But, on the other hand, it is reduced as being p^α-projective too.

We continue with two minor specifications of technical character.

Firstly, after "Lemma [9]" in [4], the sentence "... is a direct factor of $S(KA)$, hence of $V(KA)$ by [10]" should be more precise by "... is a direct factor of $S(KA)$, hence of $V(KA)$ by [10] *and [17]*". Moreover, in Proposition 9 of [4] the subgroup H should be pure in the separable p-group A, which condition on "purity" was omitted involuntarily.

Secondly, in [16], Mollov has criticized that we have not showed in Lemma 6 of [4] that $I = (1 + I(RG; C)) \cap S(RG)$ is a group whenever $C \leq G$. Of course, that this intersection I is a group follows plainly either owing to the mentioned after Conjecture 1 fact that I is the kernel of the homomorphism $S(RG) \to S(R(G/C))$ or in the following manner: Given $u_1 \in I$ and $u_2 \in I$. Hence $u_1 = 1 + v_1$ and $u_2 = 1 + v_2$, where $v_1, v_2 \in I(RG; C)$. Furthermore, $u_1u_2 = 1 + v_1 + v_2 + v_1v_2 \in I$ since it is a p-element and $I(RG; C)$ is a ring being an ideal. On the other hand, since $u_1 \in S(RG)$, there exists $k \in \mathbb{N}$ such that $u_1^{p^k} = 1$, whence $u_1^{-1} = u_1^{p^k-1} \in I$ employing inductively the previous step. So, the claim is true.

Finally, we answer two problems posed by us in [4] asked when $S(RG)$ is quasi-pure projective (q. p. p.) and quasi-pure injective (q. p. i.), provided R is the first kind field with respect to p of $char(R) \neq p$.

To this goal, we quote the following necessary and sufficient conditions argued in [1].

Criteria (Berlinghoff–Reid, 1977)

(1) The p-group G is q. p. i. $\Longleftrightarrow$ G is a coproduct of a divisible p-group and a torsion complete p-group.

(2) A non-reduced p-group G is q. p. p. $\Longleftrightarrow$ G is an algebraically compact p-group. A reduced p-group G is q. p. p. $\Longleftrightarrow$ G is a coproduct of cyclic p-groups.

And so, we proceed by proving the following.

Theorem 2. *Suppose G is an abelian p-group and R is a field of the first kind with respect to p of characteristic not equal to p. Then*

(1') $S(RG)$ is q. p. i. $\Longleftrightarrow$ G is algebraically compact, provided $\mathbb{N} \subseteq s_p(R)$.

(2') The non-reduced $S(RG)$ is q. p. p. $\Longleftrightarrow$ G is non-reduced algebraically compact. The reduced $S(RG)$ is q. p. p. $\Longleftrightarrow$ G is a coproduct of cyclic groups.

Proof. Point (1') follows directly from (1) and [5] (see [7] as well). The first part half of (2') holds in virtue of (2) combined with ([4], dependence 12) (see also [5]). The second one follows again by (2) along with [14].

The proof is completed.

3 Concluding Discussion

We conclude with certain questions and conjectures of interest. We mainly discuss here a problem related to [4] and pertaining to finding the explicit form of basic subgroups in semisimple group rings. The isomorphism classification of such basic subgroups was firstly established in ([4], Proposition 11). Nevertheless, the explicit type of these subgroups will definitely be of some importance.

We state the following:

Conjecture 2. Suppose that G is a separable abelian p-group with an arbitrary but fixed basic subgroup B and that R is the first kind field with respect to p of $char(R) \neq p$ so that $\mathbb{N} \subseteq s_p(R)$. Then $B' = [1 + I(RG; B)] \cap S(RG)$ is a basic subgroup of $S(RG)$.

We attack our claim like this: That B' is pure in $S(RG)$ and that $S(RG)/B' \cong S(R(G/B))$ is divisible follow not so hard from [4] and more especially subsequently referring to Lemma 6 and relation (8) of [4].

More complicated is the question how to derive that B' is a coproduct of cyclic groups. We show now that such a construction naturally depends on the foregoing used Generalized Direct Factor Conjecture (Problem 17 of [4]). Indeed, if we have a priori that $S(RG)/G$ is a coproduct of cyclic groups, we observe that $B'/B \cong B'G/G \subseteq S(RG)/G$ possesses this property as well. Moreover, appealing to [7], B being pure in G is pure even in $S(RG)$ whence in B'. Finally, the application of a classical theorem of L. Kulikov, used also above, ensures that $B' \cong B \times B'/B$ is a coproduct of cyclic groups, in fact, as wanted.

However, the complete proof is a theme of another research investigation, where a new simpler approach might work.

Acknowledgment: The author would like to thank the referee for the expert suggestions made.

References

[1] Berlinghoff W.P., Reid J.D. *Quasi-pure projective and injective torsion groups*. Proc. Amer. Math. Soc., 1977, **65**, N 2, p. 189–193.

[2] Danchev P.V. *Isomorphism of commutative modular group algebras*. Serdica Math. J., 1997, **23**, N 3-4, p. 211–224.

[3] Danchev P.V. *Sylow p-subgroups of modular abelian group rings*. Compt. rend. Acad. bulg. Sci., 2001, **54**, N 2, p. 5–8.

[4] Danchev P.V. *Sylow p-subgroups of abelian group rings*. Serdica Math. J., 2003, **29**, N 1, p. 33–44.

[5] Danchev P.V. *Commutative group algebras of highly torsion-complete abelian p-groups*. Comment. Math. Univ. Carolinae, 2003, **44**, N 4, p. 587–592.

[6] Danchev P.V. *Characteristic properties of large subgroups in primary abelian groups*. Proc. Indian Acad. Sci. Sect. A – Math. Sci., 2004, **114**, N 3, p. 225–233.

[7] Danchev P.V. *Torsion completeness of Sylow p-groups in semisimple group rings*. Acta Math. Sinica, 2004, **20**, N 5, p. 893–898.

[8] Danchev P.V. *Ulm-Kaplansky invariants of $S(KG)/G$*. Bull. Polish Acad. Sci. – Math., 2005, **53**, N 3, p. 147–156.

[9] Danchev P.V. *Generalized Dieudonné criterion*. Acta Math. Univ. Comenianae, 2005, **74**, N 1, p. 15–24.

[10] Danchev P.V. *Generalized Dieudonné and Honda criteria* (to appear).

[11] Diedonné J.A. *Sur les p-groupes abéliens infinis*. Portugal. Math., 1952, **11**, N 1, p. 1–5.

[12] Mollov T.Zh. *Ulm invariants of the Sylow p-subgroups of group algebras of abelian groups over a field with characteristic p*. Pliska Stud. Math. Bulgar., 1981, **2**, p. 77–82 (in Russian).

[13] Mollov T.Zh. *Invariants and unit groups of group algebras*. Dissertation – Dr. of Math. Sci., Plovdiv University "Paissii Hilendarski", Plovdiv, 1984 (in Bulgarian).

[14] Mollov T.Zh. *Sylow p-subgroups of the group of normed units of semisimple group algebras of uncountable abelian p-groups*. Pliska Stud. Math. Bulgar., 1986, **8**, p. 34–46 (in Russian).

[15] Mollov T.Zh. *Unit groups of semisimple group algebras*. Pliska Stud. Math. Bulgar., 1986, **8**, p. 54–64 (in Russian).

[16] Mollov T.Zh. Zbl. Math. 16025: (1035).

[17] Nunke R.J. *Purity and subfunctors of the identity*. Topics in Abelian Groups, Chicago, Scott. Foresman, 1963.

13 General Kutuzov Street
bl. 7, floor 2, apart. 4
4003 Plovdiv, Bulgaria
E-mail: *pvdanchev@yahoo.com*

Received July 21, 2005

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(51), 2006, Pages 53–61
ISSN 1024–7696

Commutative Moufang loops with maximum conditions for subloops

A. Babiy, N. Sandu

Abstract. It is proved that the maximum condition for subloops in a commutative Moufang loop Q is equivalent with conditions of the finite generating of different subloops of the loop Q and different subgroups of the multiplication group of the loop Q. An analogue equivalence is set for commutative Moufang ZA-loops.

Mathematics subject classification: 20N05.
Keywords and phrases: Commutative Moufang loop, multiplication group of loop, maximum condition for subloops.

It is said that the maximum condition (respect. minimum condition) for subalgebras with the property α holds in an algebra A if any ascending (respect. descending) system of subalgebras with the property α $A_1 \subseteq A_2 \subseteq \ldots$ (respect. $A_1 \supseteq A_2 \supseteq \ldots$) breaks, i.e. $A_n = A_{n+1} \ldots$ for a certain n. It is well known that the fulfillment of the maximum condition for subalgebras of an arbitrary algebra is equivalent to the fact that both the algebra and any of its subalgebras are finitely generated.

Commutative Moufang loops (CML's) with maximum condition for subloops are considered in this paper. It is proved that for a non-associative CML Q this condition is equivalent to one of the following equivalent conditions: a) if Q contains a centrally nilpotent subloop of class n, then all its subloops of this type are finitely generated; b) if Q contains a centrally solvable subloop of class s, then all its subloops of this type are finitely generated; c) all invariant subloops of Q are finitely generated; d) all non-invariant associative subloops of Q are finitely generated; e) at least one maximal associative subloop of Q is finitely generated. This list is completed with the condition of the finite generating of various subgroups of the multiplication group of Q. If Q is a ZA-loop, then the list a) – e) is completed with the condition of the finite generating of the center of Q, as well with the condition of the finite generating of other subloops of Q and various subgroups of the multiplication group of Q.

It is worth mentioning that the following statement is proved in [1, 2].

Lemma 1. *The following conditions are equivalent for an arbitrary CML Q:*
1) Q is finitely generated;
2) the maximum condition for subloops holds in Q.

In [2] the list a) – e) is completed with equivalent statements: h) the CML Q satisfies the maximum condition for invariant subloops; i) the CML Q is a subdirect product of a finite CML of exponent 3 and a finitely generated abelian group; j) the

---

CML Q possesses a finite central series, whose factors are cyclic groups of simple or infinite order.

Let us bring some notions and results on the theory of commutative Moufang loops, needed for further research.

A *commutative Moufang loop* (CML's) is characterized by the identity $x^2 \cdot yz = = xy \cdot xz$. The *multiplication group* $\mathfrak{M}(Q)$ of a CML Q is the group generated by all the *translations* $L(x)$, where $L(x)y = xy$. The subgroup $I(Q)$ of the group $\mathfrak{M}(Q)$, generated by all the *inner mappings* $L(x,y) = L(xy)^{-1}L(x)L(y)$ is called the *inner mapping group* of the CLM Q. The subloop H of a CML Q is called *normal* (*invariant*) in Q if $I(Q)H = H$.

Lemma 2 [3]. *Let Q be a commutative Moufang loop with the multiplication group $\mathfrak{M}$. Then $\mathfrak{M}/Z(\mathfrak{M})$, where $Z(\mathfrak{M})$ is the centre of the group $\mathfrak{M}$, and $\mathfrak{M}' = (\mathfrak{M}, \mathfrak{M})$ are locally finite* 3*-groups and will be finite if Q is finitely generated.*

Lemma 3. *The multiplication group $\mathfrak{M}$ of an arbitrary CML is locally nilpotent.*

Proof. Let $\overline{\mathfrak{N}}$ be the image of finitely generated subgroup of group $\mathfrak{M}$ under the homomorphism $\mathfrak{M} \to \mathfrak{M}/Z(\mathfrak{M})$. It follows from Lemma 2 that $\overline{\mathfrak{N}}$ is a finite 3-group, therefore it is nilpotent. Let us write $\overline{\mathfrak{N}}$ in the form $\mathfrak{N}Z(\mathfrak{M})/Z(\mathfrak{M})$. We have $\mathfrak{N}Z(\mathfrak{M})/Z(\mathfrak{M}) \cong \mathfrak{N}/(\mathfrak{N} \cap Z(\mathfrak{M}))$. It is obvious that $\mathfrak{N} \cap Z(\mathfrak{M}) \subseteq Z(\mathfrak{N})$. Then

$$\mathfrak{N}/Z(\mathfrak{N}) \cong (\mathfrak{N}/(\mathfrak{N} \cap Z(\mathfrak{M})))/(Z(\mathfrak{N})/(\mathfrak{N} \cap Z(\mathfrak{M}))).$$

Therefore $\mathfrak{N}/Z(\mathfrak{N})$ is nilpotent, as a homomorphic image of the nilpotent group $\mathfrak{N}/(\mathfrak{N} \cap Z(\mathfrak{M}))$. Then the group $\mathfrak{N}$ is nilpotent as well. Consequently, the group $\mathfrak{M}$ is locally nilpotent, as required.

The *center* $Z(Q)$ of a CML Q is an invariant subloop $Z(Q) = \{x \in Q | x \cdot yz = = xy \cdot z \forall y, z \in Q\}$.

Lemma 4 [3]. *The quotient loop $Q/Z(Q)$ of an arbitrary CML Q by its center $Z(Q)$ has the index three.*

Lemma 5 [3]. *A periodic CML is locally finite.*

The associator (a,b,c) of the elements a,b,c in CML Q is defined by the equality $ab \cdot c = (a \cdot bc)(a,b,c)$. We denote by Q_i (respect. $Q^{(i)}$) the subloop of the CML Q, generated by all associators of the form $(x_1, x_2, \dots, x_{2i+1})$ (respect. $(x_1, \dots, x_{3^i})^{(i)}$), where $(x_1, \dots, x_{2i-1}, x_{2i}, x_{2i+1}) = ((x_1, \dots, x_{2i-1}), x_{2i}, x_{2i+1})$ (respect. $(x_1, \dots, x_{3^i})^{(i)} = ((x_1, \dots, x_{3^{i-1}})^{(i-1)}, (x_{3^{i-1}+1}, \dots, x_{2 \cdot 3^{i-1}})^{(i-1)}, x_{2 \cdot 3^{i-1}+1}, \dots, x_{3^i})^{(i-1)})$, where $(x_1, x_2, x_3)^{(1)} = (x_1, x_2, x_3)$. The series of normal subloops $1 = Q_0 \subseteq Q_1 \subseteq \dots \subseteq Q_i \subseteq \dots$ (respect. $1 = Q^{(o)} \subseteq Q^{(1)} \subseteq \dots \dots \subseteq Q^{(i)} \subseteq \dots$) is called the *lower central series* (respect. *derived series*) of the CML Q. We will also use for associator loop the designation $Q^{(1)} = Q'$.

A CML Q is *centrally nilpotent* (respect. *centrally solvable*) *of class* n if and only if its lower central series (respect. derived series) has the form $1 \subset Q_1 \subset \dots \dots \subset Q_n = Q$ (respect. $1 \subset Q^{(1)} \subset \dots \subset Q^{(n)} = Q$) [3].

An *ascending central series* of CML Q is a linearly ordered by the inclusion system

$$1 = Q_0 \subseteq Q_1 \subseteq \ldots \subseteq Q_\alpha \subseteq \ldots \subseteq Q_\gamma = Q$$

of invariant subloops of Q, satisfying the conditions:

1) $Q_\alpha = \sum_{\beta<\alpha} Q_\beta$ for limit ordinal α;

2) $Q_{\alpha+1}/Q_\alpha \subseteq Z(Q/Q_\alpha)$.

A CML, possessing an ascending central series is called *ZA-loop*. If the ascending central series of CML is finite, then it is centrally nilpotent [3].

We will often use the following statements in our further proofs.

Lemma 6. *The following statements are equivalent for an arbitrary CML Q:*

1) Q satisfies the minimum condition for subloops;

2) Q is a direct product of a finite number of quasicyclic groups, belonging to the center CML Q, and a finite CML;

3) Q satisfies the minimum condition for invariant subloops;

4) Q satisfies the minimum condition for non-invariant associative subloops;

5) if Q contains a centrally nilpotent subloop of class n, then it satisfies the minimum condition for centrally nilpotent subloops of class n;

6) if Q contains a centrally solvable subloop of class s, then it satisfies the minimum condition for centrally solvable subloops of class s;

7) at least one maximal associative subloop of Q satisfies the minimum condition for subloops.

The equivalence of conditions 1), 2), 3) is proved in [4], the equivalence of conditions 1), 4), 5), 6) is proved in [5] and the equivalence of conditions 1), 7) is proved in [6].

Lemma 7 [4]. *The following statements are equivalent for an arbitrary non-associative CML Q with a multiplication group $\mathfrak{M}$:*

1) Q satisfies the minimum condition for subloops;

2) $\mathfrak{M}$ satisfies the minimum condition for subgroup;

3) $\mathfrak{M}$ is a product of a finite number of quasicyclic groups, lying in the center of $\mathfrak{M}$, and a finite group;

4) $\mathfrak{M}$ satisfies the minimum condition for invariant subgroup;

5) at least one maximal abelian subgroup of $\mathfrak{M}$ satisfies the minimum condition for subgroups;

6) if $\mathfrak{M}$ contains a nilpotent subgroup of class n, then $\mathfrak{M}$ satisfies the minimum condition for nilpotent subgroups of class n.

7) if $\mathfrak{M}$ contains a solvable subgroup of class s, then $\mathfrak{M}$ satisfies the minimum condition for solvable subgroups of class s.

Lemma 8 [4]. *If the center $Z(Q)$ of a commutative Moufang ZA-loop Q satisfies the minimum condition for subloops, then Q satisfies the minimum condition for subloop itself.*

Let us now consider an arbitrary non-periodic CML Q. Let $Q^3 = \{x^3 | x \in \in Q\}$. CML is di-associative [3], then it is easy to show that Q^3 is a subloop. It follows from Lemma 4 that $Q^3 \subseteq Z(Q)$, where $Z(Q)$ is the center of CML Q, therefore Q^3 is an invariant subloop of Q. Let us suppose that the subloop $Z(Q)$ is finitely generated. Then the abelian group Q^3 is also finitely generated. Therefore it decomposes into a direct product of cyclic groups $Q^3 = < r_1 > \times \ldots \times < r_k > \times \times < s_1 > \times \ldots \times < s_m > = < R > \times < S >$, where $< r_i >$ are cyclic groups of infinite order, $< s_j >$ are finite cyclic groups [7]. The group R is free abelian, therefore it is without torsion. It is shown in [3] that the associator loop Q' has the exponent three, then

$$R \cap Q' = \{1\}. \tag{1}$$

Lemma 9. *Let Q be a CML, R be its subloop, which is considered above, and let $\overline{H}$ be a subloop of CML $Q/R = \overline{Q}$. The subloop $\overline{H}$ satisfies one of the properties: 1) $\overline{H}$ is centrally nilpotent of class n; 2) $\overline{H}$ is centrally solvable of class s; 3) $\overline{H}$ is a maximal associative subloop of CML $\overline{Q}$; 4) $\overline{H}$ is the center of CML $\overline{Q}$; 5) $\overline{H}$ is a non-invariant subloop of CML Q; 6) $\overline{H}$ is an invariant subloop of CML $\overline{Q}$ if and only if the inverse image H of subloop $\overline{H}$ has the same property as the subloop $\overline{H}$, under the homomorphism $\varphi : Q \to Q/R$.*

Proof. Let us suppose that the subloop $\overline{H}$ is centrally nilpotent of class n. Let $h_1, h_2, \ldots, h_{2n+1}$ be arbitrary elements from H. Let us denote $\varphi(h_i) = \overline{h}_i, \varphi(1) = \overline{1}$. Then $\overline{h}_i = h_i R, \overline{1} = R$. We have $(\overline{h}_1, \overline{h}_2, \ldots, \overline{h}_{2n+1}) = \overline{1}, (h_1 R, h_2 R, \ldots, h_{2n+1} R) = = R$. But $R \subseteq Z(Q)$. Therefore, if $u \in R$, then $(au, b, c) = (a, b, c)$ for any elements $a, b, c \in Q$. Then $(h_1, h_2, \ldots, h_{2n+1}) = r$, where $r \in R$. It follows from (1) that $r = 1$. We have obtained that $(h_1, h_2, \ldots, h_{2n+1}) = 1$, i.e. the subloop H is centrally nilpotent of class n.

Conversely, let us suppose that the subloop H is centrally nilpotent of class n. Then there exist such elements $h_1, h_2, \ldots, h_{2n-1}$ from H that $(h_1, h_2, \ldots, h_{2n-1}) \neq 1$. It follows from (1) that $(h_1, h_2, \ldots, h_{2n-1}) \notin R$. Therefore $(\overline{h}_1, \overline{h}_2, \ldots, \overline{h}_{2n-1}) \notin \overline{1}$. Consequently $\overline{H}$, as homomorphic image of subloop H, will be a centrally nilpotent subloop of class n. It proves the statement 1). The statement 2) is proved by analogy.

Let us now suppose that $\overline{H}$ is a maximal associative subloop of CML $\overline{Q}$ and the inverse image H is not a maximal associative subloop of CML Q. Then there exists such an element $a \notin H$ that $(a, h_1, h_2) = 1$ for all $h_1, h_2 \in H$. Obviously $R \subseteq H$. Then $\varphi a = \overline{a} \notin \overline{H}$ and $(\overline{a}, \overline{h}_1, \overline{h}_2) = \overline{1}$ for all $\overline{h}_1, \overline{h}_2 \in \overline{H}$. We have obtained that the non-associative subloop $< \overline{a}, \overline{H} >$, generated by the set $\{\overline{a}, \overline{H}\}$, strictly contains $\overline{H}$, i.e. $\overline{H}$ is not a maximal associative subloop of CML $\overline{Q}$. Contradiction. Consequently, H is a maximal associative subloop of CML Q.

Conversely, let us suppose that H is a maximal associative subloop of CML Q and $\overline{H}$ is not a maximal associative subloop of CML $\overline{Q}$. Then there exists such an element $\overline{a} \notin \overline{H}$ that $(\overline{a}, \overline{h}_1, \overline{h}_2) = \overline{1}$ for all $\overline{h}_1, \overline{h}_2 \in \overline{H}$. We have obtained that $(aR, h_1R, h_2R) = R$ for all $h_1, h_2 \in H$. As $R \subseteq Z(Q)$, then $(a, h_1, h_2) = r$, where $r \in R$. It follows from (1) that $r = 1$, therefore $(a, h_1, h_2) = 1$ for all $h_1, h_2 \in H$ and $a \notin H$. It means that the subloop H is strictly contained in the associative subloop $< a, H >$. We have obtained a contradiction with the fact that the subloop H is a maximal associative subloop. This proves statement 3). Statement 4) is proved by analogy.

Statements 5), 6) follow from the fact that the natural homomorphism $Q \to Q/R$ sets a one-to-one mapping between all non-invariant (respect. invariant) subloops of CML Q, with contained R, and all non-invariant (respect. invariant) subloops of CML Q/R. This completes the proof of Lemma 9.

Theorem 1. *The following statements are equivalent for an arbitrary non-associative CML Q:*

1) Q satisfies the maximum condition for subloops;

2) if Q contains a centrally nilpotent subloop of class n, then all its subloops of this type are finitely generated;

3) if Q contains a centrally solvable subloop of class s, then all its subloops of this type are finitely generated;

4) at least one maximal associative subloop of Q is finitely generated;

5) non-invariant associative subloops of Q are finitely generated;

6) invariant subloops of Q are finitely generated.

Proof. Let us suppose that the CML Q is non-periodic. It follows from Lemma 4 that the subloop Q^3 belongs to the center of CML Q. If H is a centrally nilpotent subloop of class n either a centrally solvable subloop of class s, or a maximal associative subloop, or a non-invariant associative subloop, or an invariant subloop, then the subloop $< H, Q^3 >$ will be of this type too. Therefore it follows from the justice of one of the statements 2) – 6) of the theorem that the abelian group Q^3 is finitely generated. Then it decomposes into a direct product $Q^3 = R \times S$, where R is an abelian group without torsion, S is a finite abelian group [7]. It is obvious that CML Q/R is periodic. Then by Lemma 5 it is locally finite.

If the CML Q satisfies one of the conditions 2) – 6) of theorem, then by Lemma 9 CML Q/R satisfies this condition as well. Then all centrally nilpotent subloops of class n either all centrally solvable subloops of class s, or at least one maximal associative subloop, or all non-invariant associative subloops, or all invariant subloops are respectively finite in CML Q/R. Therefore by Lemma 6 the CML Q/R satisfies the minimum condition for subloops in any case. The center of CML Q/R is finite. Then by 2) of Lemma 6 the CML Q/R is finite. Therefore the CML Q is finitely generated and by Lemma 1, the condition 1) holds in it. It proves the implications $2) \to 1), 3) \to 1), 4) \to 1), 5) \to 1), 6) \to 1)$. The case when the

CML Q is periodic is contained in the proof of previous case. As the implications $1) \rightarrow 2), 1) \rightarrow 3), 1) \rightarrow 4), 1) \rightarrow 5), 1) \rightarrow 6)$ are obvious, the theorem is proved.

Theorem 2. *The following statements are equivalent for an arbitrary non-associative CML Q with the multiplication group $\mathfrak{M}$:*

1) Q satisfies the maximum condition for subloops;

2) $\mathfrak{M}$ is finitely generated;

3) $\mathfrak{M}$ satisfies the maximum condition for subgroups;

4) all invariant subgroups of $\mathfrak{M}$ are finitely generated;

5) at least one maximal abelian subgroup of $\mathfrak{M}$ is finitely generated;

6) if $\mathfrak{M}$ contains a nilpotent subgroup of class n, then all its subgroups of this type are finitely generated;

7) if $\mathfrak{M}$ contains a solvable subgroup of class s, then all its subgroups of this type are finitely generated.

Proof. If the CML Q satisfies the condition 1), then it is finitely generated, and by [3] the associator loop Q' is finite. By Lemma 2 the inner mapping group $I(Q)$ of Q is also finite. It is show in [8] that the relation

$$\mathfrak{M}(G/G') \cong \mathfrak{M}(G)/ < I(G), \mathbf{M}(G') >, \tag{2}$$

holds in an arbitrary CML G, where $\mathbf{M}(G')$ denotes a subgroup of the group $\mathfrak{M}(G)$, generated by the set $\{L(a)|a \in G'\}$. It is obvious that the group $< I(Q), \mathbf{M}(Q') >$ is finitely generated in our case. As the abelian group $\mathfrak{M}(Q/Q')$ is finitely generated, then it follows from (2) that the group $\mathfrak{M}$ is finitely generated as well. Consequently, $1) \rightarrow 2)$.

If the group $\mathfrak{M}$ is finitely generated, then by Lemma 3 it is nilpotent. It is known (for instance, see [7]) that the maximum condition for subgroups holds in such groups.

Let $Z(Q)$ be the center of an arbitrary CML Q, $\{Z(\mathfrak{M})\}$ be the upper central series of its multiplication group $\mathfrak{M}(Q)$. Then

$$Z(Q) \cong Z(\mathfrak{M}). \tag{3}$$

Indeed, if $\varphi \in Z(\mathfrak{M})$, then $\varphi L(x) = L(x)\varphi$ for any $x \in Q$. Further, $\varphi L(x)y = L(x)\varphi y, \varphi(xy) = x\varphi y$. Let $y = 1$. Then $\varphi x = x\varphi 1, \varphi x = L(\varphi 1)x, \varphi = L(\varphi 1)$. Now, using the equality $\varphi(xy) = x\varphi y$ we obtain that $xy \cdot \varphi 1 = \varphi(xy) = x \cdot \varphi y = x \cdot \varphi(y \cdot 1) = x \cdot y\varphi 1$. Consequently, if $\varphi \in Z(\mathfrak{M})$, then $\varphi = L(a)$ and $a \in Z(Q)$. Conversely, let $a \in Z(Q)$. Then $a \cdot xy = ax \cdot y, L(a)L(y)x = L(y)L(a)x, L(a)L(y) = L(y)L(a)$. It follows from the definition of group $\mathfrak{M}$ that $L(a) \in Z(\mathfrak{M})$. Finally, if $a, b \in Z(Q)$, then the homomorphism (3) follows from the equalities $a \cdot bx = ab \cdot x, L(a)L(b)x = L(ab)x, L(a)L(b) = L(ab)$.

In order to prove the implication $3) \rightarrow 1)$ we use the relation

$$\mathfrak{M}/Z_2(\mathfrak{M}) \cong \mathfrak{M}(Q/Z(Q)) \tag{4}$$

which takes place in an arbitrary CML [3]. By Lemma 2 the group $\mathfrak{M}/Z(\mathfrak{M})$ is periodic. Then the group $\mathfrak{M}/Z_2(\mathfrak{M})$, as an homomorphic image of the group $\mathfrak{M}/Z(\mathfrak{M})$, is also periodic. If the group $\mathfrak{M}$ satisfies the maximum condition for subgroups, then the center $Z(\mathfrak{M})$ and by (3), also the center $Z(Q)$, are finitely generated. By Lemma 3 the group $\mathfrak{M}$ is nilpotent. Then the group $\mathfrak{M}/Z_2(\mathfrak{M})$ is also nilpotent and, as it is periodic, then is finite. Hence it follows from (4) that the CML $Q/Z(Q)$ is also finite. Therefore the CML Q is finitely generated and by Lemma 1, the condition 1) holds in it. Consequently, 3) → 1).

Let us now suppose that the group $\mathfrak{M}$ is non-periodic. By Lemma 2 the group $\mathfrak{M}/Z(\mathfrak{M})$ is locally finite. It α is an element of infinite order in $\mathfrak{M}$, then $\alpha^n \in Z(\mathfrak{M})$ for a certain natural number n. We denote by $\mathfrak{R}$ the subgroup of group $\mathfrak{M}$, generated by all elements of the form α^n. It is obvious that the abelian group $Z(\mathfrak{M})$ is finitely generated if the group $\mathfrak{M}$ satisfies one of the conditions 4) – 7). Then $Z(\mathfrak{M}) = \mathfrak{N} \times \mathfrak{S}$, where $\mathfrak{N}$ is a finitely generated abelian group without torsion, $\mathfrak{S}$ is a finite abelian group [7] and $\mathfrak{R} = \mathfrak{N}$. As $\mathfrak{N} \cap \mathfrak{S} = \{1\}$, then $Z(\mathfrak{M})/\mathfrak{N} = (\mathfrak{N} \times \mathfrak{S}) \cong \mathfrak{S}$. By Lemma 2 the group $\mathfrak{M}/Z(\mathfrak{M})$ is locally finite. It follows from the relation $\mathfrak{M}/Z(\mathfrak{M}) \cong$ $\cong (\mathfrak{M}/\mathfrak{N})/(Z(\mathfrak{M})/\mathfrak{N})$ that the group $\mathfrak{M}/\mathfrak{N}$ is the extension of the finite group $Z(\mathfrak{M})/\mathfrak{N}$ by locally finite group $\mathfrak{M}/Z(\mathfrak{M})$. Therefore the group $\mathfrak{M}/\mathfrak{N}$ is locally finite.

By Lemma 2 the commutator group $\mathfrak{M}'$ is locally finite and as the group $\mathfrak{N}$ is without torsion, then

$$\mathfrak{N} \cap \mathfrak{M}' = \{1\}.$$

Let either the condition 4), or 5), or 6), or 7) hold in group $\mathfrak{M}$. By analogy with the proof of Lemma 9 we can show that in the group $\mathfrak{M}/\mathfrak{N}$ a condition analogue with either conditions 4), or 5), or 6), or 7) holds. We have already shown that group $\mathfrak{M}/\mathfrak{N}$ is locally finite. Then either all invariant subgroups, or at least one maximal abelian subgroup, or all nilpotent of class n subgroups, or all solvable of class s subgroups are finite respectively in $\mathfrak{M}/\mathfrak{N}$. The group $Z(\mathfrak{M})/\mathfrak{N}$ is finite, then it follows from the relation

$$\mathfrak{M}/Z(\mathfrak{M}) \cong (\mathfrak{M}/\mathfrak{N})/(Z(\mathfrak{M})/\mathfrak{N})$$

that in the group $\mathfrak{M}/Z(\mathfrak{M})$ the same condition as in group $\mathfrak{M}/\mathfrak{N}$ holds. Further, it follows from the relation

$$\mathfrak{M}/Z_2(\mathfrak{M}) \cong (\mathfrak{M}/Z_1(\mathfrak{M}))/(Z_2(\mathfrak{M})/Z_1(\mathfrak{M})) = (\mathfrak{M}/Z_1(\mathfrak{M}))/Z(\mathfrak{M}/Z_1(\mathfrak{M}))$$

that $\mathfrak{M}/Z_2(\mathfrak{M})$ satisfies the same condition as the group $\mathfrak{M}/Z_1(\mathfrak{M})$, and it follows from (4) that $\mathfrak{M}(Q/Z(Q))$ satisfies this condition as well, i.e. either all its invariant subgroups are finite, or at least one maximal abelian subgroup is finite, or all its nilpotent subgroups of class n are finite, or all its solvable subgroups of class s are finite. In such a case, by Lemma 7 the group $\mathfrak{M}(Q/Z(Q))$ satisfies the minimum condition for subgroups. It is obvious that the center of the group $\mathfrak{M}(Q/Z(Q))$ is finite. Then by 2) of Lemma 7 the group $\mathfrak{M}(Q)/Z(Q)$ is finite, and consequently, the CML $Q/Z(Q)$ is also finite. The center $Z(\mathfrak{M})$ of the group $\mathfrak{M}$ is finitely generated, then it follows from (3) that the center $Z(Q)$ of Q is finitely generated, too. Then the CML Q is finitely generated and by Lemma 1 it satisfies the condition 1). Consequently, if the group $\mathfrak{M}$ is non-periodic the implications $4) \rightarrow 1), 5) \rightarrow 1), 6) \rightarrow 1), 7) \rightarrow 1)$ hold.

The case when the group $\mathfrak{M}$ is periodic is proved by analogy for $\mathfrak{N} = \{1\}$. Further, as the implications $3) \rightarrow 4), 3) \rightarrow 5), 3) \rightarrow 6), 3) \rightarrow 7)$ are obvious, the theorem is proved.

Theorem 3. *The following conditions are equivalent for an arbitrary non-associative commutative Moufang ZA-loop Q with the multiplication group $\mathfrak{M}$:*

1) Q satisfies the maximum condition for subloops;

2) if Q contains a non-invariant (respect. invariant) centrally nilpotent subloop of class n, then at least one maximal non-invariant (respect. invariant) centrally nilpotent subloop of class n is finitely generated;

3) if Q contains a non-invariant (respect. invariant) centrally solvable subloop of class s, then at least one maximal non-invariant (respect. invariant) centrally solvable subloop of class s is finitely generated;

4) if Q contains a non-invariant (respect. invariant) centrally nilpotent subloop of class n, then it satisfies the maximum condition for non-invariant (respect. invariant) centrally nilpotent subloops of class n;

5) if Q contains a non-invariant (respect. invariant) centrally solvable subloop of class s, then it satisfies the maximum condition for non-invariant (respect. invariant) centrally solvable subloops of class s;

6) the center $Z(Q)$ of CML Q is finitely generated;

7) the group $\mathfrak{M}$ is finitely generated;

8) if $\mathfrak{M}$ contains a non-invariant (respect. invariant) nilpotent subgroup of class n, then at least one maximal non-invariant (respect. invariant) nilpotent subgroup of class n is finitely generated;

9) if $\mathfrak{M}$ contains a non-invariant (respect. invariant) solvable subgroup of class s, then at least one maximal non-invariant (respect. invariant) solvable subgroup of class s is finitely generated;

10) if $\mathfrak{M}$ contains a non-invariant (respect. invariant) nilpotent subgroup of class n, then it satisfies the maximum condition for non-invariant (respect. invariant) nilpotent subgroups of class n;

11) if $\mathfrak{M}$ contains a non-invariant (respect. invariant) solvable subgroup of class s, then it satisfies the maximum condition for non-invariant (respect. invariant) solvable subgroups of class s;

12) the center $Z(\mathfrak{M})$ of group $\mathfrak{M}$ is finitely generated.

Proof. The implications $1) \rightarrow 2), 1) \rightarrow 3), 1) \rightarrow 4), 1) \rightarrow 5), 1) \rightarrow 6)$ are obvious. If H is a non-invariant (respect. invariant) centrally nilpotent of class n (or centrally solvable of class s) subloop of the CML Q, then the subloop $< N, Z(Q) >$ will be of this type too. Therefore by Lemma 1 the implications $2) \rightarrow 6), 3) \rightarrow 6),$

$4) \rightarrow 6), 5) \rightarrow 6)$ hold. Let us now suppose that the condition 6) holds in the CML Q and let R be an invariant subloop, defined in Lemma 9. By 4) of Lemma 9 the center $Z(Q/R)$ is finitely generated and periodic. If follows from Lemma 5 that $Z(Q/R)$ is finite, and it follows from Lemma 8 that the CML Q/R satisfies the minimum condition for subloops. As the center $Z(Q/R)$ is finite, then by 2) of Lemma 6, the CML Q/R is finite. By its construction, the subloop L is finitely generated, therefore the CML Q is also finitely generated and the justice of condition 1) follows from Lemma 1. Consequently, the conditions 1), 2), 3), 4), 5), 6) are equivalent.

The equivalence of conditions 7), 8), 9), 10), 11), 12) is proved by analogy, using 1), 2) of theorem 2. Finally, the equivalence of conditions 6), 12) follows from (3). Therefore the theorem is fully proved.

References

[1] Evans T. *Identities and Relations in Commutative Moufang Loops.* J. Algebra, 1974, **31**, p. 508–513.

[2] Sandu N.I. *About Centrally Nilpotent Commutative Moufang Loops.* Quasigroups and Loops: Mat. issled. vyp. 51, Kishinev, Stiintsa, 1979, p. 145–155 (In Russian).

[3] Bruck R.H. *A Survey of Binary Systems.* Springer Verlag, Berlin-Heidelberg, 1958.

[4] Sandu N.I. *Commutative Moufang Loots with Minimum Condition for Subloops, I.* Buletinul Academiei de Ştiinţe a Republicii Moldova, Matematica, 2003, N 3(43), p. 25–40.

[5] Sandu N.I. *Commutative Moufang Loots with Minimum Condition for Subloops, II.* Buletinul Academiei de Ştiinţe a Republicii Moldova, Matematica, 2004, N 2(45), p. 33–48.

[6] Sandu N.I. *Commutative Moufang Loops with Finite Classes of Conjugate Subloops.* Mat. zametki, 2003, **73**, N 2, p. 269–280 (In Russian).

[7] Kurosh A.G. *Group Theory.* Moskva, Nauka, 1967 (In Russian).

[8] Sandu N.I. *Medially Nilpotent Distributive Quasigroups and CH-Quasigroups.* Sib. mat. jurnal., 1987, **XXVIII**, N 2, p. 159–170 (In Russian).

Tiraspol State University
str. Iablochkin, 5
Chişinău MD-2069
Moldova
E-mail: *sandumn@yahoo.com*

Received March 3, 2006

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(51), 2006, Pages 62–68
ISSN 1024–7696

Discontinuous term of the distribution for Markovian random evolution in R^3

Alexander D. Kolesnik

Abstract. We consider the random motion at constant finite speed in the space R^3 subject to the control of a homogeneous Poisson process and with uniform choice of directions on the unit 3-sphere. We obtain the explicit forms of the conditional characteristic function and conditional distribution when one change of direction occurs. We show that this conditional distribution represents a discontinuous term of the transition function of the motion.

Mathematics subject classification: Primary 60K99 Secondary 62G30; 60K35; 60J60; 60H30.
Keywords and phrases: Random motions, finite speed, random evolution, characteristic functions, conditional distributions.

In this note we obtain the discontinuous term of the distribution for the three-dimensional random motion at arbitrary finite speed (so-called, random evolution). This is motivated by the previous works on planar random motions by Stadje (1987), Masoliver *et al.* (1993), Kolesnik and Orsingher (2005) where the explicit form of the transition function of the process was obtained by substantially different methods. It was shown that the transition density of the motion is discontinuous on the boundary of the diffusion area, and the discontinuous term of the distribution is the Green's function to the two-dimensional wave equation. Amazingly, this discontinuous term is determined by the conditional distribution, corresponding to the case when only one change of direction occurs (see Kolesnik and Orsingher (2005, Remark 1)). The three-dimensional motion with unit speed was examined by Stadje (1989) where the transition density was derived by means of recurrent arguments. This transition density consists of two terms (see Stadje (1989, formulae (1.3) and (4.21)). The first one is a continuous function and has the form of a fairly complicated integral, which can scarcely be exactly evaluated. The second term is a logarithmic function which is discontinuous on the boundary of the diffusion area.

Here we derive this discontinuous term for a motion at arbitrary finite speed by means of characteristic functions and show that, similarly to the planar case, it is determined by the conditional distribution corresponding to the case when only one change of direction occurs. Such a behaviour of the distribution near the border of diffusion area is a very interesting feature of the two and three-dimensional motions. However, the nature of this phenomenon is not entirely clear.

Let's consider a particle starting its motion from the origin $x_1 = x_2 = x_3 = 0$ of the space R^3 at time $t = 0$. The particle is endowed with constant, finite speed c.

The initial direction is a three-dimensional random vector with uniform distribution on the unit 3-sphere

$$S_1 = \left\{(x_1, x_2, x_3) \in R^3 : \; x_1^2 + x_2^2 + x_3^2 = 1\right\}.$$

The particle changes direction at random instants which form a homogeneous Poisson process of rate $\lambda > 0$. At these moments it instantaneously takes on the new direction with uniform distribution on S_1, independently of its previous motion.

Let $X(t) = (X_1(t), X_2(t), X_3(t))$ be the position of the particle at an arbitrary time $t > 0$. Consider the conditional distributions

$$Pr\{X(t) \in dx \mid N(t) = n\} =$$

$$= Pr\{X_1(t) \in dx_1, X_2(t) \in dx_2, X_3(t) \in dx_3 \mid N(t) = n\}, \quad n \geq 1,$$

where $N(t)$ is the number of Poisson events that have occurred in the interval $(0, t)$ and $dx = dx_1 dx_2 dx_3$ is the infinitesimal volume of the space R^3.

At any time $t > 0$ the particle, with probability 1, is located in the three-dimensional ball of radius ct

$$B_{ct} = \left\{(x_1, x_2, x_3) \in R^3 : \; x_1^2 + x_2^2 + x_3^2 \leq c^2 t^2\right\}.$$

The distribution $Pr\left\{X(t) \in dx\right\}, \; x \in B_{ct}, \; t \geq 0$, consists of two components. The singular component corresponds to the case when no Poisson event occurs in the interval $(0, t)$ and is concentrated on the sphere

$$S_{ct} = \partial B_{ct} = \left\{(x_1, x_2, x_3) \in R^3 : \; x_1^2 + x_2^2 + x_3^2 = c^2 t^2\right\}.$$

In this case the particle is located on the sphere S_{ct} and the probability of this event is

$$Pr\left\{X(t) \in S_{ct}\right\} = e^{-\lambda t}.$$

If one or more than one Poisson events occur, the particle is located strictly inside the ball B_{ct}, and the probability of this event is

$$Pr\left\{X(t) \in Int\ B_{ct}\right\} = 1 - e^{-\lambda t}.$$

The part of the distribution $Pr\left\{X(t) \in dx\right\}$ corresponding to this case is concentrated in the interior

$$Int\ B_{ct} = \left\{(x_1, x_2, x_3) \in R^3 : \; x_1^2 + x_2^2 + x_3^2 < c^2 t^2\right\},$$

and forms its absolutely continuous component.

Therefore there exists the density $p(x, t) = p(x_1, x_2, x_3, x_4, t), \;\; x \in Int\ B_{ct}, \; t > 0$, of the absolutely continuous component of the distribution $Pr\left\{X(t) \in dx\right\}$.

If $N(t) = n$, the displacement of the particle $X(t)$ at any time $t > 0$ is determined by the coordinates

$$X_k(t) = c \sum_{j=1}^{n+1} (s_j - s_{j-1}) x_j^k, \qquad k = 1, 2, 3, \tag{1}$$

where x_j^k are the components of the independent random vectors $x_j = (x_j^1, x_j^2, x_j^3)$, $j = 1, ..., n+1$, uniformly distributed on the unit sphere S_1; the s_j, $j = 1, ..., n$, represent the instants at which Poisson events occur, and $s_0 = 0$, $s_{n+1} = t$.

The conditional characteristic function can be written as follows:

$$H_n(\alpha, t) = E\left\{ e^{i(\alpha, X(t))} |\ N(t) = n \right\}, \quad n \geq 1, \tag{2}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in R^3$ is the real vector of inversion parameters and $(\alpha, X(t))$ denotes the scalar (inner) product of the vectors α and $X(t)$.

By substituting (1) into (2) we have

$$H_n(\alpha, t) = E\left\{ \exp\left(ic \sum_{k=1}^{3} \alpha_k \sum_{j=1}^{n+1} (s_j - s_{j-1}) x_j^k \right) \right\} =$$

$$= E\left\{ \exp\left(ic \sum_{j=1}^{n+1} (s_j - s_{j-1})(\alpha, x_j) \right) \right\},$$

where (α, x_j) is the scalar (inner) product of the vectors α and x_j. Computing the expectation in this last equality we obtain

$$H_n(\alpha, t) = \frac{n!}{t^n} \int_0^t ds_1 \int_{s_1}^t ds_2 \dots \int_{s_{n-1}}^t ds_n \left\{ \prod_{j=1}^{n+1} \left[\frac{1}{mes\ S_1} \int_{S_1} e^{ic(s_j - s_{j-1})(\alpha, x_j)}\ dx_j \right] \right\}.$$

The surface integral over the unit sphere S_1 in the last equality can easily be computed by passing to three-dimensional polar coordinates, and it is

$$\int_{S_1} e^{ic(s_j - s_{j-1})(\alpha, x_j)}\ dx_j = 4\pi\, \frac{\sin(c(s_j - s_{j-1})\|\alpha\|)}{c(s_j - s_{j-1})\|\alpha\|}, \tag{3}$$

where $\|\alpha\| = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}$. Taking into account that $mes\ S_1 = 4\pi$, we obtain

$$H_n(\alpha, t) = \frac{n!}{t^n} \int_0^t ds_1 \int_{s_1}^t ds_2 \dots \int_{s_{n-1}}^t ds_n \left\{ \prod_{j=1}^{n+1} \frac{\sin(c(s_j - s_{j-1})\|\alpha\|)}{c(s_j - s_{j-1})\|\alpha\|} \right\}. \tag{4}$$

This expression can scarcely be explicitly computed for arbitrary $n \geq 1$. However, for the important particular case $n = 1$ this expression can be evaluated, and its inverse Fourier transform leading to the conditional distribution, corresponding to the case when only one change of direction occurs, can be explicitly given.

Theorem. *For any $t > 0$ the conditional distribution corresponding to the only change of direction has the form*

$$Pr\{X(t) \in dx \mid N(t) = 1\} = \frac{1}{4\pi (ct)^2 \|x\|}\ \ln\left(\frac{ct + \|x\|}{ct - \|x\|} \right)\ dx, \tag{5}$$

$$x = (x_1, x_2, x_3) \in Int\ B_{ct}, \quad \|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad dx = dx_1 dx_2 dx_3.$$

Proof. From (4) we have

$$H_1(\alpha, t) =$$

$$= \frac{1}{t} \int_0^t \frac{\sin(cs\|\alpha\|)}{cs\|\alpha\|} \; \frac{\sin(c(t-s)\|\alpha\|)}{c(t-s)\|\alpha\|} \, ds =$$

$$= \frac{1}{(ct\|\alpha\|)^2} \int_0^t \sin(cs\|\alpha\|) \; \sin(c(t-s)\|\alpha\|) \; \left[\frac{1}{s} + \frac{1}{t-s} \right] \, ds =$$

$$= \frac{2}{(ct\|\alpha\|)^2} \int_0^t \frac{\sin(cs\|\alpha\|) \; \sin(c(t-s)\|\alpha\|)}{s} \, ds =$$

$$= \frac{\sin(ct\|\alpha\|)}{(ct\|\alpha\|)^2} \int_0^t \frac{2\sin(cs\|\alpha\|) \; \cos(cs\|\alpha\|)}{s} \, ds - \frac{\cos(ct\|\alpha\|)}{(ct\|\alpha\|)^2} \int_0^t \frac{2\sin^2(cs\|\alpha\|)}{s} \, ds =$$

$$= \frac{\sin(ct\|\alpha\|)}{(ct\|\alpha\|)^2} \int_0^t \frac{\sin(2cs\|\alpha\|)}{s} \, ds - \frac{\cos(ct\|\alpha\|)}{(ct\|\alpha\|)^2} \int_0^t \frac{1 - \cos(2cs\|\alpha\|)}{s} \, ds =$$

$$= \frac{\sin(ct\|\alpha\|)}{(ct\|\alpha\|)^2} \; \mathrm{Si}(2ct\|\alpha\|) + \frac{\cos(ct\|\alpha\|)}{(ct\|\alpha\|)^2} \; \mathrm{Ci}(2ct\|\alpha\|), \tag{6}$$

where $\mathrm{Si}(x)$ and $\mathrm{Ci}(x)$ are the modified integral sine and cosine, respectively, given by

$$\mathrm{Si}(x) = \int_0^x \frac{\sin z}{z} \, dz, \qquad \mathrm{Ci}(x) = \int_0^x \frac{\cos z - 1}{z} \, dz. \tag{7}$$

To prove the statement of the theorem, we need to show that the inverse Fourier transform of (6) with respect to $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ leads to the conditional distribution (5). However, it is simpler to show that, inversely, the Fourier transform of function (5) in the ball B_{ct} coincides with (6). Passing to three-dimensional polar coordinates we have

$$\int_{B_{ct}} e^{i(\alpha, x)} \; Pr\{X(t) \in dx \mid N(t) = 1\} =$$

$$= \frac{1}{4\pi (ct)^2} \int_0^{ct} dr \; \left\{ r \ln \left(\frac{ct + r}{ct - r} \right) \times \right.$$

$$\left. \times \int_0^{\pi} \int_0^{2\pi} e^{ir(\alpha_1 \sin\theta_1 \sin\theta_2 + \alpha_2 \sin\theta_1 \cos\theta_2 + \alpha_3 \cos\theta_1)} \; \sin\theta_1 \; d\theta_1 \; d\theta_2 \right\}.$$

According to formula 4.624 of Gradshteyn and Ryzhik (1980)

$$\int_0^{\pi} \int_0^{2\pi} e^{ir(\alpha_1 \sin\theta_1 \sin\theta_2 + \alpha_2 \sin\theta_1 \cos\theta_2 + \alpha_3 \cos\theta_1)} \; \sin\theta_1 \; d\theta_1 \; d\theta_2 = 4\pi \; \frac{\sin(r\|\alpha\|)}{r\|\alpha\|}.$$

Therefore, applying the auxiliary Lemma (see below), we obtain

$$\int_{B_{ct}} e^{i(\alpha,x)} \; Pr\{X(t) \in dx \mid N(t) = 1\} =$$

$$= \frac{1}{(ct)^2 \|\alpha\|} \int_0^{ct} \sin(r\|\alpha\|) \; \ln\left(\frac{ct+r}{ct-r}\right) \; dr =$$

$$= \frac{1}{ct\|\alpha\|} \int_0^{1} \sin(ct\|\alpha\| z) \; \ln\left(\frac{1+z}{1-z}\right) \; dz =$$

$$= \frac{1}{(ct\|\alpha\|)^2} \Bigl[\sin(ct\|\alpha\|) \; \mathrm{Si}(2ct\|\alpha\|) + \cos(ct\|\alpha\|) \; \mathrm{Ci}(2ct\|\alpha\|) \Bigr] ,$$

and this coincides with (6). The theorem is proved.

Remark 1. Taking into account the well-known equality

$$\mathrm{Arcth}(z) = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right)$$

we can rewrite (5) as follows:

$$Pr\{X(t) \in dx \mid N(t) = 1\} = \frac{1}{2\pi (ct)^2 \|x\|} \; \mathrm{Arcth}\left(\frac{\|x\|}{ct}\right) \; dx,$$

and, for $c = 1$, this is similar to the second term of formulae (1.3) and (4.21) of Stadje (1989). The first (integral) term of these formulae, obviously, is determined by the conditional distributions corresponding to the case when more than one change of direction occurs.

Remark 2. The function (5) represents the discontinuous term of the distribution of the process $X(t)$. Formula (5) shows that the density near the border of the ball B_{ct} is large and this corresponds to the case when only one change of direction occurs. In other words, if only one change of direction occurs, the conditional density is minimal in the neighbourhood of the origin and increases as we approach to the border. This is similar to the behaviour of the analogous conditional density in the planar case (see Kolesnik and Orsingher (2005, Remark 1)).

Finally, we establish an auxiliary lemma which has been used in the proof of our theorem.

Lemma. *For arbitrary $a > 0$ the following relation holds*

$$\int_0^1 \sin(ax) \; \ln\left(\frac{1+x}{1-x}\right) \; dx = \; \frac{1}{a} \Bigl[\sin a \; \mathrm{Si}(2a) + \cos a \; \mathrm{Ci}(2a) \Bigr] , \qquad (8)$$

where the integral is treated in the improper sense.

Proof. We have

$$\int_0^1 \sin(ax)\ \ln\left(\frac{1+x}{1-x}\right)\ dx =$$

$$= \int_0^1 \sin(ax)\ \ln(1+x)\ dx - \int_0^1 \sin(ax)\ \ln(1-x)\ dx. \tag{9}$$

Let's evaluate separately the integrals in the right-hand side of (9). Integrating by parts and applying formula 2.641(2) of Gradshteyn and Ryzhik (1980) we obtain for the first integral of (9):

$$\int_0^1 \sin(ax)\ \ln(1+x)\ dx =$$

$$= -\frac{1}{a}\left(\cos a\ \ln 2 - \int_0^1 \frac{\cos(ax)}{1+x}\ dx\right) =$$

$$= -\frac{\cos a}{a}\ln 2 + \frac{1}{a}\Bigg[\cos a\,(\mathrm{ci}(2a) - \mathrm{ci}(a)) + \sin a\,(\mathrm{si}(2a) - \mathrm{si}(a))\Bigg]. \tag{10}$$

Here si(x) and ci(x) are the standard integral sine and cosine, respectively, given by

$$\mathrm{si}(x) = -\frac{\pi}{2} + \mathrm{Si}(x), \qquad \mathrm{ci}(x) = \mathbf{C} + \ln x + \mathrm{Ci}(x), \tag{11}$$

and $\mathbf{C} = 0.5772\ldots$ being the Euler constant. From (11) we easily obtain

$$\mathrm{ci}(2a) - \mathrm{ci}(a) = \ln 2 + \mathrm{Ci}(2a) - \mathrm{Ci}(a),$$

$$\mathrm{si}(2a) - \mathrm{si}(a) = \mathrm{Si}(2a) - \mathrm{Si}(a).$$

Substituting these expressions into (10) we obtain the first integral of (9):

$$\int_0^1 \sin(ax)\ \ln(1+x)\ dx =$$

$$= \frac{\cos a}{a}\Big(\mathrm{Ci}(2a) - \mathrm{Ci}(a)\Big) + \frac{\sin a}{a}\Bigg(\mathrm{Si}(2a) - \mathrm{Si}(a)\Bigg), \tag{12}$$

where Ci(x) and Si(x) are given by (7).

The function in the second integral of (9) is unbounded at the point $x = 1$. Therefore we can evaluate this integral in the improper sense only. Similarly, integrating by parts and applying formula 2.641(2) of Gradshteyn and Ryzhik (1980) we have

$$\int_0^1 \sin(ax)\ \ln(1-x)\ dx = \lim_{\varepsilon\to 0^+}\int_0^{1-\varepsilon} \sin(ax)\ \ln(1-x)\ dx =$$

$$= -\frac{1}{a}\lim_{\varepsilon\to 0^+}\left\{\cos\left(a(1-\varepsilon)\right)\ \ln\varepsilon + \int_0^{1-\varepsilon}\frac{\cos(ax)}{1-x}\ dx\right\} =$$

$$= -\frac{1}{a} \lim_{\varepsilon\to 0^+} \left\{ \cos\left(a(1-\varepsilon)\right) \ \ln\varepsilon - \right.$$

$$\left. - \left[\cos a \ \left(\mathrm{ci}(-a\varepsilon) - \mathrm{ci}(-a)\right) - \sin a \ \left(\mathrm{si}(-a\varepsilon) - \mathrm{si}(-a)\right)\right]\right\}. \tag{13}$$

Using (11) and the well-known equalities

$$\mathrm{si}(-x) = -\mathrm{si}(x) - \pi, \qquad \mathrm{Ci}(-x) = \mathrm{Ci}(x),$$

we get

$$\mathrm{ci}(-a\varepsilon) - \mathrm{ci}(-a) = \ln\varepsilon + \mathrm{Ci}(a\varepsilon) - \mathrm{Ci}(a),$$

$$\mathrm{si}(-a\varepsilon) - \mathrm{si}(-a) = \mathrm{Si}(a) - \mathrm{Si}(a\varepsilon).$$

Substituting this into (13) we obtain the second integral in the right-hand side of (9):

$$\int_0^1 \sin(ax) \ \ln(1-x) \, dx =$$

$$= -\frac{1}{a} \lim_{\varepsilon\to 0^+} \left\{ \left(\cos\left(a(1-\varepsilon)\right) - \cos a\right) \ \ln\varepsilon - \right.$$

$$\left. - \cos a \ \left(\mathrm{Ci}(a\varepsilon) - \mathrm{Ci}(a)\right) + \sin a \ \left(\mathrm{Si}(a) - \mathrm{Si}(a\varepsilon)\right)\right\} =$$

$$= -\frac{1}{a} \ \left[\sin a \, \mathrm{Si}(a) + \cos a \ \mathrm{Ci}(a)\right]. \tag{14}$$

Substituting now (12) and (14) into (9) we obtain (8). The lemma is proved.

References

[1] Gradshteyn I.S., Ryzhik I.M. *Tables of Integrals, Series and Products.* Academic Press, NY, 1980.

[2] Kolesnik A.D., Orsingher E. *A planar random motion with an infinite number of directions controlled by the damped wave equation.* J. Appl. Prob., 2005, **42**, p. 1168–1182.

[3] Masoliver J., Porrá J.M., Weiss G.H. *Some two and three-dimensional persistent random walks.* Physica A, 1993, **193**, p. 469–482.

[4] Stadje W. *The exact probability distribution of a two-dimensional random walk.* J. Stat. Phys., 1987, **46**, p. 207–216.

[5] Stadje W. *Exact probability distributions for non-correlated random walk models.* J. Stat. Phys., 1989, **56**, p. 415–435.

Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
Academiei str. 5, MD-2028 Kishinev
Moldova
E-mail: *kolesnik@math.md*

Received May 30, 2006

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(51), 2006, Pages 69–74
ISSN 1024–7696

On an algebraic method in the study of integral equations with shift

Vasile Neaga

Abstract. The work is centred on the sdudy of algebra $\mathfrak{A}$ generated by singular integral operators with shifts with continuous coefficients. We determine the set of maximal ideals of quotient algebra $\hat{\mathfrak{A}}$, $\hat{\mathfrak{A}} = \mathfrak{A}/\mathfrak{T}$, with respect to the ideal of compact operators. Prove that the bicompact of maximal ideals of $\hat{\mathfrak{A}}$ is isomorphic to the topological product $(\Gamma \times j) \times (\Gamma \times k)$, where $j = \pm 1$ and $k = \pm 1$. Necessary and sufficient condition are established for operators of $\mathfrak{A}$ to be noetherian and to admit equivalent regularization in space $L_p(\Gamma, \rho)$, regularizators for noetherian operators are constructed. The study is done in the space $L_p(\Gamma, \rho)$ with weight $\rho(t) = \prod\limits_{k=1}^{n} |t - t_k|^{\beta^k}$ and is based on the theory of Ghelfand [1] concerning Banach algebras.

Mathematics subject classification: 45E05.
Keywords and phrases: Banach algebras, noetherian singular operators, regularization of operator.

I. We remind that an operator $A \in L(\mathfrak{B})$ admits a regularization if there exists an operator $M \in L(\mathfrak{B})$ so that $AM = I + T_1, MA = I + T_2$, where T_1 and T_2 are compact operators in the space $\mathfrak{B}$. The class of operators which admit a regularization is of special interest due to the fact that operators of this class have the fallowing properties (E.Noether theorems):

1) Ecuations $Ax = 0$ and $A^*\varphi = 0$ have a finite number of linear independent solutions.

2) Ecuation $Ax = y$ is solvable if and ouly if the right hand part is orthogonal to all solutions of equation $A^*\varphi = 0$.

Operators with properties 1) and 2) are called *noetherian* and are essential generalizations of the class of operators of the form $I + T, T$ compact, for which theorems similar to that of Fredholm are true.

Let $A \in L(\mathfrak{B})$ be a noetherian operator. If it is known the operator M, the regularizator for A, then the problem of solvability of the equation

$$Ax = y \tag{1}$$

can be reduced to the solvability of the equation

$$MAx = My, \tag{2}$$

where the operator $MA - I$ is compact. To the last equation different methods to solve equations of the form $(I + T)x = y$ can be applied, where T is a compact

---

operator. Of special interest is the case when the ecuations $Ax = y$ and $MAx = My$ are equivalent[1] for every y. This is true if and only if $\ker M = \{0\}$.

We say that an operator A admits an equivalent regularization if it possesses a regularizator M for which equations (1) and (2) are equivalent for every $y \in \mathfrak{B}$. In this case the operator M is called *equivalent regularizator* for the operator A. From what we said above it results that operator M is an equivalent regularizator for A if it is a regularizator for A and is left invertible.

It is well known [2] that a singular integral operator[2] $A = aI + bS + T$ admits a regularization if and only if $a^2(t) - b^2(t) \neq 0$ for all $t \in T$. For example, as a regularizator one can take the operator $R = \frac{a}{a^2 - b^2} I - \frac{b}{a^2 - b^2} S$. Under these conditions the operator A^*, obviously, also admits a regularization and thus for A and A^* E.Noether theorems hold.

The main result of this work is given by (see [3])

Theorem 1. *The operator*

$$A = aI + bS + (cI + dS)V + T, a, b, c, d \in C_\alpha(\Gamma) \tag{3}$$

admits a regularization in $L_p(\Gamma, \rho)$ *if and only if*

$$(a(t) + b(t))^2 - (c(t) + d(t))^2 \neq 0, (a(t) - b(t))^2 - (c(t) - d(t))^2 \neq 0 \tag{4}$$

for every $t \in T$. *Under conditions (4) the operator*

$$R = \frac{\alpha}{\alpha^2 - \delta^2} P + \frac{\beta}{\beta^2 - \gamma^2} Q - (\frac{\delta}{\alpha^2 - \delta^2} P + \frac{\gamma}{\beta^2 - \gamma^2} Q) V, \tag{5}$$

where $\alpha = a + b, \beta = a - b, \delta = c + d, \gamma = c - d, P = \frac{1}{2}(I + S), Q = \frac{1}{2}(I - S)$, *is a regularizator for* A.

II. Denote by $C_\omega(\Gamma) (\subset C(\Gamma)$ the set of functions $a(t)$ continuous on Γ and satifying the condition $a(\omega(t)) = a(t)$. Evidently, this set forms a commutative algebra with identity and the norm $||a||_{C_\omega(\Gamma)} = ||a||_{C(\Gamma)}$. It is also obvious that every function of the form $a(t) = b(t) \cdot b(\omega(t))$, where $b \in C(\Gamma)$, is contained in $C_\omega(\Gamma)$. The converse of this statement is also true: every function $a \in C_\omega(\Gamma)$ may be represented in the form $a(t) = b(t) \cdot b(\omega(t))$ where $b \in C(\Gamma)$. We can join these remarks in the assertion that the algebra $C_\omega(\Gamma)$ is characterized by the relation

$$C_\alpha(\Gamma) = \{b(t) \cdot b(\omega(t)) | b \in C(\Gamma)\}.$$

Representation of functions from $C_\omega(\Gamma)$ in the form $a(t) = b(t) \cdot b(\omega(t))$ is unique up to some constant factors c_1 and $c_2, c_1 \cdot c_2 = 1$. Later on we shall assume that $c_1 = c_2 = 1$. Thus, for example, if Γ is the unit circle and $\omega(t) = -t$, then the

---

[1] Equations $Ax = y$ and $MAx = My$ are called equivalent if they have the same set of solutions.

[2] By T with indices we denote compact operators.

functions $a_1(t) = -t^2, a_2(t) = t^2$ belong to $C_\omega(\Gamma)$ and they can be represented as $a_1(t) = t \cdot (-t)$ and, respectively $a_2(t) = it \cdot (-it)$.

Let Γ be a closed Liapunov type contour, S a singular integral operator with Cauchy kernel and V an operator of shifting, $(V\varphi)(t) = \varphi(\alpha(t))$, where the function $\omega : \Gamma \to \Gamma$ satisfies conditions:

a) $\omega(\omega(t)) \equiv \omega(t), (\omega(t) \neq t)$;

b) there exists derivative $\omega^{'}(t) \neq 0$;

c) $\omega^{'}(t) \in H_\mu(\Gamma)$.

Denote by $\mathfrak{A}$ the algebra generated by S, V and the set of alle operators of multiplication by functions $a(t), a \in C_\omega(\Gamma)$. $\mathfrak{A}$ is a subalgebra of algebra $L(L_p(\Gamma, \rho))$ formed by the set of linear and bounded operators acting in the space $L_p(\Gamma, \rho)$.

Theorem 2. $\mathfrak{A}$ *is a closed algebra.*

In the proof of this theorem we use properties of S and V, caracterization of algebra $C_\omega(\Gamma)$ and the following result [2].

Lemma 1. *If the operator* $(M\varphi) = a(t)\varphi(t)$ *of multiplication by function* $a(t)$ *continuuous on* Γ, *can be represented in the form* $M = B + T$, *where* B *is invertible and* T *is an operator compact in* $L_p(\Gamma, \rho)$, *then* $a(t)$ *is not vanished on* Γ.

Remark. The norm in algebra $\mathfrak{A}$, defined as operator norm, is topologically equivalent to the norm

$$||A||_1 = \max|a(t)| + \max|b(t)| + \max|c(t)| + \max|d(t)| + ||T||.$$

The set $\mathfrak{T} = \mathfrak{T}(L(L_p(\Gamma, \rho)))$ of compact operators in the space $L_p(\Gamma, \rho)$ is included in $\mathfrak{A}$ and form a twosided closed ideal. Consider the quotient algebra $\hat{\mathfrak{A}} = \mathfrak{A}/\mathfrak{T}$, which is also a Banach algebra. Four continuous functions $a(t), b(t), c(t)$ and $d(t)$ define uniquely a coset $\hat{\mathfrak{A}}$ and; conversely, every element belonging to some coset of $\hat{\mathfrak{A}}$ is of the form $aI + bS + (cI + dS)V + T$, where T is a compact operator. Really, if the elements $aI+bS+(cI+dS)V+T$ and $a_1I+b_1+(c_1I+d_1S)V+T_1$ are in some coset, then their difference $(a-a_1)I+(b-b_1)S+((c-c_1)I+(d-d_1)S)V+T-T_1$ must be a compact operator. Under these condutions from Theorem 2 one can deduce that the operators $(a-a_1)I, (b-b_1)I, (c-c_1)I, (d-d_1)I$ are compact, but from Lemma 1 this is possible if and only if $a(t) \equiv a_1(t), b(t) \equiv b_1(t), c(t) \equiv c_1(t), d(t) \equiv d_1(t)$.

Let us return to algebra $\hat{\mathfrak{A}}$. The element of $\hat{\mathfrak{A}}$ determined by the functions $a(t), b(t), c(t)$ and $d(t)$ is denoted by $\{aI + bS + (cI + dS)V\}$. From properties of operators S and V [4–6] and by direct calculations we get

Theorem 3. *The algebra* $\hat{\mathfrak{A}}$ *is commutative and, besides, the equality*

$$\{aI + bS + (cI + dS)V\} \cdot \{a_1I + b_1S + (c_1I + d_1S)V\} =$$

$$= \{aa_1 + bb_1 + cc_1 + dd_1)I + (ab_1 + a_1b + cd_1 + c_1d)S+ \tag{6}$$

$$+((ac_1 + a_1c + bd_1 + b_1d)I + (ad_1 + a_1d + bc_1 + b_1c)S)V\}$$

is true.

The norm in $\hat{\mathfrak{A}}$ is defined by the equality

$$|\{aI + bS + (cI + dS)V\}| = \inf_{T\in\mathfrak{T}} ||aI + bS + (cI + dS)V||$$

and it is topologically equivalent to the norm

$$|\{aI + bS + (cI + dS)V\}|_1 = \max|a(t)| + \max|b(t)| + \max|c(t)| + \max|d(t)|.$$

III. Further, elements of algebra $\hat{\mathfrak{A}}$ will be expressed in the form

$$\{aP + bQ + (cP + dQ)V\}, a, b, c, d \in C_\omega(\Gamma), \tag{7}$$

where $P = \frac{1}{2}(I + S)$ and $Q = \frac{1}{2}(I - S)$.

We shall describe all maximal ideals of $\hat{\mathfrak{A}}$. This result will enable us to establish necessary and sufficient condition under which elements of $\hat{\mathfrak{A}}$ are invertible. Using this result we shall also construct regularizators for noetherian operators.

Theorem 4. *The set of elements* $\{ap + bQ + (cP + dQ)V\} \in \hat{\mathfrak{A}}$ *does form a maximal ideal of* $\hat{\mathfrak{A}}$ *if the function* $a(t) + c(t)$ *is vanished at the same point* $t_0 \in \Gamma$. *The set of elements* $\{aP + bQ + (cP + dQ)V\} \in \hat{\mathfrak{A}}$ *for which one of the functions* $a(t) - c(t), b(t) + d(t)$, *or* $b(t) - d(t)$ *is vanished at the same point (every function at its own point) also form a maximal ideal. There are no other maximal ideals.*

By virtue of I.Ghelfand [1] results, according to which an element of some Banach algebra is invertible if and only if it does not belong to ony maximal ideal, we obtain the following

Theorem 5. *An element* $\{aP + bQ + (cP + dQ)V\} \in \hat{\mathfrak{A}}$ *is invertible in* $\hat{\mathfrak{A}}$ *if and only if the functions* $a(t) \pm c(t)$ *and* $b(t) \pm d(t)$ *are not vanished on contour* Γ.

We shall establish some other properties of algebra $\hat{\mathfrak{A}}$. Observe that the intersection of all maximal ideals of $\hat{\mathfrak{A}}$ coincides to the null ideal. In fact, by Theorem 4, if $\{aP + bQ + (cP + dQ)V\} \in \cap M_1$, then $a(t) + c(t) \equiv 0$, $a(t) - c(t) \equiv 0$, $b(t) + d(t) \equiv 0$ and $b(t) - d(t) \equiv 0$, that is $\{aP + bQ + (cP + dQ)V\} = \{0\}$. Consequently,

1°. Algebra has no radical.

2°. $\hat{\mathfrak{A}}$ is an involution algebra.

Define involution by

$$\{aP + bQ + (cP + dQ)V\}' = \{\bar{a}P + \bar{b}Q + (\bar{c}P + \bar{d}Q)V\}.$$

All properties of involution are evident. We shall show only that for every element $\{aP + bQ + (cP + dQ)V\} \in \hat{\mathfrak{A}}$ there exists in $\hat{\mathfrak{A}}$ the element

$$[I + \{(aP + bQ + (cP + dQ)V) \cdot (\bar{a}P + \bar{b}Q + (\bar{c}P + \bar{d}Q)V\}]^{-1}.$$

Compute

$$[I+\{ap+bQ+(cP+dQ)V)\cdot(\bar{a}P+\bar{b}Q+(\bar{c}P+\bar{d}Q)V\}]=$$

$$=\{(1+|a|^2+|c|^2)P+(1+|b|^2+|d|^2)Q+((a\bar{c}+\bar{a}c)P++(b\bar{d}+\bar{b}d)Q)V\},$$

$$I+|b(t)|^2+|d(t)|^2\pm(b(t)\bar{d}(t)+\bar{b}(t)d(t)=1+|b(t)\pm d(t)|^2>0.$$

Hence, there exists

$$[I+\{aP+bQ+(cP+dQ)V)\cdot(\bar{a}P+\bar{b}Q+(\bar{c}P+\bar{d}Q)V\}]^{-1}=$$

$$=\left\{\begin{array}{c} \dfrac{1+|a|^2+|c|^2}{(1+|a-c|^2)(1+|a+c|^2)}P+\dfrac{1+|b|^2+|d|^2}{(1+|b-d|^2)(1+|b+d|^2)}Q- \\ -\dfrac{a\bar{c}+\bar{a}c}{(1+|a-c|^2)(1+|a+c|^2)}P+\dfrac{b\bar{d}+\bar{b}d}{(1+|b-d|^2)(1+|b+d|^2)}Q)V \end{array}\right\}$$

and this element belougs to $\hat{\mathfrak{A}}$. Property 2^o is proved.

Denote by $\mathfrak{M}$ the bicompact of maximal ideals of $\hat{\mathfrak{A}}$.

3°. $\mathfrak{M}$ is isomorphic to the topological product $(\Gamma\times j)\times(\Gamma\times k):\mathfrak{M}=(\Gamma\times j)\times(\Gamma\times k)$, where $j=\pm1$ and $k=\pm1$.

It is known [1] that every commutative Banach algebra without radical is isomorphically mapped into an algebra of functions defined on bicompact of maximal ideals. It is easy to observe that in our case to the element $A=\{aP+bQ+(cP+dQ)V\}\in\hat{\mathfrak{A}}$ the function $A(M)=(a(t)+jc(t))(b(t)+kf(t))$ corresponds.

4°. Algebra $\hat{\mathfrak{A}}$ is a symmetric algebra without radical.

In commutative and symmetric algebra $\mathfrak{R}$ every element is invertible or is a generalized zero divisor (see [1]), that is, there exists a sequence $(y_n), y_n\in\mathfrak{R}$, $|y_n|=1$ and $\lim\limits_{n\to\infty}||y_nx||=0$. Thus, every element $A=\{aP+bQ+(cP+dQ)V\}$, for which one of the functions $a(t)+c(t), a(t)-c(t), b(t)+d(t)$ or $b(t)-d(t)$ is vanished on Γ, is a generalized zero divisor.

Obviously, $L(L_p(\Gamma,\rho))\setminus\mathfrak{T}$ is a (noncommutative) Banach algebra including $\hat{\mathfrak{A}}$.

5°. An element $A\in\mathfrak{A}$ is invertible in $L(L_p(\Gamma,\rho))\setminus\mathfrak{T}$ if and only if it is invertible in $\hat{\mathfrak{A}}$.

In fact, let A be invertible in $L(L_p(\Gamma,\rho))\setminus\mathfrak{T}$ and suppose it is not invertible in $\hat{\mathfrak{A}}, A^{-1}\notin\hat{\mathfrak{A}}$. Then, by virtue of 4^o, A is a generalized zero divisor. But this is impossible, since in this case the invertible operator A should be a generalized zero divisor in $L(L_p(\Gamma,\rho))\setminus\mathfrak{T}$.

IV. Let us approach the problem of regularization of singular integral operators with shift ω, $A=aI+bS=(cI+dS)V+T$. It is easy to observe that the operator A admits a regularization in algebra $L(L_p(\Gamma,\rho))$ if and only if the element $\{aI+bS+(cI+dS)V\}\in\hat{\mathfrak{A}}$ is invertibile in $L(L_p(\Gamma,\rho))\setminus\mathfrak{T}$. In order to apply assertions of Theorem 5 and property 5^o we use the operators $P=\frac{1}{2}(I+S)$, $Q=\frac{1}{2}(I-S)$, $I=P+Q$ and $S=P-Q$. Then the operators A is transcribed as, $A=\alpha P+\beta Q+$

$(\delta P+\gamma Q)V+T$, where $\alpha = a+b, \beta = a-b, \delta = c+d, \gamma = c-d$. From Theorem 5 and property 5^o it results that $\{\alpha P + \beta Q + (\delta P + \gamma Q)V\}$ is invertible in $L(L_p(\Gamma,\rho)) \setminus \mathfrak{T}$ if and only if the functions $\alpha^2(t) - \delta^2(t)$ and $\beta^2(t) - \gamma^2(t)$ do not vanish on Γ. In other words, a singular integral operator A with shift, $A = ai + bS + (cI + dS)V + T$, admits a regularization in $L(L_p(\Gamma,\rho))$ if only if

$$\alpha^2(t) - \delta^2(t) = (a(t) + b(t))^2 - (c(t) + d(t))^2 \neq 0,$$

$$\beta^2(t) - \gamma^2(t) = (c(t) + d(t))^2 - (c(t) + d(t))^2 \neq 0.$$

Thus, condition (4) of Theorem 1 are satisfied. With the help of judgements used in the proof of Theorem 5 it is supplementary obtained that $AR = I + T_1$ and $RA = I + T_2$, where R is defined by relation (5) and T_1, T_2 are compact operators.

Theorem 6. *The operator $A = \alpha P + \beta Q + (\delta P + \gamma Q)V + T$ admits an equivalent regularization if and only if the following conditions*

$$\alpha^2(t) - \delta^2(t) \neq 0, \beta^2(t) - \gamma^2(t) \neq 0, ind\frac{\alpha^2(t) - \delta^2(t)}{\beta^2(t) - \gamma^2(t)} \leq 0$$

are verified. Under these conditions

$$IndA = -\frac{1}{2} ind\frac{\alpha^2(t) - \delta^2(t)}{\beta^2(t) - \gamma^2(t)}.$$

For $IndA < 0$ all solutions to equation $Ax = y$ are obtained from the relation $x = Rz$, where z runs all solutions to equation $RAz = y$ and R is defined by (5).

Cases when the function of shifting, ω, changes the orientation of contour Γ and systems of singular integral equation with shift will be approached, possibly, in other works of the author.

References

[1] Gelfand I. *Commutative normed rings.* Chelsea, New York, 1964.

[2] Gohberg I. *On an application of the theory of normed rings to singular integral equations.* Uspehi mat. nauk, 1964, **19**, vyp. 1, p. 71–124 (In Russian).

[3] Gahov F. *Boundary value problems.* Moskva, Nauka, 1977 (In Russian).

[4] Krupnik N. *Banach algebras with symbol and singular integral operators.* Birkhauser, Basel-Boston, 1987.

[5] Litvinchiuk G. *Introduction to the Theory of Singular Integral Operators with Shift.* Kluwer, 2001.

[6] Neagu V. *Banach algebras generated by singular integral operators.* CEP USM, 2005.

State University of Moldova
str. A.Mateevici 60
MD-2009 Chishinau
Moldova
E-mail: *neagu@usm.md*

Received May 25, 2006

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(51), 2006, Pages 75–86
ISSN 1024–7696

On a family of Hamiltonian cubic planar differential systems

Gheorghe Tigan

Abstract. A class of planar perturbed Hamiltonian systems are studied in the present work in order to identify the limit cycles. The closed curves of the unperturbed associated Hamiltonian system are described. Using the Abelian integral method we find the detection functions. Numerical explorations are presented to illustrate the distribution of the limit cycles.

Mathematics subject classification: 34C07, 37G15.
Keywords and phrases: Hamiltonian systems, limit cycles, Abelian integral.

1 Introduction

In [1] the authors study a family of dynamical systems given by

$$\dot{x} = yP_0(x), \ \dot{y} = -x + P_2(x)y^2 + P_3(x)y^3, \tag{1}$$

for some particular polynomials P_0, P_2, P_3 proving the existence of eight limit cycles. Considering first a more general class of dynamical systems of the form

$$\dot{x} = yP_0(x,y), \ \dot{y} = P_1(x,y) + P_2(x,y)y^2 + P_3(x,y)y^3, \tag{2}$$

and particularizing $P_0(x,y), P_1(x,y), P_2(x,y), P_3(x,y)$ we get a system given by

$$\dot{x} = 4by\left(-y^2 + ax^2 + 1\right), \ \dot{y} = 4ax\left(x^2 - by^2 - 1\right). \tag{3}$$

This system is a Hamiltonian system with the Hamilton function given by

$$H\left(x,y\right) = -(ax^4 + by^4) + 2abx^2y^2 + 2\left(ax^2 + by^2\right). \tag{4}$$

Indeed, given the Hamilton function (4), the associated Hamiltonian system is

$$\dot{x} = \frac{\partial H}{\partial y}, \ \dot{y} = -\frac{\partial H}{\partial x}, \tag{5}$$

which easily leads to (3). A similar Hamiltonian system has been studied in [2–4]. In this work we study the existence of the limit cycles of some perturbations of the system (5). The problem of the existence of limit cycles in a planar differential system of a given degree is known as the Hilbert‘s 16th problem. This problem is

---

still unsolved even for the quadratic polynomial differential systems. A method used to deal with such problems is the Abelian integral method [5]. Other methods can be found in [6, 7] and [8]. Further reading on limit cycles can be found in [9–15]

This paper is organized as follows. In Section 2, we give some preliminary results. In Section 3, we study the global portrait of the unperturbed system. In the last Section 4, we find the numerical values of the detection functions of the perturbed system. Finally, we present the distribution of the limit cycles in a particular representative case.

2 Preliminary results

It is known that one way to produce limit cycles is by perturbing an Hamiltonian system which has one or more centers, in such a way that limit cycles bifurcate in the perturbed system from some of the periodic orbits in the original system.

The following perturbed Hamiltonian system

$$\begin{cases} \dot{x} = y(1 + x^2 - ay^2) + \varepsilon x(ux^n + vy^n - \lambda), \\ \dot{y} = -x(1 - cx^2 + y^2) + \varepsilon y(ux^n + vy^n - \lambda) \end{cases} \tag{6}$$

where $ac > 1, a > c > 0, 0 < \varepsilon \ll 1$, u, v, λ are the real parameters and $n = 2k, k$ integer positive, has been studied in [16–19].

The following result is reported in [17].

Theorem 1. *Consider the perturbed Hamiltonian system*

$$\dot{x} = \frac{\partial H}{\partial y} + P(x, y, \varepsilon), \quad \dot{y} = -\frac{\partial H}{\partial x} + Q(x, y, \varepsilon). \tag{7}$$

Assume that $P(x, y, \varepsilon)$, $Q(x, y, \varepsilon)$ are polynomials of given degree (perturbations of the Hamiltonian system), $P(x, y, 0) = Q(x, y, 0) = 0$, the closed curve $\Gamma^h : H(x, y) = h$ defined by the Hamiltonian $H(x, y)$ of the system (7) is a periodic orbit that extends outside as h increasing, and $\Gamma^h(D)$ is the area inside Γ^h. If there exists h_0 such that function

$$A(h) = \int\limits_{\Gamma^h(D)} [P''_{x\varepsilon}(x, y, 0) + Q''_{y\varepsilon}(x, y, 0)]dxdy \tag{8}$$

satisfies $A(h_0) = 0, A'(h_0) \neq 0, \varepsilon A'(h_0) < 0 (> 0)$, then system (7) has only one stable (unstable) limit cycle nearby Γ^{h_0} for ε very small. If Γ^h constricts inside as h increasing, the stability of the limit cycle is opposite with above. If $A(h) \neq 0$, then system (7) has no limit cycle.

The integral $A(h)$ is called the *Abelian integral* [5].
If the form of the system (7) is:

$$\left\{ \begin{array}{l} \dot{x} = \dfrac{\partial H}{\partial y} + \varepsilon x(p(x,y) - \lambda), \\ \dot{y} = -\dfrac{\partial H}{\partial x} + \varepsilon y(q(x,y) - \lambda) \end{array} \right. \tag{9}$$

where $p(0,0) = q(0,0) = 0$, then, using the above Theorem 1, from $A(h) = 0$, we get:

$$\lambda = \lambda(h) = \frac{\int\limits_{\Gamma^h(D)} f(x,y)dxdy}{2 \int\limits_{\Gamma^h(D)} dxdy} \tag{10}$$

where $f(x,y) = xp'_x(x,y) + yp'_y(x,y) + p(x,y) + q(x,y)$.

This function $\lambda(h)$ is called *the detection function* of the system (9).

From Theorem 1 and using the detection function $\lambda(h)$ we get the following result:

Proposition 2. *a) If* $(h_0, \lambda(h_0))$ *is an intersecting point of line* $\lambda = \lambda_0$ *and the detection curve* $\lambda = \lambda(h)$, *and* $\lambda'(h_0) > 0 (< 0)$, *then system (9) has only one stable (unstable) limit cycle nearby* Γ^{h_0} *when* $\lambda = \lambda_0$; *b) If line* $\lambda = \lambda_0$ *and the detection curve* $\lambda = \lambda(h)$ *have no intersecting point, then the system (9) has no limit cycle when* $\lambda = \lambda_0$. *If the* Γ^h *constricts inside as* h *increasing, the stability of the limit cycle is opposite with above.*

Consider in the following the perturbed Hamiltonian system given by:

$$\left\{ \begin{array}{l} \dot{x} = 4by\left(-y^2 + ax^2 + 1\right) + P\left(x,y,\varepsilon\right), \\ \dot{y} = 4ax\left(x^2 - by^2 - 1\right) + Q\left(x,y,\varepsilon\right) \end{array} \right. \tag{11}$$

where $P\left(x,y,\varepsilon\right) = \varepsilon x((n+2)vy^n - c\dfrac{s+1}{r+1}x^r y^s - ux^2y^2 - \lambda)$, $Q\left(x,y,\varepsilon\right) = \varepsilon y((n+2)ux^n + cx^r y^s - ux^2y^2 - \lambda)$, $r + s = n$, $0 < a < b$, $ab < 1$, $0 < \varepsilon \ll 1$ u, v, λ, c are the real parameters and $n = 2k, k$ positive integer.

3 The behavior of the unperturbed system

The unperturbed system corresponding to system (11) is the system (11) in the case $\varepsilon = 0$.

System (3) has nine finite singular points and they are:

$$A_1\left(\frac{1}{ab-1}\sqrt{(1-ab)(1+b)}, \quad \frac{1}{ab-1}\sqrt{(1-ab)(a+1)}\right),$$

$$A_2\left(\frac{1}{ab-1}\sqrt{(1-ab)(1+b)}, \quad \frac{1}{ab-1}\sqrt{(1-ab)(a+1)}\right),$$

$$A_3\left(-\frac{1}{ab-1}\sqrt{(1-ab)(1+b)}, \quad \frac{1}{ab-1}\sqrt{(1-ab)(a+1)}\right),$$

$$A_4\left(-\frac{1}{ab-1}\sqrt{(1-ab)\,(1+b)},\quad \frac{1}{ab-1}\sqrt{(1-ab)\,(a+1)}\right),$$
$$B_{1,2}\,(0,\pm1)\,;\quad C_{1,2}\,(\pm1,0)\quad \text{and}\quad O\,(0,0)\,.$$

By computing eigenvalues at each singular point we have that O, A_1, A_2, A_3, A_4 are centers while the other singular points B_1, B_2, C_1, C_2 are hyperbolic saddle points.

As we said above, the Hamiltonian of the system (3) is

$$H\,(x,y) = -(ax^4+by^4)+2abx^2y^2+2\left(ax^2+by^2\right)=h. \tag{12}$$

Hence $H\,(A_i)=\dfrac{a+b+2ab}{1-ab},\;\; i=\overline{1,4},\;\; H\,(C_k)=a,\;\; H\,(B_k)=b,\;\; k=1,2$ and $H\,(O)=0$.

Because $0<a<b$ we get that: $H\,(O)<H\,(B_1)<H\,(C_1)<H\,(A_1)\,.$

In polar coordinates, $x=r\cos\theta, y=r\sin\theta$, the system (3) becomes:

$$r'=-r^3p'(\theta)+rq'(\theta),\quad \theta'=-q\,(\theta)+r^2p(\theta) \tag{13}$$

and the Hamiltonian (12)

$$H(r,\theta)=-r^4p(\theta)+2r^2q(\theta)=h, \tag{14}$$

where

$$p(\theta)=a\cos^4\theta+b\sin^4\theta-2ab\cos^2\theta\sin^2\theta,\quad q(\theta)=a\cos^2\theta+b\sin^2\theta. \tag{15}$$

Remark. The equilibrium points A_1, A_2, A_3, A_4 lie on the lines $d_\pm\,:\,\theta=\pm\arctan\sqrt{\dfrac{a+1}{b+1}}$.

Theorem 3. *([2, 20])*

As h varies on the real line, the level curves defined by Hamiltonian (14) can be divided as follows:

1. $\Gamma_1^h : -\infty<h<0$, this corresponds to an orbit that surrounds all critical points, Fig. 1 a).

2. $\Gamma_2^h\cup\Gamma_1^h : 0<h<a$, this corresponds to an orbit $\left(\Gamma_2^h\right)$ that surrounds only the origin and a curve of type $\left(\Gamma_1^h\right)$, Fig. 1 b)-a).

3. $\Gamma_3^h : a<h<b$, this corresponds to two symmetric orbits that do not intersect the Ox axis but encircle the rest of critical points, Fig. 2 b).

If $h=a$ we get four heteroclinic orbits connecting two critical fixed points C_1 and C_2, Fig.2 a).

4. $\Gamma_4^h\;:\; b<h<\dfrac{a+b+2ab}{1-ba}$, this corresponds to four orbits that surround respectively the $A_i, i=\overline{1,4}$, equilibrium points, Fig.3 b). If $h=b$ we get four homoclinic orbits connecting two critical fixed points B_1 and B_2, Fig.3 a).

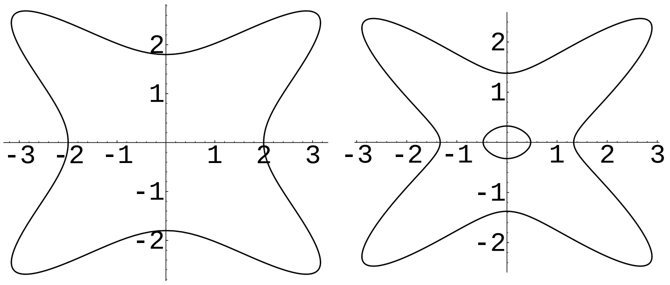

Figure 1. Orbit of type a) L_1 (left) b) L_2 and L_1 (right) .

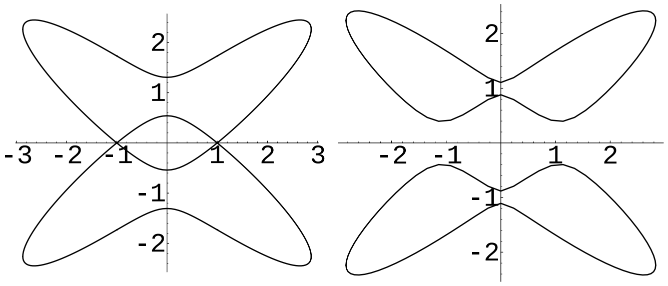

Figure 2. a) Four heteroclinic orbits connecting two critical points C_1 and C_2 (left) b) Two orbits of type L_3 (right) .

4 Numerical explorations

In this section for precisely chosen parameters a and b we numerically compute the detection curves. The four detection curves for a given h depend on the parameters u and v, (see Tables 1-4). Then for two given values of u and v, the detection curves can be plotted on the (h, λ)-plane, as can be seen in Figs.4, 5. By the Proposition 2 and the detection function graphs, we deduce the number and distribution of limit cycles. We consider here the case $n = 8$, that corresponds to perturbation of nine order.

From (14), we get

$$r_{1,2} = r_{\pm}^2(\theta, h) = \frac{1}{p(\theta)} \left(q(\theta) \pm \sqrt{q^2(\theta) - hp(\theta)} \right) \tag{16}$$

and from $\dot{\theta} = -1 + r^2 p(\theta) = 0$ we have:

$$\theta_1(h) = \frac{1}{2} \arccos \left[\left(-B + \sqrt{B^2 - AC} \right) / A \right],$$

$$\theta_2(h) = \frac{1}{2} \arccos \left[\left(-B - \sqrt{B^2 - AC} \right) / A \right]$$

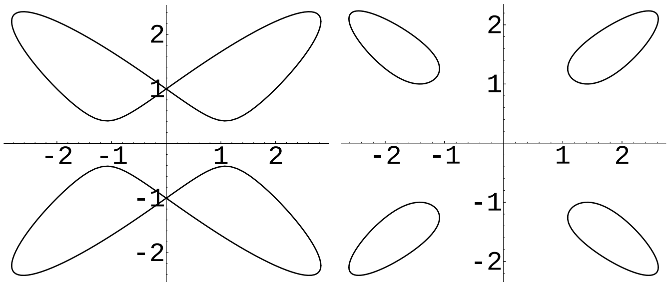

Figure 3. a) Four homoclinic orbits connecting the critical points B_1 and B_2 (left) b) Four orbits of type L_4 (right) .

where $A = a^2 - ah - 2ab\,(h+1) + b^2 - bh$, $B = a^2 - ah - b^2 + bh$, $C = a^2 - ah + 2ab\,(h+1) + b^2 - bh$.

From the form of the perturbation terms

$P\,(x,y,\varepsilon) = \varepsilon x((n+2)vy^n - c\dfrac{s+1}{r+1}x^r y^s - ux^2y^2 - \lambda)$, $Q\,(x,y,\varepsilon) = \varepsilon y((n+2)ux^n + cx^r y^s - ux^2y^2 - \lambda)$ we have $\dfrac{\partial^2 P\,(x,y,\varepsilon)}{\partial x \partial \varepsilon} + \dfrac{\partial^2 Q\,(x,y,\varepsilon)}{\partial y \partial \varepsilon} = u\,(n+2)\,x^n + v\,(n+2)\,y^n - 6ux^2y^2 - 2\lambda$.

Therefore, the four detection functions corresponding to the four closed curves Γ_j^h, $j = \overline{1,4}$, for the above perturbations are :

$$\lambda_j(h) = \frac{\int\limits_{\Gamma_j^h(D)} \left[(n+2)\,(ux^n + vy^n) - 6ux^2y^2\right] dxdy}{2 \int\limits_{\Gamma_j^h(D)} dxdy}, j = \overline{1,4} \tag{17}$$

In polar coordinates and for $a = \dfrac{1}{3}, b = 2$ and $n = 8$, (17) leads to:

$$\lambda_1(h) = \frac{\int\limits_0^{2\pi} \left(r_1^4\,(\theta,h)\,g\,(\theta) - r_1^3\,(\theta,h)\,g_1\,(\theta)\right) d\theta}{\int\limits_0^{2\pi} r_1\,(\theta,h)\,d\theta},\ -\infty < h < 1/3,$$

$$\lambda_2(h) = \frac{\int\limits_0^{2\pi} \left(r_2^4\,(\theta,h)\,g\,(\theta) - r_2^3\,(\theta,h)\,g_1\,(\theta)\right) d\theta}{\int\limits_0^{2\pi} r_2\,(\theta,h)\,d\theta},\ 0 < h < 1/3,$$

$$\lambda_3(h) = \frac{\int\limits_{\theta_2(h)}^{\pi-\theta_2(h)} \left[\left(r_1^4(\theta,h) - r_2^4(\theta,h)\right) g(\theta) - \left(r_1^3(\theta,h) - r_2^3(\theta,h)\right) g_1(\theta)\right] d\theta}{\int\limits_{\theta_1(h)}^{\pi-\theta_1(h)} \left(r_1(\theta,h) - r_2(\theta,h)\right) d\theta},$$

$$1/3 < h < 2,$$

$$\lambda_4(h) = \frac{\int\limits_{\theta_1(h)}^{\theta_2(h)} \left[\left(r_1^4(\theta,h) - r_2^4(\theta,h)\right) g(\theta) - \left(r_1^3(\theta,h) - r_2^3(\theta,h)\right) g_1(\theta)\right] d\theta}{\int\limits_{\theta_1(h)}^{\theta_2(h)} \left(r_1(\theta,h) - r_2(\theta,h)\right) d\theta},$$

$$2 < h < 11,$$

where $g(\theta) = u\cos^8\theta + v\sin^8\theta, g_1(\theta) = u\cos^2\theta\sin^2\theta$ and $r_{1,2}(\theta,h) = r_{\pm}^2(\theta,h)$.

Table 1
The values of the detection function $\lambda_1(h)$ when $a = 1/3,\ b = 2,\ n = 8$

h	$\lambda_1(h)$	h	$\lambda_1(h)$	h	$\lambda_1(h)$
-20	6.87u+1800.40v	-19	6.21u+1676.58v	-18	5.58u+1556.51v
-17	4.98u+1440.23v	-16	4.41u+1327.76v	-15	3.87u+1219.13v
-14	3.35u+1114.37v	-13	2.87u+1013.53v	-12	2.43u+916.64v
-11	2.01u+823.75v	-10	1.62u+734.91v	-9	1.27u+650.16v
-8	0.95u+569.56v	-7	0.66u+493.20v	-6	0.41u+421.13v
-5	0.19u+353.45v	-4	0.007u+290.27v	-3	-0.139u+231.71v
-2	-0.247u+177.93v	-1	-0.314u+129.12v	0.	-0.335u+85.66v
0.01	-0.335u+ 85.26v	0.04	-0.335u+ 84.05v	0.07	-0.335u+ 82.85v
0.1	-0.335u+ 81.65v	0.13	-0.334u+ 80.47v	0.16	-0.334u+ 79.29v
0.19	-0.334u+ 78.12v	0.22	-0.334u+ 76.96v	0.25	-0.333u+ 75.81v
0.31	-0.332u+ 73.56v	0.32	-0.332u+ 73.20v	0.33	-0.332u+72.85v

Table 2
The values of the detection function $\lambda_2(h)$ when $a = 1/3,\ b = 2,\ n = 8$

h	$\lambda_2(h)$	h	$\lambda_2(h)$
0.	0	0.033	$-3.32^{-6}u + 2.77^{-8}v$
0.066	$-0.0000129u + 4.49^{-7}v$	0.099	$-0.0000274u + 2.30^{-6}v$
0.132	$-0.0000432u + 7.37^{-6}v$	0.165	-0.0000530u +0.0000182v
0.198	-0.0000935u+ 0.0000305v	0.231	-0.0003127u+ 0.0000558v
0.264	-0.0003479u+ 0.0000938v	0.297	-0.0002976u+0.0001508v
0.33	0.0001670u+0.0002286v		

Table 3
The values of the detection function $\lambda_3(h)$ when $a = 1/3,\ b = 2,\ n = 8$

h	$\lambda_3(h)$	h	$\lambda_3(h)$	h	$\lambda_3(h)$
0.33	-0.348u+ 76.25v	0.43	-0.352u+ 73.63v	0.53	-0.353u+ 70.60v
0.63	-0.353u+ 67.42v	0.73	-0.352u+ 64.15v	0.83	-0.350u+ 60.84v
0.93	-0.347u+ 57.48v	1.03	-0.344u+ 54.10v	1.13	-0.340u+ 50.69v
1.23	-0.335u+ 47.25v	1.33	-0.329u+ 43.78v	1.43	-0.323u+ 40.27v
1.53	-0.315u+ 36.70v	1.63	-0.306u+ 33.03v	1.73	-0.296u+ 29.20v
1.83	-0.283u+ 25.07v	1.93	-0.269u+ 20.24v	1.98	-0.261u+ 17.021v

Table 4
The values of the detection function $\lambda_4(h)$ when $a = 1/3,\ b = 2,\ n = 8$

h	$\lambda_4(h)$	h	$\lambda_4(h)$	h	$\lambda_4(h)$
2	-0.259u+ 15.35v	2.1	-0.223u+ 9.40v	2.12	-0.213u+ 8.71v
2.14	-0.202u+ 8.10v	2.16	-0.191u+ 7.55v	2.2	-0.168u+ 6.61v
2.4	-0.055u+ 3.76v	2.6	0.044u+2.393v	2.8	0.126u+1.641v
3	0.193u+1.189v	3.2	0.246u+0.898v	3.6	0.320u+0.563v
4	0.365u+0.386v	4.4	0.391u+0.281v	4.8	0.403u+0.215v
5	0.405u+0.190v	5.2	0.405u+0.170v	5.4	0.404u+0.152v
5.6	0.401u+0.138v	5.8	0.397u+0.125v	6.	0.392u+0.114v
6.4	0.380u+ 0.096v	6.8	0.364u+0.082v	7.2	0.347u+ 0.071v
7.6	0.327u+0.062v	8.	0.307u+0.054v	8.4	0.285u+0.048v
8.8	0.263u+ 0.042v	9.2	0.240u+0.038v	9.6	0.216u+0.034v
10	0.192u+0.030v	10.4	0.168u+0.027v	10.8	0.143u+0.024v

From Tables 1–4 we have the discrete values of the four detection functions, which can now be plotted as shown in Figs. 4, 5. Using Proposition 2 and from these Figures (4, 5), one gets the following theorem:

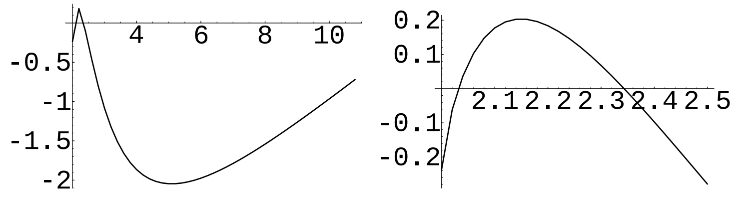

Figure 4. Detection curve λ_4 of system (11) for $a = 1/3,\ b = 2,\ n = 8,\ u = -5$ and $v = -0.1$.

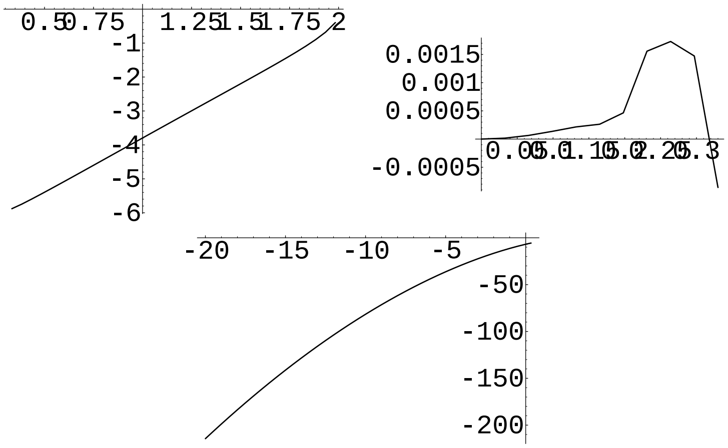

Figure 5. Detection curve λ_3 (left), λ_2 (middle), λ_1 (right) of system (11) for $a = 1/3,\ b = 2,\ n = 8,\ u = -5$ and $v = -0.1$.

Theorem 4. *For $a = 1/3,\ b = 2,\ n = 8,\ u = -5,\ v = -0.1$ and $0 < \varepsilon \ll 1$, we have the following distribution of limit cycles:*

a) If $\lambda < -5.88$, the system (11) has at least one limit cycle in the neighborhood of the orbit of type Γ_1^h, Fig.6a),

b) If $-5.88 < \lambda < -2.045$, the system (11) has at least two limit cycles in the neighborhood of each orbit of type Γ_3^h, Fig.6b),

c) If $-2.045 < \lambda < -0.72$, the system (11) has at least ten limit cycles, two of which in the neighborhood of each orbit of type Γ_4^h and one in the neighborhood of each orbit of type Γ_3^h, Fig.6c),

d) If $-0.72 < \lambda < -0.236$, the system (11) has at least six limit cycles, one of which in the neighborhood of each orbit of type Γ_4^h and Γ_3^h, Fig.7a),

e) If $-0.236 < \lambda < -0.000858$, the system (11) has at least eight limit cycles, two of which in the neighborhood of each orbit of type Γ_4^h, Fig.7b),

f) If $-0.000858 < \lambda < 0$, the system (11) has at least nine limit cycles, two of which in the neighborhood of each orbit of type Γ_4^h and one in the neighborhood of the orbit of type Γ_2^h, Fig.7c),

g) If $0 < \lambda < 0.00173$, the system (11) has at least ten limit cycles, two of which in the neighborhood of each orbit of type Γ_4^h and Γ_2^h, Fig.8, *h) If* $0.00173 < \lambda < 0.202$*, the system (11) has at least eight limit cycles, two of which in the neighborhood of each orbit of type* Γ_4^h*, Fig.7b),*

i) If $\lambda > 0.202$*, the system (11) has no limit cycles,*

where the last point of λ_1 *is* $(0.33, -5.88)$*, the first point of* λ_2 *is* $(0, 0)$*, the maximum of* λ_2 *is* $(0.264, 0.00173)$*, the last point of* λ_2 *is* $(0.33, -0.000858)$*, the first point of* λ_3 *is* $(0.33, -5.88)$*, the last point of* λ_3 *is* $(2, -0.236)$*, the first point of* λ_4 *is* $(2, -0.236)$*, the maximum of* λ_4 *is* $(2.14, 0.202)$*, the minimum of* λ_4 *is* $(5, -2.04)$*, and the last point computed of* λ_4 *is* $(10.8, -0.72)$*.*

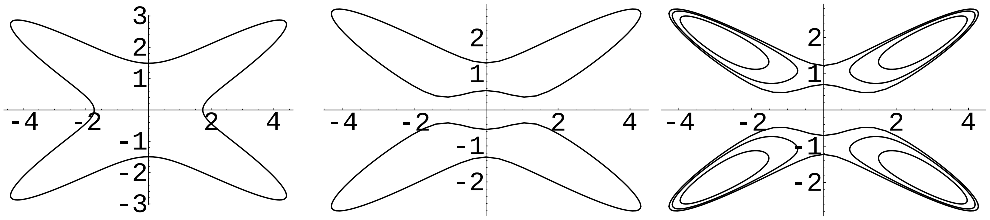

Figure 6. Distribution diagram corresponding to: a) one (left), b) two (middle) c) ten (right) limit cycles of system (11).

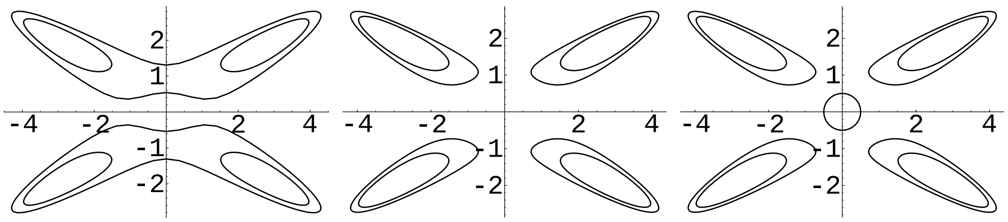

Figure 7. Distribution diagram corresponding to: a) six (left), b) eight (middle) c) nine (right) limit cycles of system (11).

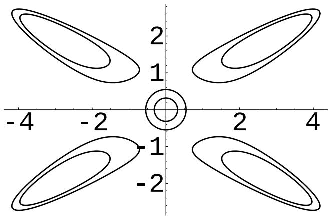

Figure 8. Distribution diagram corresponding to ten limit cycles of system (11).

5 Conclusion

In this paper we investigated a planar perturbed Hamiltonian system. The system possesses nine equilibria, five of type centers and four of type hyperbolic. The unperturbed system displays four different level curves, depending on the values of the parameter h on the real line. The Abelian integral method was employed to study the perturbed Hamiltonian system. By numerical explorations we illustrated the existence, number and distribution of limit cycles. In further papers, naturally we intend to deal with perturbations of higher order as $n = 10$ and $n = 12$.

6 Acknowledgements

This work was partially supported through a Romanian Government Research Studentship offered by CNBSS and a European Community Marie Curie Fellowship, in the framework of the CTS, contract number HPMT-CT-2001-00278.
The author is grateful to the anonymous referee for his eyes of eagle.

References

[1] Bondar Y.L., Sadovskii A.P. *Variety of the center and limit cycles of a cubic sistem, which is reduced to Lienard form.* Buletinul Academiei de Stiinte a Republicii Moldova, Matematica, 2004, N 3(46), p. 71–90.

[2] Tigan G. *Thirteen limit cycles for a class of Hamiltonian systems under seven-order perturbed terms.* Chaos, Soliton and Fractals, 2007, **31**, p. 480–488.

[3] Tigan G. *Existence and distribution of limit cycles in a Hamiltonian system.* Applied Mathematics E-Notes, 2006, **6**, p. 176-185.

[4] Tigan G. *Detecting the limit cycles for a class of Hamiltonian systems under thirteen-order perturbed terms.* http://arxiv.org/abs/math/0512342.

[5] Blows T.R., Perko L.M. *Bifurcation of limit cycles from centers and separatrix cycles of planar analytic systems.* SIAM Rev., 1994, **36**, p. 341–376.

[6] Chows S.N., Li C., Wang D. *Normal Forms and Bifurcation of Planar Vector Fields.* Cambridge University Press, 1994.

[7] Andronov A.A. *Theory of bifurcations of dynamical systems on a plane.* Israel program for scientific translations, Jerusalem 1971.

[8] Giacomini H., Llibre J., Viano M. *On the shape of limit cycles that bifurcate from Hamiltonian centers.* Nonlinear Anal. Theory Methods Appl., 1997, **41**, p. 523–537.

[9] Yanqian Y. *Theory of Limit Cycles.* Translations of Math. Monographs, vol. 66, Amer. Math. Soc.,Providence, RI, 1986.

[10] Li C.F., Li J.B. *Distribution of limit cycles for planar cubic Hamiltonian systems.* Acta Math Sinica, 1985, **28**, p. 509–521.

[11] Li J., Huang Q. *Bifurcation of limit cycles forming compound eyes in the cubic system.* Chinese Ann. Math., 1987, N 8B(4), p. 391–403.

[12] Viano M., Llibre J., Giacomini H.. *Arbitrary order bifurcations for perturbed Hamiltonian planar systems via the reciprocal of an integrating factor.* Nonlinear Analysis, 2002, **48**, p. 117–136.

[13] Li J.B., Liu Z.R. *On the connection between two parts of Hilbert's 16-th problem and equvariant bifurcation problem.* Ann. Diff. Eqs., 1998, N 14(2), p. 224–235.

[14] Giacomini H., Llibre J., Viano M. *On the nonexistence, existence and uniqueness of limit cycles.* Nonlinearity, 1996, **9**, p. 501–516.

[15] Hong Z., Chen W.,Tonghua Z.. *Perturbation from a cubic Hamiltonian with three figure eight-loops.* Chaos, Solitons and Fractals, 2004, **22**, p. 61–74.

[16] Li J.B., Liu Z.R. *Bifurcation set and limit cycles forming compound eyes in a perturbed Hamiltonian system.* Publ. Math., 1991, **35**, p. 487–506.

[17] Cao H., Liu Z., Jing Z. *Bifurcation set and distribution of limit cycles for a class of cubic Hamiltonian system with higher-order perturbed terms.* Chaos, Solitons and Fractals, 2000, **11**, p. 2293–2304.

[18] Tang M., Hong X. *Fourteen limit cycles in a cubic Hamiltonian system with nine-order perturbed term.* Chaos, Solitons and Fractals 2002, **14**, p. 1361–1369.

[19] Tigan G. *Eleven limit cycles in a Hamiltonian system.* Differential Geometry–Dynamical Systems Journal, 2006, **8**, p. 268–277.

[20] Tigan G. *Limit cycles in a perturbed Hamiltonian system.* Preprint.

Department of Mathematics
Politehnica University of Timisoara
P-ta Victoriei, Nr.2
300006, Timisoara, Timis, Romania
E-mail: *gheorghe.tigan@mat.upt.ro*

Received March 16, 2006

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(51), 2006, Pages 87–94
ISSN 1024–7696

The solvability and properties of solutions of an integral convolutional equation

A.G. Scherbakova

Abstract. The work defines the conditions of solvability of one integral convolutional equation with degreely difference kernels. This type of integral convolutional equations was not studied earlier, and it turned out that all methods used for the investigation of such equations with the help of Riemann boundary problem at the real axis are not applied there. The investigation of such type equations is based on the investigation of the equivalent singular integral equation with the Cauchy type kernel at the real axis. It is determined that the equation is not a Noetherian one. Besides, there shown the number of the linear independent solutions of the homogeneous equation and the number of conditions of solvability for the heterogeneous equation. The general form of these conditions is also shown and there determined the spaces of solutions of that equation. Thus the convolutional equation that wasn't studied earlier is presented at that work and the theory of its solvability is built there. So some new and interesting theoretical results are got at that paper.

Mathematics subject classification: 45E05, 45E10.
Keywords and phrases: Integral convolutional equation, singular integral equation, Cauchy type kernel, a Noetherian equation, conditions of solvability, index, the number of the linear independent solutions, spaces of solutions..

The present work is devoted to defining conditions of solvability and some properties of solutions of the next integral equation

$$P_m(x)\varphi(x)+\frac{1}{\sqrt{2\pi}}\int\limits_0^{+\infty} k(t,x-t)\varphi(t)\,dt+\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{0} n(t,x-t)\varphi(t)\,dt = h(x), x\in \mathbf{R}, \quad (1)$$

where $\mathbf{R}$ is the real axis;

$$k(t,x-t)=\sum_{j=0}^{n} k_j(x-t)t^j, n(t,x-t)=\sum_{\nu=0}^{s} n_\nu(x-t)t^\nu,$$

where

$$P_m(x)=\sum_{k=0}^{m} A_k x^k$$

is the known polynomial with degree m and $k_j(x), n_\nu(x)\in \mathbf{L}$, $j=\overline{1,n}$, $\nu=\overline{1,s}$, $h(x)\in \mathbf{L}_2$ are known functions.

Let $D^+ = \{z \in \mathbf{C} : \mathbf{Imz} > \mathbf{0}\}$ be an upper half plane and $D^- = \{z \in \mathbf{C} : \mathbf{Imz} < \mathbf{0}\}$ be a lower half plane of the complex plane $\mathbf{C}$. According to the properties of Fourier transformation [1, p. 77], [2, p. 16] the investigation of the equation (1) reduces to the investigation of the following differential boundary problem

$$\left[\sum_{k=0}^{m} A_k(-1)^k\Phi^{+(k)}(x) + \sum_{j=0}^{n}(-1)^j K_j(x)\Phi^{+(j)}(x)\right] -$$
$$-\left[\sum_{k=0}^{m} A_k(-1)^k\Phi^{-(k)}(x) + \sum_{\nu=0}^{s}(-1)^\nu N_\nu(x)\Phi^{-(\nu)}(x)\right] = H(x), x \in \mathbf{R}, \qquad (2)$$

where $K_j(x), N_\nu(x), H(x)$ are the Fourier transformations of functions $k_j(x), n_\nu(x)$, $h(x)$, $j = \overline{1,n}$, $\nu = \overline{1,s}$ accordingly. $\Phi^{+(p)}(x)$ and $\Phi^{-(q)}(x)$ are the boundary values at $\mathbf{R}$ of the functions $\Phi^{+(p)}(z)$ and $\Phi^{-(q)}(z)$ accordingly, where $\Phi^+(z)$, $\Phi^-(z)$ are unknown functions, which are analytical at the domains D^+ and D^- accordingly. As all the transformations of the differential boundary problem (2) and the equation (1) are identical, then the problem and the equation are equivalent in such a sense that they are solvable or unsolvable at the same time, and there is one and only one solution $\Phi^\pm(x)$ of the differential boundary problem (2) for every solution $\varphi(x)$ of the equation (1) and vice versa. The solutions of the equation (1) are expressed over solutions of the problem (2) according to the formula

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbf{R}} \left[\Phi^+(t) - \Phi^-(t)\right] e^{-ixt}\, dt, x \in \mathbf{R}. \qquad (3)$$

Later on we will consider that the functions $K_j(x), N_\nu(x) \in \mathbf{H}_\alpha^{(r)}$, $r \geq 0, 0 < \alpha \leq 1$, $\mathbf{H}_\alpha^{(0)} = \mathbf{H}_\alpha$, $j = \overline{1,n}$, $\nu = \overline{1,s}$ and the function $H(x) \in \mathbf{L}_2^{(r)}$, $r \geq 0$, $\mathbf{L}_2^{(0)} = \mathbf{L}_2$. As the functions $k_j(x), n_\nu(x) \in \mathbf{L}$, $j = \overline{1,n}$, $\nu = \overline{1,s}$, then according to Riemann-Lebesgue theorem [1, p. 42] $\lim\limits_{x\to\infty} K_j(x) = 0$, $\lim\limits_{x\to\infty} N_\nu(x) = 0$, $j = \overline{1,n}$, $\nu = \overline{1,s}$. The equation (1) is a generalization of a convolutional type equation "with two kernels", and we will study it basing on the investigation of the differential boundary problem (2). The investigation of the solvability of the problem (2) we will do basing on the investigation of the singular integral equation at the real axis. The investigation of the differential boundary problem (2) reduces to the investigation of the singular integral equation with the help of integral representations for the functions and derivatives of them built in [4]. Let construct functions $\Phi^+(z)$ and $\Phi^-(z)$ such that they are analytical at the domains D^+, D^- accordingly and disappearing at infinity. Besides, the boundary values at $\mathbf{R}$ of functions $\Phi^{+(p)}(z)$ and $\Phi^{-(q)}(z)$ satisfy the following condition $\Phi^{+(p)}(x), \Phi^{-(q)}(x) \in \mathbf{L}_2^{(r)}$, $r \geq 0, p \geq 0, q \geq 0$. According to [4] these conditions satisfy such functions as:

$$\Phi^\pm(z) = (2\pi\imath)^{-1} \int\limits_{\mathbf{R}} P^\pm(x,z)\rho(x)\, dx, z \in D^\pm, \qquad (4)$$

where

$$P^+(x,z) = \frac{(-1)^p(x+\imath)^{-p}}{(p-1)!}\times$$

$$\times\left[(x-z)^{p-1}\ln\left(1-\frac{x+\imath}{z+\imath}\right)-\sum_{k=0}^{p-2}d_{p-k-2}(x+\imath)^{k+1}(z+\imath)^{p-k-2}\right],$$

$$x\in\mathbf{R},\mathbf{z}\in\mathbf{D}^+;$$

$$P^-(x,z) = \frac{(-1)^q(x-\imath)^{-q}}{(q-1)!}\times$$

$$\times\left[(x-z)^{q-1}\ln\left(1-\frac{x-\imath}{z-\imath}\right)-\sum_{k=0}^{q-2}l_{q-k-2}(x-\imath)^{k+1}(z-\imath)^{q-k-2}\right],$$

$$x\in\mathbf{R},\mathbf{z}\in\mathbf{D}^-;$$

$$d_{p-k-2}=(-1)^{k+1}\sum_{j=0}^{k}C_{p-1}^{p-1-j}(k-j+1)^{-1},$$

$$l_{q-k-2}=(-1)^{k+1}\sum_{j=0}^{k}C_{q-1}^{q-1-j}(k-j+1)^{-1},$$

where C_n^m are binomial coefficients and the function $\ln\left[1-\frac{x+\imath}{z+\imath}\right]$ is the main branch ($\ln 1 = 0$) of the logarithmic function in the complex plane with the cut connecting such points as $z=-\imath$ and $z=\infty$, following the negative direction of the axis of ordinate. The function $\ln\left[1-\frac{x-\imath}{z-\imath}\right]$ is the main branch ($\ln 1 = 0$) of the logarithmic function in the complex plane with the cut connecting such points as $z=\imath$ and $z=\infty$, following the positive direction of the axis of ordinate. It's easy to verify, that defined by (4) functions $\Phi^+(z)$ and $\Phi^-(z)$ are unique analytical functions in the domains D^+, D^- accordingly. According to the method from the work [4], it's easy to check that the function $\rho(x)\in\mathbf{L}_2$ or the density of the integral representations, is defined uniquely by the functions $\Phi^+(z)$ and $\Phi^-(z)$ and vice versa, so with the help of the given function $\rho(x)\in\mathbf{L}_2$ both functions $\Phi^+(z)$ and $\Phi^-(z)$ are constructing uniquely. According to the work [4] the following representations take place:

$$\Phi^{+(p)}(z) = (2\pi\imath)^{-1}\int_{\mathbf{R}}(z+\imath)^{-p}(x-z)^{-1}\rho(x)\,dx, z\in D^+,$$

$$\Phi^{-q)}(z) = (2\pi\imath)^{-1}\int_{\mathbf{R}}(z-\imath)^{-q}(x-z)^{-1}\rho(x)\,dx, z\in D^-. \qquad (5)$$

We consider the case, when $m=n=s$. Using the properties [4] of partial derivatives of functions $P^{\pm}(x,z)$ with respect to z and Sohotski formulas for derivatives from

[7, p. 42], with the help of the representations (4), (5), we will transform the differential boundary problem (2) into the following singular integral equation and later on investigate it. The singular integral equation is

$$A(x)\rho(x) + B(x)(\pi\imath)^{-1}\int\limits_{\mathbf{R}}(t-x)^{-1}\rho(t)\,dt + (T\rho)(x) = H(x), x \in \mathbf{R}, \tag{6}$$

where

$$A(x) = 0,5(-1)^m\left\{[A_m + K_m(x)](x+\imath)^{-m} + [A_m + N_m(x)](x-\imath)^{-m}\right\},$$
$$B(x) = 0,5(-1)^m\left\{[A_m + K_m(x)](x+\imath)^{-m} - [A_m + N_m(x)](x-\imath)^{-m}\right\}, \tag{7}$$
$$(T\rho)(x) = \int\limits_{\mathbf{R}} K(x,t)\rho(t)\,dt, x \in \mathbf{R}, \tag{8}$$
$$K(x,t) = \frac{1}{2\pi\imath}\left[\sum_{j=0}^{m-1}(-1)^j\left[[A_j + K_j(x)]\frac{\partial^j P^+(t,x)}{\partial x^j} - \ [A_j + N_j(x)]\frac{\partial^j P^-(t,x)}{\partial x^j}\right]\right], \tag{9}$$

and $\dfrac{\partial^j P^{\pm}(t,x)}{\partial x^j}$ is a limiting value at $\mathbf{R}$ of the function $\dfrac{\partial^j P^{\pm}(t,z)}{\partial z^j}$, $j = \overline{0,m-1}$.

1 Lemma. *If the functions* $K_j(x), N_\nu(x) \in \mathbf{H}_\alpha^{(r)}$, $j = \overline{1,n}$, $\nu = \overline{1,s}$, *then the operator*

$$T : \mathbf{L_2^{(r)}} \to \mathbf{L_2^{(r)}},$$

$r \geq 0$, *defined by the formula (8) is a compact operator.*

The proof of lemma follows from Frechet-Kolmogorov-Riesz criterion of compactness of integral operators at the real axis in the space $\mathbf{L_p}, \mathbf{p} > \mathbf{1}$, the properties of functions $P^{\pm}(x,z)$ from [4] and the results of the work [8].

According to the work [9, p. 406], the problem (2) and the singular integral equation (6) are equivalent in such a sense that they are solvable or unsolvable at the same time, and for every solution $\rho(x)$ of the equation (6) there exists maybe unique solution $\Phi^{\pm}(x)$ of the problem (2) and vice versa. In order to make this accord unique it is necessary to set initial conditions for the problem (2). As its solutions $\Phi^{\pm}(x)$ are found in spaces of disappearing at infinity functions, then according to the properties of Cauchy type integral the solutions of the problem (2) are such that $\Phi^{\pm(j)}(\infty) = 0$, $j = \overline{0,m-1}$, that is the initial conditions of (2) are trivial and set automatically. Thus it follows that the differential boundary problem (2) and the singular integral equation (6) are equivalent in such a sense that they are solvable or unsolvable at the same time, and there is one and only one solution $\rho(x)$ of the equation (6) for every solution $\Phi^{\pm}(x)$ of the problem (2) and vice versa. By the force of formula (4), the solutions of the problem (2) are expressed over solutions of the equation (6) according to the formula

$$\Phi^{\pm}(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{\mathbf{R}} P^{\pm}(t,x)\rho(t)\,dt, x \in \mathbf{R}, \tag{10}$$

where $p = q = m$; $P^{\pm}(t, x)$ are the boundary values at $x \in \mathbf{R}$ of functions $P^{\pm}(t, z)$, and $\rho(x)$ is the solution of the equation (6). As the equation (1) and the problem (2) are equivalent, the problem (2) and the singular integral equation (6) are equivalent, too, it follows that the equation (1) and the equation (6) are equivalent in such a sense that they are solvable or unsolvable at the same time, and there is one and only one solution $\varphi(x)$ of the equation (1) for every solution $\rho(x)$ of the equation (6) and vice versa. Thus the solutions of the equation (1) are expressed over solutions of the equation (6) according to the formulas (10), (3). That is why the equation (1) we will call a Noetherian if the equation (6) is a Noetherian one.

2 Theorem. *The equation (1) is not a Noetherian one.*

Proof. According to the work [7, p. 208–212] the equation (6) is a Noetherian one if and only if when $A(x) + B(x) \neq 0$, $A(x) - B(x) \neq 0$ at $x \in \mathbf{R}$. From the formula (7) it follows that

$$A(x) + B(x) = (-1)^m [A_m + K_m(x)](x + \imath)^{-m},$$

$$A(x) - B(x) = (-1)^m [A_m + N_m(x)](x - \imath)^{-m}.$$

So we have got that the functions $A(x)+B(x)$, $A(x)-B(x)$ have a null at least with order m in infinity. It means that the equation (6) is not a Noetherian one. Then as the equations (1) and (6) are equivalent, the equation (1) is not a Noetherian one, too. The theorem is proved. $\square$

Firstly we consider the case when $A_m + K_m(x) \neq 0$, $A_m + N_m(x) \neq 0$ at $\mathbf{R}$. Let determine $\varkappa = ind \dfrac{A_m + N_m(x)}{A_m + K_m(x)}$.

3 Theorem. *Let the functions $k_j(x), n_\nu(x) \in \mathbf{L}$, $j, \nu = \overline{1, m}$, $h(x) \in \mathbf{L}_2$; the functions $K_j(x), N_\nu(x) \in \mathbf{H}_\alpha^{(r)}$, $j, \nu = \overline{1, m}$, $H(x) \in \mathbf{L}_2^{(r)}$, $r \geq m$; $A_m + K_m(x) \neq 0$, $A_m + N_m(x) \neq 0$ at $\mathbf{R}$. If $\varkappa - m \geq 0$, then the homogeneous equation (1) has not less than $\varkappa - m$ linearly independent solutions; the heterogeneous equation (1) is an unconditionally solvable one and its general solution depends upon not less than $\varkappa - m$ arbitrary constants. If $\varkappa - m < 0$, then generally speaking the heterogeneous equation (1) is an unsolvable one. It will be a solvable one when not less than $m - \varkappa$ conditions of solvability*

$$\int_{\mathbf{R}} H(x)\psi_j(x)\, dt = 0, \tag{11}$$

will be executed. Here $H(x)$ is a right part of the equation (6), and $\psi_j(x)$ are linearly independent solutions of the homogeneous equation

$$A(x)\psi(x) - (\pi\imath)^{-1} \int_{\mathbf{R}} (t - x)^{-1} B(t)\psi(t)\, dt + \int_{\mathbf{R}} K(t, x)\psi(t)\, dt = 0,$$

allied to the equation (6), where the functions $A(x), B(x), K(x, t)$ are defined by the formulas (7), (9) accordingly.

Proof. According to the work [7, p. 208–212] if $\varkappa - m \geq 0$, then the homogeneous equation (6) has not less than $\varkappa - m$ linearly independent solutions; the heterogeneous equation (6) is an unconditionally solvable one and its general solution depends upon not less than $\varkappa - m$ arbitrary constants. If $\varkappa - m < 0$, then generally speaking the heterogeneous equation (6) is an unsolvable one. It will be a solvable one when not less than $m - \varkappa$ conditions of solvability (11) will be executed. As the equations (6) and (1) are equivalent, then theorem is proved. □

According to the work [2, p. 262], let define by $\mathbf{L}_2[-\mu;0]$ the space of functions $\varphi(x) \in \mathbf{L}_2$ which satisfy such condition as $(x+\imath)^{\mu}\varphi(x) \in \mathbf{L}_2$.

4 Theorem. *Let the functions* $k_j(x), n_\nu(x) \in \mathbf{L}$, $j,\nu = \overline{1,m}$, $h(x) \in \mathbf{L}_2$; *the functions* $K_j(x), N_\nu(x) \in \mathbf{H}_\alpha^{(r)}$, $j,\nu = \overline{1,m}$, $H(x) \in \mathbf{L}_2^{(r)}$, $r \geq m$; $A_m + K_m(x) \neq 0$, $A_m + N_m(x) \neq 0$ *at* $\mathbf{R}$ *and the equation (1) is a solvable one. Then its solutions belong the space* $\mathbf{L}_2[-r;0]$, $r \geq m$.

Proof. According to the work [3, p. 139] the solutions of the equation (6) $\rho(x) \in \mathbf{L}_2^{(r-m)}$, $r \geq m$ in conditions of the theorem. Then in virtue of the representations (5) and the properties of Cauchy type integral the limiting values $\Phi^{\pm(m)}(x)$ at $\mathbf{R}$ of functions $\Phi^{\pm(m)}(z)$ belong the space $\mathbf{L}_2^{(r-m)}$, $r \geq m$. From the properties of Fourier transformation [2, p. 262] we have that the solutions of the equation (1) given by the formula (3) belong the space $\mathbf{L}_2[-r;0]$, $r \geq m$. The theorem is proved. □

Let the conditions $A_m + K_m(x) \neq 0$, $A_m + N_m(x) \neq 0$ at $\mathbf{R}$ are not executed. Then we suppose that the functions $A_m + K_m(x)$, $A_m + N_m(x)$ go to zero at the real axis in such points as $a_1, a_2, \ldots, a_u$ and $b_1, b_2, \ldots, b_\omega$ with accordingly integer orders $\gamma_1, \gamma_2, \ldots, \gamma_u$, $\mu_1, \mu_2, \ldots, \mu_\omega$. Then in virtue of the work [3, p. 199] the following representations take place

$$A(x) + B(x) = (x+\imath)^{-m} M(x)\rho_+(x), A(x) - B(x) = (x-\imath)^{-m} N(x)\rho_-(x), \quad (12)$$

where the functions $M(x) \neq 0, N(x) \neq 0$ at $\mathbf{R}$, $M(x), N(x) \in \mathbf{H}_\alpha^{(r)}$ and the functions $\rho_+(x), \rho_-(x)$ look as

$$\rho_+(x) = \prod_{k=1}^{u} \left(\frac{x-a_k}{x+\imath}\right)^{\gamma_k}, \rho_-(x) = \prod_{k=1}^{\omega} \left(\frac{x-b_k}{x-\imath}\right)^{\mu_k}. \quad (13)$$

Let

$$r_0 = \max\{\gamma_1, \gamma_2, \ldots, \gamma_u, \mu_1, \mu_2, \ldots, \mu_\omega, m\}, \quad (14)$$

$$\gamma = \sum_{k=1}^{u} \gamma_k, \mu = \sum_{j=1}^{\omega} \mu_k, \varkappa = ind \frac{N(x)}{M(x)}. \quad (15)$$

5 Theorem. *Let the functions* $k_j(x), n_\nu(x) \in \mathbf{L}$, $j,\nu = \overline{1,m}$, $h(x) \in \mathbf{L}_2$; *the functions* $K_j(x), N_\nu(x) \in \mathbf{H}_\alpha^{(r)}$, $j,\nu = \overline{1,m}$, $H(x) \in \mathbf{L}_2^{(r)}$, $r \geq r_0$, *where the number* r_0 *is defined by the formula (14) and the representations (12) take place. If*

$\varkappa - m - \gamma - \mu \geq 0$, where the numbers $\varkappa, \gamma, \mu$ are defined by the formulas (15), then the homogeneous equation (1) has not less than $\varkappa - m - \gamma - \mu$ linearly independent solutions; the heterogeneous equation (1) is an unconditionally solvable one and its general solution depends upon not less than $\varkappa - m - \gamma - \mu$ arbitrary constants. If $\varkappa - m - \gamma - \mu < 0$, then generally speaking the heterogeneous equation (1) is an unsolvable one. It will be a solvable one when not less than $m + \gamma + \mu - \varkappa$ conditions of solvability (11) will be executed.

Proof. According to the work [3, p. 248–278] if $\varkappa - m - \gamma - \mu \geq 0$, then the homogeneous equation (6) has not less than $\varkappa - m - \gamma - \mu$ linearly independent solutions; the heterogeneous equation (6) is an unconditionally solvable one and its general solution depends upon not less than $\varkappa - m - \gamma - \mu$ arbitrary constants. If $\varkappa - m - \gamma - \mu < 0$, then generally speaking the heterogeneous equation (6) is an unsolvable one. It will be a solvable one when not less than $m + \gamma + \mu - \varkappa$ conditions of solvability (11) will be executed. As the equations (6) and (1) are equivalent, then theorem is proved. □

6 Theorem. *Let the functions $k_j(x), n_\nu(x) \in \mathbf{L}$, $j, \nu = \overline{1, m}$, $h(x) \in \mathbf{L}_2$; the functions $K_j(x), N_\nu(x) \in \mathbf{H}_\alpha^{(r)}$, $j, \nu = \overline{1, m}$, $H(x) \in \mathbf{L}_2^{(r)}$, $r \geq r_0$, where the number r_0 is defined by the formula (14). Besides the representations (12) take place and the equation (1) is a solvable one. Then its solutions belong the space $\mathbf{L}_2[-r - m + r_0; 0]$, $r \geq r_0$.*

The proof follows from the work [3, p. 248–298] because the equation's (6) solutions belong the space $\mathbf{L}_2[r - r_0; 0]$ at that case.

Let consider the other cases of numbers' m, n, s correlation.

If $m > n, m > s$, then it will be $p = q = m$ in formulas (4), (5). So we have $A(x) + B(x) = (-1)^m A_m (x + \imath)^{-m}$, $A(x) - B(x) = (-1)^m A_m (x - \imath)^{-m}$ and theorems like 3, 4 take place.

If $m < n = s$, then it will be $p = q = n$ in formulas (4), (5). The representations

$$A(x) + B(x) = (x + \imath)^{-n} M(x) \rho_+(x), A(x) - B(x) = (x - \imath)^{-n} N(x) \rho_-(x), \quad (16)$$

where

$$\rho_+(x) = (x + \imath)^{-\gamma_0} \prod_{k=1}^{u} \left(\frac{x - a_k}{x + \imath} \right)^{\gamma_k}, \rho_-(x) = (x - \imath)^{-\mu_0} \prod_{k=1}^{\omega} \left(\frac{x - b_k}{x - \imath} \right)^{\mu_k} \quad (17)$$

take place. Besides,

$$r_0 = \max\{\gamma_1, \gamma_2, \ldots, \gamma_u, \gamma_0 + n, \mu_1, \mu_2, \ldots, \mu_\omega, \mu_0 + n\}, \gamma = \sum_{k=0}^{u} \gamma_k, \mu = \sum_{j=0}^{\omega} \mu_k. \quad (18)$$

The numbers γ_0, μ_0 are integer orders of nulls in infinity of functions $K_n(x)$ and $N_n(x)$ accordingly.

Thus theorems like 5, 6 take place.

If $m < n < s$, then it will be $p = n, q = s$ in formulas (4), (5). The representations (16) take place where $\rho_+(x), \rho_-(x)$ are like in (17). The numbers γ, μ are defined by (18) and the number r_0 is

$$r_0 = \max\{\gamma_1, \gamma_2, \dots, \gamma_u, \gamma_0 + n, \mu_1, \mu_2, \dots, \mu_\omega, \mu_0 + s\}. \tag{19}$$

Theorems like 5, 6 take place there.

If $m = n < s$, then it will be $p = n, q = s$ in formulas (4), (5). The representations (16) take place where $\rho_+(x)$ is like in (13) and $\rho_-(x)$ is like in (17). The number r_0 is:

$$r_0 = \max\{\gamma_1, \gamma_2, \dots, \gamma_u, \mu_1, \mu_2, \dots, \mu_\omega, \mu_0 + s\}. \tag{20}$$

The number μ we define by the formula (18), and the number γ – by the formula (15). Theorems like 5, 6 take place there.

If $m < s < n$, then it will be $p = s, q = n$ in formulas (4), (5). The representations

$$A(x) + B(x) = (x + \imath)^{-s} M(x)\rho_+(x), A(x) - B(x) = (x - \imath)^{-n} N(x)\rho_-(x) \tag{21}$$

take place, where $\rho_+(x), \rho_-(x)$ are like in (17). The numbers γ, μ are defined by (18) and the number r_0 - by the formula

$$r_0 = \max\{\gamma_1, \gamma_2, \dots, \gamma_u, \gamma_0 + s, \mu_1, \mu_2, \dots, \mu_\omega, \mu_0 + n\}. \tag{22}$$

Theorems like 5, 6 take place there.

If $m = s < n$, then it will be $p = n, q = s$ in formulas (4), (5). The representations (16) take place where $\rho_+(x)$ 1s like in (17), and $\rho_-(x)$ is like in (13).The number r_0 is defined by the formula

$$r_0 = \max\{\gamma_1, \gamma_2, \dots, \gamma_u, \gamma_0 + n, \mu_1, \mu_2, \dots, \mu_\omega\}. \tag{23}$$

The number μ is defined by (15) and the number γ – by the formula (18). Theorems like 5, 6 take place there.

References

[1] Knyazev P.P. *The integral transformations.* Minsk, Visheyshaya Shkola, 1969.

[2] Gakhov F.D., Cherski Y.I. *Convolutional type equations.* Moskva, Nauka, 1978.

[3] Presdorf Z. *Some classes of singular integral equations.* Moskva, Mir, 1979.

[4] Tikhonenko N.Y., Melnik A.S. Different. Equations, 2002, **38**, N 9, p.1218–1224.

[5] Azamatova V.I., Lizunova I.V. Izvestiya AN BSSR, Seria fiz.-mat. nauk, 1971, N 2, p. 43–50.

[6] Lizunova I.V. Izvestiya AN BSSR, Seria fiz.-mat. nauk, 1976, N 4, p. 40–46.

[7] Gakhov F.D. *Boundary problems.* Moskva, Nauka, 1977.

[8] Tikhonenko N.Y., Svyagina N.N. Izvestiya Vuzov, Mathematika, 1996, N 9, p. 83–87.

[9] Mushelishvili N.I. *Singular integral equations.* Moskva, Nauka, 1968.

1st May street, 38/68
Tiraspol, Moldova
E-mail: *alexma@inbox.ru*

Received March 25, 2006

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(51), 2006, Pages 95–101
ISSN 1024–7696

On commutative Moufang loops with some restrictions for subgroups of its multiplication groups

N.T. Lupashco

Abstract. Let $\mathfrak{M}$ be the multiplication group of a commutative Moufang loop Q. In this paper it is proved that if all infinite abelian subgroups of $\mathfrak{M}$ are normal in $\mathfrak{M}$, then Q is associative. If all infinite nonabelian subgroups of $\mathfrak{M}$ are normal in $\mathfrak{M}$, then all nonassociative subloops of Q are normal in Q, all nonabelian subgroups of $\mathfrak{M}$ are normal in $\mathfrak{M}$ and the commutator subgroup $\mathfrak{M}'$ is a finite 3-group.

Mathematics subject classification: 20N05.
Keywords and phrases: Commutative Moufang loop, minimum condition, multiplication IH-group, multiplication $\overline{IH}$-group, metahamiltonian group.

While considering different classes of algebras (groups, rings, loops) it is of crucial importance to analyze the existence of their subalgebras with certain predefined features. To this end, it is advisable to consider the construction of algebras under condition that they don't have any subalgebras with predefined features. Before providing the findings that we'll need further, we will remind some notions from group's theory and commutative Moufang loops (abbreviated CMLs), found in [1] and [2] respectively.

A finite nonabelian group is called a *Miller-Moreno's group* if all its proper subgroups are abelian. An *IP-group* (respect. $\overline{IH}$*-group*) is an infinite group if all its proper infinite abelian (respect. nonabelian) groups are normal within. But if all its nonabelian subgroups are normal, then such a group is called a *metagamiltonian group*.

In [1] the construction of IH-groups is described, elements of infinite order and periodic IH-group, which does not satisfy the minimum condition for abelian subgroups. It also describes the solvable $\overline{IH}$-groups with finite or infinite commutator subgroup and the (solvable) metagamiltonian or the non-metagamiltonian $\overline{IH}$-groups are characterized.

A *commutative Moufang loop* (abbreviated CMLs) is characterized by the identity $x^2 \cdot yz = xy \cdot xz$.

The *multiplicative group* $\mathfrak{M}(Q)$ of the CML Q is the group generated by all the *translations* $L(x)$, where $L(x)y = xy$. The subgroup $I(Q)$ of the group $\mathfrak{M}(Q)$, generated by all the *inner mappings* $L(x,y) = L^{-1}(xy)L(x)L(y)$ is called the *inner mapping group* of the CLM Q. The inner mappings are automorphisms in the CML. A subloop H of the CML Q is called *normal* in Q if $x \cdot yH = xy \cdot$ for all $x, y \in Q$. The subloop H is normal in Q if $\mathfrak{I}(Q)H = H$.

The *associator* (a, b, c) of the elements a, b, c of the CML Q is defined by the equality $ab \cdot c = (a \cdot bc)(a, b, c)$. The *associator subloop* Q' of CML Q is generated by all associators (x, y, z), $x, y, z \in Q$. The *centre* $Z(Q)$ of the CML Q is a normal subloop $Z(Q) = \{x \in Q | (x, y, z) = 1 \forall y, z \in Q\}$.

Let Q be an arbitrary CML and let H be a subset of the set Q. Let $\mathbf{M}(H)$ denote a subgroup of the multiplicative group $\mathfrak{M}(Q)$ of the CML Q, generated by the set $\{L(x) | \forall x \in H\}$. Takes place

Lemma 1 [3]. *Let the commutative Moufang loop Q with the multiplicative group $\mathfrak{M}$, $Z(\mathfrak{M})$, which is the centre of the group $\mathfrak{M}$ and the centre $Z(Q)$ decompose into a direct product $Q = D \times H$, moreover, $D \subseteq Z(Q)$. Then $\mathfrak{M} = \boldsymbol{M}(D) \times \boldsymbol{M}(H)$, besides, $\boldsymbol{M}(D) \subseteq Z(\mathfrak{M}), \boldsymbol{M}(D) \cong D$.*

Further, in papers [3–5] the CML is characterized with the help of various systems of subloops of CML, and also various systems of multiplication groups of CML. In particular, is proved

Lemma 2. *The following statements are equivalent for an arbitrary non-associative CML Q with the multiplication group $\mathfrak{M}$:*

1) Q satisfies the minimum condition for subloops;

2) $\mathfrak{M}$ satisfies the minimum condition for subgroups;

3) Q is a direct product of a finite number of quasicyclic groups, lying in the centre of Q, and a finite loop;

4) $\mathfrak{M}$ is a direct product of a finite number of quasicyclic groups, lying in the centre of $\mathfrak{M}$, and a finite group;

5) Q satisfies the minimum condition for non-normal subloops;

6) $\mathfrak{M}$ satisfies the minimum condition for non-normal subgroups;

7) $\mathfrak{M}$ satisfies the minimum condition for nonabelian subgroups;

8) $\mathfrak{M}$ satisfies the minimum condition for abelian subgroups.

The following are a natural reducing of statements 5) and 6):

i) all infinite associative subloops of Q are normal in Q;

ii) all infinite nonassociative subloops of Q are normal in Q;

iii) $\mathfrak{M}$ is an IH-group;

iv) $\mathfrak{M}$ is an $\overline{IH}$-group;

The structure of CML with conditions i), ii) is examined in [5]. It is proved that the CML with condition i) is associative, the CML with condition ii) has a finite associator subloop and in such a CML any nonassociative (finite or infinite) subloops are normal. In this paper it is proved that the CML with condition iii) is associative. It is proved also that the CML with condition iv) satisfies the condition ii), its multiplication group $\mathfrak{M}$ is metagamiltonian and it commutators subgroup $\mathfrak{M}'$ is a finite 3-group.

Lemma 3. *If the element a of an infinite order or of order three of the multiplication group $\mathfrak{M}$ of arbitrary CML generates a normal subgroup, then it belongs to the centre $Z(\mathfrak{M})$ of the group $\mathfrak{M}$.*

Proof. We denote $(x, y) = x^{-1}y^{-1}xy$, $x^y = y^{-1}xy$. Then $x^y = x(x, y)$. If the element $1 \neq a \in \mathfrak{M}$ generates a normal subgroup, then $a^b = a^k$ for a certain natural number k and for arbitrary fixed element $b \in \mathfrak{M}$. We have $a(a, b) = a^k$, $(a, b) = a^{k-1}$. If $k = 1$, then $(a, b) = 1$. Hence $a \in Z(\mathfrak{M})$. Let us now suppose that $k > 1$. Let $a^3 = 1$. Then $k = 2$ and $a = (a, b)$. The multiplication group $\mathfrak{M}$ is locally nilpotent [3]. Then $a = (a, b) = ((a, b), b) = (((a, b), b), b) = \ldots = 1$. We have obtained a contradiction, as $a \neq 1$. By [2, Theorem 11.4] the commutator subgroup of the group $\mathfrak{M}$ is locally finite 3-group. If a has an infinite order, then for a certain natural number n $(a^{k-1})^{3^n} = (a, b)^{3^n} = 1$, $a = 1$. We have obtained a contradiction again. Therefore the case of $k > 1$ is impossible. This completes the proof of Lemma 3.

Theorem 1. *If the multiplication group $\mathfrak{M}$ of CML Q is a IH-group, then $\mathfrak{M}$ is abelian and, consequently, the CML Q is associative.*

Proof. We suppose that the group $\mathfrak{M}$ is nonabelian. In this cases $\mathfrak{M}$ must be periodic. We suppose the contrary, that the group $\mathfrak{M}$, then the CML Q, is not periodic as well. Let a be an element of infinite order in Q. By [2] the element a^3 belongs to the centre $Z(Q)$ of CML Q. Then it is easy to show, considering the definition of group $\mathfrak{M}$, that the element $\alpha = L(a^3)$ belongs to the centre $Z(\mathfrak{M})$ of group $\mathfrak{M}$. Hence, the group $A =< \alpha >$ is an infinite abelian normal subgroup of group $\mathfrak{M}$. Let β be an arbitrary periodic element of group $\mathfrak{M}$ and let $B =< \beta >$. As $A \subseteq Z(\mathfrak{M})$, then it is easy to show that the product AB will be an infinite abelian subgroup of the group $\mathfrak{M}$. By supposition the subgroup AB is normal in $\mathfrak{M}$, hence if φ is an inner automorphism of group $\mathfrak{M}$, then $AB = \varphi(AB) = \varphi(A) \cdot \varphi(B) = A \cdot \varphi(B)$. Consequently, $AB = A \cdot \varphi(B)$. Let β_1 be an arbitrary element in B. Then there exists such elements $\alpha_1 \in A$, $\beta_2 \in B$ that $\varphi(\beta_1) = \alpha_1\beta_2$ or $\varphi(\beta_1)\beta_2^{-1} = \alpha_1$. As β_1 is a periodic element then $\varphi(\beta_1)$ also is a periodic element and α_1 as an element of infinite cyclic group is not periodic. Further, as $\varphi(\beta_1), \beta_2 \in \mathfrak{M}$, then let $\varphi(\beta_1) = L(u_1) \ldots L(u_k)$, $\beta_2^{-1} = L(v_1) \ldots L(v_n)$, where $u_i, v_j \in Q$. We denote by H the subloop of CML Q generated by the set $\{a, u_1, \ldots, u_k, v_1, \ldots, v_n\}$. The CML H is finitely generated, then by Bruck-Slaby's Theorem [2, Theorem 10.1] it is centrally nilpotent. Again by [2, Theorem 11.5] its multiplication group $\mathfrak{M}(H)$ is nilpotent. Further, we denote by $\varphi(\overline{\beta}_1), \overline{\beta}_2^{-1}, \overline{\alpha}_1$ the restriction on H of mappings $\varphi(\beta_1), \beta_2^{-1}, \alpha_1$ of set Q. It is obvious that $\varphi(\overline{\beta}_1), \overline{\beta}_2^{-1}, \overline{\alpha}_1 \in \mathfrak{M}(H)$, that $\varphi(\overline{\beta}_1), \overline{\beta}_2^{-1}$ are periodic elements and $\overline{\alpha}_1$ is an element of infinite order. The periodic elements form a subgroup in nilpotent groups, hence the product $\varphi(\overline{\beta}_1) \cdot \overline{\beta}_2^{-1}$ is a periodic element. From the equality $\varphi(\beta_1)\beta_2^{-1} = \alpha_1$ follows the equality $\varphi(\overline{\beta}_1) \cdot \overline{\beta}_2^{-1} = \overline{\alpha}_1$ and further $\overline{\alpha}_1 = 1$, $\varphi(\overline{\beta_1}) = \overline{\beta}_2$, $\varphi(B) = B$. We get that any element in $\mathfrak{M}$ generates a normal subgroup. Hence any subgroup from $\mathfrak{M}$ is normal in $\mathfrak{M}$. Then $\mathfrak{M}$ is a hamiltonian group.

Indeed, arbitrary hamiltonian groups are described by the next theorem. A *hamiltonian group* can be decomposed into a direct product of the group of quaternions and abelian groups, whose each element's order is not higher than 2. Conversely, a group that has such a decomposition is hamiltonian. A *group of quaternions* is the group generated by the generators a, b and that satisfies the identical relations $\alpha^4 = 1, \alpha^2 = \beta^2, \beta^{-1}\alpha\beta = \alpha^{-1}$. In [2, Theorem 11.4] it is proved that the quotient group $\mathfrak{M}/Z(\mathfrak{M})$ is a locally finite 3-group. Then from $\alpha^4 = 1$ it follows $\alpha \in Z(\mathfrak{M})$, from $\beta^{-1}\alpha\beta = \alpha^{-1}$ it follows $\alpha^2 = 1$, from $\alpha^2 = \beta^2$ it follows $\beta^2 = 1$ and, finally, from $\beta^2 = 1$ it follows $\beta \in Z(\mathfrak{M})$. We get that the *hamiltonian subgroup of multiplication group of CML is abelian.*

It follows from the aforementioned that the multiplication group $\mathfrak{M}$ of CML Q is abelian. But this contradicts our supposion about the nonabelian group $\mathfrak{M}$. Consequently, the group $\mathfrak{M}$ is periodic.

From Lemmas 1.4 and 3.1 of [3] it follows that the periodic multiplication group $\mathfrak{M}$ of CML Q decomposes into a direct product of its maximal p-subgroups $\mathfrak{M}_p$, in addition $\mathfrak{M}_p$ belongs to the centre $Z(\mathfrak{M})$ for $p \neq 3$. We denote $\mathfrak{M} = \mathfrak{N} \times \mathfrak{M}_3$, where $\mathfrak{N} = \prod_{p \neq 3} \mathfrak{M}_p$. We suppose that $\mathfrak{N}$ is an infinite group and let α be an arbitrary element in $\mathfrak{M}_3$. If $\mathfrak{A} =< \alpha >$ then by supposition the infinite abelian group $\mathfrak{N} \times \mathfrak{A}$ is normal in $\mathfrak{M}$. Let φ be an inner automorphism of $\mathfrak{M}$. Then $\mathfrak{N} \times \mathfrak{A} = \varphi(\mathfrak{N} \times \mathfrak{A}) = \varphi\mathfrak{N} \times \varphi\mathfrak{A} = \mathfrak{N} \times \varphi\mathfrak{A}$, i.e. $\mathfrak{N} \times \mathfrak{A} = \mathfrak{N} \times \varphi\mathfrak{A}$. Then for a certain $\alpha_1 \in \mathfrak{A}$ there exist such elements $\alpha_2 \in \mathfrak{A}$, $\beta \in \mathfrak{N}$ that $\beta\alpha_2 = \varphi\alpha_1$. Further, $\beta^3\alpha_2^3 = \varphi\alpha_1^3$, $\beta^{3^k}\alpha_2^{3^k} = \varphi\alpha_1^{3^k}$ and for a certain integer n $\alpha_2^{3^n} = \varphi\alpha_1^{3^n} = 1$. Hence $\beta^{3^k} = 1$. But the order of β doesn't divide 3. Hence $\beta = 1$ and we get $\varphi\alpha_1 = \alpha_2$, $\varphi A = A$. From here it follows that the subgroup $\mathfrak{M}_3$ is hamiltonian. Hence, as in the above case, $\mathfrak{M}_3$, then also $\mathfrak{M}$, are abelian groups. Consequently, in the decomposition $\mathfrak{M} = \mathfrak{N} \times \mathfrak{M}_3$ we should consider the case when the subgroup $\mathfrak{N}$ is finite. Further, without breaking the generality, we will consider that $\mathfrak{M}$ is a 3-group.

We suppose that the CML $\mathfrak{M}$ doesn't satisfy the minimum condition for subgroups. Then by 8) of Lemma 2 $\mathfrak{M}$ contains an abelian subgroup that doesn't satisfy the minimum condition for subgroups. Then it contains an infinite elementary abelian group $\mathfrak{H}$. Let $\mathfrak{H} = \mathfrak{H}_1 \times \mathfrak{H}_2 \times \ldots \times \mathfrak{H}_n \times \ldots$ be a decomposition of $\mathfrak{H}$ into a direct product of cyclic groups of order 3. For any element $\alpha \in \mathfrak{H}$ there will be such an infinite subgroup $\mathfrak{H}(\alpha) \subseteq \mathfrak{H}$ that $< \alpha > \cap \mathfrak{H}(x) = 1$. Let $\mathfrak{H}(\alpha) = \mathfrak{H}^1(\alpha) \times \mathfrak{H}^2(\alpha)$ be a certain decomposition of group $\mathfrak{H}(\alpha)$ in direct product of two infinite factors. As the cyclic group $< \alpha >$ is, obvious, the intersection of two infinite associative subgroups $< \alpha > \mathfrak{H}^1(\alpha)$ and $< \alpha > \mathfrak{H}^2(\alpha)$, then it is normal in $\mathfrak{M}$. Given the arbitrary element $\alpha \in \mathfrak{H}$ it follows that all factors $\mathfrak{H}_n$ are normal in $\mathfrak{M}$. Every factor $\mathfrak{H}_n$ is a cyclic group of order 3 and by Lemma 3 belongs to the centre $Z(\mathfrak{M})$. Hence $\mathfrak{H} \subseteq Z(\mathfrak{M})$. Let now β be an arbitrary element from $\mathfrak{M}$, let $\mathfrak{H}(\beta)$ be an infinite subgroup of $\mathfrak{H}$ such that $< \beta > \cap \mathfrak{H}(\beta) = 1$, and let $\mathfrak{H}(\beta) = \mathfrak{H}^1(\beta) \times \mathfrak{H}^2(\beta)$ be a certain decomposition of group $\mathfrak{H}(\beta)$ into a direct product of two infinite factors. As $\mathfrak{H}(\beta) \subseteq Z(\mathfrak{M})$, then $\mathfrak{H}^1(\beta), \mathfrak{H}^2(\beta)$ are normal in $\mathfrak{M}$ and the products $\beta\mathfrak{H}^1(\beta), \beta\mathfrak{H}^2(\beta)$ are infinite abelian subgroups. Then $\beta\mathfrak{H}^1(\beta), \beta\mathfrak{H}^2(\beta)$ are normal subgroups, hence also $< \beta >=< \beta > \mathfrak{H}^1(y) \cap \beta\mathfrak{H}^2(\beta)$ is also normal subgroup. We get that any element in $\mathfrak{M}$ generates a normal subgroup in $\mathfrak{M}$. Consequently, $\mathfrak{M}$ is a hamiltonian group and, as proved above, it is abelian. This contradicts our assumption about nonabelian group $\mathfrak{M}$. Hence $\mathfrak{M}$ satisfies the minimum condition for subgroups.

From minimum condition for subgroups for $\mathfrak{M}$ it follows by Lemma 2 that $\mathfrak{M} = \mathfrak{B} \times \mathfrak{C}$, where $\mathfrak{C} \subseteq Z(\mathfrak{M})$ and $\mathfrak{B}$ is a finite group. If γ is an arbitrary element in $\mathfrak{M}$ then $< \gamma > \mathfrak{C}$ is an infinite abelian subgroup. Further, from the normality of $< \gamma > \mathfrak{C}$ in $\mathfrak{M}$ follows the normality of $< \gamma >$ in $\mathfrak{M}$. Hence $\mathfrak{M}$ is a hamiltonian group. According to the above proofs $\mathfrak{M}$ is an abelian group. This completes the proof of Theorem 1.

Let us now consider a CML with certain restriction on nonabelian subgroups of it multiplication group. We suppose that the multiplication group $\mathfrak{M}$ of the CML Q doesn't have proper infinite nonabelian subgroups. Then by Lemma 2 $\mathfrak{M}$ satisfies the minimum condition for subgroups and $\mathfrak{M} = \mathfrak{K} \times \mathfrak{G}$, where $\mathfrak{K}$ is a direct product of a finite number of quasicyclic groups, lying in the centre $Z(\mathfrak{M})$ of the group $\mathfrak{M}$, $\mathfrak{G}$ is a finite group. But as $\mathfrak{M}$ doesn't have proper infinite nonabelian subgroups then $\mathfrak{K}$ is a quasicyclic group, $\mathfrak{G}$ is a Miller-Moreno group.

By Lemma 2 the CML Q satisfies the minimum condition for sublooops and $Q = D \times H$, where D is a direct product of a finite number of quasicyclic groups, lying in the centre $Z(Q)$ of the CML Q, H is a finite loop. Further, by Lemma 1 $\mathfrak{M} = M(D) \times M(H)$ and $M(D) \cong D$. Consequently, $M(H) \cong \mathfrak{G}$. Further, if for certain $a, b, c \in H$ $ab \cdot c \neq a \cdot bc$ then $L(c)L(a)b \neq L(a)L(c)b$, $L(c)L(a) \neq L(a)L(c)$. Hence if the CML H contains proper nonassiciative subloops, then the group $M(H)$ contains proper nonabelian subgroups. A CML is diassociative [2]. Then from the relation $M(H) \cong \mathfrak{G}$ it follows that the CML H is generated by three elements. Consequently, we proved.

Proposition 1. *A multiplication group $\mathfrak{M}$ of infinite nonassociative CML Q does not contain proper infinite nonabelian subgroups if and only if $Q = D \times H$, where D is a quasicyclic group, H is a nonassociative 3-generate loop or $\mathfrak{M} = D \times \mathfrak{G}$, where $\mathfrak{G}$ is a Miller-Moreno group.*

Theorem 2. *If the multiplication group $\mathfrak{M}$ of the CML Q is an $\overline{IH}$-group, then:*

1) $\mathfrak{M}$ is a metagamiltonian group;

2) all nonassociative sibloops of CML Q are normal in it;

3) if $\mathfrak{M}$ is non-periodic, then the commutator subgroup $\mathfrak{M}'$ of group $\mathfrak{M}$ is a finite abelian 3-group;

4) if $\mathfrak{M}$ is periodic, then the commutator subgroup $\mathfrak{M}'$ of group $\mathfrak{M}$ is solvable of a class not greater tha three finite 3-group;

Proof. By [3] the multiplication group $\mathfrak{M}$ of an arbitrary CML is locally nilpotent. Then by [1, Theorem 1.18] $\mathfrak{M}$ posed a centrally system with cyclic factors of simple orders. In this cases, if $\mathfrak{M}$ is an $\overline{IH}$-group, then by [1, Proposition 6.5] $\mathfrak{M}$ is solvable. Corollary 6.11 from [1] stipulated that a non-metagamiltonian solvable $\overline{IH}$-group satisfies the minimum condition for subgroups. Then from Lemma 2 it follows that the multiplication $\overline{IH}$-group $\mathfrak{M}$ is metagamiltonian. Consequently, the item 1) is proved.

Now let H be an arbitrary nonassociative subloop of the CML Q and let its multiplication group $\mathfrak{M}$ be an $\overline{IH}$-group. Then the subgroup $\mathfrak{N}$, generated by mappings $L(a)$, $a \in Q$, is nonabelian and by item 1) is normal in $\mathfrak{M}$. The set $\mathfrak{N}a$, $a \in Q$, partitions Q and

$$\mathfrak{N}a \cdot \mathfrak{N}b = L(\mathfrak{N}b)\mathfrak{N}a = \mathfrak{N}L(\mathfrak{N}b)a = \mathfrak{N}(\mathfrak{N}b \cdot a) =$$

$$= \mathfrak{N}(L(a)\mathfrak{N}b) = \mathfrak{N}(\mathfrak{N}L(a)b) = \mathfrak{N}L(a)b = \mathfrak{N}(ab).$$

If $\mathfrak{N}(ba) = \mathfrak{N}(ca)$, then

$$\mathfrak{N}b \cdot a = L(a)\mathfrak{N}b = \mathfrak{N}L(a)b = \mathfrak{N}(ba) = \mathfrak{N}(ca) = \mathfrak{N}L(a)c = L(a)\mathfrak{N}c = \mathfrak{N}c \cdot a,$$

so $\mathfrak{N}b = \mathfrak{N}c$. Hence the mapping φ defined by $\varphi a = \mathfrak{N}a$ is a homomorphism of the CML Q upon a loop φQ and the kernel $\mathfrak{N}1 = H$, of φ, is a normal subloop of Q. Consequently, any nonassociative subloop H of Q is normal in Q, i.e. the item 2) is proved.

By Theorem 6.3 from [1] the commutator subgroup $\mathfrak{M}'$ of non-periodic $\overline{IH}$-group is a finite abelian p-group. But the commutator subgroup of multiplication group of arbitrary CML is a 3-group [2, Theorem 11.4]. Hence the commutator subgroup $\mathfrak{M}'$ is a finite abelian 3-group, i.e. the item 3) is proved.

By Theorem 6.7 from [1] all solvable $\overline{IH}$-groups with infinite commutator subgroup satisfy the minimum condition for subgroups. But from 4) of Lemma 2 it follows that in these cases the commutator subgroup $\mathfrak{M}'$ of $\mathfrak{M}$ is finite. Hence the commutator subgroup $\mathfrak{M}'$ of $\overline{IH}$-group $\mathfrak{M}$ is finite and by [2, Theorem 11.4] is a finite 3-group.

Now let us suppose that the second commutator subgroup $\mathfrak{M}^{(2)}$ of the group $\mathfrak{M}$ is nonabelian. Then any subgroup that contains $\mathfrak{M}^{(2)}$ is nonabelian, and by item 2), it is normal in $\mathfrak{M}$. Obviously, the group $\mathfrak{M}/\mathfrak{M}^{(2)}$ is hamiltonian and as shown during the proof of Theorem 1, it is an abelian group. Therefore, $\mathfrak{M}' \subseteq \mathfrak{M}^{(2)}$, i.e. $\mathfrak{M}' = \mathfrak{M}^{(2)}$. But the commutator subgroup $\mathfrak{M}'$ is a finite 3-group, hence it is nilpotent. Therefore $\mathfrak{M}' \neq \mathfrak{M}^{(2)}$. Contradiction. Consequently, $\mathfrak{M}^{(2)}$ is an abelian subgroup, and the group $\mathfrak{M}$ is solvable of class not greater than three. This completes the proof of Theorem 2.

As by Theorem 2 the commutator subgroup $\mathfrak{M}'$ of the multiplication $\overline{IH}$-group is finite, then from [6, 7] it follows that $\mathfrak{M}$ is a group with finite classes of conjugate elements and the number of elements in each class doesn't exceed the number $\mid \mathfrak{M}' \mid$. Further, in [3] it is proved that the quotient group $\mathfrak{M}/Z(\mathfrak{M})$ of an arbitrary multiplication group $\mathfrak{M}$ by it centre $Z(\mathfrak{M})$ is a locally finite 3-group. Thus from [6, 8] it follows that if $\mathfrak{M}$ is an $\overline{IH}$-group, then any element in $\mathfrak{M}/Z(\mathfrak{M})$ is contained in a normal finite 3-group.

References

[1] Chernikov S.N. *The groups with given properties of the systems of subgroups.* Moskva, Nauka, 1980 (In Russian).

[2] Bruck R.H. *A survey of binary systems.* Springer Verlag, Berlin-Heidelberg, 1958.

[3] Sandu N.I. *Commutative Moufang loots with minimum condition for subloops I.* Buletinul Academiei de Ştiinţe a Republicii Moldova, Matematica, 2003, N 3(43), p. 25–40.

[4] Sandu N.I. *Commutative Moufang Loops with Finite Classes of Conjugate Subloops.* Mat. zametki, 2003, **73**, N 2, p. 269–280 (In Russian).

[5] Sandu N.I. *Commutative Moufang Loots with Minimum Condition for Subloops II.* Buletinul Academiei de Ştiinţe a Republicii Moldova, Matematica, 2004, N 2(45), p. 33–48.

[6] Gorchakov Iu.M. *Groups with finite classes of conlugate elements.* Moskva, Nauka, 1978 (in Russian).

[7] Neumann B.H. *Groups covered by permutable subsets.* J. London Math. Soc., 1954, **29**, p. 236–248.

[8] Baer R. *Finitiness properties of groups.* Duke Math. J., 1948, **15**, p. 1021–1032.

Tiraspol State University
str. Iablochkin 5, MD-2069 Chishinau
Moldova

Received June 5, 2006

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(51), 2006, Pages 102–105
ISSN 1024–7696

The property of universality for some monoid algebras over non-commutative rings

Elena P. Cojuhari

Abstract. We define on an arbitrary ring A a family of mappings $(\sigma_{x,y})$ subscripted with elements of a multiplicative monoid G. The assigned properties allow to call these mappings derivations of the ring A. A monoid algebra of G over A is constructed explicitly, and the universality property of it is shown.

Mathematics subject classification: Primary 16S36; Secondary 13N15, 16S10.
Keywords and phrases: Derivations, monoid algebras, free algebras.

In this note we consider monoid algebras over non-commutative rings. First, we introduce axiomatically a family of mappings $\sigma = (\sigma_{x,y})$ defined on a ring A and subscripted with elements of a multiplicative monoid G. Due to their assigned properties these mappings can be called derivations of A. Next, we construct a monoid algebra $A\langle G\rangle$ by means of the family σ, and the universality of it is shown.

1. Let A be a ring (in general non-commutative) and G a multiplicative monoid. Throughout the paper we consider $1 \neq 0$ (where 0 is the null element of A, and 1 is the unit element for multiplication), the unit element of G is denoted by e. We introduce a family of mappings of A into itself by the following assumption.

(A) For each $x \in G$ there exists a unique family $\sigma_x = (\sigma_{x,y})_{y\in G}$ of mappings $\sigma_{x,y} : A \longrightarrow A$ such that $\sigma_{x,y} = 0$ for almost all $y \in G$ (here and thereafter, almost all will mean all but a finite number, that is, $\sigma_{x,y} \neq 0$ only for a finite set of $y \in G$) and for which the following properties are fulfilled:

(i) $\sigma_{x,y}(a + b) = \sigma_{x,y}(a) + \sigma_{x,y}(b)$ $(a, b \in A; x, y \in G)$;

(ii) $\sigma_{x,y}(ab) = \sum_{z\in G} \sigma_{x,z}(a)\sigma_{z,y}(b)$ $(a, b \in A; x, y \in G)$;

(iii) $\sigma_{xy,z} = \sum_{uv=z} \sigma_{x,u} \circ \sigma_{y,v}$ $(x, y, z \in G)$;

(iv_1) $\sigma_{x,y}(1) = 0$ $(x \neq y; x, y \in G)$; (iv_2) $\sigma_{x,x}(1) = 1$ $(x \in G)$;

(iv_3) $\sigma_{e,x}(a) = 0$ $(x \neq e; x \in G)$; (iv_4) $\sigma_{e,e}(a) = a$ $(a \in A)$.

In (ii) the elements are multiplied as in the ring A, but in (iii) the symbol $\circ$ means the composition of maps.

Examples. 1. Let A be a ring and let G be a multiplicative monoid, and let σ be a monoid-homomorphism of G into $End(A)$, i.e. $\sigma(xy) = \sigma(x) \circ \sigma(y)$ $(x, y \in G)$ and $\sigma(e) = 1_A$. We define $\sigma_{x,y} : A \longrightarrow A$ such that $\sigma_{x,x} = \sigma(x)$ for $x \in G$ and $\sigma_{x,y} = 0$ for $y \neq x$. The properties $(i) - (iv_4)$ of (A) are verified at once.

---

2. Let A be a ring, and let α be an endomorphism of A and δ be an α-differentiation of A, i.e.

$$\delta(a+b)=\delta(a)+\delta(b),\ \delta(ab)=\delta(a)b+\alpha(a)\delta(b)$$

for every $a, b \in A$. Denote by G the monoid of elements x_n $(n=0,1,...)$ endowed with the law of composition defined by $x_n x_m = x_{n+m}$ $(n,m=0,1,...;\ x_0 := e)$. We write σ_{nm} instead of σ_{x_n,x_m} by defining $\sigma_{nm} : A \longrightarrow A$ as the following mappings $\sigma_{00}=1_A,\ \sigma_{10}=\delta,\ \sigma_{11}=\alpha,\ \sigma_{nm}=0$ for $m>n$ and $\sigma_{nm}=\sum_{j_1+...+j_n=m}\sigma_{1j_1}\circ ...\circ\sigma_{1j_n}$ $(m=0,1,...,n;\ n=1,2,...)$, where $j_k=0,1$ $(k=1,...,n)$. The family $\sigma=(\sigma_{nm})$ satisfies the axioms $(i)-(iv_4)$ of (A).

2. Next, we consider an algebra $A\langle G\rangle$ connected with the structure of differentiation $\sigma=(\sigma_{x,y})$. Let $A\langle G\rangle$ be the set of all mappings $\alpha : G \longrightarrow A$ such that $\alpha(x)=0$ for almost all $x \in G$. We define the addition in $A\langle G\rangle$ to be the ordinary addition of mappings into the additive group of A and define the operation of A on $A\langle G\rangle$ by the map $(a,\alpha)\longrightarrow a\alpha$ $(a \in A)$, where $(a\alpha)(x)=a\alpha(x)$ $(x \in G)$. Note that, in respect to these operations, $A\langle G\rangle$ forms a left module over A. Following notations made in [1] we write an element $\alpha \in A\langle G\rangle$ as a sum $\alpha=\sum_{x\in G} a_x \cdot x$, where by $a \cdot x$ $(a \in A,\ x \in G)$ is denoted the mapping whose value at x is a and 0 at elements different from x. Certainly, the above sum is taken over almost all $x \in G$. $A\langle G\rangle$ becomes a ring if for elements of the form $a \cdot x$ $(a \in A; x \in G)$ we define their product by the rule

$$(a\cdot x)(b\cdot y)=\sum_{z\in G} a\sigma_{x,z}(b)\cdot zy\ (a,b\in A; x,y\in G)$$

and then extend for $\alpha, \beta \in A\langle G\rangle$ by the property of distributivity. We let

$$\alpha a=\sum_{x\in G}\Big(\sum_{y\in G} a_y\sigma_{y,x}(a)\Big)\cdot x, \quad (a\in A, alpha\in A\langle G\rangle)$$

for $a \in A$ and $\alpha \in A\langle G\rangle$, and thus we obtain an operation of A on $A\langle G\rangle$ and in such a way we make $A\langle G\rangle$ into a right A-module. Thus, we may view $A\langle G\rangle$ as an algebra over A.

Remark. Let us consider the situation described in Example 1. Then the law of multiplication in $A\langle G\rangle$ is given as follows

$$\Big(\sum_{x\in G} a_x\cdot x\Big)\Big(\sum_{x\in G} b_x\cdot x\Big)=\sum_{x\in G}\sum_{y\in G} a_x\sigma_{x,x}(b_y)\cdot xy.$$

In this case, the monoid algebra $A\langle G\rangle$ represents a crossed product [2, 3] of the multiplicative monoid G over the ring A with respect to the factors $\rho_{x,y}=1$ $(x,y \in G)$. If G is a group, and $\sigma : G \longrightarrow End(A)$ is such that $\sigma(x)=1_A$ for all $x \in G$, we evidently obtain an ordinary group ring [4] (the commutative case see also [5]).

3. In this subsection we show that $A\langle G\rangle$ is a free G - algebra over A. Let B be another ring. Given a ring-homomorphism $f : A \longrightarrow B$ it can be defined on the ring B a structure of A-module, defining the operation of A on B by the map $(a, b) \longrightarrow f(a)b$ for all $a \in A$ and $b \in B$. We denote this operation by $a * b$. The axioms for a module are trivially verified. Let now $\varphi : G \longrightarrow B$ be a multiplicative monoid-homomorphism. Denote by $\langle B; f, \varphi\rangle$ the module formed by all linear combinations of elements $\varphi(x)$ $(x \in G)$ over A in respect to the operation $*$. The axioms for a left A-module are trivially verified.

We assume that the homomorphisms f and φ satisfy the following assumption.

(B) $\varphi(G)f(A) \subset \langle B; f, \varphi\rangle$.

Thus, it is postulated that an element $\varphi(x)f(a)$ $(a \in A, x \in G)$ can be written as a linear combination of the form $\sum_{b\in B, y\in G} b\,\varphi(y)$. The coefficients b depend on $\varphi(x), \varphi(y)$ and $f(a)$. To designate this fact we denote the corresponding coefficients by $\sigma_{\varphi(x),\varphi(y)}(f(a))$. Therefore, it can be considered that there are defined a family of mappings $\sigma_{\varphi(x),\varphi(y)} : B \longrightarrow B$ such that

$$\varphi(x)f(a) = \sum_{y\in G} \sigma_{\varphi(x),\varphi(y)}(f(a))\varphi(y) \ (a \in A, x \in G).$$

By these considerations, we may view $\langle B; f, \varphi\rangle$ as a right A-module. In order to make the module $\langle B; f, \varphi\rangle$ to be a ring we require the following additional assumption.

(C) The homomorphisms f and φ are such that the following diagram

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ A \arrow[u, "\sigma_{x,y}"] \arrow[r, "f"] & B \arrow[u, "\sigma_{\varphi(x),\varphi(y)}"'] \end{tikzcd}$$

is commutative for every $x, y \in G$, i.e. $\sigma_{\varphi(x),\varphi(y)} \circ f = f \circ \sigma_{x,y}$ $(x, y \in G)$.

We define multiplication in $\langle B; f, \varphi\rangle$ by the rules

$$\Big(\sum_{x\in G} a_x * \varphi(x)\Big)\Big(\sum_{x\in G} b_x * \varphi(x)\Big) = \sum_{x\in G}\sum_{y\in G}(a_x * \varphi(x))(b_y * \varphi(y)),$$

$$(a_x * \varphi(x))(b_y * \varphi(y)) = f(a_x)\sum_{z\in G}\sigma_{\varphi(x),\varphi(z)}(f(b_y))\varphi(zy).$$

The verification that $\langle B; f, \varphi\rangle$ is a ring under the above laws of composition is direct. Thus, we have made $\langle B; f, \varphi\rangle$ into an algebra over A (in general, non-commutative).

Next, we define a category $\mathcal{C}$ whose objects are algebras $\langle B; f, \varphi\rangle$ constructed as above, and whose morphisms between two objects $\langle B; f, \varphi\rangle$ and $\langle B^{'}; f^{'}, \varphi^{'}\rangle$ are ring-homomorphisms $h : B \longrightarrow B^{'}$ making the diagrams commutative:

$$\begin{tikzcd} G \arrow[r, equal] \arrow[d, "\varphi"'] & G \arrow[d, "\varphi'"] \\ B \arrow[r, "h"] & B' \\ A \arrow[r, equal] \arrow[u, "f"] & A \arrow[u, "f'"'] \end{tikzcd}$$

The axioms for a category are trivially satisfied. We call a universal object in the category $\mathcal{C}$ a free G-algebra over A, or a free (A, G)-algebra. It turns out that the monoid algebra $A\langle G\rangle$ represents a free (A, G)-algebra. To this end, we observe that the mapping $\varphi_0 : G \longrightarrow A\langle G\rangle$ given by $\varphi_0(x) = 1 \cdot x$ $(x \in G)$ is a monoid-homomorphism. The mapping φ_0 is embedding of G into $A\langle G\rangle$. In addition, we have a ring-homomorphism $f_0 : A \longrightarrow A\langle G\rangle$ given by $f_0(a) = a \cdot e$ $(a \in A)$. Obviously, f_0 is also an embedding. We identify $A\langle G\rangle$ with the triple $\langle A\langle G\rangle; f_0, \varphi_0\rangle$ and in this sense we treat $A\langle G\rangle$ as an object of the category $\mathcal{C}$. The property of the universality of $A\langle G\rangle$ is formulated by the following assertion.

Theorem 1. *Let A be a ring, and G a multiplicative monoid for which the assumptions $(A), (B)$ and (C) are satisfied. Then for every object $\langle B; f, \varphi\rangle$ of the category $\mathcal{C}$ there exists a unique ring-homomorphism $h : A\langle G\rangle \longrightarrow B$ making the following diagram commutative*

$$\begin{tikzcd}
G \arrow[r, equal] \arrow[d, "\varphi_0"'] & G \arrow[d, "\varphi"] \\
A\langle G\rangle \arrow[r, "h"] & B \\
A \arrow[r, equal] \arrow[u, "f_0"] & A \arrow[u, "f"']
\end{tikzcd}$$

The relation with the theory of skew polynomial rings [6–8] and with those obtained by Yu. M. Ryabukhin [9] (see also [10]), and further properties of the general derivation mappings $\sigma_{x,y}$ $(x, y \in G)$ will be given in a subsequent publication.

References

[1] Lang S. *Algebra.* Addison Wesley, Reading, Massuchsetts, 1970.

[2] Bovdi A.A. *Crossed products of a semigroup and a ring.* Sibirsk. Mat. Zh., 1963, **4,** p. 481–499.

[3] Passman D.S. *Infinite crossed products.* Academic Press, Boston, 1989.

[4] Bovdi A.A. *Group rings.* Kiev UMK VO, 1988 (in Russian).

[5] Karpilovsky G. *Commutative group algebras.* New-York, 1983.

[6] Cohn P.M. *Free rings and their relations.* Academic Press, London, New-York, 1971.

[7] Smits T.H.M. *Skew polynomial rings.* Indag. Math., 1968, **30**, p. 209–224.

[8] Smits T.H.M. *The free product of a quadratic number field and semifield.* Indag. Math., 1969, **31**, p. 145–159.

[9] Ryabukhin Yu.M. *Quasi-regular algebras, modules, groups and varieties.* Buletinul A.S.R.M., Matematica, 1997, N 1(23), p. 6–62 (in Russian).

[10] Andrunakievich V.A., Ryabukhin Yu.M. *Radicals of algebras and structure theory.* Moscow, Nauka, 1979 (in Russian).

State University of Moldova
A. Mateevici str. 60
Chisinau MD-2009, Moldova
E-mail: *cojuhari@usm.md*

Received August 28, 2006

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(51), 2006, Pages 106–109
ISSN 1024–7696

Junior spatial groups of $(22'\underline{1})$-symmetry

A.A. Shenesheutskaia

Abstract. All junior space groups of $(22'\underline{1})$-symmetry are obtained with the help of junior space groups of the three-fold antisymmetry.

Mathematics subject classification: 52C20, 05B45.
Keywords and phrases: P-symmetry, symmetry, coloursymmetry, antisymmetry.

I. The problem of generalization of 230 spatial Fedorov symmetry groups G_3 with 32 crystallographic P-symmetries ($P \cong G_{30}$) includes junior and middle groups. All the junior groups have already been obtained [1]. Only 2- and 3-middle groups of $(42\underline{1})$- and $(62\underline{1})$-symmetries, respectively, are not known.

Consider the generalization of k symmetry groups of any category $G_{r...}$ with 32 crystallographic P-symmetries in geometric classification. Groups $G^P_{r...}$ are divided in senior ones, among which for 1-symmetry k groups are symmetry groups of the category $G_{r...}$ (generating groups), junior and Q-middle groups. The derivation of senior groups is trivial, as $G = S \times P$, where S is a classical (generating) group, P is the permutation group of indices that characterizes the P-symmetry under consideration, and $\times$ is the symbol of the direct product of groups. The derivation of junior groups of P-symmetry in the case when S has a normal divisor H such that the factor group $P/H \cong P$ must be realized in detail. The calculation of Q-middle groups can be done using the relation between the number of Q-middle groups of some P-symmetries and the number of junior groups of others P-symmetries. To make this relation more precise A. M. Zamorzaev [1] introduced the concept of the strong isomorphism of groups.

Two transformations of a symmetry group S are called undistinguishable if they are of the same geometric type and generate groups of the same order. So, in the symmetry group of a rectangular $2 \cdot m = (1, 2, m_1, m_2)$ the elements m_1 and m_2 are undistinguishable and distinct from the element 2. In the symmetry group of a right parallelepiped $2 : m = (1, 2, m, \widetilde{2})$ all elements are distinct. In the permutation group of indices or indices with signs two elements are called undistinguishable if one of them can go over into the other by means of reindexing. For example, in the group $P_1 = \{(1,2)(3,4),\ (1,3)(2,4)\} = (I,\ (1,2)(3,4),\ (1,3)(2,4),\ (1,4)(2,3))$, which defines (22)-symmetry, all three non-identical permutations, going over into each other cyclically by a cyclic change of the indices 2, 3, 4 (or other three of indices) undistinguishable are, and in the permutation group $P_2 = \{(1+,2+)(1-,2-),\ (1+,1-)(2+,2-)\} = (I,\ (1+,2+)(1-,2-),\ (1+,1-)(2+,2-),$

$(1+, 2-)(1-, 2+))$, which defines $(2\underline{1})$-symmetry, there are no undistinguishable elements, as from one of non-identical permutations changes only indices, the other one – only signs, and the third one – both indices and signs. In the factor group S/H or P/Q cosets are undistinguishable (as factor group elements) if they contain undistinguishable elements, respectively. So, in the rotation group of a regular quadrangular prism $S = 4 : 2 = (1, 4, 2(= 4^2), 4^{-1}, 2_1, 2_2, 2_3, 2_4)$ the subgroup $H = 2 = (1, 2)$ is the normal divisor; in the factor group $S/H = (H, 4 \cdot H, 2_1 \cdot H1, 2_2 \cdot H)$ (which is isomorphic to $2 : 2$) the cosets $2_1 \cdot H = (2_1, 2_3)$ and $2_2 \cdot H = (2_2, 2_4)$ are undistinguishable and distinct from the coset $4 \cdot H = (4, 4^{-1})$.

Two elements of a group are called equally included in this group if there exists an automorphism of the group that maps one element into the other. So, in the group $4 : 2$ though the elements $2(= 4^2)$ and 2_1 are undistinguishable, they are not equally included in this group, and the elements $2_1, 2_2, 2_3, 2_4$ are undistinguishable and are equally included in $4 : 2$. Equally included elements not necessarily are undistinguishable; so, the elements $2, m$ and $\widetilde{2}$ are equally included in the group $2 : m$, but all these elements are distinct.

An isomorphism of the group G_1 onto G_2 is called strong if under this isomorphism to any undistinguishable and equally included in G_1 elements undistinguishable elements correspond, and to distinct and equally included in G_1 elements distinct elements correspond. In this case the groups G_1 and G_2 are called strong isomorphic (the designation: $G_1 \cong G_2$).

P_1-symmetry and P_2-symmetry are called isomorphic if the permutation groups P_1 and P_2, defining these P-symmetries, are strong isomorphic (the designation: $P_1 \cong P_2$).

Among 32 crystallographic P-symmetries in geometric classification only 22 are not isomorphic. Let enumerate permutation groups that define these P-symmetries, grouping them by strong isomorphism: 1)1; 2)$2 \cong \underline{1} \cong \underline{2}$; 3)3; 4)$4 \cong \underline{4}$; 5)$6 \cong 3\underline{1} \cong \underline{6}$; 6)22; 7)$2\underline{1}$; 8)$2\underline{2}$; 9)$32 \cong 3\underline{2}$; 10)$42 \cong 4\underline{2}$; 11)$\underline{4}2$; 12)$62 \cong 6\underline{2}$; 13)$32\underline{1} \cong \underline{6}2$; 14)$4\underline{1}$; 15)$6\underline{1}$; 16)$22\underline{1}$; 17)$42\underline{1}$; 18)$62\underline{1}$; 19)23; 20)$43 \cong \underline{4}3$; 21)$23\underline{1}$; 22)$43\underline{1}$.

In [1] the following affirmations are proved: 1) The number of different Q-middle groups of P-symmetry in the family is equal to the number of different junior groups of P'-symmetry with the same generating group if $P/Q \cong P'$; 2) If $P_1 \cong P_2$, then the numbers of different junior groups of P_1-symmetry and P_2-symmetry with the same generating group are equal. Hence, to calculate the number of junior and Q-middle groups of all 32 crystallographic P-symmetries by the generalization of any category of classical groups it is enough to study the groups of $2-, 3-, 4-, 6-, (22)-, (2\underline{1})-, (2\underline{2})-, (32)-, (\underline{4}2)-, (42)-, (62)-, (\underline{6}2)-, (4\underline{1})-, (6\underline{1})-, (22\underline{1})-, (42\underline{1})-, (62\underline{1})-, (23)-, (43), (23\underline{1})$- and $(43\underline{1})$-symmetry.

To finish this task it is necessary to study different junior groups of hypercrystallographic $(22'\underline{1})$-symmetry ($P \cong G_{430}$), as by means of these groups we can calculate 2-middle groups of $(42\underline{1})$-symmetry and 3-middle groups of $(62\underline{1})$-symmetry, because $42\underline{1}/2 \cong 62\underline{1}/3 \cong 22'\underline{1}$.

II. The symbol $22'\underline{1}$ is a symbol of three-dimensional point group of the CM kind generated by rotations around two two-fold antirotational and one two-fold rotational axes which are pairwise orthogonal and by antiidentical transformation $\underline{1}$. One of the hypercrystallographic P-symmetries that models junior symmetry and antisymmetry group $mm'm'$ ($22'\underline{1} \approx 22'2'\underline{1} \approx mm'm'$), generated by reflections in three pairwise orthogonal planes (one reflection plane and two antireflection planes), is denoted by this symbol ($\underline{1}$ is interpreted as reflection in a point, i.e. as an inversion).

The groups $mm'm'$ and $E_3 = \{1\} \times \{1'\} \times \{{}^*1\}$ are isomorphic, where the group E_3 is the direct product of three groups of order 2, generated by antiidentical transformations of kind 1, kind 2 and kind 3, respectively. The existence of such isomorphism makes it possible to reduce the problem of searching junior space groups of ($22'\underline{1}$)-symmetry to the problem of searching junior space groups of three-fold antisymmetry.

However, to different received junior space groups of the type M^3 from one family correspond the same groups of ($22'\underline{1}$)-symmetry, as the group $E_3 = (e,\ \underline{1},\ 1',\ {}^*1,\ \underline{1}',\ {}^*\underline{1},\ {}^*1',\ {}^*\underline{1}')$ contains 7 different kinds of antisymmetry transformations, and in the group $mm'm' = m_1m_2'm_3' = (e, m_1, m_2', m_3', m_1m_2' = 2_{12}', m_1m_3' = 2_{13}', m_2'm_3' = 2_{23}, m_1m_2'm_3' = i_{123})$ only five transformations are essentially different, for example, $m_1, m_2', 2_{12}', 2_{23}, i_{123}$, as the transformations m_2', m_3' and $2_{12}', 2_{13}'$ play the same geometric role, respectively.

Consequently, for example, to the group $\{\underline{a}, b, c\}(2' \cdot^* m : 2)$ and to five groups, received from this group by all permutations of signs –, /, * (which mean the transformations of antisymmetry of kind 1, kind 2 and kind 3, respectively),

$$\{\underline{a}, b, c\}({}^*2 \cdot m' : 2) \quad \{a', b, c\}(\underline{2} \cdot {}^*m : 2)$$

$$\{a', b, c\}({}^*2 \cdot \underline{m} : 2) \quad \{{}^*a, b, c\}(\underline{2} \cdot m' : 2) \quad \{{}^*a, b, c\}(2' \cdot \underline{m} : 2),$$

i.e. to six different junior groups of three-fold antisymmetry from family $18s$ correspond three different groups of ($22'\underline{1}$)-symmetry:

$$\{a^1, b, c\}(2^3 \cdot m^2 : 2) \qquad \{a^3, b, c\}(2^1 \cdot m^2 : 2) \qquad \{a^3, b, c\}(2^2 \cdot m^1 : 2).$$

To the group $\{\underline{a}, b, c\}({}^*2 \cdot m :\ {}^*2')$ and to two groups, received from this group by all even permutations of signs –, /, * (which mean the transformations of antisymmetry of kind 1, kind 2 and kind 3, respectively),

$$\{a', b, c\}(\underline{2} \cdot m :\ {}^*\underline{2}) \quad \{{}^*a, b, c\}(2' \cdot m : \underline{2}'),$$

i.e. to three different junior groups of three-fold antisymmetry from family $18s$ correspond two different groups of ($22'\underline{1}$)-symmetry:

$$\{a^1, b, c\}(2^3 \cdot m : 2^{23}) \qquad \{a^2, b, c\}(2^1 \cdot m : 2^{13}).$$

To the groups

$$\left\{a, b, \frac{a+b+c}{2}\right\}\left(\frac{c}{2}\,{}^*\underline{2} \cdot \frac{b}{2}\,\underline{m} : \frac{a}{2}\,{}^*2'_{\frac{b}{4}}\right)$$

and

$$\left\{a, b, \frac{a+b+c}{2}\right\}\left(\frac{c}{2}{}^*\underline{2}\cdot\frac{b}{2}{}^*m : \frac{a}{2}\underline{2}'_{\frac{b}{4}}\right),$$

i.e. to two different junior groups of three-fold antisymmetry from family $21a$ there corresponds one group of $(22'\underline{1})$-symmetry:

$$\left\{a, b, \frac{a+b+c}{2}\right\}\left(\frac{c}{2}2^{13}\cdot\frac{b}{2}m^{1} : \frac{a}{2}2^{12}{}_{\frac{b}{4}}\right).$$

Consequently, to receive all different junior groups of $(22'\underline{1})$-symmetry it is necessary to receive all different junior groups of three-fold antisymmetry, to collect them in nests and to replace 6 by 3, 3 by 2, 2 by 1 and 1 by 1.

Hence, as the number of different junior groups of three-fold antisymmetry is equal to $16937*6+2490*3+5*2+37*1 = 109139$, then the number of different junior space groups of $(22'\underline{1})$-symmetry is equal to $16937*3+2490*2+5*1+37*1 = 55833$.

To be sure of this result it is enough to use the numeric table from the work [1] in which in the second column the quantity of groups in the corresponding equivalence class (including a group-representative) is given, and in the third column – the numeric distribution of groups of type M^3 (obtained from the given group-representative) in 6, 3, 2 and 1 group.

As the number of different junior space groups of $(22'\underline{1})$-symmetry gives us the number of 2- and 3-middle groups of $(42\underline{1})$- and $(62\underline{1})$-symmetries respectively, then we obtain 55833 2-middle groups of $(42\underline{1})$-symmetry and 55833 3-middle groups of $(62\underline{1})$-symmetry.

Thus, for $(42\underline{1})$-symmetry 55833+52761=108594 groups, and for $(62\underline{1})$-symmetry 55833+55637=111470 groups are derived.

So, the number of all possible space groups G_3^P of complete 32 crystallographic P-symmetries in geometric classification is equal to 436011.

References

[1] Shenesheutskaia A.A. *Junior spatial groups of* $(22\underline{1})$*-symmetry.* Buletinul Academiei de Ştiinţe a Republicii Moldova. Matematica, 2006, N 1(50), p. 105-108.

[2] Zamorzaev A.M. *On strong isomorphism of groups and isomorphism of P-symmetries.* Izvestia AN RM. Matematika, 1994, N 1, p. 75-84 (in Russian).

State University of Moldova
A. Mateevici str. 60
Chisinau MD-2009
Moldova

Received October 28, 2005