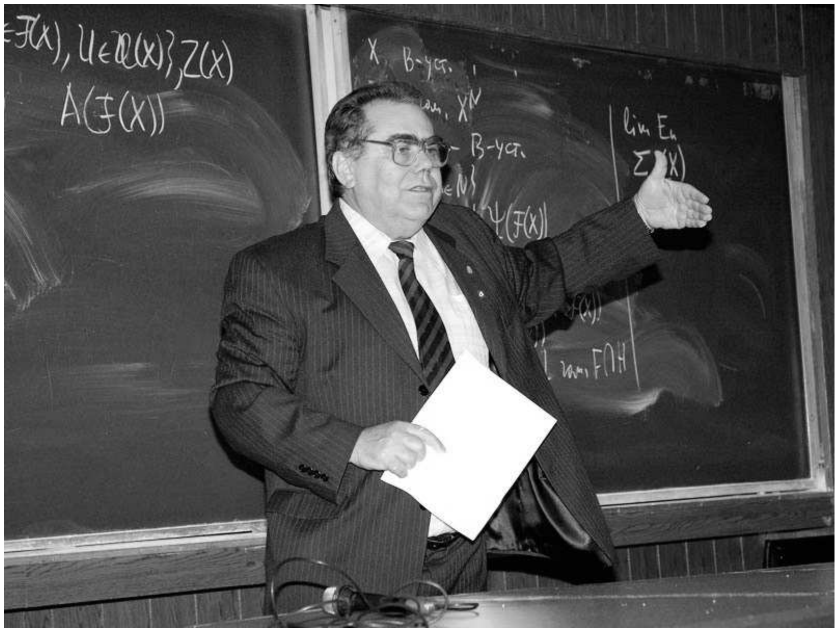

**Academician Mitrofan M. Choban**

FOUNDER OF THE MOLDAVIAN SCIENTIFIC
SCHOOL IN TOPOLOGY

On January 5, 2012, Academician Mitrofan Choban, Professor at Tiraspol State University (located in Chisinau) and President of Mathematical Society of the Republic of Moldova, was honored for his achievements, in connection with the 70th anniversary.

A work-tribute "Academicianul Mitrofan Ciobanu la a 70-a aniversare" was published in 2012 by the Academy of Sciences of Moldova and Tiraspol State University. This book includes articles (in English, Russian, or Romanian) about the scientific and social activity of M. Choban, written by scientists from different countries and by public persons from Moldova, who know Professor M. Choban. Prominent mathematicians in the area A.V. Arhangel'skii ("M. V. Lomonosov" Moscow State University, Russia; Ohio University, USA), Petar S. Kenderov (Academy of Sciences of Bulgaria, Institute of Mathematics), S. I. Nedev (Institute of Mathematics, Bulgaria), V. V. Fedorchuk ("M. V. Lomonosov" Moscow State University, Russia), R. Miron (Iasi, Romania), M. Abel (University of Tartu, Estonia) highly appreciated Mitrofan Choban's contribution in mathematics and education. They noted that Professor M. Choban was one of the first mathematicians who studied the existence of special set-valued selections for set-valued mappings and obtained important results on the existence of measurable selections of multivalued mappings. A well known problem of Hausdorff on Boolean classes was solved by M. Choban using his theory of multivalued mappings. At present one of the methods of construction of selections is known as "Choban selection procedure". A technique for the characterization of various topological invariants (topological dimension, metacompactness, etc.) was developed by him. Professor M. Choban essentially developed the general descriptive theory of topological spaces; he solved the problem of zero-dimensional representations of universal topological algebras and suggested an approximation method for such algebras. A distinct idea in Choban's research is the application of topology to the study of functional spaces where he obtained deep results on functional equivalence of spaces, on extensions of continuous functions in topological spaces. A. V. Arhangel'skii also noted that M. Choban was continuing the line of A. I. Mal'tsev in topological algebra and that "the modern theory of free universal topological algebra is his creation". The book also contains some M. Choban's memories and surveys of his scientific results.

Academician Mitrofan Choban is a leader of research in Topology and Topological Algebra and he published over 200 papers and 20 books in many branches of mathematics. The following problems were solved by M. Choban: Hausdorff's problem on Borelian classes of sets; Alexandroff's problem about the structure of compact subsets of countable pseudocharacter in topological groups; Arhangel'skii's problem on the zero-dimensional representation of topological universal algebras; two Maltsev's problems on free topological universal algebras; two Michael's problems about $G_\delta$ -sections of open mappings of compact spaces and of the $k$-coverings of open compact mappings of paracompact spaces; Phelps' problem about the structure of the set of points of Gateaux differentiability of convex functionals (with P. Kenderov and J. Revalski); Tichonoff's problem about well-possedness of optimization problems in the Banach spaces of continuous functions (with P. Kenderov and J. Revalski); Confort's problem about Baire isomorphism of compact groups; Pasynkov's problem about Raikov completion of topological groups; Arhangel'skii's problem on metrizability of $o$-metrizable topological groups (with S. Nedev); Pelczinski's and Semadeni's problems about structure of Banach spaces of continuous functions on special compact subsets of quotient spaces of topological groups.

Detailed information about biography of Professor M. Choban and his scientific activity can be found also in our journal "Buletinul Academiei de Ştiinţe a Republicii Moldova. Matematica", No. 1(38), 2002, 118–123.

Most of the articles included in this issue are based on scientific results delivered at the 20th Conference on Applied and Industrial Mathematics (see pp. 132–134) dedicated to the 70th anniversary of Academician Mitrofan M. Choban.

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# Liouville's theorem for vector-valued functions

Mati Abel *

**Abstract.** It is shown in [2] that any $X$-valued analytic map on $\mathbb{C} \cup \{\infty\}$ is a constant map in case when $X$ is a strongly galbed Hausdorff space. In [3] this result is generalized to the case when $X$ is a topological linear Hausdorff space, the von Neumann bornology of which is strongly galbed. A new detailed proof for the last result is given in the present paper. Moreover, it is shown that for several topological linear spaces the von Neumann bornology is strongly galbed or pseudogalbed.

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**Keywords and phrases:** Liouville's theorem, vector-valued analytic function, metrizable linear space, galbed space, locally pseudoconvex space, $F$-space, von Neumann bornology, strictly galbed bornology, pseudogalbed bornology.

In 1847 Joseph Liouville presented in his lecture the following result (which was published by A. L. Cauchy in 1844 but now is known as Liouville's theorem): *every bounded entire function* $f : \mathbb{C} \to \mathbb{C}$ *is a constant function.* In the theory of Banach algebras the following generalization of this result is used (see, for example, [4, Theorem 3.12]): *if* $X$ *is a complex normed space and* $f$ *a bounded weakly holomorphic* $X$*-valued map on* $\mathbb{C}$*, then* $f$ *is a constant map.*

In 1947 (see [6, Theorem 1]) Richard Arens generalized this result to the case of a locally convex Hausdorff space $X$ and later on to the case of a topological linear Hausdorff space $X$ the topological dual of which has nonzero elements. It is well-known (see, for example, [10, p. 158]) that topological linear spaces which are not locally convex could not have any nonzero continuous functionals. In this case[1] instead of $X$-valued holomorphic functions the $X$-valued analytic functions are used.

In 1973 (see [12, Corollary, p. 56]) Philippe Turpin gave the following generalization of Liouville's theorem: *if* $X$ *is an exponentially galbed Hausdorff space and* $f$ *is an analytic* $X$*-valued map on*[2] $\mathbb{C}_\infty$*, then* $f$ *is a constant map.* In 2004 (see [2, Theorem 2.1]) Mati Abel generalized this result to the case of strongly galbed Hausdorff space $X$. Moreover, in 2008 he presented in [3, Theorem 3.1] the following result:

**Theorem 1.** *Let* $X$ *be a topological linear Hausdorff space over* $\mathbb{C}$*. If the von Neumann bornology* $\mathcal{B}_N$ *of* $X$ *is strongly galbed, then every* $X$*-valued analytic map on* $\mathbb{C}_\infty$ *is a constant map.*



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[1] In 1966 Lucien Waelbroeck (see [14]) gave conditions for $X$-valued holomorphic map $f$ on $\mathbb{C}$ to be constant in case of complete pseudoconvex space $X$, generalizing for it the integral theory for such maps. Unfortunately, his results have been presented mostly without complete proofs. He gave only hints for some parts how to prove.

[2] Here and later on $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

A new detailed proof for this result is given in the present paper. Moreover, it is shown that for several topological linear spaces the von Neumann bornology is strongly galbed.

## 1 Introduction

**1.** Let $X$ be a topological linear space over $\mathbb{K}$, the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. By *$F$-seminorm* on $X$ we mean a map $q : X \to \mathbb{R}^+$ which has the following properties:

(1) $q(\lambda x) \leqslant q(x)$ for each $x \in X$ and $\lambda \in \mathbb{K}$ with $|\lambda| \leqslant 1$;

(2) $\lim_{n\to\infty} q\left(\frac{1}{n}x\right) = 0$ for each $x \in X$;

(3) $q(x + y) \leqslant q(x) + q(y)$ for each $x, y \in X$.

If from $q(x) = 0$ it follows that $x = \theta_X$ (the zero element of X), then $q$ is an *$F$-norm* on $X$. In this case $d$ with $d(x, y) = q(x - y)$ for each $x, y \in X$ defines a metric on $X$ such that $d(x + z, y + z) = d(x, y)$ for each $x, y, z \in X$.

It is well-known (see, for example, [9, p. 39, Theorem 3]) that the topology of any topological linear spaces coincides with the initial topology defined on by a collection of $F$-seminorms. A topological linear space $(X, \tau)$ topology $\tau$ of which has been defined by a $F$-norm $\| \ \|$ and $X$ is complete with respect to $\| \ \|$ is an *$F$-space*. Moreover, if $X$ is a *locally pseudoconvex space* (see, [11, p. 4], or [15, p. 4]), then $X$ has a base $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ of neighborhoods of zero consisting of balanced ($\mu U_\lambda \subset U_\lambda$ when $|\,\mu\,| \leqslant 1$) and pseudoconvex ($U_\lambda + U_\lambda \subset \mu U_\lambda$ for $\mu \geqslant 2$) sets. This base defines a set of numbers $\{k_\lambda : \lambda \in \Lambda\}$ in $(0, 1]$ (see, for example, [10, pp. 161–162] or [15, pp. 3–6]) such that

$$U_\lambda + U_\lambda \subset 2^{\frac{1}{k_\lambda}} U_\lambda$$

and

$$\Gamma_{k_\lambda}(U_\lambda) \subset 2^{\frac{1}{k_\lambda}} U_\lambda$$

for each $\lambda \in \Lambda$, where

$$\Gamma_k(U) =$$

$$= \Big\{ \sum_{\nu=1}^{n} \mu_\nu u_\nu : n \in \mathbb{N}, u_1, \ldots, u_n \in U \text{ and } \mu_1, \ldots, \mu_n \in \mathbb{K} \text{ with } \sum_{\nu=1}^{n} |\,\mu_\nu\,|^k \leqslant 1 \Big\}$$

for any subset $U$ of $X$ and $k \in (0, 1]$. The set $\Gamma_k(U)$ is the *absolutely $k$-convex hull* of $U$ in $X$. A subset $U \subset X$ is *absolutely $k$-convex* if $U = \Gamma_k(U)$ and is *absolutely pseudoconvex* if $U = \Gamma_k(U)$ for some $k \in (0, 1]$. In the case when

$$\inf\{k_\lambda : \lambda \in \Lambda\} = k > 0,$$

$X$ is a *locally $k$-convex space* and when $k = 1$, then a *locally convex space*.

It is known (see [15, pp. 3–6] or [7, pp. 189 and 195]) that the topology on a locally pseudoconvex space $X$ can be defined by a family $\mathcal{P} = \{p_\lambda : \lambda \in \Lambda\}$ of $k_\lambda$-homogeneous seminorms (that is, $p_\lambda(\mu a) = \mid \mu \mid^{k_\lambda} p_\lambda(a)$ for each $\lambda \in \Lambda$, $\mu \in \mathbb{K}$ and $a \in E$), where the power of homogeneity $k_\lambda \in (0,1]$ for each $\lambda \in \Lambda$ and every seminorm $p_\lambda$ is defined by

$$p_\lambda(a) = \inf\{|\,\mu\,|^{k_\lambda} \colon a \in \mu\Gamma_{k_\lambda}(U_\lambda)\}$$

for each $a \in A$.

Let now $l$ be the set of all $\mathbb{K}$-valued sequences $(x_n)$ for which $\sum_{k=0}^{\infty} |x_n| < \infty$, $l^0$ be the subset of $l$ of sequences with only finite number of nonzero elements and let $l_0 = l \setminus l^0$.

A topological linear space $X$ is a *galbed space* (see [2]) if there exists a sequence $(\alpha_n)$ in $l_0$ and for every neighbourhood $O$ of zero in $X$ there is another neighbourhood $U$ of zero such that[3]

$$\bigcup_{n\in\mathbb{N}_0} \Big\{ \sum_{k=0}^{n} \alpha_k u_k : u_0, \ldots, u_n \in U \Big\} \subset O.$$

In particular, when

$$\alpha_0 \neq 0 \ \text{ and } \ \alpha = \inf_{n>0} |\alpha_n|^{\frac{1}{n}} > 0, \tag{1}$$

a galbed space $X$ is *strongly galbed* and $X$ is *exponentially galbed* when $\alpha_n = \frac{1}{2^n}$ for each $n \in \mathbb{N}_0$. It is known (see [1, Proposition 2] or [3, Corollary 2.2]) that every locally pseudoconvex space is exponentially galbed (hence strongly galbed too).

**2.** A *bornology* on a set $X$ is a collection $\mathcal{B}$ of subsets of $X$ which satisfies the following conditions:

($a$) $X = \bigcup_{B\in\mathcal{B}} B$;

($b$) if $B \in \mathcal{B}$ and $C \subseteq B$, then $C \in \mathcal{B}$;

($c$) if $B_1, B_2 \in \mathcal{B}$, then $B_1 \cup B_2 \in \mathcal{B}$.

If $X$ is a linear space over $\mathbb{K}$, a bornology $\mathcal{B}$ on $X$ is called a *linear* or *vector bornology* if the following conditions are satisfied:

($d$) if $B_1, B_2 \in \mathcal{B}$, then $B_1 + B_2 \in \mathcal{B}$;

($e$) if $B \in \mathcal{B}$ and $\lambda \in \mathbb{K}$, then $\lambda B \in \mathcal{B}$;

($f$) $\bigcup_{|\lambda|\leqslant 1} \lambda B \in \mathcal{B}$ for every $B \in \mathcal{B}$.

A linear bornology $\mathcal{B}$ on a linear space $X$ is *convex* if $\Gamma_1(U) \in \mathcal{B}$ for every $U \in \mathcal{B}$ and *pseudoconvex* if there exists a number $k \in (0,1]$ such that $\Gamma_k(U) \in \mathcal{B}$ for every

[3] Here and later on $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$.

$U \in \mathcal{B}$. Moreover, a bornology $\mathcal{B}$ on a linear space $X$ over $\mathbb{K}$ is a *galbed bornology* (see [3]) if there is a sequence $(\alpha_n)$ in $l_0$ such that

$$S((\alpha_n), B) = \bigcup_{n \in \mathbb{N}_0} \Big\{ \sum_{k=0}^{n} \alpha_k b_k : b_0, \dots, b_n \in B \Big\} \in \mathcal{B} \tag{2}$$

for all $B \in \mathcal{B}$. In particular, when $(\alpha_n)$ satisfies the condition (1), $\mathcal{B}$ is a *strongly galbed bornology* on $X$, and when $\alpha_n = \frac{1}{2^n}$ for each $n \in \mathbb{N}$, $\mathcal{B}$ is *an exponentially galbed bornology* on $X$ (see [5]). Moreover, we shall say that a bornology $\mathcal{B}$ is *pseudogalbed* if for every $B \in \mathcal{B}$ there exists a sequence $(\alpha_n) \in l_0$ such that $S((\alpha_n), B) \in \mathcal{B}$. In particular, when $(\alpha_n)$ satisfies the condition (1), we shall say that the bornology $\mathcal{B}$ is *strongly pseudogalbed*.

**3.** Let $X$ be a topological linear space over $\mathbb{C}$. An $X$-valued map $f$ on $\mathbb{C}_\infty$ is *analytic at* $\lambda_0 \in \mathbb{C}$ if there exists a number $\varepsilon > 0$ and a sequence $(x_n)$ in $X$ such that

$$f(\lambda_0 + \lambda) = \sum_{k=0}^{\infty} x_k \lambda^k$$

whenever $|\lambda| < \varepsilon$, and is *analytic at* $\infty$ if there exists a number $R > 0$ and a sequence $(y_k)$ in $X$ such that

$$f(\lambda) = \sum_{k=0}^{\infty} \frac{y_k}{\lambda^k}$$

whenever $|\lambda| > R$.

If $X$ is a topological linear space, then the set of all bounded sets forms a linear bornology which is called the *von Neumann bornology* on $X$ or the *bornology on $X$ defined by the topology* of $X$.

## 2 Topological linear spaces with strongly galbed and pseudogalbed von Neumann bornology

First we describe these topological linear spaces, the von Neumann bornology $\mathcal{B}_N$ of which is strongly galbed[4].

**Proposition 1** (see [3])**.** *The von Neimann bornology of any strongly galbed space is strongly galbed.*

*Proof.* Let $X$ be a strongly galbed space. Then there exists a sequence $(\alpha_n) \in l_0$ which satisfies the condition (1), and for every neighbourhood $O$ of zero in $X$ there is another neighbourhood $U$ of zero such that $S((\alpha_n), U) \subseteq O$. Moreover, for any $B \in \mathcal{B}_N$ there is a number $\mu_B > 0$ such that $B \subseteq \mu_B U$. Since

$$S((\alpha_n), B) \subseteq S((\alpha_n), \mu_B U) \subseteq \mu_B O,$$

then $S((\alpha_n), B) \in \mathcal{B}_N$ for every $B \in \mathcal{B}_X$. Hence $\mathcal{B}_N$ is strongly galbed. □

[4]Proposition 1 is proved in [3]. A modified proof for this result is given here.

**Corollary 1.** *The von Neumann bornology of every exponentially galbed apace is strongly galbed.*

**Proposition 2.** *The von Neumann bornology of any metrizable topological linear space is pseudogalbed.*

*Proof.* Let $X$ be a metrizable topological linear space. Then $X$ has a countable base $\mathcal{L}_X = \{O_n : n \in \mathbb{N}_0\}$ of balanced neighbourhoods of zero. We can assume that $O_{n+1} + 0_{n+1} \subseteq O_n$ for each $n \in \mathbb{N}_0$ (the addition in $X$ is continuous). Let $O$ be an arbitrary neighbourhood of zero in $X$. Then there is a number $n_0 \in \mathbb{N}_0$ such that $O_{n_0} \subseteq O$ and

$$\bigcup_{n \geqslant n_0} \sum_{k=n_0}^{n} O_{k+1} \subseteq O_{n_0},$$

because

$$O_{n_0+1} + \cdots + O_{n+1} \subseteq O_{n_0+1} + \cdots + O_n + O_n \subseteq O_{n_0+1} + \cdots O_{n-1} + O_{n-1} \subseteq$$

$$\subseteq \cdots \subseteq O_{n_0+1} + O_{n_0+1} \subseteq O_{n_0}$$

for each $n \geqslant n_0$.

Let $B \in \mathcal{B}_N$ be a balanced set. Then for each $k \in \mathbb{N}_0$ there exists a number $\mu_k = \mu_k(B) > 1$ such that $B \subseteq \mu_k O_{n_0+k+1}$. Here $\mu_k \leqslant \mu_{k+1}$ because $O_{n+1} \subseteq O_n$ for each $n \in \mathbb{N}_0$. Put

$$\alpha_n = \frac{1}{\max\{\mu_n, \mu_1^n\}}$$

for each $n \in \mathbb{N}_0$. Then $|\alpha_n| \leqslant \frac{1}{\mu_1^n}$ for each $n \in \mathbb{N}_0$. Hence $(\alpha_n) \in l_0$. Since

$$\sum_{k=0}^{n} \alpha_k b_k \in \sum_{k=0}^{n} \Big(\frac{\mu_k}{\max\{\mu_k, \mu_1^k\}} O_{n_0+k+1}\Big) \subseteq \sum_{k=n_0}^{n_0+n} O_{k+1}$$

$$\subset \bigcup_{n \geqslant n_0} \sum_{k=n_0}^{n} O_{k+1} \subseteq O_{n_0} \subseteq O$$

for each $n \geqslant 0$ and each choice of elements $b_0, b_1, \ldots, b_n \in B$, then

$$S((\alpha_n), B) = \bigcup_{n \in \mathbb{N}_0} \Big\{ \sum_{k=0}^{n} \alpha_k b_k : b_0, \ldots, b_n \in B \Big\} \subseteq O.$$

Hence, $S((\alpha_n), B) \in \mathcal{B}_N$, because of which $\mathcal{B}_N$ is pseudogalbed. □

**Corollary 2.** *The von Neumann bornology of every $F$-space is pseudogalbed.*

## 3 Proof of Theorem 1

Now we give a new and detailed proof for Theorem 1.

*Proof.* Let $X$ be a topological linear Hausdorff space and $f$ an $X$-valued analytic map on $\mathbb{C}_\infty$. We can assume that $X$ is complete, otherwise we consider $X$ as a dense subset in $\tilde{X}$, the completion of $X$, and $f$ as $\tilde{X}$-valued analytic map on $\mathbb{C}_\infty$.

Let first $\lambda_0 \in \mathbb{C}$. Then there is a number $r > 0$ and a sequence $(x_n)$ in $X$ such that

$$f(\lambda_0 + \lambda) = \sum_{k=0}^{\infty} x_k \lambda^k$$

whenever $|\lambda| < r$. By assumption, the von Neumann bornology $\mathcal{B}_N$ of $X$ is strongly galbed. Therefore there exists a sequence $(\alpha_n) \in l_0$ with $\alpha < 1$ such that (2) holds for any $B \in \mathcal{B}_N$. Take $r_0 \in (0, \frac{r}{\alpha})$. Then the series

$$\sum_{k=0}^{\infty} x_k (\alpha r_0)^k$$

converges in $X$. Therefore the sequence $(x_n(\alpha r_0)^n)$ tends to zero in $X$. Hence, the set $\{x_n(\alpha r_0)^n : n \in \mathbb{N}_0\}$ is bounded in $X$. Let $U_{\lambda_0} = \{\lambda_0 + \lambda : |\lambda| < \alpha^2 r_0\}$ and

$$X_{\lambda_0} = \bigcup_{n \in \mathbb{N}_0} \Big\{ \sum_{k=0}^{n} x_k (\alpha r_0)^k t_k : (t_k) \text{ is a sequence with } |t_k| \leqslant \alpha^k \text{ for each } k \Big\}.$$

Then $X_{\lambda_0}$ is an absolutely convex and bounded set in $A$. Indeed, if $\lambda, \mu \in \mathbb{C}$ with $|\lambda| + |\mu| \leqslant 1$ and $x, y \in X_{\lambda_0}$, then there exists $n_1, n_2 \in \mathbb{N}_0$ and $t_0^x, \ldots, t_{n_1}^x$ and $t_0^y, \ldots, t_{n_2}^y$ such that $|t_k^x| \leqslant \alpha^k$ and $|t_k^y| \leqslant \alpha^k$ for each $k$,

$$x = \sum_{k=0}^{n_1} x_k (\alpha r_0)^k t_k^x$$

and

$$y = \sum_{k=0}^{n_2} x_k (\alpha r_0)^k t_k^y.$$

If $n_1 > n_2$, then we put

$$t_{n_2+1}^y = \cdots = t_{n_1}^y = 0$$

(otherwise we act similarly), then

$$\lambda x + \mu y = \sum_{k=0}^{n_1} x_k (\alpha r_0)^k (\lambda t_k^x + \mu t_k^y) \in X_{\lambda_0},$$

because

$$|\lambda t_k^x + \mu t_k^y| \leqslant |\lambda||t_k^x| + |\mu||t_k^y| \leqslant \alpha^k(|\lambda| + |\mu|) \leqslant \alpha^k$$

for each $k$. Thus $X_{\lambda_0}$ is an absolutely convex set.

To show that $X_{\lambda_0}$ is bounded, let $O$ be an arbitrary balanced neighbourdhood of zero in $X$. Because $(x_n(\alpha r_0)^n)$ is a bounded sequence in $X$, there is a number $\rho > 0$ such that $x_n(\alpha r_0)^n \in \rho O$ for each $n \in \mathbb{N}_0$. Therefore

$$x_n(\alpha r_0)^n \frac{t_n}{\alpha_n} = x_n(\alpha r_0)^n \frac{t_n}{\alpha^n}\frac{\alpha^n}{\alpha_n} \in \rho\Big(\frac{t_n}{\alpha^n}\frac{\alpha^n}{\alpha_n}O\Big) \subset \rho O$$

for all $n \in \mathbb{N}_0$ and all $(t_n)$ with $\frac{|t_n|}{\alpha^n} \leqslant 1$ for every $n$, because $\frac{\alpha^n}{|\alpha_n|} \leqslant 1$ and $O$ is balanced. Hence, the set

$$B = \Big\{x_n(\alpha r_0)^n \frac{t_n}{\alpha_n} : n \in \mathbb{N}_0, (t_n) \text{ is a sequence with } |t_n| \leqslant \alpha^n \text{ for each } n\Big\} \in \mathcal{B}_N.$$

Thus, $X_{\lambda_0} \subset S((\alpha_n), B) \in \mathcal{B}_N$, because the von Neumann bornology $\mathcal{B}_N$ is strongly galbed. Moreover, it is easy to see that $(S_n)$, where

$$S_n = \sum_{k=0}^{n} x_k(\alpha r_0)^k t_k$$

for each $n \in \mathbb{N}_0$ and fixed sequence $(t_n)$ with $|t_n| \leqslant \alpha^n$ for each $n$, is a Cauchy sequence in $X$. To show this, let $O$ be an arbitrary neighbourhood of zero in $X$ and $m \in \mathbb{N}$ a fixed number. Then there exists a balanced neighbourhood $O_1$ of zero in $X$ such that

$$\underbrace{O_1 + \cdots + O_1}_{m \text{ summands}} \subset O$$

and a positive number $\rho$ such that $x_n(\alpha r_0)^n \subset \rho O_1$ for all $n \in \mathbb{N}_0$ because the sequence $(x_n(\alpha r_0)^n)$ is bounded. Since $\alpha < 1$, then the sequence $(\alpha^n)$ vanishes. Hence, there is a number $n_0 \in \mathbb{N}_0$ such that $\alpha^n < \frac{1}{\rho}$ whenever $n > n_0$. Since

$$S_{n+m} - S_n = \sum_{k=n+1}^{n+m} x_k(\alpha r_0)^k t_k \in \rho O_1 t_{n+1} + \cdots + \rho O_1 t_{n+m} \subset \underbrace{O_1 + \cdots + O_1}_{m \text{ summands}} \subset O$$

whenever $n > n_0$ for every fixed $m \in \mathbb{N}_0$, then $(S_n)$ is a Cauchy sequence in $X$. Hence, $(S_n)$ converges in $X$. Therefore

$$\sum_{k=0}^{\infty} x_k(\alpha r_0)^k t_k \in X$$

for every fixed $(t_n)$ such that $|t_n| \leqslant \alpha^n$ for each $n$. It is easy to show that the closure $K_{\lambda_0}$ of the set $X_{\lambda_0}$ in $X$ is a closed, bounded and absolutely convex subset of $X$. Therefore (see, for example,[8, pp. 8–9]), the linear hull $A_{\lambda_0}$ in $X$, generated by $K_{\lambda_0}$, is a normed space with respect to the norm[5] $p_{\lambda_0}$ defined by

$$p_{\lambda_0}(a) = \inf\{\lambda > 0 : a \in \lambda K_{\lambda_0}\}$$

[5] Here $p_{\lambda_0}$ is a norm on $A_{\lambda_0}$ because $K_{\lambda_0}$ is bounded.

for each $a \in A_{\lambda_0}$. Taking this into account, we have

$$f(\lambda_0 + \lambda) = \sum_{k=0}^{\infty} x_k (\alpha r_0)^k \Big(\frac{\lambda}{\alpha r_0}\Big)^k \in$$

$$\in \Big\{ \sum_{k=0}^{\infty} x_k (\alpha r_0)^k t_k : (t_n) \text{ is a sequence with } |t_k| \leqslant \alpha^k \text{ for each } k \Big\} \subset K_{\lambda_0} \subset A_{\lambda_0}$$

whenever $|\lambda| < \alpha^2 r_0$. Consequently, for any point $\lambda \in \mathbb{C}$ there is an open neighbourhood $U_\lambda$ of $\lambda$ and a normed subspace $A_\lambda$ of $X$ such that the restriction $f|_{U_\lambda}$ of $f$ to $U_\lambda$ has values in $A_\lambda$.

Since $f$ is also analytic at $\infty$, then there is a sequence $(z_n)$ in $X$ and a number $R > 0$ such that

$$f(\lambda) = \sum_{k=0}^{\infty} \frac{z_k}{\lambda^k}$$

whenever $|\lambda| > R$. Let $R_0 \in (\alpha R, \infty)$. Then the series

$$\sum_{k=0}^{\infty} \frac{z_k \alpha^k}{R_0^k}$$

converges in $X$. Therefore the sequence $\big(\frac{z_n \alpha^n}{R_0^n}\big)$ is bounded in $X$.

Let $U_\infty = \{\lambda : |\lambda| > \frac{R_0}{\alpha^2}\}$ and

$$X_\infty = \bigcup_{n \in \mathbb{N}_0} \Big\{ \sum_{k=0}^{n} \frac{z_k \alpha^k}{R_0^k} t_k : (t_k) \text{ is a sequence with } |t_k| \leqslant \alpha^k \text{ for each } k \Big\}.$$

Then $X_\infty$ is an absolutely convex and bounded set in $X$. Indeed, if $\lambda, \mu \in \mathbb{C}$ with $|\lambda| + |\mu| \leqslant 1$ and $x, y \in X_\infty$, then there exist $n_1, n_2 \in \mathbb{N}_0$ and $t_0^x, \dots, t_{n_1}^x$ and $t_0^y, \dots, t_{n_2}^y$ such that $|t_k^x| \leqslant \alpha^k$ and $|t_k^y| \leqslant \alpha^k$ for each $k$,

$$x = \sum_{k=0}^{n_1} \frac{z_k \alpha^k}{R_0^k} t_k^x$$

and

$$y = \sum_{k=0}^{n_2} \frac{z_k \alpha^k}{R_0^k} t_k^y.$$

If $n_1 > n_2$, we put again

$$t_{n_2+1}^y = \cdots = t_{n_1}^y = 0$$

(otherwise we act similarly). Therefore

$$\lambda x + \mu y = \sum_{k=0}^{n_1} \frac{z_k \alpha^k}{R_0^k} (\lambda t_k^x + \mu t_k^y) \in X_\infty,$$

because

$$|\lambda t_k^x + \mu t_k^y| \leqslant |\lambda||t_k^x| + |\mu||t_k^y| \leqslant \alpha^k(|\lambda| + |\mu|) \leqslant \alpha^k$$

for each $k$.

Let $O$ be again an arbitrary balanced neighbourhood of zero in $X$. Because $\left(z_n \frac{\alpha^n}{R_0^n}\right)$ is a bounded sequence in $X$, there is a number $\pi > 0$ such that $z_n \frac{\alpha^n}{R_0^n} \in \pi O$ for each $n \in \mathbb{N}_0$. Therefore

$$\frac{z_n \alpha^n}{R_0^n} \frac{t_n}{\alpha_n} = \frac{z_n \alpha^n}{R_0^n} \frac{t_n}{\alpha^n} \frac{\alpha^n}{\alpha_n} \in \pi\Big(\frac{t_n}{\alpha^n} \frac{\alpha^n}{\alpha_n} O\Big) \subset \pi O$$

for all $n \in \mathbb{N}_0$ and all $(t_n)$ with $\frac{|t_n|}{\alpha^n} \leqslant 1$ for each $n$, because $\frac{\alpha^n}{|\alpha_n|} \leqslant 1$ and $O$ is balanced. Hence, the set

$$B' = \Big\{\frac{z_n \alpha^n}{R_0^n} \frac{t_n}{\alpha_n} : n \in \mathbb{N}_0, (t_n) \text{ is a sequence with } |t_n| \leqslant \alpha^n \text{ for each } n\Big\} \in \mathcal{B}_N.$$

Hence, $X_\infty \subset S((\alpha_n), B') \in \mathcal{B}_N$ because the von Neumann bornology $\mathcal{B}_N$ is strongly galbed. Thus, the closure $K_\infty$ of the set $X_\infty$ in $X$ is a closed, bounded and absolutely convex subset of $X$. Therefore (similarly as above) the linear hull $A_\infty$ in $X$, gnerated by $K_\infty$, is a normed space with respect to the norm $p_\infty$, defined by

$$p_\infty(a) = \inf\{\lambda > 0 : a \in \lambda K_\infty\}$$

for each $a \in A_\infty$. The same way as in the first part of the proof,

$$\sum_{k=0}^{\infty} \frac{z_k \alpha^k}{R_0^k} t_k \in X$$

for every fixed $(t_n)$ such that $|t_n| \leqslant \alpha^n$ for each $n$. Since

$$f(\lambda) = \sum_{k=0}^{\infty} \frac{z_k \alpha^k}{R_0^k} \Big(\frac{R_0}{\alpha\lambda}\Big)^k \in$$

$$\in \Big\{\sum_{k=0}^{\infty} \frac{z_k \alpha^k}{R_0^k} t_k : (t_k) \text{ is a sequence with } |t_k| \leqslant \alpha^k \text{ for each } k\Big\} \subset K_\infty \subset A_\infty$$

whenever $|\lambda| > \frac{R_0}{\alpha^2}$, there is an open neighbourhood $U_\infty$ of $\infty$ and a normed subspace $A_\infty$ of $X$ such that the restriction $f|_{U_\infty}$ of $f$ to $U_\infty$ has values in $A_\infty$.

Now $\{U_\lambda : \lambda \in \mathbb{C}\}$ and $U_\infty$ form an open cover of $\mathbb{C}_\infty$. Since $\mathbb{C}_\infty$ is compact, there are numbers $n \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that

$$\mathbb{C}_\infty = U_\infty \cup \Big(\bigcup_{k=1}^{n} U_{\lambda_k}\Big).$$

Therefore

$$f(\mathbb{C}_\infty) = f(U_\infty) \bigcup \Big(\bigcup_{k=1}^{n} f(U_{\lambda_k})\Big) \subset A_1 = A_\infty \bigcup \Big(\bigcup_{k=1}^{n} A_{\lambda_k}\Big) \subset A_0,$$

where $A_0$ is the linear hull of $A_1$. Without loss of generality we can assume that every element

$$x = \lambda_1 x_1 + \cdots + \lambda_m x_m \in A_0$$

has been presented in the form

$$x = a_1 + \cdots + a_n + a_{n+1},$$

where $a_k \in A_{\lambda_k}$ for each $k \in \{1, 2, \ldots, n\}$ and $a_{n+1} \in A_\infty$, denoting by $a_1$ the zero element if none of elements $\lambda_1 x_1, \ldots, \lambda_m x_m$ does not belong to $A_{\lambda_1}$ or the sum of all elements from $\lambda_1 x_1, \ldots, \lambda_m x_m$ which belong to $A_{\lambda_1}$; by $a_2$ the zero element if none of remainder elements from $\lambda_1 x_1, \ldots, \lambda_m x_m$ does not belong to $A_{\lambda_2}$ or the sum of all remainder elements from $\lambda_1 x_1, \ldots, \lambda_m x_m$ which belong to $A_{\lambda_2}$ and so on.

Now, for every $x \in A_0$ let

$$N(x) = \{\lambda \in \{\lambda_1, \ldots, \lambda_n, \infty\} : x \in A_\lambda\}$$

and let $p$ be the map on $A_0$, defined by

$$p(x) = \sum_{k=1}^{n+1} \max_{\lambda \in N(a_k)} p_\lambda(a_k)$$

for every $x = a_1 + \cdots + a_{n+1} \in A_0$. It is easy to check that $p$ is a norm on $A_0$. Hence $f$ maps $\mathbb{C}_\infty$ into the normed space $A_0$. Now, it is easy to show that $\varphi \circ f$ is a $\mathbb{C}$-valued analytic function on $\mathbb{C}_\infty$ for each continuous linear functional $\varphi$ on $A_0$. Hence $\varphi \circ f$ is a constant function by the classical Liouville's Theorem. Since continuous linear functionals separate the points of any normed space, then $f$ is a constant map. □

Now, by Theorem 1, Propositions 1 and Corollaries 1, we have the result of Ph. Turpin (see [12]).

**Corollary 3.** *If $X$ is an exponentially galbed (in particular a locally pseudoconvex) space, then every $X$-valued analytic map on $\mathbb{C}_\infty$ is a constant map.*

## 4 Application

Using the classical Liouville's Theorem, it is easy to prove the Gelfand-Mazur Theorem, that is, *every complex normed division algebra is topologically isomorphic to* $\mathbb{C}$. This result has many generalizations to the case of locally convex and locally pseudoconvex division algebras. Next we give a characterization of complex topological division algebras.

**Theorem 2.** *A complex Hausdorff division algebra*[6] *$A$ is topologically isomorphic to $\mathbb{C}$ if and only if*

[6]We assume here that the multiplication in topological algebras is separately continuous.

a) *every element of $A$ is bounded*[7]*;*
b) *the von Neumann bornology of $A$ is strongly galbed.*

*Proof.* Let $A$ be topologically isomorphic to $\mathbb{C}$. Then every element of $A$ has the form $\lambda e_A$, where $\lambda \in \mathbb{C}$ and $e_A$ is the unit element of $A$ and every bounded set in $A$ is in the form $Ke_A$, where $K$ is a bounded set in $\mathbb{C}$. Therefore, every element of $A$ is bounded. To show that the von Neumann bornology of $A$ is strongly galbed, let $(\alpha_n) \in l_0$ be such that the condition (1) holds, and let $L = \sum_k |\alpha_k|$, $M > 0$ and $K_M = \{\lambda \in \mathbb{C} : |\lambda| < M\}$. Moreover, let $B$ be an arbitrary bounded set in $A$. Then there is a number $M > 0$ such that $B = K_M e_A$. Since

$$\sum_{k=0}^{n} \alpha_k \mu_k e_A = \Big(\sum_{k=0}^{n} \alpha_k \mu_k\Big) e_A$$

for each $n$ and $\mu_1, \dots, \mu_n \in K_M$ and

$$\Big|\sum_{k=0}^{n} \alpha_k \mu_k\Big| \leqslant \sum_{k=0}^{n} |\alpha_k||\mu_k| \leqslant M \sum_{k=0}^{\infty} |\alpha_k| = ML,$$

then

$$\bigcup_{n\in\mathbb{N}_0} \Big\{\sum_{k=0}^{n} \alpha_k \mu_k e_A : \mu_1, \dots, \mu_n \in K_M\Big\} \subseteq K_{ML} e_A.$$

Hence, the von Neumann bornology of $A$ is strongly galbed.

Let now $A$ be a complex Hausdorff division algebra. Then (see [3, proof of Proposition 5.1]) $A$ is topologically isomorphic to $\mathbb{C}$ by Theorem 1. □

Now by Proposition 1, Corollary 1 and Theorem 2 we have

**Corollary 4.** *Every complex strongly galbed (in particular, exponentially galbed) division algebra is topologically isomorphic to $\mathbb{C}$ if and only if every element in $A$ is bounded.*

## References


[1] Abel Mart, Abel Mati. *On galbed algebras and galbed spaces.* Bull. Greek Math. Soc., 2006, **52**, 9–23.

[2] Abel Mati. *Galbed Gelfand-Mazur algebras.* Topological algebras and their applications, 17–24, Contemp. Math., **341**, Amer. Math. Soc., Providence, RI, 2004.

[3] Abel Mati. *Topological algebras with galbed von Neumann bornology.* International Conference on Topological Algebras and their Applications. ICTAA 2008, 18–28, Math. Stud. (Tartu), **4**, Est. Math. Soc., Tartu, 2008.

[4] Allan G. *Introduction to Banach spaces and algebras.* Prepared for publication and with a preface by H. Garth Dales. Oxford Graduate Texts in Math. **20**, Oxford University Press, Oxford, 2011.


[7] Element $a$ in a topological algebra $A$ is *bounded* if there exists a number $\lambda > 0$ such that the set $\Big\{\Big(\frac{a}{\lambda}\Big)^n : n \in \mathbb{N}\Big\}$ is bounded in $A$.

[5] Akkar M. *Dualité entre la bornologie d'une algèbre topologique ou bornologique et certaines propriétés particuli?res de ses fonctions puissances.* C. R. Acad. Sci. Paris Sér. **A-B 285**, 1977, No. 7, A509–A512.

[6] Arens R. *Linear topological division algebras.* Bull. Amer. Math. Soc., 1947, **53**, 623–630.

[7] Balachandran V. K. *Topological Algebras.* North-Holland Math. Studies, **185**, Elsevier, Amsterdam, 2000.

[8] Hogbe-Nlend H. *Bornologies and Functional Analysis.* North-Holland Math. Stud., **26**, North-Holland, Amsterdam, 1977.

[9] Jarchow H. *Locally Convex Spaces.* Mathematische Leitfäden, B. G. Teubner, Stuttgart, 1981.

[10] Köthe G. *Topological vector spaces, I.* Springer-Verlag New York Inc., New York, 1969.

[11] Turpin M. P. *Sur une classe d'algèbres topologiques generalisant les algèbres localement bornees.* Ph. D. Thesis Faculty of sciences, University of Grenoble, 1966.

[12] Turpin Ph. *Espaces et opérateurs exponentiellement galbés.* Séminaire Pierre Lelong (Analyse), Année 1973–1974, 48–62. Lecture Notes in Math., **474**, Springer, Berlin, 1975.

[13] Turpin Ph., Waelbroeck L. *Intégration et fonctions holomorphes dans les espaces localement pseudo-convexes.* C. R. Acad. Sci. Paris Sér. **A-B 267**, 1968, A160–A162.

[14] Waelbroeck L. *Continuous inverse locally pseudo-convex algebras.* Summer School Topol. Algebr. Theory, Bruges 1966, 128–185.

[15] Waelbroeck L. *Topological vector spaces and algebras.* Lecture Notes in Math., **230**, Springer-Verlag, Berlin-New York, 1971.

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# On the number of metrizable group topologies on countable groups

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**Abstract.** If a countable group $G$ admits a non-discrete metrizable group topology $\tau_0$, then in the group $G$, there are:
- Continuum of non-discrete metrizable group topologies stronger than $\tau_0$, and any two of these topologies are incomparable;
- Continuum of non-discrete metrizable group topologies stronger than $\tau_0$, and any two of these topologies are comparable.

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## 1 Introduction

Researches on the possibility of the definition of a Hausdorff, group topologies on countable groups were started in [1]. In this work also a method to define such group topologies on any countable group was given.

Later, in [2] it was proved that any infinite Abelian group admits a non-discrete Hausdorff group topology, and in [3] an example of a countable group which does not admit non-discrete Hausdorff group topologies was constructed.

This article is a continuation of the research in this direction. The main results of this article are Theorems 13 and 14.

## 2 Basic results

To highlight the main results we need the following well-known result (see [4], p. 203, Proposition 1, and p. 205, Corollary):

**Theorem 1.** *A set $\Omega$ of subsets of a group $G$ is a basis of the filter of neighborhoods of the unity element of a Hausdorff group topology on $G$ if and only if the following conditions are satisfied:*

*1)* $\bigcap\limits_{V\in\Omega} V = \{e\}$;

*2) For any $V_1$ and $V_2 \in \Omega$, there exists $V_3 \in \Omega$ such that $V_3 \subseteq V_1 \bigcap V_2$;*

*3) For any $V_1 \in \Omega$, there exists $V_2 \in \Omega$ such that $V_2 \cdot V_2 \subseteq V_1$;*

*4) For any $V_1 \in \Omega$, there exists $V_2 \in \Omega$ such that $V_2^{-1} \subseteq V_1$;*

*5) For any $V_1 \in \Omega$ and any element $g \in G$, there exists $V_2 \in \Omega$ such that $g \cdot V_2 \cdot g^{-1} \subseteq V_1$.*

**Remark 2.** From Theorem 1 it easily follows that if a countable group $G$ admits a non-discrete group topology $\tau_0$ such that the topological space $(G, \tau_0)$ is a Hausdorff space, then the group $G$ admits a non-discrete group topology $\tau_1$ such that the topological space $(G, \tau_1)$ is a Hausdorff space, and it has a countable basis of the filter of neighborhoods of the unity element.

**Remark 3.** From ([5], Theorem 8.1.21) it easily follows that a topology $\tau$ of topological group $(G, \tau)$ is given by a metric if and only if the topological space $(G, \tau)$ is a Hausdorff space, and it has a countable basis of the filter of neighborhoods of the unity element.

Such a topology is called *a metrizable topology.*

**Notations 4.** If $V_1, V_2, \dots$ and $S_1, S_2, \dots$ are some sequences of non-empty symmetric subsets of a group $G$, then for each natural number $k$ by induction we define a subset $F_k\Big(V_1, \dots, V_k; S_1, \dots, S_k\Big)$ of $G$ as follows: take $F_1\Big(V_1; S_1\Big) = \Big\{g \cdot V_1 \cdot g^{-1} | g \in S_1\Big\} \bigcup V_1 \cdot V_1$ and $F_{k+1} = F_1\Big(V_1 \bigcup F_k(V_2, \dots, V_{k+1}; S_2, \dots, S_{k+1}); S_1\Big)$.

**Proposition 5.** *For subsets $F_k\Big(V_1, \dots, V_k; S_1, \dots, S_k\Big)$ the following statements are true:*

5.1. *If $e \in V_1$, then $V_1 \subseteq V_1 \cdot V_1 \subseteq F_1(V_1; S_1)$ and $g \cdot V_1 \cdot g^{-1} \subseteq F_1(V_1; S_1)$ for any $g \in S_1$;*

5.2. *If $k \in N$ and the sets $S_i$ and $V_i$ are finite for $1 \le i \le k$, then $F_k\Big(V_1, \dots, V_k; S_1, \dots, S_k\Big)$ is a finite symmetric set;*

5.3. $F_k\Big(\{e\}, \dots, \{e\}; S_1, \dots, S_k\Big) = \{e\}$ *for any* $k \in N$;

5.4. *If $U_i \subseteq V_i$ and $T_i \subseteq S_i$ for any $1 \le i \le k$, then $F_k\Big(U_1, \dots, U_k; T_1, \dots, T_k\Big) \subseteq F_k\Big(V_1, \dots, V_k; S_1, \dots, S_k\Big)$;*

5.5. *If $k, p \in N$ and $e \in V_i$ for all $i \le k$ and $V_{k+j} = \{e\}$ for $1 \le j \le p$, then $F_k\Big(V_1, \dots, V_k; S_1, \dots, S_k\Big) = F_{k+p}\Big(V_1, \dots, V_{k+p}; S_1, \dots, S_{k+p}\Big)$;*

5.6. *For $k \ge 2$ the equality $F_k\Big(V_1, \dots, V_k;\ S_1, \dots, S_k\Big) = F_k\Big(V_1 \bigcup F_{k-1}\Big(V_2, \dots, V_k; S_2, \dots, S_k\Big), \dots, V_{k-1} \bigcup F_1(V_k; S_k), V_k; S_1, \dots, S_k\Big)$ is true;*

5.7. *If $e \in V_i$ for any $1 \le i \le k$, then $V_t \subseteq F_k\Big(V_1, \dots, V_k; S_1, \dots, S_k\Big)$ for any $1 \le t \le k$;*

5.8. *If $e \in V_i$ for any $1 \le i \le k$, then $F_{k+1}\Big(V_s, \dots, V_{k+s}; S_s, \dots, S_{k+s}\Big) \subseteq F_{k+s-t+1}\Big(V_t, \dots, V_{k+s}; S_1, \dots, S_{k+s}\Big)$ for any $k, s, t \in N$ and $t \le s$.*

*Proof.* Statement 5.1 follows easily from the definition of the set $F_1\Big(V_1; S_1\Big)$.

Statements 5.2, 5.3 and 5.4 can be easily proved by induction on $k$, using that the sets $S_i$ and $V_i$ for $i \in N$ are symmetric and the definition of the set $F_k\Big(V_1, \dots, V_k; S_1, \dots, S_k\Big)$.

We prove Statement 5.5 by induction on $k$.

If $k = 1$, then using Statement 5.3 we get $F_{1+p}\Big(V_1, \{e\}, \dots, \{e\}; S_1, \dots, S_{1+p}\Big) = F_1\Big(V_1 \bigcup F_p(\{e\}, \dots, \{e\}; S_2, \dots, S_{1+p}); S_1\Big) = F_1\Big(V_1 \bigcup \{e\}; S_1\Big) = F_1\Big(V_1; S_1\Big)$ for any $p \in N$.

Assume that the equality is proved for the number $k$ and all $p \in N$. Then

$$F_{k+1+p}\Big(V_1, \dots, V_{k+1}, \{e\}, \dots, \{e\}; S_1, \dots, S_{k+1+p}\Big) =$$

$$F_1\Big(V_1 \bigcup F_{k+p}(V_2, \dots, V_{k+1}, \{e\}, \dots, \{e\}; S_2, \dots, S_{k+1+p}); S_1\Big) =$$

$$F_1\Big(V_1 \bigcup F_k(V_2, \dots, V_{k+1}; S_2, \dots, S_{k+1}); S_1\Big) =$$

$$F_{k+1}\Big(V_1, V_2, \dots, V_{k+1}; S_1, S_2, \dots, S_{k+1}\Big).$$

Statement 5.5 is proved for the number $k+1$, and hence, Statement 5.5 is proved for any natural number.

We prove Statement 5.6 by induction on $k$.

If $k = 2$, then $F_2\Big(V_1,\ V_2;\ S_1,\ S_2\Big) = F_1\Big(V_1 \bigcup F_1\Big(V_2;\ S_2\Big);\ S_1\Big) = F_1\Big(V_1 \bigcup F_1\Big(V_2; S_2\Big) \bigcup F_1\Big(V_2; S_2\Big); S_1\Big) = F_2\Big(V_1 \bigcup F_1\Big(V_2; S_2\Big), V_2; S_1, S_2\Big)$.

Assume that the equality holds for the number $k \geq 2$. Then

$$F_{k+1}\Big(V_1, \dots, V_{k+1}; S_1, \dots, S_{k+1}\Big) = F_1\Big(V_1 \bigcup F_k\Big(V_2, \dots, V_k; S_2, \dots, S_k\Big); S_1\Big) =$$

$$F_1\Big(\Big(V_1 \bigcup F_k\Big(V_2, \dots, V_k; S_2, \dots, S_k\Big)\Big) \bigcup F_k\Big(V_2, \dots, V_k; S_2, \dots, S_k\Big)\Big); S_1\Big) =$$

$$F_1\Big(\ \Big(V_1 \bigcup F_k\Big(V_2 \bigcup F_{k-1}\Big(V_3, \dots, V_{k+1}; S_3, \dots, S_{k+1}\Big)\Big), \dots, V_{k-1} \bigcup$$

$$F_k\Big(V_2 \bigcup F_{k-1}\Big(V_3, \dots, V_{k+1}; S_3, \dots, S_{k+1}\Big), \dots, V_{k-1} \bigcup$$

$$F_1(V_k; S_k), V_k; S_2, \dots, S_k\Big)\Big); S_1\Big) = F_{k+1}\Big(V_1 \bigcup$$

$$F_k\Big(V_2, \dots, V_{k+1}; S_2, \dots, S_{k+1}\Big), \dots, V_k \bigcup F_1(V_{k+1}; S_{k+1}), V_{k+1}; S_1 \dots S_{k+1}\Big).$$

Statement 5.6 is proved for the number $k+1$, and hence, Statement 5.6 is proved for any integer $k \geq 2$.

We prove Statement 5.7 by induction on $k$.

If $k = 1$, then $t = 1$. Then, by Proposition 2.1, $F_1\Big(V_1; S_1\Big) \supseteq V_1$.

Assume that the required inclusion is proved for the number $k$ and all $1 \leq t \leq k$, and let $t \leq k+1$.

If $t > 1$, then considering the induction assumption, we get that

$$F_{k+1}\Big(V_1, \dots, V_{k+1}; S_1, \dots, S_{k+1}\Big) \supseteq F_1\Big(V_1 \bigcup F_k\Big(V_2, \dots, V_{k+1};$$
$$S_2, \dots, S_{k+1}\Big); S_1\Big) \supseteq F_1\Big(V_1 \bigcup V_t; S_1\Big) \supseteq V_1 \bigcup V_t \supseteq V_t.$$

If $t = 1$, then applying Statements 5.4 and 5.3, and the induction assumption, we see that

$$F_{k+1}\Big(V_1, \dots, V_{k+1}; S_1, \dots, S_{k+1}\Big) \supseteq$$
$$F_1\Big(V_1 \bigcup F_k\Big(V_2, \dots, V_{k+1}; S_2, \dots, S_{k+1}\Big); S_1\Big) \supseteq$$
$$F_1\Big(V_1 \bigcup F_k\Big(\{e\}, \dots, \{e\}; S_2, \dots, S_{k+1}\Big); S_1\Big) = F_1\Big(V_1; S_1\Big) \supseteq V_1.$$

By this Statement 5.7 is proved.

We prove Statement 5.8 by induction on the number $s - t$.

If $s - t = 0$, then $t = s$, and hence, $F_{k+1}\Big(V_s, \dots, V_{k+s}; S_s, \dots, S_{k+s}\Big) = F_{k+s-t+1}\Big(V_t, \dots, V_{k+s}; S_t, \dots, S_{k+s}\Big)$.

Assume that the required inclusion is proved for $s - t = n$ and any $k \in N$, and let $s - t = n + 1$. Then, by the inductive assumption and Statement 5.7,

$$F_{k+1}\Big(V_s, \dots, V_{k+s}; S_s, \dots, S_k\Big) \subseteq F_{k+(s-t-1)+1}\Big(V_2, \dots, V_{k+s}; S_2, \dots, S_{k+s}\Big) \subseteq$$
$$V_1 \bigcup F_{k+(s-t-1)+1}\Big(V_2, \dots, V_{k+s}; S_2, \dots, S_{k+s}\Big) \subseteq$$
$$F_1(V_1 \bigcup F_{k+s-t}\Big(V_2, \dots, V_{k+s}; S_2, \dots, S_{k+s}\Big); S_1) =$$
$$F_{k+s-t+1}\Big(V_1, \dots, V_{k+s}; S_1, \dots, S_{k+s}\Big)$$

for all $s, k \in N$.

By this Statement 5.8 is proved, and hence, Proposition 5 is proved. □

**Definition 6.** Let $G$ be a group and let $x$ be a variable. An expression of the form $g_1 \cdot x^{k_1} \cdot g_2 \cdot x^{k_2} \cdot \ldots \cdot g_s \cdot x^{k_s} \cdot g_{s+1}$, where $g_i \in G$ for $1 \leq i \leq s+1$ and $k_j$ are integers for $1 \leq j \leq s$, is called *a word on the variable $x$ over the group $G$*.

The set of all words on the variable $x$ over the group $G$ will be denoted by $G(x)$.

**Remark 7.** If we assume that $x^0 = e$, then the set $G(x)$ is a group under the multiplication of words.

Adding, if it is necessary, the unity element of the group in the expression $g_1 \cdot x^{k_1} \cdot g_2 \cdot x^{k_2} \cdot \ldots \cdot g_s \cdot x^{k_s} \cdot g_{s+1}$ we can assume that $k_i \in \{-1, 0, 1\}$.

**Definition 8.** If $f(x)$ is a word on the variable $x$ over the group $G$, then an expression of the form $f(x) = g$, where $g \in G$, is called *an equation over a group* $G$.

**Definition 9.** An element $b$ of a group $G$ is called *a root of the equation* $f(x) = g$ over the group $G$ if $f(b) = g$.

**Notations 10.** Let $G$ be a countable group, and let $G = \{e, g_1^{\pm 1}, g_2^{\pm 1}, \dots\}$ be a numbering of elements of the group $G$ (this numbering will follow throughout the article).

For each natural number $k$, we put $S_k = \{g_1^{\pm 1}, g_2^{\pm 1}, \dots, g_k^{\pm 1}\}$, for each pair of natural numbers $(i,j)$ we define subsets $V_{(i,j)}$ and $S_{(i,j)}$ of the group $G$, and for each triple of natural numbers $(i,j,k)$ such that $1 \le k \le j$ we define the set $\Phi_{(i,j,k)}(x)$ of the equations on the variable $x$ over the group $G$ as follows: $V_{(1,j)} = \{e\}$, $S_{(1,j)} = S_j$, and $\Phi_{(1,j,k)}(x) = \{x = c \mid c \in S_k\}$ for all $j,k \in N$ and $k \le j$.

Assume that the sets $V_{(i,j)}$, $S_{(i,j)}$ and $\Phi_{(i,j,k)}(x)$ for $i \le p$ and all $j,k \in N$ and $k \le j$ are defined for a natural number $p$.

If $p+1$ is even, then we take:

$V_{(p+1,j)} = \{e\}$ for any $j \ge p+1$;

$V_{(p+1,j)} = V_{(p,j)} \bigcup \{g, g^{-1}\}$, where $g$ is an element of the set $G \setminus \bigcup_{s=1}^{j} S_{(p,j)}$ [1] for any $j < p+1$;

$\Phi_{(p+1,j,k)}(x) = \Phi_{(p,j,k)}(x)$ for all $k < j \in N$;

$S_{(p+1,j)} = \{g \in G \mid g \textit{ is a root of an equation from } \bigcup_{k=1}^{j} \Phi_{(p+1,j,k)}\}$ for all $j \in N$.

If $p+1$ is odd, then we take:

$V_{(p+1,j)} = \{e\}$ for $j \ge p+1$;

$V_{(p+1,j)} = F_{p+1-j}\big(V_{(p,j+1)}, \dots, V_{(p,p+1)}; S_{j+1}, \dots, S_{p+1}\big) \bigcup V_{(p,j)}$ for $j < p+1$;

$\Phi_{(p+1,j,j)}(x) = \{x = g \mid g \in S_j\}$ for all $j \in N$ and $\Phi_{(p+1,j,k)}(x) = \Big\{f(x) = g \mid f(x) \in F_{j-k}\big(V_{(p+1,k+1)}, \dots, V_{(p+1,j-1)}, V_{(p,j)} \bigcup \{x, x^{-1}\};\ S_{k+1}, \dots, S_j\big)$ and $g \in S_k\Big\}$ for all $k < j \in N$;

$S_{(p+1,j)} = S_{(p,j)}$ for any $j \in N$.

So, we have identified the subsets $V_{(i,j)}$ and $S_{(i,j)}$ of the group $G$ for each pair of natural numbers $(i,j)$ and the set $\Phi_{(i,j,k)}(x)$ of equations over a group $G$ for each triple of natural numbers $(i,j,k)$, such that $1 \le k \le j$, respectively.

**Theorem 11.** *If a countable group $G$ admits a non-discrete Hausdorff group topology $\tau$, then for any finite set $M = \{f_1(x) = a_1, \dots, f_m(x) = a_m\}$ of equations over the group $G$ for which the unity element $e$ of the group $G$ is not a root of any of these equations, in the topological group $(G,\tau)$ there exists a neighborhood $W$ of the unity element such that its any element is not a root of any of these equations.*

---

[1] If $G \setminus \bigcup_{s=1}^{j} S_{(p,j)} = \emptyset$, then we take $V_{(p+1,j)} = V_{(p,j)}$.

*Proof.* For each positive integer $1 \le i \le m$ of the mapping $f_i : (G, \tau) \to (G, \tau)$ is a continuous mapping. Since the topological group is a Hausdorff space, then the set $\{g\}$ is a closed set in the topological group $(G, \tau)$ for any element $g \in G$. Then $V_i = G \setminus f_i^{-1}(a_i)$ is an open set, and $e \in V_i$ . If $V = \bigcap\limits_{j=1}^{m} V_j$, then $V$ is a neighborhood of the unity element and $a_i \notin f_i(V)$ for any $1 \le i \le m$, and hence any element from $V$ is not a root of any equation $f_i(x) = a_i$ for any $1 \le i \le m$.

By this the theorem is proved. □

**Proposition 12.** (see the example 3.6.18 in [5]) *There exists a set $\tilde{\mathbb{N}}$ of cardinality continuum of infinite subsets of the set $\mathbb{N}$ of natural numbers such that $A \bigcap B$ is a finite set for any distinct $A, B \in \tilde{\mathbb{N}}$*

**Theorem 13.** *If a countable group $G$ admits a non-discrete metrizable group topology $\tau_0$, then $G$ admits continuum of non-discrete metrizable group topologies stronger than $\tau_0$, and any two of these topologies are incomparable.*

*Proof.* Let $G = \Big\{e, g_1^{\pm 1}, \dots\Big\}$ be a numbering of elements of the group $G$ and $S_n = \Big\{g_1^{\pm 1}, \dots, g_n^{\pm 1}\Big\}$ for any $n \in N$. There exists a countable basis $\Big\{V_1, V_2, \dots\Big\}$ of the filter of neighborhoods of the unity element in the topological group $(G, \tau_0)$ such that $V_k^{-1} = V_k$, $V_k \bigcap S_k = \emptyset$ and $g \cdot V_{k+1} \cdot g^{-1} \subseteq V_k$ for any $g \in S_k$, $k \in N$.

By induction on $k$ one can easily prove that $F_k\Big(V_{i+1}, \dots, V_{i+k}; S_{i+1}, \dots, S_{i+k}\Big) \subseteq V_i$ for all $i, k \in N$.

Further proof of the theorem will be realized in several steps.

Step I. Construction of an auxiliary sequence of elements and a sequence of natural numbers.

By induction, we construct a sequence $k_1, k_2, \dots$ of natural numbers such that $k_i \ge i$ for all $i \in N$, and a sequence $h_1, h_2, \dots$ of elements of the set $G \backslash \{e\}$ such that $\{e, h_i, h_i^{-1}\} \subseteq V_{k_i}$ and $h_i \notin F_k\big(\{e, h_1, h_1^{-1}\}, \dots, \{e, h_{i-1}, h_{i-1}^{-1}\}, \{e\}, \{e, h_{i+1}, h_{i+1}^{-1}\}, \dots, \{e, h_k, h_k^{-1}\}; S_1, \dots, S_k\big)$ for any integers $1 \le i < k$.

We take $k_1 = 1$, and as $h_1$ we take an arbitrary element of the set $V_1 \backslash \{e\}$.

Suppose that we have already defined natural numbers $k_1, k_2, \dots, k_n$ such that $k_i \ge i$ and elements $h_1, h_2, \dots, h_n$ from the set $G \backslash \{e\}$ such that $\{e, h_i, h_i^{-1}\} \subseteq V_{k_i}$ and $h_i \notin F_n\big(\{e, h_1, h_1^{-1}\}, \dots, \{e, h_{i-1}, h_{i-1}^{-1}\}, \{e\}, \{e, h_{i+1}, h_{i+1}^{-1}\}, \dots, \{e, h_n, h_n^{-1}\}; S_1, \dots, S_n\big)$ for any $i \in N$, $1 \le i < n$ and $h_n \notin F_{n-1}\Big(\{e, h_1, h_1^{-1}\}, \dots, \{e, h_{n-1}, h_{n-1}^{-1}\}; S_1, \dots, S_{n-1}\Big)$.

For any $i \in N$, $i < n+1$ we consider the set $\Omega_{(n+1,\ i)}(x) = F_{n+1}\Big(\{e,\ h_1,\ h_1^{-1}\}, \dots, \{e,\ h_{i-1},\ h_{i-1}^{-1}\}, \{e\}, \{e,\ h_{i+1},\ h_{i+1}^{-1}\}, \dots, \{e,\ h_n,\ h_n^{-1}\}, \{x, x^{-1}\}; S_1, \dots, S_{n+1}\Big)$ of words on the variable $x$ over the group $G$ and the set of equations $\Phi'_{n+1}(x) = \bigcup\limits_{i=1}^{n} \Big\{f(x) = g \mid f(x) \in \Omega_{(n+1,i)}, g \in \{h_i, h_i^{-1}\}\Big\}$ over the group $G$.

Since (see Statement 5.5) $F_{n+1}\Big(\{e,\ h_1,\ h_1^{-1}\},\dots,\{e,\ h_{i-1},\ h_{i-1}^{-1}\},\{e\}, \{e,\ h_{i+1},\ h_{i+1}^{-1}\},\ \dots\ ,\{e,\ h_n,\ h_n^{-1}\},\ \{e\};\ S_1,\dots,S_{n+1}\Big) = F_n\Big(\{e,\ h_1,\ h_1^{-1}\},\dots, \{e,h_{i-1},h_{i-1}^{-1}\},\{e\},\{e,h_{i+1},h_{i+1}^{-1}\},\dots,\{e,h_n,h_n^{-1}\};S_1,\dots,S_n\Big)$, and by the induction assumption, $h_i\notin F_n\Big(\{e,h_1,h_1^{-1}\},\dots,\{e,h_{i-1},h_{i-1}^{-1}\},\{e\},\{e,h_{i+1},h_{i+1}^{-1}\},\dots, \{e,h_n,h_n^{-1}\};S_1,\dots,S_n\Big)$, then $f(e)\notin\{h_i,h_i^{-1}\}$ for any $i\leq n$ and for any word $f(x)$ of the set $\Omega_{(n+1,i)}(x)$. Hence, the unity element $e$ of the group $G$ is not a root of any equation of the set $\Phi'_{n+1}(x)$.

So, we have proved that $\Phi'_{n+1}(x)$ is a finite set of equations over the group $G$ and the unity element $e$ of the group $G$ is not a root of any equation of the set $\Phi'_{n+1}(x)$.

Since the topology $\tau_0$ is a non-discrete Hausdorff group topology, then by Theorem 11, the topological group $(G,\tau)$ has a neighborhood $W$ of the unity element such that any its element is not a root of any equation of the set $\Phi'_{n+1}(x)$.

The finiteness of the set $F_n(\{e,h_1,h_1^{-1}\},\dots,\{e,h_n,h_n^{-1}\};S_1,\dots,S_n)$ and the fact that $\tau_0$ is a Hausdorff topology imply that there exists a number $n+1<k_{n+1}\in N$ such that $W_{k_{n+1}}\subseteq W$ and

$$F_n(\{e,h_1,h_1^{-1}\},\dots,\{e,h_n,h_n^{-1}\};S_1,\dots,S_n)\bigcap W_{k_{n+1}}=\{e\}.$$

We take as $h_{n+1}$ any element of the set $W_{k_{n+1}}\backslash\{e\}$.

We show that these conditions are statisfied for numbers $k_1,k_2,\dots,k_{n+1}$ and for elements $h_1,h_2,\dots,h_{n+1}$ of the group $G$.

Since $h_{n+1}\in W_{k_{n+1}}\backslash\{e\}$, then $h_{n+1}\notin F_n\Big(\{e,\ h_1,\ h_1^{-1}\},\dots,\{e,h_n,h_n^{-1}\}; S_1,\dots,S_n\Big)$. Moreover, by the inductive assumption, $h_i\notin F_n\Big(\{e,h_1,h_1^{-1}\},\dots, \{e,\ h_{i-1},\ h_{i-1}^{-1}\},\ \{e\},\ \{e,\ h_{i+1},\ h_{i+1}^{-1}\},\ \dots\ ,\ \{e,\ h_n,\ h_n^{-1}\};\ S_1,\ \dots\ ,\ S_n\Big) = F_{n+1}\Big(\{e,\ h_1,\ h_1^{-1}\},\ \dots,\{e,\ h_{i-1},\ h_{i-1}^{-1}\},\ \{e\},\ \{e,\ h_{i+1},\ h_{i+1}^{-1}\},\dots,\{e,\ h_n,\ h_n^{-1}\},\{e\}; S_1,\dots,S_{n+1}\Big)$.

Since the element $h_{n+1}$ is not a root of any equation of the set $\Phi'_{n+1}(x) = \bigcup\limits_{j=1}^{n}\Big\{f(x)=g\mid f(x)\in\Omega'_{(n+1,j)},g\in\{h_j,h_j^{-1}\}\Big\}$, then $h_i\notin F_{n+1}\Big(\{e,h_1,h_1^{-1}\},\dots, \{e,h_{i-1},h_{i-1}^{-1}\},\{e\},\{e,h_{i+1},h_{i+1}^{-1}\},\dots,\{e,h_{n+1},h_{n+1}^{-1}\};S_1,\dots,S_{n+1}\Big)$.

Thus, we have constructed the sequence of natural numbers $k_1,k_2,\dots$ and the sequence $h_1,h_2,\dots$ of elements of the group $G$ such that $k_i\geq i$, $\{e,h_i,h_i^{-1}\}\subseteq W_{k_i}$ for any $i\in N$ and $h_i\notin F_k\big(\{e,h_1,h_1^{-1}\},\dots,\{e,h_{i-1},h_{i-1}^{-1}\},\{e\},\{e,h_{i+1},h_{i+1}^{-1}\},\dots, \{e,h_k,h_k^{-1}\};S_1,\dots,S_k\big)$ for any natural numbers $1\leq i<k$.

Step II. Construction of a metrizable group topology $\tau(A)$ for any infinite set $A$ of of natural numbers.

For any natural number $i$ we consider the set $U_{i,A}=\{e\}$ if $i\notin A$, and $U_{i,A}=\{e,h_i,h_i^{-1}\}$ if $i\in A$, and for any pair $(i,j)$ of natural numbers we consider the set $U_{(i,j),A} = F_j(U_{i+1,A}, \dots, U_{i+j,A}; S_{i+1}, \dots, S_{i+j})$. We will show that for the sets $U_{(i,j),A}$ the following inclusions are true:

1. From Statements 5.3 and 5.4 it follows that $e \in U_{(i,j),A}$ for any $i, j \in N$.
2. From Statement 5.5 it follows that $U_{(k,j),A} \subseteq U_{(k,n),A}$ for any $j \leq n$.
3. From Statement 5.8 it follows that $U_{(i,j),A} \subseteq U_{(k,j),A}$ for any $k \leq i$.
4. From Statement 5.2 it follows that $U_{(i,j),A}$ is a symmetric set, i. e. $\left(U_{(i,j),A}\right)^{-1} = U_{(i,j),A}$ for any $i, j \in N$.
5. By induction on $j$, we prove that $U_{(i+1,j),A} \cdot U_{(i+1,j),A} \subseteq U_{(i,j),A}$ and $g \cdot U_{(i+1,j),A} \cdot g^{-1} \subseteq U_{(i,j),A}$ for any $i, j \in N$, $j > 1$ and $g \in S_{i+1}$.

In fact, if $j = 2$, then, applying in succession the definition of the sets $U_{(i,j),A}$, Statements 5.1, 3.4 and 3.6, we obtain:

$$U_{(i+1,2),A} \cdot U_{(i+1,2),A} =$$

$$F_1(U_{i+2,A}; S_{i+2}) \cdot F_1(U_{i+2,A}; S_{i+2}) \subseteq F_1(F_1(U_{i+2,A}; S_{i+2}); S_{i+1}) \subseteq$$

$$F_1(U_{i+1,A} \bigcup F_1(U_{i+2,A}; S_{i+2}); S_{i+1}) = F_2(U_{i+1,A}, U_{i+2,A}; S_{i+1}, S_{i+2}) = U_{(i,2),A} =$$

$$U_{(i,j),A} \text{ and } g \cdot U_{(i+1,2),A} \cdot g^{-1} = g \cdot F_1\big(U_{i+2,A}; S_{i+2}\big) \cdot g^{-1} \subseteq$$

$$F_1\big(F_1(U_{i+2,A}; S_{i+2}); S_{i+1}\big) \subseteq F_1\Big(U_{i+1,A} \bigcup F_1\big(U_{i+2,A}; S_{i+2}\big); S_{i+1}\Big) =$$

$$F_2\big(U_{i+1,A}, U_{i+2,A}; S_{i+1}, S_{i+2}\big) = U_{(i,2),A}$$

for any $i \in N$.

Assume that the required inclusion is proved for $j = n \geq 2$ and any $i \in N$.

Then

$$U_{(i+1,i+n+1),A} \cdot U_{(i+1,i+n+1),A} = F_n\Big(U_{i+2,A}, \dots, U_{i+n+1,A};$$

$$S_{i+2}, \dots, S_{i+n+1}\Big) \cdot F_n\Big(U_{i+2,A}, \dots, U_{i+n+1,A}; S_{i+2}, \dots, S_{i+n+1}\Big) \subseteq$$

$$F_1\Big(F_n\big(U_{i+2,A}, \dots, U_{i+n+1,A}; S_{i+2}, \dots, S_{i+n+1}\big); S_{i+1}\Big) \subseteq$$

$$F_1(U_{i+1,A} \bigcup F_n(U_{i+2,A}, \dots, U_{i+n+1,A}; S_{i+2}, \dots, S_{i+n+1}); S_{i+1}) =$$

$$F_{n+1}(U_{i+1,A}, \dots, U_{i+n+1,A}; S_{i+1}, \dots, S_{i+n+1}) = U_{(i,n+1),A}$$

and

$$g \cdot U_{(i+1,i+n+1),A} \cdot g^{-1} = g \cdot F_n(U_{i+2,A}, \dots, U_{i+n+1,A}; S_{i+2}, \dots, S_{i+n+1}) \cdot g^{-1} \subseteq$$

$$F_1(F_n(U_{i+2,A}, \dots, U_{i+n+1,A}; S_{i+2}, \dots, S_{i+n+1}); S_{i+1}) \subseteq$$

$$F_1\Big(U_{i+1,A} \bigcup F_n(U_{i+2,A}, \dots, U_{i+n+1,A}; S_{i+2}, \dots, S_{i+n+1}); S_{i+1}\Big) =$$

$$F_{n+1}\Big(U_{i+1,A}, \dots, U_{i+n+1,A}; S_{i+1}, \dots, S_{i+n+1}\Big) = U_{(i,n+1),A}.$$

So, we have proved that $U_{(i+1,j),A} \cdot U_{(i+1,j),A} \subseteq U_{(i,j),A}$ and $g \cdot U_{(i+1,j),A} \cdot g^{-1} \subseteq U_{(i,j),A}$ for any $i, j \in N$, $j > 1$ and $g \in S_{i+1}$.

Using the inclusions 1–5 proven above, one can prove that the set $\Big\{\hat{U}_i(A) = \bigcup\limits_{j=1}^{\infty} U_{(i,j),A} \mid i \in N\Big\}$ satisfies the conditions of Theorem 1, and hence, this set is a basis of the filter of neighborhoods of the unity element for a metrizable group topology $\tau(A)$ in the group $G$.

Step III. Construction of the continuum of group topologies.

For any subset $A \in \tilde{\mathbb{N}}$ (for definition of the set $\tilde{\mathbb{N}}$, see Proposition 12) we consider the group topology $\tau(A)$, constructed in the proof of this Theorem, step II.

Since the set $\tilde{\mathbb{N}}$ has the cardinality of the continuum, then to complete the proof, it remains to show that for any sets $A, B \in \tilde{\mathbb{N}}$ the topologies $\tau(A)$ and $\tau(B)$ are incomparable.

Suppose the contrary, for definiteness assume that $\tau(A) \le \tau(B)$.

Let $n \in A$. Since $\tau(A) \le \tau(B)$ and $\hat{U}_{n,A}$ is a neighborhood of the unity element in the topological group $(G, \tau(A))$, then there exists a natural number $k \in B$ such that $\hat{U}_{k,B} \subseteq \hat{U}_{n,A}$, and since $A \bigcap B$ is a finite set, then there exists a natural number $s \in B \backslash A$, such that $s > k$ and $s > n$. Then

$$h_s \in F_{k-s}\Big(U_{k+1,B}, \dots, U_{s,B}; S_{k+1}, \dots, S_s\Big) \subseteq \hat{U}_{k,B} \subseteq \hat{U}_{n,A}.$$

From the construction of the elements $h_i$ (see step I of this proof) we have $h_s \notin F_t\Big(\{e,\ h_1,\ h_1^{-1}\}, \dots, \{e,\ h_{s-1},\ h_{s-1}^{-1}\},\ \{e\},\ \{e, h_{s+1},\ h_{s+1}^{-1}\},\ \{e,\ h_{s+t},\ h_{s+t}^{-1}\};\ S_1, \dots, S_{s+t}\Big)$ for any $t \in N$.

Since $s \notin A$, then $U_{s,A} = \{e\}$, and hence, $h_s \notin F_t\Big(U_{n+1,A}, \dots, U_{n+t,A}; S_{n+1}, \dots, S_{n+t}\Big) = U_{(n,t),A}$ for any $t \in N$. Then $h_s \notin \bigcup\limits_{t=1}^{\infty} U_{(n,t),A} = \hat{U}_{n,A}$.

We have arrived at a contradiction, so the topologies $\tau(A)$ and $\tau(B)$ are incomparable.

By this the theorem is proved. □

**Theorem 14.** *Let a countable group $G$ admit a non-discrete metrizable group topology $\tau_0$, then there exists the continuum of non-discrete metrizable group topologies on $G$ stronger than $\tau_0$, and any two of these topologies are comparable.*

*Proof.* Let $P$ be the set of all prime numbers, let $\mathbb{Q}$ be the set of all rational numbers, and let $\mathbb{R}$ be the set of all real numbers. Then there exists a bijection $\xi : \mathbb{Q} \to P$.

For each positive real number $r \in \mathbb{R}$ we consider the set $A_r = \xi(\{q \in \mathbb{Q} \mid r \le q\})$ of prime numbers, and let $\tau(A_r)$ be group topology on the group $G$, constructed in the proof of Theorem 13, step II.

We will show that the set $\{\tau(A_r) \mid r \in R\}$ is the required set of group topologies.

Since the set $\left\{\hat{U}_i(A_r) \mid i \in N\right\}$ is a basis of the filter of neighborhoods of the unity element for the group topology $\tau(A_r)$, then the topological group $\big(G, \tau(A_r)\big)$ has a countable basis of the filter of neighborhoods of the unity element.

We show that for any distinct real numbers $r, r' \in \mathbb{R}$ the topologies $\tau(A_r)$ and $\tau(A_{r'})$ are different and comparable.

In fact, if $r < r'$, then $A_r \backslash A_{r'}$ is an infinite set. Then, similarly as in the proof of Theorem 13, step III we show that $\tau(A_r) \neq \tau(A_{r'})$, and hence, the set $\{\tau(A_r) \mid r \in R\}$ has the cardinality of the continuum.

To finish the proof of the Theorem it remains to show that any two topologies from the set $\{\tau(A_r) \mid r \in \mathbb{R}\}$ are comparable.

Let $r, r' \in R$ and suppose (for definiteness) that $r < r'$. Since

$$A_{r'} = \xi(\{q \in \mathbb{Q} \mid r' \leq q\} \subseteq \xi(\{q \in \mathbb{Q} \mid r \leq q\} = A_r,$$

then (see the definition of the sets $U_{(i,j),A}$ in the proof of Theorem 13 step II) $U_{(i,j),A_{r'}} \subseteq U_{(i,j),A_r}$ for any $i, j \in \mathbb{N}$. Then $\hat{U}_{n,A_{r'}} \subseteq \hat{U}_{n,A_r}$ for any $n \in N$, and , the sets $\{U_{n,A_{r'}} \mid n \in \mathbb{N}\}$ and $\{U_{n,A_r} \mid n \in \mathbb{N}\}$ are basis of the filter of neighborhoods of the unity element in topological groups $(G, \tau(A_{r'}))$ and $(G, \tau(A_r))$, respectively. As any group topology is determined by the basis of the filter of neighborhoods of the unity element, then $\tau(A_r) < \tau(A_{r'})$.

By this the theorem is proved. □

## References


[1] Markov A. A. *On absolutely closed sets.* Mat. Sb., 1945, **18**, 3–28 (in Russian).

[2] Kertesz A., Szele T., *On existence of non-discrete topologies on an infinite Abelian groups.* Publ. Math., 1953, **3**, 187–189.

[3] Ol'shansky A. Yu. *Remark on countable non-topologizable group.* Vestnik MGU. Ser. Mat. i Mech., 3:103, 1980 (in Russian).

[4] Bourbaki N. *Topologie generale.* Moskva, 1958 (in Russian).

[5] Engelking R. *General topology.* Moskva, 1986 (in Russian).



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# Short signatures from the difficulty of factoring problem

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**Abstract.** For some practical applications there is a need of digital signature schemes (DSSes) with short signatures. The paper presents some new DSSes based on the difficulty of the factorization problem, the signature size of them being equal to 160 bits. The signature size is significantly reduced against the known DSS. The proposed DSSes are based on the multilevel exponentiation procedures. Three type of the exponentiation operations are used in the DSSes characterized in performing multiplication modulo different large numbers. As modulus prime and composite numbers are used. The latters are difficult for factoring and have relation with the prime modulus.

**Mathematics subject classification:** 11T71, 94A60.
**Keywords and phrases:** Information authentication, digital signature scheme, short signature, factorization problem, discrete logarithm problem.

## 1 Introduction

There are digital signature schemes (DSSes) which are based on different hard mathematical problems. The difficulty of factorizing a composite number $n = qr$, which is the product of two large unknown primes $q$ and $r$, is used in the design of the first DSSes which gained practical importance, the RSA cryptosystem [1], Rabin's DSS [2], and Fiat-Shamir's DSS [3]. In the mentioned DSS the digital signature length depends on the required security level estimated as the number of group operations that should be performed to forge a signature, while using sufficiently large but reasonable memory (for example up to $2^{50}$ bits). At present the $2^{80}$ operations security level can be accepted as the minimum one. The RSA and Rabin's DSS provide the minimum security level with the 1024-bit signature length [4].

An important practical problem is the develop DSS with short signature length [5]. A DSS based on the difficulty of discrete logarithm problem in the multiplicative group [6] or in group of elliptic curve points allows the reduce the signature size [7]. The DSA standard and Schnorr's DSS based on difficulty of finding discrete logarithm modulo large prime number provide comparatively short signatures which have the 320-bit length and the security level of the 1024-bit RSA. The ECDSA standard also requires the use of the 320-bit signature size [8] to get the same security.

In the present paper we consider some ways to reduce the signature length in DSS based on the difficulty of the factorization problem. In Section 2 we present

DSSes with 320-bit and 240-bit signature lengths proposed in our previous works [9, 10] using the factoring problem difficulty. In Section 3 we describe some new signature formation mechanisms which are used to reduce the signature length to the 160 bits value and present two DSSes with short signature. Section 4 presents the comparison with the known signature algorithms. Section 5 concludes the paper.

## 2 Randomized signature schemes based on factorization problem

The well known cryptosystem RSA [1] is based on the calculations modulo $n$ which is the product of two randomly chosen strong prime numbers $r$ and $q$ [11]. The public key is represented by a pair of numbers $(n,\ e)$, where $e$ is a random number that is relatively prime with the Euler phi function $\varphi(n) = (p-1)(q-1)$. The triple $(p,\ q,\ d)$, where $d = e^{-1} \mod \varphi(n)$, is the private key. Data ciphering with RSA is described as follows: $C = M^e \mod n$ (public-key encryption) and $M = C^d \mod n$ (decryption), where $M < n$ is a plaintext and $C$ is a ciphertext. The RSA signature $(S)$ generation and verification are performed as follows: $S = M^d \mod n$ and $M = S^e \mod n$, correspondingly.

Usually the signed documents are comparatively long. In such cases instead of sign the document $M$ we sign the hash function value $H = F_H(M)$ which corresponds to $M : S = H^d \mod n$. The RSA security is based on the difficulty of factoring modulus $n$, which depends on the structure of primes $p$ and $q$. At present the requirements on the primes $p$ and $q$ are well clarified [4, 12, 14]. The RSA signature size is sufficiently large, 1024 bit in the case of 80-bit security. The factoring problem is well studied, therefore it is a trusted one for designing secure DSSes. However, other DSSes based on this problem, for example Rabin's DSS [2] and Fiat-Shamir's DSS [3], also define long signatures. Using the fact that computing discrete logarithm modulo $n$ is at least as difficult as factoring $n$ the papers [9, 10] propose the DSS with the 320-bit randomized signature. The paper [11] proposes the randomized DSS with the 240-bit signature.

### 2.1 The 320-bit signature algorithm

In the DSS from [9] such strong primes $r$ and $q$ are used that the numbers $p - 1$ and $q - 1$ contain different prime divisors $\gamma'$ and $\gamma''$, respectively. The values $\gamma'$ and $\gamma''$ should have the lengths at least equal to 80 bits. The secret key is the triple $(p,\ q,\ \gamma)$. The public key is a pair of numbers $(n,\ \alpha)$, where $\alpha$ is generated as follows. Select a random number $\beta$ that is simultaneously a primitive element modulo $p$ and a primitive element modulo $q$, compute $t = \gamma'^{-1}\ \gamma''^{-1}\varphi(n) = \gamma'^{-1}\ \gamma''^{-1}(p-1)(q-1)$ and $\alpha = \beta^t \mod n$. The number $\alpha$ is generator of the $\gamma$-order group $\{\alpha \mod n,\ \ \alpha^2 \mod n, \ldots, \alpha^\gamma \mod n\}$, i. e. $\alpha^\gamma \mod n = 1$.

The generation of the $\alpha$ parameter can be performed also in the following way:

1. Choose random $\beta < n$ and calculate $\sigma = \beta^t \mod n$.

2. If $\sigma \neq 1$ and $gcd(\sigma - 1, n) = 1$, then $\alpha \leftarrow \sigma$, otherwise go to step 1.

A document or the hash value corresponding to it are interpreted as integers $M$ and $H$, correspondingly.

Therefore, the required $g$-bit sequence $H = (h_{g-1}, h_{g-2}, \dots, h_1, h_0)$ is taken as the number

$$H = h_{g-1}2^{g-1} + h_{g-2}2^{g-2} + \dots + h_2 2^2 + h_1 2^1 + h_0 2^0.$$

This scheme is described by the following verification equation:

$$g + k = (\alpha^{kgH} \mod n) \mod \delta,$$

where $(g, k)$ is the signature and $\delta > \gamma$ is a prime number which has, for example, the length of $|\delta| = |\gamma| + 4$ bits, where $|\delta|$ denotes the bit size of the value $\delta$. The signature size is $|k| + |g| = |\delta| + |\gamma| \approx 2|\gamma| = 2(|\gamma'| + |\gamma''|) \geq 320$ bits.

The signature generation is performed as follows:

1. Given the $M$ document calculate the hash value $H = F_H(M)$.

2. Check whether $H \neq 0$ and $\gcd(H, \gamma) = 1$. If H = 0 or $\gcd(H, \gamma) \neq 1$, then modify the document $M$ and go back to step 1.

3. Select a random $U < \gamma$ and calculate $Z = (\alpha^U \mod n) \mod \delta$ and $D = (Z^2/4 - U/H) \mod \gamma$.

4. Check whether $D$ is a quadratic residue modulo $\gamma$. If not, then go back to step 3.

5. Solve the following system which contains one congruence and one equation relative to the unknowns $g$ and $k$:

$$\begin{cases} kgH \equiv U \mod \gamma, \\ g + k = Z. \end{cases}$$

The solution gives the following signature generation formulas:

6. Calculate the signature using the formulas $g = Z/2 \pm \sqrt{D} \mod \gamma$ and $k = Z - g$.

The signature verification is performed as follows:

1. Calculate the hash function $H = F_H(M)$.

2. Check whether the following signature verification equation

$$g + k = (\alpha^{kgH} \mod n) \mod \delta$$

is satisfied. If $g + k \neq (\alpha^{kgH} \mod n) \mod \delta$, then reject the signature.

Proof that the signature verification works:

The left side of the signature verification equation is equal to:

$$g + k = Z = (\alpha^U \mod n) \mod \delta.$$

The right side of the signature verification equation is equal to:

$$(\alpha^{kgH} \mod n) \mod \delta = (\alpha^U \mod n) \mod \delta = g + k,$$

since we have

$$kgH \equiv (Z - Z/2 \mp sqrtD)(Z/2 \pm \sqrt{D})H \equiv (Z^2/4 - D)H \equiv \left[Z^2/4 - Z^2/4 - U/H\right] H \equiv U \mod \gamma.$$

To explain the requirements imposed on parameters $n$ and $\alpha$ it is useful to consider the case of prime value $\gamma$ (for example: $\gamma \mid p - 1$ and $\gamma \nmid q - 1$), for which we have

$$\alpha = \beta^{\varphi(n)/\gamma} = (\beta^{(q-1)})^{(p-1)/\gamma} \mod n \Rightarrow$$
$$\alpha \equiv (\beta^{(q-1)})^{(p-1)/\gamma} \equiv 1^{(p-1)/\gamma} \equiv 1 \mod q \Rightarrow$$
$$\alpha - 1 \equiv \mod q \Rightarrow q \mid \alpha - 1 \Rightarrow gcd(\alpha - 1, n) = q$$

where $\gcd(a, b)$ denotes the greatest common divisor of the numbers $a$ and $b$. Thus, in the considered case it is possible to factorize the modulus using the extended Euclidean algorithm. Therefore some restrictions imposed on generating the public key are necessary. We can prevent this attack using a prime $\gamma$ that divides both $p - 1$ and $q - 1$, but $\gamma^2$ does not divide $p - 1$ nor $q - 1$. In this case we have:

$$\alpha \equiv \beta^{\frac{(p-1)(q-1)}{\gamma^2}} \equiv \beta^{u'u''} (\mod n),$$

where $\gamma$ does not divide each of the numbers $u' = (p - 1)/\gamma$ and $u'' = (q - 1)/\gamma$. If $\beta$ is simultaneously a primitive element modulo $p$ and a primitive element modulo $q$, then we have $\alpha \mod p \neq 1$ and $\alpha \mod q \neq 1$, i. e. $gcd(\alpha - 1, n) = 1$. Unfortunately, in the case of the prime secret element $\gamma$ can be calculated factorizing the $n - 1$ value. Indeed, we have: $p = u'\gamma + 1$, $q = u''\gamma + 1$, and $n = u'u''\gamma^2 + (u' + u'')\gamma + 1$, hence $\gamma \mid (n - 1)$. Therefore the composite value $\gamma = \gamma'\gamma''$, where $\gamma'$ and $\gamma' \neq \gamma''$ are different divisors of $p - 1$ and $q - 1$, should be used. If $\beta$ is a "double primitive element", then we have

$$\alpha \equiv \beta^{\frac{(p-1)(q-1)}{\gamma'\gamma''}} \equiv \beta^{u'u''} (\mod n),$$

where $u' = (p - 1)/\gamma'$ and $u'' = (q - 1)/\gamma''$. Thus, in such way of the public key formation we also have $gcd(\alpha - 1, n) = 1$. If one of the primes $\gamma'$ and $\gamma''$, for example $\gamma'$, is small, then one can factorize $n$ trying different values $\gamma'$ and verifying the relation $\gcd(\alpha^{\gamma'} - 1, n) = p$. Therefore both values $\gamma'$ and $\gamma''$ should be sufficiently large. To define 80-bit security one has to use the 80-bit prime numbers $\gamma'$ and $\gamma''$.

## 2.2 The 240-bit signature algorithm

The paper [11] proposes some modification of the DSS described in Subsection 2.1. The main feature of the DSS introduced in [11] is to apply the "two-level" exponentiation procedure that provides the possibility to use a prime secret value $\gamma$. In the DSS from [11] the value $\gamma$ is the order of the value $\alpha$ modulo the secret value $q$. Due to hiding the modulus $q$ it becomes possible to use prime secret order $\gamma$. In this DSS the following verification equations are used:

$$R \equiv \beta^{\alpha^{kgH} \mod n} \mod p;\ k = (R^{\alpha^{g} \mod n} \mod p) \mod \delta,$$

where $p = 2n + 1$ is a prime; $n$ is the product of two 512-bit primes $q$ and $r$ ($n = rq$); $\beta$ is a number which has the order $q$ modulo $p$; and $\alpha$ is a number which has the order $\gamma$ modulo $q$. The modulus $\delta$ is a 80-bit prime. The private key is represented by the pair $(q, \gamma)$, where $\gamma$ is the 160-bit prime number such that $\gamma \,|\, q - 1$. The public key is the triple $(\alpha, \beta,\ p)$. The 240-bit signature $(k,\ g)$, where $|k| = 80$ bits and $|g| = 160$ bits, corresponds to the 160-bit hash-function value $H$ and provides the 80-bit security.

The signature generation procedure includes the following steps:

1. Generate a random value $U < \gamma$.
2. Compute the value $k$ using the formula $k = (\beta^{\alpha^U \mod q} \mod p) \mod \delta$.
3. Compute the value $g$ using the formula $g = U/(kH + 1) \mod \gamma$.

The last formula is derived from the following system of two congruences:

$$\begin{cases} t + g = U \mod \gamma \\ t = kgH \mod \gamma, \end{cases}$$

where $t$ is an auxiliary unknown.

Now prove that the signature verification works. Suppose a valid signature $(k,\ g)$ corresponding to the hash value $H$ is given. Taking into account that $\alpha$ has the order $\gamma$ modulo $q$ and substituting the values $k$ and $g$ in the verification equations we get

$$R \equiv \beta^{\alpha^{kgH} \mod n} \equiv \beta^{\alpha^{kgH} \mod q} \equiv \beta^{\alpha^{\left(\frac{kHU}{kH+1}\right)} \mod q} \mod p;$$

$$\left(R^{\alpha^g \mod n} \mod p\right) \mod \delta = \left(R^{\alpha^g \mod q} \mod p\right) \mod \delta =$$

$$= \left(\left(\beta^{\alpha^{\left(\frac{kHU}{kH+1}\right)} \mod q} \mod p\right)^{\alpha^g \mod q} \mod p\right) \mod \delta =$$

$$= \left(\beta^{\alpha^{\frac{kHU}{kH+1}+\frac{U}{kH+1}} \mod q} \mod p\right) \mod \delta = \left(\beta^{\alpha^U \mod q} \mod p\right) \mod \delta = k,$$

i. e. the signature verification result is positive, which means the DSS works correctly.

Let us consider some possible attacks. The first one includes finding the value $X = \log_\beta R \mod p$ and calculating $q$ as a divisor of the value $\left(\alpha^{kgH} \mod n\right) - X$. Then the value $\gamma$ can be determined as one of divisors of the value $q - 1$. Due to the large values $|p|$ and $|q|$ this attack is computationally infeasible.

The second attack is to find the value $X' = \log_\alpha \alpha^{kgH} \mod n$ and then to calculate $\gamma$ as one of divisors of the value $kgH - X'$. The second attack defines the following requirement: the value $\alpha$ should have a large order $\lambda$ modulo $n$. Taking into account the comments for selection of the value $\alpha$ in Subsection 2.1 the value $\lambda$ should be equal to the product of two large primes: $\lambda = \gamma\mu$, where $\mu$ divides

$p - 1$ and does not divide $q - 1$; $|\mu| \geq 160$ bits. If this requirement is satisfied, then the second attack is also computationally infeasible.

The most efficient is the third attack implementing a modification of the Baby-Step-Giant-Step algorithm to compute the value $x$ from the known value $y = \beta^{\alpha^x \bmod n} \bmod p$.

The algorithm is described as follows [10]:

1. Select a random value $U > 2^{|\gamma|+10}$ and calculate $y = \beta^{\alpha^U \bmod n} \bmod p$.

2. For $i = 0$ to $D = \left[\sqrt{\gamma}\right] + 1$ calculate $z' = \beta^{\alpha^{iD} \bmod n} \bmod p$. Save the values $z'$ in the table containing pairs $z'(i)$.

3. Order the table of pairs $(i,\ z'(i))$ according to the value $z'(i)$ and set $j = 0$.

4. Calculate $z''(j) = y^{1/\alpha^j \bmod n} \bmod p$.

5. Check if in the table there exists $z'(i_0)$ such that $z'(i_0) = z''(j)$. If $z''(j) \neq z'(i)$ for $i = 0$ to $D$, then increment the counter $j : j = j + 1$ and go to step 4.

6. Calculate the value $U' = i_0 + j$ and factorize the value $U - U'$.

7. Select a divisor $\gamma$ such that $\beta^{\alpha^\gamma \bmod n} \bmod p = \beta$.

The difficulty of this algorithm is equal to $W \approx 3\sqrt{\gamma}$ exponentiation operations. To provide the 80-bit security one should use the prime order $\gamma$ which has the size equal to 160-bits.

## 3 Proposed 160-bit signature schemes

The proposed DSS is based on the three-level exponentiation procedure that defines the following function

$$y = \Omega^{\beta^{\alpha^x \bmod n} \bmod N} \bmod p$$

where $p = eN + 1$; $N = PQ$; $P = e'n + 1$; $n = qr$; $e$ is a 16-bit even integer; $e'$ is a 100-bit even integer; $Q = 2Q' + 1$ is a prime; $r = 2r' + 1$ is a prime; $q = e''\gamma + 1$ is a prime; $Q', q$, and $r'$ are 512-bit primes; $\gamma$ is a 80-bit prime; $P$ is a 1124-bit prime. The value $\Omega$ has order $P$ modulo $p$; the value $\beta$ has order $q$ modulo $P$; the value $\alpha$ has order $\gamma$ modulo $q$. The values $P$, $Q$, $r$, and $\gamma$ are elements of the private key.

The three-level exponentiation procedure makes the Baby-Step-Giant-Step algorithm inefficient to compute the value x that defines the given value y. Indeed, we have

$$\begin{aligned} y &\equiv \Omega^{\beta^{\alpha^x \bmod n} \bmod N} \bmod p \equiv \\ &\equiv \Omega^{\beta^{\alpha^{iD+j} \bmod n} \bmod N} \bmod p \equiv \\ &\equiv \Omega^{\beta^{\alpha^{iD}\alpha^j \bmod n} \bmod N} \bmod p, \end{aligned}$$

i. e. it is not possible to transform the formula defining the function $y(x)$ in the formula the right side of which is free from the integer $j$ and the left side is free from the integer $i$. Below we propose two variants of the 160-bit signature scheme using an additional 80-bit prime $\delta$ as a specified parameter of the signature algorithm.

### 3.1 The first scheme

The public key includes the values $p, N, n, \Omega, \beta$, and $\alpha$.

The modulus $p$ is generated as follows:

1. Generate 512-bit primes $Q$, $q$, and $r$.
2. Generate a random 100-bit even integer $e'$ such that the value $P = e'rq + 1$ is prime.
3. Select such a 16-bit even integer $e$ that the value $p = ePQ + 1$ is prime.

The value $\Omega$ is generated as follows:

1. Generate a random number $\rho < p$ and compute $\Omega' = \rho^{eQ} \mod p$.
2. If $\Omega' \neq 1$, then output $\Omega = \Omega'$.

The value $\beta$ is generated as follows:

1. Generate a random number $\rho < N$ and compute $\beta' = \rho^{e'r} \mod N$.
2. If $\beta'$ is a primitive element modulo $Q$ and $\beta' \neq 1 \mod q$, then output $\beta = \beta'$.

The value $\alpha$ is generated as follows:

1. Generate a random number $\rho < n$ and compute $\alpha' = \rho^{e''} \mod n$.
2. If $\alpha'$ is a primitive element modulo $r$ and $\alpha' \neq 1 \mod q$, then output $\alpha = \alpha'$.

The first variant of the 160-bit DSS includes the following signature generation procedure.

1. Compute the hash function value $H$ from the message $M$ to be signed $H = F_H(M)$.
2. Generate a random value $t < \gamma$ and compute the value $R$:

$$R = \Omega^{\beta^{H\alpha^t \mod q} \mod P} \mod p.$$

3. Compute the first signature element $k$: $k = RH \mod \delta$.
4. Compute the second signature element $g$: $g = k^{-1}t \mod \gamma$.

The corresponding signature verification procedure is as follows:

1. Compute the hash function value $H = F_H(M)$ from the message $M$ to which the signature $(k,\ g)$ is appended.
2. Compute the value $\tilde{R}$: $\tilde{R} = \Omega^{\beta^{H\alpha^{kg} \mod n} \mod N} \mod p$.
3. Compute the value $\tilde{k}$: $\tilde{k} = \tilde{R}H \mod \delta$.

If $\tilde{k} = k$ the signature is valid. Otherwise the signature is rejected.

Correctness proof.

$$\tilde{R} = \Omega^{\beta^{H\alpha^{kg} \mod n} \mod N} \mod p =$$
$$= \Omega^{\beta^{H\alpha^{kg} \mod q} \mod P} \mod p =$$
$$= \Omega^{\beta^{H\alpha^{t} \mod q} \mod P} \mod p = R \Longrightarrow$$
$$\Longrightarrow \tilde{k} = \tilde{R}H \mod \delta = RH \mod \delta = k.$$

### 3.2 The second scheme

The second variant of the 160-bit signature scheme uses an additional element of the public key $y = \alpha^x \mod q$, where $x$ is the additional 80-bit element of the private key, and includes the following signature generation procedure.

1. Compute the hash function value $H$ from the message $M$ to be signed $H = F_H(M)$.

2. Generate a random value $t < \gamma$ and compute the value $R$:

$$R = \Omega^{H\beta^{\alpha^t} \mod q \mod P} \mod p.$$

3. Compute the first signature element $k$: $k = RH \mod \delta$.

4. Compute the second signature element $g$: $g = t - xk \mod \gamma$.

The corresponding signature verification procedure is as follows:

1. Compute the hash function value $H = F_H(M)$ from the message $M$ to which the signature $(k,\ g)$ is appended.

2. Compute the value $\tilde{R}$: $\tilde{R} = \Omega^{H\beta^{y^k\alpha^g} \mod n \mod N} \mod p$.

3. Compute the value $\tilde{k}$: $\tilde{k} = \tilde{R}H \mod \delta$.

If $\tilde{k} = k$, the signature is valid. Otherwise the signature is rejected.

Correctness proof.

$$\begin{aligned}
\tilde{R} &= \Omega^{H\beta^{y^k\alpha^g} \mod n \mod N} \mod p = \\
&= \Omega^{H\beta^{y^k\alpha^g} \mod q \mod P} \mod p = \\
&= \Omega^{H\beta^{\alpha^{xk}\alpha^g} \mod q \mod P} \mod p = \\
&= \Omega^{H\beta^{\alpha^{xk+t-xk}} \mod q \mod P} \mod p = \\
&= \Omega^{H\beta^{\alpha^t} \mod q \mod P} \mod p = R \Longrightarrow \\
&\Longrightarrow \tilde{k} = \tilde{R}H \mod \delta = RH \mod \delta = k.
\end{aligned}$$

## 4 Comparison with the known signature algorithms

The attacks against the DSS [10] which are described in Section 2 can be also considered as attacks against the proposed in this section signature algorithm, which are based on the three-level exponentiation procedure. Actually due to one added exponentiation level it becomes impossible to apply the Baby-Step-Giant-Step algorithm and it becomes possible to reduce the size of the order of the value $\alpha$, which leads to shortening the digital size to 160 bits. Table 1 illustrates the comparison of some signature schemes related to Baby-Step-Giant-Step algorithm.

Table 1. Transformation of the base function using the representation of the unknown $x$ as $x = iD + j$ where $D = \left[(\sqrt{\gamma})\right] + 1,\ i,\ j = 0, 1, 2, \ldots, D$; $\gamma$ is order of the value $\alpha$ modulo prime $p$.

| *DSS* | *Base function of DSS* | *Representation of the base formula* |
|---|---|---|
| [7] | $y = \alpha^x \mod p$ | $y\alpha^{-iD} = \alpha^j \mod p$ |
| [10] | $y = \beta^{\alpha^x \mod n} \mod p$ | $y^{\alpha^{-iD} \mod n} = = \beta^{\alpha^j \mod n} \mod p$ |
| *Propo–sed* | $y = \Omega^{\beta^{\alpha^x \mod n} \mod N} \mod p$ | ? |

The proposed DSSes are oriented to applications that require using short signatures and the performance of the signature generation and verification procedures is not of high significance. In the described signature generation and verification procedures more exponentiation operations are used than in the known DSSes with 240-bit and 320-bit signatures. For comparison see Table 2.

Table 2. The performance comparison of the proposed signature scheme with the DSSes of [7] and [10] for the case of the 80-bit security.

| Signature properties | DSS | | | |
|---|---|---|---|---|
| | The 1st proposed | The 2nd proposed | [10] | [7] |
| Signature generation performance, arbitrary units | 5 | 5 | 30 | 100 |
| Signature verification performance, arb. un. | 2 | 2 | 11 | 50 |
| Signature size, bits | 160 | 160 | 240 | 320 |

## 5 Conclusion

This paper introduces an approach to design 160-bit signature schemes based on the difficulty of factorization problem. Different variants of implementing the approach are possible applying the proposed three-level exponentiation procedure. We estimate that the signature generation and verification performance can be increased by factor $\approx 3$, however such implementations of the 160-bit signature algorithms represent an additional problem.

## References

[1] Rivest R. L., Shamir A., Adleman L. M. *A Method for Obtaining Digital Signatures and Public Key Cryptosystems.* Communications of the ACM, 1978, **21**, No. 2, 120–126.

[2] Rabin M. O. *Digitalized signatures and public key functions as intractable as factorization.* Technical report MIT/LCS/TR-212, MIT Laboratory for Computer Science, 1979.

[3] Fiat A., Shamir A. *How to prove yourself: Practical solutions to identification and signature problems.* Advances in cryptology – CRYPTO'86, Springer-Verlag LNCS, 1987, vol. 263, 186–194.

[4] Menezes A. J., Vanstone S. A. *Handbook of Applied Cryptography.* CRC Press, 1996.

[5] Pintsov L., Vanstone S. *Postal revenue collection in the digital age.* Proceedings of Financial Cryptography (ed. by Y. Frankel), Springer-Verlag LNCS, 2000, vol. 1962, 105–120.

[6] ElGamal T. *A public key cryptosystem and a signature scheme based on discrete logarithms.* IEEE Transactions on Information Theory. 1985, vol. IT-31, No. 4. 469–472.

[7] Schnorr C. P. *Efficient signature generation by smart cards.* J. Cryptology. 1991, **4,** 161–174.

[8] ANSI X9.62 and FIPS 186-2. Elliptic curve signature algorithm, 1998.

[9] Moldovyan N. A. *New Public Key Cryptosystems Based on Difficulty of Factorization and Discrete Logarithm problems.* 3d Int. Workshop IF&GIS'07 Proc. St.Petersburg, May 28–29, 2007, Springer LNGC, 2007, vol. XIV, 160–172.

[10] Moldovyan N. A. *Short Signatures from Difficulty of Factorization Problem.* International Journal of Network Security, 2009, **8,** No. 1, 90–95 (http://ijns.femto.com.tw).

[11] Moldovyan N. A. *An approach to shorten digital signature length.* Computer Science Journal of Moldova, 2006, **14,** No. 3(42), 390–396.

[12] Gordon J. *Strong primes are easy to find.* Advances in cryptology – EUROCRYPT'84, Springer-Verlag LNCS, 1985, vol. 209, 216–223.

[13] Pieprzyk J., Hardjono Th., Seberry J. *Fundumentals of Computer Security.* Springer-Verlag, Berlin, 2003.

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# Selected Old Open Problems in General Topology

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**Abstract.** We present a selection of old problems from different domains of General Topology. Formally, the number of problems is 20, but some of them are just versions of the same question, so the actual number of the problems is 15 or less. All of them are from 30 to 50 years old, and are known to have attracted attention of many topologists. A brief survey of these problems, including some basic references to articles and comments on their present status, are given.

**Mathematics subject classification:** 54A25, 54B05.
**Keywords and phrases:** Lindelöf space, $G_\delta$-point, weakly first-countable, paracompact $p$-space, symmetrizable, homogeneous, topological group, extremally disconnected, pointwise convergence.

## 1 Introduction

There is more than one way to achieve progess in mathematics. One of them is to introduce a good new concept, usually generalizing some classical concept. The next step is to consider elementary natural questions arising from it. This may lead to a success and recognition, depending on whether the new concept provides valuable new insights in or not. Another, more standard and more widely spread, way is to take a classical construction, procedure, and to modify some parameters involved in it so that the construction becomes applicable to a wider class of objects.

But there is yet another, more secure and pleasant, way to gain immediately recognition and to make an impact on mathematics: to solve a well-known, or even famous, open problem in one of its fields. Whether a certain problem can be recognized as famous depends on many objective and subjective factors. One of them is how old is the problem, another factor is the history of it, in particular, who has posed the problem, and who has worked on it.

Below I briefly survey a very finite set of inspiring open problems in General Topology. The list is very selective. All of the problems in it are rather old, aged from 30 to 50 years, and I will provide some basic references to the literature. The brief survey I offer to the reader shows in many directions, it includes very different problems. So trying to make this survey detailed and comprehensive would result in a huge and non-focused text passing by many major topics in General Topology. This is not the task I have in mind at this time, so that only very selective references to results and articles are given in the comments.

A "space" below is a Tychonoff topological space. In particular, by a compact space we mean a compact Hausdorff space. In terminology and notation we follow [24].

## 2 The cardinality of Lindelöf spaces

In the summer of 1923 P. S. Alexandroff and P. S. Urysohn, building up the theory of compact spaces, came to the question: is it true that the cardinality of every first-countable compact space is not greater than the cardinality of the set of real numbers? This question became known as *Alexandroff-Urysohn Problem*. Apparently, it has appeared in print for the first time in [3]. The work on the Problem gave a good push to developing and refining set-theoretic methods in General Topology.

It is well-known that the cardinality of every metrizable uncountable compact space is exactly $2^\omega$. Alexandroff and Urysohn have been able to considerably generalize this fact: they proved, by a nice ramification method, that if every closed subset of a compact space $X$ is a $G_\delta$-set, then $|X| \leq 2^\omega$ [3]. Alexandroff-Urysohn's Problem had been solved only in 1969, in [7]: the cardinality of every first-countable compact space is indeed not greater than $2^\omega$. In fact, it was shown in [7] that this inequality holds for Lindelöf first-countable spaces as well. The following question was raised at that time:

**Problem 1** (A. V. Arhangel'skii, 1969). *Suppose that $X$ is a Lindelöf space such that every point of $X$ is a $G_\delta$-point. Then is it true that $|X| \leq 2^\omega$?*

Of course, the author of this problem wanted to see its solution in ZFC. In this sense, the problem still remains unsolved, even though it has been shown that, consistently, the anwer to Problem 1 is "no". We are still looking for an example in $ZFC$ of a large Lindelöf space $X$ in which every point is a $G_\delta$. See [8, 34] and [32] for more about this question. It will definitely require new techniques and new ideas to answer Problem 1. However, notice that the answer is "yes" if we add the assumption that the tightness of $X$ is countable.

## 3 Weakly first-countable spaces

Suppose that $X$ is a space and $\eta_x = (V_n(x) : n \in \omega)$ is a decreasing sequence of subsets of $X$, for every $x \in X$, such that $x \in V_n(x)$ and the following condition is satisfied:

$(wfc)$ A subset $U$ of $X$ is open if and only if for every $x \in U$ there exists $n \in \omega$ such that $V_n(x) \subset U$.

In this case, we will say that the family $\{\eta_x : x \in X\}$ is a *weakly basic $wfc$-structure* on the space $X$. A space $X$ is called *weakly first-countable* if there exists a weakly basic $wfc$-structure on $X$. The concept of a weakly first-countable space was introduced in [5]. Note that the interiors of the sets $V_n(x)$ in the above definition can be empty. There are many examples of weakly first-countable spaces that are not first-countable. However, every weakly first-countable space is sequential, and therefore, the tightness of every weakly first-countable space is countable. In connection with the next question, see [5] and [8].

**Problem 2** (Arhangel'skii, 1966). *Give an example in ZFC of a weakly first-countable compact space $X$ such that $|X| > 2^\omega$.*

N. N. Jakovlev has constructed under $CH$ a weakly first-countable, but not first-countable, compact space [33]. On the other hand, it has been shown [8], also under $CH$, that every homogeneous weakly first-countable compact space is first-countable and hence, the cardinality of it doesn't exceed $2^\omega$.

However, the next question, closely related to Problem 2, remains open:

**Problem 3** (Arhangel'skii, $<$ 1978). *Give an example in ZFC of a weakly first-countable, but not first-countable, compact space $X$.*

For consistency results on the existence of a weakly first-countable compact space with the cardinality larger than $2^\omega$, see [41] and [1].

A very interesting open question about weakly first-countable spaces arises in connection with the well-known Hajnal–Juhasz theorem on the cardinality of first-countable spaces with the countable Souslin number [32, 34].

**Problem 4** (Arhangel'skii, $<$ 1980). *Suppose that $X$ is a weakly first-countable space such that the Souslin number of $X$ is countable. Then is it true that $|X| \leq 2^\omega$?*

In topological groups weak first-countability turns out to be as strong as the first-countability itself. Indeed, M. M. Choban and S. J. Nedev in [44] have shown that every weakly first-countable topological group is metrizable.

## 4 Symmetrizable spaces

For the introduction to the theory of symmetrizable spaces, see [5] and [43]. A closely related but less general concept of a semimetrizable space was introduced by Alexandroff and Niemytzkii in [2], in 1927. A *symmetric* on a set $X$ is a non-negative real-valued function $d(,)$ of two variables on $X$ such that $d(x, y) = 0$ if and only if $x = y$, and $d(x, y) = d(y, x)$ for any $x, y \in X$. Suppose that $X$ is a space and $d$ is a symmetric on $X$ such that a subset $A$ of $X$ is closed in $X$ if and only if $d(x, A) > 0$ for every $x \in X \setminus A$. Then we say that $d$ generates the topology of the space $X$. A space $X$ is *symmetrizable* if the topology of $X$ is generated by some symmetric on $X$. Symmetrizable spaces naturally arise under quotient mappings with compact fibers [5]. They constitute a much larger class of spaces than the class of metrizable spaces. It is easily seen that every symmetrizable space is weakly first-countable. Hence, every symmetrizable space is sequential. However, not every symmetrizable space is first-countable. But the next question is still open:

**Problem 5** ($<$ 1970). *Is every point in a symmetrizable space a $G_\delta$?*

If I remember correctly, this question was formulated for the first time in the correspondence between E. Michael and A. Arhangel'skii in sixties. Of course, this problem is motivated by the following fact: every metrizable space is first-countable.

However, in some situations symmetrizable spaces, indeed, behave in the same way as metrizable spaces. In particular, every symmetrizable compact space is metrizable [5]. For semimetrizable spaces (they can be characterized as first-countable symmetrizable spaces) this statement was proved by V. V. Niemytzkii in [45]. The main step in the proof of the last statement is to show that every point in a symmetrizable compact space is a $G_\delta$. After that is done, it remains to refer to Niemytzkii's result mentioned above.

It should be noted that symmetrizable first-countable spaces are quite well behaved in general. In particular, a symmetrizable space $X$ is first-countable if and only if every subspace of $X$ is symmetrizable [5]. The next case of Problem 5 is also open.

**Problem 6.** *Is every pseudocompact symmetrizable space first-countable?*

The fact that every symmetrizable countably compact space $X$ is metrizable [43] makes the last Problem especially interesting.

A pseudocompact symmetrizable first-countable space needn't be metrizable (this is witnessed by Mrowka's space), but every such space is a Moore space (see [51]). Thus, the next question is a reformulation of Problem 6:

**Problem 7.** *Is every pseudocompact symmetrizable space a Moore space?*

## 5 Topological groups and topological invariants

A typical object of topological algebra can be described as a result of a happy marriage of an algebraic structure with a topology. The ties arising from this marriage strongly influence the properties of both structures. A classical example of this situation is Birkgoff–Kakutani Theorem: a topological group $G$ is metrizable if and only if it is first-countable [18].

Every first-countable space $X$ is *Fréchet-Urysohn*, that is, a point $x \in X$ is in the closure of a subset $A$ of $X$ only if some sequence in $A$ converges to $x$. Clearly, every countable first-countable space is metrizable, but it is easy to construct a non-metrizable countable Fréchet-Urysohn space. This shows that the class of Fréchet-Urysohn spaces is much wider than the class of first-countable spaces. Therefore, it is amazing that the next problem remains unsolved:

**Problem 8** (V. I. Malykhin, $< 1980$)**.** *Construct in ZFC a non-metrizable countable Fréchet-Urysohn topological group.*

It is known, however, that under $MA + \neg CH$ one can find a dense subgroup of $D^{\omega_1}$ with these properties (see [8, 10]).

Another well-known open problem of topological algebra concerns the class of extremally disconnected topological groups. This class is apparently very narrow, unlike the class of extremally disconnected spaces. Recall that a space $X$ is *extremally* *disconnected* if the closure of every open subset of $X$ is open. These spaces seem to be quite special. In particular, none of them contains a non-trivial convergent sequence. Therefore, only discrete extremally disconnected spaces are first-countable. Nevertheless, extremally disconnected spaces are rather easy to encounter in General Topology, since every space can be represented as an image of an extremally disconnected space under an irreducible perfect mapping [48], see also [35,36]. A. Gleason has characterized extremally disconnected compacta as projective objects in the category of compact spaces [27].

On the other hand, it has been known for a long time that extremal disconnectedness is on quite bad terms with homogeneity: in 1968 Z. Frolik proved [26] that every extremally disconnected homogeneous compact space is finite (see also [8]). However, there exist non-discrete homogeneous spaces [25]. One may argue that the highest degree of homogeneity is achieved in topological groups. Indeed, a space $X$ is said to be homogeneous if for any $x, y \in X$ there exists a homeomorphism $h$ of $X$ onto itself such that $h(x) = y$. To verify homogeneity of topological groups, it is enough to use left or right translations (shifts). The next problem, which remains open today, has been posed 46 years ago in [6].

**Problem 9** (A. V. Arhangel'skii, 1967)**.** *Construct in ZFC a non-discrete extremally disconnected topological group.*

It was proved in [6] that every compact subspace of an arbitrary extremally disconnected topological group is finite. Hence, if an extremally disconnected topological group $G$ is a $k$-space, then the space $G$ is discrete. S. Sirota was the first to show that the existence of a non-discrete extremally disconnected topological group is consistent with $ZFC$ [50]. More information on the vast and delicate research around Problem 9 can be found in [37,39,40,52]. It was shown that extremal disconnectedness strongly influences the structure of a topological group. In particular, every extremally disconnected topological group $G$ has an open subgroup $H$ such that $a^2 = e$ for every $a \in H$, where $e$ is the neutral element of $G$ [40].

Another interesting and long standing open problem on topological groups I wish to recall concerns free topological groups. For the definitions and basic facts on free topological groups, see Chapter 7 in [18].

**Problem 10** (A. V. Arhangel'skii, 1981)**.** *Is the free topological group $F(X)$ of an arbitrary paracompact $p$-space $X$ paracompact?*

This question is motivated by the fact established in [9], where the above problem has been posed: the free topological group of any metrizable space is paracompact. We remind that a paracompact $p$-space is a preimage of a metrizable space under a perfect mapping [4].

The next question concerns the behaviour of topological properties of topological groups under the product operation. We give two slightly different versions of this question.

**Problem 11.** *Construct in $ZFC$ countably compact topological groups $G$ and $H$ such that their product $G \times H$ is not countably compact.*

**Problem 12.** *Construct in $ZFC$ a countably compact topological group $G$ such that its square $G \times G$ is not countably compact.*

The general idea behind the last two questions is that in topological groups many topological properties improve so that some of them, which are not productive in the general case, may become productive in the special case of topological groups. For example, pseudocompactness is a property of this kind (W. W. Comfort and K. A. Ross, see [22] and [18]). Observe that Problems 11 and 12 are not equivalent, since, in general, the free topological sum of two topological groups is not a topological group. Under $MA + \neg CH$, there exists a countably compact topological group $G$ such that its square $G \times G$ is not countably compact [23]. See also [38] and [31].

## 6 Homogeneous compacta

Some of the most natural and oldest open problems in General Topology concern homogeneous compacta. The next Problem had been posed by W. Rudin in 1956 in [49]:

**Problem 13** (W. Rudin, 1956)**.** *Is it true that every infinite homogeneous compact space contains a non-trivial convergent sequence?*

A motivation for this unusual question comes from the well-known fact, established by A. N. Tychonoff, that the Stone–Čech remainder of the discrete space of natural numbers doesn't contain non-trivial convergent sequences. In 1956 it was an open question whether this remainder (which is compact) is homogeneous or not. Obiously, a positive solution of Problem 13 would immediately provide the negative answer to the last question as well. However, Problem 13 is still open, after more than 55 years have passed since it had been published.

Of course, in the very special case of compact topological groups the answer to Problem 13 is "yes", because every compact topological group is a dyadic compactum (see [18]).

The next question came to my mind in eighties (see [10]). Later I learned from Jan van Mill that Eric van Douwen also came to this question.

**Problem 14.** *Is it possible to represent an arbitrary compact space $Y$ as an image of a homogeneous compact space $X$ under a continuous mapping?*

Observe that every nonempty metrizable compactum is a continuous image of the Cantor set. The Cantor set is not only homogeneous, it is a compact topological group. However, it is not possible to represent an arbitrary compact space $Y$ as an image of a compact topological group $X$ under a continuous mapping, since only dyadic compacta can be represented in this way.

The third open problem on homogeneity presented in this section is less known than the other two, but is also very interesting, in my opinion. A compact space $X$ is *$\omega$-monolithic* if the closure of any countable subset of $X$ is metrizable. For example, every compact LOTS, that is, every compact space whose topology is generated by a linear ordering, is $\omega$-monolithic.

**Problem 15** (Arhangel'skii, 1987)**.** *Is every homogeneous $\omega$-monolithic compact space $X$ first-countable?*

This question was posed in [10]. It was shown there that the answer to it is "yes" under the additional assumption that the tightness of $X$ is countable. The next closely related to the above question problem is also open:

**Problem 16.** *Is the cardinality of every homogeneous $\omega$-monolithic compact space $X$ not greater than $2^\omega$?*

If the answer to the last question is "yes", then the answer to the preceding question is "yes" under $CH$. It is not difficult to notice that behind the last two question is hidden the next problem which is also open now (see [10]):

**Problem 17.** *Is it true that every nonempty monolithic compact space is first-countable at some point?*

## 7 *t*-equivalence and *t*-invariants

Below $C_p(X)$ stands for the space of real-valued continuous functions with the topology of pointwise convergence on a space $X$. These spaces, studied in $C_p$-theory [14], have many applications in mathematics. In $C_p(X)$ an algebraic structure is naturally blended with a topology. To compare topological spaces $X$ and $Y$, we may use homeomorphisms between $C_p(X)$ and $C_p(Y)$ which do not necessarily preserve the algebraic operations in $C_p(X)$ and $C_p(Y)$. In particular, this approach had been adopted in [14]. Following it, we say that spaces $X$ and $Y$ are *$t$-equivalent* if there exists a homeomorphism $h$ of $C_p(X)$ onto $C_p(Y)$. Note that $h$ in the above definition needn't be a topological isomorphism, that is, $h$ needn't preserve the operations.

One of the basic general questions in $C_p$-theory is the next one: how close are the properties of $X$ and $Y$ if $X$ and $Y$ are $t$-equivalent spaces [13, 14]? For example, it has been established by S. Gul'ko and T. Khmyleva [30] that the usual space $R$ of real numbers is $t$-equivalent to the closed unit interval $I$. Amazingly, this is a very non-trivial result! The reader may find even more unexpected that the answer to the following two questions are still unknown:

**Problem 18** (A. V. Arhangel'skii, $\leq$ 1989)**.** *Is the unit segment $I$ $t$-equivalent to the square $I^2$?*

**Problem 19** (A. V. Arhangel'skii, $\leq$ 1989)**.** *Is the unit segment $I$ $t$-equivalent to the Cantor set?*

Solving the last two problems may lead to a solution of the next basic question:

**Problem 20** (A. V. Arhangel'skii, $\leq$ 1985)**.** *Is the dimension dim preserved by the $t$-equivalence, at least in the class of separable metrizable spaces or in the class of compacta?*

Gul'ko–Khmyleva's result mentioned above shows that compactness is not $t$-invariant. Local compactness, in general, is also not preserved by $t$-equivalence. On the other hand, the dimension $dim(X)$ is preserved by linear homeomorphisms between $C_p(X)$ and $C_p(Y)$. This is a deep result of V. G. Pestov [46]. In connection with Pestov's result and Problem 20, see also [29]. For more on $C_p$-theory and $t$-equivalence, see [11, 14, 15, 19, 21, 42, 47] and [12].

## References

[1] Abraham U., Gorelic I., Juhász I. *On Jakovlev spaces.* Isr. J. Math., 2006, **152:1**, 206–219.

[2] Alexandroff P. S., Niemytzkii V. V. *A condition for metrizability of a topological space and the symmetry axiom.* Mat. Sb., 1938, **3(45)**, 663–672 (in Russian).

[3] Alexandroff P. S., Urysohn P. S. *Memoire sur les espaces topologiques compacts.* Nederl. Akad. Wetensch. Proc., Ser. A 14, 1929, 1–96.

[4] Arhangel'skii A. V. *On a class of spaces containing all metric and all locally compact spaces.* Mat. Sb., 1965, **67(109)**, 55-88. English translation: Amer. Math. Soc. Transl., 1970, **92**, 1–39.

[5] Arhangel'skii A. V. *Mappings and spaces.* Russian Math. Surveys, 1966, **21**, 115-162.

[6] Arhangel'skii A. V. *Groupes topologiques extremalement discontinus.* C. R. Acad. Sci. Paris, 1967, **265**, 822–825.

[7] Arhangel'skii A. V. *The power of bicompacta with first axiom of countability.* Dokl. Akad. Nauk SSSR, 1969, **187**, 967–970. English translation: Soviet Math. Dokl., 1969, **10**, 951–955.

[8] Arhangel'skii A. V. *Structure and classification of topological spaces and cardinal invariants.* Russian Math. Surveys, 1978, **33**, 33–96.

[9] Arhangel'skii A. V. *Classes of topological groups.* Russian Math. Surveys, 1981, **36:3**, 151–174.

[10] Arhangel'skii A. V. *Topological homogeneity. Topological groups and their continuous images.* Russian Math. Surveys, 1987, **42(2)**, 83–131.

[11] Arhangel'skii A. V. *A survey of $C_p$-theory.* Questions and Answers in General Topology, 1987, **5**, 1–109.

[12] Arhangel'skii A. V. *Problems in $C_p$-theory.* J. van Mill and G. M. Reed, Ed-rs, Open Problems in Topology, Ch. 31, North-Holland, Amsterdam, 1990, 603–615.

[13] Arhangel'skii A. V. *On linear topological classification of spaces of continuous functions in the topology of pointwise convergence.* Math. USSR Sbornik, 1991, **70:1**, 129–142.

[14] Arhangel'skii A. V. *Topological function spaces*, Kluwer Academic Publishers, Mathematics and its Applications, **78**, Dordrecht, 1992 (translated from Russian).

[15] Arhangel'skii A. V. *$C_p$-theory.* In: M. Hušek, and J. van Mill, eds., Recent Progress in General Topology, Chapter 1, North-Holland, Amsterdam, 1992, 1–56.

[16] Arhangel'skii A. V. *On topological and algebraic structure of extremally disconnected semitopological groups.* Comment. Math. Univ. Carolinae, 2000, **41**, No. 4, 803–810.

[17] Arhangel'skii A. V., Ponomarev V. I. *Fundamentals of general topology in problems and exercises.* Moskva, Nauka, 1974 (in Russian). English translation in: Ser. Mathematics and its Applications, D. Reidel Publishing Co., Dordrecht-Boston, Mass., 1984. Polish translation: Panstwowe Wydawnictwo Naukowe (PWN), Warsaw, 1986.

[18] Arhangel'skii A. V., Tkachenko M. G. *Topological groups and related structures*. Atlantis Press, Amsterdam-Paris, 2008.

[19] Baars J., de Groot J. *On Topological and Linear Equivalence of certain Function Spaces*. CWI–tract 86, Centre for Mathematics and Computer Sciences, Amsterdam, 1992.

[20] Burke D. K. *Covering properties*. In: Handbook of Set-theoretic Topology, K. Kunen and J. Vaughan, eds., North-Holland, Amsterdam, 1984, 347–422.

[21] Cauty R., Dobrowolski T., Marciszewski W. *A contribution to the topological classification of the spaces $C_p(X)$*. Fund. Math., 1993, **142**, 269–301.

[22] Comfort W. W., Ross K. A. *Pseudocompactness and uniform continuity in topological groups*. Pacific J. Math., 1966, **16**, 483–496.

[23] van Douwen E. K. *The product of two countably compact topological groups*. Trans. Amer. Math. Soc., 1980, **262**, 417–427.

[24] Engelking R. *General Topology*. PWN, Warszawa, 1977.

[25] Efimov B. A. *Absolutes of homogeneous spaces*. Dokl. AN SSSR, 1968, **179:2**, 271–274 (in Russian).

[26] Frolík Z. *Fixed points of maps of extremally disconnected spaces and complete Boolean algebras*. Bull. Acad. Polon. Sci., Ser. Math., Astron., Phys., 1968, **16**, 269–275.

[27] Gleason A. M. *Projective topological spaces*. Illinois Journal Math., 1958, **2:4**, 482–489.

[28] Gruenhage G. *Generalized metric spaces*. In: Handbook of Set-theoretic Topology, K. Kunen and J. Vaughan, eds., North-Holland, Amsterdam, 1984, 423–501.

[29] Gul'ko S. P. *Uniformly homeomorphic spaces of functions*. Proc. of the Steklov Inst. of Math., 1983, **3**, 87–93 (in Russian).

[30] Gul'ko S. P., Khmyleva T. E. *Compactness is not preserved by the relation of $t$-equivalence*. Math. Notes, 1986, **39**, issue 5–6, 484–488.

[31] Hart K., van Mill J. *A countably compact group $H$ such that $H \times H$ is not countably compact*. Trans. Amer. Math. Soc., 1991, **323**, 811–821.

[32] Hodel R. E. *Cardinal Functions, 1*. In: Handbook of Set-theoretic Topology, Ch. 1. K. Kunen and J. Vaughan, eds., North-Holland, Amsterdam, 1984, 1–62.

[33] Jakovlev N. N. *On the theory of o-metrizable spaces*. Dokl. Akad. Nauk SSSR, 1976, **229**, No. 6, 1330–1331 ( Soviet Math. Dokl., 1976, **17**, No. 4, 1217–1219).

[34] Juhász I. *Cardinal Functions in Topology*. Math. Centre Tracts, No. 34, Amsterdam, 1971.

[35] Iliadis S. D. *Absolutes of Hausdorff spaces*. Dokl. Akad. Nauk SSSR, 1963, **149**, 22–25 (in Russian).

[36] Iliadis S. D., Fomin S. V. *Method of centered families of sets in the theory of topological spaces*. Uspekhi Mat. Nauk, 1966, **21(4)**, 47–76 (in Russian).

[37] Louveau A. *Sur un article de S. Sirota*. Bull. Sci. Math., 1972, **96**, 3–7.

[38] Malykhin V. I. *Nonpreservation of properties of topological groups on taking their square*. Siberian Math. J., 1987, **28**, 639–645.

[39] Malykhin V. I. *Extremally disconnected and close to extremally disconnected groups*. Soviet Math. Dokl., 1975, **16**, 21–25 (Russian original in: Dokl. Akad. Nauk SSSR, 1975, **220**, 27–30).

[40] Malykhin V. I. *On extremally disconnected topological groups*. Uspekhi Mat. Nauk, 1979, **34**, issue 6, 59–66.

[41] Malykhin V. I. *Cardinality of bicompacta with the weak first-countability axiom*. Mathematical Notes, 1982, **32(2)**, 610–614.

[42] van Mill J. *Topological equivalence of certain function spaces.* Compositio Math., 1987, **63**, 159–188.

[43] Nedev S. J. *o-metrizable spaces.* Trudy Mosk. Mat. Ob-va, 1971, **24**, 201–236.

[44] Nedev S. J., Choban M. M. *On metrization of topological groups.* Vestnik Mosk. Gos. Un-ta Mat. Mekh., 1968, **6**, 18–20 (in Russian).

[45] Niemytzki V. V. *Uber die Axiome des metrischen Raumes.* Math. Ann., 1931, 104:5, 666–671.

[46] Pestov V. G. *The coincidence of the dimension dim of $l$-equivalent topological spaces.* Soviet Math. Dokl., 1982, **26**, 380–383 (in Russian).

[47] Pol R. *For a metrizable $X$, $C_p(X)$ needn't be linearly homeomorphic to $C_p(X) \times C_p(X)$.* Mathematika, 1995, **42**, 49–55.

[48] Ponomarev V. I. *On the absolute of a topological space.* Dokl. Akad. Nauk SSSR, 1963, **149**, 26–29 (in Russian).

[49] Rudin W. *Homogeneity problems in the theory of Čech compactifications.* Duke Math. J., 1956, **23**, 409-419.

[50] Sirota S. *A product of topological groups and extremal disconnectedness.* Mat. Sb. (NS), 1969, **79(121)**, 179–192 (*in Russian*).

[51] Stephenson R. M., Jr. *Symmetrizable, $\mathcal{F}-$, and weakly first-countable spaces.* Can. J. Math., 1977, **29(3)**, 480–488.

[52] Zelenyuk Y. *On extremally disconnected topological groups.* Topol. Appl., 2006, **153**, 2382–2385.

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# Infinitely many maximal primitive positive clones in a diagonalizable algebra

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**Abstract.** We present a rather simple example of infinitely many maximal primitive positive clones in a diagonalizable algebra, which serve as an algebraic model for the provability propositional logic $GL$.

**Mathematics subject classification:** 03F45, 03G25, 06E25.
**Keywords and phrases:** Primitive positive clones, Provability logic, Diagonalizable algebra.

## 1 Introduction

The present paper deals with clones of operations of a diagonalizable algebra which are closed under definitions by existentially quantified systems of equations. Such clones are called *primitive positive clones* [1] (in [2] they are referred to as *clones acting bicentrally*, and are also called *parametrically closed classes* in [3, 4]). Diagonalizable algebras [5] are known to be algebraic models for the propositional provability logic GL [6].

The proof that there are finitely many primitive positive clones in any $k$-valued logic was given in [1]. In the case of 2-valued boolean functions, i. e. $card(A) = 2$, A. V. Kuznetsov stated there are 25 primitive positive clones [3], and A. F. Danil'čenco proved there are 2986 primitive positive clones among 3-valued functions [4]. In the present paper we construct a diagonalizable algebra, generated by its least element, which has infinitely many primitive positive clones, moreover, these primitive positive clones are maximal.

## 2 Definitions and notations

**Diagonalizable algebras.** A diagonalizable algebra [5] $\mathfrak{D}$ is a boolean algebra $\mathfrak{A} = (A; \&, \vee, \supset, \neg, \mathbb{0}, \mathbb{1})$ with an additional operator $\Delta$ satisfying the following relations:

$$\Delta(x \supset y) \leq \Delta x \supset \Delta y,$$
$$\Delta x \leq \Delta\Delta x,$$
$$\Delta(\Delta x \supset x) = \Delta x,$$
$$\Delta\mathbb{1} = \mathbb{1},$$

where $\mathbb{1}$ is the unit of $\mathfrak{A}$.

We consider the diagonalizable algebra $\mathfrak{M} = (M; \&, \vee, \supset, \neg, \Delta)$ of all infinite binary sequences of the form $\alpha = (\mu_1, \mu_2, \dots)$, $\mu_i \in \{0,1\}$, $i = 1, 2, \dots$. The boolean operations $\&, \vee, \supset, \neg$ over elements of $M$ are defined component-wise, and the operation $\Delta$ over element $\alpha$ is defined by the equality $\Delta\alpha = (1, \nu_1, \nu_2, \dots)$, where $\nu_i = \mu_1 \& \cdots \& \mu_i$. Let $\mathfrak{M}^*$ be the subalgebra of $\mathfrak{M}$ generated by its zero $\mathbb{0}$ element $(0, 0, \dots)$. Remark the unite $\mathbb{1}$ of the algebra $\mathfrak{M}^*$ is the element $(1, 1, \dots)$.

As usual, we denote by $x \sim y$ and $\Delta^2 x, \dots, \Delta^{n+1} x, \dots$ the corresponding functions $(\neg x \vee y) \& (\neg y \vee x)$ and $\Delta\Delta x, \dots, \Delta\Delta^n x, \dots$. Denote by $\Box x$ the function $x \& \Delta x$ and denote by $\nabla x$ the function $\Box\neg\Box\neg\Box x$.

**Primitive positive clones.** The term algebra $\mathcal{T}(\mathfrak{D})$ of $\mathfrak{D}$ is defined as usual, stating from constants $\mathbb{0}, \mathbb{1}$ and variables and using operations $\&, \vee, \supset, \neg, \Delta$. We consider the set $Term$ of all term operations of $\mathfrak{M}^*$, which obviously forms a clone [7].

Let us recall that a *primitive positive formula* $\Phi$ over a set of operations $\Sigma$ of $\mathfrak{D}$ is of the form

$$\Phi(x_1, \dots, x_m) = (\exists x_{m+1}) \dots (\exists x_n)((f_1 = g_1) \& \cdots \& (f_s = g_s)),$$

where $f_1, g_1, \dots, f_s, g_s \in \mathcal{T}(\mathfrak{D}) \cup Id_A$ and the formula $(f_1 = g_1) \& \cdots \& (f_s = g_s)$ contains variables only from $x_1, \dots, x_n$. An *$n$-ary term operation $f$ of $\mathcal{T}(\mathfrak{D})$ is (primitive positive) definable over* $\Sigma$ if there is a primitive positive formula $\Phi(x_1, \dots, x_n, y)$ over $\Sigma$ of $\mathcal{T}(\mathfrak{D})$ such that for any $a_1, \dots, a_n, b \in \mathfrak{D}$ we have $f(a_1, \dots, a_n) = b$ if and only if $\Phi(a_1, \dots, a_n, b)$ on $\mathfrak{D}$ [8]. Denote by $[\Sigma]$ all term operations of $\mathfrak{D}$ which are primitive positive definable over $\Sigma$ of $\mathfrak{D}$. They say also $[\Sigma]$ is a primitive positive clone on $\mathfrak{D}$ generated by $\Sigma$. If $[\Sigma]$ contains $\mathcal{T}(\mathfrak{D})$ then it is referred to as a *complete primitive positive clone on* $\mathfrak{D}$. A primitive positive clone $C$ of $\mathfrak{D}$ is *maximal in* $\mathfrak{D}$ if $\mathcal{T}(\mathfrak{D}) \not\subseteq C$ and for any $f \in \mathcal{T}(\mathfrak{D}) \setminus C$ we have $\mathcal{T}(\mathfrak{D}) \subseteq [C \cup \{f\}]$.

Let $\alpha \in \mathfrak{D}$. They say $f(x_1, \dots, x_n) \in \mathcal{T}(\mathfrak{D})$ conserves the relation $x = \alpha$ on $\mathfrak{D}$ if $f(\alpha, \dots, \alpha) = \alpha$. According to [9] the set of all functions that preserves the relation $x = \alpha$ on an arbitrary $k$-element set is a primitive positive clone.

## 3 Preliminary results

We start by presenting some useful properties of the term operations $\Delta, \Box$ and $\nabla$ of $\mathfrak{M}^*$.

**Proposition 1.** *Let $x, y$ be arbitrary elements of $\mathfrak{M}^*$. Then:*

$$\Box x \geq \Delta\mathbb{0} \text{ if and only if } \nabla x = \mathbb{1} \tag{1}$$

$$\Box x = \mathbb{0} \text{ if and only if } \nabla x = \mathbb{0} \tag{2}$$

$$\text{For any } x, y, \text{ either } \Box x \leq \Box y \text{ or } \Box y \leq \Box x \tag{3}$$

$$\Delta x = \Delta\Box x \tag{4}$$

$$\nabla\mathbb{0} = \mathbb{0},\ \nabla\mathbb{1} = \mathbb{1} \tag{5}$$

$$\Box x \geq \Delta\mathbb{0} \text{ if and only if } \Box\neg x = \mathbb{0} \tag{6}$$

$$\Box x = \mathbb{0} \textit{ if and only if } \Box\neg x \geq \Delta\mathbb{0} \tag{7}$$

*Proof.* The proof is almost obvious by construction of the algebra $\mathfrak{M}^*$. □

Let us mention the following

*Remark* 1. Any function $f$ of $\mathcal{T}(\mathfrak{D})$ is primitive positive definable on $\mathfrak{D}$ via the system of functions $x \,\&\, y,\ x \vee y,\ x \supset y,\ \neg x, \Delta y$.

Let us consider on $\mathfrak{D}$ the following functions (8) and (9) of $\mathcal{T}(\mathfrak{D})$, denoted by $f_\neg(x, y)$ and $f_\Delta(x, y)$ correspondingly, where $\alpha_i, \xi \in \mathfrak{D}$, $\alpha_i = \neg\Delta^i\mathbb{0}$, where $\xi \neq \alpha_i$ and $\eta \neq \alpha_i$:

$$(\nabla\neg(x \sim y) \,\&\, ((\neg x \sim y) \sim \xi)) \vee (\nabla(x \sim y) \,\&\, \alpha_i), \tag{8}$$
$$(\nabla y \,\&\, ((\Delta x \sim y) \sim \eta)) \vee (\neg\nabla y \,\&\, \alpha_i). \tag{9}$$

**Proposition 2.** *Let arbitrary $\alpha, \beta \in \mathfrak{M}^*$. If $\neg\alpha = \beta$ on $\mathfrak{M}^*$, then*

$$f_\neg(\alpha, \beta) = \xi$$

*on $\mathfrak{M}^*$.*

*Proof.* Since $\neg\alpha = \beta$ we get $\alpha \sim \beta = \mathbb{0}, \neg(\alpha \sim \beta) = \mathbb{1}$ and by (5) we have

$$\nabla(\alpha \sim \beta) = \mathbb{0},\ \nabla\neg(\alpha \sim \beta) = \mathbb{1},$$

which implies

$$f_\neg(\alpha, \beta) = (\mathbb{1} \,\&\, (\mathbb{1} \sim \xi)) \vee (\mathbb{0} \,\&\, \alpha_i) = \xi.$$

□

**Proposition 3.** *Let arbitrary $\alpha, \beta \in \mathfrak{M}^*$. If $\neg\alpha \neq \beta$ on $\mathfrak{M}^*$, then*

$$f_\neg(\alpha, \beta) \neq \xi$$

*on $\mathfrak{M}^*$.*

*Proof.* Since $\neg\alpha \neq \beta$ we get $\neg\alpha \sim \beta \neq \mathbb{1}, \alpha \sim \beta \neq \mathbb{0}$. We distinguish two cases: 1) $\Box(\alpha \sim \beta) = \mathbb{0}$, and 2) $\Box(\alpha \sim \beta) \geq \Delta\mathbb{0}$.

In the case 1) by (7), (1) and (2) we get $\Box\neg(\alpha \sim \beta) \geq \Delta\mathbb{0},\ \nabla\neg(\alpha \sim \beta) = \mathbb{1},$ and $\nabla(\alpha \sim \beta) = \mathbb{0}$, which implies

$$\begin{aligned} f_\neg(\alpha, \beta) &= (\nabla\neg(\alpha \sim \beta) \,\&\, ((\neg\alpha \sim \beta) \sim \xi)) \vee (\nabla(\alpha \sim \beta) \,\&\, \alpha_i \\ &= (\mathbb{1} \,\&\, ((\neg\alpha \sim \beta) \sim \xi) \vee (\mathbb{0} \,\&\, \alpha_i) = (\neg\alpha \sim \beta) \sim \xi \neq \xi, \end{aligned}$$

Thus the first case has already been examined.

Now consider the second case, when $\Box x \geq \Delta\mathbb{0}$. Again, since $\neg\alpha \neq \beta$ by (1), (2) and (6) we obtain $\Box\neg(\alpha \sim \beta) = \mathbb{0},\ \nabla\neg(\alpha \sim \beta) = \mathbb{0},\ \nabla(\alpha \sim \beta) = \mathbb{1}$. Then,

$$\begin{aligned} f_\neg(\alpha, \beta) &= (\nabla\neg(\alpha \sim \beta) \,\&\, ((\neg\alpha \sim \beta) \,\&\, \xi)) \vee (\nabla(\alpha \sim \beta) \,\&\, \alpha_i) \\ &= (\mathbb{0} \,\&\, ((\neg\alpha \sim \beta) \,\&\, \xi)) \vee (\mathbb{1} \,\&\, \alpha_i) = \alpha_i \neq \xi. \end{aligned}$$

□

**Proposition 4.** *Let arbitrary $\alpha, \beta \in \mathfrak{M}^*$ be such that $\Delta\alpha = \beta$. Then*

$$f_\Delta(\alpha, \beta) = \eta.$$

*Proof.* Since $\Delta\alpha \geq \mathbb{0}$ and $\Delta\alpha = \beta$ we have $\Box\beta \geq \Delta\mathbb{0}$, $\Delta\alpha \sim \beta = \mathbb{1}$ and by (1) we get $\nabla\beta = \mathbb{1}, \neg\nabla\beta = \mathbb{0}$. These ones imply the following relations:

$$\begin{aligned} f_\Delta(\alpha, \beta) &= (\nabla\beta \,\&\, ((\Delta\alpha \sim \beta) \sim \eta)) \vee (\neg\nabla\beta \,\&\, \alpha_i) \\ &= (\mathbb{1} \,\&\, (\mathbb{1} \sim \eta)) \vee (\mathbb{0} \,\&\, \alpha_i) = \mathbb{1} \sim \eta = \eta. \end{aligned}$$

□

**Proposition 5.** *Let arbitrary $\alpha, \beta \in \mathfrak{M}^*$ be such that $\Delta\alpha \neq \beta$. Then*

$$f_\Delta(\alpha, \beta) \neq \eta.$$

*Proof.* We consider 2 cases: 1) $\Box\beta = \mathbb{0}$, and 2) $\Box\beta \geq \Delta\mathbb{0}$.

Suppose $\Box\beta = \mathbb{0}$. In view of (2) we have $\nabla\beta = \mathbb{0}$ and $\neg\nabla\beta = \mathbb{1}$. Subsequently,

$$\begin{aligned} f_\Delta(\alpha, \beta) &= (\nabla\beta \,\&\, ((\Delta\alpha \sim \beta) \sim \eta)) \vee (\neg\nabla\beta \,\&\, \alpha_i) \\ &= (\mathbb{0} \,\&\, ((\Delta\alpha \sim \beta) \sim \eta)) \vee (\mathbb{1} \,\&\, \alpha_i) = 0 \vee \alpha_i = \alpha_i \neq \eta. \end{aligned}$$

Suppose now $\Box\beta \geq \Delta\mathbb{0}$. Let us note $\Delta\alpha \sim \beta \neq \mathbb{1}$. Then considering (1) we get

$$\begin{aligned} f_\Delta(\alpha, \beta) &= (\nabla\beta \,\&\, ((\Delta\alpha \sim \beta) \sim \eta)) \vee (\neg\nabla\beta \,\&\, \alpha_i) \\ &= (\mathbb{1} \,\&\, ((\Delta\alpha \sim \beta) \sim \eta)) \vee (\mathbb{0} \,\&\, \alpha_i) = (\Delta\alpha \sim \beta) \sim \eta \neq \eta. \end{aligned}$$

□

**Proposition 6.** *Let arbitrary $\alpha \in \mathfrak{M}^*$. Then*

$$f_\neg(\alpha, \alpha) = \alpha_i.$$

*Proof.* Let us calculate $f_\neg(\alpha, \alpha)$. By (5) we obtain immediately:

$$\begin{aligned} f_\neg(\alpha, \alpha) &= (\nabla\neg(\alpha \sim \alpha) \,\&\, ((\neg\alpha \sim \alpha) \,\&\, \xi)) \vee (\nabla(\alpha \sim \alpha) \,\&\, \alpha_i) \\ &= (\mathbb{0} \,\&\, (\mathbb{0} \,\&\, \xi)) \vee (\mathbb{1} \,\&\, \alpha_i) = \alpha_i. \end{aligned}$$

□

**Proposition 7.** *Let arbitrary $\alpha \in \mathfrak{M}^*$ and $\Box\alpha = \mathbb{0}$. Then*

$$f_\Delta(\alpha, \alpha) = \alpha_i.$$

*Proof.* Taking into account (2) we have

$$\begin{aligned} f_\Delta(\alpha, \alpha) &= (\nabla\alpha \,\&\, ((\Delta\alpha \sim \alpha) \sim \eta)) \vee (\neg\nabla\alpha \,\&\, \alpha_i) \\ &= (\mathbb{0} \,\&\, ((\Delta\alpha \sim \alpha) \sim \eta)) \vee (\mathbb{1} \,\&\, \alpha_i) = \mathbb{0} \vee \alpha_i = \alpha_i. \end{aligned}$$

□

## 4 Important properties of some primitive positive clones

Consider an arbitrary value $i$, $i = 1, 2, \ldots$. Let $K_i$ be the primitive positive clone of $\mathfrak{M}^*$ consisting of all functions of $\mathfrak{M}^*$ which preserve the relation $x = \neg\Delta^i\mathbb{0}$ on $\mathfrak{M}^*$. For example, $K_1$ is defined by the relation $x = (0, 1, 1, 1, \ldots)$.

*Remark* 2. The functions $\Box x, x \,\&\, y,\ x \vee y,\ \neg\Delta^i\mathbb{0} \in K_i$, and $\neg x, \Delta x \notin K_i$.

*Remark* 3. Since $K_i$ is a primitive positive clone it follows from the above statement the functions $\neg x$ and $\Delta x$ are not primitive positive definable via functions of $K_i$ on $\mathfrak{M}^*$, so $\mathcal{T}(\mathfrak{M}^*) \nsubseteq K_i$ and thus the clone $K_i$ is not complete in $\mathfrak{M}^*$.

*Remark* 4. By Propositions 6 and 7 we have the earlier defined functions $f_\neg(x, y)$ and $f_\Delta(x, y)$ are in $K_i$.

**Lemma 1.** *Suppose an arbitrary $f(x_1, \ldots, x_k) \in \mathcal{T}(\mathfrak{M}^*)$ and $f \notin K_i$. Then the functions $\Delta x$ and $\neg x$ are primitive positive definable via functions of $K_i \cup \{f(x_1, \ldots, x_k)\}$.*

*Proof.* Let us note since $f \notin K_i$ we have $f(\neg\Delta^i\mathbb{0}, \ldots, \neg\Delta^i\mathbb{0}) \neq \neg\Delta^i\mathbb{0}$. Now consider the next term operations $f'_\neg$ and $f'_\Delta$ defined by terms (10) and (11):

$$(\nabla\neg(x \sim y) \,\&\, ((\neg x \sim y) \sim f(\neg\Delta^i\mathbb{0}, \ldots, \neg\Delta^i\mathbb{0}))) \vee (\nabla(x \sim y) \,\&\, \neg\Delta^i\mathbb{0}) \qquad (10)$$

$$(\nabla y \,\&\, ((\Delta x \sim y) \sim f(\neg\Delta^i\mathbb{0}, \ldots, \neg\Delta^i\mathbb{0}))) \vee (\neg\nabla y \,\&\, \neg\Delta^i\mathbb{0}) \qquad (11)$$

and examine the primitive positive formulas containing only functions from $K_i \cup \{f\}$:

$$(f'_\neg(x, y) = f(\neg\Delta^i\mathbb{0}, \ldots, \neg\Delta^i\mathbb{0})) \text{ and } (f'_\Delta(x, y) = f(\neg\Delta^i\mathbb{0}, \ldots, \neg\Delta^i\mathbb{0})).$$

Let us note by Propositions 2 and 3 we have $(\neg x = y)$ if and only if $(f'_\neg(x, y) = f(\neg\Delta^i\mathbb{0}, \ldots, \neg\Delta^i\mathbb{0}))$ and according to Propositions 4 and 5 we get $(\Delta x = y)$ if and only if $(f'_\Delta(x, y) = f(\neg\Delta^i\mathbb{0}, \ldots, \neg\Delta^i\mathbb{0}))$.

Lemma is proved. □

## 5 Main result

**Theorem 1.** *There are infinitely many maximal primitive positive clones in the diagonalizable algebra $\mathfrak{M}^*$.*

*Proof.* The theorem is based on the example of an infinite family of maximal primitive positive clones presented below.

**Example 1.** The classes $K_1$, $K_2, \ldots$ of term operations of $\mathcal{T}(\mathfrak{M}^*)$, which preserve on algebra $\mathfrak{M}^*$ the corresponding relations $x = \neg\Delta\mathbb{0}$, $x = \neg\Delta^2\mathbb{0}, \ldots$, constitute a numerable collection of maximal primitive positive clones in $\mathfrak{M}^*$.

Really, it is known [9] that these classes of functions represent primitive positive clones. According to Remark 3 each clone $K_i$ is not complete in $\mathfrak{M}^*$. In virtue of Lemma 1 these primitive positive clones are maximal. It remains to show these clones are different. The last thing is obvious since

$$\neg\Delta^j\mathbb{0} \in K_j \text{ and } \neg\Delta^j\mathbb{0} \notin K_i, \text{ when } i \neq j.$$

The theorem is proved. □

## 6 Conclusions

We can consider the logic $L\mathfrak{M}^*$ of $\mathfrak{M}^*$, which happens to be an extension of the propositional provability logic $GL$, and consider primitive positive classes of formulas $M_1, M_2, \dots$ of the propositional provability calculus of $GL$ preserving on $\mathfrak{M}^*$ the corresponding relations $x = \neg\Delta\mathbb{0}, x = \neg\Delta^2\mathbb{0}, \dots$.

**Theorem 2.** *The classes of formulas $M_1, M_2, \dots$ constitute an infinite collection of primitive positive classes of formulas in the extension $L\mathfrak{M}^*$ of the propositional provability logic $GL$.*

*Proof.* The statement of the theorem is just another formulation of the Theorem 1 above in terms of formulas of the calculus of $GL$, which follows the usual terminology of [3]. □

## References

[1] Burris S., Willard R. *Finitely many primitive positive clones.* Proceedings of the American Mathematical Society, 1987, **101**, No. 3, 427–430.

[2] Szabo L. *On the lattice of clones acting bicentrally.* Acta Cybernet., 1984, **6**, 381–388.

[3] Kuznetsov A. V. *On detecting non-deducibility and non-expressibility.* Locical deduction. Nauka, Moscow, 1979, 5–33 (in Russian).

[4] Danil'čenco A. F. *Parametric expressibility of functions of three-valued logic.* Algebra i Logika, 1977, **16**, 397–416 (in Russian).

[5] Magari R. *The diagonalizable algebras (the algebraization of the theories which express Theor.: II).* Boll. Unione Mat. Ital., 1975, **12**, (suppl. fasc 3), 117–125.

[6] Solovay R. M. *Provability interpretations of modal logic.* Israel J. Math., 1975, **25**, 287–304.

[7] Szendrei Á. *Clones in universal algebra.* Séminaire de Mathématiques Supérieures, **99**, Les Presses de l'Université de Montréal, 1986.

[8] Szabó L. *On algebras with primitive positive clones.* Acta Sci. Math. (Szeged), 2007, **73**, 463–470.

[9] Danil'čenco A. F. *On parametrical expressibility of the functions of $k$-valued logic.* Colloq. Math. Soc. Janos Bolyai, **28**, North-Holland, 1981, 147–159.

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# Functional compactifications of $T_0$-spaces and bitopological structures

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**Abstract.** We study the compactifcation of $T_0$-spaces generated by families of special continuous mappings into a given standard space $E$. In this context we have introduced the notions of $E$-thin and $E$-rough $g$-compactifications. The maximal $E$-thin and $E$-rough $g$-compactifications are constructed.

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## 1 Introduction

In functional analysis and related areas of mathematics different dual pairs of topologies are used. A *bitopological structure* on a set $X$ is called a pair of topologies $\{\mathcal{T}, \mathcal{T}^d\}$ on $X$. In this case $(X, \mathcal{T}, \mathcal{T}^d)$ is a bitopological space. The general concept of a bitopological structure was introduced by J. C. Kelly [7] and applied in distinct domains by many authors (see [8, 9]).

If $(X, \mathcal{T}, \mathcal{T}^d)$ is a bitopological space, then we put $\mathcal{T}' = max\{\mathcal{T}, \mathcal{T}^d\}$ and say that $\mathcal{T}$ is the initial topology, $\mathcal{T}^d$ is the dual topology and $\mathcal{T}'$ is the final topology. In many constructions the final topology is a Hausdorff topology.

**Example 1.** Let $X$ be a set and $\mathcal{Q} = \{\rho_\alpha : \alpha \in A\}$ be a family of functions on $X \times X$ with the next properties:

– $sup\{\rho_\alpha(x, y) + \rho_\alpha(y, x) : \alpha \in A\} = 0$ if and only if $x = y$;

– $\rho_\alpha(x, y) + \rho_\alpha(y, z) \geq \rho_\alpha(x, z)$ for all $x, y, z \in X$ and $\alpha \in A$.

Then we say that $\mathcal{Q}$ is a family of pseudo-quasimetrics on $X$. We put $V(x, \rho_\alpha, r) = \{y \in X : \rho_\alpha(x, y) < r\}$ for all $x \in X$, $\alpha \in A$ and $r > 0$. The intersections of finite elements of the family $\{V(x, \rho_\alpha, r) : x \in X, \alpha \in A, r > 0\}$ form a base of the topology $\mathcal{T}(\mathcal{Q})$ on $X$. The functions $\mathcal{Q}^d = \{\rho^d_\alpha(x, y) = \rho_\alpha(y, x) : \alpha \in A\}$ form the dual family of pseudo-quasimetrics on $X$ and the dual topology $\mathcal{T}(\mathcal{Q}^d)$. The functions $\mathcal{Q}^s = \{\rho^s_\alpha(x, y) = 2^{-1}(\rho_\alpha(x, y) + \rho_\alpha(y, x)) : \alpha \in A\}$ form the final family of pseudo-metrics on $X$ and the final topology $\mathcal{T}(\mathcal{Q}^s) = sup\{\mathcal{T}(\mathcal{Q}), \mathcal{T}(\mathcal{Q}^d)\} = \mathcal{T}(\mathcal{Q} \cup \mathcal{Q}^d)$. Then $(X, \mathcal{T}(\mathcal{Q}), \mathcal{T}(\mathcal{Q}^d))$ is a bitopological space with the initial topology $\mathcal{T}(\mathcal{Q})$, the dual topology $\mathcal{T}(\mathcal{Q}^d)$ and the completely regular final topology $\mathcal{T}(\mathcal{Q}^s)$.

The first examples of bitopological spaces were constructed in this way [7–9]. In many cases the family $\mathcal{Q}$ is a singleton set, i. e. is a quasimetric on $X$.

For our aim the initial and the final topologies on $X$ are important. From this point of view, in the present article we suppose that any bitopological structure $\{\mathcal{T}, \mathcal{T}'\}$ on $X$ has the following properties:

– $\mathcal{T} \subseteq \mathcal{T}'$;

– any compact subspace of the space $(X, \mathcal{T}')$ is Hausdorff and closed;

– the space $(X, \mathcal{T})$ is a $T_0$-space.

In this case we say that $\mathcal{T}$ is the initial or weak topology on $X$ and $\mathcal{T}'$ is the final or strong topology on $X$. For any subset $A$ of $X$ consider two closures: the initial closure $clA = cl_{(X,\mathcal{T})}A$ and the strong closure $s - clA = cl_{(X,\mathcal{T}')}A$.

We use the terminology from [5, 6].

**Definition 1.** *A $g$-compactification of a space $X$ is a pair $(Y, f)$, where $Y$ is a compact $T_0$-space, $f : X \to Y$ is a continuous mapping, the set $f(X)$ is dense in $Y$. If the set $\{y\}$ is closed in $Y$ for any point $y \in Y \setminus f(X)$, then $(Y, f)$ is called a $g$-compactification of a space $X$ with a $T_1$-remainder. If $f$ is an embedding, then we say that $Y$ is a compactification of $X$ and consider that $X \subseteq Y$, where $f(x) = x$ for any $x \in X$.*

Let $(Y, f)$ and $(Z, g)$ be two $g$-compactifications of the space $X$. We consider that $(Y, f) \le (Z, g)$ if there exists a continuous mapping $\varphi : Z \longrightarrow Y$ such that $f = \varphi \circ g$, i.e. $f(x) = \varphi(g(x))$ for each $x \in X$. If $(Y, f) \le (Z, g)$ and $(X, f) \le (Y, g)$, then we say that $g$-compactifications $(Y, f)$ and $(Z, g)$ are equivalent. If $\varphi$ is a homeomorphism of $Z$ onto $Y$, then we say that the $g$-compactifications $(Y, f)$ and $(Z, g)$ are identical. We identify the identical $g$-compactifications.

The class of all compactifications of a given non-empty space is not a set (see [3]).

A family $\mathcal{L}$ of subsets of a space $X$ is called a *$WS$-ring* if $\mathcal{L}$ is a family of closed subsets of $X$, $X \in \mathcal{L}$, $\emptyset \in \mathcal{L}$ and $F \cap H, F \cup H \in \mathcal{L}$ for any $F, H \in \mathcal{L}$.

For any family $\mathcal{L}$ of closed subsets of a space $X$ denote by $r\mathcal{L}$ the minimal $WS$-ring of sets containing $\mathcal{L}$.

Fix a family $\mathcal{L}$ of closed subsets of $X$. Let $\mathcal{L}' = \{X\} \cup \mathcal{L}$. Denote by $M(\mathcal{L}, X)$ the family of all $\mathcal{L}'$-ultrafilters $\xi \in \mathcal{L}$. We put $\xi_{\mathcal{L}}(x) = \{H \in \mathcal{L}' : x \in H\}$. Let $\omega_{\mathcal{L}}X = M(\mathcal{L}, X) \cup \{\xi_{\mathcal{L}} : x \in X\}$.

Consider the mapping $\omega_{\mathcal{L}} : X \to \omega_{\mathcal{L}}X$, where $\omega_{\mathcal{L}}(x) = \xi_{\mathcal{L}}(x)$ for any $x \in X$. On $\omega_{\mathcal{L}}X$ consider the topology generated by the closed semibase $\omega\mathcal{L} = \{< H >= \{\xi \in \omega_{\mathcal{L}}X : H \in \xi\} : H \in \mathcal{L}'\}$. If $\mathcal{L}$ is a $WS$-ring, then $\omega\mathcal{L}$ is a closed base.

The pair $(\omega_{\mathcal{L}}X, \omega_{\mathcal{L}})$ is a $g$-compactification of the space $X$ with a $T_1$-remainder.

If $\mathcal{L}$ is a closed base of the space $X$, then $(\omega_{\mathcal{L}}X, \omega_{\mathcal{L}})$ is a compactification of the space $X$ with a $T_1$-remainder.

By virtue of the following theorem, it is sufficient to consider the $g$-compactifications $\omega_{\mathcal{L}}X$ for $WS$-rings $\mathcal{L}$.

**Theorem 1 (see [3]).** $\omega_{r\mathcal{L}}X = \omega_{\mathcal{L}}X$.

**Definition 2.** *A $g$-compactification $(Y, f)$ of a space $X$ is called a Wallman-Shanin $g$-compactification of the space $X$ if $(X, f) = (\omega_{\mathcal{L}}X, \omega_{\mathcal{L}})$ for some $WS$-ring $\mathcal{L}$.*

If $\mathcal{L}$ is the family of all closed subsets of a space $X$, then $\omega X = \omega_{\mathcal{L}} X$ is the *Wallman compactification* of the space $X$ and $\omega_X : X \longrightarrow \omega X$ is the identical mapping (see [3, 5]).

The compactifications of the Wallman-Shanin type were introduced by N. A. Shanin [10] and studied by many authors (see [1–4, 11–13]). There exist Hausdorff compactifications of discrete spaces which are not Wallman–Shanin compactifications [13].

## 2 Functional compactifications

A space $E$ with the topology $\mathcal{T}$ is called *a standard space* if it has the next properties:

– $E$ is a commutative additive topological semigroup with the zero element $0 \in E$;

– there exist a point $1 \in E$ and an open subset $U$ of $E$ such that $0 \in U$ and $1 \notin U$;

– on $E$ a topology $\mathcal{T}'$ is given such that the pair of topologies $\{\mathcal{T}, \mathcal{T}'\}$ is a bitopological structure on $E$.

In particular, $\mathcal{T} \subseteq \mathcal{T}'$ and any compact subspace of the space $(E, \mathcal{T}')$ is Hausdorff and closed.

Fix a standard space $E$. Let $E$ be the set $E$ with the initial topology $\mathcal{T}$ and $E_s$ be the set $E$ with the final topology $\mathcal{T}'$. Denote by $C_b(X, E)$ the space of all continuous mappings $f$ of a space $X$ into the space $(E, \mathcal{T})$ for which the set $s - clf(X)$ is a compact subset of the space $(E, \mathcal{T}')$. Since $\mathcal{T} \subseteq \mathcal{T}'$, we consider that $C_b(X, E_s) \subseteq C_b(X, E)$.

We say that a space $X$ is *$E$-regular* if for each closed subset $B$ of $X$ and any point $x_0 \in X \setminus B$ there exists a mapping $g \in C_b(X, E)$ such that $f(x_0) \notin cl_E g(B)$. A space $X$ is *$E$-completely regular* if for each closed subset $B$ of $X$ and any point $x_0 \in X \setminus B$ there exists a mapping $g \in C_b(X, E_s)$ such that $f(x_0) \notin cl_E g(B)$ (in this case $f(x_0) \notin cl_{E_s} g(B)$ too).

If the space $X$ is $E$-completely regular, then the space $X$ is a Tychonoff space. Really, $E_f = s - clf(X)$ is a Hausdorff compact subspace of the space $E_s$ and $X$ is a subspace of the Hausdorff compact space $\Pi\{E_f : f \in C_b(X, E_s)\}$.

Fix a non-empty space $X$.

Any non-empty set $\mathcal{F} \subseteq C_b(X, E)$ generates two mappings $l_{\mathcal{F}} : X \longrightarrow E_s^{\mathcal{F}}$ and $e_{(\mathcal{F},X)} : X \longrightarrow E^{\mathcal{F}}$, where $e_{\mathcal{F}}(x) = l_{\mathcal{F}}(x) = (f(x) : f \in \mathcal{F})$ for any point $x \in X$, and the identical mapping $i_{\mathcal{F}} : E_s^{\mathcal{F}} \longrightarrow E^{\mathcal{F}}$. Now we put $e_{\mathcal{F}} = e_{(\mathcal{F},X)}$.

Consider the family $B_{\mathcal{F}} = \{f^{-1}(H) : S \setminus H \in \mathcal{T}\}$ of closed subsets of $X$, the compact space $r_{\mathcal{F}} X$ which is the closure of the set $e_{\mathcal{F}}(X)$ in $E_s^{\mathcal{F}}$, the compact space $c_{\mathcal{F}} X$ which is the closure of the set $l_{\mathcal{F}}(X)$ in $E^{\mathcal{F}}$ and the compact space $s_{\mathcal{F}} X = i_{\mathcal{F}}(r_{\mathcal{F}} X)$. Let $(\omega_{\mathcal{F}} X, \omega_{\mathcal{F}}) = (\omega_{B_{\mathcal{F}}} X, \omega_{B_{\mathcal{F}}})$.

The pairs $(c_{\mathcal{F}} X, e_{\mathcal{F}})$ are called the *$E$-rough functional $g$-compactifications* of the space $X$. The pairs $(s_{\mathcal{F}} X, e_{\mathcal{F}})$ are called the *$E$-thin functional $g$-compactifications* of the space $X$.

By construction, $(c_{\mathcal{F}}X, e_{\mathcal{F}}) \leq (s_{\mathcal{F}}X, e_{\mathcal{F}})$ and $s_{\mathcal{F}}X$ is a dense subspace of the space $c_{\mathcal{F}}X$.

If $\mathcal{F} = C_b(X,E)$, then we put $(R_E X, e_E) = (c_{\mathcal{F}}X, e_{\mathcal{F}})$ and $(\beta_E X, e_E) = (s_{\mathcal{F}}X, e_{\mathcal{F}})$.

**Theorem 2.** *Let* $\emptyset \neq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq C_b(X,E)$. *Then* $(c_{\mathcal{F}_1}X, e_{\mathcal{F}_1}) \leq c_{\mathcal{F}_2}X, e_{\mathcal{F}_2})$ *and* $(s_{\mathcal{F}_1}X, e_{\mathcal{F}_1}) \leq s_{\mathcal{F}_2}X, e_{\mathcal{F}_2})$.

*Proof.* Let $p : E^{\mathcal{F}_2} \longrightarrow E^{\mathcal{F}_1}$ and $q : E_s^{\mathcal{F}_2} \longrightarrow E_s^{\mathcal{F}_1}$ be the natural projections. Then $p(c_{\mathcal{F}_2}X) \subseteq c_{\mathcal{F}_1}X$ and $q(r_{\mathcal{F}_2}X) = r_{\mathcal{F}_1}X$. These facts complete the proof. □

**Theorem 3.** *Let* $(Y,f)$ *be a compactification of a space* $X$ *and* $\mathcal{F} \subseteq \{g \circ f : g \in C_b(Y,E)\}$. *Then:*

*1.* $(c_{\mathcal{F}}X, e_{\mathcal{F}}) \leq (Y,f))$.

*2. If* $\mathcal{F} = \{g \circ f : g \in C_b(Y,E)\}$ *and the space* $Y$ *is* $E$*-completely regular, then* $(s_{\mathcal{F}}X, e_{\mathcal{F}}) \geq (Y,f))$.

*3. If* $\mathcal{H} = \{g \circ f : g \in C_b(Y,E_s)\}$ *and the space* $Y$ *is* $E$*-completely regular, then* $(s_{\mathcal{H}}X, e_{\mathcal{F}}) = (Y,f))$.

*Proof.* By virtue of Theorem 2, we can assume that $\mathcal{F} = \{g \circ f : g \in C_b(Y,E)\}$. Then there exists a continuous mapping $h : Y \longrightarrow E^{\mathcal{F}}$ such that $e_{\mathcal{F}} = h \circ f : X \longrightarrow E^{\mathcal{F}}$. Thus $h(Y) \subseteq c_{\mathcal{F}}X$. The assertion 1 is proved. □

If the space $Y$ is $E$-completely regular, then we put $H = \{g \circ f : g \in C_b(Y,E_s)\}$. Obviously, $H \subseteq F$ and $h$ is an embedding.

In this case $l_{\mathcal{H}}(X) = i_{\mathcal{H}}^{-1}(e_{\mathcal{H}}(X)) \subseteq i_{\mathcal{H}}^{-1}(e_{\mathcal{H}}(h(Y)))$ and $i_{\mathcal{H}}^{-1}(e_{\mathcal{H}}(h(Y))$ is a compact set. Then $Y = r_{\mathcal{H}}X = i_{\mathcal{H}}^{-1}(e_{\mathcal{H}}(h(Y)))$ and $(s_{\mathcal{F}}X, e_{\mathcal{F}}) \geq (s_{\mathcal{H}}X, e_{\mathcal{H}}) \geq (Y,f)$. The proof is complete.

*Remark 1.* The pair $(R_E X, e_E)$ is the unique maximal element of the set of $g$-compactifications $\{(c_{\mathcal{F}}X, e_{\mathcal{F}}) : \mathcal{F} \subseteq C_b(X,E)\}$.

*Remark 2.* The pair $(\beta_E X, e_E)$ is the unique maximal element of the set of $g$-compactifications $\{(s_{\mathcal{F}}X, e_{\mathcal{F}}) : F \subseteq C_b(X,E)\} \cup \{(c_{\mathcal{F}}X, e_{\mathcal{F}}) : \mathcal{F} \subseteq C_b(X,E)\}$.

We say that a space $X$ is an *$E$-extensible* (respectively, *a strong $E$-extensible*) *space* if for each mapping $f \in C_b(X,E)$ there exists a (respectively, exists a unique) mapping $\omega f \in C_b(\omega X, E)$ such that $f = \omega f | X$.

**Theorem 4.** *Let* $\emptyset \neq \mathcal{F} \subseteq C_b(X,E)$, $(\omega_{\mathcal{F}}X, \omega_{\mathcal{F}}) \leq (\omega X, \omega_X)$ *and* $X$ *is an* $E$*-extensible space. Then* $(c_{\mathcal{F}}X, e_{\mathcal{F}}) \leq (\omega_{\mathcal{F}}X, \omega_{\mathcal{F}})$.

*Proof.* By definition, $e_{\mathcal{F}}(x) = (f(x) : f \in \mathcal{F}) \in E^{\mathcal{F}}$ and $c_{\mathcal{F}}X$ is the closure of the set $e_{\mathcal{F}}(X)$ in $E^{\mathcal{F}}$. Fix $f \in \mathcal{F}$ and the continuous extension $\omega f : \omega X \to E$ of $f$.

We put $\Omega = \{\Pi\{U_f : f \in \mathcal{F}\} : U_f$ *is open in* $E$ *and the set* $\{f : U_f \neq E\}$ *is finite*$\}$. By construction, $\Omega$ is the standard open base of the space $E^{\mathcal{F}}$. Moreover, $U \cap V \in \Omega$ for all $U, V \in \Omega$. If $\mathcal{L} = \{X \setminus \omega_{\mathcal{F}}^{-1}(H) : H \in \Omega\}$, then $\omega_{\mathcal{F}}X = \omega_{\mathcal{L}}X$.

Consider the continuous mapping $\psi : \omega X \to E^{\mathcal{F}}$, where $\psi(z) = (\omega f(z) : f \in \mathcal{F})$ for each $z \in \omega X$. Obviously, $\psi | X = e_{\mathcal{F}}$.

There exists a continuous mapping $\varphi : \omega X \longrightarrow \omega_{\mathcal{F}} X$ such that $\varphi(x) = \omega_{\mathcal{F}}(x)$ for each $x \in X$. In this case, for $< H > = \{\xi \in \omega_{\mathcal{L}} X : H \in \xi\}$ we have $\varphi^{-1}(< H >) = cl_{\omega X}(H)$ for each $H \in \mathcal{L}$. If $z \in= \omega_{\mathcal{F}} X \setminus = \omega_{\mathcal{L}}(X)$, then $\psi(\varphi^{-1}(z))$ is a singleton set and we put $h(z) = \psi(\varphi^{-1}(z))$. The mapping $h : \omega_{\mathcal{L}}(X) \longrightarrow (c_{\mathcal{F}} X$ is continuous and $h(\omega_{\mathcal{L}}(x) = e_{\mathcal{F}}(x)$ for all $x \in X$. The proof is complete. □

*Remark 3.* $(R_E X, e_E) \leq \omega X$ for any $E$-extensible space $X$ and each standard space $E$.

*Remark 4.* Let $Y$ be a non-empty subspace of a space $X$, $\mathcal{H} \subseteq C_b(X, E)$ and $\mathcal{F} = \{g|Y : g \in \mathcal{H}\}$. Then:

1. $\mathcal{F} \subseteq C_b(Y, E)$.
2. $(s_{\mathcal{F}} Y \subseteq cl_{s_{\mathcal{H}} X} e_{\mathcal{H}}(Y) \subseteq c_{\mathcal{F}} Y = cl_{c_{\mathcal{H}} X} e_{\mathcal{H}}(Y) \subseteq c_{\mathcal{H}} X$.
3. $e_{(\mathcal{F},Y)} = e_{(\mathcal{F},X)}|X$.

**Theorem 5.** *Let $f : X \longrightarrow Y$ be a continuous mapping of a space $X$ into a space $Y$. Then there exist a continuous mapping $\omega f : R_E X \longrightarrow R_E Y$ and a unique continuous mapping $\beta f : \beta_E X \longrightarrow \beta_E Y$ such that $\beta f = \omega f | \beta_E X$ and $\beta f \circ e_{(E,X)} = e_{(E,Y)} \circ f$.*

*Proof.* By virtue of Remark 4, we can assume that $Y = f(X)$. In this case:

1. For any $E$-thin compactification $(Z, \varphi)$ of the space $Y$ the pair $(Z, \varphi \circ f)$ is a $E$-thin compactification of the space $X$. Thus we can consider that any $E$-thin compactification $(Z, \varphi)$ of the space $Y$ is a $E$-thin compactification of the space $X$. Then $(\beta_E Y, e_{(E,Y)}) \leq (\beta_E X, e_{(E,X)})$.

2. For any $E$-rough compactification $(Z, \varphi)$ of the space $Y$ the pair $(Z, \varphi \circ f)$ is a $E$-rough compactification of the space $X$. Thus we can consider that any $E$-rough compactification $(Z, \varphi)$ of the space $Y$ is a $E$-rough compactification of the space $X$. Then $(R_E Y, e_{(E,Y)}) \leq (R_E X, e_{(E,X)})$.

The proof is complete. □

**Example 2.** Let $E_1$ be an infinite countable set $0 \notin E_1$ and $E = E_1 \cup \{0\}$. Consider that $0 + x = x + 0 = 0$ for each $x \in E$ and $x + y = x$ for all $x, y \in E_1$. On $E$ consider the topology $\mathcal{T} = \{E, \emptyset\} \cup \{E \setminus F : F \textit{ is a finite set}\}$ and the topology $\mathcal{T}' = \mathcal{T} \cup \{H \subseteq E_1\}$. Then $(E, \mathcal{T}')$ is the Alexandroff one-point compactification of the discrete space $E_1$. Let $X = \{r_1, r_2, ...\}$ be the space of all rational numbers in the usual topology. The space $X$ is metrizable and $\omega X = \beta X$ is the Stone-Čech compactification of the space $X$. Fix a countable subset $A = \{a_1, a_2, ...\}$ of $E_1$ and we suppose that $a_n \neq a_m$ for $n \neq m$. Then the mapping $g : X \longrightarrow E$, where $g(r_n) = a_n$, is continuous. Since the space $E$ is countable, the mapping $g$ is not continuous extendable on $\omega X$. Thus the space $X$ is not $E$-extensible. If $\mathcal{F} = \{g\}$, then $(c_{\mathcal{F}} X, e_{\mathcal{F}}) = (E, g)$ and $s_{\mathcal{F}} X = \{0\} \cup A$. In particular, $(R_E X, e_E) \not\leq \omega X$.

**Example 3.** Let $E_1$ be an infinite set $0 \notin E_1$ and $E = E_1 \cup \{0\}$. Consider that $0 + x = x + 0 = 0$ for each $x \in E$ and $x + y = x$ for all $x, y \in E_1$. On $E$ consider the topology $\mathcal{T} = \{E, \emptyset\} \cup \{E \setminus F : F \textit{ is a finite set}\}$ and the topology $\mathcal{T}' = \mathcal{T} \cup \{H \subseteq E_1\}$. Then $(E, \mathcal{T}')$ is the Alexandroff one-point compactification of the discrete space $E_1$. Assume that the cardinality $|E| \geq exp(exp(\aleph_0))$. Let $X = \{r_1, r_2, ...\}$ be the space of all rational numbers in the usual topology. Obviously $|\omega X| \leq |E|$. Thus the space $X$ is $E$-extensible and not strong $E$-extensible. For each mapping $f \in C_b(X, E)$ we fix a mapping $\omega f : \omega X \longrightarrow E$ such that $\omega f(x) = f(x)$ for $x \in X$ and $\omega f(y) \neq \omega f(z)$ for distinct points $y, z \in \omega X \setminus X$. Then $\omega f$ is a continuous extension of $f$. There exist many extensions of this kind. Hence $(R_E X, e_E) \leq \omega X$. Since the space $X$ is countable, $(\beta_E X, e_E) \nleq \omega X$.

## 3 Examples

For any space $X$ with a topology $\mathcal{T}$ denote by $X_h$ the set $X$ with the topology generated by the open semibase $\mathcal{T} \cup \{X \setminus : U \subseteq X, U \textit{ is an open compact subset}\}$.

A space $X$ is called a spectral space if the space $X_h$ is compact and on $X$ there exists an open base $\mathcal{B}$ of open compact subsets and $U \cap V \in \mathcal{B}$ for all $U, V \in \mathcal{B}$ [3].

**Definition 3.** *A $g$-compactification $(Y, f)$ of a space $X$ is called a spectral $g$-compactification of the space $X$ if $Y$ is a spectral space and the set $f(X)$ is dense in the space $Y_h$.*

**Example 4.** Denote by $\mathbb{F}$ the set $\{0, 1\}$ by the initial topology $\mathcal{T} = \{\emptyset, \{0\}, \mathbb{F}\}$ and by the final discrete topology $\mathcal{T}' = \{\emptyset, \{0\}, \{1\}, \mathbb{F}\}$. On $F$ consider the additive operation $0 + 0 = 0$ and $0 + 1 = 1 + 0 = 1 + 1 = 1$. Then $(\mathbb{F}, \mathcal{T}, \mathcal{T}')$ is a standard space. Any $T_0$-space is $\mathbb{F}$-regular and $\mathbb{F}$-extensible. A space $X$ is a $\mathbb{F}$-completely regular space if and only if $indX = 0$, i. e. $X$ has a family of open-and-closed sets which form an open base. In this case any zero-dimensional $g$-compactification $(Y, f)$ of a $T_0$-space $X$ is a $\mathbb{F}$-thin $g$-compactification. A $g$-compactification $(Y, f)$ of a space $X$ is a $\mathbb{F}$-thin $g$-compactification if and only if the $g$-compactification $(Y, f)$ is a spectral $g$-compactification. If the space $X$ is not discrete, then the maximal $\mathbb{F}$-thin compactification $\beta_{\mathbb{F}} X$ is not completely regular. If $\mathcal{H} \subseteq C_b(X, \mathbb{F})$, $x_0 \in X$, $g_0 \in \mathcal{H}$, $g_0(X) = \{0\}$, $f(x_0) = 0$ for any $f\mathcal{H}$ and $e_{\mathcal{F}} : X \longrightarrow F^{\mathcal{H}}$ is an embedding of $X$, then the $\mathbb{F}$-rough compactification $c_{\mathcal{H}} X$ is not $\mathbb{F}$-thin. In this case $s_{\mathcal{H}} X \neq c_{\mathcal{H}} X = \mathbb{F}^{\mathcal{H}}$.

**Example 5.** Denote by $\mathbb{D}$ the set $\{0, 1\}$ by the initial and final discrete topologies $\mathcal{T} = \mathcal{T}' = \{\emptyset, \{0\}, \{1\}, F\}$. On $F$ consider the additive operation $0 + 0 = 1 + 1 = 0$ and $0 + 1 = 1 + 0 = 1$. Then $(\mathbb{D}, \mathcal{T}, \mathcal{T}')$ is a standard space. A space $X$ is a $D$-regular space if and only if $indX = 0$, i. e. $X$ has a family of open-and-closed sets which form an open base. A $g$-compactification $(Y, f)$ of a space $X$ is a $\mathbb{D}$-thin $g$-compactification if and only if the $g$-compactification $(Y, f)$ is zero-dimensional. Any $\mathbb{D}$-rough $g$-compactification is $\mathbb{D}$-thin.

**Example 6.** Denote by $\mathbb{R}$ the space of reals in the usual topology $\mathcal{T}'$ and by $\mathbb{R}_u$ the space of reals in the topology $\mathcal{T}$ generated by the open base $\{(-\infty, t) : t \in \mathbb{R}\}$. Then $(\mathbb{R}_u, \mathcal{T}, \mathcal{T}')$ is a standard space with the initial topology $\mathcal{T}$ and the final topology $\mathcal{T}'$. Any $T_0$-space is $\mathbb{R}_u$-regular space and $\mathbb{R}_u$-extensible. A space $X$ is a completely regular space if and only if $X$ is a $\mathbb{R}_u$-completely regular space. In this case any Hausdorff $g$-compactification $(Y, f)$ of a $T_0$-space $X$ is a $\mathbb{R}_u$-thin

$g$-compactification. Any $\mathbb{F}$-thin $g$-compactification is $\mathbb{R}_u$-thin. If $(Y, f)$ is a Hausdorff $g$-compactification of a $T_0$-space $X$ and $indY > 0$, then $(Y, f)$ is a $\mathbb{R}_u$-thin and not spectral $g$-compactification of the space $X$.

**Example 7.** Denote by $\mathbb{R}$ the space of reals in the usual topology $\mathcal{T}' = \mathcal{T}$. Then $(\mathbb{R}, \mathcal{T}, \mathcal{T}')$ is a standard space with the initial topology $\mathcal{T}$ and the final topology $\mathcal{T}'$. A space $X$ is a completely regular space if and only if $X$ is a $\mathbb{R}$-regular space. In this case only the Hausdorff $g$-compactifications $(Y, f)$ of a $T_0$-space $X$ are $\mathbb{R}$-thin. Any $\mathbb{R}$-rough $g$-compactification is $\mathbb{R}$-thin.

From the above examples it follows that the notions of thinness and roughness depend on the standard space $E$ and its initial and final topologies.

## 4 General case

In the present section we suppose that the bitopological structure $\{\mathcal{T}, \mathcal{T}'\}$ on a given standard space $E$ has the following property: $(\mathbb{F}, \mathcal{T})$ is a subspace of the space $(E, \mathcal{T})$.

**Theorem 6.** *Any $\mathbb{F}$-thin $g$-compactification $(s_{\mathcal{H}}X, e_{\mathcal{H}})$ of a space $X$ is an $E$-thin $g$-compactification of $X$.*

*Proof.* If $\mathcal{H} \subseteq C_b(X, \mathbb{F})$, then $\mathcal{H} \subseteq C_b(X, E)$. Obviously, $\mathbb{F}^{\mathcal{H}} \subseteq E^{\mathcal{H}}$, $\mathbb{D}^{\mathcal{H}} = \mathbb{F}_s^{\mathcal{H}} \subseteq E_s^{\mathcal{H}}$, $cl_{\mathbb{D}^{\mathcal{H}}}(l_{\mathcal{H}}(X)) = cl_{E_s^{\mathcal{H}}}(l_{\mathcal{H}}(X))$ and $cl_{\mathbb{F}^{\mathcal{H}}}(e_{\mathcal{H}}(X)) \subseteq cl_{E^{\mathcal{H}}}(e_{\mathcal{H}}(X))$. The proof is complete. □

## References


[1] Alo R. A., Shapiro H. L. *Wallman compact and realcompact spaces.* Contribs. Extens. Theory Topol. Struct., Proceed. Sympos. Berlin 1967, Berlin, 1969, 9–14.

[2] Arhangel'skii A. V., Choban M. M. *Remainders of rectifiable spaces.* Topology and Appl., 2010, **157**, 789–799.

[3] Choban M. M., Calmuţchi L. I. *Some problems of the theory of compactifications of topological spaces.* ROMAI Journal, 2006, **2:1**, 25–51.

[4] Choban M. M., Calmuţchi L. I. *On Wallman compactifications of $T_0$-spaces and related questions.* Bul. Acad. Ştiinţe Repub. Moldova, Mat., 2011, No. 2(66), 102–111.

[5] Engelking R. *General Topology.* PWN, Warszava, 1977.

[6] Gillman L., Jerison M. *Rings of continuous functions.* Springer-Verlag, New York, 1976.

[7] Kelly J. C. *Bitopological spaces.* Proc. London Math. Soc., 1963, **13(3)**, 71—-89.

[8] Kopperman R. *Asymmetry and duality in topology.* Topology Appl., 1995, **66(1)**, 1–39.

[9] Salbany S. *Bitopological spaces, compactifications and completions.* Department of Mathematics, University of Cape Town, Cape Town, 1974.

[10] Shanin N. A. *On special extension of topological spaces*. Dokl. AN SSSR, 1943, **38**, 7–11 (in Rusian).

[11] Shapiro L. B. *Reduction of the basic problem on bicompact extension of the Wallman type*. Soviet Math. Dokl., 1974, **15**, 1020–1023 (Russian original: Dokl. AN SSSR, 1974, **217**, 38–41).

[12] Steiner E. F. *Wallman spaces and compactifications*. Fund. Math., 1968, **61**, 295–304.

[13] Uljanov V. M.. *Solution of a basic problem on compactification of Wallman type*. Soviet Math. Dokl., 1977, **18**, 567–571 (Russian original: Dokl. AN SSSR, 1977, **233**, 1056–1059).

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# On free groups in classes of groups with topologies

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**Abstract.** We study properties of free groups in distinct classes of groups with topologies. The conditions under which the quasi-metric on the space of generators $X$ is extended to an invariant quasi-metric on a free group $F(X, \mathcal{V})$ in the fixed quasi-variety $\mathcal{V}$ of groups with topologies are given. This result is applied to the study:
– of free paratopological groups;
– of free quasitopological groups;
– of free semitopological groups;
– of free left topological groups.

**Mathematics subject classification:** 22F30, 29J15, 54H11, 54E15.
**Keywords and phrases:** Paratopological group, semitopological group, qusitopological group, quasi-variety of groups with topologies, quasi-metric.

## 1 Introduction

By a space we understand a topological $T_0$-space. We use the terminology from [3, 9]. Let $\mathbb{N} = \{1, 2, ...\}$. By $cl_X H$ we denote the closure of a set $H$ in a space $X$, $|A|$ is the cardinality of a set $A$.

A *paratopological* group is a group endowed with a topology such that the multiplication is jointly continuous. Recall that a *semitopological* group is a group with a topology such that the multiplication is separately continuous. Every paratopological group is a semitopological group. A semitopological group with a continuous inverse operation $x \to x^{-1}$ is called a *quasitopological* group. A *topological* group is a paratopological group with a continuous inverse operation $x \to x^{-1}$.

The space $S$ of reals $\mathbb{R}$ with the topology generated by the open base consisting of the sets $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$, where $a, b \in \mathbb{R}$ and $a < b$, is called the Sorgenfrey line [9]. The Sorgenfrey line has the following properties [3]:

– $S$ is an Abelian paratopological group with the Baire property;

– $S$ is a hereditarily Lindelöf first-countable hereditarily separable non-metrizable space;

– $S$ does not admit a structure of a topological group.

In this paper we study properties of free paratopological groups in a given quasi-variety of paratopological groups $\mathcal{W}$. The general theorem of existence of free paratopological (semitopological, quasitopological) groups in distinct classes of groups with topologies was proved in [7]. We follow [5, 7, 8, 11, 12] for the concept of a free object. The paratopological topology on a free group $F(X, \mathcal{W})$ is constructed by the Markov–Graev method [10, 13] developed in [15] for pseudo-quasi-metrics. We develop this method for free groups in the non-Burnside quasi-varieties of paratopological groups. In [15] the authors use the method of left (right) invariant pseudo-quasi-metrics. Since the topology generated by left (right) invariant pseudo-quasi-metrics may not be a paratopological topology [3, 4, 14, 15], this point of view may create dangerous moments. For this we use the method of invariant pseudo-quasi-metrics. The method of invariant pseudo-metrics on free objects was developed in [6, 10].

There exist distinct conditions under which a paratopological topology on a group is topological (see the references in [1–3, 15]). If $G$ is a paratopological group and $x^n = e$ for some natural number $n$, then $G$ is a topological group. By virtue of this fact, the method of invariant pseudo-quasi-metrics is useful in the non-Burnside quasi-varieties of paratopological groups. In the Burnside quasi-varieties of paratopological groups any invariant pseudo-quasi-metric is a pseudo-metric.

## 2 Quasi-metrics on groups

A function $\rho : X \times X \to R$ is called a *pseudo-quasi-metric* if $\rho(x, x) = 0$ and $0 \leq \rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for all $x, y, z \in X$. If $\rho$ is a pseudo-quasi-metric and $\rho(x, y) + \rho(y, x) > 0$ for all distinct $x, y \in X$, then $\rho$ is called a *quasi-metric*.

Any pseudo-quasi-metric $\rho$ generates a topology $\mathcal{T}(\rho)$ with the open base $\{B(x, \rho, r) = \{y \in X : \rho(x, y) < r\} : x \in X, r > 0\}$. The family $\mathcal{P}$ of pseudo-quasi-metric generates the topology $\mathcal{T}(P) = sup\{\mathcal{T}(\rho) : \rho \in \mathcal{P}\}$. If $\mathcal{P} = \emptyset$, then $\mathcal{T}(P) = \{\emptyset, X\}$. The topology $\mathcal{T}(\mathcal{P})$ is a $T_0$-topology if and only if for any two distinct points $x, y \in X$ we have $\rho(x, y) + \rho(y, y) > 0$ for some $\rho \in \mathcal{P}$.

If $\rho$ is a pseudo-quasi-metric on a space $X$ and the sets from $\mathcal{T}(\rho)$ are open in $X$, then we say that $\rho$ is a *continuous pseudo-quasi-metric*.

Let $U$ be an open subset of the space $X$. We put $\rho_U(x, y) = 1$ if $x \in U$ and $y \in X \setminus U$, and $\rho_U(x, y) = 0$ otherwise. Then $\mathcal{T}(\rho_U) = \{\emptyset, U, X\}$. Hence, any topology is generated by some family of pseudo-quasi-metrics.

Let $G$ be a group and $\rho$ be a pseudo-quasi-metric on $G$. The pseudo-quasi-metric $\rho$ is called:

– *left* (respectively, *right*) *invariant* if $\rho(xa, xb) = \rho(a, b)$ (respectively, $\rho(ax, bx) = \rho(a, b)$) for all $x, a, b \in G$;

– *invariant* if it simultaneously is both left and right invariant.

If $\rho$ is a left (or right) invariant pseudo-quasi-metric on a paratopological group $G$, then $\rho$ is continuous if and only if the set $B(e, \rho, r)$ is open in $G$ for any $r > 0$.

If $\rho$ is an invariant pseudo-quasi-metric on the group $G$, then $(G, \mathcal{T}(\rho))$ is a paratopological group and $\rho(x^{-1}, y^{-1}) = \rho(y, x)$ for any $x, y \in G$. Thus any family $\mathcal{P}$ of invariant pseudo-quasi-metrics generates a paratopological topology $\mathcal{T}(\mathcal{P})$ on the group.

A pseudo-quasi-metric $\rho$ on a group $G$ is called *a stable* pseudo-quasi-metric if $\rho(x_1x_2, y_1y_2) \leq \rho(x_1y_1) + \rho(x_2y_2)$ for all $x_1, x_2, y_1, y_2 \in G$ [6].

**Proposition 1.** *Let $\rho$ be a pseudo-quasi-metric on a group $G$. The next assertions are equivalent:*

*1. $\rho$ is invariant.*

*2. $\rho$ is stable.*

*Proof.* Is obvious.

If $\rho$ is a pseudo-quasi-metric on a group $G$ and $\rho(x,y) = \rho(y,x)$ for all $x, y \in X$, then $\rho$ is *a pseudo-metric*. The pseudo-metric $\rho$ is invariant if and only if $\rho(y^{-1}, x^{-1}) = \rho(x,y) = \rho(zx, zy) = \rho(xz, yz)$ for all $x, y, z \in G$.

**Definition 1.** A subset $H$ of a group $G$ is called invariant if $xHx^{-1} = H$ for any $x \in G$.

**Proposition 2.** *Let $U$ be an invariant open subset of a paratopological group $G$ with a topology $\mathcal{T}$ and $e \in U$. We put $d_U(x,y) = 0$ if $x^{-1}y \in U$ and $d_U(x,y) = 1$ if $x^{-1}y \notin U$. Then $d_U$ is an invariant pseudo-quasi-metric and $\mathcal{T}(d_U) \subseteq \mathcal{T}$.*

*Proof.* If $x \in U$, then $d_U(e,x) = 0$ and $d_U(e,y) = 1$ if $y \notin U$. Thus $B(e, d_U, r) = U$ for $0 < r \leq 1$ and $B(e, d_U, r) = G$ for $r > 1$. By construction, $d_U(x,y) = d_U(e, x^{-1}y) = d_U(e, (x^{-1}z^{-1})(zy)) = d_U(zx, zy)$ for all $x, y, z \in G$. Let $x, y \in G$. Then $x^{-1}y \in U$ if and only if $(z^{-1}x^{-1})(yz) \in U$ for any $z \in G$. Thus $d_U(xz, yz) = d_U(x,y)$. The proof is complete. □

**Corollary 1.** *For a paratopological group $G$ the following assertions are equivalent:*

*1. The topology on $G$ is generated by a family of invariant pseudo-quasi-metrics.*

*2. There exists an open base $\mathcal{B}$ of $G$ at $e$ such that any $U \in \mathcal{B}$ is invariant.*

*Remark 1.* Let $U$ be an open subset of a paratopological group $G$ with a topology $\mathcal{T}$ and $e \in U$. We put $d_{lU}(x,y) = 0$ if $x^{-1}y \in U$, and $d_{lU}(x,y) = 1$ if $x^{-1}y \notin U$, $d_{rU}(x,y) = 0$ if $xy^{-1} \in U$, and $d_U(x,y) = 1$ if $xy^{-1} \notin U$. Then $d_{lU}$ is a continuous left invariant pseudo-quasi-metric on $G$ and $d_{rU}$ is a continuous right invariant pseudo-quasi-metric on $G$. Thus:

- the topology of a paratopological group $G$ is generated by a family of left invariant pseudo-quasi-metrics;

- the topology of a paratopological group $G$ is generated by a family of right invariant pseudo-quasi-metrics.

As was established by A. S. Mishchenko [14] (see also [4]), the topology, generated by a family of left (or right) invariant pseudo-metrics, may not be a paratopological topology.

## 3 Free paratopological groups

A class $\mathcal{V}$ of groups with topologies is called *a quasi-variety* of groups if:

(F1) the class $\mathcal{V}$ is multiplicative;

(F2) if $G \in \mathcal{V}$ and $A$ is a subgroup of $G$, then $A \in \mathcal{V}$;

(F3) every space $G \in \mathcal{V}$ is a $T_0$−space.

Let $\mathcal{S}$ be a set of multiplicative and hereditary properties of groups with topologies. A class $\mathcal{V}$ of groups with topologies is called *an $\mathcal{S}$-complete* quasi-variety of groups with topologies if it is a quasi-variety with the next property:

(F4) if $G \in \mathcal{V}$, then $G$ is a group with topology with the properties $\mathcal{S}$.

(F5) if $G \in \mathcal{V}$ and $T$ is a $T_0$-topology on $G$ with the properties $\mathcal{S}$, then $(G, T) \in \mathcal{V}$ too.

A quasi-variety $\mathcal{V}$ of paratopological groups is called *an $\mathcal{S}$-complete variety* of paratopological groups if it is an $\mathcal{S}$-complete quasi-variety with the next property:

(F6) if $g : A \longrightarrow B$ is a continuous homomorphism of a paratopological group $A \in \mathcal{V}$ onto a $T_0$-paratopological group $B$ with the property $\mathcal{S}$, then $B \in \mathcal{V}$.

Denote by $I_p$ the property to be a paratopological group with an invariant bases at the identity $e$. If $\mathcal{S}_p$ is the property to be a paratopological group, then an $\mathcal{S}_p$-complete variety is called a complete variety and an $\mathcal{S}_p$-complete quasi-variety is called a complete quasi-variety of paratopological groups.

Let $X$ be a non-empty topological space and $\mathcal{V}$ be a quasi-variety of groups with topologies. In any space $X$ the basic point $p_X \in X$ is fixed, i.e. any space is pointed.

A free group of a space $X$ in a class $\mathcal{V}$ is a pair $(F(X, \mathcal{V}), e_X)$ with the properties:

– $F(X, \mathcal{V}) \in \mathcal{V}$, $e_X : X \to F(X, \mathcal{V})$ is a continuous mapping and $e = e_X(p_X)$ is the neutral element of the group $F(X, \mathcal{V})$;

– the set $e_X(X)$ generates the group $F(X, \mathcal{V})$;

– for any continuous mapping $f : X \to G \in \mathcal{V}$, where $f(p_X) = e$, there exists a unique continuous homomorphism $\bar{f} : F(X, \mathcal{V}) \to G$ such that $f = \bar{f} \circ e_X$.

An abstract free group of a space $X$ in the class $\mathcal{V}$ is a pair $(F^a(X, \mathcal{V}), a_X)$ with the properties:

– $F^a(X, \mathcal{V}) \in X, a_X : X \to F^a(X, \mathcal{V})$ is a mapping and $e = a_x(p_X)$;

– the set $a_X(X)$ generates the group $F^a(X, \mathcal{V})$;

– for any mapping $g : X \to G \in \mathcal{V}$, where $f(p_X) = e$, there exists a unique continuous homomorphism $\widehat{g} : F^a(X, \mathcal{V}) \to G$ such $g = \widehat{g} \circ a_X$.

In the proof of the following assertion we use the Kakutani's method [11].

**Theorem 1** (see [7]). *Let $\mathcal{V}$ be a quasi-variety of groups with topologies. Then for each space $X$ there exist:*

*- a unique free group $(F(X, \mathcal{V}), e_X)$;*

*- a unique abstract free group $(F^a(X, \mathcal{V}), a_X)$;*

*- a unique continuous homomorphism $r_X : F^a(X, \mathcal{V}) \to F(X, \mathcal{V})$ of $F^a(X, \mathcal{V})$ onto $F(X, \mathcal{V})$ such that $e_X = r_X \circ a_X$.*

*Proof.* Let $\tau$ be an infinite cardinal number and $|X| \leq \tau$. Then the class $\{f_\alpha : X \to G_\alpha : \alpha \in A\}$ of all mappings $f_\alpha : X \to G_\alpha$ with $G_\alpha \in \mathcal{V}$ and $|G_\alpha| \leq \tau$ is a set.

Let $B = \{\beta \in A : f_\beta : X \to G_\beta \textit{ is continuous}\}$. Consider the diagonal product $a_X = \Delta\{f_\alpha : \alpha \in A\} : X \to H_1 = \prod\{G_\alpha : \alpha \in A\}$ and the diagonal product $e_X = \Delta\{f_\alpha : \alpha \in A\} : X \to H_2 = \prod\{G_\alpha : \alpha \in B\}$. Let $F^a(X, \mathcal{V})$ be the subgroup of $H_1$ generated by the set $a_X(X)$ and $F(X, \mathcal{V})$ be the subgroup of $H_2$ generated by the set $e_X(X)$. Since $B \subseteq A$ there exists a unique continuous projection

$r_X : F^a(X, \mathcal{V}) \to F(X, \mathcal{V})$ such that $e_X = r_X \circ a_X$. The objects $(F(X, \mathcal{V}), e_X)$, $(F^a(X, V), a_X)$ and $r_X$ are constructed. The proof is complete. □

The group $F(X, \mathcal{V})$ is called *abstract free* if $r_X$ is a continuous isomorphism. The next problems are important in the theory of universal algebras with topologies (see [5, 7, 12]) .

**Problem 1.** Under which conditions the free group $F(X, V)$ is abstract free?

**Problem 2.** Under which conditions the mapping $e_X : X \to F(X, V)$ is an embedding?

These problems for varieties of topological algebras were posed by A. I. Mal'cev [12].

*Remark 2* (see [5,6,8]). A quasi-variety $\mathcal{V}$ is non-trivial if in $\mathcal{V}$ there exists an infinite group $G$. If the variety $V$ is non-trivial, then:

– $a_X$ is a one-to-one mapping of $X$ onto $a_X(X)$.

Moreover, if $\mathcal{V}$ is a non-trivial $I_p$-complete quasi-variety or a non-trivial $S_p$-complete quasi-variety, then:

– for any completely regular space $X$ the mapping $e_X$ is an embedding of $X$ into $F(X, \mathcal{V})$ and the free group $F(X, V)$ is abstract free (see [5, 7]).

**Proposition 3.** *Let $G$ be a paratopological group, $n \in \mathbb{N}$ and $x^n = e$ for any $x \in G$. Then $G$ is a topological group.*

*Proof.* Since $x^{-1} = x^{n-1}$ and the mapping $x \to x^{n-1}$ is continuous, the mapping $x \to x^{-1}$ is continuous. The proof is complete. □

Let $\mathcal{V}$ be a quasi-variety of paratopological groups, $X$ be a space and $e \in X$. On the free group $F^a(X, \mathcal{V}, e)$ with the identity $e$ consider the maximal paratopological topology $\mathcal{T}(X, \mathcal{V}, e)$ for which the identical mapping $a_X : X \to F^a(X, \mathcal{V}, e)$ is continuous.

**Proposition 4.** *Let $\mathcal{V}$ be a quasi-variety of semitopological groups, $X$ be a space and $e, e_1 \in X$. Then:*

*1. The semitolopological groups $F(X, \mathcal{V}, e)$ and $F(X, \mathcal{V}, e_1)$ are topologically isomorphic.*

*2. The semitolopological groups $(F^a(X, \mathcal{V}, e), a_X, \mathcal{T}(X, \mathcal{V}, e))$ and $(F^a(X, \mathcal{V}, e_1), b_X, \mathcal{T}(X, \mathcal{V}, e_1))$ are topologically isomorphic.*

*Proof.* Consider the natural continuous mappings $e_X : X \to F(X, \mathcal{V}, e)$ and $l_X : X \to F(X, \mathcal{V}, e_1)$. We can assume that $e_X$ and $l_X$ are embeddings and $e_X(x) = l_X(x) = x$ for any $x \in X$. There exist two continuous homomorphisms $\varphi : F(X, \mathcal{V}, e) \to F(X, \mathcal{V}, e_1)$ and $\psi : F(X, \mathcal{V}, e_1) \to F(X, \mathcal{V}, e)$ such that $\varphi(x)) = xe^{-1}$ and $\psi(x) = e_X(xe_1^{-1})$ for any $x \in X$. Since $\psi$ is a homomorphism, $\psi(\varphi(x)) = \psi(x \cdot e^{-1}) = \psi(x) \cdot \psi(e^{-1}) = \psi(x) \cdot \psi(e)^{-1} = (xe_1^{-1}) \cdot (e \cdot e_1^{-1})^{-1} = x$ for $x \in X \subseteq F(X, \mathcal{V}, e)$. Hence the composition $\varphi \circ \psi$ is a continuous homomorphism such that $(\psi \circ \varphi)(x) = x$ for any $x \in X$. Thus $\psi \circ \varphi$ is the identical isomorphism and $\psi = \varphi^{-1}$. The assertion 1 is proved. The proof of the assertion 2 is similar. The proof is complete. □

## 4 Construction of the group $F^a(X, \mathcal{V})$

Fix a non-trivial quasi-variety $\mathcal{V}$ of paratopological groups.

Consider a space $X$. Then we can assume that $X \subseteq F^a(X, \mathcal{V})$ as a subset and $a_X(x) = x$ for each $x \in X$. In particular $e = p_X$ is the neutral element of the group $F^a(X, \mathcal{V})$. In this case $e \in X \subseteq F^a(X, \mathcal{V})$. The set $X$ is called an alphabet.

Let $\widetilde{X} = X \cup X^{-1}$. Obviously, if $x = p_X$, then $x^{-1} = x = e$.

If $n \geq 1$ and $x_1, x_2, ..., x_n \in \widetilde{X}$, then the symbol $x_1x_2...x_n$ is called a word of the length $n$ in the alphabet $X$.

Any word $x_1x_2...x_n$, where $x_1, x_2, ..., x_n \in \widetilde{X}$, represents a unique element $[x_1x_2...x_n] = x_1 \cdot x_2 \cdot ... \cdot x_n \in F^a(X, \mathcal{V})$.

A given element $b \in F^a(X, \mathcal{V})$ is represented by many words. There exists a word of the minimal length which represents the given element $b$. The length $n$ of this word is called the length of the element $b$ and we put $l(b) = n$.

If an element $b \in F^a(X, \mathcal{V})$ is represented by the words $x_1x_2...x_n$, $y_1y_2...y_m$ of the minimal length, then $n = m$ and $\{x_1, x_2, ..., x_n\} = \{y_1, y_2, ..., y_m\}$. In this case we say that the word $x_1x_2...x_n$ is irreducible and that $Sup(b) = X \cap \{x_1, x_1^{-1}, x_2, x_2^{-1}, ..., x_n, x_n^{-1}\}$ is the support of the element $b$. The set $Sup^*(b) = \{e, x_1, x_1^{-1}, x_2, x_2^{-1}, ..., x_n, x_n^{-1}\}$ is the generalized support of the element $b$. Obviously, $Sup(e) = \{e\}$ and $e \notin Sup(b)$ if $b \neq e$. If $e \in Y \subseteq X$, $b \in F^a(X, \mathcal{V})$ and $F^a(Y, \mathcal{V})$ is the subgroup of $F^a(X, \mathcal{V})$ generated by the set $Y$, then $b \in F^a(Y, \mathcal{V})$ if any only if $Sup(b) \subseteq Y$. If $\mathcal{V}$ is the variety of all paratopological groups, then any $b \in F^a(X, \mathcal{V})$ is represented by a unique word of the minimal length. Moreover, in this case any irreducible word is of the minimal length.

Let $\mathcal{V}_a$ be the variety of all $T_0$-paratopological Abelian groups and $\mathcal{V}_g$ be the variety of all $T_0$-paratopological groups.

For any $n \in \mathbb{N}$ denote by $\mathcal{B}_n$ the Burnside variety of all $T_0$-paratopological groups of the exponent (index) $n$: $G \in \mathcal{B}_n$ if and only if $x^n = e$ for each $x \in G$. The variety $\mathcal{B}_1$ is the unique trivial variety of paratopological groups. If $\mathcal{V}$ is an $I_p$-variety of Abelian paratopological groups, then either $\mathcal{V} = \mathcal{V}_a$, or $\mathcal{V} = \mathcal{V}_a \cap \mathcal{B}_n$ for some $n \in \mathbb{N}$.

If $\mathcal{V}$ is a quasi-variety of paratopological groups and $\mathbb{Z} \in \mathcal{V}$, where $\mathbb{Z}$ is the group of integers, then $\mathcal{V}$ is a quasi-variety of the exponent 0. Obviously, if $\mathcal{V}$ is an $I_p$-complete variety of the exponent 0, then $\mathcal{V}_a \subseteq \mathcal{V}$.

A class of paratopological Abelian groups is $I_p$-complete if and only if it is complete.

## 5 Extension of pseudo-quasi-metrics on free groups

Fix a non-trivial $I_p$-complete quasi-variety $\mathcal{V}$ of paratopological groups. Consider a non-empty set $X$ with a fixed point $e \in X$. We assume that $e \in X \subseteq F^a(X, \mathcal{V})$ and $e$ is the identity of the group $F^a(X, \mathcal{V})$.

Let $\rho$ be a pseudo-quasi-metric on the set $X$. Denote by $Q(\rho)$ the set of all invariant pseudo-quasi-metrics $d$ on $F^a(X, \mathcal{V})$ for which $d(x, y) \leq \rho(x, y)$ for all

$x, y \in X$. The set $Q(\rho)$ is non-empty, since it contains the trivial pseudo-quasi-metric $d(x, y) = 0$ for all $x, y$. For all $a, b \in F^a(X, \mathcal{V})$ we put $\widehat{\rho}(a, b) = sup\{d(a, b) : d \in Q(\rho)\}$. We say that $\widehat{\rho}$ is the maximal extension of $\rho$ on $F^a(X, \mathcal{V})$.

**Property 1.** $\widehat{\rho}(a, a) = 0$ and $\widehat{\rho}(a, b) \leq \widehat{\rho}(a, c) + \widehat{\rho}(c, b)$ for all $a, b, c \in F^a(X, \mathcal{V})$.

*Proof.* We assume that $\infty + \infty = t + \infty = \infty + t = \infty \leq \infty$ and $t < \infty$ for any real number $t$. In these conditions the assertion of Property 1 follows from the construction of $\widehat{\rho}$. □

**Property 2.** $\widehat{\rho}(x, y) \leq \rho(x, y)$ for all $x, y \in X$.

*Proof.* Follows from the constructions of $\widehat{\rho}$. □

**Property 3.** $\widehat{\rho}(xa, xb) = \widehat{\rho}(ax, bx) = \widehat{\rho}(a, b)$ for all $x, a, b \in F^a(X, \mathcal{V})$.

*Proof.* Follows from the invariance of the pseudo-quasi-metrics $Q(\rho)$. □

**Property 4.** $\widehat{\rho}(a, b) = \widehat{\rho}(ab^{-1}, e) = \widehat{\rho}(e, a^{-1}b)$.

*Proof.* Follows from Property 3. □

**Property 5.** $\widehat{\rho}(a_1a_2, b_1b_2) \leq \widehat{\rho}(a_1, b_1) + \widehat{\rho}(a_2, b_2)$.

*Proof.* Follows from Proposition 1 and Property 3. □

**Property 6.** $\widehat{\rho}(a^{-1}, b^{-1}) = \widehat{\rho}(b, a)$ for all $a, b \in F^a(X, \mathcal{V})$.

*Proof.* We have $\widehat{\rho}(a^{-1}, b^{-1}) = \widehat{\rho}(aa^{-1}b, ab^{-1}b) = \widehat{\rho}(b, a)$. □

**Property 7.** $\widehat{\rho}(a, b) < \infty$ for all $a, b \in F^a(X, \mathcal{V})$.

*Proof.* For some $n \in N$ we have $a = x_1^{\varepsilon_1}x_2^{\varepsilon_2}...x_n^{\varepsilon_n}$ and $b = y_1^{\delta_1}y_2^{\delta_2}...y_n^{\delta_n}$. Fix $i \leq n$. If $\varepsilon_i = \delta_i = 1$, then $\widehat{\rho}(x_i^{\varepsilon_i}, y_i^{\delta_i}) = \widehat{\rho}(x_i, y_i) \leq \rho(x, y)$. If $\varepsilon_i = \delta_i = -1$, then $\widehat{\rho}(x_i^{\varepsilon_i}, y_i^{\delta_i}) = \widehat{\rho}(x_i^{-1}, y_i^{-1}) = \widehat{\rho}(y_i, x_i) \leq \rho(y_i, x_i)$. If $\varepsilon_i = -\delta_i$, then $\widehat{\rho}(x_i^{\varepsilon_i}, y_i^{\delta_i}) \leq \widehat{\rho}(x_i, y_i^{-1}) + \widehat{\rho}(x_i^{-1}, y_i) \leq \rho(e, x_i) + \rho(x_i, e) + \rho(e, y_i) + \rho(y_i, e)$. Hence $\widehat{\rho}(x_i^{\varepsilon_i}, y_i^{\delta_i}) \leq \rho(x_i, y_i) + \rho(y_i, x_i) + \rho(e, x_i) + \rho(x_i, e) + \rho(e, y_i) + \rho(y_i, e) < \infty$. Then $\widehat{\rho}(a, b) \leq \sum\{\widehat{\rho}(x_i^{\varepsilon_i}, y_i^{\delta_i}) : i \leq n\} < \infty$. □

**Example 1.** Consider the variety $\mathcal{B}_2$ of paratopological groups. Any group $G \in \mathcal{B}_2$ is commutative. Fix a space $X$ with the fixed point $e \in X$.

Let $\rho(z, z) = \rho(z, e) = 0$ for any $z \in X$ and $\rho(e, x) = \rho(x, y) = 1$ for all $x, y \in X, x \neq y, x \neq e \neq y$. Then $\rho$ is a quasi-metric on $X$. In this case $x^{-1} = x$ for $x \in X$. If $x_1x_2...x_n$ is an irreducible word, then $|\{x_1, x_2, ..., x_n\}| = n$, i.e. $x_i \neq x_j$ for distinct $i, j \leq n$. Consider the maximal extension $\widehat{\rho}$ of the quasi-metric $\rho$ on $G = F^a(X, \mathcal{B}_2)$. Then $\widehat{\rho}(x, y) = \widehat{\rho}(y^{-1}, x^{-1}) = \widehat{\rho}(y, x)$, i.e. $\widehat{\rho}$ is a pseudo-metric. Hence $\widehat{\rho}(x, e) = \widehat{\rho}(e, x) = \rho(x, e) = 0$ for any $x \in X$. Thus $0 \leq \widehat{\rho}(x, y) \leq \widehat{\rho}(x, e) + \rho(e, y) = 0$ for all $x, y \in X$. Therefore $\widehat{\rho}(a, b) = 0$ for all $a, b \in G$. We proved that the pseudo-quasi-metric $\widehat{\rho}$ is trivial.

**Example 2.** Let $\mathcal{A}_3$ be the variety of all paratopological Abelian groups with the identity $x^3 = e$. Fix a space $X$ with the basic point $e \in X$. Let $b \in X \setminus \{e\}$. Then the words $bb$ and $b^{-1}$ are irreducible, $bb = b^{-1}$, the word $bb$ is not of the minimal length and the word $b^{-1}$ is of the minimal length.

**Proposition 5.** *Let $\rho$ be a quasi-metric on $X, \rho(x^{-1}, y^{-1}) = \rho(y, x)$, $\rho(x, y^{-1}) = max\{\rho(x, e), \rho(e, y^{-1})\}$ and $\rho(y^{-1}, x) = max\{\rho(y^{-1}, e), \rho(e, x)\}$ for all $x, y \in X$. Then $\widehat{\rho}(a, b) = inf\{\sum\{\widehat{\rho}(x_i^{\varepsilon_i}, y_i^{\delta_i}) : i \leq n\} : n \in N, a \equiv x_1^{\varepsilon_1} \cdot x_2^{\varepsilon_2} \cdot ... \cdot x_n^{\varepsilon_n}, b \equiv y_1^{\delta_1} \cdot y_2^{\delta_2} \cdot ... \cdot y_n^{\delta_n}\}$ for all $a, b \in F^a(X, \mathcal{V})$.*

*Proof.* Obviously $\widehat{\rho_1}(a, b) = inf\{\sum\{\widehat{\rho}(x_i^{\varepsilon_i}, y_i^{\delta_i}) : i \leq n, n \in N, a \equiv x_1^{\varepsilon_1} \cdot x_2^{\varepsilon_2} \cdot ... \cdot x_n^{\varepsilon_n}, b \equiv y_1^{\delta_1} \cdot y_2^{\delta_2} \cdot ... \cdot y_n^{\delta_n}\}$ is an invariant pseudo-quasi-metric on $F^a(X, \mathcal{V})$ and $\widehat{\rho_1}(x, y) \leq \widehat{\rho}(x, y)$ for all $x, y \in F^a(X, \mathcal{V})$. If $a = x_1^{\varepsilon_1} \cdot x_2^{\varepsilon_2} \cdot ... \cdot x_n^{\varepsilon_n}$ and $b = y_1^{\delta_1} \cdot y_2^{\delta_2} \cdot ... \cdot y_n^{\delta_n}$, then $\widehat{\rho}(a, b) \leq \sum\{\widehat{\rho}(x_i^{\varepsilon_i}, t_i^{\delta_i}) : i \leq n\}$. Thus $\widehat{\rho}(a, b) \leq \widehat{\rho_1}(a, b)$. The proof is complete. □

## 6 Elementary spaces and free groups

Fix a non-trivial quasi-variety $\mathcal{V}$ of paratopological groups. Consider the space $E_\infty = \{0, 1, -1, 2, -2, \ldots n, -n, \ldots\}$ with the topology generated by the quasi-metric $\rho_\infty(x, y) = 1$ if $x < y$, and $\rho_\infty(x, y) = 0$ if $x \leq y$. Let $E_n = \{0, 1, -1, 2, -2, \ldots n, -n\}$ and $\rho_n(x, y) = \rho_\infty(x, y)$ for all $x, y \in E_n$. Then $(E_n, \rho_n)$ is a subspace of the quasi-metric space $(E_\infty, \rho_\infty)$. Assume that $p_{E_\infty} = p_{E_n} = 0 \in E_n$ for each $n$. For any $n \in N$ consider the continuous retraction $r_n : E_\infty \to E_n$, where $r_n(x) = x$ for any $x \in E_n$, $r_n(x) = -n$ for any $x \leq -n$ and $r_n(x) = n$ for any $x \geq n$.

**Proposition 6.** *The following assertions are equivalent:*

1. *$e_{E_1} : E_1 \to F(E_1, \mathcal{V})$ is an embedding;*
2. *$e_{E_\infty} : E_\infty \to F(E_\infty, \mathcal{V})$ is an embedding;*
3. *$e_X : X \to F(X, \mathcal{V})$ is an embedding for any space $X$.*

*Proof.* Implications $3 \to 2 \to 1$ are obvious. Assume that $e_{E_1}$ is an embedding. Fix a $T_0$-space $X$. There exist a cardinal number $\tau$ and an embedding $f : X \to E_1^\tau$, where $f(p_X) = 0$. We assume that $E_1 \subseteq F(E_1, \mathcal{V})$. Then $E_1^\tau \subseteq F(E_1, \mathcal{V})^\tau$. Consider the continuous homomorphism $\widehat{f} : F(X, \mathcal{V}) \to F(E_1, \mathcal{V})^\tau$ generated by the mapping $f$. Since $f = \widehat{f} \circ e_X$ is an embedding, $e_X$ is an embedding too. The proof is complete. □

**Lemma 1.** *Let $F$ be a finite set of a space $X$ and $|F| = n \geq 1$. Then there exists a continuous mapping $s_F : X \to E_n \subseteq E_\infty$ such that $s_F(p_X) = 0$ and $s_F(x) \neq s_F(y)$ for all distinct $x, y \in F$.*

*Proof.* We can assume that $p_X \in F$. In any non-empty finite space $Y$ there exists a point $y$ such that the set $\{y\}$ is closed in $Y$. Thus in $F$ there exists a well-ordering $F = \{x_1, x_2, ..., x_n\}$ such that the set $\{x_1, x_2, ..., x_i\}$ is closed in $F$ for any $i \leq n$. Assume that $p_X = x_k$, where $1 \leq k \leq n$. We put $F_1 = cl_X\{x_1\}, F_2 = cl_X\{x_1, x_2\} \setminus cl_X\{x_1\}$,..., $F_{n-1} = cl_X\{x_1, x_2, ..., x_{n-1}\} \setminus cl_X\{x_1, x_2, ..., x_{n-2}\}$, $F_n = X \setminus cl_X\{x_1, x_2, ..., x_{n-1}\}$. Obviously, there exists a continuous mapping $s_F : X \to E_n$ such that $s_F(p_X = 0$, $s_F(x_i) = i - k < s_F(x_j) = j - k$ for $1 \leq j < i \leq n$ and $s_F(F_i) = s_F(\{x_i\})$ for any $i \leq n$. The proof is complete. □

**Proposition 7.** *For a quasi-variety $\mathcal{V}$ the following assertions are equivalent:*

1. *The free group $F(E_\infty, \mathcal{V})$ is abstract free.*

*2. For each $n \in N$ the free group $F(E_n, \mathcal{V})$ is abstract free.*
*3. For each $T_0$-space $X$ the free group $F(X, \mathcal{V})$ is abstract free.*

*Proof.* Implications $3 \to 2 \to 1 \to 2$ are obvious. Implication $1 \to 3$ follows from Lemma 1. □

The next assertion is obvious.

**Proposition 8.** *For an $I_p$-complete quasi-variety $\mathcal{V}$ the following assertions are equivalent:*

*1. The maximal extension $d_\infty$ of the quasi-metric $\rho_\infty$ on $F^a(E_\infty, \mathcal{V})$ is a quasi-metric and $\rho(x, y) = d_\infty(x, y)$ for all $x, y \in E_\infty$;*

*2. For any $n \in N$ the maximal extension $d_n$ of the quasi-metric $\rho_n$ on $F^a(E_n, \mathcal{V})$ is a quasi-metric and $\rho_n(x, y) = d_n(x, y)$ for all $x, y \in E_n$.*

**Proposition 9.** *Let $\mathcal{V}$ be an $I_p$-complete quasi-variety, $n \in \mathbb{N}$ and the maximal extension $d_n$ of the quasi-metric $\rho_n$ on $F^a(E_n, \mathcal{V})$ is a quasi-metric. Then:*
*1. $F(E_n, \mathcal{V})$ is an abstract free group;*
*2. $e_{E_n} : E_n \to F(E_n, \mathcal{V})$ is an embedding;*
*3. $d_n(x, y) = \rho_n(x, y)$ for all $x, y \in E_n$.*

*Proof.* There exists $r > 0$ such that $r \leq 1$ and $1 = \rho_n(x, y) \geq d_n(x, y) \geq r$ for any $x, y \in E_n$ for which $x < y$. Then $d'(x, y) = min\{1, r^{-1}d_n(x, y)\}$ is an invariant quasi-metric on $F^a(E_n, V)$ and $d'(x, y) = \rho_n(x, y)$ for all $x, y \in E_n$. Since $d'(x, y) \leq d_n(x, y)$ for all $x, y \in F^a(E_n, \mathcal{V})$, we have $d_n(x, y) = \rho_n(x, y)$ for all $x, y \in E_n$. The proof is complete. □

**Corollary 2.** *If $\mathcal{V}$ is an $I_p$-complete quasi-variety and $d_\infty$ is a quasi-metric on $F^a(E_\infty, \mathcal{V})$, then $d_\infty(x, y) = \rho_\infty(x, y)$ for all $x, y \in E_\infty$.*

**Corollary 3.** *Let $\mathcal{V}$ be an $I_p$-complete quasi-variety. Assume that $d_\infty$ is a quasi-metric on $F^a(E_\infty, \mathcal{V})$. Then for any $T_0$-space $X$ the free group $F(X, \mathcal{V})$ is abstract free, $e_X : X \to F(X, \mathcal{V})$ is an embedding and on $F(X, \mathcal{V})$ there exists a $T_0-$topology $T$ which is generated by some family of invariant pseudo-quasi-metrics and $e_X$ is an embedding of $X$ into $(F(X, \mathcal{V}), T)$.*

## 7 Free Abelian groups of spaces

**Proposition 10.** *The maximal extension $d_\infty$ of the quasi-metric $\rho_\infty$ on $F^a(E_\infty, \mathcal{V}_a)$ is a quasi-metric.*

*Proof.* On the group $\mathbb{Z}$ of integers consider the topology generated by the quasi-metric $\rho(x, y) = 1$ for $x < y$ and $\rho(x, y) = 0$ for $y \leq x$. Obviously, $\rho$ is an invariant quasi-metric and $\mathbb{Z} \in \mathcal{V}_a$.

The group $G = \{(x_n : n \in \mathbb{Z}) \in \mathbb{Z}^{\mathbb{Z}} : \textit{the set } \{n \in \mathbb{Z} : x_n \neq 0\} \textit{ is finite}\}$, $G \in \mathcal{V}_a$ is Abelian and on $G$ consider the topology generated by the invariant quasi-metric $d((x_n : n \in \mathbb{Z}), (y_n : n \in \mathbb{Z})) = sup\{d(x_n, y_n) : n \in \mathbb{Z}\}$. We put $a_0 = (x_n : n \in \mathbb{Z})$ and $x_n = 0$ for any $n \in \mathbb{Z}$. If $n \in \mathbb{Z}$ and $n \geq 1$, then $a_n : (x_n : n \in \mathbb{Z})$, where $x_i = 1$ for $i \in \{0, 1, 2, ..., n-1\}$ and $x_j = 0$ for each $j \in \mathbb{Z} \setminus \{0, 1, 2, ..., n-1\}$. If $n \in \mathbb{Z}$ and

$n \le -1$, then $a_n = (x_n : n \in \mathbb{Z})$, where $x_i = -1$ for each $i \in \{-1, -2, ..., -n\}$ and $x_j = 0$ for any $j \in \mathbb{Z} \setminus \{-1, -2, ..., -n\}$. Consider the mapping $h : E_\infty \to \mathbb{Z}^\mathbb{Z}$, where $d(n) = a_n$ for any $n \in E_\infty$. By construction, $\rho_\infty(x, y) = d(h(x), h(y))$ for all $x, y \in E_\infty$. Thus $h$ is an isometrical embedding of $E_\infty$ in $\mathbb{Z}^\mathbb{Z}$. The set $d_\infty(E_\infty)$ generated the group $G$ and the pair $(G, h)$ is the abstract free group $(F^a(E_\infty, \mathcal{V}_a), a_{E_\infty})$ of the space $E_\infty$. In this case $d(x, y) \le d_\infty(x, y)$ for any $x, y \in G = F^a(E_\infty, V_\infty)$. Thus $d_\infty$ is a quasi-metric. The proof is complete. □

**Corollary 4.** *For any $T_0$-space $X$ the free group $F(X, \mathcal{V}_a)$ is abstract free, $e_X : X \to F(X, \mathcal{V}_a)$ is an embedding and the topology of the space $F(X, \mathcal{V}_a)$ is generated by some family of invariant pseudo-quasi-metric.*

## 8 On the non-Burnside quasi-varieties

Let $\mathcal{V}$ be an $I_p$-complete quasi-variety of paratopological groups. Assume that $\mathbb{Z} \in \mathcal{V}$, i. e. $\mathcal{V}_a \subseteq \mathcal{V} \subseteq \mathcal{V}_g$.

We put $F(X) = F(X, \mathcal{V})$ and $F^a(X) = F^a(X, \mathcal{V})$ for any space $X$.

Fix two words $x_1x_2...x_n$ and $y_1y_2...y_m$, where $x_1, x_2, ..., x_n, y_1, y_2, ..., y_m \in \widetilde{X}$.

If $x_1x_2...x_n$ and $y_1y_2...y_m$ are irreducible, then $[x_1x_2...x_n] = [y_1y_2...y_m]$ if and only if $n = m$ and there exists a bijection $h = \{1, 2, ..., n\} \longrightarrow \{1, 2, ..., m\}$ such that $x_i = y_{h(i)}$ for any $i \le n$.

The words $x_1x_2...x_n$ and $y_1y_2...y_m$ are called equivalent if $[x_1x_2...x_n] = [y_1y_2...y_m]$.

The words $x_1x_2...x_n$ and $y_1y_2...y_m$ are called strongly equivalent if $[x_1x_2...x_n] = [y_1y_2...y_m]$, $n = m$ and there exists a bijection $h : \{1, 2, ..., n\} \longrightarrow \{1, 2, ..., m\}$ such that $x_i = y_{h(i)}$ for any $i \le n$.

Let $\mathbb{N}_n = \{1, 2, ..., n\}$ for any $n \in \mathbb{N}$. If $i < j$, then we put $[i, j] = [j, i] = \{k \in \mathbb{N} : i \le k \le j\}$. If $i, j \in A \subseteq \mathbb{N}$, then $[i, j]_A = [i, j] \cap A$.

A scheme for an element $b \in F^a(X)$ is a word $x_1x_2...x_n$ and a mapping $s : \mathbb{N}_n \to \mathbb{N}_n$ such that:

1. $b = [x_1x_2...x_n]$;
2. $s(i) \ne i$ and $s(s(i)) = i$ for any $i \le n$;
3. There exist a word $y_1y_2...y_n$ and a bijection $h : \mathbb{N}_n \longrightarrow \mathbb{N}_n$ such that:

– $b = [y_1y_2...y_n]$ and $y_i = x_{h(i)}$ for any $i \le n$;

– if $\sigma(i) = h^{-1}(s(h(i)))$ then for any $i, j \in \{1, 2, ..., n\}$ the sets $[i, \sigma(i)]$, $[j, \sigma(j)]$ are either disjoint or one contains the other.

A mapping $\sigma$ from the definition of the scheme has the following properties:

4. There are no $i, j \in \mathbb{N}_n$ such that $i < j < \sigma(i) < \sigma(j)$.
5. For some $i < n$ we have $\sigma(i) = i + 1$.
6. The mappings $s$ and $\sigma$ are bijections and involutions without fixed points.
7. The number $n$ is even.

The method of scheme for pseudo-metric and $\mathcal{V} \in \{\mathcal{V}_a, \mathcal{V}_g\}$ are due to the work of M. I. Graev [10]. The problem of the extension of pseudo-metrics on $F^a(X, \mathcal{V})$ for any quasi-variety $\mathcal{V}$ of topological algebras was examined in [6]. In the case $\mathcal{V} \in \{\mathcal{V}_a, \mathcal{V}_g\}$ the notion of the scheme for pseudo-quasi-metrics was defined in [15].

We use the method of scheme from [15], in the general case, for any non-Burnside quasi-variety.

On a space $X$ fix a continuous pseudo-quasi-metric $\rho$. Assume that $e \in X \subseteq F^a(X)$ and $e$ is the identity of the group $F^a(X)$. For any $x, y \in X$ we put $\rho^*(x^{-1}, y^{-1}) = \rho(y, x)$, $\rho^*(x^{-1}, y) = \rho^*(x^{-1}, e) + \rho^*(e, y)$ and $\rho^*(x, y^{-1}) = \rho^*(y^{-1}, x) = \rho^*(x, e) + \rho^*(e, y^{-1})$. Obviously, $\rho^*$ is a pseudo-quasi-metric on $\widetilde{X}$.

For any $b \in F^a(X)$ we put $N_\rho(b) = inf\{\frac{1}{2}\sum\{\rho^*(x_i^{-1}, x_{s(i)}) : i \leq 2m\} : m \in \mathbb{N}, s : N_m \to N_m \textit{ is a scheme}, b = [x_1x_2...x_m]\}$.

As in [15] we say that the word $x_1x_2...x_n$ is almost irreducible if:

– $x_i \in Sup^*([x_1, x_2, ..., x_n])$ for any $i \leq n$;

– any word $y_1y_2...y_n$ which is strongly equivalent with the word $x_1x_2...x_n$ does not contain two consecutive symbols of the form $u^{-1}u$, $u \in \widetilde{X} \setminus \{e\}$.

If $b \in F^a(X)$ and $2m \geq l(b) \geq 1$, then $b = [x_1, x_2, ..., x_{2m}]$ for some almost irreducible word $x_1x_2...x_{2m}$. The next property of the function $N_\rho$ is important.

**Lemma 2** (see [15], Claim 2). *If $b \in F^a(X)$ and $l(b) = n$, then there exist an almost irreducible word $x_1x_2...x_{2m}$ and a scheme $s : \mathbb{N}_m \to \mathbb{N}_m$ such that:*

*1. $b = [x_1x_2...x_{2m}]$ and $n \leq 2m \leq 2n$;*

*2. $2N_\rho(b) = \sum\{\rho^*(x_i^{-1}, x_{s(i)}) : i \leq 2m\}$.*

*Proof.* Obviously, we can assume that $b \neq e$. Let $b = [x_1, x_2...x_{2m}]$ and $s : \mathbb{N}_{2m} \to \mathbb{N}_{2m}$ be a scheme.

Assume that the word $x_1x_2...x_m$ is not almost irreducible. Then we can suppose that there exist $i < 2m$ and $u \in \widetilde{X}$ such that $x_i = u$ and $x_{i+1} = u^{-1}$. If $h(i) = i+1$, then we put $A = \{1, ..., i-1, i+2, ..., 2m\}$ and $\sigma = s|A$. Then $\sigma$ is a scheme for the element $x = [x_1x_2...x_{i-1}x_{i+2}...x_{2m}]$ and respective word $x_1x_2...x_{i-1}x_{i+2}...x_{2m}$, $|A| = 2m-2$ and $\sum\{\rho^*(x_j^{-1}, x_{\varphi(j)}) : j \in A\} \leq \sum\{\rho^*(x_i^{-1}, x_{h(i)}) : i \in \mathbb{N}_{2m}\}$. If $r = s(i) \neq i+1$ and $t = s(i+1)$, then $A = \{1, ..., i-1, i+2, ..., 2m\}, \sigma(j) = s(j)$ for $j \in \mathbb{N}_{2m} \setminus \{i, i+1, r, t\}$ and $\sigma(r) = t$, $\sigma(t) = r$. Since $\rho^*(x_r^{-1}, x_t) + \rho^*(x_t^{-1}, x_r) \leq \rho^*(x_1^{-1}, u^{-1}) + \rho^*(u^{-1}, x_r) + \rho^*(x_r^{-1}, u) + \rho^*(u, x_1) = \rho^*(x_t^{-1}, x_{i+1}) + \rho^*(x_i^{-1}, x_r) + \rho^*(x_r^{-1}, x_i) + \rho^*(x_{i+1}^{-1}, x_t)$, $\sigma$ is a scheme and $\sum\{\rho^*(x_j^{-1}, x_{\varphi(j)}) : j \in A\} \leq \sum\{\rho^*(x_i^{-1}, x_{h(i)}) : i \in \mathbb{N}_{2m}\}$. Thus we can assume that the word $x_1x_2...x_{2m}$ is almost irreducible and for any $i \leq 2m$ we have $x_{i+1} \neq x_i^{-1}$. In particular, if $i < 2m$, then $x_i \cdot x_{i+1} \neq e$. In this conditions, the word $x_1x_2...x_{2m}$ is almost irreducible and $2m \leq 2n = 2l(a)$. Since there exists a finite set of almost irreducible words of the length $\leq 2l(b)$ which represents the given element $b \in F^a(X)$, the proof is complete. □

**Lemma 3** (see [15], Claim 4). *$N_\rho(x^{-1}y) = \rho(x, y)$ for all $x, y \in X$.*

*Proof.* Fix $x, y \in X$. If $x = y$, then $x^{-1} \cdot y = y \cdot x^{-1} = e, N_\rho(e) = 0 = \rho(x, y)$. Assume that $x \neq y$. Then $l(x^{-1}y) = 2$ and for the element $b = x^{-1}y$ there exist only the next possible almost irreducible words of the length $\leq 4$: $x^{-1}y$, $ex^{-1}ey$, $x^{-1}eye$, $ex^{-1}ye$, $y^{-1}x$, $ey^{-1}ex$, $y^{-1}exe$, $ey^{-1}xe$. If $\mathcal{V} \neq \mathcal{V}_a$, then there exist only the next possible almost irreducible words of the length $\leq 4$: $x^{-1}y$, $ex^{-1}ey$, $x^{-1}eye$, $ex^{-1}ye$. The direct calculation permits to obtain $N_\rho(x^{-1}y) = \rho(x, y)$. □

**Lemma 4** (see [15], Claim 3). *The function $N_\rho$ has the the next properties:*

1. $N_\rho(e) = 0$ *and* $N_\rho(b) \geq 0$ *for any* $b \in F^a(X)$.
2. $N_\rho(a \cdot b) \leq N_\rho(a) + N_\rho(b)$ *for any* $a, b \in F^a(X)$.
3. $N_\rho(xbx^{-1}) = N_\rho(b)$ *for any* $b, x \in F^a(X)$.

*Proof.* Assertions (1) and (2) are obvious. Let $b_1 b_2 ... b_{2m}$ be an almost irreducible word, $b = [b_1 b_2 ..., b_{2m}]$, $s : \mathbb{N}_{2m} \to \mathbb{N}_{2m}$ be a scheme and $2\mathbb{N}_\rho(b) = \sum\{\rho^*(b_i^{-1}, b_{s(i)}) : i \in \mathbb{N}_{2m}\}$. Fix the irreducible word $x_1 x_2 ... x_k$. Put $x = [x_1, x_2, ..., x_k]$, $y_{2m+1} y_{2m+2} ... y_{2m+k} = x_1 x_2 ... x_k$, $y_{2m+k+1} ... y_{2m+2k} = x_k^{-1}$, $A = \{1, 2, ..., 2m, ..., 2m + 2k\}$, $\varphi(i) = s(i)$ for $i \leq 2m$ and $\varphi(2m + i) = 2m + 2k - i + 1$ for $i \leq k$.

Let $y_1, y_2, ..., y_{2m} = b_1, b_2, ..., b_{2m}$. Then $\varphi$ is a scheme on $A$ for the element $x^{-1}bx$, $x_{-1}bx = [y_{2m+k+1} ... y_{2m+2k} y_1 ... y_{2m} y_{2m+1} ... y_{2m+k}]$ and $\sum\{\rho^*(y_j^{-1}, y_{h(j)}) : j \in A\} = \sum\{\rho^*(b_i^{-1}, b_{h(j)}) : i \in \mathbb{N}_m\}$. Hence $N_\rho(x^{-1}bx) \leq \mathbb{N}_\rho(b)$ and $N_\rho(b) = M_\rho((xx^{-1})b(xx^{-1})) \leq \mathbb{N}_\rho(x^{-1}bx)$. The property (3) is proved. □

**Lemma 5.** *The function $d(x, y) = N_\rho(x^{-1}y)$ is an invariant pseudo-quasi-metric on $F^a(X)$. Moreover, $d(x, y) \leq \widehat{\rho}(x, y)$ and $N_\rho(b) \leq \widehat{\rho}(e, b)$, where $\widehat{\rho}$ is the maximal extension of $\rho$ on $F^a(X)$, for any $x, y, b \in F^a(X)$.*

*Proof.* Really, $d(xa, xb) = N_\rho(a^{-1}x^{-1}xb) = N_\rho(a^{-1}b) = d(a, b)$ and $d(ax, bx) = N_\rho(x^{-1}a^{-1}bx) = N_\rho(a^{-1}b) = d(a, b)$. Since $d(x, y) = \rho(x, y)$ for $x, y \in X$, we have $d(x, y) \leq \widehat{\rho}(x, y)$ for all $x, y \in F^a(X)$. The proof is complete. □

**Proposition 11.** *Let $r > 0$ and $X$ be a linear ordered space with the topology generated by the quasi-metric $\rho(x, y) = r$ if $x < y$ and $\rho(x, y) = 0$ if $y \leq x$. Then the maximal extension $d$ of $\rho$ on $F^a(X, e)$ is a quasi-metric for any point $e \in X$.*

*Proof.* We can assume that $r = 1$.

Let $e \in X$ and $e_1 \notin X$. We put $Y = X \cup \{e_1\}$, $\rho(e_1, e_1) = 0$ and $\rho(x, e_1) = \rho(e_1, x) = 1$ for any $x \in X$. Then $(X, \rho)$ is a quasi-metric subspace of the quasi-metric space $(Y, \rho)$. On $F^a(Y) = F^a(Y, \mathcal{V}, e_1)$ consider the function $\mathbb{N}_\rho(y)$. □

**Claim 1.** *If $b \in F^a(Y) \setminus \{e_1\}$, then $\mathbb{N}_\rho(b) + \mathbb{N}_\rho(b^{-1}) \neq 0$.*

*Proof.* Assume that $\mathbb{N}_\rho(b) + \mathbb{N}_\rho(b^{-1}) = 0$, $b = [b_1 b_2 ... b_n]$ and the word $b_1 b_2 ... b_n$ is irreducible. Then $b_1, b_2, ..., b_n \in \widetilde{X} \subseteq \widetilde{Y}$. Since $\mathbb{N}_\rho(b) = 0$, there exists an almost irreducible word $x_1 x_2 ... x_{2m}$ of the minimal length and a scheme $s : \mathbb{N}_{2m} \to \mathbb{N}_{2m}$ such that $b = [x_1 x_2 ... x_{2m}]$ and $\sum\{\rho^*(x_i^{-1}, x_{s(i)}) : i \leq 2m\} = 0$. From the minimality of the length of the word $x_1 x_2 ... x_n$ it follows that $x_i \neq e_1$ for any $i \leq 2m$. Really, if $x_i = e_1$, then $x_{s(i)} \neq e_1$ and $\rho^*(x_{s(i)}^{-1}, x_i) = 1$, a contradiction. Thus the words $x_1 x_2 ... x_{2m}$ and $b_1 b_2 ... b_n$ are equivalent, $n = 2m$ is an even number and $s$ is a scheme on $\mathbb{N}_{2m}$ for the element $b$. We can assume that $x_i = b_i$ for any $i \leq n$. Therefore $\sum\{\rho^*(b_i, b_{s(i)}) : i \leq n\} = 0$. We can assume that for any $i, j \in \{1, 2, ..., n\}$ the sets $[i, s(i)]$, $[j, s(j)]$ are either disjoint or one contains the other.

Since $\mathbb{N}_\rho(b^{-1}) = 0$ and $b = [b_n^{-1}...b_2^{-1}b_1^{-1}]$, there exists a scheme $q : \mathbb{N}_n \to \mathbb{N}_n$ such that $\sum\{\rho^*(b_i, b_{q(i)}^{-1}) : i \leq n\} = 0$. Since the word $b_1b_2...b_n$ is irreducible, $b_{i+1} \neq b_i^{-1}$ for any $i < n$.

We affirm that $s = q$. Assume that $i_1 \in \mathbb{N}$ and $s(i_1) \neq q(i_1)$. Put $j_1 = s(i_1)$. Since $s(j_1) \neq q(j_1)$ and $\rho^*(x_i^{-1}, x_{j_1}) = \rho^*(j_i^{-1}, x_{i_1}) = 0$, $X \cap \{x_{i_1}, x_{j_1}\} \neq \emptyset$ and $X^{-1}\{x_{i_1}, x_{j_1}\} \neq \emptyset$. We can assume that $x_{i_1} \in X$ and $x_{j_1} \in X^{-1}$. For any $k \geq 1$ we put $i_{k+1} = q(j_k)$ and $j_{k+1} = s(i_{k+1})$. For any $k < 1$ we have $x_{j_k} \neq x_k \neq x_{i_{k+1}}$, $\rho^*(x_k^{-1}, x_{k_k}) = \rho^*(x_{j_k}^{-1}, x_{i_k}) = \rho^*(x_{j_k}, x_{i_{k+1}}^{-1}) = \rho^*(x_{j_{k+1}}, x_{j_k}^{-1})$. Let $k$ be the first number for which $\{i_{k+1}, j_{k+1}\} \cap \{i_1, j_1, i_2, j_2, ..., i_k, j_k\} \neq \emptyset$. Suppose that $i_{k+1} = q(j_k) \in \{i_1, j_1, ..., i_k, j_k\}$. If $i_{k+1} = i_p$ for some $p \leq k$, then $j_k = q(i_{k+1}) = q(i_p) = j_{p-1}$, a contradiction. If $i_{k+1} = i_p$ for some $p \leq k$, then $j_k = s(i_{k+1}) = h(j_p) = i_{p+1}$. Since $i_k \neq j_k$, we have $p + 1 < k$ and $j_k \in \{i_{p+1}, j_{p+1}\}$, a contradiction. Now suppose that $j_{k+1} \in \{i_1, j_1, ..., i_k, j_k\}$. If $j_{k+1} = i_p$ for some $p \leq k$, then $i_{k+1} = s(j_{k+1}) = s(i_p) = j_p$, a contradiction. If $j_{k+1} = j_p$ for some $p \leq k$, then $i_{k+1} = s(i_{k+1}) = s(j_p) = i_p$, a contradiction. Therefore $q(i) = s(i)$ for any $i \leq n$. There exists $i < n$ such that $h(i) = q(i) = i+1$. Let $x_i \in X$. Then $x_{i+1} \in X^{-1}$. Since $\rho^*(x_{i+1}^{-1}, x_i) = \rho^*(x_{h(i)}^{-1}) = 0$, we have $x_{i+1}^{-1} \leq x_i$. Since $\rho^*(x_i, x_{i+1}^{-1}) = \rho^*(x_i, x_{q(i)}^{-1}) = 0$, we have $x_i \leq x_{i+1}^{-1}$. Hence $x_i = x_{i+1}^{-1}$ and $x_i \cdot x_{i+1} = e$, a contradiction with the condition of irreducibility of the word $b_1b_2...b_n$. Claim 1 is proved. □

**Claim 2.** *On $F^a(Y, e_1)$ there exists an invariant quasi-metric such that:*
- *$d_1(x, y) = \rho(x, y)$ for any $x, y \in X$;*
- *$d_1(x, y) \in \{0, 1\}$ for any $x, y \in F^a(Y, e_1)$.*

*Proof.* Let $d_2(x, y) = N_\rho(x^{-1}y)$ for all $x, y \in F^a(Y, e_1)$. By construction, $d_2(x, y) \geq 1$ provided $d_2(x, y) > 0$. From this fact and the Claim 1 it follows that $d_1(x, y) = min\{1, d_2(x, y)\}$ is the desired quasi-metric. □

**Claim 3.** *Let $e \in X \subseteq Y$. Then on $F^a(Y, e)$ there exists a quasi-metric $\rho_1$ such that:*
- *$\rho_1(x, y) = \rho(x, y)$ for any $x, y \in X$;*
- *$\rho_1(x, y) \in \{0, 1\}$ for any $x, y \in F^a(Y, e)$.*

*Proof.* Let $d_1$ be quasi-metric with the properties from Claim 2. In the proof of Proposition 4 it was established that there exists an isomorphism $\varphi : F^a(X, e) \to F^a(Y, e_1)$ such that $\varphi(x) = xe^{-1}$ for any $x \in Y$. We put $\rho_1(x, y) = d_1(\varphi(x), \varphi(y))$ for any $x, y \in F^a(Y, e)$. Since the quasi-metric $d_1$ is continuous on the space $(F^a(Y, e_1), T(y, e_1))$, the quasi-metric $\rho_1$ is continuous on the space $(F^a(Y, e), T(Y, e))$. For any $x, y \in Y$ we have $\rho_1(x, y) \in \{0, 1\}$. Let $x, y \in X$ and $x < y$. Then $\rho(y, x) = 0$, $\rho(x, y) = 1$ and $1 \in \{\rho_1(x, y), \rho(y, x)\}$. Since the quasi-metric $\rho_1$ is continuous, we have $\rho_1(x, y) = 1 = \rho(x, y)$ and $\rho_1(y, x) = 0 = \rho(y, x)$. Claim 3 is proved. □

Since $F^a(Y, e)$ is a subgroup of the group $F^a(Y, e)$, the proof is complete.

**Corollary 5.** *For any $n \in \mathbb{N}$ the maximal extension $d_n$ of the quasi-metric $\rho_n$ on $F^a(E_n)$ is a quasi-metric.*

**Corollary 6.** *The maximal extension $d_\infty$ of the quasi-metric $\rho_\infty$ on $F^a(E_\infty)$ is a quasi-metric.*

From Corollary 5 it follows

**Corollary 7.** *For any pointed $T_0$-space $X$ the free group $F(X)$ is abstract free and $e_X : X \to F(X)$ is an embedding. Moreover on $F(X)$ there exists a topology $T$ which is generated by some family of almost invariant pseudo-quasi-metrics and $e_X$ is an embedding of $X$ into $(F(X), T)$.*

## 9 On quasi-varities of paratopological groups

Let $\mathcal{S}$ be a set of properties of paratopological groups, any paratopological group with invariant base has the properties $\mathcal{S}$, $\mathcal{W}$ be a non-trivial $\mathcal{S}$-complete quasi-variety of paratopoligical groups. Denote by $\mathcal{V}$ the $\mathcal{S}$-complete variety of paratopoligical groups generated by the quasi-variety $\mathcal{W}$. We say that $\mathcal{W}$ is a Burnside quasi-variety if $\mathcal{V}$ is a Burnside variety.

A quasi-variety $\mathcal{W}$ is a non-Burnside quasi-variety if and only if $\mathbb{Z} \in \mathcal{W}$.

The next assertions affirm that the free objects of spaces in quasi-varieties are the same as in varieties.

**Proposition 12.** *For any pointed space $X$:*

*1. There exists an isomorphism $\varphi : F^a(X, \mathcal{V}) \longrightarrow F^a(X, \mathcal{W})$ such that $\varphi(x) = x$ for any $x \in X$.*

*2. There exists a topological isomorphism $\varphi : F(X, \mathcal{V}) \longrightarrow F(X, \mathcal{W})$ such that $\psi(e_{(X,\mathcal{V})}(x)) = e_{(X,\mathcal{W})}(x)$ for any $x \in X$.*

*Proof.* The assertion 1 is obvious.

Fix a space $X$. Let $(F(X, \mathcal{V}), e_{(X,\mathcal{V})})$ be the free object of the space $X$ in the class $\mathcal{V}$ and $(F(X, \mathcal{W}), e_{(X,\mathcal{W})})$ be the free object of the space $X$ in the class $\mathcal{W}$. There exists a continuous homomorphism $\varphi : F(X, \mathcal{V}) \longrightarrow F(X, \mathcal{W})$ such that $\varphi(x) = x$ for any $x \in X$.

*Case 1.* $\mathbb{Z} \notin \mathcal{W}$.

In this case $\mathcal{W} \subseteq \mathcal{B}_n$ for some $n \in \mathbb{N}$. By virtue of Proposition 3, $\mathcal{W}$ is a quasi-variety of topological groups. For quasi-varieties of topological groups the assertions of Proposition 12 are known (see [5, 8]).

*Case 2.* $\mathbb{Z} \in \mathcal{W}$.

In this case the variety $\mathcal{V}$ is not a Burnside variety. Then, by virtue of Corollary 7, the free objects $F(X, \mathcal{V})$, $F(X, \mathcal{W})$ are abstract free and the mappings $e_{(X,\mathcal{V})}$ and $e_{(X,\mathcal{W})}$ are embeddings. The proof is complete. □

**Theorem 2.** *If $\mathbb{Z} \in \mathcal{W}$, then for any pointed space $X$ we have:*

*1. The free topological group $(F(X, \mathcal{W}), e_X)$ is abstract free and the mapping $e_X$ is an embedding.*

*2. If $\rho$ is a continuous pseudo-quasi-metric on the space $X$, then:*

*(2a) the maximal extension $\widehat{\rho}$ of the pseudo-quasi-metric $\rho$ on $F(X, \mathcal{W})$ is a continuous invariant pseudo-quasi-metric;*

*(2b)* $\rho(x,y) = \widehat{\rho}(e_X(x), e_X(y))$ *for all* $x, y \in X$;

*(2c) if* $x, y \in F(X, \mathcal{W})$ *and* $\rho$ *is a quasi-metric on* $Sup^*(x) \cup Sup^*(y)$*, then* $\widehat{\rho}(x,y) + \widehat{\rho}(y,x) > 0$;

*(2d) if* $\rho$ *is a quasi-metric, then* $\widehat{\rho}$ *is a quasi-metric too.*

*Proof.* We can assume that $e_X(x) = x$ for any $x \in X$ and $X \subseteq (F(X, \mathcal{W})$.

On $(F(X, \mathcal{W})$ consider the function $N_\rho(b)$ and the pseudo-quasi-metric $d(x,y) = N_\rho(x^{-1}y)$. By virtue of Lemma 5, $d$ is an invariant pseudo-quasi-metric. From Lemma 3 it follows that $d(x,y) = N_\rho(x^{-1}y) = \rho(x,y)$ for all $x, y \in X$. Thus $d(x,y) \leq \widehat{\rho}(x,y)$ for all $x, y \in F(X, \mathcal{W})$ and $d(x,y) = \widehat{\rho}(x,y)$ for all $x, y \in X$.

Let $x, y \in F(X, \mathcal{W})$ and $\rho$ be a quasi-metric on $Z = Sup^*(x) \cup Sup^*(y)$. We put $b = x^{-1}y$. Then $Sup^*(b) \subseteq Z$. Let $r = min\{\rho(u,v) : u, v \in Z, \rho(u,v) > 0\}$. Since the space $Z$ is finite, we have $r > 0$ and there exists an ordering on $Z$ such that $\rho(u,v) > 0$ provided $u < v$. We have $Z \subseteq F^a(Z, \mathcal{W}) \subseteq F(X, \mathcal{W})$. By virtue of Proposition 11, $N_\rho(c) + N_\rho(c^{-1}) > 0$ for each $c \in F^a(Z, \mathcal{W})$. Since $b \in F^a(Z, \mathcal{W})$, $0 < N_\rho(b) + N_\rho(b^{-1}) = d(x,y) + d(y,x) \leq \widehat{\rho}(x,y) + \widehat{\rho}(y,x)$. The assertions 1, (2a), (2b) and (2c) are proved. The assertion (2d) follows from the assertion (2c). The proof is complete. □

## 10 Free groups of quasi-uniform spaces

A quasi-uniformity on a set $X$ is a family $\mathcal{U}$ of entourages of the diagonal $\Delta(X) = \{(x,x) : x \in X\}$ and a family $\mathcal{P}$ of the pseudo-quasi-metrics on $X$, which satisfies the following conditions:

(QU1) If $V \in \mathcal{U}$ and $V \subseteq W \subseteq X \times X$, then $V \in \mathcal{U}$.

(QU2) If $V, W \in \mathcal{U}$, then $V \cap W \in \mathcal{U}$.

(QU3) If $V \in \mathcal{U}$, then there exist $\rho \in \mathcal{P}$ and $r > 0$ such that $\{(x,y) \in X \times X : \rho(x,y) < r\} \subseteq V$.

(QU4) $\{(x,y) \in X \times X : \rho(x,y) < r\} \in \mathcal{U}$ for all $\rho \in \mathcal{P}$ and $r > 0$.

(QU5) If $\rho_1, \rho_2 \in \mathcal{P}$, then there exists $\rho \in \mathcal{P}$ such that $max\{\rho_1(x,y), \rho_2(x,y)\} \leq \rho(x,y)$ for all $x, y \in X$.

(QU6) If $x, y \in X$ and $x \neq y$, then $\rho(x,y) + \rho(y,x) > 0$ for some $\rho \in \mathcal{P}$.

Obviously, the quasi-uniformity $\mathcal{U}$ is generated by a family of pseudo-quasi-metrics $\mathcal{P}$.

Fix a non-trivial $I_p$-complete quasi-variety $\mathcal{W}$ of paratopological groups.

Let $G \in \mathcal{W}$. Denote by $QP(G)$ the family of all continuous pseudo-quasi-metrics on the space $G$, $LQP(G) = \{d \in QP(G) : d \textit{ is left invariant}\}$, $RQP(G) = \{d \in QP(G) : d \textit{ is right invariant}\}$ and $IQP(G) = LQP(G) \cap RQP(G)$.

The pseudo-quasi-metrics $LQP(G)$ generate the left quasi-uniformity $\mathcal{U}_l$ on $G$ and the pseudo-quasi-metrics $RQP(G)$ generate the left quasi-uniformity $\mathcal{U}_r$ on $G$. These quasi-uniformities generate the topology of the space $G$. If $G$ is a paratopological group with the invariant base at the identity $e$, then $\mathcal{U}_l = \mathcal{U}_r$.

Assume that $\mathcal{W}$ is not a Burnside quasi-variety. Fix a quasi-uniformity pointed space $(X, \mathcal{U})$ generated by the pseudo-quasi-metrics $\mathcal{P}$. For any $\rho \in \mathcal{P}$ denote by $\widehat{\rho}$

its maximal extension on $F(X, \mathcal{W})$. We put $\widehat{\mathcal{P}} = \{\widehat{\rho} : \rho \in \mathcal{P}\}$. The family $\widehat{\mathcal{P}}$ generates an invariant quasi-uniformity on $F(X, \mathcal{W})$.

## 11 Free quasitopological groups

A class $\mathcal{V}$ of quasitopological groups is called *a $C$-complete quasi-variety* of quasitopological groups if:

(QF1) the class $\mathcal{V}$ is multiplicative;

(QF2) if $G \in \mathcal{V}$ and $A$ is a subgroup of $G$, then $A \in \mathcal{V}$;

(QF3) every space $G \in \mathcal{V}$ is a $T_0$-space;

(QF4) if $G \in \mathcal{V}$, $\mathcal{T}$ is a compact $T_0$-topology on $G$ and $(G, \mathcal{T})$ is a quasitopological group, then $(G, \mathcal{T}) \in \mathcal{V}$.

**Lemma 6.** *Let $G$ be a quasitopological group. If $G$ is a $T_0$-space, then $G$ is a $T_1$-space.*
*Proof.* It is obvious. □

On any set $X$ there exists the profinite topology $\mathcal{T}_{pf}(X) = \{X, \emptyset\} \cup \{X \setminus F : F$ *is a finite set*$\}$. The space $(X, \mathcal{T}_{pf}(X))$ is a compact $T_1$-space.

**Lemma 7.** *Let $G$ be a group. Then $(G, \mathcal{T}_{pf}(G))$ is a quasitopological group.*

*Proof.* It is obvious. □

**Theorem 3.** *Let $\mathcal{W}$ be a non-trivial $C$-complete quasi-variety of quasitopological groups. For any $T_1$-space $X$ the free group $F(X, \mathcal{W})$ is abstract free and $e_X : X \to F(X, \mathcal{W})$ is an embedding.*

*Proof.* For any infinite cardinal $\tau$ we fix a group $G_\tau \in \mathcal{W}$ of the cardinality $\tau$ with the profinite topology $\mathcal{T}_{pf}(G_\tau)$. Further we fix an infinite group $G_0 \in \mathcal{W}$. Let $F$ be a non-empty closed subset of the space $X$ and $b \notin F$.

*Case 1.* The set $X \setminus F$ is finite.

In this case the sets $F$ and $X \setminus F$ are open-and-closed. There exists a mapping $g : X \longrightarrow G_0$ such that $g(p_X) = e$, $g(F)$ and $g(X \setminus F)$ are singletons and $g^{-1}(g(F)) = F$. Then the mapping $g$ is continuous and $g(b) \notin cl_{G_0} g(F) = g(F)$.

*Case 2.* The set $X \setminus F$ is infinite.

Let $\tau = |X \setminus F|$. There exists a mapping $g : X \longrightarrow G$ such that $g(X) = G_\tau$, $g^{-1}(g(F)) = F$, $g(F)$ is a singleton $g(p_X) = e$ and $g(x) \neq g(y)$ for distinct points $x, y \in X \setminus F$. Since $X$ is a $T_1$-space, the mapping $g$ is continuous and $g(b) \notin cl_{G_0} g(F) = g(F)$.

Therefore the mapping $e_X : X \to F(X, \mathcal{W})$ is an embedding. Thus we can assume that $e = p_X \in X \subseteq F(X, \mathcal{W})$.

Assume that $e = p_X \in X \subseteq F^a(X, \mathcal{W})$. On $F(X, \mathcal{W})$ we consider the profinite topology $\mathcal{T}_{pf}(F^a(X, \mathcal{W}))$. Then the mapping $a_X : X \longrightarrow F^a(X, \mathcal{W})$ is a continuous injection. Therefore there exists a continuous homomorphism $\psi : F(X, \mathcal{W}) \to F^a(X, \mathcal{W})$ such that $\psi(x) = x$ for any $x \in X$. Hence $\psi$ is an isomorphism. The proof is complete. □

## 12 Free left topological groups

A group $G$ with topology is called a *left* (respectively, *right*) topological group if the left translation $L_a(x) = ax$ (respectively, the right translation $R_a(x) = xa$) is continuous for any $a \in G$.

A class $\mathcal{V}$ of left topological groups is called *an LI-complete quasi-variety* of left topological groups if:

(LF1) the class $\mathcal{V}$ is multiplicative;

(LF2) if $G \in \mathcal{V}$ and $A$ is a subgroup of $G$, then $A \in \mathcal{V}$;

(LF3) every space $G \in \mathcal{V}$ is a $T_0$-space;

(LF4) if $G \in \mathcal{V}$, $\mathcal{T}$ is a compact $T_0$-topology on $G$ and $(G, \mathcal{T})$ is a left topological group, then $(G, \mathcal{T}) \in \mathcal{V}$;

(SF5) if $G \in \mathcal{V}$, $\mathcal{T}$ is a $T_0$-topology on $G$ and $(G, \mathcal{T})$ is a paratopological group with an invariant base, then $(G, \mathcal{T}) \in \mathcal{V}$.

From Theorem 2 it follows

**Corollary 8.** *Let $\mathcal{W}$ be a non-trivial LI-complete quasi-variety of left topological groups and $\mathbb{Z} \in \mathcal{W}$. Then for any pointed space $X$ the free left topological group $(F(X, \mathcal{W}), e_X)$ is abstract free and the mapping $e_X$ is an embedding.*

From Theorem 3 it follows

**Corollary 9.** *Let $\mathcal{W}$ be a non-trivial LI-complete quasi-variety of left topological groups, $n \in \mathbb{N}$ and $x^n = e$ for any $x \in G$ and $G \in \mathcal{W}$. Then for any pointed $T_1$-space $X$ the free left topological group $(F(X, \mathcal{W}), e_X)$ is abstract free and the mapping $e_X$ is an embedding.*

The following assertion completes Corollary 9.

**Lemma 8.** *Let $G$ be a left topological group and for any $x \in G$ there exists $n(x) \in \mathbb{N}$ such that $x^{n(x)} = e$. Then $G$ is a $T_1$-space.*

*Proof.* Any finite $T_0$-space contains a closed one-point subset. Thus any finite left topological group is a $T_1$-space. By conditions, any point $a \in G$ is contained in a finite subgroup $G(a) = \{a^i : 0 \leq i \leq n(a)\}$. Thus $\{e\}$ is a closed subset of the group $G$ and $G$ is a $T_1$-space. □

*Remark 3.* The similar assertions are true for classes of right topological groups.

## 13 Free semitopological groups

A class $\mathcal{V}$ of semitopological groups is called *a CI-complete quasi-variety* of semitopological groups if:

(SF1) the class $\mathcal{V}$ is multiplicative;

(SF2) if $G \in \mathcal{V}$ and $A$ is a subgroup of $G$, then $A \in \mathcal{V}$;

(SF3) every space $G \in \mathcal{V}$ is a $T_0$-space;

(SF4) if $G \in \mathcal{V}$, $\mathcal{T}$ is a compact $T_0$-topology on $G$ and $(G, \mathcal{T})$ is a quasitopological group, then $(G, \mathcal{T}) \in \mathcal{V}$;

(SF5) if $G \in \mathcal{V}$, $\mathcal{T}$ is a $T_0$-topology on $G$ and $(G, \mathcal{T})$ is a paratopological group with an invariant base, then $(G, \mathcal{T}) \in \mathcal{V}$.

From Theorem 2 it follows

**Corollary 10.** *Let $\mathcal{W}$ be a non-trivial $CI$-complete quasi-variety of semitopological groups and $\mathbb{Z} \in \mathcal{W}$. Then for any pointed space $X$ the free topological group $(F(X, \mathcal{W}), e_X)$ is abstract free and the mapping $e_X$ is an embedding.*

From Theorem 3 it follows

**Corollary 11.** *Let $\mathcal{W}$ be a non-trivial $CI$-complete quasi-variety of semitopological groups, $n \in \mathbb{N}$ and $x^n = e$ for any $x \in G$ and $G \in \mathcal{W}$. Then for any pointed $T_1$-space $X$ the free topological group $(F(X, \mathcal{W}), e_X)$ is abstract free and the mapping $e_X$ is an embedding.*

Lemma 6 completes Corollary 11.

## References


[1] Arhangel'skii A. V., Choban M. M. *Completeness type properties of semitopological groups, and the theorems of Montgomery and Ellis.* Topology Proceed., 2011, **37**, 33–60.

[2] Arhangel'skii A. V., Choban M. M., Kenderov P. S. *Topological games and continuity of group operations.* Topol. Appl., 2010, **157**, 2542–2552.

[3] Arhangelskii A. V., Tkachenko M. G. *Topological groups and related structures.* Atlantis Press. Amsterdam-Paris, 2008.

[4] Bel'nov V. K. *Dimension of topological homogeneous spaces and free homogeneous spaces.* Dokl. Akad. Nauk SSSR, 1978, **238:4**, 781–784.

[5] Choban M. *On the theory of topological algebraic systems.* Trans. Moscow Math. Soc., 1986, **48**, 115–159 (Russian original: Trudy Mosk. Matem. Ob-va, 1985, **48**, 106–149).

[6] Choban M. *The theory of stable metrics.* Math. Balkanica, 1988, **2**, 357–373.

[7] Choban M. *General conditions of the existence of free algebras.* Acta Comment. Univ. Tartuensis, 1989, 836, 157–171.

[8] Choban M. *Some topics in topological algebra.* Topol. Appl., 1993, **54**, 183–202.

[9] Engelking R. *General Topology.* PWN. Warszawa, 1977.

[10] Graev M. I. *Free topological groups.* Trans. Moscow Math. Soc., 1962, **8**, 303–364 (Russian original: Izvestia Akad. Nauk SSSR, 1948, **12**, 279–323).

[11] Kakutani Sh. *Free topological groups and infinite direct product topological groups.* Proceed. Imp. Acad. Tokyo, 1944, **20**, 595–598.

[12] Mal'cev A. I. *Free topological algebras.* Trans. Moscow Math. Soc., 1961, (2) 17, 173–200. (Russian original: Izvestia Akad. Nauk SSSR 21 (1957), 171-198).

[13] Markov A. A. *On free topological groups.* Trans. Moscow Math. Soc., 1962, **8**, 195–272 (Russian original: Izvestia Akad. Nauk SSSR 9 (1945), 3-64).

[14] Mishchenko A. S. *On the dimension of groups with left-invariant topologies.* Dokl. Akad. Nauk. SSSR, 1964, **159**, 753–754.

[15] Romaguera S., Sanchis M., Tkachenko M. *Free paratopological groups.* Topology Proceed., 2003, **27:2**, 613–640 (Topology Atlas, Preprint, 529, 2003, 21 p.).

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# On a four-dimensional hyperbolic manifold with finite volume

I. S. Gutsul

**Abstract.** In article [1] the authors construct and classify all the hyperbolic space-forms $H^n/\Gamma$ where $\Gamma$ is a torsion-free subgroup of minimal index in the congruence two subgroup $\Gamma_2^n$ for $n = 3, 4$. In the present paper some hyperbolic 3- and 4-manifolds are constructed that are absent in [1].

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In the works [1] and [2] some four-dimensional hyperbolic manifolds with finite volume were constructed. They were obtained by identifying the faces of the regular 24-cells in $H^4$ with all vertices being on the absolute. The present article is devoted to the construction of a four-dimensional hyperbolic manifold with finite volume by identifying the faces of a four-dimensional hyperbolic polyhedron with all the vertices being on the absolute. This polyhedron is not regular and its construction is non-trivial.

## 1. The construction of a four-dimensional polyhedron

In the four-dimensional space $H^4$ consider the regular 24-cells $R$. As it is known this polyhedron has 24 three-dimensional faces, 96 two-dimensional faces, 96 edges, and 24 vertices. The three-dimensional faces of the polyhedron are regular octahedra, two-dimensional faces are regular triangles. Inscribe in the polyhedron $R$ a three-dimensional sphere $S^3$. Denote its radius by $r$. If we begin to enlarge the radius $r$ of the sphere $S^3$, the polyhedron $R$ will increase, but its dihedral angles at the two-dimensional faces will decrease. Continuing the process, we ultimately come to the case when for some $r_0$ all the vertices of the polyhedron $R$ become infinitely removed, i. e. they get out on the absolute. In this case the three-dimensional faces are regular octahedra with all the vertices being on the absolute. Then the dihedral angles at the two-dimensional faces will be equal to $\pi/2$. Indeed, consider a three-dimensional horosphere centered at a vertex of the polyhedron $R$. Choose the radius of the horosphere such that the horosphere intersects only one-dimensional edges of the polyhedron which go to the center of the horosphere. Then the intersection of the horosphere and the polyhedron $R$ is a cube. But the dihedral angles at the two-dimensional faces of the polyhedron $R$ are equal to the dihedral angles at the edges of the obtained cube. Since the metric on the horosphere is Euclidean, the dihedral angles of the cube are equal to $\pi/2$, i. e. the dihedral angles at the two-dimensional faces of the polyhedron $R$ are equal to $\pi/2$. If we continue to enlarge the radis of the three-dimensional sphere, the vertices of the polyhedron $R$ will get out on the absolute. We obtain a four-dimensional polyhedron $R_2$ with all the vertices being infinitely removed. The polyhedron $R_2$ has three-dimensional faces of two kinds: 24 cubes with all the vertices being on the absolute and 24 truncated octahedra with all the vertices being infinitely removed. The polyhedron $R_2$ has 94 infinitely removed vertices, 288 one-dimensional edges, two-dimensional faces of two kinds: 144 squares with all the vertices being on the absolute and 96 triangles with all the vertices being infinitely removed. Dihedral angles at two-dimensional faces of this polyhedron are of two kinds: dihedral angles at the squares are equal to $\pi/2$, dihedral angles at the triangles are equal to $\pi/3$, both facts can be easy proved. Label infinitely removed vertices of the polyhedron $R_2$ by the numbers from 1 to 94. Write all three-dimensional faces of the obtained polyhedron. First write cubes with all the vertices being on the absolute:

$K_{25}(1, 7, 8, 6, 30, 16, 15, 22)$ $K_{26}(13, 21, 91, 39, 5, 1, 2, 12)$
$K_{27}(7, 2, 10, 3, 28, 17, 19, 44)$ $K_{28}(3, 9, 4, 8, 24, 26, 49, 35)$
$K_{29}(4, 6, 5, 11, 54, 33, 31, 40)$ $K_{30}(18, 17, 27, 68, 92, 14, 15, 23)$
$K_{31}(9, 10, 12, 11, 52, 48, 42, 41)$ $K_{32}(13, 20, 64, 38, 36, 16, 14, 93)$
$K_{33}(18, 19, 21, 20, 67, 71, 45, 46)$ $K_{34}(22, 23, 25, 24, 34, 29, 94, 73)$
$K_{35}(50, 26, 25, 96, 69, 51, 28, 27)$ $K_{36}(29, 30, 31, 32, 75, 95, 36, 37)$
$K_{37}(74, 56, 35, 34, 32, 76, 55, 33)$ $K_{38}(37, 38, 39, 40, 59, 77, 65, 58)$
$K_{39}(66, 46, 91, 58, 57, 63, 43, 41)$ $K_{40}(45, 70, 51, 44, 42, 43, 60, 47)$
$K_{41}(47, 48, 49, 50, 72, 61, 53, 56)$ $K_{42}(53, 62, 78, 55, 54, 52, 57, 59)$
$K_{43}(61, 60, 86, 89, 90, 62, 63, 87)$ $K_{44}(66, 67, 64, 65, 85, 87, 81, 80)$
$K_{45}(70, 71, 81, 86, 83, 69, 68, 79)$ $K_{46}(83, 82, 88, 89, 72, 96, 73, 74)$
$K_{47}(78, 90, 85, 77, 75, 76, 88, 84)$ $K_{48}(92, 94, 82, 79, 80, 93, 95, 84)$

The polyhedron $R_2$ has also 24 truncated octahedra with all the vertices being on the absolute. Write these faces:

$O_1(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)$
$O_2(15, 16, 30, 22, 23, 14, 36, 29, 92, 93, 95, 94)$
$O_3(1, 7, 15, 16, 13, 2, 17, 14, 20, 21, 19, 18)$
$O_4(7, 8, 22, 15, 17, 3, 24, 23, 25, 26, 28, 27)$
$O_5(8, 6, 30, 22, 24, 4, 31, 29, 34, 35, 33, 32)$
$O_6(6, 1, 16, 30, 36, 13, 5, 31, 37, 38, 39, 40)$
$O_7(12, 10, 42, 41, 91, 2, 44, 43, 45, 46, 21, 19)$
$O_8(10, 9, 48, 42, 44, 3, 49, 47, 50, 51, 28, 26)$
$O_9(9, 11, 52, 48, 49, 4, 54, 53, 55, 56, 35, 33)$
$O_{10}(11, 12, 41, 52, 54, 57, 91, 5, 39, 40, 59, 58)$
$O_{11}(62, 61, 60, 63, 57, 53, 47, 43, 52, 48, 42, 41)$
$O_{12}(13, 21, 91, 39, 38, 20, 46, 58, 65, 66, 67, 64)$

$$O_{13}(17, 19, 44, 28, 27, 18, 45, 51, 69, 68, 71, 70)$$
$$O_{14}(72, 74, 73, 96, 50, 56, 34, 25, 26, 24, 35, 49)$$
$$O_{15}(78, 76, 75, 77, 59, 55, 32, 37, 40, 54, 33, 31)$$
$$O_{16}(18, 20, 67, 71, 68, 14, 64, 81, 80, 79, 92, 93)$$
$$O_{17}(25, 27, 69, 96, 73, 83, 68, 23, 92, 94, 82, 79)$$
$$O_{18}(38, 37, 77, 65, 64, 36, 75, 85, 84, 80, 93, 95)$$
$$O_{19}(45, 46, 67, 71, 70, 43, 66, 81, 86, 60, 63, 87)$$
$$O_{20}(32, 34, 74, 76, 75, 29, 73, 88, 82, 84, 95, 94)$$
$$O_{21}(50, 51, 69, 96, 83, 70, 47, 72, 89, 86, 60, 61)$$
$$O_{22}(55, 56, 74, 76, 78, 53, 72, 88, 90, 62, 61, 89)$$
$$O_{23}(58, 59, 77, 65, 66, 57, 78, 85, 90, 62, 63, 87)$$
$$O_{24}(89, 88, 85, 81, 83, 86, 90, 87, 80, 84, 79, 82)$$

## 2. The construction of a four-dimensional hyperbolic manifold

Indicate motions ( isometries ) that identify faces of the polyhedron:

$$\varphi_1 : \begin{array}{llllllllllll} (1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & 12) \\ (89, & 88, & 85, & 81, & 83, & 86, & 90, & 87, & 80, & 84, & 79, & 82); \end{array}$$

$$\varphi_2 : \begin{array}{llllllllllll} (15, & 16, & 30, & 22, & 23, & 14, & 36, & 29, & 92, & 93, & 95, & 94) \\ (62, & 61, & 60, & 63, & 57, & 53, & 47, & 43, & 52, & 48, & 42, & 41); \end{array}$$

$$\varphi_3 : \begin{array}{llllllllllll} (1, & 7, & 15, & 16, & 13, & 2, & 17, & 14, & 20, & 21, & 19, & 18) \\ (55, & 56, & 74, & 76, & 78, & 53, & 72, & 88, & 90, & 62, & 61, & 89); \end{array}$$

$$\varphi_4 : \begin{array}{llllllllllll} (7, & 8, & 22, & 15, & 17, & 3, & 24, & 23, & 25, & 26, & 28, & 27) \\ (58, & 59, & 77, & 65, & 66, & 57, & 78, & 85, & 90, & 62, & 63, & 87); \end{array}$$

$$\varphi_5 : \begin{array}{llllllllllll} (8, & 6, & 30, & 22, & 24, & 4, & 31, & 29, & 34, & 35, & 33, & 32) \\ (45, & 46, & 67, & 71, & 70, & 43, & 66, & 81, & 86, & 60, & 63, & 87); \end{array}$$

$$\varphi_6 : \begin{array}{llllllllllll} (6, & 1, & 16, & 30, & 36, & 13, & 5, & 31, & 37, & 38, & 39, & 40) \\ (50, & 51, & 69, & 96, & 83, & 70, & 47, & 72, & 89, & 86, & 60, & 61); \end{array}$$

$$\varphi_7 : \begin{array}{llllllllllll} (12, & 10, & 42, & 41, & 91, & 2, & 44, & 43, & 45, & 46, & 21, & 19) \\ (32, & 34, & 74, & 76, & 75, & 29, & 73, & 88, & 82, & 84, & 95, & 94); \end{array}$$

$$\varphi_8 : \begin{array}{llllllllllll} (10, & 9, & 48, & 42, & 44, & 3, & 49, & 47, & 50, & 51, & 28, & 26) \\ (38, & 37, & 77, & 65, & 64, & 36, & 75, & 85, & 84, & 80, & 93, & 95); \end{array}$$

$$\varphi_9 : \begin{array}{llllllllllll} (9, & 11, & 52, & 48, & 49, & 4, & 54, & 53, & 55, & 56, & 35, & 33) \\ (18, & 20, & 67, & 71, & 68, & 14, & 64, & 81, & 80, & 79, & 92, & 93); \end{array}$$

$$\varphi_{10} : \begin{array}{llllllllllll} (11, & 12, & 41, & 52, & 54, & 57, & 91, & 5, & 39, & 40, & 59, & 58) \\ (25, & 27, & 69, & 96, & 73, & 83, & 68, & 23, & 92, & 94, & 82, & 79); \end{array}$$

$$\varphi_{11}:\quad \begin{matrix} (13, & 21, & 91, & 39, & 38, & 20, & 46, & 58, & 65, & 66, & 67, & 64) \\ (72, & 74, & 73, & 96, & 50, & 56, & 34, & 25, & 26, & 24, & 35, & 49); \end{matrix}$$

$$\varphi_{12}:\quad \begin{matrix} (17, & 19, & 44, & 28, & 27, & 18, & 45, & 51, & 69, & 68, & 71, & 70) \\ (78, & 76, & 75, & 77, & 59, & 55, & 32, & 37, & 40, & 54, & 33, & 31); \end{matrix}$$

$$\varphi_{13}:\quad \begin{matrix} (1, & 7, & 8, & ,6, & 30, & 16, & 15, & 22) \\ (66, & 46, & 91, & 58, & 57, & 63, & 43, & 41); \end{matrix}$$

$$\varphi_{14}:\quad \begin{matrix} (13, & 21, & 91, & 39, & 5, & 1, & 2, & 12) \\ (66, & 67, & 64, & 65, & 85, & 87, & 81, & 80); \end{matrix}$$

$$\varphi_{15}:\quad \begin{matrix} (7, & 2, & 10, & 3, & 28, & 17, & 19, & 44) \\ (50, & 26, & 25, & 96, & 69, & 51, & 28, & 27); \end{matrix}$$

$$\varphi_{16}:\quad \begin{matrix} (3, & 9, & 4, & 8, & 24, & 26, & 49, & 35) \\ (83, & 82, & 88, & 89, & 72, & 96, & 73, & 74); \end{matrix}$$

$$\varphi_{17}:\quad \begin{matrix} (4, & 6, & 5, & 11, & 54, & 33, & 31, & 40) \\ (74, & 56, & 35, & 34, & 32, & 76, & 55, & 33); \end{matrix}$$

$$\varphi_{18}:\quad \begin{matrix} (18, & 17, & 27, & 68, & 92, & 14, & 15, & 23) \\ (45, & 70, & 51, & 44, & 42, & 43, & 60, & 47); \end{matrix}$$

$$\varphi_{19}:\quad \begin{matrix} (9, & 10, & 12, & 11, & 52, & 48, & 42, & 41) \\ (13, & 20, & 64, & 38, & 36, & 16, & 14, & 93); \end{matrix}$$

$$\varphi_{20}:\quad \begin{matrix} (18, & 19, & 21, & 20, & 67, & 71, & 45, & 46) \\ (70, & 71, & 81, & 86, & 83, & 69, & 68, & 79); \end{matrix}$$

$$\varphi_{21}:\quad \begin{matrix} (22, & 23, & 25, & 24, & 34, & 29, & 94, & 73) \\ (61, & 60, & 86, & 89, & 90, & 62, & 63, & 87); \end{matrix}$$

$$\varphi_{22}:\quad \begin{matrix} (29, & 30, & 31, & 32, & 75, & 95, & 36, & 37) \\ (53, & 62, & 78, & 55, & 54, & 52, & 57, & 59); \end{matrix}$$

$$\varphi_{23}:\quad \begin{matrix} (37, & 38, & 39, & 40, & 59, & 77, & 65, & 58) \\ (78, & 90, & 85, & 77, & 75, & 76, & 88, & 84); \end{matrix}$$

$$\varphi_{24}:\quad \begin{matrix} (47, & 48, & 49, & 50, & 72, & 61, & 53, & 56) \\ (92, & 94, & 82, & 79, & 80, & 93, & 95, & 84). \end{matrix}$$

Consider cycles of two-dimensional faces of the polyhedron $R_2$. As the dihedral angles at the quadrangular faces are equal to $\pi/2$, in order that the cycles of these faces be inessential each of them must contain four faces. Write cycles of these faces. We will present cycles of faces as follows: write a face, then write the motion that transfers this face into another face, then again a face, again a motion, and so on.

$(O_1 \cap K_{25})(1, 7, 8, 6)$ $\varphi_{13}$ $(O_{12} \cap K_{39})(66, 46, 91, 58)$ $\varphi_{11}$ $(O_{14} \cap K_{34})$
$(24, 34, 73, 25)$ $\varphi_{21}$ $(O_{24} \cap K_{43})(89, 90, 87, 86)$ $\varphi_1^{-1}$ $(O_1 \cap K_{25})(1, 7, 8.6)$;

$(O_3 \cap K_{25})(1,7,15,16)\ \varphi_{13}\ (O_{19} \cap K_{39})(66,46,43,63)\ \varphi_5^{-1}\ (O_5 \cap K_{29})$
$(31,6,4,33)\ \varphi_{17}\ (O_{22} \cap K_{37})(55,56,74,76)\ \varphi_3^{-1}\ (O_3 \cap K_{25})(1,7,15,16);$

$(O_6 \cap K_{25})(1,6,30,16)\ \varphi_{13}\ (O_{23} \cap K_{39})(66,58,57,63)\ \varphi_4^{-1}\ (O_4 \cap O_{27})$
$(17,7,3,28)\ \varphi_{15}\ (O_{21} \cap K_{35})(51,50,96,69)\ \varphi_6^{-1}\ (O_6 \cap K_{25})(1,6,30,16);$

$(O_5 \cap K_{25})(8,6,30,22)\ \varphi_{13}\ (O_10 \cap K_{39})(91,58,57,41)\ \varphi_{10}\ (O_{17} \cap K_{45})$
$(68,79,83,69)\ \varphi_{20}^{-1}\ (O_{19} \cap K_{33})(45,46,67,71)\ \varphi_5^{-1}\ (O_5 \cap K_{25})(8,6,30,22);$

$(O_4 \cap K_{25})(8,7,15,22)\ \varphi_{13}\ (O_7 \cap K_{39})(91,46,43,41)\ \varphi_7\ (O_{20} \cap K_{47})$
$(75,84,88,76)\ \varphi_{23}^{-1}\ (O_{23} \cap K_{38})(59,58,65,77)\ \varphi_4^{-1}\ (O_4 \cap K_{25})(8,7,15,22);$

$(O_2 \cap K_{25})(15,16,30,22)\ \varphi_{13}\ (O_{11} \cap K_{39})(43,63,57,41)\ \varphi_2^{-1}\ (O_2 \cap K_{34})$
$(29,22,23,94)\ \varphi_{21}\ (O_{11} \cap K_{43})(62,61,60,63)\ \varphi_2^{-1}\ (O_2 \cap K_{25})(15,16,30,22);$

$(O_1 \cap K_{26})(1,2,12,5)\ \varphi_{14}\ (O_{24} \cap K_{44})(87,81,80,85)\ \varphi_1^{-1}\ O_1 \cap K_{28})$
$(8,4,9,3)\ \varphi_{16}\ (O_{24} \cap K_{46})(89,88,82,83)\ \varphi_1^{-1}\ (O_1 \cap K_{26})(1,2,12,5);$

$(O_3 \cap K_{26})(1,2,21,13)\ \varphi_{14}\ (O_{19} \cap K_{44})(87,81,67,66)\ \varphi_5^{-1}\ (O_5 \cap K_{36})$
$(32,29,30,31)\ \varphi_{22}\ (O_{22} \cap K_{42})(55,53,62,78)\ \varphi_3^{-1}\ (O_3 \cap K_{26})(1,2,21,13);$

$(O_6 \cap K_{26})(1,5,39,13)\ \varphi_{14}\ (O_{23} \cap K_{44})(87,85,65,66)\ \varphi_4^{-1}\ (O_4 \cap K_{30})$
$(27,23,15,17)\ \varphi_{18}\ (O_{21} \cap K_{40})(51,47,60,70)\ \varphi_6^{-1}\ (O_6 \cap K_{26})(1,5,39,13);$

$(O_7 \cap K_{26})(91,21,2,12)\ \varphi_{14}\ (O_{16} \cap K_{44})(64,67,81,80)\ \varphi_9^{-1}\ (O_9 \cap K_{42})$
$(54,52,53,55)\ \varphi_{22}^{-1}\ (O_{20} \cap K_{36})(75,95,29,32)\ \varphi_7^{-1}\ (O_7 \cap K_{26})(91,21,2,12);$

$(O_{12} \cap K_{26})(91,21,13,39)\ \varphi_{14}\ (O_{12} \cap K_{44})(64,67,66,65)\ \varphi_{11}\ (O_{14} \cap K_{28})$
$(49,35,24,26)\ \varphi_{16}\ (O_{14} \cap K_{46})(73,74,72,96)\ \varphi_{11}^{-1}\ (O_{12} \cap K_{26})(91,21,13,39);$

$(O_{10} \cap K_{26})(91,12,5,39)\ \varphi_{14}\ (O_{18} \cap K_{44})(64,80,85,65)\ \varphi_8^{-1}\ (O_8 \cap K_{40})$
$(44,51,47,42)\ \varphi_{18}^{-1}\ (O_{17} \cap K_{30})(68,27,23,92)\ \varphi_{10}^{-1}\ (O_{10} \cap K_{26})(91,12,5,3);$

$(O_1 \cap K_{27})(2,7,3,10)\ \varphi_{15}\ (O_{14} \cap K_{35})(26,50,96,25)\ \varphi_{11}^{-1}\ (O_{12} \cap K_{38})$
$(65,38,39,58)\ \varphi_{23}\ (O_{24} \cap K_{47})(88,90,85,84)\ \varphi_1^{-1}\ (O_1 \cap K_{27})(2,7,3,10);$

$(O_3 \cap K_{27})(2,7,17,19)\ \varphi_{15}\ (O_8 \cap K_{35})(26,50,51,28)\ \varphi_8\ (O_{18} \cap K_{48})$
$(95,84,80,93)\ \varphi_{24}^{-1}\ (O_{22} \cap K_{41})(53,56,72,61)\ \varphi_3^{-1}\ (O_3 \cap K_{27})(2,7,17,19);$

$(O_7 \cap K_{27})(2,10,44,19)\ \varphi_{15}\ (O_4 \cap K_{35})(26,25,27,28)\ \varphi_4\ (O_{23} \cap K_{43})$
$(62,90,87,63)\ \varphi_{21}^{-1}\ (O_{20} \cap K_{34})(29,34,73,94)\ \varphi_7^{-1}\ (O_7 \cap K_{27})(2,10,44,19);$

$(O_8 \cap K_{27})(28,3,10,44)\ \varphi_{15}\ (O_{17} \cap K_{35})(69,96,25,27)\ \varphi_{10}^{-1}\ (O_{10} \cap K_{31})$
$(41,52,11,12)\ \varphi_{19}\ (O_{18} \cap K_{32})(93,36,38,64)\ \varphi_8^{-1}\ (O_8 \cap K_{27})(28,3,10,44);$

$(O_{13} \cap K_{27})(28,17,19,44)\ \varphi_{15}\ (O_{13} \cap K_{35})(69,51,28,27)\ \varphi_{12}\ (O_{15} \cap K_{38})$
$(40,37,77,59)\ \varphi_{23}\ (O_{15} \cap K_{47})(77,78,76,75)\ \varphi_{12}^{-1}\ (O_{13} \cap K_{27})(28,17,19,44);$

$(O_8 \cap K_{28})(3,9,49,26)\ \varphi_{16}\ (O_{17} \cap K_{46})(83,82,73,96)\ \varphi_{10}^{-1}\ (O_{10} \cap K_{42})$
$(57,59,54,52)\ \varphi_{22}^{-1}\ (O_{18} \cap K_{36})(36,37,75,95)\ \varphi_8^{-1}\ (O_8 \cap K_{28})(3,9,49,26);$

$(O_5 \cap K_{28})(4,8,24,35)\ \varphi_{16}\ (O_{22} \cap K_{46})(88,89,72,74)\ \varphi_3^{-1}\ (O_3 \cap K_{30})$
$(14,18,17,15)\ \varphi_{18}\ (O_{19} \cap K_{40})(43,45,70,60)\ \varphi_5^{-1}\,;(O_5 \cap K_{28})(4,8,24,35);$

$(O_9 \cap K_{28})(4,9,49,35)\ \varphi_{16}\ (O_{20} \cap K_{46})(88,82,73,74)\ \varphi_7^{-1}\ (O_7 \cap K_{40})$
$(43,45,44,42)\ \varphi_{18}^{-1}\ (O_{16} \cap K_{30})(14,18,68,92)\ \varphi_9^{-1}\ (O_9 \cap K_{28})(4,9,49,35);$

$(O_4 \cap K_{28})(3,8,24,26)\ \varphi_{16}\ (O_{21} \cap K_{46})(83,89,72,96)\ \varphi_6^{-1}\ (O_6 \cap K_{36})$
$(36,37,51,30)\ \varphi_{22}\ (O_{23} \cap K_{42})(57,59,78,62)\ \varphi_4^{-1}\ (O_4 \cap K_{28})(3,8,24,26);$

$(O_1 \cap K_{29})(4,6,5,11)\ \varphi_{17}\ (O_{14} \cap K_{37})(74,56,35,34)\ \varphi_{11}^{-1}\ (O_{12} \cap K_{33})$
$(21,20,67,46)\ \varphi_{20}\ (O_{24} \cap K_{45})(81,86,83,79)\ \varphi_1^{-1}\ (O_1 \cap K_{29})(4,6,5,11);$

$(O_9 \cap K_{29})(4,11,54,33)\ \varphi_{17}\ (O_{20} \cap K_{37})(74,34,32,76)\ \varphi_7^{-1}\ (O_7 \cap K_{31})$
$(42,10,12,41)\ \varphi_{19}\ (O_{16} \cap K_{32})(14,20,64,93)\ \varphi_9^{-1}\ (O_9 \cap K_{29})(4,11,54,33);$

$(O_6 \cap K_{29})(5,6,31,40)\ \varphi_{17}\ (O_9 \cap K_{37})(35,56,55,33)\ \varphi_9\ (O_{16} \cap K_{48})$
$(92,79,80,93)\ \varphi_{24}^{-1}\ (O_{21} \cap K_{41})(47,50,72,61)\ \varphi_6^{-1}\ (O_6 \cap K_{29})(5,6,31,40);$

$(O_{10} \cap K_{29})(5,11,54,40)\ \varphi_{17}\ (O_5 \cap K_{37})(35,34,32,33)\ \varphi_5\ (O_{19} \cap K_{43})$
$(60,86,87,63)\ \varphi_{21}^{-1}\ (O_{17} \cap K34)(23,25,73,94)\ \varphi_{10}^{-1}\ (O_{10} \cap K_{29})(5,11,54,40);$

$(O_{15} \cap K_{29})(31,40,54,33)\ \varphi_{17}\ (O_{15} \cap K_{37})(55,33,32,76)\ \varphi_{12}^{-1}\ (O_{13} \cap K_{33})$
$(18,71,45,19)\ \varphi_{20}\ (O_{13} \cap K_{45})(70,69,68,71)\ \varphi_{12}\ (O_{15} \cap K_{29})(31,40,54,33);$

$(O_2 \cap K_{30})(14,15,23,92)\ \varphi_{18}\ (O_{11} \cap K_{40})(43,60,47,42)\ \varphi_2^{-1}\ (O_2 \cap K_{36})$
$(29,30,36,95)\ \varphi_{22}\ (O_{11} \cap K_{42})(53,62,57,52)\ \varphi_2^{-1}\ (O_2 \cap K_{30})(14,15,23,92);$

$(O_{13} \cap K_{30})(17,18,68,27)\ \varphi_{18}\ (O_{13} \cap K_{40})(70,45,44,51)\ \varphi_{12}\ (O_{15} \cap K_{36})$
$(31,32,75,37)\ \varphi_{22}\ (O_{15} \cap K_{42})(78,55,54,59)\ \varphi_{12}^{-1}\ (O_{13} \cap K_{30})(17,18,68,27);$

$(O_1 \cap K_{31})(9,10,12,11)\ \varphi_{19}\ (O_{12} \cap K_{32})(13,20,64,38)\ \varphi_{11}\ (O_{14} \cap K_{41})$
$(72,56,49,50)\ \varphi_{24}\ (O_{24} \cap K_{48})(80,84,82,79)\ \varphi_1^{-1}\ (O_1 \cap K_{31})(9,10,12,11);$

$(O_8 \cap K_{31})(9,10,42,48)\ \varphi_{19}\ (O_3 \cap K_{32})(13,20,14,16)\ \varphi_3\ (O_{22} \cap K_{47})$
$(78,90,88,76)\ \varphi_{23}^{-1}\ (O_{18} \cap K_{38})(37,38,65,77)\ \varphi_8^{-1}\ (O_8 \cap K_{31})(9,10,42,48);$

$(O_9 \cap K_{31})(9,11,52,48)\ \varphi_{19}\ (O_6 \cap K_{32})(13,38,36,16)\ \varphi_6\ (O_{21} \cap K_{45})$
$(70,86,83,69)\ \varphi_{20}^{-1}\ (O_{16} \cap K_{33})(18,20,67,71)\ \varphi_9^{-1}\ (O_9 \cap K_{31})(9,11,52,48);$

$(O_{11} \cap K_{31})(41,42,48,52)\ \varphi_{19}\ (O_2 \cap K_{32})(93,14,16,36)\ \varphi_2\ (O_{11} \cap K_{41})$
$(48,53,61,47)\ \varphi_{24}\ (O_2 \cap K_{48})(94,95,93,92)\ \varphi_2\ (O_{11} \cap K_{31})(41,42,48,52);$

$(O_3 \cap K_{33})(18,19,21,20)\ \varphi_{20}\ (O_{19} \cap K_{45})(70,71,81,86)\ \varphi_5^{-1}\ (O_5 \cap K_{34})$
$(24,22,29,34)\ \varphi_{21}\ (O_{22} \cap K_{43})(89,61,62,90)\ \varphi_3^{-1}\ (O_3 \cap K_{33})(18,19,21,20);$

$(O_7 \cap K_{33})(19,21,46,45)\ \varphi_{20}\ (O_{16} \cap K_{45})(71,81,79,68)\ \varphi_9^{-1}\ (O_9 \cap K_{41})$
$(48,53,56,49)\ \varphi_{24}\ (O_{20} \cap K_{48})(94,95,84,82)\ \varphi_7^{-1}\ (O_7 \cap K_{33})(19,21,46,45);$

$(O_4 \cap K_{34})(22,23,25,24)\ \varphi_{21}\ (O_{21} \cap K_{43})(61,60,86,89)\ \varphi_6^{-1}\ (O_6 \cap K_{38})$
$(40,39,38,37)\ \varphi_{23}\ (O_{23} \cap K_{47})(77,85,90,78)\ \varphi_4^{-1}\ (O_4 \cap K_{34})(22,23,25,24);$

$(O_{10} \cap K_{38})(40,39,58,59)\ \varphi_{23}\ (O_{18} \cap K_{47})(77,85,84,75)\ \varphi_8^{-1}\ (O_8 \cap K_{41})$
$(48,47,50,49)\ \varphi_{24}\ (O_{17} \cap K_{48})(94,92,79,82)\ \varphi_{10}\ (O_{10} \cap K_{38})(40,39,58,59).$

The dihedral angles at the triangular faces of the polyhedron are equal to $\pi/3$. Therefore a cycle of these faces will be inessential if it contains six faces. Write cycles of these faces:

$(O_1 \cap O_3)(1,2,7)\ \varphi_1\ (O_{22} \cap O_{24})(89,88,90)\ \varphi_3^{-1}\ (O_3 \cap O_{16})$
$(18,14,20)\ \varphi_9^{-1}\ (O_1 \cap O_9)(9,4,11)\ \varphi_1\ (O_{16} \cap O_{24})(80,81,79)$
$\varphi_9^{-1}\ (O_9 \cap O_{22})(55,53,56)\ \varphi_3^{-1}\ (O_1 \cap O_3)(1,2,7);$

$(O_1 \cap O_6)(1,5,6)\ \varphi_1\ (O_{21} \cap O_{24})(89,83,86)\ \varphi_6^{-1}\ (O_6 \cap O_{18})$
$(37,36,38)\ \varphi_8^{-1}\ (O_1 \cap O_8)(9,3,10)\ \varphi_1\ (O_{18} \cap O_{24})(80,85,84)$
$\varphi_8^{-1}\ (O_8 \cap O_{21})(51,47,50)\ \varphi_6^{-1}\ (O_1 \cap O_3)(1,5,6);$

$(O_1 \cap O_5)(8,6,4)\ \varphi_1\ (O_{19} \cap O_{24})(87,86,81)\ \varphi_5^{-1}\ (O_5 \cap O_{20})$
$(32,34,29)\ \varphi_7^{-1}\ (O_1 \cap O_7)(12,10,2)\ \varphi_1\ (O_{20} \cap O_{24})(82,84,88)$
$\varphi_7^{-1}\ (O_7 \cap O_{19})(45,46,43)\ \varphi_5^{-1}\ (O_1 \cap O_5)(8,6,4);$

$(O_1 \cap O_4)(8,7,3)\ \varphi_1\ (O_{23} \cap O_{24})(87,90,85)\ \varphi_4^{-1}\ (O_4 \cap O_{17})$
$(27,25,23)\ \varphi_{10}^{-1}\ (O_1 \cap O_{10})(12,11,5)\ \varphi_1\ (O_{17} \cap O_{24})(82,79,83)$
$\varphi_{10}^{-1}\ (O_{10} \cap O_{23})(59,58,57)\ \varphi_4^{-1}\ (O_1 \cap O_4)(8,7,3);$

$(O_2 \cap O_3)(15,16,14)\ \varphi_2\ (O_{11} \cap O_{23})(62,61,53)\ \varphi_3^{-1}\ (O_3 \cap O_7)$
$(21,19,2)\ \varphi_7\ (O_2 \cap O_{20}(95,94,29)\ \varphi_2\ (O_7 \cap O_{11})(41,42,43)$
$\varphi_7\ (O_{20} \cap O_{22}(76,74,88)\ \varphi_3^{-1}\ (O_2 \cap O_3)(15,16,14);$

$(O_2 \cap O_4)(15,22,23)\ \varphi_2\ (O_{11} \cap O_{23})(62,63,57)\ \varphi_4^{-1}\ (O_4 \cap O_8)$
$(26,28,3)\ \varphi_8\ (O_2 \cap O_{18})(95,93,36)\ \varphi_2\ (O_8 \cap O_{11})(42,48,47)$
$\varphi_8\ (O_{18} \cap O_{23})(65,77,85)\ \varphi_4^{-1}\ (O_2 \cap O_4)(15,22,23);$

$(O_2 \cap O_4)(22,29,30)\ \varphi_2\ (O_{11} \cap O_{19})(63,43,60)\ \varphi_5^{-1}\ (O_5 \cap O_9)$
$(33,4,35)\ \varphi_9\ (O_2 \cap O_{16})(93,14,92)\ \varphi_2\ (O_9 \cap O_{11})(48,53,52)$
$\varphi_9\ (O_{16} \cap O_{19})(71,81,67)\ \varphi_5^{-1}\ (O_2 \cap O_4)(22,29,30);$

$(O_2 \cap O_6)(16,36,30)\ \varphi_2\ (O_{11} \cap O_{21})(61,47,60)\ \varphi_6^{-1}\ (O_6 \cap O_{10})$
$(40,5,39)\ \varphi_{10}\ (O_2 \cap O_{17})(94,23,92)\ \varphi_2\ (O_{10} \cap O_{11}(41,57,52)$
$\varphi_{10}\ (O_{17} \cap O_{21})(69,83,96)\ \varphi_6\ (O_2 \cap O_6)(16,36,30);$

$(O_3 \cap O_6)(1,16,13)\ \varphi_3\ (O_{15} \cap O_{22})(55,76,78)\ \varphi_{12}^{-1}\ (O_3 \cap O_{13})$
$(18,19,17)\ \varphi_3\ (O_{21} \cap O_{22})(89,61,72)\ \varphi_6^{-1}\ (O_6 \cap O_{15})(37,40,31)$
$\varphi_{12}^{-1}\ (O_{13} \cap O_{21})(51,69,70)\ \varphi_6^{-1}\ (O_3 \cap O_6)(1,16,13);$

$(O_3 \cap O_4)(15,17,7)\ \varphi_3\ (O_{14} \cap O_{22})(74,72,56)\ \varphi_{11}^{-1}\ (O_3 \cap O_{12})$
$(21,13,20)\ \varphi_3\ (O_{22} \cap O_{23})(62,78,90)\ \varphi_4^{-1}\ (O_4 \cap O_{14})(26,24,25)$
$\varphi_{11}^{-1}\ (O_{12} \cap O_{23})(65,66,58)\ \varphi_4^{-1}\ (O_3 \cap O_4)(15,17,7);$

$(O_4 \cap O_5)(8,22,24)\ \varphi_4\ (O_{15} \cap O_{23})(59,77,78)\ \varphi_{12}^{-1}\ (O_4 \cap O_{13})$
$(27,28,17)\ \varphi_4\ (O_{19} \cap O_{23})(87,63,66)\ \varphi_5^{-1}\ (O_5 \cap O_{15})(32,33,31)$
$\varphi_{12}^{-1}\ (O_{13} \cap O_{19})(45,71,70)\ \varphi_5^{-1}\ (O_4 \cap O_5)(8,22,24);$

$(O_5 \cap O_{14})(34,35,24)\ \varphi_5\ (O_{19} \cap O_{21})(86,60,70)\ \varphi_6^{-1}\ (O_6 \cap O_{12})$
$(38,39,13)\ \varphi_{11}\ (O_{14} \cap O_{21})(50,96,72)\ \varphi_6^{-1}\ (O_5 \cap O_6)(6,30,31)$
$\varphi_5\ (O_{12} \cap O_{19})(46,67,66)\ \varphi_{11}\ (O_5 \cap O_{14})(34,35,24);$

$(O_7 \cap O_8)(10,42,44)\ \varphi_7\ (O_{14} \cap O_{20})(34,74,73)\ \varphi_{11}^{-1}\ (O_7 \cap O_{12})$
$(46,21,91)\ \varphi_7\ (O_{18} \cap O_{20})(84,95,75)\ \varphi_8^{-1}\ (O_8 \cap O_{14})(50,26,49)$
$\varphi_{11}^{-1}\ (O_{12} \cap O_{18})(38,65,64)\ \varphi_8^{-1}\ (O_7 \cap O_8)(10,42,44);$

$(O_7 \cap O_{10})(12, 41, 91)\ \varphi_7\ (O_{15} \cap O_{20})(32, 76, 75)\ \varphi_{12}^{-1}\ (O_7 \cap O_{13})$
$(45, 19, 44)\ \varphi_7\ (O_{17} \cap O_{20})(82, 94, 73)\ \varphi_{10}^{-1}\ (O_{10} \cap O_{15})(59, 40, 54)$
$\varphi_{12}^{-1}\ (O_{13} \cap O_{17})(27, 69, 68)\ \varphi_{10}^{-1}\ (O_7 \cap O_{10})(12, 41, 91);$

$(O_8 \cap O_9)(9, 48, 49)\ \varphi_8\ (O_{15} \cap O_{18})(37, 77, 75)\ \varphi_{12}^{-1}\ (O_8 \cap O_{13})$
$(51, 28, 44)\ \varphi_8\ (O_{16} \cap O_{18})(80, 93, 64)\ \varphi_9^{-1}\ (O_9 \cap O_{15})(55, 33, 54)$
$\varphi_{12}^{-1}\ (O_{13} \cap O_{16})(18, 71, 68)\ \varphi_9^{-1}\ (O_8 \cap O_9)(9, 48, 49);$

$(O_9 \cap O_{10})(54, 52, 11)\ \varphi_9\ (O_{12} \cap O_{16})(64, 67, 20)\ \varphi_{11}\ (O_9 \cap O_{14})$
$(49, 35, 56)\ \varphi_9\ (O_{16} \cap O_{17})(68, 92, 79)\ \varphi_{10}^{-1}\ (O_{10} \cap O_{12})(91, 39, 58)$
$\varphi_{11}\ (O_{14} \cap O_{17})(73, 96, 25)\ \varphi_{10}^{-1}\ (O_9 \cap O_{10})(54, 52, 11).$

Finally write cycles of one-dimensional edges of the polyhedron $R_2$:

$(1, 2)\ \varphi_{14}\ (87, 81)\ \varphi_1^{-1}\ (8, 4)\ \varphi_{16}\ (89, 88)\ \varphi_3^{-1}\ (18, 14)$
$\varphi_{18}\ (45, 43)\ \varphi_7\ (82, 88)\ \varphi_{16}^{-1}\ (9, 4)\ \varphi_1\ (80, 81)$
$\varphi_{14}^{-1}\ (12, 2)\ \varphi_7\ (32, 29)\ \varphi_{22}\ (55, 53)\ \varphi_3^{-1}\ (1, 2);$

$(1, 5)\ \varphi_{14}\ (87, 85)\ \varphi_1^{-1}\ (8, 3)\ \varphi_{16}\ (89, 83)\ \varphi_6^{-1}\ (37, 36)$
$\varphi_{22}\ (59, 57)\ \varphi_{10}\ (82, 83)\ \varphi_{16}^{-1}\ (9, 3)\ \varphi_1\ (80, 85)$
$\varphi_{14}^{-1}\ (12, 5)\ \varphi_{10}\ (27, 23)\ \varphi_{18}\ (51, 47)\ \varphi_6^{-1}\ (1, 5);$

$(1, 6)\ \varphi_{13}\ (66, 58)\ \varphi_{11}\ (24, 25)\ \varphi_{21}\ (89, 86)\ \varphi_6^{-1}\ (37, 38)$
$\varphi_{23}\ (78, 90)\ \varphi_3^{-1}\ (13, 20)\ \varphi_{19}^{-1}\ (9, 10)\ \varphi_1\ (80, 84)$
$\varphi_{24}^{-1}\ (72, 56)\ \varphi_3^{-1}\ (17, 7)\ \varphi_{15}\ (51, 50)\ \varphi_6^{-1}\ (1, 6);$

$(1, 7)\ \varphi_{13}\ (66, 46)\ \varphi_{11}\ (24, 34)\ \varphi_{21}\ (89, 90)\ \varphi_3^{-1}\ (18, 20)$
$\varphi_{20}\ (70, 86)\ \varphi_6^{-1}\ (13, 38)\ \varphi_{19}^{-1}\ (9, 11)\ \varphi_1\ (80, 79)$
$\varphi_{24}^{-1}\ (72, 50)\ \varphi_6^{-1}\ (31, 6)\ \varphi_{17}\ (55, 56)\ \varphi_3^{-1}\ (1, 7);$

$(1, 16)\ \varphi_{13}\ (66, 63)\ \varphi_5^{-1}\ (31, 33)\ \varphi_{17}\ (55, 76)\ \varphi_{12}^{-1}\ (18, 19)$
$\varphi_{20}\ (70, 71)\ \varphi_5^{-1}\ (24, 22)\ \varphi_{21}\ (89, 61)\ \varphi_6^{-1}\ (37, 40)$
$\varphi_{23}\ (78, 77)\ \varphi_{12}^{-1}\ (17, 28)\ \varphi_{15}\ (51, 69)\ \varphi_6^{-1}\ (1, 16);$

$(1, 13)\ \varphi_{14}\ (87, 66)\ \varphi_4^{-1}\ (27, 17)\ \varphi_{18}\ (51, 70)\ \varphi_{12}\ (37, 31)$
$\varphi_{22}\ (59, 78)\ \varphi_4^{-1}\ (8, 24)\ \varphi_{16}\ (89, 72)\ \varphi_3^{-1}\ (18, 17)$
$\varphi_{18}\ (45, 70)\ \varphi_1 2\ (32, 31)\ \varphi_{22}\ (55, 78)\ \varphi_3^{-1}\ (1, 13);$

$(2, 10)\ \varphi_{15}\ (26, 25)\ \varphi_4\ (62, 90)\ \varphi_{21}^{-1}\ (29, 34)\ \varphi_5 (81, 86)\ \varphi_{20}^{-1}$
$(21, 20)\ \varphi_{11}\ (74, 56)\ \varphi_{17}^{-1}\ (4, 6)\ \varphi_5\ (43, 46)\ \varphi_{13}^{-1}$
$(15, 7)\ \varphi_4\ (65, 58)\ \varphi_{23}\ (88, 84)\ \varphi_1^{-1}\ (2, 10);$

$(2, 7)\ \varphi_{15}\ (26, 50)\ \varphi_8\ (95, 84)\ \varphi_{24}^{-1}\ (53, 56)\ \varphi_9\ (81, 79)$
$\varphi_{20}^{-1}\ (21, 46)\ \varphi_{11}\ (74, 34)\ \varphi_{17}^{-1}\ (4, 11)\ \varphi_9\ (14, 20)$
$\varphi_{19}^{-1}\ (42, 10)\ \varphi_8\ (65, 38)\ \varphi_{23}\ (88, 90)\ \varphi_1^{-1}\ (2, 7);$

$(2, 19)\ \varphi_{15}\ (26, 28)\ \varphi_4\ (62, 63)\ \varphi_{21}^{-1}\ (29, 94)\ \varphi_2\ (43, 41)$
$\varphi_{13}^{-1}\ (15, 22)\ \varphi_4\ (65, 77)\ \varphi_{23}\ (88, 76)\ \varphi_3^{-1}\ (14, 16)$
$\varphi_{19}^{-1}\ (42, 48)\ \varphi_2^{-1}\ (95, 93)\ \varphi_{24}^{-1}\ (53, 61)\ \varphi_3^{-1}\ (2, 19);$

$(2,21)\ \varphi_{14}\ (81,67)\ \varphi_5^{-1}\ (29,30)\ \varphi_{22}\ (53,62)\ \varphi_2^{-1}\ (14,15)$
$\varphi_{18}\ (43,60)\ \varphi_5^{-1}\ (4,35)\ \varphi_{16}\ (88,74)\ \varphi_7^{-1}\ (43,42)$
$\varphi_{18}^{-1}\ (14,92)\ \varphi_2\ (53,52)\ \varphi_{22}^{-1}\ (29,95)\ \varphi_7^{-1}\ (2,21);$

$(3,7)\ \varphi_{15}\ (96,50)\ \varphi_{11}^{-1}\ (39,38)\ \varphi_{23}\ (85,90)\ \varphi_4^{-1}\ (23,25)$
$\varphi_{21}\ (60,86)\ \varphi_5^{-1}\ (35,34)\ \varphi_{17}^{-1}\ (5,11)\ \varphi_1\ (83,79)$
$\varphi_{20}^{-1}\ (67,46)\ \varphi_5^{-1}\ (30,6)\ \varphi_{13}\ (57,58)\ \varphi_4^{-1}\ (3,7);$

$(3,10)\ \varphi_{15}\ (96,25)\ \varphi_{11}^{-1}\ (39,58)\ \varphi_{23}\ (85,84)\ \varphi_8^{-1}\ (47,50)$
$\varphi_{24}\ (92,79)\ \varphi_9^{-1}\ (35,56)\ \varphi_{17}^{-1}\ (5,6)\ \varphi_1\ (83,86)$
$\varphi_{20}^{-1}\ (67,20)\ \varphi_9^{-1}\ (51,11)\ \varphi_{19}\ (36,38)\ \varphi_8^{-1}\ (3,10);$

$(3,26)\ \varphi_{16}\ (83,96)\ \varphi_6^{-1}\ (36,30)\ \varphi_{22}\ (57,62)\ \varphi_2^{-1}\ (23,15)$
$\varphi_{18}\ (47,60)\ \varphi_6^{-1}\ (5,39)\ \varphi_{14}\ (85,65)\ \varphi_8^{-1}\ (47,42)$
$\varphi_{18}^{-1}\ (23,92)\ \varphi_2\ (57,52)\ \varphi_{22}^{-1}\ (36,95)\ \varphi_8^{-1}\ (3,26);$

$(3,28)\ \varphi_{15}\ (96,69)\ \varphi_{10}^{-1}\ (52,41)\ \varphi_{19}\ (36,93)\ \varphi_2\ (47,48)$
$\varphi_{24}\ (92,94)\ \varphi_{10}^{-1}\ (39,40)\ \varphi_{23}\ (85,77)\ \varphi_4^{-1}\ (23,22)$
$\varphi_{21}\ (60,61)\ \varphi_2^{-1}\ (30,16)\ \varphi_{13}\ (57,63)\ \varphi_4^{-1}\ (3,28);$

$(4,33)\ \varphi_{17}\ (74,76)\ \varphi_3^{-1}\ (15,16)\ \varphi_{13}\ (43,63)\ \varphi_2^{-1}\ (29,22)$
$\varphi_{21}\ (62,61)\ \varphi_3^{-1}\ (21,19)\ \varphi_{20}\ (81,71)\ \varphi_9^{-1}\ (53,48)$
$\varphi_{24}\ (95,94)\ \varphi_2\ (42,41)\ \varphi_{19}\ (14,93)\ \varphi_9^{-1}\ (4,33);$

$(5,40)\ \varphi_{17}\ (35,33)\ \varphi_5\ (60,63)\ \varphi_{21}^{-1}\ (23,94)\ \varphi_2\ (57,41)$
$\varphi_{13}^{-1}\ (30,22)\ \varphi_5\ (67,71)\ \varphi_{20}\ (83,69)\ \varphi_6^{-1}\ (36,16)$
$\varphi_{19}^{-1}\ (52,48)\ \varphi_2^{-1}\ (92,93)\ \varphi_{24}^{-1}\ (47,61)\ \varphi_6^{-1}\ (5,40);$

$(6,8)\ \varphi_{13}\ (58,91)\ \varphi_{13}\ (25,73)\ \varphi_{21}\ (86,87)\ \varphi_5^{-1}$
$(34,32)\ \varphi_{17}^{-1}\ (11,54)\ \varphi_9\ (20,64)\ \varphi_{19}^{-1}\ (10,12)\ \varphi_1(84,82)$
$\varphi_{24}^{-1}\ (56,49)\ \varphi_9(79,68)\ \varphi_{20}^{-1}\ (46,45)\ \varphi_5^{-1}\ (6,8);$

$(7,8)\ \varphi_{13}\ (46,91)\ \varphi_7\ (84,75)\ \varphi_{23}^{-1}\ (58,59)\ \varphi_{10}\ (79,82)$
$\varphi_{24}^{-1}\ (50,49)\ \varphi_{11}^{-1}\ (38,64)\ \varphi_{19}^{-1}\ (11,12)\ \varphi_{10}\ (25,27)$
$\varphi_{15}^{-1}\ (10,44)\ \varphi_7\ (34,73)\ \varphi_{21}\ (90,87)\ \varphi_1^{-1}\ (7,8);$

$(8,22)\ \varphi_{13}\ (91,41)\ \varphi_7\ (75,76)\ \varphi_{23}^{-1}\ (59,77)\ \varphi_{12}^{-1}\ (27,28)$
$\varphi_{15}^{-1}\ (44,19)\ \varphi_7\ (73,94)\ \varphi_{21}\ (87,63)\ \varphi_5^{-1}\ (32,33)$
$\varphi_{17}^{-1}\ (54,40)\ \varphi_{12}^{-1}\ (68,69)\ \varphi_{20}^{-1}\ (45,71)\ \varphi_5^{-1}\ (8,22);$

$(9,49)\ \varphi_{16}\ (82,73)\ \varphi_7^{-1}\ (45,44)\ \varphi_{18}^{-1}\ (18,68)\ \varphi_{12}\ (55,54)$
$\varphi_{22}^{-1}\ (32,75)\ \varphi_7^{-1}\ (12,91)\ \varphi_{14}\ (80,64)\ \varphi_8^{-1}\ (51,44)$
$\varphi_{18}^{-1}\ (27,68)\ \varphi_{12}\ (59,54)\ \varphi_{22}^{-1}\ (37,75)\ \varphi_8^{-1}\ (9,49);$

$(9,48)\ \varphi_{19}\ (13,16)\ \varphi_3\ (78,76)\ \varphi_{23}^{-1}\ (37,77)\ \varphi_{12}^{-1}\ (51,28)$
$\varphi_{15}^{-1}\ (17,19)\ \varphi_3\ (72,61)\ \varphi_{24}\ (80,93)\ \varphi_9^{-1}\ (55,33)$
$\varphi_{17}^{-1}\ (31,40)\ \varphi_{12}^{-1}\ (70,69)\ \varphi_{20}{-1}\ (18,71)\ \varphi_9^{-1}\ (9,48);$

$(12,41)\ \varphi_{19}\ (64,93)\ \varphi_9^{-1}\ (54,33)\ \varphi_{17}\ (32,76)\ \varphi_{12}^{-1}\ (45,19)$
$\varphi_{20}\ (68,71)\ \varphi_9^{-1}\ (49,48)\ \varphi_{24}\ (82,94)\ \varphi_{10}^{-1}\ (59,40)$
$\varphi_{23}\ (75,77)\ \varphi_{12}^{-1}\ (44,28)\ \varphi_{15}\ (27,69)\ \varphi_{10}^{-1}\ (12,41);$

$(13,21)\ \varphi_{14}\ (66,67)\ \varphi_{11}\ (24,35)\ \varphi_{16}\ (72,74)\ \varphi_3^{-1}\ (17,15)$
$\varphi_{18}\ (70,60)\ \varphi_6^{-1}\ (13,39)\ \varphi_{14}\ (66,65)\ \varphi_{11}\ (24,26)$
$\varphi_{16}\ (72,96)\ \varphi_6^{-1}\ (31,30)\ \varphi_{22}\ (78,62)\ \varphi_3^{-1}\ (13,21);$

$(95,75)\ \varphi_{22}\ (52,54)\ \varphi_9\ (67,64)\ \varphi_{14}^{-1}\ (21,91)\ \varphi 11\ (74,73)$
$\varphi_{16}^{-1}\ (35,49)\ \varphi_9\ (92,68)\ \varphi_{18}\ (42,44)\ \varphi_8\ (65,64)$
$\varphi_{14}^{-1}\ (39,91)\ \varphi_{11}\ (96,73)\ \varphi_{16}^{-1}\ (26,49)\ \varphi_8\ (95,75).$

As each cycle contains 12 edges, we have shown that cycles of one-dimensional edges are inessential, too. Thus we have shown that identifying the faces of the polyhedron $R_2$ by the motions $\varphi_1, \varphi_2, \ldots, \varphi_{24}$, the cycles both of two-dimensional faces and one-dimensional edges are inessential. Therefore the group $\Gamma$ generated by the motions $\varphi_1, \varphi_2, \ldots, \varphi_{24}$ does not contain elements of finite order, i. e. $\Gamma$ is torsion-free. Then the quotient space of the space $H^4$ by the group $\Gamma$ is a four-dimensional hyperbolic manifold $M$ with finite volume which is not closed. The manifold $M$ has four cusps, i. e. four ends of the form $T^3 \times [0,\infty)$, where $T^3$ is a three-dimensional torus.

## References


[1] Ratcliffe I. G., Tschantz S. T. *The Volume Spectrum of Hyperbolic 4-Manifolds.* Experimental Math., 2000, **9**, 101–125.

[2] Gutsul I. S. *Some hyperbolic manifolds.* Bul. Acad. Ştiinţe Repub. Moldova, Matematica, 2004, No. 3(46), 63–70.



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# On spaces of densely continuous forms

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**Abstract.** We study the structure of the domain of the minimal upper semicontinuous extension of the set-valued mapping. It is proved that the set of all compact-valued upper semicontinuous mappings is closed in the space of all set-valued mappings. A similar assertion is true for the space of densely continuous forms.

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## 1 Introduction

Let $(Y,\mathcal{U})$ be a uniform space and $X$ be a topological space. By $\exp(Y)$ or $2^Y$ we denote the space of all closed subsets of $Y$. The uniformity $\mathcal{U}$ is generated by a family of uniformly continuous pseudometrics $P(\mathcal{U})$. Consider that $\rho(y,z) \leq 1$ for all $\rho \in P(\mathcal{U})$ and $y, z \in Y$.

Let $\rho \in P(\mathcal{U})$. If $y \in Y$ and $L \subseteq Y$, then $N(y,\rho,r) = \{z \in Y : \rho(y,z) < r\}$ and $N(L,\rho,r) = \cup\{N(y,\rho,r) : y \in L\}$ for a real number $r > 0$. We put $\varphi(L,M) = \inf\{r : L \subseteq N(M,\rho,r)$, $M \subseteq N(L,\rho,r)$. If $\emptyset \in \{L,M\}$ and $L \neq M$, then $h\rho(L,M) = 1$. The families $hP(\mathcal{U}) = \{h\rho : \rho \in P(\mathcal{U})\}$ generate the uniformity $h(\mathcal{U})$ on $\exp(Y)$.

A set-valued mapping $g : X \to Y$ assigns to each point $x \in X$ a closed subset $g(x)$ of $Y$.

Let $g : X \to Y$ be a set-valued mapping. The mapping $g$ is called:

– upper semicontinuous (us-continuous) at a point $x_0 \in X$ if for every open $V \subseteq Y$ with $g(x_0) \subseteq V$ there exists an open set $U$ of $X$ such that $x_0 \in U$ and $F(x) \subseteq V$ for any $x \in U$;

– lower semicontinuous (ls-continuous) at a point $x_0 \in X$ if for every open $V \subseteq Y$ with $g(x_0) \cap V \neq \emptyset$ there exists an open set $U$ of $X$ with $g(x) \cap V \neq \emptyset$ for each $x \in U$;

– continuous at a point $x \in X$ if $g$ simultaneous by is us-continuous and ls-continuous at the point $x$;

– weakly continuous ($w$-continuous) if the graphic $Gr(g) = \cup\{\{x\}\times g(x) : x \in X\}$ is a closed subset of the space $X \times Y$;

– minimal if the graphic $Gr(g)$ is closed in $X \times Y$, the set $Dom(g) = \{x \in X : g(x) \neq \emptyset\}$ is dense in $X$ and for each closed subset $F$ of $Gr(g)$ such that $F \neq Gr(g)$ there exists a point $x \in Dom(g)$ such that $F \cap (\{x\} \times g(x)) = \emptyset$.

*Remark 1.* The set $Dom(g) = \{x \in X : g(x) \neq \emptyset\}$ is the domain of the mapping $g : X \to Y$. If the set $Dom(g)$ is dense in $X$ and $x \in X \setminus Dom(g)$ then $g$ is not us-continuous and not ls-continuous at the point $x \in X$.

A mapping $g : X \to Y$ is called:
– us-continuous if it is us-continuous at any point $x \in Dom(g)$;
– ls-continuous if it is ls-continuous at any point $x \in Dom(g)$;
– continuous if it is continuous at any point $x \in Dom(g)$.

Denote by $F(X, Y)$ the set of all single-valued mappings of the space $X$ into the space $Y$, by $F(X, 2^Y)$ the set of all set-valued mappings of $X$ into $Y$, by $C(X, Y) = \{g \in F(X, Y) : g \text{ is continuous}\}$ the set of all continuous mappings of $X$ into $Y$.

Let $\mathcal{A}$ be a family of subsets of $X$ which is closed under finite union and which covers $X$.

We define on $F(X, 2^Y)$ the topology of uniform convergence on sets in $\mathcal{A}$ as follows.

For any pseudometric $\rho \in P(\mathcal{U})$ and each $B \in \mathcal{A}$ on $F(X, 2^Y)$ define the pseudometric $\rho_B(f, g) = \sup\{h\rho(f(x), g(x)) : x \in B)\}$.

Then $F(X, 2^Y)$ has the topology generated by the family of pseudometrics $\mathcal{A}(\mathcal{U}) = \{\rho_B : B \in \mathcal{A}, \rho \in P(\mathcal{U})\}$. The pseudometrics $\mathcal{A}(\mathcal{U})$ form on $F(X, 2^Y)$ a Hausdorff uniform structure and the space $F_{\mathcal{A}}(X, 2^Y)$ with this topology is completely regular and Hausdorff [3].

Whenever $\mathcal{A}$ consists of the all finite subsets of $X$, the topology generated by the uniform structure $\mathcal{A}(\mathcal{U})$ is the topology of pointwise convergence on $F(X, 2^Y)$ and this space is denoted by $F_p(X, 2^Y)$.

Since $\mathcal{A}$ consists of the all compact subsets of $X$, then the topology generated by the uniform structure $\mathcal{A}(\mathcal{U})$ is the topology of uniform convergence on compact sets and this space is denoted by $F_c(X, 2^Y)$.

Whenever $X \in \mathcal{A}$, then this topology is called the topology of uniform convergence and this space is denoted by $F_u(X, 2^Y)$.

On subspaces of the space $F_{\mathcal{A}}(X, 2^Y)$ we consider the topology generated by the uniform structure $\mathcal{A}(\mathcal{U})$ too.

## 2 Extensions of mappings

Fix a space $X$, a uniform space $(Y, \mathcal{U})$ with the uniformity $\mathcal{U}$ generated by the pseudometrics $P(\mathcal{U})$ and a compactification $cY$ of $Y$.

Let $g : X \to Y$ be a set-valued mapping.

A set-valued mapping $\varphi : X \to Y$ is said to be an usc-extension of the mapping $g$ if $\varphi$ is a compact-valued us-continuous mapping and $g(x) \subseteq \varphi(x)$ for any $x \in X$.

A set-valued mapping $\varphi : X \to Y$ is said to be a minimal usc-extension of the mapping $g$ if $\varphi$ is an usc-extension of the mapping $g$ and for any usc-extension $\psi : X \to Y$ of $g$ we have $\varphi(x) \subseteq \psi(x)$ for any $x \in X$.

*Remark 2.* One can say that a set-valued mapping $\varphi : X \to Y$ is a maximal usc-extension of the mapping $g$ if $\varphi$ is an usc-extension of $g$ and $Dom(\psi) \subseteq Dom(\varphi)$ for any usc-extension $\psi : X \to Y$ of $g$. If the space $Y$ is compact, then the mapping $g_{max} : X \longrightarrow Y$, where $g_{max}(x) = Y$ for any $x \in X$, is the maximal usc-extension of any mapping $g : X \longrightarrow Y$. If the space $Y$ is not compact and $\psi : X \longrightarrow Y$ is an usc-extension of the mapping $g : X \longrightarrow Y$, then $\psi(x) \neq Y$ for any $x \in X$. Fix a compact subset $F$ of $Y$ and put $\psi_F(x) = \psi(x) \cup F$ for each $\cup x \in X$. Then $\psi_F : X \longrightarrow Y$ is a usc-extension of the mapping $g$ and $Gr(\psi) \subseteq Gr(\psi_F)$. Hence, for a non-compact space $Y$ for each mapping $g : X \longrightarrow Y$ does not exist maximal usc-extension.

**Proposition 1**. *Let $g : X \to Y$ be a set-valued mapping and the domain $Dom(g)$ is dense in $X$. The following assertions are equivalent:*

1. *The mapping $g$ has some usc-extension.*
2. *For $g$ there exists a unique minimal usc-extension $m_c g : X \to Y$.*

*Proof.* The implication $2 \to 1$ is obvious. Assume that $\varphi : X \to Y$ is an usc-extension of $g$. Then $\varphi$ is an us-continuous mapping of $X$ into $cY$. Denote by $\pi_X : X \times cY \to X$ the projection $\pi_X(x, y) = x$ for all $(x, y) \in X \times cY$. Since $cY$ is a compact space, the projection $\pi_X$ is a perfect mapping. Denote by $\pi_{cY} : X \times cY \to cY$ the projection onto $cY$. The mapping $\pi_{cY}$ is continuous.

Every subset $M \subseteq X \times cY$ is the graphic of some concrete set-valued mapping $\theta_M : X \to cY$, where $\theta_M(x) = \prod_{cY}(M \cap (\{x\} \times CY))$ for any $x \in X$. The mapping $\theta_M$ is us-continuous if and only if the set $M$ is closed in the subspace $\pi_X(M) \times cY$.

In particular, if $\psi : X \to Y$ is an usc-extension of $g$, then $Gr(g) \subseteq Gr(\psi)$ and the set $Gr(\psi)$ is closed in the subspace $Dom(\psi) \times cY$. Hence $Gr(g) \subseteq Gr(\varphi)$ and the set $Gr(\varphi)$ is closed in $Dom(\varphi) \times cY$.

Denote by $\Phi$ the closure of the set $Gr(g)$ in the space $X \times cY$.

Then the set $\Phi_1 = \Phi \cap (Dom(\varphi) \times cY)$ is the closure of $Gr(g)$ in $Dom(\varphi) \times cY$. The mapping $w : X \to cY$, where $Gr(w) = \Phi$, is us-continuous. Moreover, $g(x) \subseteq w(x) \subseteq \varphi(x) \subseteq Y$ for any $x \in Dom(g)$. Let $H = \{x \in X : w(x) \subseteq Y\}$. By construction, $Dom(g) \subseteq Dom(\varphi) \subseteq H$. Denote by $m_c g : X \to Y$ the mapping with the domain $Dom(m_c g) = H$ and $m_c g(x) = w(x)$ for any $x \in H$. Since $Dom(w) = X$, the mapping $m_c g$ is correctly defined. Obviously, $m_c g$ is an usc-extension of $g$.

Let $\psi : X \to Y$ be a usc-extension of $g$. Since $Gr(g) \subseteq Gr(\psi)$, the set $\Phi \cap Gr(\psi)$ is the closure of the set $Gr(g)$ in $Dom(\psi) \times cY$. Thus $m_c g(x) = w(x) \subseteq \psi(x)$ for any $x \in H \cap Dom(\psi) = Dom(m_c g) \cap Dom(\psi)$ and $Dom(\psi) \subseteq Dom(m_c g)$. Hence $m_c g$ is the minimal usc-extension of $g$. The existence of the minimal usc-extension is proved. The uniqueness of the minimal usc-extension is obvious. The proof is complete. □

Let $g : X \to Y$ be a set-valued mapping. The mapping $m_e g : X \to Y$ with the graphic $Gr(m_e g) = cl_{X \times Y} Gr(g)$ is called the minimal $w$-continuous extension of the mapping $g$.

**Proposition 2.** *Let $g : X \to Y$ be a set-valued mapping and the set $Dom(g)$ is dense in $X$. Then:*

*1. If $\varphi : X \to Y$ is a w-continuous mapping and $g(x) \subseteq \varphi(x)$ for any $x \in Dom(g)$, then $m_e g(x) \subseteq \varphi(x)$ for any $x \in X$.*

*2. If $m_c g$ is the minimax usc-extension of $g$, then $m_c g(x) \subseteq m_e g(x)$ for any $x \in X$.*

*Proof.* Follows from the coincidence of the closures of the sets $Gr(g)$, $Gr(m_c g)$, and $Gr(m_e g)$ in $X \times cY$. □

**Corollary 1.** *If the space $Y$ is compact, then $m_c g = m_e g$ for any set-valued mapping $g : X \to Y$ with the dense domain $Dom(g)$ in $X$.*

*Remark 3.* Let $m_c g : X \to Y$ be the minimax usc-extension of a set-valued mapping $g : X \to Y$ with the dense domain $Dom(g)$ in $X$. If $x \notin Dom(g)$, then we say that $X$ is an essential point of usc-discontinuity of the mapping $g$. If $x \in Dom(m_c g) \setminus Dom(g)$, then $x$ is inessential point of usc-discontinuits of the mapping $g$.

## 3 $m$-metric and $m$-Baire spaces

Let $m$ be an infinite cardinal number.

A uniform space $(Y,\mathcal{U})$ is an $m$-metric space if the uniform structure $U$ is generated by a family $P(\mathcal{U})$ of pseudometrics of cardinality $\leq m$. In this case we assume that the cardinality $|P(\mathcal{U})| \leq m$ and for any $\rho_1, \rho_2 \in P(U)$ there exists $\rho \in P(U)$ such that $\sup\{\rho_1(x,y), \rho_2(x,y)\} \leq \rho(x,y)$ for all $x, y \in Y$.

A set $L$ of a space $X$ is called a $G_m$-set if $L$ is the intersection of $m$ open subsets of $X$. For $m = \aleph_0$ the $G_m$-set is called a $G_\delta$-set. The complement of a $G_m$-set is an $F_m$-set and of $G_\delta$-set is an $F_\sigma$-set.

A subset $A$ of a space $X$ is called $m$-meager if $A$ is the union of $m$ nowhere dense subsets of $X$. The space $X$ is called an $m$-Baire space if every non-empty open subset of $X$ is not $m$-meager.

For a space $X$ the next three assertions are equivalent:

1 bm) $X$ is an $m$-Baire space.

2 bm) The intersection of $m$ open and dense subsets of $X$ is dense in $X$.

3 bm) The intersection of $m$ dense $G_m$-subsets is dense in $X$.

A space $X$ is a Baire space if it is an $\aleph_0$-Baire space.

A space $X$ is called $m$-complete if $X$ is a $G_m$-subset of some compactification $cX$ of $X$.

**Proposition 3.** *Let $(Y,\mathcal{U})$ be an $m$-complete $m$-metric space, $g : X \to Y$ be a set-valued mapping with a dense domain $Dom(g)$ in $X$ and $m_c g : X \to Y$ be the usc-extension of $g$. Then $Dom(m_c g)$ is a dense $G_m$-set of $X$.*

*Proof.* Let $\Phi$ be the closure of the set $Gr(g)$ in $X \times cY$, where $cY$ is a compactification of $Y$, and $\omega : X \to cY$ be the us-continuous mapping with the graphic $Gr(\omega) = \Phi$. Then $\Phi$ is the closure of the set $Gr(m_c g)$ in $X \times cY$ too. By construction, $Dom(m_c g) = \{x \in X : \omega(x) \subseteq Y\}$ and $m_c g(x) = \omega(x)$ for all $x \in Dom(m_c g)$. Fix a family $\{U_\alpha : \alpha \in A\}$ of open subsets of $Y$ for which $|A| \leq m$ and $Y = \cap\{U_\alpha : \alpha \in A\}$. For any $\alpha \in A$ the set $V_\alpha = \{x \in X : \omega(x) \subseteq U_\alpha\}$ is open in $X$ and $Dom(m_c g) \subseteq V_\alpha$. Let $L = \cap\{V_\alpha : \alpha \in A\}$. By construction, $Dom_c(g) \subseteq L$ and $L$ is a $G_m$-set of $X$. Suppose that $x \notin Dom_c(g)$. Then there exist a point $y \in \omega(x) \setminus Y$ and $\alpha \in A$ for which $y \notin U_\alpha$. Then $x \notin V_\alpha$ and $x \notin L$. Therefore $L = Dom_c(g)$. The proof is complete. □

**Proposition 4.** *Let $g : X \to Y$ be a minimal us-continuous mapping of a space $X$ into an $m$-metric space $(Y,\mathcal{U})$. Then $Dom_s(g) = \{x \in X : g(x)$ is a singleton set$\}$ is a $G_m$-subset of $Dom(g)$.*

*Proof.* We can assume that $X = Dom(g)$. Consider the pseudometrics $P(\mathcal{U}) = \{\rho_\alpha : \alpha \in A\}$ which generate the uniformity $\mathcal{U}$ on $Y$. Assume that $|A| \leq m$.

For every $n \in N = \{1, 2, \ldots\}$, $\alpha \in A$ and $y \in Y$ we put $V(y, \alpha, n) = \{x \in X : g(x) \subseteq N(y, \rho_\alpha, 2^{-n})\}$ and $V(\alpha, n) = \cup\{V(y, \alpha, n) : y \in Y\}$. Since the set $N(y, \rho_\alpha, 2^{-n})$ is open in $Y$ and the mapping $g$ is us-continuous, the set $V(\alpha, n)$ is open in $X$. The set $L = \cap\{V(\alpha, n) : \alpha \in A, n \in N\}$ is a $G_m$-set of $X$.

If $x \in Dom_s(g)$ and $g(x) = y \in Y$, then $x \in V(y, \alpha, n)$ for all $\alpha \in A$ and $n \in N$. Hence $Dom_s(g) \subseteq L$. Let $x \notin Dom_s(g)$. Then there exist two distinct points $y_0, z_0 \in g(x), \alpha \in A$ and $n \in N$ for which $\rho_\alpha(y_0, z_0) > 2^{-n} > 0$. In this case $g(x) \setminus V(y, \alpha, n) \neq \emptyset$ for any $y \in Y$. Thus $x \notin V(\alpha, h)$. Therefore $L = Dom_s(g)$. The proof is complete. □

**Corollary 2** *Let $g : X \to Y$ be a us-continuous mapping of an $m$-Baire space $X$ into an $m$-complete $m$-metric uniform space $(Y,\mathcal{U})$. The following assertions are equivalent:*

*1. The mapping $m_e g : X \to Y$, $Y$ is minimal, i.e. $g : Dom(g) \to Y$ is a minimal mapping.*

*2. $Dom_s(g)$ is a dense $G_m$-set of $X$.*

*Proof.* Implication 2 → 1 is obvious. Let $g : Dom(g) \to Y$ be minimal. Then the mapping $m_c g : X \to Y$ is minimal. Proposition 4 completes the proof. □

## 4 Spaces of dense forms

Fix an infinite cardinal number $m$, an $m$-Baire space $X$ and an $m$-complete $m$-metric space $(Y,\mathcal{U})$ with uniformity $\mathcal{U}$ generated by the family $P(\mathcal{U}) = \{\rho_\alpha : \alpha \in A\}$ of pseudometrics, where $|A| \leq m$.

A set-valued mapping $g : X \to Y$ is called a dence set-valued continuous form from $X$ to $Y$ if $Dom(g)$ is a dense subset of $X$ and $g = m_c g$.

A set-valued mapping $g : X \to Y$ is called a dense continuous form from $X$ to $Y$ if $Dom_s(g)$ is a dense subset of $X$ and $g = m_c g$.

*Remark 4.* From Corollary 2 it follows that for a set-valued mapping $g : X \to Y$ the following assertions are equivalent:

1. $g$ is a dense continuous form from $X$ to $Y$.

2. There exists a dense subspace $Z$ of $X$ and a continuous single-valued mapping $f : Z \to Y$ such that $g = m_c f$.

Hence our definition of a dense continuous form coincides with the definition of a dense continuous form from [5].

Denote by $DUC(X, Y)$ the family of all us-continuous compact-valued mappings $g : X \to Y$ for which the domain $Dom(g)$ is dense in $X$, by $DU(X, Y)$ the family of all dense set-valued continuous forms from $X$ to $Y$, by $DC(X, Y)$ the family of all single-valued mappings $g \in DUC(X, Y)$ and by $D(X, Y)$ the family of all dense continuous forms from $X$ to $Y$. It is obvious that $D(X, Y) \subseteq DU(X, Y) \subseteq DUC(X, Y)$.

There exists a single-valued mapping $e : DUC(X, Y) \to DU(X, Y)$, where $e(g) = m_c g$ for any $g \in DUC(X, Y)$.

## 5 Completeness of the spaces of set-valued dense continuous forms

Fix a space $X$ and a complete uniform space $(Y, \mathcal{U})$ with the uniformity $\mathcal{U}$ generated by the family of pseudometrics $P(\mathcal{U}) = \{\rho_\alpha : \alpha \in A\}$.

Let $FC(X, Y)$ be the set of all compact-valued us-continuous mappings of $X$ into $Y$. On $X$ fix a family $\Gamma$ of subsets which is closed under finite union and which covers $X$.

On $F(X, exp(Y))$ consider the topology and the uniformity generated by the pseudometrics $\Gamma(\mathcal{U}) = \{\rho_{\alpha B} : \alpha \in A, B \in \Gamma\}$.

**Theorem 1.** *Let $X \in \Gamma$. Then the set $FC(X, Y)$ is closed in the space $F_\Gamma(X, exp(Y))$.*

*Proof.* Let $g \in F(X, exp(Y) \setminus FC(X, Y)$.

Case 1. $g(x_0)$ is not a compact set for some point $x_0 \in X$.

Since $Y$ is a complete uniform space, in this case there exist $\alpha \in A$, $\varepsilon > 0$ and an infinite sequence $\{y_n \in g(x_0) : n \in N\}$ such that $\rho_\alpha(y_n, y_m) \geq \varepsilon$ for all $n, m \in N$ and $n \neq m$. Fix $B \in \Gamma$ for which $x_0 \in B$ and $\delta < e^{-1}$ such that $0 < 3\delta < \varepsilon$.

Let $f \in F(X : \exp(Y))$ and $\rho_{\alpha B}(f, g) < \delta$. Then for any $n \in N$ there exists a point $z_n \in f(x_0)$ such that $\rho_\alpha(y_n, z_n) < \delta$. In this case $\rho_\alpha(z_{n,} z_m) \geq \delta$ for all $n, m \in N$ and $n \neq m$. Thus the set $f(x_0)$ is not precompact in $Y$. Since $Y$ is complete, the set $f(x_0)$ is not compact. Therefore the set $V = \{f \in F(X, \exp(Y)) : \rho_{\alpha B}(g, f) < \delta\}$ is open in $F_\Gamma(X, \exp(Y))$, $g \in V$ and $V \cap FC(X, Y) = \emptyset$.

Case 2. $g(x)$ is a compact set of $Y$ for each $x \in X$.

In this case there exists a point $x_0 \in Dom(g)$ such that $g$ is not us-continuous at $x_0$. Thus there exists an open subset $U$ of $Y$ such that $g(x_0) \subseteq U$ and for any neighborhood $W$ of $x_0$ in $X$ there exists a point $x \in W \cap Dom(g)$ for which $g(x) \setminus U \neq \emptyset$.

Since the set $g(x_0)$ is compact there exists $\varepsilon > 0$ and $\alpha \in A$ such that $N(g(x_0), \rho_\alpha, 4\varepsilon) \subseteq U$. Suppose that $\rho_{\alpha x}(g, f) < \varepsilon$ and $f \in FC(X, Y)$. In this case $f(x_0) \subseteq N(g(x_0), p_\alpha, \varepsilon)$, the set $W = \{x \in X : f(x) \subseteq N(g(x_0), p_\alpha, \varepsilon)\}$ is open in $X$ and $x_0 \in W$. There exists $x \in W$ such that $g(x) \setminus U \neq \emptyset$. Fix $y \in g(x) \setminus U$.

Since $y \in N(g(x_0), p_\alpha, 4\varepsilon)$, then $N(y, p_\alpha, 4\varepsilon) \cap g(x_0) = \emptyset$. Since $\rho_{\alpha x}(g, f) < \varepsilon$, there exists $z \in f(x)$ such that $\rho_{\alpha X}(g, f) < \varepsilon$. Hence $z \notin N(g(x_0), \rho_\alpha, \varepsilon)$ and $x \notin W$, a contradiction. Therefore $N(g(x_0), \rho_\alpha, \varepsilon) \cap FC(X, Y) = \emptyset$ and the set $FC(X, Y)$ is closed. □

In the case 1 we have proved the following assertion.

**Proposition 5**. *The set $F^c(X, \exp Y)$ of all compact-valued mappings is closed in the space $F_\Gamma(X, \exp Y)$ for any family $\Gamma$.*

**Proposition 6.** *The set $DU(X, Y)$ is dense in the space $F^c_p(X, \exp Y)$ of all compact–valued mappings in the topology of pointwise convergence.*

*Proof.* Fix a mapping $g \in F^c(X, \exp Y), \alpha \in A, \varepsilon > 0$ and a finite subset $F = \{x_1, x_2 \ldots x_m\}$ of $X$. Fix a point $b \in Y$ and the open subsets $\{v_1, v_2, \ldots v_n\}$ of $X$ such that $x_i \in V_i$ and $V_i \cap V_j = \emptyset$ for all $i, j \leq n$ and $i \neq j$.

We put $f(x) = g(x_i)$ for all $i \leq n$ and $x \in V_i$, and $f(x) = \cup\{g(x_i) : i \leq n\}$ for any $x \in (X \setminus U\{V_i : i \leq n\})$. Then $f$ is us-continuous, $Dom(f) = X$ and $\rho_{\alpha F}(g, f) = 0 < \varepsilon$. The proof is complete. □

**Proposition 7.** *The set $F^d(X, \exp Y)$ of all set-valued mappings $g : X \to Y$ with a dense domain $Dom(g)$ in $X$ is closed in $F_u(X, \exp(Y))$ in the topology of uniform convergence.*

*Proof.* Let $g : X \to Y$ be a set-valued mapping and the set $Dom(g)$ be not dense in $X$. Then the set $V = X \setminus cl_X\, Dom(g)$ is open and non–empty.

Fix $\alpha \in A$. If $L = Y$ and $L \neq \emptyset$, $h\rho_\alpha(\emptyset, L) = 1$. The set $U = \{f \in F(X, \exp(Y)) : h\rho_\alpha(g, f) < 1\}$ is open in $F_u(X, \exp(Y))$ and $g \in U$. Let $f \in F^d(X, \exp(Y))$. Since the set $Dom(f)$ is dense in $X$, there exists a point $x \in V \cap Dom(f)$. In this case $f(x) \neq \emptyset$ and $g(x) = \emptyset$. Hence $h\rho_\alpha(f(x), g(x)) = 1$ and $f \notin U$. Therefore $U \cap F^d(X, \exp Y) = \emptyset$. The proof is complete. □

**Corollary 3.** *The set $F^{cd}(X, \exp(Y))$ of all compact-valued mappings with the dense domain is dense in the space $F_u(X, \exp(Y))$.*

*Proof.* By virtue of Propositions 5 and 7, the set $F^{ed}(X, \exp(Y)) = F^c(X : \exp Y) \cap F^d(X, \exp(Y))$ is closed in $F_u(X, \exp(Y))$. □

**Theorem 2.** *The set $FC(X, Y)$ is closed in the space $F_u(X, \exp(Y))$.*

*Proof.* Let $\exp_c(Y)$ be the spaces of all compact subsets of $Y$ in the topology generated by the pseudometrics $hP(U)$. The uniform space $\exp_c(Y)$ is complete [9]. Fix a Cauchy sequence $\{g_\mu : \mu \in M\}$, where $M$ is a directed set. Since the space $\exp_c(Y)$ is complete, for any $x \in X$ in $\exp_c(Y)$ there exists the limit $g(x) = lim\{g_\mu(x) : \mu \in M$. In this case $g = \lim\{g_\mu : \mu \in M\}$ in the space $F_n(X, \exp(Y))$. Fix $\alpha \in A$. There exists $\lambda \in M$ such that $h\rho_\alpha(g(x), g_\mu(x)) < 1$ for all $\mu \geq \lambda$ and all $x \in X$. Thus

$Dom(g) = Dom(g_\mu)$ for all $\mu \geq \lambda$. We can assume that $Dom(g) = Dom(g_\mu) = X$ for all $\mu \in M$.

We affirm that the mapping $g : X \to Y$ is us-continuous. Fix $x_0 \in X$ and an open subset $U$ of $Y$ for which $g(x_0) \subseteq U$. There exist $\alpha \in A$ and $0 < \varepsilon < 1$ such that $N(g(x_0), \rho_\lambda, 4\varepsilon) \subseteq U$. Fix now $\mu \in M$ for which $h\rho_\alpha(g(x), g_\mu(x)) < \varepsilon$ for all $x \in X$. The set $V = \{x \in X : g_\mu(x) \subseteq N(g_\mu(x_0), \rho_\alpha, \varepsilon)\}$ is open in $X$ and $x_0 \in V$. If $x \in V$, then $h\rho_\alpha(g_\mu(x_0), g_\mu(x)) < \varepsilon$, $h\rho_\alpha(g(x_0), \rho_\mu(x_0)) < \varepsilon$ and $h\rho_\alpha(g(x_0), g_\mu(x)) < 2\varepsilon$. Sice $h\rho_\alpha(g(x), g_\mu(x)) < \varepsilon$, then $h\rho_\alpha(g(x_0), g(x)) < 3\varepsilon$ and $g(x) \subseteq N(g(x_0), \rho_\alpha, 4\varepsilon) \subseteq U$. Hence $g$ is us-continuous at the point $x_0$. The proof is complete. □

**Corollary 4.** *The set $DU(X, Y)$ of all set-valued $\alpha$ continuous forms $M$ which are closed in the space $F_u(X, \exp(Y))$ and in the uniformity of uniform convergence is a complete uniform space.*

## 6 Completeness of the space of dense continuous forms

Fix an infinite cardinal number $m$, an $m$-Baire space $X$ and an $m$-complete $m$-metric space $(Y,\mathcal{U})$ with a complete uniformity $\mathcal{U}$ generated by the pseudometrics $P(\mathcal{U}) = \{\rho_\alpha : \alpha \in A\}$, where $|A| \leq m$.

**Theorem 3.** *The set $D(X, Y)$ is closed in the space $F_u(X, \exp(Y))$.*

*Proof.* Since $D(X, Y) \subseteq DU(X, Y)$ and the set is closed in $F_u(X, \exp Y)$, then it is sufficient to prove that the set $D(X, Y)$ is closed in the space $DU_u(X, Y)$. Let $\{g_\mu \in D(X, Y) : \mu \in M\}$ be a Cauchy sequence where $M$ is a directed set. Since $Y$ is an $m$-metric space we can assume that $|M| \leq m$. Let $g = \lim\{g_\mu : \mu \in M\}$. From Theorem 2 it follows that $g$ is a compact-valued us-continuous mapping. If $\alpha \in A$, then there exists $\lambda \in M$ such that $h\rho_\alpha(g(x), g_\mu(x)) < 1$ for all $x \in X$ and $\mu \geq \alpha$. Thus $Dom(g) = Dom(g_\mu)$ for all $\mu \geq \lambda$.

Therefore $g \in DU(X, Y)$ and we can assume that $Dom(g) = Dom(g_\mu) = X$ for all $\mu \in M$. From Corollary 2 it follows that $Dom_s(g_\mu) = \{x \in X : g_\mu(x)$ is a singleton set$\}$ is a dense $G_m$-set of $X$ for any $\mu \in M$. Since $|M| \leq m$ and $X$ is an $m$-Baire space, the subspace $Z = \cap\{Dom_s(g_\mu) : \mu \in M\}$ is a dense $G_m$-set of $X$. Thus $f_\mu = g_\mu|Z : Z \to Y$ is a single-valued continuous mapping of $Z$ into $Y$ for any $\mu \in M$.

Let $f = g|Z : Z \to Y$. Then $f = \lim\{f_\mu : \mu \in M\}$ and the uniform limit of single–valued mappings is a single–valued mappings. Thus $Z \subseteq Dom_s(g)$ and $Dom_s(g)$ is a dense subset of $X$. From Remark 4 it follows that $g \in D(X; Y)$. The proof is complete. □

**Corollary 5.** *The space $D(X; Y)$ in the uniformity of uniform convergence is complete.*

## References

[1] Arens R. *A topology for spaces of transformations.* Ann. of Math., 1946, **47**, 480–495.

[2] Beer G. *Topologies on closed and closed convex sets.* Kluwer Acad. Publ., Dordrecht, 1993.

[3] Engelking R. *General Topology.* Heldermann Verlag, Berlin, 1989.

[4] Gilman L., Jerison M. *Rings of Continuous Functions.* D. Van Nostrum, Princeton, 1960.

[5] Hammer S. T., McCoy R. A. *Spaces of Densely Continuous Forms.* Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, July 1997, 247–266.

[6] Ipate D. M. *Lattices on topological spaces.* Mat. Issled., vyp. 44, Chişinău, Ştiinţa, 1977, 58–74.

[7] Kuratovski K. *Applications of set-valued mappings to various spaces of Continuous functions.* Gen. Top. Apl., 1971, No. 2, 155–161.

[8] McCoy R., Ntantu I. *Topological Properties of Space of Continuous Functions.* Lecture Notes in Math., Vol. 1315, Springer-Verlug, New York, 1988.

[9] Michael E. *Topologies on spaces of subsets.* Tran. Amer. Math. Soc., 1951, **71**, 152–182.

[10] Oxtoby J. *Measure and Category Graduate Texts in Mathematics.* Springer-Verlag, New York, 1971.

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# A selection theorem for set-valued maps into normally supercompact spaces

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**Abstract.** The following selection theorem is established:
Let $X$ be a compactum possessing a binary normal subbase $\mathcal{S}$ for its closed subsets. Then every set-valued $\mathcal{S}$-continuous map $\Phi\colon Z \to X$ with closed $\mathcal{S}$-convex values, where $Z$ is an arbitrary space, has a continuous single-valued selection. More generally, if $A \subset Z$ is closed and any map from $A$ to $X$ is continuously extendable to a map from $Z$ to $X$, then every selection for $\Phi|A$ can be extended to a selection for $\Phi$.
This theorem implies that if $X$ is a $\kappa$-metrizable (resp., $\kappa$-metrizable and connected) compactum with a normal binary closed subbase $\mathcal{S}$, then every open $\mathcal{S}$-convex surjection $f\colon X \to Y$ is a zero-soft (resp., soft) map. Our results provide some generalizations and specifications of Ivanov's results (see [5–7]) concerning superextensions of $\kappa$-metrizable compacta

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## 1 Introduction

In this paper we assume that all topological spaces are Tychonoff and all single-valued maps are continuous.

Recall that supercompact spaces and superextensions were introduced by de Groot [4]. A space is *supercompact* if it possesses a binary subbase for its closed subsets. Here, a collection $\mathcal{S}$ of closed subsets of $X$ is *binary* provided any linked subfamily of $\mathcal{S}$ has a non-empty intersection (we say that a system of subsets of $X$ is *linked* provided any two elements of this system intersect). The supercompact spaces with binary *normal* subbase will be of special interest for us. A subbase $\mathcal{S}$ which is closed both under finite intersections and finite unions is called normal if for every $S_0, S_1 \in \mathcal{S}$ with $S_0 \cap S_1 = \varnothing$ there exists $T_0, T_1 \in \mathcal{S}$ such that $S_0 \cap T_1 = \varnothing = T_0 \cap S_1$ and $T_0 \cup T_1 = X$. A space $X$ possessing a binary normal subbase $\mathcal{S}$ is called *normally supercompact* [9] and will be denoted by $(X, \mathcal{S})$.

The *superextension* $\lambda X$ of $X$ consists of all maximal linked systems of closed sets in $X$. The family

$$U^+ = \{\eta \in \lambda X : F \subset U \text{ for some } F \in \eta\},$$

$U \subset X$ is open, is a subbase for the topology of $\lambda X$. It is well known that $\lambda X$ is normally supercompact. Let $\eta_x$, $x \in X$, be the maximal linked system of all closed


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sets in $X$ containing $x$. The map $x \to \eta_x$ embeds $X$ into $\lambda X$. The book of van Mill [9] contains more information about normally supercompact space and superextensions, see also Fedorchuk-Filippov's book [3].

If $\mathcal{S}$ is a closed subbase for $X$ and $B \subset X$, let $I_{\mathcal{S}}(B) = \bigcap\{S \in \mathcal{S} : B \subset S\}$. A subset $B \subset X$ is called *$\mathcal{S}$-convex* if for all $x, y \in B$ we have $I_{\mathcal{S}}(\{x, y\}) \subset B$. An *$\mathcal{S}$-convex map* $f\colon X \to Y$ is a map whose fibers are $\mathcal{S}$-convex sets. A set-valued map $\Phi\colon Z \to X$ is said to be *$\mathcal{S}$-continuous* provided for any $S \in \mathcal{S}$ both sets $\{z \in Z : \Phi(z) \cap (X\backslash S) \neq \varnothing\}$ and $\{z \in Z : \Phi(z) \subset X\backslash S\}$ are open in $Z$.

**Theorem 1.** *Let $(X, \mathcal{S})$ be a normally supercompact space and $Z$ an arbitrary space. Then every $\mathcal{S}$-continuous set-valued map $\Phi\colon Z \to X$ has a single-valued selection provided all $\Phi(z)$, $z \in Z$, are $\mathcal{S}$-convex closed subsets of $X$. More generally, if $A \subset Z$ is closed and every map from $A$ to $X$ can be extended to a map from $Z$ to $X$, then every selection for $\Phi|A$ is extendable to a selection for $\Phi$.*

**Corollary 1.** *Let $\Phi\colon Z \to X$ be an $\mathcal{S}$-continuous set-valued map such that each $\Phi(z) \subset X$ is closed, where $X$ is a space with a binary normal closed subbase $\mathcal{S}$ and $Z$ arbitrary. Then the map $\Psi\colon Z \to X$, $\Psi(z) = I_{\mathcal{S}}\big(\Phi(z)\big)$, has a continuous selection.*

A map $f\colon X \to Y$ is invertible if for any space $Z$ and a map $g\colon Z \to Y$ there exists a map $h\colon Z \to X$ with $f \circ h = g$. If $X$ has a closed subbase $\mathcal{S}$, we say $f\colon X \to Y$ is *$\mathcal{S}$-open* provided $f(X\backslash S) \subset Y$ is open for every $S \in \mathcal{S}$. Theorem 1 yields next corollary.

**Corollary 2.** *Let $X$ be a space possessing a binary normal closed subbase $\mathcal{S}$. Then every $\mathcal{S}$-convex $\mathcal{S}$-open surjection $f\colon X \to Y$ is invertible.*

Another corollary of Theorem 1 is a specification of Ivanov's results [7] (see also [5] and [6]). Here, a map $f\colon X \to Y$ is $\mathcal{A}$-soft, where $\mathcal{A}$ is a class of spaces, if for any $Z \in \mathcal{A}$, its closed subset $A$ and any two maps $k\colon Z \to Y$, $h\colon A \to X$ with $f \circ h = k|A$ there exists a map $g\colon Z \to X$ extending $h$ such that $f \circ g = k$. When $\mathcal{A}$ is the family of all (0-dimensional) paracompact spaces, then $\mathcal{A}$-soft maps are called (0-)soft [11].

**Corollary 3.** *Let $\mathcal{A}$ be a given class of spaces and $X$ be an absolute extensor for all $Z \in \mathcal{A}$. If $X$ has a binary normal closed subbase $\mathcal{S}$, then any $\mathcal{S}$-convex $\mathcal{S}$-open surjection $f\colon X \to Y$ is $\mathcal{A}$-soft.*

Theorem 1 is also applied to establish the following proposition:

**Proposition 1.** *Let $X$ be a $\kappa$-metrizable (resp., $\kappa$-metrizable and connected) compactum with a normal binary closed subbase $\mathcal{S}$. Then every open $\mathcal{S}$-convex surjection $f\colon X \to Y$ is a zero-soft (resp., soft) map.*

**Corollary 4** (see [5,6]). *Let $X$ be a $\kappa$-metrizable (resp., $\kappa$-metrizable and connected) compactum. Then $\lambda f\colon \lambda X \to \lambda Y$ is a zero-soft (resp., soft) map for any open surjection $f\colon X \to Y$.*

## 2 Proof of Theorem 1 and Corollaries 1–3

Recall that a set-valued map $\Phi : Z \to X$ is lower semi-continuous (br., lsc) if the set $\{z \in Z : \Phi(z) \cap U \neq \varnothing\}$ is open in $Z$ for any open $U \subset X$. $\Phi$ is upper semi-continuous (br., usc) provided that the set $\{z \in Z : \Phi(z) \subset U\}$ is open in $Z$ whenever $U \subset X$ is open. Upper semi-continuous and compact-valued maps are called usco maps. If $\Phi$ is both lsc and usc, it is said to be continuous. Obviously, every continuous set-valued map $\Phi : Z \to X$ is $\mathcal{S}$-continuous, where $\mathcal{S}$ is a binary closed normal subbase for $X$. Let $C(X, Y)$ denote the set of all (continuous single-valued) maps from $X$ to $Y$.

*Proof of Theorem 1.* Suppose $X$ has a binary normal closed subbase $\mathcal{S}$ and $\Phi \colon Z \to X$ is a set-valued $\mathcal{S}$-continuous map with closed $\mathcal{S}$-convex values. Let $A \subset Z$ be a closed set such that every $f \in C(A, X)$ can be extended to a map $\bar{f} \in C(Z, X)$. Fix a selection $g \in C(A, X)$ for $\Phi|A$ and its extension $\bar{g} \in C(Z, X)$. By [9, Theorem 1.5.18], there exists a (continuous) map $\xi : X \times \exp X \to X$, defined by

$$\xi(x, F) = \bigcap\{I_{\mathcal{S}}(\{x, a\}) : a \in F\} \cap I_{\mathcal{S}}(F),$$

where $\exp X$ is the space of all closed subsets of $X$ with the Vietoris topology. This map has the following properties for any $F \in \exp X$: (i) $\xi(x, F) = x$ if $x \in I_{\mathcal{S}}(F)$; (ii) $\xi(x, F) \in I_{\mathcal{S}}(F)$, $x \in X$. Because each $\Phi(z)$, $z \in Z$, is a closed $\mathcal{S}$-convex set, $I_{\mathcal{S}}(\Phi(z)) = \Phi(z)$, see [9, Theorem 1.5.7]. So, for all $z \in Z$ we have $h(z) = \xi\big(\bar{g}(z), \Phi(z)\big) \in \Phi(z)$. Therefore, we obtain a map $h \colon Z \to X$ which is a selection for $\Phi$ and $h(z) = g(z)$ for all $z \in A$. It remains to show that $h$ is continuous. We can show that the subbase could be supposed to be invariant with respect to finite intersections. Because $\xi$ is continuous, this would imply continuity of $h$. But instead of that, we follow the arguments from the proof of [9, Theorem 1.5.18].

Let $z_0 \in Z$ and $x_0 = h(z_0) \in W$ with $W$ being open in $X$. We may assume that $W = X \backslash S$ for some $S \in \mathcal{S}$. Because $x_0$ is the intersection of a subfamily of the binary family $\mathcal{S}$, there exists $S^* \in \mathcal{S}$ containing $x_0$ and disjoint from $S$. Since $\mathcal{S}$ is normal, there exist $S_0, S_1 \in \mathcal{S}$ such that $S \subset S_1 \backslash S_0$, $x_0 \in S^* \subset S_0 \backslash S_1$ and $S_0 \cup S_1 = X$. Hence, $x_0 \in (X \backslash S_1) \cap \Phi(z_0)$. Because $\Phi$ is $\mathcal{S}$-continuous, there exists a neighborhood $O_1(z_0)$ of $z_0$ such that $\Phi(z) \cap (X \backslash S_1) \neq \varnothing$ for every $z \in O_1(z_0)$. Observe that $\bar{g}(z_0) \in X \backslash S_1$ provided $\Phi(z_0) \cap S_1 \neq \varnothing$, otherwise $x_0 \in I_{\mathcal{S}}(\{\bar{g}(z_0), a\}) \subset S_1$, where $a \in \Phi(z_0) \cap S_1$. Consequently, we have two possibilities: either $\Phi(z_0) \subset X \backslash S_1$ or $\Phi(z_0)$ intersects both $S_1$ and $X \backslash S_1$. In the first case there exists a neighborhood $O_2(z_0)$ with $\Phi(z) \subset X \backslash S_1$ for all $z \in O_2(z_0)$, and in the second one take $O_2(z_0)$ such that $\bar{g}\big(O_2(z_0)\big) \subset X \backslash S_1$ (recall that in this case $\bar{g}(z_0) \in X \backslash S_1$). In both cases let $O(z_0) = O_1(z_0) \cap O_2(z_0)$. Then, in the first case we have $h(z) \in \Phi(z) \subset X \backslash S_1 \subset X \backslash S$ for every $z \in O(z_0)$. In the second case let $a(z) \in \Phi(z) \cap (X \backslash S_1)$, $z \in O(z_0)$. Consequently, $h(z) \in I_{\mathcal{S}}(\{\bar{g}(z), a(z)\}) \subset X \backslash S_1 \subset S_0 \subset X \backslash S$ for any $z \in O(z_0)$. Hence, $h$ is continuous.

When the set $A$ is a point $a$ we define $g(a)$ to be an arbitrary point in $\Phi(a)$ and $\bar{g}(x) = g(a)$ for all $x \in X$. Then the above arguments provide a selection for $\Phi$. $\square$

*Proof of Corollary 1.* Since each $\Psi(z)$ is $\mathcal{S}$-convex, by Theorem 1 it suffices to show that $\Psi$ is $\mathcal{S}$-continuous. To this end, suppose that $F_0 \in \mathcal{S}$ and $\Psi(z_0)\cap(X\backslash F_0) \neq \varnothing$ for some $z_0 \in Z$. Then $\Phi(z_0) \cap (X\backslash F_0) \neq \varnothing$, for otherwise $\Phi(z_0) \subset F_0$ and $\Psi(z_0)$, being intersection of all $F \in \mathcal{S}$ containing $\Phi(z_0)$, would be contained in $F_0$. Since $\Phi$ is $\mathcal{S}$-continuous, there exists a neighborhood $O(z_0) \subset Z$ of $z_0$ such that $\Phi(z)\cap(X\backslash F_0) \neq \varnothing$ for all $z \in O(z_0)$. Consequently, $\Psi(z) \cap (X\backslash F_0) \neq \varnothing$, $z \in O(z_0)$.

Suppose now that $\Psi(z_0) \subset X\backslash F_0$. Then $\Psi(z_0) \cap F_0 = \varnothing$, so there exists $S_0 \in \mathcal{S}$ with $\Phi(z_0) \subset S_0$ and $S_0 \cap F_0 = \varnothing$ (recall that $\mathcal{S}$ is binary). Since $\mathcal{S}$ is normal, we can find $S_1, F_1 \in \mathcal{S}$ such that $S_0 \subset S_1\backslash F_1$, $F_0 \subset F_1\backslash S_1$ and $F_1 \cup S_1 = X$. Using again that $\Phi$ is $\mathcal{S}$-continuous to choose a neighborhood $U(z_0) \subset Z$ of $z_0$ with $\Phi(z) \subset X\backslash F_1 \subset S_1$ for all $z \in U(z_0)$. Hence, $\Psi(z) \subset S_1 \subset X\backslash F_0$, $z \in U(z_0)$, which completes the proof. □

*Proof of Corollary 2.* Let $X$ possess a binary normal closed subbase $\mathcal{S}$, $f\colon X \to Y$ be an $\mathcal{S}$-open $\mathcal{S}$-convex surjection, and $g : Z \to Y$ be a map. Since $f$ is both $\mathcal{S}$-open and closed (recall that $X$ is compact as a space with a binary closed subbase), the map $\phi\colon Y \to X$, $\phi(y) = f^{-1}(y)$, is $\mathcal{S}$-continuous and $\mathcal{S}$-convex valued. So is the map $\Phi = \phi \circ g : Z \to X$. Then, by Theorem 1, $\Phi$ admits a continuous selection $h : Z \to X$. Obviously, $g = f \circ h$. Hence, $f$ is invertible. □

*Proof of Corollary 3.* Suppose $X$ is a compactum with a normal binary closed subbase $\mathcal{S}$ such that $X$ is an absolute extensor for all $Z \in \mathcal{A}$. Let us show that every $\mathcal{S}$-open $\mathcal{S}$-convex surjection $f\colon X \to Y$ is $\mathcal{A}$-soft. Take a space $Z \in \mathcal{A}$, its closed subset $A$ and two maps $k\colon Z \to Y$, $h\colon A \to X$ such that $k|A = f \circ h$. Then $h$ can be continuously extended to a map $\bar{h}\colon Z \to X$. Moreover, the set-valued map $\Phi : Z \to X$, $\Phi(z) = f^{-1}(k(z))$, is $\mathcal{S}$-continuous and has $\mathcal{S}$-convex values. Hence, by Theorem 1, there is a selection $g\colon Z \to X$ for $\Phi$ extending $h$. Then $f \circ g = k$. So, $f$ is $\mathcal{A}$-soft. □

## 3 Proof of Proposition 1 and Corollary 4

*Proof of Proposition 1.* According to Corollary 3, it suffices to show that $X$ is a Dugundji space (resp., an absolute retract) provided $X$ is a $\kappa$-metrizable (resp., $\kappa$-metrizable and connected) compactum with a normal binary closed subbase $\mathcal{S}$ (recall that the class of Dugundji spaces coincides with the class of compact absolute extensors for 0-dimensional spaces, see [8]). To this end, we follow the arguments from the proof of [12, Proposition 3.2]. Suppose first that $X$ is a $\kappa$-metrizable compactum with a normal binary closed subbase $\mathcal{S}$. Consider $X$ as a subset of a Tyhonoff cube $\mathbb{I}^\tau$. Then, by [10] (see also [12] for another proof), there exists a function $\mathrm{e}\colon \mathcal{T}_X \to \mathcal{T}_{\mathbb{I}^\tau}$ between the topologies of $X$ and $\mathbb{I}^\tau$ such that:

(e1) $\mathrm{e}(\varnothing) = \varnothing$ and $\mathrm{e}(U) \cap X = U$ for any open $U \subset X$;

(e2) $\mathrm{e}(U) \cap \mathrm{e}(V) = \varnothing$ for any two disjoint open sets $U, V \subset X$.

Consider the set valued map $r\colon \mathbb{I}^\tau \to X$ defined by

$$r(y) = \bigcap\{I_{\mathcal{S}}(\overline{U}) : y \in e(U), U \in \mathcal{T}_X\} \text{ if } y \in \bigcup\{e(U) : U \in \mathcal{T}_X\} \qquad (1)$$

$$\text{and } r(y) = X \text{ otherwise,}$$

where $\overline{U}$ is the closure of $U$ in $X$. According to condition (e2), the system $\gamma_y = \{U \in \mathcal{T}_X : y \in e(U)\}$ is linked for every $y \in \mathbb{I}^\tau$. Consequently, $\omega_y = \{S \in \mathcal{S} : \overline{U} \subset S$ for some $U \in \gamma_y\}$ is also linked. This implies $r(y) = \bigcap\{S : S \in \omega_y\} \neq \varnothing$ because $\mathcal{S}$ is binary.

**Claim.** *$r(x) = \{x\}$ for every $x \in X$.*

Suppose there is another point $z \in r(x)$. Then, by normality of $\mathcal{S}$, there exist two elements $S_0, S_1 \in \mathcal{S}$ such that $x \in S_0 \backslash S_1$, $z \in S_1 \backslash S_0$ and $S_0 \cup S_1 = X$. Choose an open neighborhood $V$ of $x$ with $\overline{V} \subset S_0 \backslash S_1$. Observe that $x \in \mathrm{e}(V)$, so $z \in I_{\mathcal{S}}(\overline{V}) \subset S_0$, a contradiction.

Finally, we can show that $r$ is upper semi-continuous. Indeed, let $r(y) \subset W$ with $y \in \mathbb{I}^\tau$ and $W \in \mathcal{T}_X$. Then there exist finitely many $U_i \in \mathcal{T}_X$, $i = 1, 2, .., k$, such that $y \in \bigcap_{i=1}^{i=k} e(U_i)$ and $\bigcap_{i=1}^{i=k} I_{\mathcal{S}}(\overline{U}_i) \subset W$. Obviously, $r(y') \subset W$ for all $y' \in \bigcap_{i=1}^{i=k} e(U_i)$. So, $r$ is an usco retraction from $\mathbb{I}^\tau$ onto $X$. According to [1], $X$ is a Dugundji space.

Suppose now, that $X$ is connected. By [9], any set of the form $I_{\mathcal{S}}(F)$ is $\mathcal{S}$-convex, so is each $r(y)$. According to [9, Corollary 1.5.8], all closed $\mathcal{S}$-convex subsets of $X$ are also connected. Hence, the map $r$, defined by (1), is connected-valued. Consequently, by [1], $X$ is an absolute extensor in dimension 1, and there exists a map $r_1 : \mathbb{I}^\tau \to \exp X$ with $r_1(x) = \{x\}$ for all $x \in X$, see [2, Theorem 3.2]. On the other hand, since $X$ is normally supercompact, there exists a retraction $r_2$ from $\exp X$ onto $X$, see [9, Corollary 1.5.20]. Then the composition $r_2 \circ r_1 \colon \mathbb{I}^\tau \to X$ is a (single-valued) retraction. So, $X \in AR$. □

*Proof of Corollary 4.* It is well known that $\lambda$ is a continuous functor preserving open maps, see [3]. So, $\lambda X$ is $\kappa$-metrizable. Moreover, $\lambda X$ is connected if so is $X$. On the other hand, the family $\mathcal{S} = \{F^+ : F \text{ is closed in } X\}$, where $F^+ = \{\eta \in \lambda X : F \in \eta\}$, is a binary normal subbase for $\lambda X$. Observe that $\lambda f$ is $\mathcal{S}$-convex because $(\lambda f)^{-1}(\nu) = \bigcap\{f^{-1}(H)^+ : H \in \nu\}$ for every $\nu \in \lambda Y$. Then, Proposition 1 completes the proof. □

The next proposition shows that the statements from Proposition 1 and Corollary 4 are actually equivalent. At the same time it provides more information about validity of Corollary 3.

**Proposition 2.** *For any class $\mathcal{A}$ the following statements are equivalent:*

- *(i) If $X$ is a compactum possessing a normal binary closed subbase $\mathcal{S}$, then any open $\mathcal{S}$-convex surjection $f\colon X \to Y$ is $\mathcal{A}$-soft.*
- *(ii) The map $\lambda f\colon \lambda X \to \lambda Y$ is $\mathcal{A}$-soft for any compactum $X$ and any open surjection $f\colon X \to Y$.*

*Proof.* $(i) \Rightarrow (ii)$ Let $X$ be a compactum and $f\colon X \to Y$ be an open surjection. It is easily seen that $\lambda f$ is an open surjection too. We already noted that $\mathcal{S} = \{F^+ : F \subset X \text{ is closed}\}$ is a normal binary closed subbase for $\lambda X$ and $\lambda f$ is a $\mathcal{S}$-convex and open map. Hence, by (i), $\lambda f$ is $\mathcal{A}$-soft.

$(ii) \Rightarrow (i)$. Suppose $X$ is a compactum possessing a normal binary closed subbase $\mathcal{S}$, and $f\colon X \to Y$ is an $\mathcal{S}$-convex open surjection. To show that $f$ is $\mathcal{A}$-soft, take a space $Z \in \mathcal{A}$, its closed subset $A$ and two maps $h : A \to X$, $g\colon Z \to Y$ with $f \circ h = g|A$. So, we have the following diagram, where $i_X$ and $i_Y$ are embeddings defined by $x \to \eta_x$ and $y \to \eta_y$, respectively.

$$\begin{array}{ccccc} A & \xrightarrow{h} & X & \xrightarrow{i_X} & \lambda X \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \lambda f} \\ Z & \xrightarrow{g} & Y & \xrightarrow{i_Y} & \lambda Y \end{array}$$

Since, by (ii), $\lambda f$ is $\mathcal{A}$-soft, there exists a map $g_1\colon Z \to \lambda X$ such that $h = g_1|A$ and $\lambda f \circ g_1 = g$. The last equality implies that $g_1(Z) \subset (\lambda f)^{-1}(Y)$. According to [9, Corollary 2.3.7], there exists a retraction $r\colon \lambda X \to X$, defined by

$$r(\eta) = \bigcap\{F \in \mathcal{S} : F \in \eta\}. \tag{2}$$

Consider now the map $\bar{g} = r \circ g_1\colon Z \to X$. Obviously, $\bar{g}$ extends $h$. Let us show that $f \circ \bar{g} = g$. Indeed, for any $z \in Z$ we have

$$g_1(z) \in (\lambda f)^{-1}(g(z)) = (f^{-1}(g(z)))^+.$$

Since $f$ is $\mathcal{S}$-convex, $I_{\mathcal{S}}\big(f^{-1}(g(z))\big) = f^{-1}(g(z))$, see [9, Theorem 1.5.7]. Hence, $f^{-1}(g(z))$ is the intersection of the family $\{F \in \mathcal{S} : f^{-1}(g(z)) \subset F\}$ whose elements belong to any $\eta \in (\lambda f)^{-1}(g(z))$. It follows from (2) that $r(\eta) \in f^{-1}(g(z))$, $\eta \in (\lambda f)^{-1}(g(z))$. In particular, $\bar{g}(z) \in f^{-1}(g(z))$. Therefore, $f \circ \bar{g} = g$. □

The following corollary follows from Corollary 3 and Proposition 2.

**Corollary 5.** *If $X$ is a compactum with a binary normal closed subbase $\mathcal{S}$ such that $\lambda X$ is an absolute extensor for a given class $\mathcal{A}$, then any open $\mathcal{S}$-convex surjection $f\colon X \to Y$ is $\mathcal{A}$-soft.*

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## References

[1] Dranishnikov A. *Multivalued absolute retracts and absolute extensors in dimnesions* 0 *and* 1. Usp. Mat. Nauk, 1984, **39**, issue 5(239), 241–242 (in Russian).

[2] Fedorchuk V. *Some geometric properties of covariant functors*. Usp. Mat. Nauk, 1984, **39**, issue 5(239), 169–208 (in Russian).

[3] Fedorchuk V., Filippov V. *General Topology: Basic constructions*, Moscow State University, Moscow 1988 (in Russian).

[4] de Groot J. *Supercompactness and superextensions*. Contributions to extension theory of topological structures, Symp. Berlin 1967, Deutscher Verlag Wiss., Berlin, 1969, 89–90.

[5] Ivanov A. *Superextension of openly generated compacta*, Dokl. Akad. Nauk SSSR, 1981, **259**, No. 2, 275–278 (in Russian).

[6] IVANOV A. *Solution of van Mill's problem on the characterization of compacta, the superextension of which are absolute retracts*. Dokl. Akad. Nauk SSSR, 1982, **262**, No. 3, 526–528 (in Russian).

[7] IVANOV A. *Mixers, functors, and soft mappings*. Proceedings of Steklov Inst. of Math., 1993, **193**, No. 3, 139–142.

[8] HAYDON R. *On a problem of Pelczynski: Milutin spaces, Dugundji spaces and* $AE(0-dim)$. Studia Math., 1974, **52**, 23–31.

[9] VAN MILL J. *Supercompactness and Wallman spaces*. Math. Centre Tracts, **85**, Amsterdam 1977.

[10] SHIROKOV L. *External characterization of Dugundji spaces and* $\kappa$*-metrizable compacta*. Dokl. Acad. Nauk SSSR, 1982, **263**, No. 5, 1073–1077 (in Russian).

[11] SHCHEPIN E. *Topology of limit spaces of uncountable inverse spectra*. Russian Math. Surveys, 1976, **315**, 155–191.

[12] VALOV V. *Extenders and* $\kappa$*-metrizable compacta*. Math. Notes, 2011, **89**, No. 3, 331–341.

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# Minimal m-handle decomposition of three-dimensional handlebodies

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**Abstract.** For the 3-dimensional handlebody we build an m-handle decomposition with minimal number of handles and prove a criterion of minimality. It is proved that two functions can be connected by a path in the m-function space without inner critical points on the solid torus if they have the same number of critical points of each index.

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Let $M$ be a three-dimensional handlebody, i. e. a closed bounded domain in the Euclidean space whose boundary is a smooth closed surface $F = \partial M$. In this paper, we consider m-functions without inner critical points on $M$. For such functions, the restriction of the function on the boundary is a Morse function. The index of critical points of the Morse function is defined as the index of the quadratic form of Hesse (Hessian matrix of this form consists of the second partial derivatives at the critical point). In addition, the direction of the gradient field is given by the sign ($\epsilon = \pm 1$) at the critical point. The index of a critical point of an m-function is the pair (the index of the restriction to the boundary, the number of $\epsilon$). $\epsilon = -1$ if the gradient field is directed to inside of the manifold and $\epsilon = +1$ if it is directed outside. Note that, similar to a Morse function on a closed manifold, m-functions exist and form an open set in the space of all functions.

V. Sharko [1] and S. Maksymenko [2] proved that two Morse functions can be connected by a path in the space of Morse functions on a closed two-dimensional manifold if and only if the functions have the same number of critical points of each index.

Topological properties of the m-functions and the m-handle decomposition were investigated in [3, 4]. In [5] using m-handles the authors give a criterion of the existence of a path between two m-functions on the three-dimensional body without inner critical points.

The aim of this work is to construct an m-handle decomposition of the handlebody with a minimal number of m-handles of each index, to study the conditions when the decomposition is minimal, and to apply the minimal handle decompositions for the homotopy classification of m-functions without inner critical points on the solid torus.

## 1 m-handle decompositions

Let us start with the handle decomposition of a closed surface $F$. A handle of index $\lambda$ is the product $H^\lambda = D^\lambda \times D^{2-\lambda}$. The curve $\partial D^\lambda \times D^{2-\lambda}$ is called a gluing curve, and $D^\lambda \times \partial D^{2-\lambda}$ is called an inner curve. Thus, the gluing curve is 1) $\emptyset$ for a handle of index 0, 2) a pair of segments for a handle of index 1, and 3) a circle for a handle of index 2. It is known from Morse Theory that if a function $g : F \to \mathbb{R}$ has one critical point on the interval $[y, z]$ in the inner segment and only one critical point of index $\lambda$ takes this value, then $g^{-1}(z) \cong g^{-1}(y) \cup_\varphi H^\lambda$ is obtained from $g^{-1}(y)$ by attaching a handle of index $\lambda$ for some embedding $\varphi : \partial D^\lambda \times D^{2-\lambda} \to \partial g^{-1}(y)$.

m-handles can be obtained from ordinary handles by the multiplication with the interval $[0, 1]$. Denote them $H^\lambda_+$ and $H^\lambda_-$. Thus, $H^\lambda_+ \cong H^\lambda_- \cong D^\lambda \times D^{2-\lambda} \times [0, 1]$.

The boundary $\partial H^\lambda_-$ of a handle of index $(\lambda, -1)$ is divided into three parts:

1) the outside region $D^\lambda \times D^{2-\lambda} \times 0$,

2) the attaching region $\partial D^\lambda \times D^{2-\lambda} \times [0, 1]$,

3) the inside region $D^\lambda \times \partial D^{2-\lambda} \times [0, 1] \cup D^\lambda \times D^{2-\lambda} \times 1$.

The boundary $\partial H^\lambda_+$ of a handle of index $(\lambda, +1)$ is divided into two parts:

1) the outside region $D^\lambda \times D^{2-\lambda} \times 1$,

2) the attaching region $\partial(D^\lambda \times D^{2-\lambda}) \times [0, 1] \cup D^\lambda \times D^{2-\lambda} \times 0$.

As a result of m-handle attaching, the boundary consists of inside and outside regions. Their common boundary is called a corner of the manifold. The attaching region of next handles is embedded in the inside region. Moreover, $\partial D^\lambda \times D^{2-\lambda} \times 0$ is embedded in the corner for handles of index $(\lambda, -1)$ and $\partial D^\lambda \times D^{2-\lambda} \times 1$ is embedded in the corner for handles of index $(\lambda, +1)$. Thus, outside regions of m-handles give a handle decomposition of the surface $F$. Moreover, the union of attaching regions is equal to the union of inside regions.

Like regular handle decompositions, one can perform the following operations with m-handles:

1. A permutation of handles – if two handles are disjoint, they can be attached in any order.

2. An isotopy of the attaching map of handles, if one of $(1, \pm 1)$ handle slides over another $(1, \pm 1)$ handle, in this case we say that it is added to this handle.

3. A reduction of pairs of additional handles – if a handle of index $(1, -1)$ intersects a $(0, -1)$- or $(2, 1)$-handle along a 2-disk, then the pair of handles can be reduced (we can build another handle decomposition without these two handles). Similarly, a pair consisting of a $(1, +1)$-handle and a $(0, +1)$- or a $(2, +1)$-handle whose intersection is an interval can be reduced. The inverse operation to the reduction is the introduction of pairs of additional handles.

Note that m-handles will be additional if they have the same sign of $\epsilon$ and that additional handles are on the edge of their limits.

A criterion for homotopy equivalence of functions was proved in [5]:

**Theorem 1.** *Two functions on the three-dimensional handlebody are homotopy equivalent if and only if they have the same number of handles for each index and the m-handle decomposition of one manifold can be obtained from the m-handle decomposition of another one using isotopy, permutations, additions, reductions and the introduction of pairs of additional handles.*

Our next task will be for an arbitrary handle decomposition using operations 1) − 3) to build a minimal handle decomposition and investigate its topological properties.

## 2 Minimal m-handle decomposition

In the beginning, from an arbitrary handle decomposition we construct a decomposition with minimal number of handles for each index. Since the boundary of a manifold is connected, then for each (0, ± 1)-handle, except the first one, there exists an additional (1, ± 1)-handle. If they are of the same sign, then this pair of handles can be reduced. Two $(0, -1)$-handles can not be connected by a $(1, +1)$-handle. However, a $(0, +1)$-handle can be connected by a $(1, -1)$-handle with other handles. In this case, this pair of handles is equivalent to a simple 1-handle on a 3-manifold. Similarly, $(2, \pm 1)$-handles, exept one $(2, +1)$-handle, can be connected by $(1, \pm 1)$-handles. Pairs of the same sign are reduced, and the pair of $(1, +1)$- and $(2, -1)$-handles is equivalent to a simple 2-handle. If the pair of $(0, +1)$- and $(1, -1)$-handles forms a simple 1-handle, the unglued part of the border of the $(0, +1)$-handle in the inside region is on the border of the surface. Then every $(1, +1)$-handle, with attached at least one end to the boundary components, can be made additional $(0, 1)$-handles. We do the same in the case of a simple 2-handle. Thus, we construct an m-handle decomposition with one $(0, -1)$-handle, without $(0, +1)$- and $(2, -1)$-handles and one $(2, +1)$-handle. Obviously, this decomposition has no additional pairs of handles. A handle decomposition that does not contain pairs of additional handles or handles which can be made additional after an isotopy, is called minimal.

**Theorem 2.** *A handle decomposition is minimal if and only if it contains by one* $(0, -1)$- *and* $(2, +1)$-*handles and no* $(0, +1)$- *and* $(2, -1)$-*handles.*

*Proof. Necessity.* It follows from the previous discussion that if a handle decomposition has more than one $(0, -1)$- or $(2, +1)$-handle or has $(0, +1)$- and $(2, -1)$-handles, all these handles can be reduced. At the same time as the boundary of a manifolds is compact and the restriction of any function on the boundary has a minimum point (of index 0) and a maximum point (of index 2), so the corresponding handle decomposition has handles of index 0 and 2 and the m-handle decomposition has handles of indexes $(0, -1)$ and $(2, +1)$.

*Sufficiency.* Let an m-handle decomposition have by one $(0, -1)$- and $(2, +1)$-handles and no $(0, +1)$- and $(2, -1)$-handles. Since $(0, -1)$- and $(2, +1)$-handles can not be reduced, they are not additional for other handles. The remaining $(1, \pm 1)$-handles can not be reduced because they can not have additional handles. Thus, the handle decomposition is minimal. ☐

## 3 m-functions on the solid torus

On the solid torus we fix a parallel $u$, which is a curve on the boundary that defines the generators of the fundamental group of the solid torus. We also fix a meridian $v$ which is a curve on the boundary that intersects transversally one parallel at one point and is the boundary of a 2-disk on the solid torus. We fix the orientation of these curves.

**Theorem 3.** *Two m-functions without inner critical points on the solid torus can be connected by a m-function space without inner critical points if they have the same number of critical points of each index.*

*Proof.* *Necessity* follows from Theorem 1.

*Sufficiency.* Let the functions have the same number of points of each index. Construct from them a minimal m-handle decomposition. Theorem 2 implies that such a decomposition has four m-handles whose indexes are $(0,-1)$, $(1,-1)$, $(1,1)$ and $(2,+1)$.

Consider the union of $(0,1)$- and $(1,-1)$-handles for the first function. Let $L$ be the intersection of the boundary of their union with the inside region of the union of $(0,1)$- and $(1,1)$-handles. Two components of the boundary $\partial L$ are homotopic to the meridian $u$ in the solid torus. Let $w_1$ be one of the two components. We choose its orientation to be parallel to the meridian. Then $[w_1] = [u] + n_1[v]$ in the one-dimensional homology group of the torus. For the second function, by analogy, we have $[w_2] = [u] + n_2[v]$.

Since $L$ is homeomorphic to a cylinder $S^1 \times [0,1]$, and the attaching points of $(1,+1)$-handle are on different bases of the cylinder (because after removing attaching area of this handle from $L$ it should remain a two-dimensional disk) an isotopy of the attaching point of a $(1,1)$-handle can ensure that the intersection of the $(1,-1)$- and $(1,+1)$-handles is empty.

Then we change the order of attaching handles so that a $(1,+1)$-handle be the first attached one. The inside region of the union of $(0,-1)$- and $(1,1)$-handles is homeomorphic to two 2-disks the boundaries of which $\gamma_1$ and $\gamma_2$ are homotopic to the meridian $v$ on the torus. Slipping one of the two attaching points of $(1,-1)$-handle $n_2 - n_1$ times in one of two directions along $\gamma_1$ and $\gamma_2$, achieve that $[w_1] = [w_2]$. We have that the curves of $w_1$ and $w_2$ are isotopic. Then the m-handle decompositions for two functions are isotopic, too. Applying Theorem 1 completes the proof. □

## 4 Conclusion

The m-handle decomposition expansion with the minimum number of handles has been built and a criterion of minimality has been proved for the m-handle decomposition of the 3-dimensional handlebody. This construction allowed us to prove that two functions can be connected by a path in the m-function space without inner critical points on the solid torus if and only if they have the same number of critical points of each index.

The authors expect that the minimal m-handle decomposition can be used for homotopy classification of m-functions on other handlebodies. However, in this case one may have a lot of non-isotopic minimal m-handle decompositions.

## References


[1] Sharko V. *Functions on the surfaces, I.* Some contemporary problems in mathematics. Kiev: Proc. In-t of Math. NASU, 1998, **25**, 408–434.

[2] Maksymenko S. I. *Path-components of Morse mappings spaces of surface.* Comment. Math. Helv., 2005, **80**, 655–690.

[3] Hajduk B. *Minimal m-functions.* Fund. math., 1981, **110**, 178–200.

[4] Jankowski A., Rubinsztein R. *Functions with non-degenerated critical points on manifolds with boundary.* Comm. Math., 1972, **16**, 99–112.

[5] Prishlyak O. O., Prishlyak K. O., Vyatchaninova O. N. *Homotopic classification of noncritical m-functions on 3-disk.* J. Num. Appl. Math., 2012, **110**, 113–119.



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# Examples of quasitopological groups

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**Abstract.** In this paper we construct several examples of completely regular submetrizable quasitopological groups with slightly different combinations of properties, in particular, a countable quasitopological group $G$ with countable $\pi$-weight, countable tightness, countable $\delta$-character, but not first-countable, and a countable quasitopological group $P$ with countable $\pi$-weight, countable tightness, but of uncountable $\delta$-character.

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## 1 Introduction

All spaces considered below are assumed to be Tychonoff. In terminology and notations we follow [7] and [8]. A space is submetrizable if its topology contains a metrizable topology.

A group $G$ with a topology $\mathcal{T}$ is a semitopological (paratopological, respectively) group if the multiplication is separately continuous (jointly continuous, respectively).

If $G$ is a semitopological and the inverse operation $x \to x^{-1}$ is continuous, then $G$ is said to be a quasitopological group.

Recall that a $\pi$-base of a space $X$ is a family $\beta$ of non-empty open subsets of $X$ such that every open non-empty set $U$ contains some member of $\beta$. A $\pi$-base of a space $X$ at a point $x \in X$ is a family $\beta$ of non-empty open subsets of $X$ such that every open neighborhood of $x$ contains at least one element of $\beta$.

We will say that the $\delta$-character of a space $X$ at a point $x \in X$ is countable, if there exists a sequence $\gamma = \{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of $X$ converging to $x$.

## 2 The topologies $\mathcal{T}^*$, $\mathcal{T}^{**}$ and $\mathcal{T}^{\triangle}$ on $\mathbb{R}^2$

Let $\mathbb{R}$ be the usual topological group of reals. Consider the group $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ with the Euclidean topology $\mathcal{T}_E$.

For any $(x, y) \in G$ we put:

– $V((x, y), r) = \{(x, y)\} \cup \{(u, v) : u \neq x, |u - x| < r, 0 < (v - y)/(u - x) < r\}$, where $0 < r$;

– $W((x, y), r) = \{(x, y)\} \cup \{(u, v) : u \neq x, |u - x| < r, -r < (v - y)/(u - x) < r\}$, where $0 < r$;

– $\mathcal{S}((x,y),r) = \{U \in \mathcal{T}_E : U, -U + (x,y) = U - (x,y), x < u < x+r\} \subset U \subseteq W((x,y),r)\}$, where $0 < r$;

– $\mathcal{S}(x,y) = \cup\{\mathcal{S}((x,y),r) : 0 < r < \infty\}$.

In particular, $U \in \mathcal{S}(0,0)$ if and only if $U$ is open in $\mathbb{R}^2$, $-U = U$ and $\{(t,0) : 0 < t < r\} \subseteq U \subseteq W((0,0),r)$ for some $r > 0$. In this case, since $U = -U$, we have $\{(t,0) : -r < t < 0\} \subseteq U$ too.

By construction, the sets $V(x,y) \setminus \{(x,y)\}$ and $W(x,y) \setminus \{(x,y)\}$ are open in $\mathbb{R}^2$.

Now, we put $O((x,y),r,U) = V((x,y),r) \cup U$, $\mathcal{B}^*(x,y) = \{O((x,y),r,U) : U \in \mathcal{S}((x,y),r), 0 < r < \infty\}$ and $\mathcal{B}^* = \cup\{\mathcal{B}^*(x,y) : (x,y) \in \mathbb{R}^2\}$.

The family $\mathcal{B}^*$ is an open base of a new topology $\mathcal{T}^*$ on the set $\mathbb{R}^2$. In particular, $(\mathbb{R}^2, \mathcal{T}^*)$ is a submetrizable space, and hence, any compact subset of $(\mathbb{R}^2, \mathcal{T}^*)$ is metrizable.

A sequence $s = \{s_n : n \in \mathbb{N}\}$ of real numbers is called an $r$-basic sequence if $0 < -s_{n+1} < -s_n < n^{-1}$ and $ns_n > -r$ for each $n \in \mathbb{N}$. Consider an $r$-basic sequence $s = \{s_n : n \in \mathbb{N}\}$. We construct the continuous function $h_s : [0,1] \to \mathbb{R}$, where $h_s(x) = (s_{n+1} - s_n)((n+1)^{-1} - n^{-1})(x - n^{-1}) + s_n$ for each $x \in [(n+1)^{-1}, n^{-1}]$ and $n \in \mathbb{N}$. We put $D^+((x,y),r,s) = \{\{(u,v) : u - x < r, x + (1+n)^{-1} \leq u < x + n^{-1}, h(x) < v \leq y\} : n \in \mathbb{N}\}$, $D^-((x,y),s) = -D^+((x,y),s)$ and $D((x,y),s) = D^+((x,y),s) \cup D^-((x,y),s)$.

Now we put $H((x,y),r,s) = V((x,y),r) \cup D((x,y),r,s)$ for each $r > 0$ and each $r$-basic sequence $s = \{s_n : n \in \mathbb{N}\}$.

**Property 2.1**. The group $\mathbb{R}^2$ with the topology $\mathcal{T}^*$ is a quasitopological group.

*Proof.* By construction, $O((0,0),r,U) = -O((x,y),r,U)$, $O((0,0),r,U) + (x,y) = O((x,y),r,U) + (x,y)$ and $U + (x,y) \in \mathcal{S}((x,y),r)$ for all $U \in \mathcal{S}((0,0),r)$ and $0 < r < \infty\}$.

**Property 2.2**. The family $\mathcal{H}(x,y) = \{H((x,y),r,s)) : 0 < r \leq 1, s$ *is an $r$-basic sequence*$\}$ is an open base of the space $(\mathbb{R}^2, \mathcal{T}^*)$ at the point $(x,y)$.

*Proof.* Fix $O((0,0),r,U) = V((0,0),r) \cup U$, where $r > 0$ and $U \in \mathcal{S}((x,y),r)$. Let $k$ be the first natural number for which $1/k < r$. We put $r_1 = 1/k$. The set $U$ is open and the sets $F_n = \{(t,0) : 1/(n+1) \leq t \leq 1/n\}$ are compact in the space $(\mathbb{R}^2, \mathcal{T}_E)$. For each $n \geq k$ we have $F_n \subseteq U$. Hence, there exists $\delta_n > 0$ such that $\{(u,v) : 1/(n+1) \leq u \leq 1/n, -\delta_n < v \leq 0\} \subseteq U$. We can assume that $\delta_{n+1} < \delta_n \leq 1/n$ for each $n \geq k$, $\delta_m < 1/m$ for $i < k$ and $\delta = \{\delta_n : n \in \mathbb{N}\}$ is an $r_1$-basic sequence. By construction, $H((x,y),r_1) \subseteq O((x,y),r,U)$ and $H((x,y),r_1,\delta) \in \mathcal{T}_1(0,0)$.

**Property 2.3**. If $r_2 < r_1 \leq 1$, $s = \{s_n : n \in \mathbb{N}\}$ is an $r_1$-basic sequence and $\delta = \{\delta_n : n \in \mathbb{N}\}$ is an $r_2$-basic and $\delta_n < r_n$ for each $n \in \mathbb{N}$, then the closure of the set $H((x,y),r_2,\delta)$ in the space $(\mathbb{R}^2, \mathcal{T}^*)$ is a subset of the set $H((x,y),r_1,s)$.

*Proof.* It is obvious.

**Property 2.4**. The space $(\mathbb{R}^2, \mathcal{T}^*)$ is completely regular.

*Proof.* Fix an $r > 0$, an $r$-basic sequence $s = \{s_n : n \in \mathbb{N}\}$ and the neighborhood $H = H((0,0),r,s)$ of the point $(0,0)$.

Consider the function $f : \mathbb{R}^2 \to [0,1]$, where:
(1) $f((0,0)) = 1$ and $f((-x,-y)) = f((x,y))$ for any point $(x,y) \in \mathbb{R}^2$;
(2) $f((0,y)) = 0$ for each $y \in \mathbb{R} \setminus \{0\}$;
(3) if $(x,y) \in \mathbb{R}^2$ and $x \geq r$, then $f((x,y)) = 0$;
(4) if $(x,y) \in \mathbb{R}^2$, $x > 0$ and $y/x \geq r$, then $f((x,y)) = 0$;
(5) if $(x,y) \in \mathbb{R}^2$, $0 < x < r$ and $y/x \leq r$, then $f((x,y)) = r^{-2}x^{-1}(r-x)(rx-y)$;
(6) if $(x,y) \in \mathbb{R}^2$, $n \in \mathbb{N}$, $(n+1)^{-1} \leq x \leq n^{-1}, x < r$ and $y \leq 0$, then $f((0,y)) = 0$ for $y \leq h_s(x)$ and $f((x,y)) = r^{-1}h_s(x)^{-1}(r-x)(h_s(x) - y)$ for $y > h_s(x)$.

By construction, $f((0,0)) = 1$ and $\mathbb{R}^2 \setminus H = f^{-1}(0)$. Moreover, if $Z = \mathbb{R}^2 \setminus \{(0,y) : y \in \mathbb{R}\}$ is a a subspace of the space $(\mathbb{R}^2, \mathcal{T}_E)$, the function $f|Z : Z \to [0,1]$ is continuous on $Z$. From this fact, the condition $H \subseteq W((x,y),r)$ and the construction (5) it follows that the function $f$ is continuous on the space $(\mathbb{R}^2, \mathcal{T}^*)$. Hence, the space $(\mathbb{R}^2, \mathcal{T}^*)$ is completely regular.

The family $\mathcal{B}^\triangle = \{W((x,y),r) : (x,y) \in \mathbb{R}^2, r > 0\}$ is an open base of the topology $\mathcal{T}^\triangle$ on $\mathbb{R}^2$.

**Property 2.5**. The group $\mathbb{R}^2$ with the topology $\mathcal{T}^\triangle$ satisfies the following conditions:
1. It is a completely regular quasitopological group.
2. It is a first countable space with a countable $\pi$-base.
3. It is a not normal space and has the Baire property.
4. It is submetrizable and Dieudonné complete.
5. It is not a topological group.

Denote by $\mathcal{T}^{**}$ the topology on the space $\mathbb{R}^2$ generated by the open base $\mathcal{B}^{**} = \{U \cup \{(x,y)\} : (x,y) \in \mathbb{R}^2, U \in \mathcal{S}(x,y)\}$. By construction, $\mathcal{T}_E \subseteq \mathcal{T}^\triangle \subset \mathcal{T}^* \subset \mathcal{T}^{**}$. In particular, $(\mathbb{R}^2, \mathcal{T}^{**})$ is a submetrizable space and any compact subset of $(\mathbb{R}^2, \mathcal{T}^{**})$ is metrizable. Consider $Z = \mathbb{R}^2 \setminus \{(0,y) : y \in \mathbb{R}\}$ as a subspace of the space $(\mathbb{R}^2, \mathcal{T}_E)$.

**Property 2.6**. The group $\mathbb{R}^2$ with the topology $\mathcal{T}^{**}$ is a quasitopological group.

*Proof.* By construction, if $U \in \mathcal{T}(0,0)$, then $U = -U$ and $U + (x,y) \in \mathcal{T}(x,y)$.

**Property 2.7**. The space $(\mathbb{R}^2, \mathcal{T}^{**})$ is completely regular. *Proof.* Fix $U \in \mathcal{S}(0,0)$.

The set $U$ is open in $X$ and $F = Z \cap \{(x,0) : -r \leq x \leq r\} \subseteq U$ for some $r > 0$. Since the set $F$ is closed in $Z$ and the space $Z$ is metrizable, there exists a continuous function $g : Z \to [0,1]$ such that $X \setminus U = g^{-1}(0)$ and $F = g^{-1}(1)$. Put $f((0,0)) = 1$, $f((0,y)) = 0$ for any $y \neq 0$ and $f((x,y)) = g((x,y))$ for any $(x,y) \in Z$. By definition of the topology $\mathcal{T}^{**}$, the function $f$ is continuous on $G$, $f((0,0)) = 1$ and $G \setminus (U \cup \{(0,0)\}) = f^{-1}(0)$.

## 3 Some subgroups of the group $(\mathbb{R}^2, \mathcal{T}^*)$

Fix two dense subgroups $A$ and $B$ of the topological group $\mathbb{R}$ in the Euclidean topology.

Put $G = A \times B$. We will consider $G$ as a subspace and subgroup of the quasitopological group $(\mathbb{R}^2, \mathcal{T}^*)$.

**Property 3.1**. $G$ is a quasitopological group.

*Proof.* Use Property 2.1.

**Property 3.2**. The space $G$ is completely regular, not first-countable.

*Proof.* The space $G$ is completely regular, by Property 2.4.

Fix an infinite sequence $\{s^k = \{s_{kn} : n \in \mathbb{N}\} : k \in \mathbb{N}\}$ of $r_n$-basic sequences. For each $n \in \mathbb{N}$ fix a number $s_n$ such that $max\{-n^{-1}, s_{nn}\} < s_n < 0$. Then $s = \{s_n : n \in \mathbb{N}\}$ is a 1-basic sequence. Obviously $(G \cap H((0,0), r_n, s^n)) \setminus H((0,0), 1, s) \neq \emptyset$ for each $n \in \mathbb{N}$. Thus, the space $G$ is not first-countable.

**Property 3.3**. If $indA = indB = 0$, then $indG = 0$.

*Proof.* Assume that $indA = indB = 0$. Fix $r > 0$, $U \in \mathcal{S}((0,0), r)$ and $O((0,0), r, U) = V((0,0), r) \cup U$. Let $G^+ = \{(x,y) \in G : x > 0\}$ be a subspace of the space $(\mathbb{R}^2, \mathcal{T}_E)$, $F = \{(x,y) \in G^+ : 2x \leq r, 0 \leq 2y/x \leq r\}$ and $H = O((0,0), r, U)$. Then $G^+$ is a separable metrizable space, $dimG^+ = 0$, the set $H$ is open in $G^+$, the set $F$ is closed in $G^+$ and $F \subseteq H$. Thus there exists an open-and-closed subset $H_1$ of the space $G^+$ such that $F \subseteq H_1 \subseteq H$. Then the set $H_2 = H_1 \cup (-H_1) \cup \{(0,0)\}$ is an open-and-closed subset of the space $G$ such that $(0,0) \in H_2 \subseteq H$.

**Property 3.4**. $G$ is a space with a countable $\pi$-base.

*Proof.* If $\mathcal{L}$ is a base of $(\mathbb{R}^2, \mathcal{T}_E)$, then $\{U \cap G : U \in \mathcal{L}\}$ is a $\pi$-base of $G$.

**Property 3.5**. $G$ is not a topological group.

*Proof.* Any topological group with a countable $\pi$-base is metrizable (see [7]). Property 3.2 completes the proof.

**Property 3.6**. Any point of $G$ has a countable $\delta$-character in $G$.

*Proof.* The family $\{\{(u,v) \in G : u^2 + v^2 < 2^{-n}, 0 < v < 2^{-n}u\} : n \in \mathbb{N}\}$ is a strong $\pi$-base of the space $G$ at the point $(0,0)$.

**Property 3.7**. If $(a,b) \in G$, then:

1. The subspace $\{a\} \times B$ of $G$ is discrete.
2. The subspace $A \times \{b\}$ of $G$ is separable, metrizable and a subspace of the space $(\mathbb{R}^2, \mathcal{T}_E)$.

**Property 3.8**. If the set $B$ is countable, then the space $G$ is Lindelöf and has a countable network. Moreover, if the groups $A$ and $B$ are countable, the the group $G$ is countable too.

*Proof.* Clearly, $G$ is a union of a countable family of separable metrizable subspaces. Hence, $G$ has a countable network.

**Property 3.9**. If $B = \mathbb{R}$, then the space $G$ is not normal.

*Proof.* The proof is similar to the proof for the Niemytski plane ([8], Example 1.5.9).

**Property 3.10**. The tightness of the space $G$ is countable.

*Proof.* Let $M \subseteq \{(x,y) : x > 0, y < 0\}$ and $(0,0) \in cl_G M$. We put $K = (cl_{(\mathbb{R}^2, \mathcal{T}_E)} M \cap \{(x,0) \in G : x \in \mathbb{R}\}) \setminus \{(0,0)\}$. We have two possible cases.

**Case 1.** $(0,0) \notin cl_{(\mathbb{R}^2, \mathcal{T}_E)} K$.

There exists $k \in \mathbb{N}$ such that $\{(x,y) : x \leq k^{-1}\} \cap K = \infty$. Fix $0 < r < (2k)^{-1}$. and $O > s_i > -(2i)^{-1}r$ for each $i < k$. Since the sets $F_n = \{(u,0) : (n+1)^{-1} \leq u \leq n^{-1}\}$ are compact, there exists a sequence $\{s_n : n \geq k\}$ such that $s_k < -(2n)^{-1}r \leq s_n < s_{n+1} < 0$ and $M \cap \{(u,v) : u - x < r, x + (1+n)^{-1} \leq u < x + n^{-1}, h_s(x) < v \leq 0\} = \emptyset$ for each $n \geq k$.

The sequence $s = \{s_n : n \in \mathbb{N}\}$ is an $r$-basic sequence, $M \cap D^+((x,y), r, s) = M \cap H((x,y), r, s) = \emptyset$. Thus, $(0,0) \notin cl_G M$. Hence, Case 1 is impossible.

**Case 2.** $(0,0) \in cl_{(\mathbb{R}^2, \mathfrak{T}_E)} K$.

For each $n \in \mathbb{N}$ fix a point $(a_n, 0) \in K$ such that $0 < a_n < 2^{-n}$. Since $(a_n, 0) \in K$, there exists a sequence $\{(a_{nm}, b_{nm}) \in M : m \in \mathbb{N}\}$ such that $|a_{nm} - a_n| - b_{nm} < 2^{-n-m}$ for each $m \in \mathbb{N}$. By construction, the set $\{(a_{nm}, b_{nm}) : n, m \in \mathbb{N}\}$ is countable, $L \subseteq M$ and $(0,0) \in cl_G L$. The proof is complete.

A space $X$ is Dieudonné complete if there exists a complete uniformity on the space $X$, i. e the universal uniformity on $X$ is complete [8].

**Property 3.11**. The space $G$ is Dieudonné complete.

*Proof.* Any submetrizable space is Dieudonné complete.

**Property 3.12**. If the space $A \times B$ has the Baire property, then the space $G$ has the Baire property too.

*Proof.* Any dense open subset of $G$ contains a dense open subset of the space $A \times B$ and any dense subset of $A \times B$ is dense in $G$ too.

**Property 3.13**. If $bG$ is a Hausdorff compactification of the space $G$, then the remainder $bG \setminus G$ is not Lindelöf and is not pseudocompact.

*Proof.* A space $X$ is of countable type if every compact subset of $X$ is contained in a compact subset of countable character. M. Henriksen and J. R. Isbel [9] have proved that a space $X$ is of countable type if and only if any remainder of $X$ is Lindelöf. The character of any non-empty compact subset of $G$ in $G$ is uncountable. Therefore, the remainders of $G$ are not Lindelöf.

Since the $\delta$-character of the space $G$ in $G$ is countable at some point, then any remainder of $G$ is not pseudocompact (see [3]).

## 4 Some subgroups of the group $(\mathbb{R}^2, \mathfrak{T}^{**})$

Fix two dense subgroups $A$ and $B$ of the topological group $\mathbb{R}$ in the Euclidean topology.

Denote $P = A \times B$. We consider $P$ as a subspace and subgroup of the quasitopological group $(\mathbb{R}^2, \mathfrak{T}^{**})$.

**Property 4.1**. $G$ is a quasitopological group.

*Proof.* Use Property 2.6.

**Property 4.2**. The space $P$ is completely regular and the $\delta$-character of $P$ is not countable.

*Proof.* From Property 2.7 it follows that the space $P$ is completely regular. If the space $P$ has countable $\delta$-character at the $(0,0)$, then there exists a sequence $S =$

$\{(a_n, b_n) \in P : n \in \mathbb{N}\}$ such that $a_n \cdot b_n \neq 0$ for each $n \in \mathbb{N}$ and $\{(0,0)\} = cl_P S \setminus S$. Then the set $Z \setminus S$ is open in $Z$ and $\{(0,0)\} \cup (Z \setminus S)$ is open in $P$, a contradiction.

**Property 4.3**. If $indA = indB = 0$, then $indP = 0$.

*Proof.* The proof is similar to the proof of Property 3.3.

**Property 4.4**. $P$ is a space with a countable $\pi$-base.

*Proof.* If $\mathcal{L}$ is a base of $(\mathbb{R}^2, \mathcal{T}_E)$, then $\{U \cap P : U \in \mathcal{L}\}$ is a $\pi$-base of $P$.

**Property 4.5**. $P$ is not a topological group.

*Proof.* Any topological group with a countable $\pi$-base is metrizable (see [7]). Property 4.4 completes the proof.

**Property 4.6**. If $(a, b) \in P$, then:

1. The subspace $\{a\} \times B$ of $P$ is discrete.
2. The subspace $A \times \{b\}$ of $P$ is separable, metrizable and a subspace of the space $(\mathbb{R}^2, \mathcal{T}_E)$.

**Property 4.7**. If the set $B$ is countable, then the space $P$ is Lindelöf and has a countable network. Moreover, if the groups $A$ and $B$ are countable, the the group $P$ is countable too.

*Proof.* It is similar to the proof of Property 3.8.

**Property 4.8**. If $B = \mathbb{R}$, then the space $P$ is not normal.

*Proof.* The proof is as for the Niemytski plane ([8], Example 1.5.9).

**Property 4.9**. The tightness of the space $P$ is countable.

*Proof.* It is similar to the proof of Property 3.10.

**Property 4.10**. The space $P$ is Dieudonné complete.

*Proof.* Any submetrizable space is Dieudonné complete.

**Property 4.11**. If the space $A \times B$ has the Baire property, then the space $P$ has the Baire property too.

*Proof.* Any dense open subset of $P$ contains a dense open subset of the space $A \times B$ and any dense subset of $A \times B$ is dense in $P$ too. The proof is complete.

## 5 General construction

Let $E$ be a metrizable additive commutative topological group without isolated points, $dimE = 0$ and in $E$ there exists an infinite sequence $\{c_n : n \in \mathbb{N}\}$ of distinct points of $E$ such that $lim_{n\to\infty} c_n = 0$, where $0$ is the neutral element of $E$. Fix a sequence $\{O_n : n \in \mathbb{N}\}$ of open-and-closed subsets of the space $E$ such that:

– $(O_n \cup (-O_n)) \cap (O_m \cup (-O_m)) = \emptyset$ for $n, m \in \mathbb{N}$ and $n \neq m$;

– if $U$ is open in $E$ and $0 \in U$, then there exists $n \in \mathbb{N}$ such that $O_m \subseteq U$ for all $m \geq n$.

Fix an open base $\{U_n : n \in \mathbb{N}\}$ of the space $E$ at the point 0. We can assume that $O_{n+1} \subseteq U_{n+1} \subseteq U_{n+1} + U_{n+1} \subseteq U_n = -U_n$ and $U_n$ is open-and-closed in $E$ for each $n \in \mathbb{N}$.

In $E \times E$ consider the family $\mathcal{B}_1 = \{V : V$ *is open-and-closed in* $E \times E \setminus \{0\} \times E, U = -U$, *there exists* $m \in \mathbb{N}$ *such that* $\cup\{(O_{2n-1} \times U_n) \cup ((U_m \setminus \{0\}) \times \{0\}) : n \in \mathbb{N}, n \geq m\} \subseteq U\}$ and the family $\mathcal{B}_2 = \{V : V$ *is open-and-closed in* $E \times E \setminus \{0\} \times E, U = -U$, *there exists* $m \in \mathbb{N}$ *such that* $(U_m \setminus \{0\}) \times \{0\} \subseteq U\}$.

The family $\mathcal{B}^\circ = \{\{z\} \cup (U + z) : z \in E \times E, U \in \mathcal{B}_1\}$ is a base of the topology $\mathcal{T}^\circ$ on $E \times E$ and the family $\mathcal{B}^{\circ\circ} = \{\{z\} \cup (U + z) : z \in E \times E, U \in \mathcal{B}_2\}$ is a base of the topology $\mathcal{T}^{\circ\circ}$ on $E \times E$ . The sets from $\mathcal{B}^\circ$ are open-and-closed in $(E \times E, \mathcal{T}^\circ)$ and the sets from $\mathcal{B}^{\circ\circ}$ are open-and-closed in $(E \times E, \mathcal{T}^{\circ\circ})$. Thus the spaces $(E \times E, \mathcal{T}^\circ)$ and $(E \times E, \mathcal{T}^{\circ\circ})$ are zero-dimensional and completely regular. By construction, $\mathcal{T}^\circ \subseteq \mathcal{T}^{\circ\circ})$.

Fix two subgroups $A$ and $B$ without isolated points of the topological group $E$. Assume that $\{c_n : n \in \mathbb{N}\} \subseteq cl_E A$

We consider $C$ the set $A \times B$ as a subspace of the space $(E \times E, \mathcal{T}^\circ)$ and $D$ the set $A \times B$ as a subspace of the space $(E \times E, \mathcal{T}^{\circ\circ})$.

**Property 5.1**. The group $C$ with the topology $\mathcal{T}^\circ|C$ and the group $D$ with the topology $\mathcal{T}^{\circ\circ}|D$ satisfy the following conditions:

1. Are completely regular zero-dimensional qvasitopological groups.
2. $C$ is a space of the countable $\delta$-character and $D$ is a space of the countable $\pi$-character.
3. The space $C$ is not first-countable and the $\delta$-character of $D$ is uncountable.
4. The tightnesses of $C$ and $D$ are countable.
5. If the space $A \times B$ has the Baire property, then $C$ and $D$ have the Baire property, too.
6. Are submetrizable, Dieudonné complete and with $\sigma$-discrete $\pi$-bases.
7. The $\pi$-weights of $C$ and $D$ are equal with the weight of the space $E$.
8. Are not topological groups.
9. Any remainder of $C$ is not Lindelöf and it is not pseudocompact, and any remainder of $D$ is pseudocompact and not Lindelöf.
10. If the space $B$ is $\sigma$-discrete, then $C$ and $D$ are paracompact $F_\sigma$-metrizable spaces. In particular, $C$ and $D$ are paracompact $\sigma$-spaces.

*Proof.* The proofs of the properties of $C$ are similar to the proof of Properties 3.1–3.13 and the proofs of the properties of $D$ are similar to the proof of Properties 4.1–4.12.

## 6 Open Problems

In [9] M. Henriksen and J. R. Isbel have proved that a space $X$ is of countable type if and only if any remainder of $X$ is Lindelöf. In [3] Arhangel'skii proved that any remainder of a topological group is either pseudocomact or Lindelöf. Various properties of remainders have been studied in [2–6]. Examples constructed in this paper motivate the following open questions:

**Problem 6.1**. Is it true that there exists a completely regular sequential (Fréchet-Urysohn) quasitopological group with countable $\delta$-character, but not first-countable?

**Problem 6.2**. Is it true that there exists a completely regular bisequential non-first-countable quasitopological group with a first-countable remainder?

**Problem 6.3**. Is it true that there exists a completely regular quasitopological group $G$ with countable $\pi$-character, but without countable $\delta$-character and such that $G$, in addition, satisfies at least one of one of the following properties:

1) $G$ is sequential;
2) $G$ is Fréchet-Urysohn;
3) any remainder of $G$ is not Lindelöf and it is not pseudocompact;
4) $G$ has a first-countable remainder in some compactification.

## References


[1] Arhangel'skii A. V. *An addition theorem for the weight of sets lying in bicompacta*. Dokl. Akad. Nauk SSSR, 1959, **126**, 239–241.

[2] Arhangel'skii A. V. *More on remainders close to metrizable spaces*, Topology and Appl., 2007, **154**, 1084–1088.

[3] Arhangel'skii A. V. *Two types of remainders of topological groups*. Comment. Math. Univ. Carolinae, 2008, **49**, 119–126.

[4] Arhangel'skii A. V. *The Baire property in remainders of topological groups and other results*. Comment. Math. Univ. Carolinae., 2009, **50**, No. 2, 273–279.

[5] Arhangel'skii A. V., Choban M. M. *Remainders of rectifiable spaces*. Topology and its Appl., 2010, **157**, 789–799.

[6] Arhangel'skii A. V., Choban M. M. *Completeness type properties of semitopological groups, and the theorems of Montgomery and Ellis*. Topology Proceedings, 2011, **37**, No. 1, 33–60.

[7] Arhangel'skii A. V., Tkachenko M. G. *Topological groups and related structures*. Atlantis Press. Amsterdam-Paris, 2008.

[8] Engelking R. *General Topology*. PWN. Warszawa, 1977.

[9] Henriksen M., Isbel J. R. *Some properties of compactifications*. Duke Math. Jour., 1958, **25**, 83–106.

[10] Shapirovskij B. E. *On $\pi$-character and $\pi$-weight in bicompacta*. Dokl. Akad. Nauk SSSR, 1975, **223**, 799–802.



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# Certain differential superordinations using a multiplier transformation and Ruscheweyh derivative

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**Abstract.** In the present paper we define a new operator, by means of convolution product between Ruscheweyh derivative and the multiplier transformation $I(m, \lambda, l)$. For functions $f$ belonging to the class $\mathcal{A}$ we define the differential operator $IR^m_{\lambda,l}$ : $\mathcal{A} \to \mathcal{A}$, $IR^m_{\lambda,l} f(z) := (I(m, \lambda, l) * R^m) f(z)$, where $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \dots, z \in U\}$ is the class of normalized analytic functions, with $\mathcal{A}_1 = \mathcal{A}$. We study some differential superordinations regarding the operator $IR^m_{\lambda,l}$.

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**Keywords and phrases:** Differential superordination, convex function, best subordinant, differential operator.

## 1 Introduction

Denote by $U$ the unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$ and by $\mathcal{H}(U)$ the space of all holomorphic functions in $U$.

Let

$$\mathcal{A}(p, n) = \{f \in \mathcal{H}(U) : f(z) = z^p + \sum_{j=p+n}^{\infty} a_j z^j, \quad z \in U\},$$

with $\mathcal{A}(1, n) = \mathcal{A}_n$, $\mathcal{A}(1, 1) = \mathcal{A}_1 = \mathcal{A}$ and

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U), \ f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots, \ z \in U\}$$

for $a \in \mathbb{C}$ and $p, n \in \mathbb{N}$.

If $f$ and $g$ are analytic functions in $U$, we say that $f$ is superordinate to $g$, written $g \prec f$, if there is an analytic in $U$ function $w$ , with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $g(z) = f(w(z))$ for all $z \in U$. If $f$ is univalent, then $g \prec f$ if and only if $f(0) = g(0)$ and $g(U) \subseteq f(U)$.

Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ and $h$ be an analytic function in $U$. If $p$ and $\psi(p(z), zp'(z); z)$ are univalent in $U$ and satisfy the (first-order) differential superordination

$$h(z) \prec \psi(p(z), zp'(z); z), \qquad \text{for} \quad z \in U, \tag{1}$$

then $p$ is called a solution of the differential superordination. The analytic function $q$ is called a subordinant of the solutions of the differential superordination, or more simply a subordinant, if $q \prec p$ for all $p$ satisfying (1).

A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants $q$ of (1) is said to be the best subordinant of (1). The best subordinant is unique up to a rotation of $U$.

**Definition 1 [7].** For $f \in \mathcal{A}(p,n)$, $p,n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $\lambda, l \geq 0$, the operator $I_p(m,\lambda,l) f(z)$ is defined by the following infinite series

$$I_p(m,\lambda,l) f(z) := z^p + \sum_{j=p+n}^{\infty} \left( \frac{p+\lambda(j-1)+l}{p+l} \right)^m a_j z^j.$$

*Remark 1.* It follows from the above definition that

$$I_p(0,\lambda,l) f(z) = f(z),$$

$$(p+l) I_p(m+1,\lambda,l) f(z) = [p(1-\lambda)+l] I_p(m,\lambda,l) f(z) + \lambda z \left(I_p(m,\lambda,l) f(z)\right)',$$

for $z \in U$.

*Remark 2.* If $p=1$ and $n=1$, then we have $\mathcal{A}(1,1) = \mathcal{A}_1 = \mathcal{A}$, $I_1(m,\lambda,l) f(z) = I(m,\lambda,l)$ and

$$(l+1) I(m+1,\lambda,l) f(z) = [l+1-\lambda] I(m,\lambda,l) f(z) + \lambda z \left(I(m,\lambda,l) f(z)\right)',$$

for $z \in U$.

*Remark 3.* If $f \in \mathcal{A}$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $I(m,\lambda,l) f(z) = z + \sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m a_j z^j$, for $z \in U$.

*Remark 4.* For $l = 0$ and $\lambda \geq 0$, the operator $D_\lambda^m = I(m,\lambda,0)$ was introduced and studied by Al-Oboudi [6]. The study of this operator is reduced to the Sălăgean differential operator $S^m = I(m,1,0)$ [10] for $\lambda = 1$.

**Definition 2 [9].** For $f \in \mathcal{A}$ and $m \in \mathbb{N}$ the operator $R^m$ is defined by $R^m : \mathcal{A} \to \mathcal{A}$,

$$\begin{aligned} R^0 f(z) &= f(z) \\ R^1 f(z) &= z f'(z) \\ &\dots \\ (m+1) R^{m+1} f(z) &= z(R^m f(z))' + m R^m f(z), \quad z \in U. \end{aligned}$$

*Remark 5.* If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $R^m f(z) = z + \sum_{j=2}^{\infty} C_{m+j-1}^m a_j z^j$, $z \in U$.

**Definition 3 [8].** We denote by $Q$ the set of all functions that are analytic and injective on $\overline{U} \backslash E(f)$, where $E(f) = \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\}$, and $f'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$. The subclass of $Q$ for which $f(0) = a$ is denoted by $Q(a)$.

We will use the following lemmas.

**Lemma 1 [8].** *Let $h$ be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C}\backslash\{0\}$ be a complex number with $Re\gamma \geq 0$. If $p \in \mathcal{H}[a,n] \cap Q$, $p(z) + \frac{1}{\gamma}zp'(z)$ is univalent in $U$ and*

$$h(z) \prec p(z) + \frac{1}{\gamma}zp'(z), \quad \text{for} \quad z \in U,$$

then

$$q(z) \prec p(z), \quad \text{for} \quad z \in U,$$

*where $q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{\gamma/n-1}dt$, for $z \in U$. The function $q$ is convex and is the best subordinant.*

**Lemma 2 [8].** *Let $q$ be a convex function in $U$ and let $h(z) = q(z) + \frac{1}{\gamma}zq'(z)$, for $z \in U$, where $Re\ \gamma \geq 0$. If $p \in \mathcal{H}[a,n] \cap Q$, $p(z) + \frac{1}{\gamma}zp'(z)$ is univalent in $U$ and*

$$q(z) + \frac{1}{\gamma}zq'(z) \prec p(z) + \frac{1}{\gamma}zp'(z), \quad \text{for} \quad z \in U,$$

*then*

$$q(z) \prec p(z), \quad \text{for} \quad z \in U,$$

*where $q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{\gamma/n-1}dt$, for $z \in U$. The function $q$ is the best subordinant.*

## 2 Main Results

**Definition 4 [3].** Let $m, \lambda, l \in \mathbb{N}$. Denote by $IR_{\lambda,l}^m$ the operator given by the Hadamard product (the convolution product) of the operator $I(m,\lambda,l)$ and the Ruscheweyh operator $R^m$, $IR_{\lambda,l}^m : \mathcal{A} \to \mathcal{A}$,

$$IR_{\lambda,l}^m f(z) = (I(m,\lambda,l) * R^m) f(z).$$

*Remark 6.* If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $IR_{\lambda,l}^m f(z) = z + \sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m \cdot C_{m+j-1}^m a_j^2 z^j$, for $z \in U$.

*Remark 7.* For $l = 0$, $\lambda \geq 0$, we obtain the Hadamard product $DR_\lambda^n$ [2] of the generalized Sălăgean operator $D_\lambda^n$ and Ruscheweyh operator $R^n$.

For $l = 0$ and $\lambda = 1$, we obtain the Hadamard product $SR^n$ [1] of the Sălăgean operator $S^n$ and Ruscheweyh operator $R^n$.

**Theorem 1.** *Let $h$ be a convex function, $h(0) = 1$. Let $m, \lambda, l \in \mathbb{N}$, $f \in \mathcal{A}$ and suppose that $\frac{l+1}{[\lambda(l-m+2)-(l+1)]z} \cdot \left[(m+1) IR_{\lambda,l}^{m+1} f(z) - (m-2) IR_{\lambda,l}^m f(z)\right] + \left(1 - \frac{l+1}{\lambda(l-m+2)-(l+1)}\right) - \frac{2(l+1)(m-1)-2\lambda m}{\lambda(l-m+2)-(l+1)} \int_0^z \frac{IR_{\lambda,l}^m f(t)-t}{t^2} dt$ is univalent and $\left(IR_{\lambda,l}^m f(z)\right)' \in \mathcal{H}[1,1] \cap Q$. If*

$$h(z) \prec \frac{l+1}{[\lambda(l-m+2)-(l+1)]z} \left[(m+1) IR_{\lambda,l}^{m+1} f(z) - (m-2) IR_{\lambda,l}^m f(z)\right] +$$

$$\left(1 - \frac{l+1}{\lambda(l-m+2)-(l+1)}\right) - \frac{2(l+1)(m-1)-2\lambda m}{\lambda(l-m+2)-(l+1)} \int_0^z \frac{IR_{\lambda,l}^m f(t)-t}{t^2} dt, \quad (2)$$

*for* $z \in U$, *then*

$$q(z) \prec \left(IR^m_{\lambda,l} f(z)\right)', \qquad \text{for} \quad z \in U,$$

*where* $q(z) = \frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1) z^{\frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1)}}} \int_0^z h(t)\, t^{-\frac{\lambda(m-1)+(l+1)}{\lambda(l+1)}} dt$. *The function* $q$ *is convex and it is the best subordinant.*

*Proof.* With notation $p(z) = \left(IR^m_{\lambda,l} f(z)\right)' = 1 + \sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C^m_{m+j-1} j a_j^2 z^{j-1}$ and $p(0) = 1$, we obtain for $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$,

$p(z) + zp'(z) = m1 + \sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C^m_{m+j-1} j a_j^2 z^{j-1} +$

$\sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C^m_{m+j-1} j(j-1) a_j^2 z^{j-1} =$

$1 + \sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C^m_{m+j-1} j^2 a_j^2 z^{j-1} =$

$\frac{1}{z}\Bigg(z + \sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m+1} C^{m+1}_{m+j} \frac{m+1}{\lambda} a_j^2 z^j -$

$\sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C^m_{m+j-1} \frac{\lambda(m-1)-(l+1)}{\lambda(l+1)} j a_j^2 z^j -$

$\sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C^m_{m+j-1} \frac{m-2}{\lambda} a_j^2 z^j -$

$\sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C^m_{m+j-1} \frac{1}{j-1} \frac{2(l+1)(m-1)-2\lambda m}{\lambda(l+1)} a_j^2 z^j\Bigg) =$

$\frac{1}{z}\Bigg[\frac{m+1}{\lambda}\left(z + \sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m+1} C^{m+1}_{m+j} a_j^2 z^j\right) -$

$\frac{m-2}{\lambda}\left(z + \sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C^m_{m+j-1} a_j^2 z^j\right)\Bigg] +$

$\left(1 - \frac{m+1}{\lambda} - \frac{m-2}{\lambda}\right) + \left(1 + \sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C^m_{m+j-1} a_j^2 j z^{j-1}\right) \frac{\lambda(m-1)-(l+1)}{\lambda(l+1)} -$

$\frac{\lambda(m-1)-(l+1)}{\lambda(l+1)} - \sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C^m_{m+j-1} \frac{1}{j-1} \frac{2(l+1)(m-1)-2\lambda m}{\lambda(l+1)} a_j^2 z^{j-1} =$

$\frac{1}{z}\left(\frac{m+1}{\lambda} IR^{m+1}_{\lambda,l} f(z) - \frac{m-2}{\lambda} IR^m_{\lambda,l} f(z)\right) + \frac{\lambda(m-1)-(l+1)}{\lambda(l+1)} \left(IR^m_{\lambda,l} f(z)\right)' +$

$\frac{\lambda l - \lambda m + 2\lambda - 2l - 2}{\lambda(l+1)} - \frac{2(l+1)(m-1)-2\lambda m}{\lambda(l+1)} \sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C^m_{m+j-1} \frac{1}{j-1} a_j^2 z^{j-1} =$

$\frac{1}{z}\left(\frac{m+1}{\lambda} IR^{m+1}_{\lambda,l} f(z) - \frac{m-2}{\lambda} IR^m_{\lambda,l} f(z)\right) + \frac{\lambda(m-1)-(l+1)}{\lambda(l+1)} \left(IR^m_{\lambda,l} f(z)\right)' +$

$\left(1 - \frac{m-1}{l+1} - \frac{2}{\lambda}\right) - \frac{2(l+1)(m-1)-2\lambda m}{\lambda(l+1)} \int_0^z \frac{IR^m_{\lambda,l} f(t) - t}{t^2} dt.$

Therefore $p(z) + \frac{\lambda(l+1)}{\lambda(l-m+2)-(l+1)} zp'(z) =$

$\frac{l+1}{[\lambda(l-m+2)-(l+1)]z}\left[(m+1)\, IR^{m+1}_{\lambda,l} f(z) - (m-2)\, IR^m_{\lambda,l} f(z)\right] +$

$\left(1 - \frac{l+1}{\lambda(l-m+2)-(l+1)}\right) - \frac{2(l+1)(m-1)-2\lambda m}{\lambda(l-m+2)-(l+1)} \int_0^z \frac{IR^m_{\lambda,l} f(t) - t}{t^2} dt.$

Then (2) becomes

$$h(z) \prec p(z) + \frac{\lambda(l+1)}{\lambda(l-m+2)-(l+1)} zp'(z), \quad \text{for} \ \ z \in U.$$

By using Lemma 1 for $\gamma = 1 - \frac{m-1}{l+1} - \frac{1}{\lambda}$ and $n = 1$, we have

$$q(z) \prec p(z), \quad \text{for} \ \ z \in U, \quad \text{i.e.} \quad q(z) \prec \left(IR^m_{\lambda,l} f(z)\right)', \quad \text{for} \ \ z \in U,$$

where $q(z) = \frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1) z^{\frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1)}}} \int_0^z h(t)\, t^{-\frac{\lambda(m-1)+(l+1)}{\lambda(l+1)}} dt$. The function $q$ is convex and it is the best subordinant. □

**Corollary 1** [5]. *Let $h$ be a convex function and $h(0) = 1$. Let $\lambda \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}$ and suppose that $\frac{m+1}{(m\lambda+1)z} DR^{m+1}_{\lambda} f(z) - \frac{m(1-\lambda)}{(m\lambda+1)z} DR^m_{\lambda} f(z)$ is univalent and $(DR^m_{\lambda} f(z))' \in \mathcal{H}[1,1] \cap Q$. If*

$$h(z) \prec \frac{m+1}{(m\lambda+1)z} DR^{m+1}_{\lambda} f(z) - \frac{m(1-\lambda)}{(m\lambda+1)z} DR^m_{\lambda} f(z), \quad \textit{for} \ \ z \in U, \tag{3}$$

*then*

$$q(z) \prec (DR^m_{\lambda} f(z))', \quad \textit{for} \ \ z \in U,$$

*where $q(z) = \frac{m+\frac{1}{\lambda}}{z^{m+\frac{1}{\lambda}}} \int_0^z h(t)\, t^{m-1+\frac{1}{\lambda}} dt$. The function $q$ is convex and it is the best subordinant.*

**Corollary 2** [4]. *Let $h$ be a convex function and $h(0) = 1$. Let $n \in \mathbb{N}$, $f \in \mathcal{A}$ and suppose that $\frac{1}{z} SR^{n+1} f(z) + \frac{n}{n+1} z (SR^n f(z))''$ is univalent and $(SR^n f(z))' \in \mathcal{H}[1,1] \cap Q$. If*

$$h(z) \prec \frac{1}{z} SR^{n+1} f(z) + \frac{n}{n+1} z (SR^n f(z))'', \quad \textit{for} \ \ z \in U, \tag{4}$$

*then*

$$q(z) \prec (SR^n f(z))', \quad \textit{for} \ \ z \in U,$$

*where $q(z) = \frac{1}{z} \int_0^z h(t) dt$. The function $q$ is convex and it is the best subordinant.*

**Theorem 2**. *Let $q$ be convex in $U$ and let $h$ be defined by $h(z) = q(z) + \frac{\lambda(l+1)}{\lambda(l-m+2)-(l+1)} zq'(z)$, $m, \lambda, l \in \mathbb{N}$. If $f \in \mathcal{A}$, suppose that $\frac{l+1}{[\lambda(l-m+2)-(l+1)]z} \left[(m+1) IR^{m+1}_{\lambda,l} f(z) - (m-2) IR^m_{\lambda,l} f(z)\right] + \left(1 - \frac{l+1}{\lambda(l-m+2)-(l+1)}\right)$ $- \frac{2(l+1)(m-1)-2\lambda m}{\lambda(l-m+2)-(l+1)} \int_0^z \frac{IR^m_{\lambda,l} f(t) - t}{t^2} dt$ is univalent, $\left(IR^m_{\lambda,l} f(z)\right)' \in \mathcal{H}[1,1] \cap Q$ and satisfies the differential superordination*

$$h(z) = q(z) + \frac{\lambda(l+1)}{\lambda(l-m+2)-(l+1)} zq'(z) \prec$$

$$\frac{l+1}{[\lambda(l-m+2)-(l+1)]z} \left[(m+1) IR^{m+1}_{\lambda,l} f(z) - (m-2) IR^m_{\lambda,l} f(z)\right]$$

$$+\left(1-\frac{l+1}{\lambda\left(l-m+2\right)-\left(l+1\right)}\right)-\frac{2\left(l+1\right)\left(m-1\right)-2\lambda m}{\lambda\left(l-m+2\right)-\left(l+1\right)}\int_0^z\frac{IR_{\lambda,l}^m f\left(t\right)-t}{t^2}dt, \tag{5}$$

*for* $z\in U$*. Then*

$$q(z)\prec\left(IR_{\lambda,l}^m f\left(z\right)\right)', \quad \textit{for} \quad z\in U,$$

*where* $q(z)=\frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1)z^{\frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1)}}}\int_0^z h\left(t\right)t^{-\frac{\lambda(m-1)+(l+1)}{\lambda(l+1)}}dt$*. The function* $q$ *is the best subordinant.*

*Proof.* Let $p\left(z\right)=\left(IR_{\lambda,l}^m f\left(z\right)\right)'=1+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C_{m+j-1}^m j a_j^2 z^{j-1}$.

Differentiating, we obtain $p\left(z\right)+zp'\left(z\right)=\frac{1}{z}\left(\frac{m+1}{\lambda}IR_{\lambda,l}^{m+1}f\left(z\right)-\frac{m-2}{\lambda}IR_{\lambda,l}^m f\left(z\right)\right)+\frac{\lambda(m-1)-(l+1)}{\lambda(l+1)}\left(IR_{\lambda,l}^m f\left(z\right)\right)'+\left(1-\frac{m-1}{l+1}-\frac{2}{\lambda}\right)-\frac{2(l+1)(m-1)-2\lambda m}{\lambda(l+1)}\int_0^z\frac{IR_{\lambda,l}^m f(t)-t}{t^2}dt$,

$p\left(z\right)+\frac{\lambda(l+1)}{\lambda(l-m+2)-(l+1)}zp'\left(z\right)=\frac{l+1}{[\lambda(l-m+2)-(l+1)]z}\left[\left(m+1\right)IR_{\lambda,l}^{m+1}f\left(z\right)-\left(m-2\right)IR_{\lambda,l}^m f\left(z\right)\right]+\left(1-\frac{l+1}{\lambda(l-m+2)-(l+1)}\right)-\frac{2(l+1)(m-1)-2\lambda m}{\lambda(l-m+2)-(l+1)}\int_0^z\frac{IR_{\lambda,l}^m f(t)-t}{t^2}dt$,
for $z\in U$, and (2) becomes

$$q(z)+\frac{\lambda\left(l+1\right)}{\lambda\left(l-m+2\right)-\left(l+1\right)}zq'(z)\prec p(z)+\frac{\lambda\left(l+1\right)}{\lambda\left(l-m+2\right)-\left(l+1\right)}zp'\left(z\right),$$

for $z\in U$.

Using Lemma 2 for $\gamma=1-\frac{m-1}{l+1}-\frac{1}{\lambda}$ and $n=1$, we have $q(z)\prec p(z)$, $z\in U$, i.e.

$$q(z)=\frac{\lambda\left(l-m+2\right)-\left(l+1\right)}{\lambda\left(l+1\right)z^{\frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1)}}}\int_0^z h\left(t\right)t^{-\frac{\lambda(m-1)+(l+1)}{\lambda(l+1)}}dt\prec\left(IR_{\lambda,l}^m f\left(z\right)\right)', \quad z\in U,$$

and $q$ is the best subordinant. □

**Corollary 3 [5]**. *Let* $q$ *be convex in* $U$*,* $h$ *be defined by* $h\left(z\right)=q\left(z\right)+\frac{\lambda}{m\lambda+1}zq'\left(z\right)$*,* $\lambda\geq 0$*,* $m\in\mathbb{N}$ *and* $f\in\mathcal{A}$*. Suppose that* $\frac{m+1}{(m\lambda+1)z}DR_{\lambda}^{m+1}f\left(z\right)-\frac{m(1-\lambda)}{(m\lambda+1)z}DR_{\lambda}^m f\left(z\right)$ *is univalent,* $\left(DR_{\lambda}^m f\left(z\right)\right)'\in\mathcal{H}\left[1,1\right]\cap Q$ *and satisfies the differential superordination*

$$h(z)=q\left(z\right)+\frac{\lambda}{m\lambda+1}zq'\left(z\right)\prec\frac{m+1}{\left(m\lambda+1\right)z}DR_{\lambda}^{m+1}f\left(z\right)-\frac{m\left(1-\lambda\right)}{\left(m\lambda+1\right)z}DR_{\lambda}^m f\left(z\right), \tag{6}$$

*for* $z\in U$*. Then*

$$q(z)\prec\left(DR_{\lambda}^m f\left(z\right)\right)', \quad \textit{for} \quad z\in U,$$

*where* $q(z)=\frac{m+\frac{1}{\lambda}}{z^{m+\frac{1}{\lambda}}}\int_0^z h\left(t\right)t^{m-1+\frac{1}{\lambda}}dt$*. The function* $q$ *is the best subordinant.*

**Corollary 4 [4]**. *Let $q$ be convex in $U$ and let $h$ be defined by $h(z) = q(z) + zq'(z)$. If $n \in \mathbb{N}$ and $f \in \mathcal{A}$, suppose that $\frac{1}{z}SR^{n+1}f(z) + \frac{n}{n+1}z(SR^n f(z))''$ is univalent, $(SR^n f(z))' \in \mathcal{H}[1,1] \cap Q$ and satisfies the differential superordination*

$$h(z) = q(z) + zq'(z) \prec \frac{1}{z}SR^{n+1}f(z) + \frac{n}{n+1}z(SR^n f(z))'', \quad \text{for} \ \ z \in U. \tag{7}$$

*Then*

$$q(z) \prec (SR^n f((z)))', \quad \text{for} \ \ z \in U,$$

*where $q(z) = \frac{1}{z}\int_0^z h(t)dt$. The function $q$ is the best subordinant.*

**Theorem 3**. *Let $h$ be a convex function and $h(0) = 1$. Let $m, \lambda, l \in \mathbb{N}$, $f \in \mathcal{A}$ and suppose that $\left(IR^m_{\lambda,l} f(z)\right)'$ is univalent and $\frac{IR^m_{\lambda,l}f(z)}{z} \in \mathcal{H}[1,1] \cap Q$. If*

$$h(z) \prec \left(IR^m_{\lambda,l} f(z)\right)', \quad \text{for} \ \ z \in U, \tag{8}$$

*then*

$$q(z) \prec \frac{IR^m_{\lambda,l} f(z)}{z}, \quad \text{for} \ \ z \in U,$$

*where $q(z) = \frac{1}{z}\int_0^z h(t)dt$. The function $q$ is convex and it is the best subordinant.*

*Proof.* Consider $p(z) = \frac{IR^m_{\lambda,l} f(z)}{z} = \frac{z + \sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C^m_{m+j-1} a_j^2 z^j}{z} = 1 + \sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C^m_{m+j-1} a_j^2 z^{j-1}$. Evidently $p \in \mathcal{H}[1,1]$.

We have $p(z) + zp'(z) = \left(IR^m_{\lambda,l} f(z)\right)'$, for $z \in U$. Then (8) becomes

$$h(z) \prec p(z) + zp'(z), \quad \text{for} \ \ z \in U.$$

By using Lemma 1 for $\gamma = 1$ and $n = 1$, we have

$$q(z) \prec p(z), \quad \text{for} \ \ z \in U, \quad \text{i.e.} \quad q(z) \prec \frac{IR^m_{\lambda,l} f(z)}{z}, \quad \text{for} \ \ z \in U,$$

where $q(z) = \frac{1}{z}\int_0^z h(t)dt$. The function $q$ is convex and it is the best subordinant. □

**Corollary 5 [5]**. *Let $h$ be a convex function, $h(0) = 1$. Let $\lambda \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}$ and suppose that $(DR^m_\lambda f(z))'$ is univalent and $\frac{DR^m_\lambda f(z)}{z} \in \mathcal{H}[1,1] \cap Q$. If*

$$h(z) \prec (DR^m_\lambda f(z))', \quad \text{for} \ \ z \in U. \tag{9}$$

*Then*

$$q(z) \prec \frac{DR^m_\lambda f(z)}{z}, \quad \text{for} \ \ z \in U,$$

*where $q(z) = \frac{1}{z}\int_0^z h(t)dt$. The function $q$ is convex and it is the best subordinant.*

**Corollary 6 [4]**. *Let $h$ be a convex function, $h(0) = 1$. Let $n \in \mathbb{N}$, $f \in \mathcal{A}$ and suppose that $(SR^n f(z))'$ is univalent and $\frac{SR^n f(z)}{z} \in \mathcal{H}[1,1] \cap Q$. If*

$$h(z) \prec (SR^n f(z))', \quad \text{for } z \in U, \tag{10}$$

*then*

$$q(z) \prec \frac{SR^n f(z)}{z}, \quad \text{for } z \in U,$$

*where $q(z) = \frac{1}{z}\int_0^z h(t)dt$. The function $q$ is convex and it is the best subordinant.*

**Theorem 4**. *Let $q$ be convex in $U$, $h$ be defined by $h(z) = q(z) + zq'(z)$, $m, \lambda, l \in \mathbb{N}$ and $f \in \mathcal{A}$. Suppose that $\left(IR^m_{\lambda,l} f(z)\right)'$ is univalent, $\frac{IR^m_{\lambda,l} f(z)}{z} \in \mathcal{H}[1,1] \cap Q$ and satisfies the differential superordination*

$$h(z) = q(z) + zq'(z) \prec \left(IR^m_{\lambda,l} f(z)\right)', \quad \text{for } z \in U. \tag{11}$$

*Then*

$$q(z) \prec \frac{IR^m_{\lambda,l} f(z)}{z}, \quad \text{for } z \in U,$$

*where $q(z) = \frac{1}{z}\int_0^z h(t)dt$. The function $q$ is the best subordinant.*

*Proof.* Let $p(z) = \frac{IR^m_{\lambda,l} f(z)}{z} = \frac{z + \sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C^m_{m+j-1} a_j^2 z^j}{z} = 1 + \sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C^m_{m+j-1} a_j^2 z^{j-1}$. Evidently $p \in \mathcal{H}[1,1]$.

Differentiating, we obtain $p(z) + zp'(z) = \left(IR^m_{\lambda,l} f(z)\right)'$, for $z \in U$ and (11) becomes

$$q(z) + zq'(z) \prec p(z) + zp'(z), \quad \text{for } z \in U.$$

Using Lemma 2 for $\gamma = 1$ and $n = 1$, we have

$$q(z) \prec p(z), \quad \text{for } z \in U, \quad \text{i.e.} \quad q(z) = \frac{1}{z}\int_0^z h(t)dt \prec \frac{IR^m_{\lambda,l} f(z)}{z}, \quad \text{for } z \in U,$$

and $q$ is the best subordinant. □

**Corollary 7 [5]**. *Let $q$ be convex in $U$, $h$ be defined by $h(z) = q(z) + zq'(z)$, $\lambda \geq 0$, $m \in \mathbb{N}$ and $f \in \mathcal{A}$. Suppose that $(DR^m_\lambda f(z))'$ is univalent, $\frac{DR^m_\lambda f(z)}{z} \in \mathcal{H}[1,1] \cap Q$ and satisfies the differential superordination*

$$h(z) = q(z) + zq'(z) \prec (DR^m_\lambda f(z))', \quad \text{for } z \in U. \tag{12}$$

*Then*

$$q(z) \prec \frac{DR^m_\lambda f(z)}{z}, \quad \text{for } z \in U,$$

*where $q(z) = \frac{1}{z}\int_0^z h(t)dt$. The function $q$ is the best subordinant.*

**Corollary 8** [4]. *Let $q$ be convex in $U$ and let $h$ be defined by $h(z) = q(z) + zq'(z)$. If $n \in \mathbb{N}$, $f \in \mathcal{A}$, suppose that $(SR^n f(z))'$ is univalent, $\frac{SR^n f(z)}{z} \in \mathcal{H}[1,1] \cap Q$ and satisfies the differential superordination*

$$h(z) = q(z) + zq'(z) \prec (SR^n f(z))', \quad \textit{for} \ \ z \in U. \tag{13}$$

*Then*

$$q(z) \prec \frac{SR^n f(z)}{z}, \quad \textit{for} \ \ z \in U,$$

*where $q(z) = \frac{1}{z}\int_0^z h(t)dt$. The function $q$ is the best subordinant.*

**Theorem 5**. *Let $h(z) = \frac{1+(2\beta-1)z}{1+z}$ be a convex function in $U$, where $0 \le \beta < 1$. Let $m, \lambda, l \in \mathbb{N}$, $f \in \mathcal{A}$ and suppose that $\left(IR^m_{\lambda,l} f(z)\right)'$ is univalent and $\frac{IR^m_{\lambda,l} f(z)}{z} \in \mathcal{H}[1,1] \cap Q$. If*

$$h(z) \prec \left(IR^m_{\lambda,l} f(z)\right)', \quad \textit{for} \ \ z \in U, \tag{14}$$

*then*

$$q(z) \prec \frac{IR^m_{\lambda,l} f(z)}{z}, \quad \textit{for} \ \ z \in U,$$

*where $q$ is given by $q(z) = 2\beta - 1 + 2(1-\beta)\frac{\ln(1+z)}{z}$, for $z \in U$. The function $q$ is convex and it is the best subordinant.*

*Proof.* Following the same steps as in the proof of Theorem 3 and considering $p(z) = \frac{IR^m_{\lambda,l} f(z)}{z}$, the differential superordination (14) becomes

$$h(z) = \frac{1+(2\beta-1)z}{1+z} \prec p(z) + zp'(z), \quad \text{for} \ \ z \in U.$$

By using Lemma 1 for $\gamma = 1$ and $n = 1$, we have $q(z) \prec p(z)$, i. e.,

$$q(z) = \frac{1}{z}\int_0^z h(t)dt = \frac{1}{z}\int_0^z \frac{1+(2\beta-1)t}{1+t}dt = 2\beta - 1 + 2(1-\beta)\frac{1}{z}\ln(z+1) \prec \frac{IR^m_{\lambda,l} f(z)}{z},$$

for $z \in U$.
The function $q$ is convex and it is the best subordinant. □

**Theorem 6**. *Let $h$ be a convex function, $h(0) = 1$. Let $m, \lambda, l \in \mathbb{N}$, $f \in \mathcal{A}$ and suppose that $\left(\frac{zIR^{m+1}_{\lambda,l} f(z)}{IR^m_{\lambda,l} f(z)}\right)'$ is univalent and $\frac{IR^{m+1}_{\lambda,l} f(z)}{IR^m_{\lambda,l} f(z)} \in \mathcal{H}[1,1] \cap Q$. If*

$$h(z) \prec \left(\frac{zIR^{m+1}_{\lambda,l} f(z)}{IR^m_{\lambda,l} f(z)}\right)', \quad \textit{for} \ \ z \in U, \tag{15}$$

*then*

$$q(z) \prec \frac{IR^{m+1}_{\lambda,l} f(z)}{IR^m_{\lambda,l} f(z)}, \quad \textit{for} \ \ z \in U,$$

*where $q(z) = \frac{1}{z}\int_0^z h(t)dt$. The function $q$ is convex and it is the best subordinant.*

*Proof.* Consider $p(z) = \frac{IR_{\lambda,l}^{m+1}f(z)}{IR_{\lambda,l}^{m}f(z)} = \frac{z+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m+1}C_{m+j}^{m+1}a_j^2z^j}{z+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}C_{m+j-1}^{m}a_j^2z^j} = \frac{1+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m+1}C_{m+j}^{m+1}a_j^2z^{j-1}}{1+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}C_{m+j-1}^{m}a_j^2z^{j-1}}$. Evidently $p \in \mathcal{H}[1,1]$.

We have $p'(z) = \frac{\left(IR_{\lambda,l}^{m+1}f(z)\right)'}{IR_{\lambda,l}^{m}f(z)} - p(z) \cdot \frac{\left(IR_{\lambda,l}^{m}f(z)\right)'}{IR_{\lambda,l}^{m}f(z)}$. Hence $p(z) + zp'(z) = \left(\frac{zIR_{\lambda,l}^{m+1}f(z)}{IR_{\lambda,l}^{m}f(z)}\right)'$.

Then (15) becomes

$$h(z) \prec p(z) + zp'(z), \quad \text{for} \quad z \in U.$$

By using Lemma 1 for $\gamma = 1$ and $n = 1$, we have

$$q(z) \prec p(z), \quad \text{for} \quad z \in U, \quad \text{i.e.} \quad q(z) \prec \frac{IR_{\lambda,l}^{m+1}f(z)}{IR_{\lambda,l}^{m}f(z)}, \quad \text{for} \quad z \in U,$$

where $q(z) = \frac{1}{z}\int_0^z h(t)dt$. The function $q$ is convex and it is the best subordinant. □

**Corollary 9** [**5**]. *Let $h$ be a convex function, $h(0) = 1$. Let $\lambda \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}$ and suppose that $\left(\frac{zDR_{\lambda}^{m+1}f(z)}{DR_{\lambda}^{m}f(z)}\right)'$ is univalent and $\frac{DR_{\lambda}^{m+1}f(z)}{DR_{\lambda}^{m}f(z)} \in \mathcal{H}[1,1] \cap Q$. If*

$$h(z) \prec \left(\frac{zDR_{\lambda}^{m+1}f(z)}{DR_{\lambda}^{m}f(z)}\right)', \quad \textit{for} \quad z \in U, \tag{16}$$

*then*

$$q(z) \prec \frac{DR_{\lambda}^{m+1}f(z)}{DR_{\lambda}^{m}f(z)}, \quad \textit{for} \quad z \in U,$$

*where $q(z) = \frac{1}{z}\int_0^z h(t)dt$. The function $q$ is convex and it is the best subordinant.*

**Corollary 10** [**4**]. *Let $h$ be a convex function, $h(0) = 1$. Let $n \in \mathbb{N}$, $f \in \mathcal{A}$ and suppose that $\left(\frac{zSR^{n+1}f(z)}{SR^{n}f(z)}\right)'$ is univalent and $\frac{SR^{n+1}f(z)}{SR^{n}f(z)} \in \mathcal{H}[1,1] \cap Q$. If*

$$h(z) \prec \left(\frac{zSR^{n+1}f(z)}{SR^{n}f(z)}\right)', \quad \textit{for} \quad z \in U, \tag{17}$$

*then*

$$q(z) \prec \frac{SR^{n+1}f(z)}{SR^{n}f(z)}, \quad \textit{for} \quad z \in U,$$

*where $q(z) = \frac{1}{z}\int_0^z h(t)dt$. The function $q$ is convex and it is the best subordinant.*

**Theorem 7**. *Let $q$ be convex in $U$, $h$ be defined by $h(z) = q(z) + zq'(z)$, $m, \lambda, l \in \mathbb{N}$ and $f \in \mathcal{A}$. Suppose that $\left(\frac{zIR_{\lambda,l}^{m+1}f(z)}{IR_{\lambda,l}^{m}f(z)}\right)'$ is univalent, $\frac{IR_{\lambda,l}^{m+1}f(z)}{IR_{\lambda,l}^{m}f(z)} \in \mathcal{H}[1,1] \cap Q$ and* *satisfies the differential superordination*

$$h(z) = q(z) + zq'(z) \prec \left(\frac{zIR_{\lambda,l}^{m+1}f(z)}{IR_{\lambda,l}^{m}f(z)}\right)', \quad \textit{for} \quad z \in U, \tag{18}$$

*then*

$$q(z) \prec \frac{IR_{\lambda,l}^{m+1}f(z)}{IR_{\lambda,l}^{m}f(z)}, \quad \textit{for} \quad z \in U,$$

*where* $q(z) = \frac{1}{z}\int_0^z h(t)dt$. *The function* $q$ *is the best subordinant.*

*Proof.* Let $p(z) = \frac{IR_{\lambda,l}^{m+1}f(z)}{IR_{\lambda,l}^{m}f(z)} = \frac{z+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m+1}C_{m+j}^{m+1}a_j^2z^j}{z+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}C_{m+j-1}^{m}a_j^2z^j} = \frac{1+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m+1}C_{m+j}^{m+1}a_j^2z^{j-1}}{1+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}C_{m+j-1}^{m}a_j^2z^{j-1}}$. Evidently $p \in \mathcal{H}[1,1]$.

Differentiating, we obtain $p(z) + zp'(z) = \left(\frac{zIR_{\lambda,l}^{m+1}f(z)}{IR_{\lambda,l}^{m}f(z)}\right)'$, for $z \in U$ and (18) becomes

$$q(z) + zq'(z) \prec p(z) + zp'(z), \quad \text{for} \quad z \in U.$$

Using Lemma 2 for $\gamma = 1$ and $n = 1$, we have

$$q(z) \prec p(z), \quad \text{for} \quad z \in U, \quad \text{i.e.} \quad q(z) = \frac{1}{z}\int_0^z h(t)dt \prec \frac{IR_{\lambda,l}^{m+1}f(z)}{IR_{\lambda,l}^{m}f(z)}, \quad \text{for} \quad z \in U,$$

and $q$ is the best subordinant. □

**Corollary 11** [**5**]. *Let* $q$ *be convex in* $U$, $h$ *be defined by* $h(z) = q(z) + zq'(z)$, $\lambda \geq 0$, $m \in \mathbb{N}$ *and* $f \in \mathcal{A}$. *Suppose that* $\left(\frac{zDR_{\lambda}^{m+1}f(z)}{DR_{\lambda}^{m}f(z)}\right)'$ *is univalent,* $\frac{DR_{\lambda}^{m+1}f(z)}{DR_{\lambda}^{m}f(z)} \in \mathcal{H}[1,1] \cap Q$ *and satisfies the differential superordination*

$$h(z) = q(z) + zq'(z) \prec \left(\frac{zDR_{\lambda}^{m+1}f(z)}{DR_{\lambda}^{m}f(z)}\right)', \quad \textit{for} \quad z \in U. \tag{19}$$

*Then*

$$q(z) \prec \frac{DR_{\lambda}^{m+1}f(z)}{DR_{\lambda}^{m}f(z)}, \quad \textit{for} \quad z \in U,$$

*where* $q(z) = \frac{1}{z}\int_0^z h(t)dt$. *The function* $q$ *is the best subordinant.*

**Corollary 12** [**4**]. *Let* $q$ *be convex in* $U$, $h$ *be defined by* $h(z) = q(z) + zq'(z)$, $n \in \mathbb{N}$, $f \in \mathcal{A}$. *Suppose that* $\left(\frac{zSR^{n+1}f(z)}{SR^{n}f(z)}\right)'$ *is univalent,* $\frac{SR^{n+1}f(z)}{SR^{n}f(z)} \in \mathcal{H}[1,1] \cap Q$ *and satisfies the differential superordination*

$$h(z) = q(z) + zq'(z) \prec \left(\frac{zSR^{n+1}f(z)}{SR^{n}f(z)}\right)', \quad \textit{for} \quad z \in U. \tag{20}$$

*Then*

$$q(z) \prec \frac{SR^{n+1}f(z)}{SR^{n}f(z)}, \quad \textit{for} \ \ z \in U,$$

*where* $q(z) = \frac{1}{z}\int_0^z h(t)dt$. *The function* $q$ *is the best subordinant.*

**Theorem 8**. *Let* $h(z) = \frac{1+(2\beta-1)z}{1+z}$ *be a convex function in* $U$, *where* $0 \le \beta < 1$. *Let* $m, \lambda, l \in \mathbb{N}$, $f \in \mathcal{A}$ *and suppose that* $\left(\frac{zIR_{\lambda,l}^{m+1}f(z)}{IR_{\lambda,l}^{m}f(z)}\right)'$ *is univalent,* $\frac{IR_{\lambda,l}^{m+1}f(z)}{IR_{\lambda,l}^{m}f(z)} \in \mathcal{H}[1,1] \cap Q$. *If*

$$h(z) \prec \left(\frac{zIR_{\lambda,l}^{m+1}f(z)}{IR_{\lambda,l}^{m}f(z)}\right)', \quad \textit{for} \ \ z \in U, \tag{21}$$

*then*

$$q(z) \prec \frac{IR_{\lambda,l}^{m+1}f(z)}{IR_{\lambda,l}^{m}f(z)}, \quad \textit{for} \ \ z \in U,$$

*where* $q$ *is given by* $q(z) = 2\beta - 1 + 2(1-\beta)\frac{\ln(1+z)}{z}$, *for* $z \in U$. *The function* $q$ *is convex and it is the best subordinant.*

*Proof.* Following the same steps as in the proof of Theorem 2 and considering $p(z) = \frac{IR_{\lambda,l}^{m+1}f(z)}{IR_{\lambda,l}^{m}f(z)}$, the differential superordination (21) becomes

$$h(z) = \frac{1+(2\beta-1)z}{1+z} \prec p(z) + zp'(z), \quad \text{for} \ \ z \in U.$$

By using Lemma 1 for $\gamma = 1$ and $n = 1$, we have $q(z) \prec p(z)$, i.e.,

$$q(z) = \frac{1}{z}\int_0^z h(t)dt = \frac{1}{z}\int_0^z \frac{1+(2\beta-1)t}{1+t}dt =$$

$$2\beta - 1 + 2(1-\beta)\frac{1}{z}\ln(z+1) \prec \frac{IR_{\lambda,l}^{m+1}f(z)}{IR_{\lambda,l}^{m}f(z)}, \quad \text{for} \ \ z \in U.$$

The function $q$ is convex and it is the best subordinant. □

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## References

[1] Alb Lupaş A. *Certain differential subordinations using Salagean and Ruscheweyh operators.* Acta Universitatis Apulensis, 2012, No. 29, 125–129 .

[2] Alb Lupaş A. *Certain differential subordinations using a generalized Salagean operator and Ruscheweyh operator.* Journal of Mathematics and Applications, 2010, No. 33, 67–72.

[3] Alb Lupaş Alina. *A note on a certain subclass of analytic functions defined by multiplier transformation.* Journal of Computational Analysis and Applications, 2010, **12**, No. 1-B, 369–373.

[4] Alb Lupaş Alina. *Certain differential superordinations using Sălăgean and Ruscheweyh operators.* Analele Universităţii din Oradea, Fascicola Matematica, 2010 **XVII**, No. 2, 203–210.

[5] Alb Lupaş Alina. *Certain differential superordinations using a generalized Sălăgean and Ruscheweyh operators.* Acta Universitatis Apulensis 2011, **25**, 31–40.

[6] Al-Oboudi F. M. *On univalent functions defined by a generalized Sălăgean operator*, Ind. J. Math. Math. Sci., 2004, **25**, 1429–1436.

[7] Cătaş A. *On certain class of p-valent functions defined by new multiplier transformations.* Proceedings Book of the International Symposium on Geometric Function Theory and Applications, August 20–24, 2007, TC Istanbul Kultur University, Turkey, 2007, 241–250.

[8] Miller S. S., Mocanu P. T. *Subordinants of Differential Superordinations.* Complex Variables, 2003, **48**, No. 10, 815–826.

[9] Ruscheweyh St. *New criteria for univalent functions.* Proc. Amer. Math. Soc., 1975, **49**, 109–115.

[10] Sălăgean G. St. *Subclasses of univalent functions.* Lecture Notes in Math., Springer Verlag, Berlin, 1983, **1013**, 362–372.

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## The 20th Conference on Applied and Industrial Mathematics (CAIM-2012) dedicated to the 70th anniversary of Academician Mitrofan M. Choban

"The 20th Conference on Applied and Industrial Mathematics" (CAIM 2012) took place in Chişinău (Republic of Moldova) from August 22nd to 25th, 2012. It was organized with the financial support of Romanian Society of Applied and Industrial Mathematics (ROMAI), Mathematical Society of Moldova, Academy of Sciences of Moldova, Tiraspol State University (Chişinău), Academy of Economic Studies of Moldova, Institute of Mathematics and Computer Science of the Academy of Sciences of Moldova, Moldova State University. The conference was dedicated to the 70th anniversary of Professor Mitrofan Choban, President of the Mathematical Society of Moldova and Vice-President of the Romanian Society of Applied and Industrial Mathematics. This event was conceived to provide a discussion forum on the achievements of the last decades in the fields of Mathematics, Computer Sciences, Physics and its Applications in retrospective analysis, as well as to highlight the present state of investigations and education in these fields.

CAIM 2012 was hosted by the Faculty of Physics, Mathematics and Information Technologies of the Tiraspol State University, Chişinău.

During the conference over 190 reports were delivered at plenary (16) and parallel sessions (175).

Sessions of the conference were held in five plenary sessions and eight parallel scientific sessions:

1. Mathematical Analysis.
2. Differential Equations.
3. Algebra and Logic.
4. Geometry and Topology.

5. Analytical and Numerical Methods in Partial Differential Equations.
6. Computer Science.
7. Mathematical Models in Industry, Physics and Biology.
8. Education. Didactics of Mathematics, Physics and Informatics.

The conference was attended by over 100 participants from abroad including Romania, Ukraine, Bulgaria, Canada, Armenia, Estonia, Germany, Israel, Italy, Russian Federation, Spain, Tajikistan, Belarus.

During the plenary sessions a number of well-known scholars presented their papers, among them: Mati Abel (Estonia) *"Main classes of Gelfand-Mazur algebras"*, Alexander Arhangel'skii (Russia) *"A nice class of topological spaces"*, Vasile Berinde (Romania) *"On the stability of multi-step fixed point iteration procedures"*, Ilie Burdujan (Romania) *"Automorphisms and derivations of homogeneous quadratic differential systems on $R^3$"*, Adrian Carabineanu (Romania) *"A complex boundary element method for the study of the potential flow past submerged profiles"*, Sergiu Cataranciuc (Moldova) *"Algebraic topology of the multi-ary relation in the applications"*, Svetlana Cojocaru and Constantin Gaindric (Moldova) *"Research in Computer Science and Information Technology at the Institute of Mathematics and Computer Science"*, Adrian Constantinescu (Romania) *"Some topological aspects of the finite generation of subalgebras. I: Variations on a Theorem of Goodman and Landman"*, Ion Crăciun (Romania) *"Some problems of the linear theory of piezoelectric micropolar thermoelasticity"*, Peter Kenderov (Bulgaria) (in collaboration with Mitrofan Choban (Moldova) and W. B. Moors (New Zeland)) *"Eberlein Theorem for Sequences of Sets and Fragmentability of Function Spaces"*, Mario Lefebvre (Canada) *"First passage to a semi-infinite line for a two-dimensional Wiener process"*, Boris Loginov (Russia) *"Branching Equations and Branching Equations in the root-subspaces potentiality conditions for Andronov-Hopf bifurcation II"*, Radu Miron (Romania) *"The generalized Lagrangian mathematical systems"*, Gheorghe Paun (Romania) *"Membrane Computing Basics, Recent Developments, Applications"*, Vesco Valov (Canada) *"Homogeneous compacta"*, Nicolae Vulpe (Moldova) *"Global analysis of infinite singularities of quadratic vector fields"*.

A number of papers presented at different sessions aroused valuable and insightful discussions: Yaroslav Bihun, Inessa Berezovska and Nataliya Romanenko (Ukraine) *"Averaging of a multifrequency boundary-value problem with constant delay and linearly transformed argument"*, Iurie Calin and Valeriu Baltag (Moldova) *"Invariant center conditions for quadratic differential system with degenerate infinity perturbed by cubic nonliniarities"*, Dumitru Cozma (Moldova) *"Darboux integrability in cubic systems with two invariant straight lines"*, Florin Damian (Moldova) *" Involution without fixed points on hiperbolic manifolds"*, Vasile Glavan (Moldova) *"Horseshoes as viable sets in set-valued dynamics"*, Valeriu Guţu (Moldova) *"The Pythagoras tree and Borsuk's conjecture"*, Anca Veronica Ion and Raluca Mihaela Georgescu (Romania) *"Numerical investigation of the Bautin-type bifurcation for a delay differential equation"*, Stelian Ion (Romania) *"A soft package to estimate the parameters in an ecological model"*, Vladimir Izbash (Moldova) *"Polynomial morphisms of medial quasigroups"*, Nicolae Jitaraşu (Moldova) *"On the boundary value problem for elliptic and parabolic equations"*, Alexandru Lazari ( Romania, Moldova) *"Polynomial algorithms for probabilistic characterization of composed stochastic sys-*

*tems with final critical state"*, Vadim E. Levit and Eugen Mandrescu (Israel) *"Critical sets in almost unicyclic Konig-Egervary graphs"*, Dmitrii Lozovanu and Maria Capcelea (Moldova) *"Determining the optimal stationary strategies for stochastic positional games"*, Ekaterina Mihaylova (Bulgaria) *"Co-homogenity and Klebanov Spaces"*, Gheorghe Mishkoy, L. Mitev and D. Begenari (Moldova) *"Numerical results for probability of states with PH distribution for Polling models"*, Marcelina Mocanu (Romania) *"A unifying approach to Sobolev-type spaces on metric measure spaces"*, Vasile Neagu (Moldova) *"Symbol of singular integral operators on piecewise Lyapunov contours"*, Andrei Perjan and Galina Rusu (Moldova) *"Some convergence estimates for abstract second order singularly perturbed Cauchy problems with monotone nonlinearities"*, Mihail Popa (Moldova) *"Applications of algebraic methods to the center-focus problem"*, Mefodie Raţiu (Moldova) *"Expressibility of implication in intuitionistic logic with the method of the formula realization of algebras"*, Vladislav Seichuc (Moldova) *"On appoximate solving of some nonlinear mixed singular integral equations"*, Fidir Sokhatsky and Iren Fryz (Ukraine) *"About orthogonality of multiary operations"*, Alexandru Suba and Vadim Repesco (Moldova) *"Cubic systems with degenerate infinity and a triplet of parallel invariant straight lines"*, Parascovia Syrbu (Moldova) *"Recursively differentiable quasigroups"*, Marcel Teleuca, Ilie Lupu, and Larisa Sali (Moldova) *"Didactical aspects of the organization of investigation activities in mathematics"*, Alexandra Tkachenko (Moldova) *"Fuzzy multicriteria transportation model"*, Inga Ţiţchiev (Moldova) *"Soundness and Equivalence of Workflow Nets and Finite State Automata"*, and other.

The morning plenary session of August 24, 2012 was dedicated to the presentation of the book "Academicianul M. Ciobanu la a 70-a aniversare" (Academicin M. Choban at the 70th anniversary).

The closing session of the conference took place on August 24, 2012. It was dedicated to a broad discussion concerning the major present-day problems in the field of Mathematics, Computer Science, Physics and its Applications encountered by Moldovan and foreign researchers. The final session was followed by the traditional General Assembly of ROMAI.

The present edition of the journal "Buletinul Academiei de Ştiinţe a Republicii Moldova. Matematica" comprises some works presented at the Conference.

The Programme of the Conference and the research papers announced by the participants (one book of communications with the volume of 242 pages in the domains of Mathematics, Computer Science and its Applications and other book of communications with the volume of 266 pages in the domains of Education) were published in advance and distributed during the official opening of the Conference.

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*Council of the Mathematical Society of the Republic of Moldova*