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The Second Hankel Determinant for k-symmetrical Functions

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Abstract. In this article, we find the upper bound of the second Hankel determinant $|a_2a_4 - a_3^2|$ for subclasses of starlike and convex functions with respect to k-symmetric points.

Mathematics subject classification: 30C45, 30C50.
Keywords and phrases: Hankel determinant problems, starlike function, k-symmetric points, subordination, coefficient bound.

1 Introduction

The work here is considering the class $\mathcal{F}$ of functions analytic in $\widetilde{\mathcal{U}} = \{w : w \in \mathbb{C}, |w| < 1\}$, and of the form

$$f(w) = w + \sum_{n=2}^{\infty} a_n w^n, \tag{1}$$

and suppose $\widetilde{\mathcal{S}}$ denotes the subclass of $\mathcal{F}$ consisting of all functions that are univalent in $\widetilde{\mathcal{U}}$. For $f, g \in \mathcal{F}$, we say that f is subordinate to g written as $f \prec g$ if there exists a holomorphic map h of the unit disk $\widetilde{\mathcal{U}}$ into itself with $h(0) = 0$ such that $f = g \circ h$. Note that if $g \in \widetilde{\mathcal{S}}$, then $f \prec g$ is equivalent to the condition that $f(0) = g(0)$ and $f(\widetilde{\mathcal{U}}) \subset g(\widetilde{\mathcal{U}})$. Let $\mathcal{P}$ be the family of analytic functions p in $\widetilde{\mathcal{U}}$ with $\Re\{p(w)\} > 0$ which have the form $p(w) = 1+q_1w+q_2w^2+...$ $(w \in \widetilde{\mathcal{U}})$. The class $\mathcal{P}$ of functions with positive real part plays a crucial role in geometric function theory. Its significance can be seen from the fact that simple subclasses are class $\widetilde{\mathcal{S}}^*$ of starlike functions and class $\widetilde{\mathcal{K}}$ of convex functions.

Definition 1. [10] For $k \in \mathbb{N} = \{1, 2, ...\}$, let $\varepsilon = e^{\left(\frac{2\pi i}{k}\right)}$ denote the k^{th} root of unity for $f \in \mathcal{F}$. Its k-weighted mean function is

$$M_{f,k}(w) = \sum_{v=1}^{k-1} \varepsilon^{-v} f(\varepsilon^v w).\frac{1}{\sum_{v=1}^{k-1} \varepsilon^{-v}}.$$

A function f in $\mathcal{F}$ is called k-symmetrical function for each $w \in \widetilde{\mathcal{U}}$ if $f(\varepsilon w) = \varepsilon f(w)$. The family of all k-symmetrical functions will be denoted by $\mathcal{F}^k$.

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A function f in $\mathcal{F}$ is said to belong to the class $\widetilde{\mathcal{S}}_k^*$ of functions starlike with respect to k-symmetric points if for every r close to 1, $r<1$, the angular velocity of f about the point $M_{f_k}(w_0)$ is positive at $w=w_0$ as z traverses the circle $|w|=r$ in the positive direction, that is $\Re\left(\dfrac{zf'(w)}{f(w)-M_{f,k}(w_0)}\right)>0$ for $w=w_0$, $|w_0|=r$.

Definition 2. [27] For a positive integer k, let $\mathcal{S}_k^*$ denote the family of starlike functions with respect to k-symmetric points $f\in\mathcal{F}$ which satisfy

$$\Re\left\{\frac{wf'(w)}{f_k(w)}\right\}>0, \quad w\in\widetilde{\mathcal{U}}, \tag{2}$$

where

$$f_k(w)=\frac{1}{k}\left[f(w)-M_{f,k}(w)\right]. \tag{3}$$

Remark 1. Equivalently, (3) can be written as

$$f_k(w)=\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{-v}f(\varepsilon^v w), \tag{4}$$

or

$$f_k(w)=w+\sum_{n=2}^{\infty}\psi_n a_n w^n \quad \text{where} \quad \psi_n=\begin{cases}1 & \text{if} \quad n=lk+1, \qquad l\in\mathbb{N}_0. \\ 0 & \text{if} \quad n\neq lk+1\end{cases}. \tag{5}$$

Let $\widetilde{\mathcal{K}}_k$ denote the subclass of functions $f\in\mathcal{F}$ which satisfies

$$f\in\widetilde{\mathcal{K}}_k \Leftrightarrow wf'\in\widetilde{\mathcal{S}}_k^*. \tag{6}$$

For more details, some interesting properties of the classes of functions with respect to k-symmetric points have been discussed by the authors in [1, 2].

One of the most fundamental problems in geometric function theory is to find the coefficient bounds for a certain class of functions. In this work, we study the Hankel determinant $\widetilde{\mathcal{H}}_{\vartheta,n}(f)$ $(\vartheta,n\in\mathbb{N})$ for the well-known class of starlike functions $\widetilde{\mathcal{S}}^*$ which was introduced by Pommerenke [23, 24], and is defined as follows:

$$\widetilde{\mathcal{H}}_{\vartheta,n}(f)=\begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+\vartheta-1} \\ a_{n+1} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n+\vartheta-1} & \dots & \dots & a_{n+2\vartheta-2}\end{vmatrix}.$$

We can easily note that $\widetilde{\mathcal{H}}_{2,1}(f)=a_3-a_2^2$, $\widetilde{\mathcal{H}}_{2,2}(f)=a_2a_4-a_3^2$ and

$$\widetilde{\mathcal{H}}_{3,1}(f)=2a_2a_3a_4-a_3^3-a_4^2+a_3a_5-a_2^2a_5.$$

Many authors have studied and investigated the Hankel determinants for various subclasses of $\mathcal{F}$. The famous problem solved by using the Loewner technique to determine the greatest value of the coefficient was investigated by Fekete and Szegö in [9], they generalized the estimate $|a_3 - \mu a_2^2|$ where μ is real and $f \in \widetilde{\mathcal{S}}^*$. Later, Jae Ho Choi et al.[6] provided a new method for solving the Fekete-Szegö problem which opened up a lot of new opportunities for research in the related fields. This determinant was studied for other classes of functions by many other authors like Noor [21], Ehrenborg [8], and Layman [15]. For some other related works for subclasses regarding symmetric points, one can look up Janteng et al.[11–13] who have considered the functional $|\widetilde{\mathcal{H}}_{2,2}(f)|$ and studied the second Hankel determinant and have shown that $|\widetilde{\mathcal{H}}_{2,2}(f)| \leq 4/9$, $|\widetilde{\mathcal{H}}_{2,2}(f)| \leq 1$, $|\widetilde{\mathcal{H}}_{2,2}(f)| \leq 1/8$ and $|\widetilde{\mathcal{H}}_{2,2}(f)| \leq 1$, $|\widetilde{\mathcal{H}}_{2,2}(f)| \leq 1/9$, respectively, for the classes of analytic, starlike, convex, close-to-starlike and close-to-convex functions concerning symmetric points.

The third-order Hankel determinant $|\widetilde{\mathcal{H}}_{3,1}(f)|$ for subclasses of $\mathcal{F}$ was studied for the first time by Babalola [3]. In 2017, Zaprawa [28] improved the results of Babalola [3] by proving $|\widetilde{\mathcal{H}}_{3,1}(f)| \leq 1$, $|\widetilde{\mathcal{H}}_{3,1}(f)| \leq 49/540$, $|\widetilde{\mathcal{H}}_{3,1}(f)| \leq 41/60$ for the classes of starlike, convex and bounded turning functions respectively.

The estimation of the fourth Hankel determinant $|\widetilde{\mathcal{H}}_{4,1}(f)|$ for the bounded turning functions has been obtained by Arif et al.[16] and they proved $|\widetilde{\mathcal{H}}_{4,1}(f)| \leq 0.78050$.

Recently, Barukab et al.[4] obtained the sharp bounds of $|\widetilde{\mathcal{H}}_{3,1}(f)|$ for a collection of bounded turning functions associated with the petal-shaped domain. Khan et al.[14] investigated the third Hankel determinant for a class of starlike functions with respect to two symmetric points with a sine function. Other interesting topics have been discussed in 2021 and 2022; see [25, 26].

The aim of the present work is to determine the upper bound of the Hankel determinants of order two for the functions belonging to the classes $\widetilde{\mathcal{S}}_k^*$ and $\widetilde{\mathcal{K}}_k$.

2 Preliminary Results

Lemma 1. [7] *If $p \in \mathcal{P}$, then $|q_n| \leq 2, (n = 1, 2, ...)$.*

Lemma 2. [17, 18] *If $p \in \mathcal{P}$, then*

$$2q_2 = q_1^2 + (4 - q_1^2)x,$$

$$4q_3 = q_1^3 + 2q_1(4 - q_1^2)x - q_1(4 - q_1^2)x^2 + 2(4 - q_1^2)(1 - |x|^2)w,$$

for some x and w satisfying $|x| \leq 1$, $|w| \leq 1$ and $p_1 \in [0, 2]$.

3 Main Results

Theorem 1. *Let $f \in \widetilde{\mathcal{S}}^*_k$, then*

$$|a_2a_4 - a_3^2| \leq \frac{4}{(3-\psi_3)^2}, \tag{7}$$

where ψ_n is defined by (5).

Proof. Since $f \in \widetilde{\mathcal{S}}^*_k$, then there exists $p \in \mathcal{P}$ such that

$$\frac{wf'(w)}{f_k(w)} = p(w),$$

or

$$\frac{1+\sum_{n=2}^{\infty} na_nw^{n-1}}{\sum_{n=1}^{\infty} \psi_n a_n w^{n-1}} = 1 + q_1w + q_2w^2 + q_3w^3 + \dots. \tag{8}$$

Equating coefficients in (8) yields

$$\psi_1 = 1, \quad a_2 = \frac{q_1}{2-\psi_2}, \quad a_3 = \frac{1}{3-\psi_3}\left[q_2 + \frac{\psi_2 q_1^2}{2-\psi_2}\right], \tag{9}$$

$$a_4 = \frac{1}{4-\psi_4}\left[q_3 + \frac{\psi_2 q_1 q_2}{2-\psi_2} + \frac{\psi_3 q_1}{3-\psi_3}\left(q_2 + \frac{\psi_2 q_1^2}{2-\psi_2}\right)\right]. \tag{10}$$

By (9) and (10) we get

$$|a_2a_4 - a_3^2| =$$

$$\left|\frac{q_1}{(2-\psi_2)(4-\psi_4)}\left[q_3 + \frac{\psi_2 q_1 q_2}{2-\psi_2} + \frac{\psi_3 q_1}{3-\psi_3}\left(q_2 + \frac{\psi_2 q_1^2}{2-\psi_2}\right)\right] - \frac{1}{(3-\psi_3)^2}\left[q_2 + \frac{\psi_2 q_1^2}{2-\psi_2}\right]^2\right|.$$

Using Lemma (1) and Lemma (2) in the above equation we get

$$|a_2a_4 - a_3^2| =$$

$$\left|\frac{q_1}{4(2-\psi_2)(4-\psi_4)}\left[q_1^3 + 2p_1(4-q_1^2)x - q_1(4-q_1^2)x^2 + 2(4-q_1^2)(1-|x|^2)w\right] + \right.$$

$$\frac{q_1}{(2-\psi_2)(4-\psi_4)}\left[\frac{\psi_2 q_1}{2(2\psi_2)}\{q_1^2 + (4-q_1^2)x\} + \frac{\psi_3 q_1}{2(3-\psi_3)}\{q_1^2 + (4-q_1^2)x + \frac{2\psi_2 q_1^2}{2-\psi_2}\}\right]$$

$$-\frac{1}{(3-\psi_3)^2}\left[\frac{1}{4}\{q_1^4 + 2q_1^2(4-q_1^2)x + (4-q_1^2)^2x^2\}\right]$$

$$\left. -\frac{1}{(3-\psi_3)^2}\left[(q_1^2 + (4-q_1^2)x)\frac{\psi_2 q_1^2}{2-\psi_2} + \frac{\psi_2^2 q_1^4}{(2-\psi_2)^2}\right]\right|$$

$$= \left|\left[\frac{1}{2(2-\psi_2)(4-\psi_4)}\left\{1 + \frac{\psi_2}{2-\psi_2} + \frac{\psi_3}{3-\psi_3}\right\} - \frac{1}{2(3-\psi_3)^2} - \frac{\psi_2}{(2\psi_2)(3\psi_3)^2}\right] q_1^2(4-q_1^2)x\right.$$

$$-\left[\frac{q_1^2}{4(2-\psi_2)(4-\psi_4)}+\frac{(4-q_1^2)}{4(3-\psi_3)^2}\right](4-q_1^2)x^2+\frac{q_1(4-q_1^2)(1-|x|^2)w}{2(2-\psi_2)(4-\psi_4)}$$
$$+\frac{q_1^4}{(2-\psi_2)(4-\psi_4)}\left\{\frac{1}{4}+\frac{\psi_2}{2(2-\psi_2)}+\frac{\psi_3}{2(3-\psi_3)}+\frac{\psi_2\psi_3}{(2-\psi_2)(3-\psi_3)}\right\}$$
$$-\frac{q_1^4}{4((3-\psi_3)^2}-\frac{q_1^4\psi_2}{(2-\psi_2)(3-\psi_3)^2}-\frac{q_1^4\psi_2^2}{(2-\psi_2)^2(3-\psi_3)^2}\Bigg|.$$

Let $q_1=q$ and $0\le q\le 2$, and utilizing the assumption $|w|\le 1$, we obtain

$$|a_2a_4-a_3^2|\le \mathcal{R}_1(q)+\mathcal{R}_2(q)\mu+\mathcal{R}_3(q)\mu^2=G(q,\mu), \tag{11}$$

where $\mu=|x|\le 1$ with

$$\mathcal{R}_1(q)=q^4\left[\frac{1}{(2-\psi_2)(4-\psi_4)}\left\{\frac{1}{4}+\frac{\psi_2}{2(2-\psi_2)}+\frac{\psi_3}{2(3-\psi_3)}+\frac{\psi_2\psi_3}{(2-\psi_2)(3-\psi_3)}\right\}\right]$$
$$+q^4\left[\frac{\psi_2}{(2\psi_2)(3-\psi_3)^2}-\frac{1}{4(3-\psi_3)^2}-\frac{\psi_2^2}{(2-\psi_2)^2(3-\psi_3)^2}\right]+\frac{q(4-q^2)}{2(2-\psi_2)(4-\psi_4)},$$
$$\mathcal{R}_2(q)=\left[\frac{1}{2(2-\psi_2)(4-\psi_4)}\left\{1+\frac{\psi_2}{2-\psi_2}+\frac{\psi_3}{3-\psi_3}\right\}+\frac{\psi_2}{(2-\psi_2)(3-\psi_3)^2}-\frac{1}{2(3-\psi_3)^2}\right]q^2(4-q^2),$$
$$\mathcal{R}_3(q)=\left[\frac{q(q+2)}{4(2-\psi_2)(4-\psi_4)}+\frac{(4-q^2)}{4(3-\psi_3)^2}\right](4-q^2).$$

Now, we have to maximize $G(q,\mu)$ on the closed square $[0,2]\times[0,1]$.
By taking partial derivative of $G(q,\mu)$ in (11) with respect to μ, we get

$$\frac{\partial G(q,\mu)}{\partial\mu}=\left[\frac{q(q+2)}{4(2-\psi_2)(4-\psi_4)}+\frac{(4-q^2)}{4(3-\psi_3)^2}\right]2(4-q^2)\mu \tag{12}$$
$$+\frac{q^2(4-q^2)}{2(2-\psi_2)(4-\psi_4)}\left\{1+\frac{\psi_2}{2-\psi_2}+\frac{\psi_3}{3-\psi_3}\right\}$$
$$+\frac{q^2(4-q^2)\psi_2}{(2-\psi_2)(3-\psi_3)^2}-\frac{q^2(4-q^2)}{2(3-\psi_3)^2}.$$

For $\mu\in(0,1)$ and for fixed $q\in(0,1)$, from (12), we observe that $\frac{\partial G(q,\mu)}{\partial\mu}>0$, and then $G(q,\mu)$ is increasing in μ, for fixed $p\in[0,2]$, the maximum of $G(q,\mu)$ occurs at $\mu=1$ and

$$\max_{0\le\mu\le1}G(q,\mu)=G(q,1)=F(q). \tag{13}$$

From (11) and (13), upon simplification, we get

$$F(q)=G(q,1)=\frac{\psi_2\psi_3q^4}{(2-\psi_2)^2(3-\psi_3)(4-\psi_4)}+\frac{q^4}{2(3-\psi_3)^2}-\frac{\psi_2^2q^4}{(2-\psi_2)^2(3-\psi_3)^2}$$
$$-\frac{q^4}{2(2-\psi_2)(4-\psi_4)}-q^3+\frac{2q^2}{4-\psi_4}\left\{1+\frac{\psi_2}{2-\psi_2}+\frac{\psi_3}{3-\psi_3}\right\}-\frac{q^2}{(3-\psi_3)^2} \tag{14}$$

$$+\frac{4q^2\psi_2}{(2-\psi_2)(3-\psi_3)^2}+\frac{q^2}{(2-\psi_2)(4-\psi_4)}+\frac{4}{(2-\psi_2)(4-\psi_4)}q+\frac{4}{(3-\psi_3)^2}.$$

Suppose that $F(q)$ has a maximum value at $q \in (0,2)$. Now by differentiating with respect to q and after some simple calculations we find

$$F'(q)=\frac{4\psi_2\psi_3q^3}{(2-\psi_2)^2(3-\psi_3)(4-\psi_4)}+\frac{4q^3}{2(3-\psi_3)^2}-\frac{4q^3\psi_2^2}{2-\psi_2)^2(3-\psi_3)^2}$$

$$-\frac{4q^3}{2(2-\psi_2)(4-\psi_4)}-3q^2+\frac{4q}{(2-\psi_2)(4-\psi_4)}\left\{1+\frac{\psi_2}{2-\psi_2}+\frac{\psi_3}{3-\psi_3}\right\}$$

$$-\frac{2q}{(3-\psi_3)^2}+\frac{8q\psi_2}{(2-\psi_2)(3-\psi_3)^2}+\frac{2q}{(2-\psi_2)(4-\psi_4)}+\frac{4}{(2-\psi_2)(4-\psi_4)}.$$

Clearly, $F'(q)=0$ has no optimal solutions in $(0,2)$. Thus, $F(q)$ achieves its maximum value outside the interval, which contradicts our assumption of having the maximum value at the interior point of $q \in [0,2]$. Thus any maximum point of F must be on the boundary of $[0,2]$.
It is clear that $F(0) > F(2)$. Hence the maximum is achieved at $q=0$. Therefore the upper bound for (11) corresponds to $\mu = 1$ and $q = 0$. Hence from (11) we obtain (7).

□

For $k=1$ in Theorem 7, we have the following result proved by Janteng [12].

Corollary 1. *If $f(w) \in \widetilde{\mathcal{S}}^*$, then*

$$|a_2a_4-a_3^2| \le 1.$$

We can prove on similar lines the following theorem.

Theorem 2. *Let $f \in \widetilde{\mathcal{K}}_k$, then*

$$|a_2a_4-a_3^2| \le \frac{4}{9(3-\psi_3)^2}, \tag{15}$$

where ψ_n is defined by (5).

Replacing k by 1 in Theorem 2, we have the following result proved by Janteng [12].

Corollary 2. *If $f(w) \in \widetilde{\mathcal{K}}$, then*

$$|a_2a_4-a_3^2| \le \frac{1}{9}.$$

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Relative Separation Axioms via Semi-Open Sets

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Abstract. The concept of relative topological properties was introduced by Arhangel'skii and Gennedi and was subsequently investigated by many authors for different notions of general topology. In this paper few semi-separation axioms in relative sense are introduced and studied by utilizing semi-open sets. Characterizations and preservation under mapping of these newly defined notions are provided. Relationship that exists between these notions, with some of the absolute properties and with the existing relative separation axioms are investigated.

Mathematics subject classification: 54D15.
Keywords and phrases: semi-open, s-regular, s-normal, relative topological properties, relatively regular, relatively normal.

1 Introduction and Preliminaries

The notions of semi-open set and semi-continuity were introduced by N. Levine [14] and were subsequently utilized by several researchers in different settings. A set A is said to be semi-open in a topological space X if there is an open set U such that $U \subset A \subset cl(U)$, where $cl(U)$ is the closure of U in X. The condition of being semi-open is weaker than the condition of being open. A function $f : X \to Y$ is said to be semi-continuous if the inverse image of every open set is semi-open. Semi-closed sets, semi-interior and semi-closure were defined by S. Gene Crossley and S. K. Hildebrand in a manner analogous to the corresponding concepts of closed set, interior and closure [6]. Semi-open and semi-closed functions were defined by Biswas [5] and Noiri [19]. According to them a function $f : X \to Y$ is semi-open if the image of every open set is semi-open and $f : X \to Y$ is said to be semi-closed if the image of every closed set is semi-closed. Various separation axioms have been defined using semi-open sets. Maheshwari and Prasad in [15–17] defined semi-T_i, $i = 0, 1, 2$, s-regular, and s-normal spaces respectively just by replacing open sets by semi-open sets in definition of T_i, $i = 0, 1, 2$, regular, and normal space. Charles Dosett in [10] further investigated these separation axioms and established relationships with each other and with other notions. Crossley and Hildebrand gave the concept of semi-homeomorphism [7] and stated that a property of topological spaces is defined to be a semi-topological property if it is preserved by semi-homeomorphism. They showed that some of the topological properties like first category, Hausdorffness, separability and connectedness are semi-topological properties. Hamlett showed that the property of a topological space being a Baire space is semi-topological [11]. Nayar

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and Arya [18] developed techniques which help to establish whether a topological property is semi-topological or not. Till now a lot of work has been done in general topology using semi-open sets.

In this paper we study some relative versions of semi-separation axioms. We establish the relationship of relative semi-separation axioms with the absolute properties and with the existing relative separation axioms. Characterizations of relatively s-regular and relatively s-normal are also given. Behavior of these spaces under mapping is also studied. In this paper we proved that for $Y \subset X$, Y is relatively s-regular in X iff $\pi_R(Y)$ is relatively s-regular in X_R, where R is an equivalence relation on X, X_R denotes the quotient space X/R and $\pi_R : X \to X/R$ is canonical projection map defined by $\pi_R(x) = [x]$. This result is similar to Dosett's Theorem 3.3. [10] generalized in relative sense. A number of examples and counter examples are also provided in support of various statements.

Let $Y \subset X$. Y is said to be T_1 in X or *relatively* T_1 [2] if for every $y \in Y$, $\{y\}$ is closed in X. Y is said to be T_2 in X or *relatively* T_2 [2] if for every pair of distinct points in Y there exist disjoint open sets in X separating them. Y is said to be regular in X [2] or *relatively regular* if for every closed set A of X and a point $y \in Y$ such that $y \notin A$, there exist disjoint open subsets U and V of X such that $A \cap Y \subset U$ and $y \in V$. Y is said to be normal in X or *relatively normal* [2] if for every pair of disjoint closed sets A and B of X, there exist disjoint open subsets U and V of X such that $A \cap Y \subset U$ and $B \cap Y \in V$.

Throughout this paper the semi-closure of A in X is denoted by $scl(A)$ and the semi-interior of A in X is denoted by $sint(A)$.

2 Relative Semi-Separation Axioms

Semi-T_i, for $i = 0, 1, 2$, in relative sense can be defined in the same manner as relatively T_i just by replacing open sets by semi-open sets. It is clear from the definitions that the condition of being semi-T_i is stronger than the condition of being relatively semi-T_i. Also the condition of being relatively T_i is stronger than the condition of being relatively semi-T_i.

Definition 1. $Y \subset X$ is said to be relatively s-regular if for every closed set A in X and a point $y \in Y$ such that $y \notin A$, there exist disjoint semi-open sets U and V in X such that $y \in U$ and $A \cap Y \subset V$.

Definition 2. $Y \subset X$ is said to be relatively s-normal if for every pair of closed sets A and B in X, there exist disjoint semi-open sets U and V in X such that $A \cap Y \subset U$ and $B \cap Y \subset V$.

It is clear from the definitions that Y is relatively s-normal if X is s-normal and Y is relatively s-normal if Y is relatively normal. Similarly Y is relatively s-regular if X is s-regular and Y is relatively s-regular if Y is relatively regular. The following are some examples showing that none can be reversed.

Example 1. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, X, \{b, c, d\}, \{a, b, d\}, \{b, d\}, \{d\}, \{b\}\}$. Let $Y = \{a, c\}$. Semi-open sets of X other than all open sets includes $\{a, d\}$, $\{c, d\}$, $\{a, b\}$ and $\{a, c\}$. Clearly Y is relatively s-normal in X but Y is not relatively normal in X as $\{a\}$ and $\{c\}$ are closed in X which cannot be separated by disjoint semi-open sets in X.

Example 2. Let $X = \{a, b, c, \}$ and $\tau = \{\{a\}, \{a, b\}, \{a, c\}, X, \phi\}$. Let $Y = \{a\}$. The only semi-open sets of X are all open sets. X is not s-normal because $\{b\}$ and $\{c\}$ are disjoint closed sets in X which cannot be separated by disjoint semi-open sets in X. But Y is relatively s-normal.

Theorem 1. *$Y \subset X$ is said to be relatively s-regular if and only if for every $y \in Y$ and every open set O in X containing y, there exists semi-open set U in X such that $y \in U \subset scl(U) \subset O \cap X \setminus Y$.*

Proof. Let Y be a relatively s-regular space. Let $y \in Y$ and O be an open set in X such that $y \in O$. $X \setminus O$ is closed in X and $y \notin X \setminus O$. Since Y is relatively s-regular, there exist disjoint semi-open sets U and V in X such that $y \in U$ and $(X \setminus O) \cap Y \subset V$, thus $y \in U \subset X \setminus V \subset O \cup X \setminus Y$. $X \setminus V$ being semi-closed implies $y \in U \subset scl(U) \subset X \setminus V \subset O \cup X \setminus Y$.

Conversely let $y \in Y$ and A be a closed set in X such that $y \notin A$. $X \setminus A$ is open in X and $y \in X \setminus A$, there exists semi-open set U such that $y \in U \subset scl(U) \subset (X \setminus A) \cap X \setminus Y$, which implies $y \in U \subset scl(U) \subset X \setminus (A \cap Y)$. Now let $V = X \setminus scl(U) = sint(X \setminus U)$, therefore V is the largest semi-open set contained in $X \setminus U$. Also $scl(U) \subset X \setminus (A \cap Y)$ which implies $A \cap Y \subset X \setminus scl(U) = V$. Hence U and V are disjoint semi-open sets such that $y \in U$ and $A \cap Y \subset V$. Thus Y is relatively s-regular space. □

Theorem 2. *$Y \subset X$ is relatively s-normal if and only if for every closed set A of X and every open set B of X containing A, there exists semi-open set U in X such that $A \cap Y \subset U \subset scl(U) \subset B \cap X \setminus Y$.*

Proof. Let Y be a relatively s-normal space. Let A be a closed set in X and B be an open set in X containing A. Then A and $X \setminus B$ are disjoint closed sets in X. Since Y is relatively s-normal, there exist disjoint semi-open sets U and V in X such that $A \cap Y \subset U$ and $(X \setminus B) \cap Y \subset V$, thus $A \cap Y \subset U \subset X \setminus V \subset B \cup X \setminus Y$. $X \setminus V$ being semi-closed implies $A \cap Y \subset U \subset scl(U) \subset X \setminus V \subset B \cup X \setminus Y$.

Conversely let A and B be disjoint closed sets in X. Since $A \subset X \setminus B$ which is open in X, there exists a semi-open set U in X such that $A \cap Y \subset U \subset scl(U) \subset (X \setminus B) \cap X \setminus Y \subset X \setminus (B \cap Y)$. Now let $V = X \setminus scl(U) = sint(X \setminus U)$, therefore V is the largest semi-open set contained in $X \setminus U$. Also $scl(U) \subset X \setminus (B \cap Y)$ which implies $B \cap Y \subset X \setminus scl(U) = V$. Here U and V are disjoint semi-open sets such that $A \cap Y \subset U$ and $B \cap Y \subset V$. Hence Y is relatively s-normal space. □

Proof of the following theorem is obvious from definitions.

Theorem 3. *Every relatively T_0, relatively s-regular space is relatively semi-T_2.*

Definition 3. A topological space X is said to be R_0 if for every open set G in X, $x \in G$ implies $cl(\{x\}) \subset G$.

Theorem 4. *In an R_0 space every relatively s-normal subset is relatively s-regular.*

Proof. Let Y be relatively s-normal and X be an R_0 space. Let A be a closed set in X and $y \in Y$ be such that $y \notin A$. X being R_0, $cl\{y\} \cap A = \phi$. Now A and $cl\{y\}$ are two disjoint open sets in X and Y is relatively s-normal, there exist disjoint semi-open sets U and V in X such that $cl\{y\} \subset U$ and $A \cap Y \subset V$. □

The following corollary follows from the fact that every T_1 space is R_0 space.

Corollary 1. *In a T_1 space any relatively s-normal space is relatively s-regular.*

Theorem 5. *Every relatively T_1, relatively s-normal space is relatively s-regular.*

In the above Corollary and Theorem, T_1 and relatively T_1 cannot be replaced by semi-T_2 and relatively semi-T_2 respectively as is evident from the following example.

Example 3. Let $X_1 = \{a, b, c\}$ and $X_2 = [0, 1]$, $T = \{X_1, \phi, \{a\}, \{a, b\}, \{c\}, \{a, c\}\}$ be a topology on X_1 and S be the usual topology on X_2. Then $(X_1 \times X_2, P)$, where P denotes the product topology on $X_1 \times X_2$, is s-normal, semi-T_2 [10]. Let $Y = \{a, b\} \times X_2$. Y is not relatively s-regular since $C = \{b\} \times X_2$ is closed in X and $y = (a, 1/2) \in Y$, $y \notin C$, and there do not exist disjoint semi-open sets containing y and C, respectively.

Definition 4. [20] $Y \subset X$ is said to be relatively almost normal if for any two disjoint closed subsets A and B of X such that one of them is regularly closed, there exist disjoint open sets U and V in X such that $A \cap Y \subset U$ and $B \cap Y \subset V$.

Definition 5. [20] $Y \subset X$ is said to be relatively κ-normal if for any two disjoint regularly closed subsets A and B of X, there exist disjoint open sets U and V in X such that $A \cap Y \subset U$ and $B \cap Y \subset V$.

Example 4. Relative almost normality does not imply relative s-normality.
Consider Example 2. Let $Y = \{b, c\}$. Y is relatively almost normal because the only regularly closed sets are ϕ and X. But Y is not relatively s-normal.

Example 5. Relative s-normality does not imply relative κ-normality.
Let X be the set of integers. Define a topology τ on X, where every odd integer is open and a set U is open if for even integer $p \in U$ the successor and the predecessor of p also belong to U. This topology is called odd-even topology. Let Y be the set of all even integers. $A = \{4, 5, 6\}$ and $B = \{8, 9, 10\}$ are regularly closed in X. But $A \cap Y = \{4, 6\}$ and $B \cap Y = \{8, 10\}$, which cannot be separated by disjoint open sets in X. Hence Y is not relatively κ-normal in X. If we denote any even integer by e and any odd integer by o, then the semi-open sets of X are of the form $\{o, o, ...e\}$, $\{e, o, o.., o\}$, $\{e, o, o, ...e\}$, $\{o, o, ..., e, o, ..., o\}$, $\{o, o, ..., o\}$ and the sets which are not semi-open are the sets containing two consecutive even numbers and no odd number between them, like $\{2, 4, 5\}$ is not semi-open. Here in this case we can easily check that Y is relatively s-normal.

From the above examples we conclude that the concept of relatively s-normal is independent of relatively almost normal and relatively κ-normal.

Definition 6. X is said to be *β-normal* [4] if for any two disjoint closed subsets A and B of X, there exist open subsets U and V of X such that $A \cap U$ is dense in A and $B \cap V$ is dense in B and $cl(U) \cap cl(V) = \phi$.

Definition 7. [9] $Y \subset X$ is said to be relatively super β-normal if for any two disjoint subsets A and B closed in X, there exist open subsets U and V of X such that $(A \cap Y) \cap U$ is dense in A and $(B \cap Y) \cap V$ is dense in B and $cl(U) \cap cl(V) = \phi$.

Definition 8. [9] $Y \subset X$ is said to be relatively strong by β-normal if for any two disjoint subsets A and B closed in Y, there exist open subsets U and V of X such that $A \cap U$ is dense in A and $B \cap V$ is dense in B and $cl(U) \cap cl(V) = \phi$.

Theorem 6. *[9] In the class of relatively super β-normal spaces, every κ-normal space is normal.*

Theorem 7. *[9] In the class of β-normality (relative β-normality or relative strong β-normality) every κ-normal space is relatively normal.*

From above results the following results are obvious.

Theorem 8. *In the class of relative super β-normality every κ-normal space is s-normal.*

Theorem 9. *In the class of β-normality (relative β-normality or relative strong β-normality) every κ-normal space is relatively s-normal.*

Definition 9. [3] $Y \subset X$ is said to be relatively superregular if for every closed set A in X and a point $y \in Y$ such that $y \notin A$, there exist disjoint open sets U and V in X such that $A \subset U$ and $y \in V$.

Theorem 10. *The image of a relatively superregular space under continuous, semi-closed, semi-open and onto map is relatively s-regular.*

Proof. Let $f : (X_1, T) \to (X_2, S)$ be a continuous, semi-open, semi-closed and onto map. Let $Y \subset X$ be relatively superregular. Let O be an open set in X_2 and $y_2 \in f(Y)$. Let $y_1 \in f^{-1}(y_2)$. Since f is continuous, $f^{-1}(O)$ is open in X_1 and $y_1 \in f^{-1}(O)$. Since Y is relatively superregular, there exists an open set U in X_1 such that $y_1 \in U \subset cl(U) \subset O$ which implies $y_2 \in f(U) \subset f(cl(U)) \subset f(O)$. Since f is semi-open, $cl(U)$ is closed in X_1, $f(cl(U))$ is semi-closed in X_2 and $scl(f(U))$ is the smallest semi-closed set containing $f(U)$. Therefore $y_2 \in f(U) \subset scl(f(U)) \subset f(cl(U)) \subset f(O) \subset f(O) \cup X_2 \setminus f(Y)$. Hence $f(Y)$ is relatively s-regular. □

Remark 1. Relative supperregularity cannot be replaced by relative s-regularity in the above theorem.

Let $X_1 = \{a, b, c, d\}$ with topology $T = \{\{a, b, d\}, \{b, c, d\}, \{b, d\}, \{d\}, \{b\}, X_1, \phi\}$ and

$X_2 = \{e, f, g\}$ with topology $S = \{\{e, f\}, \{e, g\}, \{e\}, X_2, \phi\}$. Define $f : X_1 \to X_2$ as $f(a) = f$, $f(b) = e$, $f(c) = g$ and $f(d) = e$. Then this map is continuous, onto, semi-open and semi-closed. Let $Y = \{a, c\}$. Clearly Y is relatively s-regular (not relatively superregular) and $f(Y)$ is not relatively s-regular.

Theorem 11. *$Y \subset X$ is relatively s-normal if for every pair of disjoint closed sets A and B in X, there exists a semi-continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

The Theorem stated above is actually one-sided Urysohn's Lemma type result and can be proved easily.

Let X be a topological space and R be an equivalence relation on X defined as xRy iff $cl(\{x\}) = cl(\{y\})$. The resulting quotient space X/R is actually T_0 and is called T_0-identification of X. Also this quotient space is a decomposition space as X/R is a set of equivalence classes of R which forms a partition of X. The canonical projection map $\pi_R : X \to X/R$ defined by $\pi_R(x) = [x]$ is the decomposition map. For simplicity we are using X_R instead of X/R.

Theorem 12. *$Y \subset X$ is relatively s-regular in X iff $\pi_R(Y)$ is relatively s-regular in X_R.*

Proof. Let $Y \subset X$ be relatively s-regular. Let $\mathcal{C}$ be a closed set in X_R and $C \in \pi_R$ such that $C \notin \mathcal{C}$. Let $y \in C$, then $[y] = C$. Since π_R is continuous, $\pi_R(\mathcal{C})$ is closed in X. Also $y \notin \pi_R^{-1}(\mathcal{C})$ and $y \in Y$. By relative s-regularity of Y in X, there exist disjoint semi-open sets A and B in X such that $y \in A$ and $\pi_R^{-1}(\mathcal{C}) \cap Y \subset B$. Since A and B are semi-open sets, there exist open sets U and V in X such that $U \subset A \subset cl(U)$ and $V \subset B \subset cl(V)$ which implies that $U \cup \{y\}$ and $V \cup \pi_R^{-1}(\mathcal{C}) \cap Y$ are disjoint semi-open sets in X. Now since π_R is open and continuous and $\pi_R^{-1}(\pi_R(O)) = O$ for all O open in X, $D = \pi_R(U \cup Y)$ and $E = \pi_R(V \cup \pi_R^{-1}(\mathcal{C}) \cap Y)$ are disjoint semi-open sets in X_R containing $[y]$ and $\mathcal{C} \cap \pi_R(Y)$. Hence $\pi_R(Y)$ is s-regular.

Conversely suppose that $\pi_R(Y)$ is relatively s-regular in X_R. Let C be a closed set in X and $y \in Y$ such that $y \notin C$. Then $[y] \cap \pi_R(C) = \phi$. Since π_R is closed, $\pi_R(C)$ is closed in X_R. By relative s-regularity of $\pi_R(Y)$ for $[y] \in \pi_R(Y)$ and as $\pi_R(C)$ is closed in X_R and $[y] \cap \pi_R(C)$, there exist semi-open sets U and V in X_R such that $[y] \in U$ and $\pi_R(C) \cap \pi_R(Y) \subset V$. Since π_R is continuous and open, $\pi_R^{-1}(U)$ and $\pi_R^{-1}(V)$ are disjoint semi-open sets in X containing y and $C \cap Y$. Hence Y is relatively s-regular in X. □

Remark 2. The space X_R in the above theorem is T_0, $\pi_R(Y)$ is relatively T_0. By Theorem 3 if $\pi_R(Y)$ is relatively s-regular then $\pi_R(Y)$ is relatively semi-T_2.

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Growth Properties of Solutions to Higher Order Complex Linear Differential Equations with Analytic Coefficients in the Annulus

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Abstract. In this paper, by using the Nevanlinna value distribution theory of meromorphic functions on an annulus, we deal with the growth properties of solutions of the linear differential equation $f^{(k)} + B_{k-1}(z) f^{(k-1)} + \cdots + B_1(z) f' + B_0(z) f = 0$, where $k \geq 2$ is an integer and $B_{k-1}(z), ..., B_1(z), B_0(z)$ are analytic on an annulus. Under some conditions on the coefficients, we obtain some results concerning the estimates of the order and the hyper-order of solutions of the above equation. The results obtained extend and improve those of Wu and Xuan in [16].

Mathematics subject classification: 30D10, 30D20, 30B10, 34M05.
Keywords and phrases: linear differential equations, analytic solutions, annulus, hyper order.

1 Introduction and results

Throughout this article, we shall assume that the reader is familiar with the standard notations and fundamental results of the Nevanlinna value distribution theory of meromorphic functions in the complex plane and in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, see [3, 4, 10, 14, 17].

Nevanlinna theory has appeared to be powerful tool in the field of complex differential equations in the complex plane and in the unit disc which are simple connected domains. In the year 2000, Heittokangas [5] firstly investigated the growth and oscillation theory of second and higher order linear differential equations when the coefficients are analytic functions in the unit disc $\mathbb{D}$, by introducing the definition of the function spaces. Recently, Wu [15], Long [11], Belaïdi [2], Zemirni and Belaïdi [18] have obtained some results about the growth of analytic solutions of higher order linear differential equations in a sector of the unit disc. It is well-known that Nevanlinna theory of meromorphic functions in the complex plane and in the unit disc can be extended in a modified form to multiply-connected plane domains, in particular in the annulus [6–9, 12, 13] which is a doubly-connected domain. In 2005, Khrystiyanyn and Kondratyuk [6, 7] gave an extension of the Nevanlinna value distribution theory for meromorphic functions in annuli. In their extension the main characteristics of meromorphic functions are one-parameter and possessing the same

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properties as in the classical case of a simply connected domain. From the doubly-connected mapping theorem [1], we can get that each doubly-connected domain is conformally equivalent to the annulus $\{z : r < |z| < R,\ 0 \leq r < R \leq +\infty\}$. We consider only two cases: $r = 0$, $R = +\infty$ simultaneously and $0 \leq r < R \leq +\infty$. In the latter case, the homothety $z \longmapsto \frac{z}{\sqrt{rR}}$ reduces the given domain to the annulus $\frac{1}{R_0} < |z| < R_0$, where $R_0 = \sqrt{\frac{R}{r}}$. Thus, every annulus is invariant with respect to the inversion $z \longmapsto \frac{1}{z}$ in two cases.

Before stating our main results, we give some notations and basic definitions of meromorphic functions in the annulus $\mathcal{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $1 < R_0 \leq +\infty$. Let f be a meromorphic function in the complex plane, we define

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left|f\left(re^{i\varphi}\right)\right| d\varphi,$$

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r$$

and

$$T(r, f) = m(r, f) + N(r, f) \ \ (r > 0)$$

is the Nevanlinna characteristic function of f, where $\log^+ x = \max(0, \log x)$ for $x \geq 0$, and $n(t,\ f)$ is the number of poles of f lying in $\{z : |z| \leq t\}$, counted according to their multiplicity. Now, we give the Nevanlinna theory in the annulus $\mathcal{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $1 < R_0 \leq +\infty$. Set

$$N_1(r, f) = \int_{\frac{1}{r}}^1 \frac{n_1(t, f)}{t} dt, \quad N_2(r, f) = \int_1^r \frac{n_2(t, f)}{t} dt,$$

$$m_0(r, f) = m(r, f) + m\left(\frac{1}{r}, f\right) - 2m(1, f),$$

$$N_0(r, f) = N_1(r, f) + N_2(r, f),$$

where $n_1(t, f)$ and $n_2(t, f)$ are the counting functions of poles of f lying in $\{z : t < |z| \leq 1\}$ and $\{z : 1 < |z| \leq t\}$ respectively, counted according to their multiplicity. The Nevanlinna characteristic of f in the annulus $\mathcal{A}$ is defined by

$$T_0(r, f) = m_0(r, f) + N_0(r, f).$$

Definition 1. ([16]) Let f be a nonconstant meromorphic function in the annulus $\mathcal{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $1 < R_0 \leq +\infty$. The function f is called a transcendental or admissible in $\mathcal{A}$ provided that

$$\limsup_{r \to +\infty} \frac{T_0(r, f)}{\log r} = +\infty \text{ if } 1 < r < R_0 = +\infty$$

or

$$\limsup_{r\to R_0^-}\frac{T_0(r,f)}{\log\frac{1}{R_0-r}}=+\infty \text{ if } 1<r<R_0<+\infty$$

respectively. The order of f is defined as

$$\rho_{\mathcal{A}}(f)=\limsup_{r\to+\infty}\frac{\log T_0(r,f)}{\log r} \text{ if } 1<r<R_0=+\infty$$

or

$$\rho_{\mathcal{A}}(f)=\limsup_{r\to R_0^-}\frac{\log T_0(r,f)}{\log\frac{1}{R_0-r}} \text{ if } 1<r<R_0<+\infty$$

respectively. The hyper-order of f is defined as

$$\rho_{2,\mathcal{A}}(f)=\limsup_{r\to+\infty}\frac{\log\log T_0(r,f)}{\log r} \text{ if } 1<r<R_0=+\infty$$

or

$$\rho_{2,\mathcal{A}}(f)=\limsup_{r\to R_0^-}\frac{\log\log T_0(r,f)}{\log\frac{1}{R_0-r}} \text{ if } 1<r<R_0<+\infty$$

respectively.

Now, we introduce the concepts of lower order, hyper lower order, type and lower type of a meromorphic function f in the annulus $\mathcal{A}$.

Definition 2. Let f be a meromorphic function in $\mathcal{A}$. The lower order of f is defined as

$$\mu_{\mathcal{A}}(f)=\liminf_{r\to+\infty}\frac{\log T_0(r,f)}{\log r} \text{ if } 1<r<R_0=+\infty$$

or

$$\mu_{\mathcal{A}}(f)=\liminf_{r\to R_0^-}\frac{\log T_0(r,f)}{\log\frac{1}{R_0-r}} \text{ if } 1<r<R_0<+\infty$$

respectively. The hyper lower order of f is defined as

$$\mu_{2,\mathcal{A}}(f)=\liminf_{r\to+\infty}\frac{\log\log T_0(r,f)}{\log r} \text{ if } 1<r<R_0=+\infty$$

or

$$\mu_{2,\mathcal{A}}(f)=\liminf_{r\to R_0^-}\frac{\log\log T_0(r,f)}{\log\frac{1}{R_0-r}} \text{ if } 1<r<R_0<+\infty$$

respectively.

Definition 3. Let f be a meromorphic function in $\mathcal{A}$ with order $0<\rho_{\mathcal{A}}(f)<+\infty$. Then, the type of f is defined by

$$\tau_{\mathcal{A}}(f)=\limsup_{r\to+\infty}\frac{T_0(r,f)}{r^{\rho_{\mathcal{A}}(f)}} \text{ if } 1<r<R_0=+\infty$$

or

$$\tau_{\mathcal{A}}(f) = \limsup_{r \to R_0^-} \frac{T_0(r,f)}{\left(\frac{1}{R_0 - r}\right)^{\rho_{\mathcal{A}}(f)}} \text{ if } 1 < r < R_0 < +\infty$$

respectively. Similarly, let f be a meromorphic function in $\mathcal{A}$ with lower order $0 < \mu_{\mathcal{A}}(f) < +\infty$. Then, the lower type of f is defined by

$$\underline{\tau}_{\mathcal{A}}(f) = \liminf_{r \to R_0^-} \frac{T_0(r,f)}{r^{\mu_{\mathcal{A}}(f)}} \text{ if } 1 < r < R_0 = +\infty$$

or

$$\underline{\tau}_{\mathcal{A}}(f) = \liminf_{r \to R_0^-} \frac{T_0(r,f)}{\left(\frac{1}{R_0 - r}\right)^{\mu_{\mathcal{A}}(f)}} \text{ if } 1 < r < R_0 = +\infty$$

respectively.

For $k \geq 2$, we consider the linear differential equation

$$f^{(k)} + B_{k-1}(z) f^{(k-1)} + \cdots + B_1(z) f' + B_0(z) f = 0, \tag{1}$$

where $B_{k-1}(z), ..., B_1(z)$ and $B_0(z)$ are analytic in the annulus

$$\mathcal{A} = \left\{ z : \frac{1}{R_0} < |z| < R_0 \right\} \ (1 < R_0 \leq +\infty).$$

Recently in [16], Wu and Xuan have studied the growth of solutions of higher order linear complex differential equations in $\mathcal{A}$ and obtained the following result.

Theorem 1. *([16]) Let $B_{k-1}(z), ..., B_1(z)$, $B_0(z)$ be analytic functions in the annulus $\mathcal{A} = \left\{ z : \frac{1}{R_0} < |z| < R_0 \right\}$ $(1 < R_0 \leq +\infty)$ such that*

$$\max\{\rho_{\mathcal{A}}(B_j) : j = 1, 2, ..., k-1\} < \rho_{\mathcal{A}}(B_0).$$

Then every solution $f \not\equiv 0$ of equation (1) satisfies $\rho_{\mathcal{A}}(f) = +\infty$ and $\rho_{2,\mathcal{A}}(f) \geq \rho_{\mathcal{A}}(B_0)$.

Note that the result of Theorem 1 occurs when there exists only one dominant coefficient. Thus, the following question arises naturally: Whether the results similar to Theorem 1 can be obtained in $\mathcal{A}$ if there are more than one dominant coefficients? In this paper, we give some answers to the above question. In fact, by using the concepts of the type and the lower type, we obtain some results which indicate growth estimate of every non-trivial analytic solution of equation (1) by the growth estimate of the coefficient $B_0(z)$. We mainly obtain the following results.

Theorem 2. *Let $B_{k-1}(z), ..., B_1(z),\ B_0(z)\ (k \geq 2)$ be analytic functions in the annulus $\mathcal{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$ $(1 < R_0 \leq +\infty)$. Suppose that there exist three positive real numbers α, β and μ with $0 \leq (k-1)\beta < \alpha$, $\mu > 0$, such that we have*

$$T_0(r, B_0) \geq \alpha r^{\mu} \tag{2}$$

and

$$T_0(r, B_j) \leq \beta r^{\mu},\ j = 1, \ldots, k-1 \tag{3}$$

if $1 < r < R_0 = +\infty$ as $|z| = r \to +\infty$ for $r \in E_r$ which satisfies $\int\limits_{E_r} \frac{dr}{r} = +\infty$, or

$$T_0(r, B_0) \geq \frac{\alpha}{(R_0 - r)^{\mu}} \tag{4}$$

and

$$T_0(r, B_j) \leq \frac{\beta}{(R_0 - r)^{\mu}}\ \ (j = 1, ..., k-1) \tag{5}$$

if $1 < r < R_0 < +\infty$ as $|z| = r \to R_0^-$ for $r \in F_r$ which satisfies $\int\limits_{F_r} \frac{dr}{R_0 - r} = +\infty$. Then every solution $f \not\equiv 0$ of equation (1) satisfies $\rho_{\mathcal{A}}(f) = +\infty$ and $\rho_{2,\mathcal{A}}(f) \geq \mu$.

Theorem 3. *Let $B_{k-1}(z), ..., B_1(z),\ B_0(z)\ (k \geq 2)$ be analytic functions in the annulus $\mathcal{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$ $(1 < R_0 \leq +\infty)$ such that*

$$\max\{\rho_{\mathcal{A}}(B_j) : j = 1, 2, ..., k-1\} \leq \rho_{\mathcal{A}}(B_0) = \rho\ \ (0 < \rho < \infty)$$

and

$$\sum_{\rho_{\mathcal{A}}(B_j) = \rho_{\mathcal{A}}(B_0)} \tau_{\mathcal{A}}(B_j) < \tau_{\mathcal{A}}(B_0) = \tau\ \ (\ 0 < \tau < \infty).$$

Then every solution $f \not\equiv 0$ of equation (1) satisfies $\rho_{\mathcal{A}}(f) = +\infty$ and $\rho_{2,\mathcal{A}}(f) \geq \rho_{\mathcal{A}}(B_0)$.

Theorem 4. *Let $B_{k-1}(z), ..., B_1(z),\ B_0(z)\ (k \geq 2)$ be analytic functions in the annulus $\mathcal{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$ $(1 < R_0 \leq +\infty)$ such that $0 < \mu_{\mathcal{A}}(B_0) = \mu \leq \rho_{\mathcal{A}}(B_0) < \infty$. Assume that*

$$\max\{\rho_{\mathcal{A}}(B_j) : j = 1, 2, ..., k-1\} \leq \mu_{\mathcal{A}}(B_0) = \mu$$

and

$$\sum_{\rho_{\mathcal{A}}(B_j) = \mu_{\mathcal{A}}(B_0)} \tau_{\mathcal{A}}(B_j) < \underline{\tau}_{\mathcal{A}}(B_0) = \underline{\tau}\ \ (\ 0 < \underline{\tau} < \infty).$$

Then every solution $f \not\equiv 0$ of equation (1) satisfies $\rho_{\mathcal{A}}(f) = \mu_{\mathcal{A}}(f) = +\infty$ and $\rho_{2,\mathcal{A}}(f) \geq \mu_{2,\mathcal{A}}(f) \geq \mu_{\mathcal{A}}(B_0)$.

Theorem 5. *Let $B_{k-1}(z), ..., B_1(z),\ B_0(z)\ (k \geq 2)$ be analytic functions in the annulus $\mathcal{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}\ (1 < R_0 \leq +\infty)$ such that $B_0(z)$ is admissible in $\mathcal{A}$ and*

$$\limsup_{r \to +\infty} \frac{\sum_{j=1}^{k-1} m_0(r, B_j)}{m_0(r, B_0)} < 1 \text{ if } 1 < r < R_0 = +\infty$$

or

$$\limsup_{r \to R_0^-} \frac{\sum_{j=1}^{k-1} m_0(r, B_j)}{m_0(r, B_0)} < 1 \text{ if } 1 < r < R_0 < +\infty.$$

Then every solution $f \not\equiv 0$ of equation (1) satisfies $\rho_{\mathcal{A}}(f) = +\infty$ and $\rho_{2,\mathcal{A}}(f) \geq \rho_{\mathcal{A}}(B_0)$.

2 Some Preliminary Lemmas

We need the following lemmas to prove our results.

Lemma 1. *Let f be a meromorphic function with finite order $0 < \rho_{\mathcal{A}}(f) < +\infty$ and finite type $0 < \tau_{\mathcal{A}}(f) < +\infty$. Then for any given $\eta < \tau_{\mathcal{A}}(f)$, there exists a subset E_r of $(1, +\infty)$ with $\int_{E_r} \frac{dr}{r} = +\infty$ such that for all $r \in E_r$*

$$T_0(r, f) > \eta r^{\rho_{\mathcal{A}}(f)} \text{ if } 1 < r < R_0 = +\infty$$

holds or there exists a subset $E_r^{/}$ of $(1, R_0)$ with $\int_{E_r^{/}} \frac{dr}{R_0 - r} = +\infty$ such that for all $r \in E_r^{/}$ holds

$$T_0(r, f) > \frac{\eta}{(R_0 - r)^{\rho_{\mathcal{A}}(f)}} \text{ if } 1 < r < R_0 < +\infty.$$

Proof. Case $R_0 = +\infty$: By Definition 3, there exists an increasing sequence $\{r_m\}_{m=1}^{\infty}$ ($r_m \to +\infty$, $m \to +\infty$) satisfying $(1 + \frac{1}{m})r_m < r_{m+1}$ and

$$\lim_{m \to +\infty} \frac{T_0(r_m, f)}{r_m^{\rho_{\mathcal{A}}(f)}} = \tau_{\mathcal{A}}(f).$$

So, there exists a positive integer m_0 such that for all $m \geq m_0$ and for any given $0 < \varepsilon < \tau_{\mathcal{A}}(f) - \eta$, we have

$$T_0(r_m, f) > (\tau_{\mathcal{A}}(f) - \varepsilon) r_m^{\rho_{\mathcal{A}}(f)}. \tag{6}$$

Since

$$\lim_{m \to +\infty} \left(\frac{m}{m+1}\right)^{\rho_{\mathcal{A}}(f)} = 1,$$

then for any given $\eta < \tau_{\mathcal{A}}(f) - \varepsilon$, there exists a positive integer m_1 such that for all $m \geq m_1$, we have

$$\left(\frac{m}{m+1}\right)^{\rho_{\mathcal{A}}(f)} > \frac{\eta}{\tau_{\mathcal{A}}(f) - \varepsilon}. \tag{7}$$

Take $m \geq m_2 = \max\{m_1, m_0\}$. By (6) and (7), for any $r \in \left[r_m, (1 + \frac{1}{m})r_m\right]$

$$T_0(r, f) \geq T_0(r_m, f) > (\tau_{\mathcal{A}}(f) - \varepsilon) r_m^{\rho_{\mathcal{A}}(f)}$$

$$\geq (\tau_{\mathcal{A}}(f) - \varepsilon)\left(\frac{m}{m+1} r\right)^{\rho_{\mathcal{A}}(f)} > \eta r^{\rho_{\mathcal{A}}(f)}.$$

Set $E_r = \bigcup_{m=m_2}^{+\infty} [r_m, (1 + \frac{1}{m})r_m]$. Then there holds

$$\int_{E_r} \frac{dr}{r} = \sum_{m=m_2}^{+\infty} \int_{r_m}^{(1+\frac{1}{m})r_m} \frac{dt}{t} = \sum_{m=m_2}^{+\infty} \log(1 + \frac{1}{m}) = +\infty.$$

Case $R_0 < +\infty$: By Definition 3, there exists an increasing sequence $\{r_m\}_{m=1}^{\infty} \subset (1, R_0)$ ($r_m \to R_0^-$, $m \to +\infty$) satisfying $R_0 - \left(1 - \frac{1}{m}\right)(R_0 - r_m) < r_{m+1}$ and

$$\lim_{m \to +\infty} \frac{T_0(r_m, f)}{\left(\frac{1}{R_0 - r_m}\right)^{\rho_{\mathcal{A}}(f)}} = \tau_{\mathcal{A}}(f).$$

So, there exists a positive integer m_3 such that for all $m \geq m_3$ and for any given $0 < \varepsilon < \tau_{\mathcal{A}}(f) - \eta$, we have

$$T_0(r_m, f) > (\tau_{\mathcal{A}}(f) - \varepsilon)\left(\frac{1}{R_0 - r_m}\right)^{\rho_{\mathcal{A}}(f)}. \tag{8}$$

Since

$$\lim_{m \to +\infty} \left(1 - \frac{1}{m}\right)^{\rho_{\mathcal{A}}(f)} = 1,$$

then for any given $\eta < \tau_{\mathcal{A}}(f) - \varepsilon$, there exists a positive integer m_4 such that for all $m \geq m_4$, we have

$$\left(1 - \frac{1}{m}\right)^{\rho_{\mathcal{A}}(f)} > \frac{\eta}{\tau_{\mathcal{A}}(f) - \varepsilon}. \tag{9}$$

Take $m \geq m_5 = \max\{m_3, m_4\}$. By (8) and (9), for any $r \in [r_m, R_0 - \left(1 - \frac{1}{m}\right)(R_0 - r_m)]$, we obtain

$$T_0(r, f) \geq T_0(r_m, f) > (\tau_{\mathcal{A}}(f) - \varepsilon)\left(\frac{1}{R_0 - r_m}\right)^{\rho_{\mathcal{A}}(f)}$$

$$\geq (\tau_{\mathcal{A}}(f)-\varepsilon)\left(\frac{1-\frac{1}{m}}{R_0-r}\right)^{\rho_{\mathcal{A}}(f)} > \frac{\eta}{(R_0-r)^{\rho_{\mathcal{A}}(f)}}.$$

Set $E_r^{/} = \bigcup\limits_{m=m_5}^{+\infty} [r_m, R_0 - \left(1-\frac{1}{m}\right)(R_0-r_m)]$. Then there holds

$$\int\limits_{E_r^{/}} \frac{dr}{R_0-r} = \sum_{m=m_5}^{+\infty} \int\limits_{r_m}^{R_0-\left(1-\frac{1}{m}\right)(R_0-r_m)} \frac{dt}{R_0-t} = \sum_{m=m_5}^{+\infty} \log\frac{m}{m-1} = +\infty.$$

□

Lemma 2. *([7],[16]) (The lemma of the logarithmic derivative). Let f be a non-constant meromorphic function in the annulus $\mathcal{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $1 < r < R_0 \leq +\infty$, and $k \geq 1$ be an integer. Then*

$$m_0\left(r, \frac{f^{(k)}}{f}\right) = \begin{cases} O(\log r), \ R_0 = +\infty \text{ and } \rho_{\mathcal{A}}(f) < +\infty, \\ O\left(\log \frac{1}{R_0-r}\right), \ R_0 < +\infty \text{ and } \rho_{\mathcal{A}}(f) < +\infty, \\ O(\log r + \log T_0(r,f)), \ r \notin \Delta_r, \ R_0 = +\infty, \\ O\left(\log\frac{1}{R_0-r} + \log T_0(r,f)\right), \ r \notin \Delta_r', \ R_0 < +\infty, \end{cases}$$

where Δ_r and Δ_r' are sets with $\int\limits_{\Delta_r} \frac{dr}{r} < +\infty$ and $\int\limits_{\Delta_r'} \frac{dr}{R_0-r} < +\infty$ respectively.

Lemma 3. *Let f be a meromorphic function with finite order $\rho_{\mathcal{A}}(f) < +\infty$. Then, there exists a subset E_r of $(1,+\infty)$ with $\int\limits_{E_r} \frac{dr}{r} = +\infty$ such that for all $r \in E_r$ holds*

$$\rho_{\mathcal{A}}(f) = \lim_{r\to+\infty} \frac{\log T_0(r,f)}{\log r} \ \textit{if } 1 < r < R_0 = +\infty$$

or there exists a subset $E_r^{/}$ of $(1, R_0)$ with $\int\limits_{E_r^{/}} \frac{dr}{R_0-r} = +\infty$ such that for all $r \in E_r^{/}$ holds

$$\rho_{\mathcal{A}}(f) = \lim_{r\to R_0^-} \frac{\log T_0(r,f)}{\log\frac{1}{R_0-r}} \ \textit{if } 1 < r < R_0 < +\infty.$$

Proof. Case $R_0 = +\infty$. The definition of $\rho_{\mathcal{A}}(f)$ implies that there exists a sequence $\{r_n\}_{n=1}^{\infty}$ ($r_n \to +\infty$, $n \to +\infty$) satisfying $\left(1+\frac{1}{n}\right) r_n < r_{n+1}$ and

$$\lim_{n\to+\infty} \frac{\log T_0(r_n,f)}{\log r_n} = \rho_{\mathcal{A}}(f).$$

Then, there exists an integer number n_1 such that for all $n \geq n_1$ and for any $r \in \left[r_n, \left(1+\frac{1}{n}\right) r_n\right]$, we have

$$\frac{\log T_0(r_n,f)}{\log\left(1+\frac{1}{n}\right) r_n} = \frac{\log T_0(r_n,f)}{\log\left(1+\frac{1}{n}\right) + \log r_n} \leq \frac{\log T_0(r,f)}{\log r}$$

$$\leq \frac{\log T_0\left(\left(1+\frac{1}{n}\right)r_n, f\right)}{\log r_n} = \frac{\log T_0\left(\left(1+\frac{1}{n}\right)r_n, f\right)}{\log\left(1+\frac{1}{n}\right)r_n} \cdot \frac{\log\left(1+\frac{1}{n}\right)+\log r_n}{\log r_n}.$$

Setting $E_r = \bigcup\limits_{n=n_1}^{+\infty}\left[r_n, \left(1+\frac{1}{n}\right)r_n\right]$, then for any $r \in E_r$, we get

$$\lim_{r\to+\infty}\frac{\log T_0(r,f)}{\log r} = \lim_{n\to+\infty}\frac{\log T_0(r_n,f)}{\log r_n} = \rho_{\mathcal{A}}(f),$$

where

$$\int\limits_{E_r}\frac{dr}{r} = \sum_{n=n_1}^{+\infty}\int\limits_{r_n}^{\left(1+\frac{1}{n}\right)r_n}\frac{dt}{t} = \sum_{n=n_1}^{+\infty}\log\left(1+\frac{1}{n}\right) = +\infty.$$

Case $R_0 < +\infty$: By definition of $\rho_{\mathcal{A}}(f)$, there exists an increasing sequence $\{r_n\}_{n=1}^{\infty} \subset (1, R_0)$ ($r_n \to R_0^-$, $n \to +\infty$) satisfying $R_0 - \left(1-\frac{1}{n}\right)(R_0 - r_n) < r_{n+1}$ and

$$\lim_{n\to+\infty}\frac{\log T_0(r_n,f)}{\log\frac{1}{R_0-r_n}} = \rho_{\mathcal{A}}(f).$$

So, there exists a positive integer n_2 such that for all $n \geq n_2$ and for any $r \in \left[r_n, R_0 - \left(1-\frac{1}{n}\right)(R_0-r_n)\right]$, we have

$$\frac{\log T_0(r_n,f)}{\log\frac{1}{\left(1-\frac{1}{n}\right)(R_0-r_n)}} \leq \frac{\log T_0(r,f)}{\log\frac{1}{R_0-r}} \leq \frac{\log T_0\left(R_0-\left(1-\frac{1}{n}\right)(R_0-r_n), f\right)}{\log\frac{1}{R_0-r_n}}.$$

It follows that

$$\frac{\log T_0(r_n,f)}{\log\frac{n}{n-1}+\log\frac{1}{R_0-r_n}} \leq \frac{\log T_0(r,f)}{\log\frac{1}{R_0-r}}$$

$$\leq \frac{\log T_0\left(R_0-\left(1-\frac{1}{n}\right)(R_0-r_n), f\right)}{\log\frac{1}{\left(1-\frac{1}{n}\right)(R_0-r_n)}} \cdot \frac{\log\frac{1}{\left(1-\frac{1}{n}\right)(R_0-r_n)}}{\log\frac{1}{R_0-r_n}}.$$

Set $E_r^{/} = \bigcup\limits_{n=n_2}^{+\infty}[r_n, R_0-\left(1-\frac{1}{n}\right)(R_0-r_n)]$. Then for any $r \in E_r^{/}$, we get

$$\lim_{r\to R_0^-}\frac{\log T_0(r,f)}{\log\frac{1}{R_0-r}} = \lim_{n\to+\infty}\frac{\log T_0(r_n,f)}{\log\frac{1}{R_0-r_n}} = \rho_{\mathcal{A}}(f),$$

where

$$\int\limits_{E_r^{/}}\frac{dr}{R_0-r} = \sum_{n=n_2}^{+\infty}\int\limits_{r_n}^{R_0-\left(1-\frac{1}{n}\right)(R_0-r_n)}\frac{dt}{R_0-t} = \sum_{n=n_2}^{+\infty}\log\frac{n}{n-1} = +\infty.$$

$\square$

3 Proofs of the Theorems

Proof of Theorem 2

Proof. Let $f \not\equiv 0$ be a solution of (1). We divide through equation (1) by f to get

$$-B_0(z) = \frac{f^{(k)}(z)}{f(z)} + \sum_{j=1}^{k-1} B_j(z) \frac{f^{(j)}(z)}{f(z)}. \tag{10}$$

By (10) and Lemma 2, it follows that

$$m_0(r, B_0) = T_0(r, B_0) \leq \sum_{j=1}^{k-1} m_0(r, B_j) + \sum_{j=1}^{k} m_0\left(r, \frac{f^{(j)}}{f}\right) + O(1)$$

$$\leq \sum_{j=1}^{k-1} m_0(r, B_j) + \begin{cases} O(\log r + \log T_0(r, f)), \ R_0 = +\infty, \ r \notin \Delta_r \\ O\left(\log \frac{1}{R_0 - r} + \log T_0(r, f)\right), \ R_0 < +\infty, \ r \notin \Delta'_r \end{cases}$$

$$= \sum_{j=1}^{k-1} T_0(r, B_j) + \begin{cases} O(\log r + \log T_0(r, f)), \ R_0 = +\infty, \ r \notin \Delta_r, \\ O\left(\log \frac{1}{R_0 - r} + \log T_0(r, f)\right), \ R_0 < +\infty, \ r \notin \Delta'_r, \end{cases} \tag{11}$$

where Δ_r and Δ'_r are sets with $\int_{\Delta_r} \frac{dr}{r} < +\infty$ and $\int_{\Delta'_r} \frac{dr}{R_0 - r} < +\infty$ respectively.
Case $R_0 = +\infty$. By substituting (2) and (3) into (11), we conclude for $r \in E_r \backslash \Delta_r$ sufficiently large

$$\alpha r^{\mu} \leq (k-1)\beta r^{\mu} + O(\log r + \log T_0(r, f)). \tag{12}$$

From (12), we obtain

$$(\alpha - (k-1)\beta) r^{\mu} \leq O(\log r + \log T_0(r, f))$$

and since $\alpha > (k-1)\beta$, this leads to $\rho_{\mathcal{A}}(f) = +\infty$ and $\rho_{2,\mathcal{A}}(f) \geq \mu$.

Case $R_0 < +\infty$. Let $f \not\equiv 0$ be a solution of (1). By substituting (4) and (5) into (11), we conclude for $r \in F_r \backslash \Delta'_r$, $r \to R_0^-$

$$\frac{\alpha}{(R_0 - r)^{\mu}} \leq (k-1) \frac{\beta}{(R_0 - r)^{\mu}} + O(\log \frac{1}{R_0 - r} + \log T_0(r, f)). \tag{13}$$

Then by (13), we obtain

$$\frac{\alpha - (k-1)\beta}{(R_0 - r)^{\mu}} \leq O(\log \frac{1}{R_0 - r} + \log T_0(r, f))$$

and since $\alpha > (k-1)\beta$, this leads to $\rho_{\mathcal{A}}(f) = +\infty$ and $\rho_{2,\mathcal{A}}(f) \geq \mu$. $\square$

Proof of Theorem 3

Proof. Let $f \not\equiv 0$ be a solution of (1). If $\rho_{\mathcal{A}}(B_j) < \rho_{\mathcal{A}}(B_0)$ for all $1, 2, ..., k-1$, then Theorem 3 reduces to Theorem 1. Thus, we assume that at least one of B_j $(1, 2, ..., k-1)$ satisfies $\rho_{\mathcal{A}}(B_j) = \rho_{\mathcal{A}}(B_0) = \rho$. So, there exists a set $J \subseteq \{1, 2, ..., k-1\}$ such that for $j \in J$, we have $\rho_{\mathcal{A}}(B_j) = \rho_{\mathcal{A}}(B_0) = \rho$ with $\sum\limits_{j \in J} \tau_{\mathcal{A}}(B_j) < \tau_{\mathcal{A}}(B_0) = \tau$ and for $j \in \{1, 2, ..., k-1\} \backslash J$, we have $\rho_{\mathcal{A}}(B_j) < \rho_{\mathcal{A}}(B_0) = \rho$. Hence, we can choose α_1, α_2 satisfying $\sum\limits_{j \in J} \tau_{\mathcal{A}}(B_j) < \alpha_1 < \alpha_2 < \tau$ such that for sufficiently large r and any given ε $\left(0 < \varepsilon < \frac{\alpha_2 - \alpha_1}{k-1}\right)$, we have

$$T_0(r, B_j) = m_0(r, B_j) \leq (\tau_{\mathcal{A}}(B_j) + \varepsilon) r^{\rho_{\mathcal{A}}(B_j)} = (\tau_{\mathcal{A}}(B_j) + \varepsilon) r^{\rho}, \ j \in J \tag{14}$$

and

$$T_0(r, B_j) = m_0(r, B_j) \leq r^{\rho_0}, \ j \in \{1, 2, ..., k-1\} \backslash J, \tag{15}$$

where $0 < \rho_0 < \rho$. For $r \to R_0^-$ and any given ε $\left(0 < \varepsilon < \frac{\alpha_2 - \alpha_1}{k-1}\right)$, we obtain

$$\begin{aligned} T_0(r, B_j) = m_0(r, B_j) &\leq (\tau_{\mathcal{A}}(B_j) + \varepsilon) \left(\frac{1}{R_0 - r}\right)^{\rho_{\mathcal{A}}(B_j)} \\ &= (\tau_{\mathcal{A}}(B_j) + \varepsilon) \left(\frac{1}{R_0 - r}\right)^{\rho}, \ j \in J \end{aligned} \tag{16}$$

and

$$T_0(r, B_j) = m_0(r, B_j) \leq \left(\frac{1}{R_0 - r}\right)^{\rho_0}, \ j \in \{1, 2, ..., k-1\} \backslash J, \tag{17}$$

where $0 < \rho_0 < \rho$. By applying Lemma 1, there exists a subset E_r of $(1, \infty)$ with $\int\limits_{E_r} \frac{dr}{r} = +\infty$ such that for all $r \in E_r$, we have

$$T_0(r, B_0) = m_0(r, B_0) > \alpha_2 r^{\rho} \text{ if } 1 < r < R_0 = +\infty \tag{18}$$

or there exists a subset $E_r^{/}$ of $(1, R_0)$ with $\int\limits_{E_r^{/}} \frac{dr}{R_0 - r} = +\infty$ such that for all $r \in E_r^{/}$ holds

$$T_0(r, B_0) = m_0(r, B_0) > \alpha_2 \left(\frac{1}{R_0 - r}\right)^{\rho} \text{ if } 1 < r < R_0 < +\infty. \tag{19}$$

Case $R_0 = +\infty$: By substituting the assumptions (14), (15) and (18) into (11), for all sufficiently large $r \in E_r \backslash \Delta_r$ and any given ε $\left(0 < \varepsilon < \frac{\alpha_2 - \alpha_1}{k-1}\right)$, we obtain

$$\begin{aligned} \alpha_2 r^{\rho} &\leq \sum_{j \in J} (\tau_{\mathcal{A}}(B_j) + \varepsilon) r^{\rho} + \sum_{j \in \{1, ..., k-1\} \backslash J} r^{\rho_0} + O(\log r + \log T_0(r, f)) \\ &\leq (\alpha_1 + (k-1)\varepsilon) r^{\rho} + (k-1) r^{\rho_0} + O(\log r + \log T_0(r, f)). \end{aligned}$$

It follows that

$$(\alpha_2 - \alpha_1 - (k-1)\varepsilon) r^{\rho} \leq (k-1) r^{\rho_0} + O(\log r + \log T_0(r,f)). \tag{20}$$

Since $\varepsilon \left(0 < \varepsilon < \frac{\alpha_2 - \alpha_1}{k-1}\right)$, then from (20), we get $\rho_{\mathcal{A}}(f) = +\infty$ and $\rho_{2,\mathcal{A}}(f) \geq \rho_{\mathcal{A}}(B_0) = \rho$.

Case $R_0 < +\infty$: By substituting the assumptions (16), (17) and (19) into (11), for all $r \in E'_r \backslash \Delta'_r$ with $r \to R_0^-$ and any given $\varepsilon \left(0 < \varepsilon < \frac{\alpha_2 - \alpha_1}{k-1}\right)$, we obtain

$$\alpha_2 \left(\frac{1}{R_0 - r}\right)^{\rho} \leq \sum_{j \in J} (\tau_{\mathcal{A}}(B_j) + \varepsilon) \left(\frac{1}{R_0 - r}\right)^{\rho}$$

$$+ \sum_{j \in \{1,...,k-1\} \backslash J} \left(\frac{1}{R_0 - r}\right)^{\rho_0} + O\left(\log \frac{1}{R_0 - r} + \log T_0(r,f)\right)$$

$$\leq (\alpha_1 + (k-1)\varepsilon) \left(\frac{1}{R_0 - r}\right)^{\rho} + (k-1) \left(\frac{1}{R_0 - r}\right)^{\rho_0}$$

$$+ O\left(\log \frac{1}{R_0 - r} + \log T_0(r,f)\right).$$

It follows that

$$(\alpha_2 - \alpha_1 - (k-1)\varepsilon) \left(\frac{1}{R_0 - r}\right)^{\rho} \leq (k-1) \left(\frac{1}{R_0 - r}\right)^{\rho_0}$$

$$+ O\left(\log \frac{1}{R_0 - r} + \log T_0(r,f)\right). \tag{21}$$

Since $\varepsilon \left(0 < \varepsilon < \frac{\alpha_2 - \alpha_1}{k-1}\right)$, then from (21), we obtain $\rho_{\mathcal{A}}(f) = +\infty$ and $\rho_{2,\mathcal{A}}(f) \geq \rho_{\mathcal{A}}(B_0) = \rho$. $\square$

Proof of Theorem 4

Proof. Let $f \not\equiv 0$ be a solution of (1). First, we suppose that $b = \max\{\rho_{\mathcal{A}}(B_j) : j = 1, 2, ..., k-1\} < \mu_{\mathcal{A}}(B_0) = \mu$. Then, for any given ε $(0 < 2\varepsilon < \mu - b)$ and sufficiently large r, we have

$$T_0(r, B_0) = m_0(r, B_0) \geq r^{\mu - \varepsilon} \tag{22}$$

and

$$T_0(r, B_j) = m_0(r, B_j) \leq r^{b+\varepsilon}, \quad j = 1, 2, ..., k-1. \tag{23}$$

For $r \to R_0^-$ and any given ε $(0 < 2\varepsilon < \mu - b)$, we obtain

$$T_0(r, B_0) = m_0(r, B_0) \geq \left(\frac{1}{R_0 - r}\right)^{\mu - \varepsilon} \tag{24}$$

and

$$T_0(r, B_j) = m_0(r, B_j) \leq \left(\frac{1}{R_0 - r}\right)^{b+\varepsilon}, \quad j = 1, 2, ..., k-1. \tag{25}$$

Case $R_0 = +\infty$: By substituting the assumptions (22) and (23) into (11), for any given ε $(0 < 2\varepsilon < \mu - b)$ and sufficiently large $r \notin \Delta_r$, we obtain

$$r^{\mu-\varepsilon} \leq (k-1) r^{b+\varepsilon} + O(\log r + \log T_0(r, f)). \tag{26}$$

Since ε $\left(0 < \varepsilon < \frac{\mu-b}{2}\right)$, then from (26) we get $\rho_{\mathcal{A}}(f) = \mu_{\mathcal{A}}(f) = +\infty$ and $\rho_{2,\mathcal{A}}(f) \geq \mu_{2,\mathcal{A}}(f) \geq \mu_{\mathcal{A}}(B_0) = \mu$.

Case $R_0 < +\infty$: By substituting the assumptions (24) and (25) into (11), for any given ε $(0 < 2\varepsilon < \mu - b)$ and $r \to R_0^-$, $r \notin \Delta_r'$ we obtain

$$\left(\frac{1}{R_0 - r}\right)^{\mu-\varepsilon} \leq (k-1)\left(\frac{1}{R_0 - r}\right)^{b+\varepsilon} + O\left(\log \frac{1}{R_0 - r} + \log T_0(r, f)\right). \tag{27}$$

Since ε $\left(0 < \varepsilon < \frac{\mu-b}{2}\right)$, then from (27) we have $\rho_{\mathcal{A}}(f) = \mu_{\mathcal{A}}(f) = +\infty$ and $\rho_{2,\mathcal{A}}(f) \geq \mu_{2,\mathcal{A}}(f) \geq \mu_{\mathcal{A}}(B_0) = \mu$.

Assume

$$\max\{\rho_{\mathcal{A}}(B_j) : j = 1, 2, ..., k-1\} = \mu_{\mathcal{A}}(B_0) = \mu$$

and $\tau_1 = \sum_{\rho_{\mathcal{A}}(B_j)=\mu_{\mathcal{A}}(B_0)} \tau_{\mathcal{A}}(B_j) < \underline{\tau}_{\mathcal{A}}(B_0) = \underline{\tau}$. Then, there exists a set $J \subseteq \{1, 2, ..., k-1\}$ such that for $j \in J$, we have $\rho_{\mathcal{A}}(B_j) = \mu_{\mathcal{A}}(B_0) = \mu$ with $\tau_1 = \sum_{j \in J} \tau_{\mathcal{A}}(B_j) < \underline{\tau}_{\mathcal{A}}(B_0) = \underline{\tau}$ and for $j \in \{1, 2, ..., k-1\} \backslash J$, we have $\rho_{\mathcal{A}}(B_j) < \mu_{\mathcal{A}}(B_0) = \mu$. Hence, we can choose β_1, β_2 satisfying $\sum_{j \in J} \tau_{\mathcal{A}}(B_j) < \beta_1 < \beta_2 < \underline{\tau}$ such that for sufficiently large r and any given ε $\left(0 < \varepsilon < \frac{\beta_2 - \beta_1}{k-1}\right)$, we have

$$T_0(r, B_j) = m_0(r, B_j) \leq (\tau_{\mathcal{A}}(B_j) + \varepsilon) r^{\rho_{\mathcal{A}}(B_j)} = (\tau_{\mathcal{A}}(B_j) + \varepsilon) r^{\mu}, \ j \in J \tag{28}$$

and

$$T_0(r, B_j) = m_0(r, B_j) \leq r^{\rho_1}, \ j \in \{1, 2, ..., k-1\} \backslash J, \tag{29}$$

where $0 < \rho_1 < \mu$. By the definition of lower type for sufficiently large r, we have

$$T_0(r, B_0) = m_0(r, B_0) \geq \beta_2 r^{\mu}. \tag{30}$$

For $r \to R_0^-$ and any given ε $\left(0 < \varepsilon < \frac{\beta_2 - \beta_1}{k-1}\right)$, we obtain

$$T_0(r, B_j) = m_0(r, B_j) \leq (\tau_{\mathcal{A}}(B_j) + \varepsilon)\left(\frac{1}{R_0 - r}\right)^{\rho_{\mathcal{A}}(B_j)}$$

$$= (\tau_{\mathcal{A}}(B_j) + \varepsilon) \left(\frac{1}{R_0 - r} \right)^{\mu}, \ j \in J \tag{31}$$

and

$$T_0(r, B_j) = m_0(r, B_j) \leq \left(\frac{1}{R_0 - r} \right)^{\rho_1}, \ j \in \{1, 2, ..., k-1\} \setminus J, \tag{32}$$

where $0 < \rho_1 < \mu$. By the definition of lower type, for $r \to R_0^-$, we have

$$T_0(r, B_0) = m_0(r, B_0) \geq \beta_2 \left(\frac{1}{R_0 - r} \right)^{\mu}. \tag{33}$$

Case $R_0 = +\infty$: By substituting the assumptions (28), (29) and (30) into (11), for all sufficiently large $r \notin \Delta_r$ any given $\varepsilon \left(0 < \varepsilon < \frac{\beta_2 - \beta_1}{k-1} \right)$, we obtain

$$\beta_2 r^{\mu} \leq \sum_{j \in J} (\tau_{\mathcal{A}}(B_j) + \varepsilon) r^{\mu} + \sum_{j \in \{1, \dots, k-1\} \setminus J} r^{\rho_1} + O(\log r + \log T_0(r, f))$$

$$\leq (\beta_1 + (k-1)\varepsilon) r^{\mu} + (k-1) r^{\rho_1} + O(\log r + \log T_0(r, f)).$$

It follows that

$$(\beta_2 - \beta_1 - (k-1)\varepsilon) r^{\mu} \leq (k-1) r^{\rho_1} + O(\log r + \log T_0(r, f)). \tag{34}$$

From (34), since $\varepsilon \left(0 < \varepsilon < \frac{\beta_2 - \beta_1}{k-1} \right)$, we have $\rho_{\mathcal{A}}(f) = \mu_{\mathcal{A}}(f) = +\infty$ and $\rho_{2,\mathcal{A}}(f) \geq \mu_{2,\mathcal{A}}(f) \geq \mu_{\mathcal{A}}(B_0) = \mu$.

Case $R_0 < +\infty$: By substituting the assumptions (31), (32) and (33) into (11), for all $r \notin \Delta_r'$ with $r \to R_0^-$ and any given $\varepsilon \left(0 < \varepsilon < \frac{\beta_2 - \beta_1}{k-1} \right)$, we obtain

$$\beta_2 \left(\frac{1}{R_0 - r} \right)^{\mu} \leq \sum_{j \in J} (\tau_{\mathcal{A}}(B_j) + \varepsilon) \left(\frac{1}{R_0 - r} \right)^{\mu}$$

$$+ \sum_{j \in \{1, \dots, k-1\} \setminus J} \left(\frac{1}{R_0 - r} \right)^{\rho_1} + O\left(\log \frac{1}{R_0 - r} + \log T_0(r, f) \right)$$

$$\leq (\beta_1 + (k-1)\varepsilon) \left(\frac{1}{R_0 - r} \right)^{\mu} + (k-1) \left(\frac{1}{R_0 - r} \right)^{\rho_1}$$

$$+ O\left(\log \frac{1}{R_0 - r} + \log T_0(r, f) \right).$$

It follows that

$$(\beta_2 - \beta_1 - (k-1)\varepsilon) \left(\frac{1}{R_0 - r} \right)^{\mu} \leq (k-1) \left(\frac{1}{R_0 - r} \right)^{\rho_1}$$

$$+ O\left(\log \frac{1}{R_0 - r} + \log T_0(r, f) \right). \tag{35}$$

From (35), since $\varepsilon \left(0 < \varepsilon < \frac{\beta_2 - \beta_1}{k-1} \right)$, we get $\rho_{\mathcal{A}}(f) = \mu_{\mathcal{A}}(f) = +\infty$ and $\rho_{2,\mathcal{A}}(f) \geq \mu_{2,\mathcal{A}}(f) \geq \mu_{\mathcal{A}}(B_0) = \mu$. □

Proof of Theorem 5

Proof. Let $f \not\equiv 0$ be a solution of (1). Suppose that

$$\limsup_{r\to+\infty}\frac{\sum_{j=1}^{k-1} m_0(r,B_j)}{m_0(r,B_0)} < 1 \text{ if } 1<r<R_0=+\infty \tag{36}$$

or

$$\limsup_{r\to R_0^-}\frac{\sum_{j=1}^{k-1} m_0(r,B_j)}{m_0(r,B_0)} < 1 \text{ if } 1<r<R_0<+\infty. \tag{37}$$

Then for sufficiently large r or $r \to R_0^-$, we have

$$\sum_{j=1}^{k-1} m_0(r,B_j) < \gamma m_0(r,B_0), 0<\gamma<1. \tag{38}$$

Thus, by substituting (38) into (11), we obtain for sufficiently large r or $r \to R_0^-$

$$m_0(r,B_0) \le \gamma m_0(r,B_0) + \begin{cases} O(\log r + \log T_0(r,f)), \ R_0=+\infty, \ r\notin \Delta_r, \\ O\left(\log\frac{1}{R_0-r} + \log T_0(r,f)\right), \ R_0<+\infty, \ r\notin \Delta'_r. \end{cases} \tag{39}$$

From (39), it follows that

$$\begin{aligned}&(1-\gamma)m_0(r,B_0) = (1-\gamma)T_0(r,B_0)\\ &\le \begin{cases} O(\log r + \log T_0(r,f)), \ R_0=+\infty, \ r\notin \Delta_r, \\ O\left(\log\frac{1}{R_0-r} + \log T_0(r,f)\right), \ R_0<+\infty, \ r\notin \Delta'_r. \end{cases}\end{aligned} \tag{40}$$

Case $R_0=+\infty$: By (40), we obtain for r sufficiently large

$$(1-\gamma)\frac{T_0(r,B_0)}{\log r} \le O\left(1+\frac{\log T_0(r,f)}{\log r}\right), \ r\notin\Delta_r \tag{41}$$

and

$$\frac{\log(1-\gamma)}{\log r} + \frac{\log T_0(r,B_0)}{\log r} \le \frac{\log\log r}{\log r} + \frac{\log\log T_0(r,f)}{\log r} + \frac{O(1)}{\log r}, \ r\notin\Delta_r. \tag{42}$$

Since $B_0(z)$ is an admissible analytic function in the annulus $\mathcal{A}$, then from (41) and (42), we get $\rho_{\mathcal{A}}(f)=+\infty$ and $\rho_{2,\mathcal{A}}(f) \ge \rho_{\mathcal{A}}(B_0)$.

Case $R_0<+\infty$: By (40), we have for $r\to R_0^-$

$$(1-\gamma)\frac{T_0(r,B_0)}{\log\frac{1}{R_0-r}} \le O\left(1+\frac{\log T_0(r,f)}{\log\frac{1}{R_0-r}}\right), \ r\notin\Delta'_r \tag{43}$$

and

$$\frac{\log(1-\gamma)}{\log\frac{1}{R_0-r}}+\frac{\log T_0(r,B_0)}{\log\frac{1}{R_0-r}}\leq\frac{\log\log\frac{1}{R_0-r}}{\log\frac{1}{R_0-r}}+\frac{\log\log T_0(r,f)}{\log\frac{1}{R_0-r}}+\frac{O(1)}{\log\frac{1}{R_0-r}},\ r\notin\Delta'_r. \tag{44}$$

Since $B_0(z)$ is an admissible analytic function in the annulus $\mathcal{A}$, then from (43) and (44), we obtain $\rho_{\mathcal{A}}(f)=+\infty$ and $\rho_{2,\mathcal{A}}(f)\geq\rho_{\mathcal{A}}(B_0)$. □

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On T-nilpotence of a matrix set

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Abstract. Let R be a ring and I be an arbitrary right T-nilpotent subset of R. In the paper it is proved that in this case the set of all $n \times n$-matrices with entries in I is a right T-nilpotent subset of the ring of $n \times n$-matrices with entries in R, where $n \in \mathbb{N}$. It is also showed that it is impossible to generalize this result for rings of matrices of infinite dimension.

Mathematics subject classification: 16D99, 16D90.
Keywords and phrases: T-nilpotent, matrix, ring.

Dedicated to the memory of Professor M. Ya. Komarnytskyi

1 Introduction

All rings are considered to be associative with $1 \neq 0$. The category of left K-modules is denoted by $K - Mod$. The set of all $n \times n$-matrices with entries in a set I will be denoted by $M_n(I)$, where $n \in \mathbb{N}$.

Definition 1. ([5, p. 291]) A set A of elements of a ring R is called left (resp. right) T-nilpotent, if for every family

$$(a_1, a_2, a_3, ...), a_i \in A$$

a $k \in \mathbb{N}$ exists with

$$a_k a_{k-1} ... a_1 = 0, (a_1 a_2 ... a_k = 0).$$

(See also [1, p. 313].)

The notion of T-nilpotence has the important applications in certain areas of the Ring and Module Theory, especially in theory of perfect and semiartinian rings, but not only (for example, see [8, p. 183–184, 189], [6, p. 60], [7, p. 67], [3, p. 86, 87]).

Recall the definition of the equivalence. Let C and D be categories. A functor $S : C \to D$ is an equivalence if there exist a functor $T : D \to C$ and natural equivalences $TS \to 1_C$ and $ST \to 1_D$. (See [8, p. 82].)

In the paper [4] the following corollaries are obtained:

Corollary 1. *(See Corollary 11 [4, p. 52]) Let R, S be equivalent rings, via an equivalence $F : R - Mod \to S - Mod$. If I is a right T-nilpotent two-sided ideal of R, then so is the two-sided ideal $\{s \in S \,|\forall x \in F(R/I) : sx = 0\}$ of S.*

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Corollary 2. *(See Corollary 12 [4, p. 52]) Let R be a ring and let $n \in \mathbb{N}$. If I is a right T-nilpotent ideal of R, then $M_n(I)$ is a right T-nilpotent ideal of $M_n(R)$.*

The aim of our paper is to obtain the stronger statement than Corollary 2. Indeed, in this corollary an arbitrary subset of a ring instead of a two-sided ideal can be considered.

2 Preliminaries

Lemma 1. *(König's Graph Lemma, [2, p. 40]) Start with a countable sequence $\{F_n \,|n = 1, 2, ...\}$ of finite sets, and for each n, assume that there is a map Φ_n of F_n into $Pow(F_{n+1})$. In order to simplify notation, denote Φ_n by Φ, $\forall n$, and the union of the given family of finite sets by F. A path in (the ordered pair) (F, Φ) is a finite or infinite sequence of elements $b_1, ..., b_n, ...$ of F such that $b_i \in F_i$ and $b_{i+1} \in \Phi(b_i)$, $i = 1, 2, ...$ The length of a finite path $b_1, b_2, ..., b_m$ is m; the length of the infinite path $b_1, b_2, ...$is infinite. Then if (F, Φ) has paths of ever greater length, then it has a path of infinite length.*

3 Main result

Theorem 1. *Let R be a ring and I be a right T-nilpotent subset of R. Then $M_n(I)$ is a right T-nilpotent subset of $M_n(R)$, where $n \in \mathbb{N}$.*

Proof. Let I be a right T-nilpotent subset of R and $n \in \mathbb{N}$.

Assume $M_n(I)$ is not right T-nilpotent. Then there is an infinite sequence of matrices

$$||a_{ij}^{(1)}||, ||a_{ij}^{(2)}||, \dots, ||a_{ij}^{(k)}||, \dots$$

belonging to $M_n(I)$ such that for each $k \in \mathbb{N}$

$$A_k \neq O, \tag{1}$$

where $A_k = ||a_{ij}^{(1)}||||a_{ij}^{(2)}||...||a_{ij}^{(k)}||$.

Let $A_k = ||A_{ij}^{(k)}||$ for each $k \in \mathbb{N}$.

Then it is obvious that

$$A_{ij}^{(1)} = a_{ij}^{(1)} \quad \text{and} \quad A_{ij}^{(k)} = \sum_{t_1=1}^{n} \sum_{t_2=1}^{n} ... \sum_{t_{k-1}=1}^{n} a_{it_1}^{(1)} a_{t_1 t_2}^{(2)} ... a_{t_{k-1} j}^{(k)} \quad \text{for} \quad k \geq 2. \tag{2}$$

Consider the sets $F_1, F_2, ..., F_k, ...$ defined as follows:

$$F_1 = \{(\lambda_1, \lambda_2) | \lambda_1, \lambda_2 \in \{1, 2, ..., n\}, a_{\lambda_1 \lambda_2}^{(1)} \neq 0\},$$

$$F_2 = \{(\lambda_1, \lambda_2, \lambda_3) | \lambda_1, \lambda_2, \lambda_3 \in \{1, 2, ..., n\}, a_{\lambda_1 \lambda_2}^{(1)} a_{\lambda_2 \lambda_3}^{(2)} \neq 0\},$$

$$\vdots \quad \vdots \quad \vdots$$

$$F_k = \{(\lambda_1, \lambda_2, ..., \lambda_{k+1}) | \lambda_1, \lambda_2, ..., \lambda_{k+1} \in \{1, 2, ..., n\}, a^{(1)}_{\lambda_1\lambda_2} a^{(2)}_{\lambda_2\lambda_3} ... a^{(k)}_{\lambda_k\lambda_{k+1}} \neq 0\},$$

$$\vdots \quad \vdots \quad \vdots$$

(1)-(2) imply

$$\forall k \in \mathbb{N} : F_k \neq \emptyset.$$

Hence for each $k \in \mathbb{N}$ it is possible to consider the following mapping:

$$\Phi_k : \begin{cases} F_k \to Pow(F_{k+1}), \\ (\lambda_1, \lambda_2, ..., \lambda_{k+1}) \mapsto \{(\lambda_1, \lambda_2, ..., \lambda_{k+1}, \lambda_{k+2}) | a^{(1)}_{\lambda_1\lambda_2} ... a^{(k+1)}_{\lambda_{k+1}\lambda_{k+2}} \neq 0\}. \end{cases}$$

Let u be an arbitrary integer greater than 0. Then $A_u \neq O$. It follows from this that for some $i, j \in \{1, 2, ..., n\}$ $A^{(u)}_{ij} \neq 0$. It follows from (2) that for some $t_1, ..., t_{u-1} \in \{1, 2, ..., n\}$

$$a^{(1)}_{it_1} a^{(2)}_{t_1t_2} ... a^{(u)}_{t_{u-1}j} \neq 0. \tag{3}$$

Whence

$$\begin{aligned} a^{(1)}_{it_1} &\neq 0, \\ a^{(1)}_{it_1} a^{(2)}_{t_1t_2} &\neq 0, \\ &\vdots \quad \vdots \quad \vdots \\ a^{(1)}_{it_1} a^{(2)}_{t_1t_2} ... a^{(u-1)}_{t_{u-2}t_{u-1}} &\neq 0. \end{aligned} \tag{4}$$

Put

$$\begin{aligned} b_1 &= (i, t_1), \\ b_2 &= (i, t_1, t_2), \\ &\vdots \quad \vdots \quad \vdots \\ b_{u-1} &= (i, t_1, t_2, ..., t_{u-1}), \\ b_u &= (i, t_1, t_2, ..., t_{u-1}, j). \end{aligned}$$

(3)-(4) imply that $b_1 \in F_1$, $b_2 \in F_2$, $\ldots$, $b_u \in F_u$.

It is clear that

$$\begin{aligned} b_2 &\in \Phi_1(b_1), \\ b_3 &\in \Phi_2(b_2), \\ &\vdots \quad \vdots \quad \vdots \\ b_u &\in \Phi_{u-1}(b_{u-1}). \end{aligned}$$

The length of the path $b_1, b_2, ..., b_u$ is u. Since u is an arbitrary integer greater than 0, we have paths of ever greater length. Therefore, by König's Graph Lemma, there exists a path of infinite length.

It means that there exists an infinite sequence of numbers $\lambda_1, \lambda_2, ..., \lambda_p, ...$ belonging to $\{1, 2, ..., n\}$ satisfying the following conditions:

$$\begin{aligned} (\lambda_1, \lambda_2) &\in F_1, \\ (\lambda_1, \lambda_2, \lambda_3) &\in F_2, \\ \vdots \;\; &\vdots \;\; \vdots \\ (\lambda_1, \lambda_2, ..., \lambda_{p+1}) &\in F_p, \\ \vdots \;\; &\vdots \;\; \vdots \end{aligned} \tag{5}$$

Consider the sequence

$$a^{(1)}_{\lambda_1\lambda_2}, a^{(2)}_{\lambda_2\lambda_3}, ..., a^{(s)}_{\lambda_s\lambda_{s+1}}, ... \; .$$

It follows from (5) that for an arbitrary $p \in \mathbb{N}$

$$a^{(1)}_{\lambda_1\lambda_2} a^{(2)}_{\lambda_2\lambda_3} ... a^{(p)}_{\lambda_p\lambda_{p+1}} \neq 0.$$

Hence I is not right T-nilpotent, which is a contradiction.

$\square$

Now we will see that it is impossible to generalize our result for rings of matrices of infinite dimension.

Example 1. Let K be a ring and S be a subset of K. Let $\mathbb{RFM}_{\mathbb{N}}(S)$ be the set of all mappings $f : \mathbb{N} \times \mathbb{N} \to S$, where for each $\alpha \in \mathbb{N}$ the set $\{f(\alpha, \beta) \neq 0 | \beta \in \mathbb{N}\}$ is finite. Then $\mathbb{RFM}_{\mathbb{N}}(K)$ is a natural generalization of the matrix rings $M_n(K)$ (see [1, p. 19]).

Let k be a field. Consider the polynomial ring $k[x_1, x_2, ..., x_n, ...]$ in a countable number of variables. Let M be the ideal of this ring spanned by the following elements: x_1^2, x_2^3, ..., x_n^{n+1}, ..., $x_i x_j$, where $i \neq j$ and $i, j \in \mathbb{N}$. Denote the elements $a + M$ of the factor ring $K := k[x_1, x_2, ..., x_n, ...]/M$ by $\bar{a}$.

And now let I be the ideal of K spanned by the elements $\bar{x}_1$, $\bar{x}_2$,..., $\bar{x}_n$,... . It is obvious that I is right T-nilpotent.

Define a function $g : \mathbb{N} \times \mathbb{N} \to K$ as follows:

$$g(i, i) = \bar{x}_i, g(i, j) = \bar{0},$$

for all $i, j \in \mathbb{N}$, where $i \neq j$. It is clear that $g \in \mathbb{RFM}_{\mathbb{N}}(I)$, but g is not nilpotent.

Therefore $\mathbb{RFM}_{\mathbb{N}}(I)$ is not right T-nilpotent, although I is right T-nilpotent.

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On the Existence of Stationary Nash Equilibria for Mean Payoff Games on Graphs

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Abstract. In this paper we extend the classical concept of positional strategies for a mean payoff game to a general mixed stationary strategy approach, and prove the existence of mixed stationary Nash equilibria for an arbitrary m-player mean payoff game on graphs. Traditionally, a positional strategy represents a pure stationary strategy in a classical mean payoff game, where a Nash equilibrium in pure stationary strategies in general may not exist. Based on a constructive proof of the existence of specific equilibria for an m-player mean payoff game we propose a new approach for determining the optimal mixed stationary strategies. Additionally we characterize and extend the general problem of the existence of pure stationary Nash equilibria for some special classes of mean payoff games.

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Keywords and phrases: mean payoff game, pure stationary strategy, mixed stationary strategy, stationary Nash equilibrium.

1 Introduction

In [3,5,11] the following game of two players on a graph has been considered: Let $G = (X, E)$ be a finite directed graph in which every vertex $x \in X$ has at least one outgoing directed edge $e = (x, y) \in E$. On the edge set E a function $c : E \to R$ is given which assigns a value $c(e)$ to each edge $e \in E$. Furthermore, the vertex set X is divided into two disjoint subsets X_1 and X_2 $(X = X_1 \cup X_2,\ X_1 \cap X_2 = \emptyset)$ which are regarded as position sets of the two players. The game starts in a given position $x_0 \in X$. If $x_0 \in X_1$ then the move is done by the first player, otherwise it is done by second one. Move means the passage from position x_0 to a neighbor position x_1 through the directed edge $e_0 = (x_0, x_1) \in E$. After that if $x_1 \in X_1$ then the move is done by the first player, otherwise it is done by the second one and so on indefinitely. The first player has the aim to maximize $\lim_{t\to\infty} \inf \frac{1}{t} \sum_{\tau=0}^{t-1} c(e_\tau)$ while the second player has the aim to minimize $\lim_{t\to\infty} \sup \frac{1}{t} \sum_{\tau=0}^{t-1} c(e_\tau)$. In [3] it has been proven that for this game there exists a value $v(x_0)$ such that the first player has a strategy (of moves) that insures $\lim_{t\to\infty} \inf \frac{1}{t} \sum_{\tau=0}^{t-1} c(e_\tau) \geq v(x_0)$ and the second player has a strategy that

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insures $\lim_{t\to\infty} \sup \frac{1}{t}\sum_{\tau=0}^{t-1} c(e_\tau) \leq v(x_0)$. Furthermore it has been shown that players in this games can achieve the values $v(x_0)$ applying the strategies of moves which do not depend on t but depend only on the vertex (position) from which the player is able to move. Therefore, in [3, 11] such strategies are called positional strategies and the game sometimes is called positional game; in [5, 10] these strategies are called *stationary strategies*. More precisely the stationary strategies can be specified as *pure stationary strategies* because each move through a directed edge at a vertex of the game is chosen from the set of feasible strategies of moves by the corresponding player with the probability equal to 1 and in each position such a strategy does not change in time.

A generalization of a zero-sum mean payoff game to a non-zero-sum m-player positional game, where $m \geq 2$, is now the following: Consider a finite directed graph $G = (X, E)$ in which every vertex has at least one outgoing directed edge. Assume that the vertex set X is divided into m disjoint subsets $X_1, X_2, \ldots, X_m$ ($X = X_1 \cup X_2 \cup \cdots \cup X_m;\ X_i \cap Xj = \emptyset, i \neq j$) which we regard as position sets of the m players. Additionally, we assume that on the edge set m functions $c^i : F \to R,\ i = 1, 2, \ldots, m$, are defined that assign to each directed edge $e = (x, y) \in E$ the values $c^1_e, c^2_e, \ldots, c^m_e$ that are regarded as the rewards for the corresponding players $1, 2, \ldots, m$.

On G we consider the following m-person dynamic game: The game starts at a given position $x_0 \in X$ at the moment of time $t = 0$ where the player $i \in \{1, 2, \ldots, m\}$ who is the owner of the starting position x_0 makes a move from x_0 to a neighbor position $x_1 \in X$ through the directed edge $e_0 = (x_0, x_1) \in E$. After that players $1, 2, \ldots, m$ receive the corresponding rewards $c^1_{e_0}, c^2_{e_0}, \ldots, c^m_{e_0}$. Then at the moment of time $t = 1$ the player $k \in \{1, 2, \ldots, m\}$ who is owner of position x_1 makes a move from x_1 to a position $x_2 \in V$ through the directed edge $e_1 = (x_1, x_2) \in E$, players $1, 2, \ldots, m$ receive the corresponding rewards $c^1_{e_1}, c^2_{e_1}, \ldots, c^m_{e_1}$, and so on, indefinitely. Such a play of the game on G produces the sequence of positions $x_0, x_1, x_2, \ldots, x_t, \ldots$ where each x_t is the position at the moment of time t.

An m-player mean payoff game on G is the game with payoffs

$$\omega^i_{x_o} = \lim_{t\to\infty} \inf \frac{1}{t}\sum_{\tau=0}^{t-1} c^i_{e_\tau}, \quad i = 1, 2, \ldots, m.$$

The positional game on graph G formulated above in the cas $m = 2$ and $c^1_e = -c^2_e = c_e,\ \forall e \in E$, is transformed into a a two-player zero-sum mean payoff game on graph G for which Nash equilibria in pure stationary strategies exist. In general, a non-zero-sum mean payoff game on a graph may have no Nash equilibrium in pure stationary strategies. This fact has been shown in [5], where an example of two-player non-zero-sum mean payoff game that has no Nash equilibria in pure strategies is constructed. A pure stationary Nash equilibrium may exist only for some special cases of non-zero mean payoff games (see [1, 5, 10]).

In this contribution we consider the non-zero-sum positional games in mixed stationary strategies. We define a mixed stationary strategy of moves in a position $x \in X_i$ for the player $i \in \{1, 2, \ldots, m\}$, as a probability distribution over the set of feasible moves from x. We show that an arbitrary m-player mean payoff game on a graph possesses a Nash equilibrium in mixed stationary strategies. Based on a constructive proof of this result we propose an approach for determining the optimal mixed stationary strategies of the players.

The paper is organized as follows: In Section 2 an average stochastic positional game that generalizes non-zero-sum mean payoff games is formulated. Then in Sections 3 the known results of the existence of stationary Nash equilibria for an average stochastic positional game and an approach for determining the optimal strategies of players in such a game are presented. In Sections 4, 5, based on results from the Sections 3 the existence of Nash equilibria in mixed stationary strategies for non-zero-sum mean payoff games is proven and an approach for determining the optimal strategies of the players is proposed.

2 A Generalization of Mean Payoff Game on Graphs to Average Stochastic Positional Games

The problem of determining Nash equilibria in mixed stationary strategies for mean payoff games on graphs leads to a special class of stochastic games from [7–9] called *average stochastic positional games*. In [8] it is shown that such class of games possesses Nash equilibria in mixed stationary strategies. Therefore in the paper we shall use the average stochastic positional games for studying the existence of mixed stationary Nash equilibria in non-zero-sum mean payoff games. An m-player average stochastic positional game consists of the following elements:

– a state space X (which we assume to be finite);

– a partition $X = X_1 \cup X_2 \cup \cdots \cup X_m$ where X_i represents the position set of player $i \in \{1, 2, \ldots, m\}$;

– a finite set $A(x)$ of actions in each state $x \in X$;

– a step reward $f^i(x, a)$ with respect to each player $i \in \{1, 2, \ldots, m\}$ in each state $x \in X$ and for an arbitrary action $a \in A(x)$;

– a transition probability function $p : X \times \prod_{x \in X} A(x) \times X \to [0, 1]$ that gives the probability transitions $p^a_{x,y}$ from an arbitrary $x \in X$ to an arbitrary $y \in X$ for a fixed action $a \in A(x)$, where $\sum_{y \in X} p^a_{x,y} = 1, \forall x \in X, a \in A(x)$;

– a starting state $x_0 \in X$.

The game starts at the moment of time $t = 0$ in the state x_0 where the player $i \in \{1, 2, \ldots, m\}$ who is the owner of the state position x_0 $(x_0 \in X_i)$ chooses an action $a_0 \in A(x_0)$ and determines the rewards $f^1(x_0, a_0)$, $f^2(x_0, a_0)$, $\ldots$, $f^m(x_0, a_0)$ for the corresponding players $1, 2, \ldots, m$. After that the game passes to a state $y = x_1 \in X$ according to a certain probability distribution $\{p^{a_0}_{x_0,y}\}$. At the moment of time $t = 1$ the player $k \in \{1, 2, \ldots, m\}$ who is the owner of the state position x_1 $(x_1 \in X_k)$ chooses an action $a_1 \in A(x_1)$ and players $1, 2, \ldots, m$ receive the corresponding rewards $f^1(x_1, a_1), f^2(x_1, a_1), \ldots, f^m(x_1, a_1)$. Then the game passes to a state $y = x_2 \in X$ according to a probability distribution $\{p^{a_1}_{x_1,y}\}$ and so on indefinitely. Such a play of the game produces a sequence of states and actions $x_0, a_0, x_1, a_1, \ldots, x_t, a_t, \ldots$ that defines a stream of stage rewards $f^1(x_t, a_t), f^2(x_t, a_t), \ldots, f^m(x_t, a_t), \quad t = 0, 1, 2, \ldots$.

The average stochastic positional game is the game with payoffs of the players

$$\omega^i_{x_0} = \lim_{t\to\infty} \inf \ \mathsf{E}\Big(\frac{1}{t}\sum_{\tau=0}^{t-1} f^i(x_\tau, a_\tau)\Big), \quad i = 1, 2, \ldots, m$$

where E is the expectation operator with respect to the probability measure in the Markov process induced by actions chosen by players in their position sets and given starting state x_0.

In the following we will consider the stochastic positional game when the players use pure and mixed stationary strategies of choosing the actions in the states.

3 Existence and Determining Mixed Stationary Nash Equilibria for Average Stochastic Positional Games

In this section we present the main results concerned with the existence of stationary Nash equilibria for stochastic positional games with average payoffs. Note that in general for an average stochastic game a stationary Nash equilibrium may not exist (see [4]).

3.1 Stochastic Positional Games in Pure and Mixed Stationary Strategies

A strategy of player $i \in \{1, 2, \ldots, m\}$ in a stochastic positional game is a mapping s^i that gives for every state $x_t \in X_i$ a probability distribution over the set of actions $A(x_t)$. If these probabilities take only values 0 and 1, then s^i is called *a pure strategy*, otherwise s^i is called *a mixed strategy*. If these probabilities depend only on the state $x_t = x \in X_i$ (i.e. s^i does not depend on t), then s^i is called *a stationary strategy*, otherwise s^i is called a *non-stationary strategy.*

Thus, we can identify the set of mixed stationary strategies $\mathbf{S}^i$ of player i with the set of solutions of the system

$$\begin{cases} \displaystyle\sum_{a\in A(x)} s^i_{x,a} = 1, & \forall x \in X_i; \\ \qquad\quad s^i_{x,a} \geq 0, & \forall x \in X_i, \quad \forall a \in A(x). \end{cases} \tag{1}$$

Each basic solution s^i of this system corresponds to a pure stationary strategy of player $i \in \{1, 2, \ldots, m\}$. So, the set of pure stationary strategies S^i of player i corresponds to the set of basic solutions of system (1).

Let $\mathbf{s}= (s^1, s^2, \dots, s^m) \in \mathbf{S} = \mathbf{S}^1 \times \mathbf{S}^2 \times \cdots \times \mathbf{S}^m$ be a profile of stationary strategies (pure or mixed strategies) of the players. Then the elements of probability transition matrix $P^{\mathbf{s}} = (p^{\mathbf{s}}_{x,y})$ in the Markov process induced by $\mathbf{s}$ can be calculated as follows:

$$p^{\mathbf{s}}_{x,y} = \sum_{a \in A(x)} s^i_{x,a} p^a_{x,y} \quad \text{for} \quad x \in X_i, \quad i = 1, 2, \dots, m. \tag{2}$$

If we denote by $Q^{\mathbf{s}} = (q^{\mathbf{s}}_{x,y})$ the limiting probability matrix of matrix $P^{\mathbf{s}}$ then the average payoffs per transition $\omega^1_{x_0}(\mathbf{s}),\ \omega^2_{x_0}(\mathbf{s}), \dots, \omega^m_{x_0}(\mathbf{s})$ for the players induced by profile $\mathbf{s}$ are determined as follows

$$\omega^i_{x_0}(\mathbf{s}) = \sum_{k=1}^{m} \sum_{y \in X_k} q^{\mathbf{s}}_{x_0,y} f^i(y, s^k), \quad i = 1, 2, \dots, m, \tag{3}$$

where

$$f^i(y, s^k) = \sum_{a \in A(y)} s^k_{y,a} f^i(y, a), \quad \text{for} \ \ y \in X_k,\ k \in \{1, 2, \dots, m\} \tag{4}$$

expresses the average reward (step reward) of player i in the state $y \in X_k$ when player k uses the strategy s^k.

The functions $\omega^1_{x_0}(\mathbf{s}),\ \omega^2_{x_0}(\mathbf{s}),\ \dots,\ \omega^m_{x_0}(\mathbf{s})$ on $\mathbf{S} = \mathbf{S}^1 \times \mathbf{S}^2 \times \cdots \times \mathbf{S}^m$, defined according to (10), (11), determine a game in normal form that we denote by $\langle \{\mathbf{S}^i\}_{i=\overline{1,m}},\ \{\omega^i_{x_0}(\mathbf{s})\}_{i=\overline{1,m}} \rangle$. This game corresponds to the *average stochastic positional game in mixed stationary strategies* that in extended form is determined by the tuple $(\{X_i\}_{i=\overline{1,m}},\ \{A(x)\}_{x\in X},\ \{f^i(x, a)\}_{i=\overline{1,m}},\ p,\ x_0)$. The functions $\omega^1_{x_0}(\mathbf{s}),\ \omega^2_{x_0}(\mathbf{s}), \dots, \omega^m_{x_0}(\mathbf{s})$ on $S = S^1 \times S^2 \times \cdots \times S^m$, determine the game $\langle \{S^i\}_{i=\overline{1,m}}, \{\omega^i_{x_0}(\mathbf{s})\}_{i=\overline{1,m}} \rangle$ that corresponds to the *average stochastic positional game in pure strategies*. In the extended form this game is also determined by the tuple $(\{X_i\}_{i=\overline{1,m}},\ \{A(x)\}_{x\in X},\ \{f^i(x, a)\}_{i=\overline{1,m}},\ p,\ x_0)$.

A stochastic positional game can be considered also for the case when the starting state is chosen randomly according to a given distribution $\{\theta_x\}$ on X. So, for a given stochastic positional game we may assume that the play starts in the state $x \in X$ with probability $\theta_x > 0$ where $\sum\limits_{x\in X} \theta_x = 1$. If the players use mixed stationary strategies then the payoff functions

$$\psi^i_\theta(\mathbf{s}) = \sum_{x \in X} \theta_x \omega^i_x(\mathbf{s}), \quad i = 1, 2, \dots, m$$

on $\mathbf{S}$ define a game in normal form $\langle \{\mathbf{S}^i\}_{i=\overline{1,m}},\ \{\psi^i_\theta(\mathbf{s})\}_{i=\overline{1,m}} \rangle$ that in extended form is determined by $(\{X_i\}_{i=\overline{1,m}}, \{A(x)\}_{x\in X}, \{f^i(x, a)\}_{i=\overline{1,m}}, p, \{\theta_x\}_{x\in X})$. In the case $\theta_x = 0, \forall x \in X \setminus \{x_0\},\ \theta_{x_o} = 1$ the considered game becomes a stochastic positional game with a fixed starting state x_0.

3.2 Stationary Nash Equilibria for an Average Stochastic Positional Game and Determining the Optimal Strategies of the Players

We present a Nash equilibria existence result and an approach for determining the optimal mixed stationary strategies of the players for the average stochastic positional game when the starting state of the game is chosen randomly according to a given distribution $\{\theta_x\}$ on the set of states X. In this case for the game in normal form $\langle \{\mathbf{S}^i\}_{i=\overline{1,m}},\ \{\psi^i_\theta(\mathbf{s})\}_{i=\overline{1,m}}\ \rangle$, the set of strategies $\mathbf{S}^i$ and the payoff functions $\psi^i_\theta(\mathbf{s}),\ \ i=1,2,\dots,m$, can be specified as follows:

Let $\mathbf{S}^i,\ i\in\{1,2,\dots m\}$ be the set of solutions of the system

$$\left\{\begin{array}{lll} \sum\limits_{a\in A(x)} s^i_{x,a}=1, & & \forall x\in X_i;\\ \qquad\ s^i_{x,a}\ge 0, & & \forall x\in X_i,\ \ \forall a\in A(x). \end{array}\right. \tag{5}$$

On $\mathbf{S}=\mathbf{S}^1\times\mathbf{S}^2\times\cdots\times\mathbf{S}^m$ we define m payoff functions

$$\psi^i_\theta(s^1,s^2,\dots,s^m)=\sum_{k=1}^m\sum_{x\in X_k}\sum_{a\in A(x)} s^k_{x,a}f^i(x,a)q_x,\qquad i=1,2,\dots,m, \tag{6}$$

where q_x for $x\in X$ are determined uniquely from the following system of linear equations

$$\left\{\begin{array}{ll} q_y-\sum\limits_{k=1}^m\sum\limits_{x\in X_k}\sum\limits_{a\in A(x)} s^k_{x,a}\,p^a_{x,y}\,q_x=0, & \forall y\in X;\\ q_y+w_y-\sum\limits_{k=1}^m\sum\limits_{x\in X_k}\sum\limits_{a\in A(x)} s^k_{x,a}\,p^a_{x,y}\,w_x=\theta_y, & \forall y\in X \end{array}\right. \tag{7}$$

for an arbitrary fixed profile $\mathbf{s}=(s^1,s^2,\dots,s^m)\in\mathbf{S}$.

The functions $\psi^i_\theta(s^1,s^2,\dots,s^m),\ i=1,2,\dots,m,$ represent the payoff functions for the average stochastic game in normal form $\langle \{\mathbf{S}^i\}_{i=\overline{1,m}},\ \{\psi^i_\theta(\mathbf{s})\}_{i=\overline{1,m}}\ \rangle$ determined by the tuple $(\{X_i\}_{i=\overline{1,m}},\ \{A(x)\}_{x\in X},\ \{f^i(x,a)\}_{i=\overline{1,m}},\ p,\ \{\theta_y\}_{y\in X})$ where θ_y for $y\in X$ are given positive values such that $\sum_{y\in X}\theta_y=1$. If $\theta_y=0,\ \forall y\in X\setminus\{x_0\}$ and $\theta_{x_0}=1$, then we obtain an average stochastic game in normal form $\langle\{\mathbf{S}^i\}_{i=\overline{1,m}},\ \{\omega^i_{x_0}(\mathbf{s})\}_{i=\overline{1,m}}\ \rangle$ when the starting state x_0 is fixed, i.e. $\psi^i_\theta(s^1,s^2,\dots,\ s^m)=\omega^i_{x_0}(s^1,s^2,\ \dots,\ s^m),\ i=1,2,\dots,m$. So, in this case the game is determined by $(\{X_i\}_{i=\overline{1,m}},\ \{A(x)\}_{x\in X},\ \{f^i(x,a)\}_{i=\overline{1,m}},\ p,\ x_0)$.

In [8] it has been shown that if $\theta_x>0,\forall x\in X,\sum_{x\in X}=1$ then each payoff function $\psi^i_\theta(s),\ i\in\{1,2,\dots,m\}$ in the game $\langle\{\mathbf{S}^i\}_{i=\overline{1,m}},\ \{\psi^i_\theta(\mathbf{s})\}_{i=\overline{1,m}}\ \rangle$ is quasi-monotonic (quasi-convex and quasi-concave) with respect to $\mathbf{s}^i$ on a convex and compact set $\mathbf{S}^i$ for fixed $\mathbf{s}^1,\mathbf{s}^2,\dots,\mathbf{s}^{i-1},\mathbf{s}^{i+1},\dots,\mathbf{s}^m$. Moreover for the game $\langle\{\mathbf{S}^i\}_{i=\overline{1,m}},\ \{\psi^i_\theta(\mathbf{s})\}_{i=\overline{1,m}}\ \rangle$ it has been shown that each payoff function $\psi^i_\theta(s),\ i\in\{1,2,\dots,m\}$, is graph-continuous in the sense of Dasgupta and Maskin [2]. Based on these properties in [8] the following theorem is proved.

Theorem 1. *The game* $\langle\{\mathbf{S}^i\}_{i=\overline{1,m}}, \{\psi^i_\theta(s)\}_{i=\overline{1,m}}\rangle$ *with* $\theta_x > 0, \forall x \in X, \sum_{x\in X} = 1$ *possesses a Nash equilibrium* $\mathbf{s}^*= (s^{1*}, s^{2*}, \ldots, s^{m*}) \in \mathbf{S}$ *which is a Nash equilibrium in mixed stationary strategies for the average stochastic positional game determined by the tuple* $(\{X_i\}_{i=\overline{1,m}},\ \{A(x)\}_{x\in X}, \{f^i(x,a)\}_{i=\overline{1,m}},\ p,\ \{\theta_y\}_{y\in X})$. *Moreover,* $\mathbf{s}^* = (s^{1*}, s^{2*}, \ldots, s^{m*})$ *is a Nash equilibrium in mixed stationary strategies for the average stochastic positional game* $\langle\{\mathbf{S}^i\}_{i=\overline{1,m}}, \{\omega^i_y(\mathbf{s})\}_{i=\overline{1,m}}\ \rangle$ *with an arbitrary starting state* $y \in X$.

Thus, for an average stochastic positional game a Nash equilibrium in mixed stationary strategies can be found using the noncooperative static game model $\langle\{\mathbf{S}^i\}_{i=\overline{1,m}}, \{\psi^i_\theta(\mathbf{s})\}_{i=\overline{1,m}}\rangle$, where $\mathbf{S}^i$ and $\psi^i_\theta(\mathbf{s})$, $i = 1, 2, \ldots, m$, are determined according to (5)–(7). In the case $m = 2$, $f(x, a) = f^1(x, a) = -f^2(x, a)$, $\forall x \in X$, $\forall a \in A(x)$ this game corresponds to a two-player zero-sum average stochastic positional game. In [7] it is shown that for a two-player zero-sum average stochastic game there exist pure stationary equilibria. The proof of this results is similar to the proof of the existence of pure stationary equilibria for two-player zero-sum mean payoff games from [5]. Algorithms for determining the optimal stationary strategies in such games are proposed in [5, 6, 9, 11].

4 Formulation of Mean Payoff Games in Mixed Stationary strategies

Let us consider an m-player mean payoff game determined by the tuple $(G, \{X_i\}_{i=\overline{1,m}}, \{c^i\}_{i=\overline{1,m}}, x_0)$, where $G = (X, E)$ is a finite directed graph with a vertex set X and an edge set E, $X = X_1 \cup X_2 \cup \cdots \cup X_m$ $(X_i \cap Xj = \emptyset, i \neq j)$ is a partition of X that determines the corresponding position sets of players and $c^i : E \to R^1$, $i = 1, 2, \ldots, m$, are the real functions that determine the rewards on edges of graph G and x_0 is the starting position of the game.

The pure and mixed stationary strategies in the mean payoff game on G can be defined in a similar way as for the average stochastic positional game. We identify the set of mixed stationary strategies S^i of player $i \in \{1, 2, \ldots, m\}$ with the set of solutions of the system

$$\begin{cases} \displaystyle\sum_{y\in X(x)} s^i_{x,y} = 1, \quad \forall x \in X_i; \\ \qquad\qquad s^i_{x,y} \geq 0, \quad \forall x \in X_i,\ y \in X(x) \end{cases} \tag{8}$$

where $X(x)$ represents the set of neighbor vertices for the vertex x, i.e. $X(x) = \{y \in X | e = (x, y) \in E\}$.

Let $\mathbf{s} = (s^1, s^2, \ldots, s^m)$ be a profile of stationary strategies (pure or mixed strategies) of the players. This means that the moves in the mean payoff game from an arbitrary $x \in X$ to $y \in X$ induced by $\mathbf{s}$ are made according to probabilities of the stochastic matrix $P^s = (s_{x,y})$, where

$$\mathbf{s}_{x,y} = \begin{cases} s^i_{x,y} & \text{if } \ e = (x, y) \in E, x \in X_i, y \in X;\ i = 1, 2, \ldots, m; \\ 0 & \text{if } \ e = (x, y) \notin E. \end{cases} \tag{9}$$

Thus, for a given profile $\mathbf{s}$ we obtain a Markov process with the probability transition matrix $P^{\mathbf{s}} = (\mathbf{s}_{x,y})$ and the corresponding rewards $c^i_{x,y},\ i = 1, 2, \dots, m,$ on edges $(x, y) \in E$. Therefore, if $Q^{\mathbf{s}} = (q^{\mathbf{s}}_{x,y})$ is the limiting probability matrix of $P^{\mathbf{s}}$ then the average rewards per transition $\omega^1_{x_0}(\mathbf{s}),\ \omega^2_{x_0}(\mathbf{s}),\ \dots, \omega^m_{x_0}(\mathbf{s})$ for the players can be determined as follows

$$\omega^i_{x_0}(\mathbf{s}) = \sum_{k=1}^{m} \sum_{y \in X_k} q^{\mathbf{s}}_{x_0,y} \mu^i(y, s^k), \quad i = 1, 2, \dots, m, \tag{10}$$

where

$$\mu^i(y, s^k) = \sum_{z \in X(y)} s^k_{y,z} c^i(y, z), \quad \text{for} \ \ y \in X_k,\ k \in \{1, 2, \dots, m\} \tag{11}$$

expresses the average step reward of player i in the state $y \in X_k$ when player k uses the mixed stationary strategy s^k. The functions $\omega^1_{x_0}(\mathbf{s}), \omega^2_{x_0}(\mathbf{s}), \dots, \omega^m_{x_0}(\mathbf{s})$ on $\mathbf{S} = \mathbf{S}^1 \times \mathbf{S}^2 \times \dots \times \mathbf{S}^m$, defined according to (10), (11), determine a game in normal form that we denote by $\langle \{\mathbf{S}^i\}_{i=\overline{1,m}},\ \{\omega^i_{x_0}(\mathbf{s})\}_{i=\overline{1,m}} \rangle$. This game corresponds to the *mean payoff game in mixed stationary strategies* on G with a fixed starting position x_0. So this game is determined by the tuple $(G, \{X_i\}_{i=\overline{1,m}}, \{c^i\}_{i=\overline{1,m}}, x_0)$.

In a similar way as for an average stochastic game here we can consider the mean payoff game on G when the starting state is chosen randomly according to a given distribution $\{\theta_x\}$ on X. So, for such a game we will assume that the play starts in the states $x \in X$ with probabilities $\theta_x > 0$ where $\sum_{x \in X} \theta_x = 1$. If the players in such a game use mixed stationary strategies of moves in their positions then the payoff functions

$$\psi^i_\theta(\mathbf{s}) = \sum_{x \in X} \theta_x \omega^i_x(\mathbf{s}), \quad i = 1, 2, \dots, m$$

on $\mathbf{S}$ define a game in normal form $\langle \{\mathbf{S}^i\}_{i=\overline{1,m}},\ \{\psi^i_\theta(\mathbf{s})\}_{i=\overline{1,m}} \rangle$ that is determined by the following tuple $(G,\ \{X_i\}_{i=\overline{1,m}},\ \{c^i\}_{i=\overline{1,m}},\ \{\theta_x\}_{x \in X})$. In the case $\theta_x = 0$, $\forall x \in X \setminus \{v_0\}$, $\theta_{v_0} = 1$ this game becomes a mean payoff game with fixed starting state x_0.

5 Nash Equilibria in Mixed Stationary Strategies for Mean Payoff Games and Determining the Optimal Strategies of the Players

In this section we show how the results from the previous sections can be applied for determining Nash equilibria and the optimal mixed stationary strategies of the players for mean payoff games.

Let $\langle \{\mathbf{S}^i\}_{i=\overline{1,m}},\ \{\psi^i_\theta(s)\}_{i=\overline{1,m}} \rangle$ be the game in normal form for the mean payoff game determined by $(G,\ \{X_i\}_{i=\overline{1,m}},\ \{c^i\}_{i=\overline{1,m}},\ \{\theta_x\}_{x \in X})$. We show that $\mathbf{S}^i$ and $\psi^i_\theta(s)$ for $i \in \{1, 2, \dots, m\}$ can be defined as follows: $\mathbf{S}^i$ represents a set of the solutions of the system

$$\begin{cases} \displaystyle\sum_{y \in X(x)} s^i_{x,y} = 1, & \forall x \in X_i; \\ s^i_{x,y} \geq 0, & \forall x \in X_i,\ y \in X(x) \end{cases} \tag{12}$$

and

$$\psi_\theta^i(s^1, s^2, \dots, s^m) = \sum_{k=1}^{m} \sum_{y \in X_k} \sum_{y \in X(x)} s_{x,y}^k c^i(x,y) q_x, \qquad (13)$$

where q_x for $x \in X$ are determined uniquely (via $s_{x,y}^k$) from the following system of equations

$$\begin{cases} q_y - \sum\limits_{k=1}^{m} \sum\limits_{x \in X_k} s_{x,y}^k \; q_x = 0, & \forall y \in X; \\ q_y + w_y - \sum\limits_{k=1}^{m} \sum\limits_{x \in X_k} s_{x,y}^k \; w_x = \theta_y, & \forall y \in X. \end{cases} \qquad (14)$$

Here θ_y for $y \in X$ represent arbitrary fixed positive values where $\sum\limits_{y \in X} \theta_y = 1$.

Using Theorem 1 we can prove now the following result.

Theorem 2. *For a mean payoff game on graph G the corresponding game in normal form $\langle \{\mathbf{S}^i\}_{i=\overline{1,m}}, \{\psi_\theta^i(s)\}_{i=\overline{1,m}} \rangle$ possesses a Nash equilibrium $\mathbf{s}^* = (s^{1^*}, s^{2^*}, \dots, s^{m*}) \in \mathbf{S}$ which is a Nash equilibrium in mixed stationary strategies for the mean payoff game on G with an arbitrary starting position $x_0 \in X$.*

Proof. To prove the theorem it is sufficient to show that the functions $\psi_\theta^i(s)$, $i \in \{1, 2, \dots, m\}$, defined according to (13), (14) represent the payoff functions for the mean payoff game determined by $(G,\ \{X_i\}_{i=\overline{1,m}},\ \{c^i\}_{i=\overline{1,m}},\ \{\theta_x\}_{x \in X})$. This is easy to verify because if we replace in (6) the rewards $f^i(x,a)$ for $x \in X$ and $a \in A(x)$ by rewards $c_{x,y}^i$ for $(x,y) \in E$ and in (6), (7) we replace the probabilities $p_{x,y}^a$, $x \in X_k$, $a \in A(x)$ for the corresponding players $k = 1, 2, \dots, m$ by $p_{x,y}^k \in \{0,1\}$ according to the structure of graph G then we obtain that (6), (7) are transformed into (13), (14). If we apply Theorem 1 after that then obtain the proof of the theorem. □

So, the optimal mixed stationary strategies of the players in a mean payoff game can be found if we determine the optimal stationary strategies of the players for the game $\langle \{\mathbf{S}^i\}_{i=\overline{1,m}},\ \{\psi_\theta^i(s)\}_{i=\overline{1,m}} \rangle$ where $\mathbf{S}^i$ and $\psi_\theta^i(s)$ for $i \in \{1, 2, \dots, m\}$ are defined according to (12)–(14). If $m = 2$, $c_{x,y} = c_{x,y}^1 = -c_{x,y}^2$, $\forall (x,y) \in E$ then we obtain a game-theoretic model in normal form for the zero-sum two-player mean payoff on graph G. In this case the equilibrium exists in pure stationary strategies and the considered game model allows us to determine the optimal pure stationary strategies of the players. The results from [9] related to antagonistic average stochastic positional games can be also extended to antagonistic mean payoff games on graphs if we take into account the transformations mentioned above in the proof of Theorem 2, i.e. we should change the rewards $f^i(x,a)$ for $x \in X$, $a \in A(x)$ by rewards $c_{x,y}^i$ for $(x,y) \in E$ and replace the probabilities $p_{x,y}^a$, $x \in X_k$, $a \in A(x)$, $k = 1, 2, \dots, m$ by probabilities $p_{x,y}^k \in \{0,1\}$ according to the structure of the graph G.

6 Conclusion

The considered m-player non-zero mean payoff games on graphs generalize the zero-sum two-player mean payoff games on graphs considered in [3, 5, 11]. For zero-sum two-player mean payoff games on graphs there exist Nash equilibria in pure stationary strategies that can be determined based on results from [5, 11]. For the case of non-zero-sum mean payoff games on graphs Nash equilibria in pure stationary strategies may not exist, however there exist Nash equilibria in mixed stationary strategies. Such equilibria can be determined and characterized as Nash equilibria for the noncooperative static game models from Sections 5, 6.

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Global Asymptotic Stability of Generalized Homogeneous Dynamical Systems

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Abstract. The goal of the paper is to study the relationship between asymptotic stability and exponential stability of the solutions of generalized homogeneous nonautonomous dynamical systems. This problem is studied and solved within the framework of general non-autonomous (cocycle) dynamical system. The application of our general results for differential and difference equations is given.

Mathematics subject classification: 34C11, 34C14, 34D05, 34D23, 37B25, 37B55, 37C75.
Keywords and phrases: uniform asymptotic stability; global attractor; homogeneous dynamical system.

1 Introduction

This paper is dedicated to the study of the problem of asymptotic stability of a class of nonautonomous dynamical systems with some property of symmetry. Namely, we study this problem for so-called generalized homogeneous nonautonomous dynamical systems, that is, a class of nonautonomous dynamical systems invariant with respect to a group of transformations called dilations. We establish our main results in the framework of general nonautonomous (cocycle) dynamical systems.

The motive for writing of this article was the works of A. Bacciotti and L. Rosier [2], A. Polyakov [16], V. I. Zubov [24] (see also the bibliography therein) and the work of the author [5]. We prove the equivalence of uniform asymptotic stability and exponential stability for this class of nonautonomous dynamical systems. If the phase space Y of driving system $(Y,\mathbb{T},\sigma)$ for the cocycle dynamical systems $\langle\mathbb{R}^n,\varphi,(Y,\mathbb{T},\sigma)\rangle$ is compact, then we prove that the asymptotic stability and uniform asymptotic stability are equivalent. If additionally the driving system $(Y,\mathbb{S},\sigma)$ with compact phase space Y is minimal, then for asymptotic stability the uniform stability and the existence of a positive number a and an element $y_0 \in Y$ such that $\lim\limits_{t\to+\infty}|\varphi(t,u,y_0)| = 0$ for any $u \in B[0,a]$ are sufficient. We apply these results for differential and difference equations.

The paper is organized as follows. In the second Section, we collect some known notions and facts from dynamical systems that we use in this paper. Namely, we

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present the construction of shift dynamical systems, definitions of Poisson stable motions and some facts about compact global attractors of dynamical systems. In the third Section we establish the relation between uniformly asymptotic stability and exponential stability for general nonautonomous (cocycle) dynamical systems. The fourth Section is dedicated to the relation between asymptotic stability and exponential stability for the nonautonomous dynamical systems with the compact phase space of their driving system. In the fifth Section, we study the nonautonomous dynamical system with driving system $(Y, \mathbb{S}, \sigma)$, when Y is a compact and minimal set. Finally, in the sixth Section we apply our general results, obtained in Sections 3-5 to differential/difference equations.

2 Preliminaries

Throughout the paper, we assume that X and Y are metric spaces and for simplicity we use the same notation ρ to denote the metrics on them, which we think would not lead to confusion. Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$, $\mathbb{S} = \mathbb{R}$ or $\mathbb{Z}$, $\mathbb{S}_+ := \{s \in \mathbb{S} |\ s \geq 0\}$ and $\mathbb{T} \subseteq \mathbb{S}$ be a sub-semigroup of $\mathbb{S}$ such that $\mathbb{S}_+ \subseteq \mathbb{T}$.

Let $(X, \mathbb{T}, \pi)$ be a dynamical system on X and $\mathfrak{M}$ be some family of subsets from X.

Definition 1. A dynamical system $(X, \mathbb{T}, \pi)$ is said to be $\mathfrak{M}$-dissipative if for every $\varepsilon > 0$ and $M \in \mathfrak{M}$ there exists $L(\varepsilon, M) > 0$ such that $\pi^t M \subseteq B(K, \varepsilon)$ for any $t \geq L(\varepsilon, M)$, where K is a subset from X depending only on $\mathfrak{M}$. In this case we will call K an attractor for $\mathfrak{M}$.

The most important for applications are the cases when K is a bounded or compact set and $\mathfrak{M} = \{\{x\} \mid x \in X\}$ or $\mathfrak{M} = C(X)$, or $\mathfrak{M} = \{B(x, \delta_x) \mid x \in X,\ \delta_x > 0\}$, where

1. $C(X)$ is the family of all compact subsets of X;
2. $B(x_0, \delta) := \{x \in X |\ \rho(x, x_0) < \delta\}$.

Definition 2. The system $(X, \mathbb{T}, \pi)$ is called:

- pointwise dissipative if there exists $K \subseteq X$ such that for every $x \in X$

$$\lim_{t \to +\infty} \rho(xt, K) = 0; \tag{1}$$

- compactly dissipative if the equality (1) takes place uniformly w.r.t. x on the compact subsets from X;
- locally dissipative if for any point $p \in X$ there exists $\delta_p > 0$ such that the equality (1) takes place uniformly w.r.t. $x \in B(p, \delta_p)$.

Let $(X,\mathbb{T},\pi)$ be compactly dissipative and K be a nonempty compact set that is an attractor for compact subsets X. Then for every compact $M\subseteq X$ the equality

$$\lim_{t\to+\infty}\sup_{x\in M}\rho(xt,K)=0$$

holds. It is possible to show [7, Ch.I] that the set J defined by the equality

$$J:=\omega(K)$$

does not depend on the choice of the set K attracting all compact subsets of the space X.

Lemma 1. *[7, Ch.I] If the dynamical system $(X,\mathbb{T},\pi)$ is pointwise dissipative, $\Omega_X\neq\emptyset$ and it is compact, then $\Omega_X\subseteq J^+(\Omega_X)$.*

Theorem 1. *[7, Ch.I] For the dynamical systems $(X,\mathbb{T},\pi)$ with the locally compact phase space X the pointwise, compact and local dissipativity are equivalent.*

Definition 3. (Cocycle on the state space E with the base $(Y,\mathbb{S},\sigma)$.) A triplet $\langle E,\phi,(Y,\mathbb{S},\sigma)\rangle$ (or briefly ϕ if no confusion) is said to be a *cocycle* on state space (or fibre) E with base $(Y,\mathbb{S},\sigma)$ (or driving system $(Y,\mathbb{S},\sigma)$) if the mapping $\phi:\mathbb{S}_+\times Y\times E\to E$ satisfies the following conditions:

1. $\phi(0,u,y)=u$ for all $u\in E$ and $y\in Y$;
2. $\phi(t+\tau,u,y)=\phi(t,\phi(\tau,u,y),\sigma(\tau,y))$ for all $t,\tau\in\mathbb{S}_+,u\in E$ and $y\in Y$;
3. the mapping ϕ is continuous.

Remark 1. If $\varphi(t_0,u_1,y_0)=\varphi(t_0,u_2,y_0)$ ($t_0>0$, $u_1,u_2\in E$ and $y_0\in Y$), then $\varphi(t,u_1,y_0)=\varphi(t,u_2,y_0)$ for any $t\geq t_0$.

Condition (C). (Strong uniqueness condition.) If $\varphi(t_0,u_1,y_0)=\varphi(t_0,u_2,y_0)$ ($t_0>0$, $u_1,u_2\in E$ and $y_0\in Y$), then $\varphi(t,u_1,y_0)=\varphi(t,u_2,y_0)$ for any $t\in\mathbb{T}_+$.

Everywhere below in this paper we consider the cocycles φ satisfying Condition (C).

Definition 4. (Skew-product dynamical system.) Let $\langle E,\phi,(Y,\mathbb{S},\sigma)\rangle$ be a cocycle on E, $X:=E\times Y$ and π be a mapping from $\mathbb{S}_+\times X$ to X defined by $\pi:=(\phi,\sigma)$, i.e., $\pi(t,(u,y))=(\phi(t,u,y),\sigma(t,y))$ for all $t\in\mathbb{S}_+$ and $(u,y)\in E\times Y$. The triplet $(X,\mathbb{S}_+,\pi)$ is an autonomous dynamical system and is called *skew-product dynamical system*.

Let $x\in X$. Denote by $\Sigma_x^+:=\{\pi(t,x):\ t\geq 0\}$ (respectively, $\Sigma_x:=\{\pi(t,x_0):t\in\mathbb{T}\}$) the positive semi-trajectory (respectively, the trajectory) of the point x and $H^+(x):=\overline{\Sigma_x^+}$ (respectively, $H(x):=\overline{\Sigma_x}$) the semi-hull of x (respectively, the hull of x), where by bar the closure of Σ_x^+ (respectively, Σ_x) in X is denoted.

Let $(X,\mathbb{S},\pi)$ be a dynamical system. Let us recall the classes of Poisson stable motions we study in this paper, see [20, 23] for details.

Definition 5. A point $x \in X$ is called *stationary* (respectively, *τ-periodic*) if $\pi(t,x) = x$ (respectively, $\pi(t+\tau,x) = \pi(t,x)$) for all $t \in \mathbb{S}$.

Definition 6. For given $\varepsilon > 0$, a number $\tau \in \mathbb{R}$ is called an *ε-shift of x* (respectively, *ε-almost period of x*) if $\rho(\pi(\tau,x),x) < \varepsilon$ (respectively, $\rho(\pi(\tau+t,x),\pi(t,x)) < \varepsilon$ for all $t \in \mathbb{R}$).

Definition 7. A point $x \in X$ is called *almost recurrent* (respectively, *almost periodic*) if for any $\varepsilon > 0$ there exists a positive number l such that any segment of length l contains an ε-shift (respectively, ε-almost period) of x.

Definition 8. If a point $x \in X$ is almost recurrent and its trajectory Σ_x is precompact, then x is called *(Birkhoff) recurrent*.

Remark 2. It is easy to see that every almost periodic point $x \in X$ is recurrent, but the reverse statement generally speaking is not true.

Denote by $C(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n)$ the family of all continuous functions $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ equipped with the compact-open topology. This topology can be generated by Bebutov distance (see, e.g., [3], [23, ChIV])

$$d(f,g) := \sup_{L>0} \min\{\max_{|t|+|x|\le L} \rho(f(t,x),g(t,x)), 1/L\}.$$

Denote by $(C(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n), \mathbb{T}, \sigma)$ the shift dynamical system (or called Bebutov dynamical system), i.e., $\sigma(\tau, f) := f^\tau$, where $f^\tau(t,x) := f(t+\tau,x)$ for any $(t,x) \in \mathbb{T} \times \mathbb{R}^n$.

We will say that the function $f \in C(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n)$ possesses the property (A) if the motion $\sigma(t.f)$ possesses this property in the shift dynamical system $(C(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n), \mathbb{T}, \sigma)$. As the property (A) we will consider the Lagrange stability, periodicity in time (respectively, almost periodicity, recurrence and so on).

Note that the function $f \in C(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n)$ is Lagrange stable if and only if the function $f_K := f_{\big|\mathbb{T}\times K}$ is bounded and uniformly continuous on $\mathbb{T} \times K$ for any compact subset K from $\mathbb{R}^n$ (see, e.g., [21], [23, ChIV]).

Definition 9. Let $(\mathbb{R}^n, \mathbb{T}, \lambda)$ be a linear dynamical system on $\mathbb{R}^n$ [7, Ch.II]. A function $F \in C(Y \times \mathbb{R}^n, \mathbb{R}^n)$ is said to be λ-homogeneous if

$$F(y, \lambda(\tau,w)) = \lambda(\tau, F(y,w)) \text{ (or equivalently } F(y, \lambda^\tau w) = \lambda^\tau F(y,w))$$

for any $(y,\tau,w) \in Y \times \mathbb{T} \times \mathbb{R}^n$.

Example 1. Let $(Y, \mathbb{T}, \sigma)$ be a dynamical system on the metric space Y and $\mathbb{T} = \mathbb{R}_+$ or $\mathbb{R}$. Consider a differential equation

$$u' = F(\sigma(t,y),u), \ (y \in Y) \tag{2}$$

where $F \in C(Y \times \mathbb{R}^n, \mathbb{R}^n)$ is a regular function, i.e., for any $(u,y) \in \mathbb{R}^n \times Y$ there exists a unique solution $\varphi(t,u,y)$ of equation (2) defined on $\mathbb{R}_+$ with initial data

$\varphi(0,u,y) = u$. Then (see, for example, [4], [20]-[22] the continuous mapping $\varphi : \mathbb{R}_+ \times \mathbb{R}^n \times Y \to \mathbb{R}^n$ satisfying the condition $\varphi(t+\tau,u,y) = \varphi(t,\varphi(\tau,u,y),\sigma(\tau,y))$ for any $t,\tau \in \mathbb{R}_+$ and $(u,y) \in \mathbb{R}^n \times Y$ is well defined . Then the triplet $\langle \mathbb{R}^n, \varphi, (Y,\mathbb{T},\sigma)\rangle$ is a cocycle over $(Y,\mathbb{T},\sigma)$ with the fibre $\mathbb{R}^n$ (shortly φ) generated by (2).

Lemma 2. *Assume that the function $F \in C(Y \times \mathbb{R}^n, \mathbb{R}^n)$ is λ-homogeneous, then the cocycle φ generated by (2) is also λ-homogeneous.*

Proof. To prove this statement we consider the function $\psi(t) := \lambda^\tau \varphi(t,u,y)$. It is easy to check that

$$\psi^{'}(t) = \lambda^\tau \varphi^{'}(t,u,y) = \lambda^\tau F(\sigma(t,y),\varphi(t,u,y)) = \\ F(\sigma(t,y),\lambda^\tau \varphi(t,u,y)) = F(\sigma(t,y),\psi(t))$$

for any $t \in \mathbb{T}$. Since $\psi(0) = \lambda^\tau u$, then we obtain $\psi(t) = \varphi(t,\lambda^\tau u,y)$, i.e., $\lambda^\tau \varphi(t,u,y) = \varphi(t,\lambda^\tau u,y)$ for any $t,\tau \in \mathbb{T}$ and $(u,y) \in \mathbb{R}^n \times Y$. Lemma is proved. □

3 Uniformly Asymptotical Stability of Nonautonomous Generalized Homogeneous Dynamical Systems: General Case

Let $X := \mathbb{R}^n$ with euclidian norm $|x| := \sqrt{x_1^2 + \ldots + x_n^2}$. Denote by

$$|x|_{r,p} := \left(\Sigma_{i=1}^n |x_i|^{\frac{p}{r_i}}\right)^{\frac{1}{p}},$$

where $r := (r_1,\ldots,r_n)$, $r_i > 0$ for any $i = 1,\ldots,n$ and $p \geq 1$.

Denote by

1. $\rho(x) := |x|_{r,p}$;
2. $S_{r,p} := \{x \in \mathbb{R}^n |\ \rho(x) = 1\}$;
3. $\mathcal{K} := \{\alpha \in C(\mathbb{R}_+,\mathbb{R}_+)|\ \alpha(0) = 0$ and α is strictly increasing$\}$ and
4. $\mathcal{K}_\infty := \{\alpha \in \mathcal{K}|\ \alpha(t) \to +\infty$ as $t \to +\infty\}$.

There exist $a, b \in \mathcal{K}_\infty$ such that

$$a(|x|_{r,p}) \leq |x| \leq b(|x|_{r,p}) \tag{3}$$

for any $x \in \mathbb{R}^n$ (see for example [10]).

A generalized weight is a vector $r = (r_1,\ldots,r_n)$ with $r_i > 0$ for any $i = 1,\ldots,n$. The dilation associated to the generalized weight r is the action of the multiplicative group $\mathbb{R}_+ \setminus \{0\}$ on $\mathbb{R}^n$ given by:

$$\Lambda^r : \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}^n \quad \big((\mu,x) \to \Lambda^r_\mu x\big),$$

where $\Lambda^r_\mu := diag(\mu^{r_i})_{i=1}^n$.

Remark 3. The following statements hold:

1. $\Lambda^r_1 = I$, where $I := diag(1,\dots,1)$;
2. $\Lambda^r_{\mu_1}\Lambda^r_{\mu_2} = \Lambda^r_{\mu_1\mu_2}$ for any $\mu_1,\mu_2 \in \mathbb{R}_+ \setminus \{0\}$;
3. the matrix Λ^r_μ $(\mu > 0)$ is invertible and $\Lambda^r_{\mu^{-1}}$ is its inverse, i.e., $\Lambda^r_{\mu^{-1}} = \left(\Lambda^r_\mu\right)^{-1}$, because $L^r_\mu\Lambda^r_{\mu^{-1}} = \Lambda^r_1 = I$ for any $\mu > 0$;
4. $||\Lambda^r_\mu|| \to 0$ as $\mu \to 0$;
5. $$|\Lambda^r_\mu x| \ge \mu^\nu |x| \tag{4}$$
   for any $x \in \mathbb{R}^n$ and $\mu > 0$, where $\nu := \min\{r_i |\ i = 1,\dots,n\} > 0$;
6. $$\rho(\Lambda^r_\mu x) = \mu\rho(x) \tag{5}$$
   for any $(\mu, x) \in (0,+\infty) \times \mathbb{R}^n$, where $\rho(x) := |x|_{r,p}$;
7. $\Lambda^{(1,\dots,1)}_\mu = diag(\mu,\dots,\mu) = \mu I$ for any $\mu > 0$.

Lemma 3. *[7, Ch.II] Let $\mathfrak{D}$ be a family of functions $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the conditions:*

a. there exists $M > 0$ such that $0 < \eta(t) \le M$ for all $t \ge 0$ and $\eta \in \mathfrak{D}$;

b. $\eta(t) \to 0$ as $t \to +\infty$ uniformly in $\eta \in \mathfrak{D}$, i.e., for any $\varepsilon > 0$ there exists $L(\varepsilon) > 0$ such that $\eta(t) < \varepsilon$ for any $t \ge L(\varepsilon)$ and $\eta \in \mathfrak{D}$.

Then we have the following statements:

1. if $\eta(t+\tau) \le \eta(t)\eta(\tau)$ for any $t,\tau \ge 0$ and $\eta \in \mathfrak{D}$, then there exit positive numbers $\mathcal{N}$ and ν such that

$$\eta(t) \le \mathcal{N}e^{-\nu t}$$

for any $t \ge 0$ and $\eta \in \mathfrak{D}$;

2. if $\eta(t+\tau) \le \eta(t)\eta(\tau\eta^m(t))$ $(m > 0)$ for any $t,\tau \ge 0$ and $\eta \in \mathfrak{D}$, then there exist positive numbers a and b such that

$$\eta(t) \le M(a+bt)^{-\frac{1}{m}}$$

for any $t \ge 0$ and $\eta \in \mathfrak{D}$.

Definition 10. Following [13, 16, 18, 24] a cocycle $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ over dynamical system $(Y, \mathbb{T}, \sigma)$ (driving system) with the fibre $\mathbb{R}^n$ is said to be r-homogeneous of degree $m \in \mathbb{R}$ if

$$\varphi(t, \Lambda^r_\mu u, y) = \mu^m\Lambda^r_\mu\varphi(t,u,y) \tag{6}$$

for any $\mu > 0$ and $(t,u,y) \in \mathbb{T}_+ \times \mathbb{R}^n \times Y$.

In this subsection we suppose that the phase space Y of the driving system $(Y, \mathbb{R}, \sigma)$, generally speaking, is not compact.

Definition 11. The trivial motion $u = 0$ of the cocycle φ is said to be:

1. uniformly stable if for arbitrary positive number ε there exists a positive number $\delta = \delta(\varepsilon)$ such that $|u| < \delta$ implies
$$|\varphi(t, u, y)| < \varepsilon$$
for any $(t, y) \in \mathbb{T}_+ \times Y$;
2. uniformly attracting if there exists a positive number γ such that
$$\lim_{t\to+\infty} \sup_{|u|\leq\gamma, y\in Y} |\varphi(t, u, y)| = 0;$$
3. uniformly asymptotically stable if it is uniformly stable and uniformly attracting.

Lemma 4. *The trivial motion $u = 0$ of the r-homogeneous cocycle φ of the degree zero is uniformly stable if and only if for arbitrary $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\rho(u) < \delta$ implies $\rho(\varphi(t, u, y)) < \varepsilon$ for any $(t, y) \in \mathbb{T}_+ \times Y$.*

Proof. Let $u = 0$ be uniformly stable motion of φ, $\mu > 0$ and $\Delta(\mu) > 0$ be a positive number figuring in the definition of the uniform stability of $u = 0$. For any $\varepsilon > 0$ we put $\delta(\varepsilon) := b^{-1}(\Delta(a(\varepsilon))) > 0$, where a and b are some functions from $\mathcal{K}_\infty$ figuring in (3). If $\rho(u) < \delta$, then we have $|u| \leq b(\rho(u)) < \Delta(a(\varepsilon))$ and, consequently, $|\varphi(t, u, y)| < a(\varepsilon)$ for any $t \in \mathbb{T}_+$. Note that $\rho(\varphi(t, u, y)) \leq a^{-}(|\varphi(t, u, y)|) < a^{-1}(a(\varepsilon)) = \varepsilon$ for any $t \geq 0$.

The inverse statement can be proved using the same arguments as above. Lemma is proved. □

Lemma 5. *If the trivial motion $u = 0$ of the cocycle φ is uniformly stable, then there exists a positive number M such that*
$$|\varphi(t, u, y)| \leq \tilde{M}$$
for any $|u| \leq 1$ and $(t, y) \in \mathbb{T}_+ \times Y$.

Proof. Since the trivial motion $u = 0$ of the cocycle φ is uniformly stable, then there exists a positive number $\delta_0 = \delta(1)$ such that $|u| \leq \delta_0$ implies
$$|\varphi(t, u, y)| \leq 1$$
for any $|u| \leq \delta_0$ and $(t, y) \in \mathbb{T}_+ \times Y$. Since $\|\Lambda^r_{\mu^{-1}}\| \to 0$ as $\mu \to +\infty$, then there exists a positive number μ_0 such that
$$\|\|\Lambda^r_{\mu^{-1}}\| \leq \delta_0$$

for any $\mu \geq \mu_0$. Note that

$$|\varphi(t,u,y)| = |\varphi(t,\Lambda^r_\mu\Lambda^r_{\mu^{-1}}u,y)| = |\Lambda^r_\mu\varphi(t,\Lambda^r_{\mu^{-1}}u,y)| \leq \|\Lambda^r_\mu\||\varphi(t,\Lambda^r_{\mu^{-1}}u,y)| \quad (7)$$

for any $\mu \geq \mu_0$ and $(t,u,y) \in \mathbb{T}_+ \times \mathbb{R}^n \times Y$. By (7) we have

$$|\Lambda^r_{\mu_0^{-1}}u| \leq \delta_0$$

for any $|u| \leq 1$ and, consequently,

$$\sup_{|u|\leq 1} |\varphi(t,\Lambda^r_{\mu_0^{-1}}u,y)| \leq 1 \quad (8)$$

for any $(t,y) \in \mathbb{T}_+ \times Y$. Finally, we note that from (7) and (8) we obtain

$$|\varphi(t,u,y)| \leq \|\Lambda^r_{\mu_0^{-1}}\| := \widetilde{M}$$

for any $|u| \leq 1$ and $(t,y) \in \mathbb{T}_+ \times Y$. Lemma is proved. □

Corollary 1. *Under the conditions of Lemma 5 for any $R > 0$ there exists a positive constant $M(R)$ such that*

$$|\varphi(t,u,y)| \leq M(R)$$

for any $u \in \mathbb{R}^n$ with $|u| \leq R$ and $(t,y) \in \mathbb{T}_+ \times Y$.

Proof. Let R be an arbitrary positive number. Since $\|\Lambda^r_{\mu^{-1}}\| \to 0$ as $\mu \to +\infty$, then there exists a positive number $\mu_0 = m_0(R)$ such that

$$\|\Lambda^r_{\mu^{-1}}\| \leq R^{-1} \quad (9)$$

for any $\mu \geq \mu_0$ and, consequently,

$$|\Lambda^r_{\mu^{-1}}u| \leq \|\Lambda^r_{\mu_0^{-1}}\||u| \leq R^{-1}R = 1 \quad (10)$$

for any $|u| \leq R$. Note that

$$|\varphi(t,u,y)| = |\varphi(t,\Lambda^r_{\mu_0}\Lambda^r_{\mu_0^{-1}}u,y)| = |\Lambda^r_{\mu_0}\varphi(t,\lambda^r_{\mu_0^{-1}}u,y)| \leq \\ \|\Lambda^r_{\mu_0}\||\varphi(t,\Lambda^r_{\mu_0^{-1}}u,y)| \quad (11)$$

for any $(t,u,y) \in \mathbb{T}_+ \times \mathbb{R}^n \times Y$. According to (9)-(11) we obtain

$$|\varphi(t,u,y)| \leq \|\Lambda^r_{\mu_0}\||\varphi(t,\Lambda^r_{\mu_0^{-1}}u,y)| \leq \|\Lambda^r_{\mu_0^{-1}}\|\widetilde{M} := M(R)$$

for any $|u| \leq R$ and $(t,y) \in \mathbb{T}_+ \times Y$. □

Corollary 2. *Under the conditions of Lemma 5 there exists a positive constant M such that*

$$\rho(\varphi(t,u,y)) \leq M$$

for any $u \in \mathbb{R}^n$ with $\rho(u) \leq 1$ and $(t,y) \in \mathbb{T}_+ \times Y$.

Proof. Let $u \in \mathbb{R}^n$ with $\rho(u) \leq 1$ and $a, b \in \mathcal{K}_\infty$ be the function from (3), then we have

$$|u| \leq b(\rho(u)) \leq b(1)$$

and

$$a(\rho(\varphi(t,u,y)) \leq |\varphi(t,u,y)| \leq M(b(1)) \tag{12}$$

for any $(t,y) \in \mathbb{T}_+ \times Y$. From (12) we obtain

$$\rho(\varphi(t,u,y)) \leq a^{-1}(M(b(1)) := M$$

for any $\rho(u) \leq 1$ and $(t,y) \in \mathbb{T}_+ \times Y$. □

Lemma 6. *Let $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{T}, \sigma) \rangle$ be a cocycle over $(Y, \mathbb{T}, \sigma)$ with the fibre $\mathbb{R}^n$. Assume that φ is an r-homogeneous of the degree zero cocycle.*

Then

1. $$\rho(\varphi(t+\tau,u,y)) = \rho(\varphi(\tau,u,y))\rho(\varphi(t, \Lambda^r_{\mu^{-1}}\varphi(\tau,u,y), \sigma(\tau,y)) \tag{13}$$

   for any $t, \tau \in \mathbb{T}_+$, where $\mu := \rho(\varphi(\tau,u,y))$;

2. $$\rho(\varphi(t,u,y)) = \rho(u)\rho(\varphi(t, \Lambda^r_{\rho(u)^{-1}}u, y))$$

   for any $u \in \mathbb{R}^n \setminus \{0\}$, $t \in \mathbb{T}_+$ and $y \in Y$.

Proof. Note that

$$\begin{gathered}\rho(\varphi(t+\tau,u,y)) = \rho(\varphi(t,\varphi(\tau,u,y),\sigma(\tau,y)) = \\ \rho(\varphi(t, \Lambda^r_\mu \Lambda^r_{\mu^{-1}}\varphi(\tau,u,y), \sigma(\tau,y)) = \rho(\Lambda^r_\mu \varphi(t, \Lambda^r_{\mu^{-1}}\varphi(\tau,u,y),\sigma(\tau,y)) = \\ \mu\rho(\varphi(t,\Lambda^r_{\mu^{-1}}\varphi(\tau,u,y),\sigma(\tau,y))\end{gathered} \tag{14}$$

for any $\mu > 0$, $t, \tau \in \mathbb{T}_+$ and $(u,y) \in \mathbb{R}^n \times Y$. In particular for $\mu = \rho(\varphi(\tau,u,y)) > 0$ we obtain from (14) the following equality

$$\rho(\varphi(t+\tau,u,y)) = \rho(\varphi(\tau,u,y))\rho(\varphi(t,\Lambda^r_{\mu^{-1}}\varphi(\tau,u,y),\sigma(\tau,y).$$

The second statement of Lemma follows from the first one if we take $\tau = 0$. □

Theorem 2. *Let $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{T}, \sigma) \rangle$ be an r-homogeneous cocycle of the degree zero. The following statements are equivalent:*

1. *the trivial motion $u = 0$ of the cocycle φ is uniformly stable;*

2. *there exists a positive number M such that*

   $$\rho(\varphi(t,u,y)) \leq M\rho(u) \tag{15}$$

   for any $(t,u,y) \in \mathbb{T}_+ \times \mathbb{R}^n \times Y$.

Proof. To prove this Theorem it is sufficient to show (i) implies (ii) because the inverse implication, taking into account Lemma 4, is evident.

Let M be the positive number from Corollary 2 and (t,u,y) be an arbitrary element from $\mathbb{T}_+ \times \mathbb{R}^n \times Y$ with $u \neq 0$, then by Lemma 6 (item (ii)) we have

$$\rho(\varphi(t,u,y)) = \rho(u)\rho(\varphi(t,\Lambda^r_{\rho(u)^{-1}}u,y)). \tag{16}$$

Since $\rho(\Lambda^r_{\rho(u)^{-1}}u) = \rho(u)^{-1}\rho(u) = 1$, then by Corollary 2 we have

$$\rho(\varphi(t,\Lambda^r_{\rho(u)^{-1}}u,y)) \le M. \tag{17}$$

From (16) and (17) we obtain (15). Theorem is proved. □

Lemma 7. *If the trivial motion $u = 0$ of the cocycle φ is uniformly attracting, then*

$$\lim_{t\to+\infty} \sup_{|u|\le 1, y\in Y} |\varphi(t,u,y)| = 0. \tag{18}$$

Proof. Since the trivial motion $u = 0$ of the cocycle φ is uniformly attracting, then there exists a positive number γ such that

$$\lim_{t\to+\infty} \sup_{|u|\le \gamma, y\in Y} |\varphi(t,u,y)| = 0. \tag{19}$$

Since $\|\Lambda^r_{\mu^{-1}}\| \to 0$ as $\mu \to +\infty$, then there exists a positive number μ_0 such that

$$\||\Lambda^r_{\mu^{-1}}\| \le \gamma \tag{20}$$

for any $\mu \ge \mu_0$ and, consequently,

$$|\Lambda^r_{\mu_0^{-1}}u| \le \|\Lambda^r_{\mu_0^{-1}}\||u| \le \gamma \tag{21}$$

for any $|u| \le 1$. From (7) we have

$$|\varphi(t,u,y)| \le \|\Lambda^r_{\mu_0}\||\varphi(t,\Lambda^r_{\mu_0^{-1}}u,y)| \tag{22}$$

and taking into account (19)-(22) we obtain (18). Lemma is proved. □

Corollary 3. *Assume that the trivial motion $u = 0$ of the cocycle φ is uniformly attracting, then*

$$\lim_{t\to+\infty} \sup_{|u|\le R, y\in Y} |\varphi(t,u,y)| = 0 \tag{23}$$

for any $R > 0$.

Proof. Let R be an arbitrary (fixed) positive number. Since $\|\Lambda^r_{\mu^{-1}}\| \to 0$ as $\mu \to +\infty$, then there exists a positive number μ_0 such that

$$\||\Lambda^r_{\mu^{-1}}\| \le R^{-1} \tag{24}$$

for any $\mu \geq \mu_0$ and, consequently,

$$|\Lambda^r_{\mu_0^{-1}} u| \leq \|\Lambda^r_{\mu_0^{-1}}\||u| \leq R^{-1}R = 1 \tag{25}$$

for any $|u| \leq R$. Taking into account (23)-(25) we obtain

$$|\varphi(t,u,y)| = |\varphi(t,\Lambda^r_{\mu_0}\Lambda^r_{\mu_0^{-1}}u,y)| = |\Lambda^r_{\mu_0}\varphi(t,\Lambda^r_{\mu_0^{-1}}u,y)| \leq$$
$$\|\Lambda^r_{\mu_0}\||\varphi(t,\Lambda^r_{\mu_0^{-1}}u,y)| \leq R^{-1} \sup_{|v|\leq 1, y\in Y} |\varphi(t,v,y)| \to 0$$

as $t \to +\infty$ uniformly with respect to $|u| \leq R$ and $y \in Y$. □

Corollary 4. *Under the conditions of Lemma 7 we have*

$$\lim_{t\to+\infty} \sup_{\rho(u)\leq 1, y\in Y} \rho(\varphi(t,u,y)) = 0. \tag{26}$$

Proof. Let $u \in \mathbb{R}^n$ with $\rho(u) \leq 1$, then $|u| \leq b(1)$. Since

$$a(\rho(\varphi(t,u,y)) \leq |\varphi(t,u,y)| \leq \sup_{|u|\leq b(1), y\in Y} |\varphi(t,u,y)| := \eta(t), \tag{27}$$

and by Corollary 3

$$\lim_{t\to+\infty} \eta(t) = 0. \tag{28}$$

From (27) we obtain

$$\sup_{\rho(u)\leq 1, y\in Y} \rho(\varphi(t,u,y)) \leq a^{-1}(\eta(t)) \tag{29}$$

for any $t \in \mathbb{T}_+$. Passing to the limit in (29) and taking into account (28) we obtain (26). □

Theorem 3. *Let φ be an r-homogeneous cocycle over dynamical system $(Y, \mathbb{T}, \sigma)$ with the fibre. The following statements are equivalent:*

1. *the trivial motion $u = 0$ of the cocycle φ is uniformly asymptotically stable;*
2. *there are positive numbers $\mathcal{N}$ and ν such that*

$$\rho(\varphi(t,u,y)) \leq \mathcal{N}e^{-\nu t}\rho(u) \tag{30}$$

for any $(t,u,y) \in \mathbb{T}_+ \times \mathbb{R}^n \times Y$.

Proof. It is evident that 2. implies 1.

Now we will establish that 1. implies 2. Indeed, denote by

$$m(t) := \sup_{\rho(u)\leq 1, y\in Y} \rho(\varphi(t,u,y)) \tag{31}$$

for every $t \in \mathbb{T}_+$. By (31) the mapping $m : \mathbb{T}_+ \to \mathbb{R}_+$ is well defined possessing the following properties:

a. $0 \le m(t) \le M$ for any $t \in \mathbb{T}_+$, where $M := a^{-1}(M(b(1)))$ from Corollary 2;

b. $m(t) \to 0$ as $t \to +\infty$;

c. $m(t+\tau) \le m(t)m(\tau)$ for any $t, \tau \in \mathbb{T}_+$.

The statement a. (respectively, statement b.) follows from Corollary 2 (respectively, Corollary 4). To prove the statement c. we note that

$$\begin{aligned} m(t+\tau) = \sup_{\rho(u)\le 1, y\in Y} \rho(\varphi(t+\tau,u,y) = \\ \sup_{\rho(u)\le 1, y\in Y} \rho(\varphi(t,\varphi(\tau,u,y),\sigma(\tau,y))) = \\ \sup_{\rho(u)\le 1, y\in Y} \rho(\varphi(t,\Lambda^r_\mu \Lambda^r_{\mu^{-1}}\varphi(\tau,u,y),\sigma(\tau,y))) = \\ \sup_{\rho(u)\le 1, y\in Y} \rho(\Lambda^r_\mu \varphi(t,\Lambda^r_{\mu^{-1}}\varphi(\tau,u,y),\sigma(\tau,y))), \end{aligned} \tag{32}$$

where

$$\mu := \rho(\varphi(\tau,u,y)). \tag{33}$$

By the equality (5) we have

$$\begin{aligned} \sup_{\rho(u)\le 1, y\in Y} \rho(\Lambda^r_\mu \varphi(t,\Lambda^r_{\mu^{-1}}\varphi(\tau,u,y),\sigma(\tau,y))) = \\ \mu\rho(\varphi(t,\Lambda^r_{\mu^{-1}}\varphi(\tau,u,y),\sigma(\tau,y))). \end{aligned} \tag{34}$$

Note that

$$\rho(\Lambda^r_{\mu^{-1}}\varphi(\tau,u,y)) = \mu^{-1}\rho(\varphi(\tau,u,y) = 1 \tag{35}$$

and, consequently,

$$\rho(\varphi(t,\Lambda^r_{\mu^{-1}}\varphi(\tau,u,y),\sigma(\tau,y))) \le \sup_{\rho(v)\le 1, q\in Y} \rho(\varphi(t,v,q)) = m(t). \tag{36}$$

From (32)-(36) we obtain

$$m(t+\tau) \le m(\tau)m(t)$$

for any $t, \tau \in \mathbb{T}_+$.

According to Lemma 6 (item (ii)) we have

$$\rho(\varphi(t,u,y)) = \rho(u)\rho(\varphi(t,\Lambda^r_{\rho(u)^{-1}}u,y)) \le m(t)\rho(u)$$

for any $u \in \mathbb{R}^n \setminus \{0\}$ and $(t,y) \in \mathbb{T}_+ \times Y$ because $\rho(\Lambda^r_{\rho(u)^{-1}}u) = 1$ and, consequently,

$$\rho(\varphi(t,\Lambda^r_{\rho(u)^{-1}}u,y)) \le \sup_{\rho(v)\le 1, y\in Y} \rho(\varphi(t,v,y)) = m(t). \tag{37}$$

By Lemma 3 there are positive numbers $\mathcal{N}$ and ν such that $m(t) \le \mathcal{N}e^{-\nu t}$ for any $t \in \mathbb{T}_+$, and taking into account (37) we obtain (30). Theorem is proved. $\square$

4 Asymptotic Stability of Nonautonomous Generalized Homogeneous Dynamical Systems: The Case of the Compact Phase Space of Driving System

Let $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a cocycle over $(Y, \mathbb{T}, \sigma)$ with the fibre $\mathbb{R}^n$ and Y be a compact metric space. Assume that the cocycle φ admits the trivial motion 0, i.e., $\varphi(t, 0, y) = 0$ for any $(t, y) \in \mathbb{T}_+ \times Y$.

Remark 4. Let $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be r homogeneous of order m, then φ admits the trivial motion.

Denote by

$$W^s_y(0) := \{u \in \mathbb{R}^n | \lim_{t\to+\infty} |\varphi(t, u, y)| = 0\}.$$

Definition 12. A trivial motion 0 of the cocycle φ is said to be:

1. uniformly stable if for arbitrary $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|u| < \delta$ implies $|\varphi(t, u, y)| < \varepsilon$ for any $t \in \mathbb{T}_+$ and $y \in Y$;
2. attracting if there exists $\gamma > 0$ such that $\lim_{t\to+\infty} |\varphi(t, u, y)| = 0$ for any $|u| < \gamma$ and $y \in Y$;
3. asymptotically stable if it is uniformly stable and attracting;
4. globally asymptotically stable if it is asymptotically stable and $W^s_y(0) = \mathbb{R}^n$ for any $y \in Y$.

Let $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a cocycle over $(Y, \mathbb{T}, \sigma)$ with the fiber $\mathbb{R}^n$ and $\varphi(t, 0, y) = 0$ for any $(t, y) \in \mathbb{T}_+ \times Y$.

Lemma 8. *Let $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be an $r \in (0, +\infty)^n$ homogeneous (of the degree zero) cocycle over $(Y, \mathbb{T}, \sigma)$ with the fiber $\mathbb{R}^n$. Assume that $W^s_y(0)$ is neighborhood of 0, then $W^s_y(0) = \mathbb{R}^n$.*

Proof. Let $u \in \mathbb{R}^n$ be an arbitrary point. Under the condition of Lemma there exists a positive number δ_y such that $B(0, \delta_y) \subseteq W^s_y(0)$, where $B(0, \delta) := \{u \in \mathbb{R}^n | \, |u| < \delta\}$. Since the cocycle φ is r homogeneous of the degree zero, then there exists a positive number $\mu_0 < 1$ such that

$$\Lambda^r_\mu u \in B(0, \delta_y) \tag{38}$$

for any $0 < \mu < \mu_0$. Note that

$$\varphi(t, u, y) = \varphi(t, \Lambda^r_{\mu^{-1}} \Lambda^r_\mu u, y)) = \Lambda^r_{\mu^{-1}} \varphi(t, \Lambda^r_\mu u, y). \tag{39}$$

From (38)-(39) we obtain $u \in W^s_y(0)$, that is, $\mathbb{R}^n = W^s_y(0)$. Lemma is proved. ∎

Theorem 4. *Let $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a cocycle over $(Y, \mathbb{T}, \sigma)$ with the fibre $\mathbb{R}^n$ and $r \in (0, +\infty)^n$. Assume that the cocycle φ is r homogeneous of the degree zero and Y is compact. Then the following conditions are equivalent:*

1. *the trivial motion* $u = 0$ *of the cocycle* φ *is attracting;*
2. *the skew-product dynamical system* $(X, \mathbb{T}_+, \sigma)$ *generated by* φ *is pointwise dissipative.*

Proof. To prove this statement it is sufficient to show that (i) implies (ii). Let $x = (u, y) \in X = E \times Y$ be an arbitrary point. By Lemma 8 we have $W^s_y(0) = \mathbb{R}^n$ and, consequently $u \in W^s_y(0)$, i.e.,

$$\lim_{t\to+\infty} |\varphi(t, u, y)| = 0.$$

Since the space Y is compact, then the motion $\pi(t, x)$ ($x = (u, y)$ and $\pi(t, x) = (\varphi(t, u, y), \sigma(t, y))$) is positively Lagrange stable and $\emptyset \neq \omega_x \subseteq \Theta := \{0\} \times Y$. Thus $\Omega_X \subseteq \Theta$ and, consequently, the dynamical system $(X, \mathbb{T}, \sigma)$ is pointwise dissipative. Theorem is proved. □

Theorem 5. *Let* φ *be an* r *homogeneous cocycle over* $(Y, \mathbb{T}, \sigma)$ *of the degree zero and* Y *be a compact metric space. Then the following statements are equivalent:*

1. *the trivial motion of* φ *is attracting;*
2. *the skew-product dynamical system* $(X, \mathbb{T}_+, \pi)$ *generated by cocycle* φ *(*$X := \mathbb{R}^n \times Y$ *and* $\pi = (\varphi, \sigma)$*) and its Levinson center* $J \subseteq \Theta := \{0\} \times Y$*.*

Proof. To prove this statement it is sufficient to show that 1. implies 2. Indeed, by Lemma 8 we have $W^s_y(0) = \mathbb{R}^n \times Y$ for any $y \in Y$. Since the space Y is compact, then the skew-product dynamical system $(X, \mathbb{T}_+, \pi)$ ($X = \mathbb{R}^n \times Y$ and $\pi = (\varphi, \sigma)$) is pointwise dissipative. Since the phase space $X = \mathbb{R}^n \times Y$ is locally compact, then by Theorem 1 the dynamical system $(X, \mathbb{T}_+, \pi)$ is compactly dissipative. Denote by J its Levinson center. Since J is a compact subset of X, then there exists a positive number γ_0 such that $J \subseteq B[0, \gamma_0] \times Y$, where $B[0, \gamma_0] := \{u |\ |u| \leq \gamma_0\}$. Now we will show that $J \subseteq \Theta$. If we suppose that it is not true, then there exists a point $x_0 = (u_0, y_0) \in J \setminus \Theta$. This means that $u_0 \neq 0$ and through the point x_0 passes a full trajectory $\{\pi(t, x_0) = (\varphi(t, u_0, y_0), \sigma(t, y_0)|\ t \in \mathbb{S}\}$ which belongs to J. Since the cocycle φ is r-homogeneous of the degree zero, then

$$\varphi(t, \Lambda^r_\mu u_0, y_0) = \Lambda^r_\mu \varphi(t, u_0, y_0) \tag{40}$$

for any $t \in \mathbb{S}$. From (40) it follows that the full trajectory $\{(\varphi(t, \Lambda^r_\mu u_0, y_0), \sigma(t, y_0)|\ t \in \mathbb{S}\}$ is precompact and, consequently,

$$(\Lambda^r_\mu u_0, y_0) \in J$$

for any $\varepsilon \in (0, +\infty)$. Note that

$$|\Lambda^r_\mu u_0| \geq \mu^\nu |u_0| \tag{41}$$

for any $\mu > 0$, where $\nu = \min\{r_1, \ldots, r_n\} > 0$. Passing to the limit in (41) as $\mu \to +\infty$ we conclude that the set J is not compact. This contradicts the fact that the Levinson center is the maximal compact invariant set of $(X, \mathbb{T}_+, \pi)$. The obtained contradiction proves our statement. Theorem is completely proved. □

Theorem 6. *Let φ be an r homogeneous cocycle over $(Y,\mathbb{T},\sigma)$ of the degree zero and Y be a compact metric space. Then the trivial motion $u=0$ of the cocycle φ is asymptotically stable if and only if it is uniformly asymptotically stable.*

Proof. To prove this statement it is sufficient to show that the asymptotic stability of the trivial motion $u=0$ of φ implies its uniformly asymptotic stability. Assume that the trivial motion $u=0$ of the cocycle φ is asymptotically stable. Then by Theorem 5 the skew-product dynamical system $(X,\mathbb{T}_{+},\pi)$ generated by cocycle φ ($X:=\mathbb{R}^n\times Y$, $\pi=(\varphi,\sigma)$) and its Levinson center $J\subseteq\Theta:=\{0\}\times Y$. Let γ be an arbitrary positive number, then

$$\lim_{t\to+\infty}\sup_{|u|\le\gamma,y\in Y}|\varphi(t,u,y)|=0.$$

Suppose that it is not true, then there exist positive numbers ε_0, γ_0 and sequences $\{u_k\}$ (with $|u_k|\le\gamma_0$ for any $k\in\mathbb{N}$), $\{y_k\}\subset Y$ and $t_k\ge k$ such that

$$|\varphi(t_k,u_k,y_k)|\ge\varepsilon_0 \tag{42}$$

for any $k\in\mathbb{N}$. Since the set $K_0:=B[0,\gamma_0]\times Y$ is compact and the skew-product dynamical system $(X,\mathbb{T}_{+},\pi)$ is compactly dissipative, then without loss of generality we may assume that the sequences $\{u_k\}$, $\{y_k\}$, $\{\sigma(t_k,y_k)\}$ and $\{\varphi(t_k,u_k,y_k)\}$ are convergent. Denote by $\bar{y}=\lim\limits_{k\to\infty}\sigma(t_k,y_k)$ and

$$\bar{u}=\lim_{k\to\infty}\varphi(t_k,u_k,y_k). \tag{43}$$

It is clear $\pi(t_k,(u_k,y_k))=(\varphi(t_k,u_k,y_k),\sigma(t_k,y_k))\in\Sigma^{+}_{K_0}:=\bigcup\{\pi(t,K_0)|\ t\ge0\}$ and $(\bar{u},\bar{y})\in\omega(K_0)\subseteq J\subseteq\Theta:=\{0\}\times Y$. This means, in particular, that

$$|\bar{u}|=0. \tag{44}$$

On the other hand passing to the limit in (42) as $k\to\infty$ and taking into account (43) we obtain

$$|\bar{u}|\ge\varepsilon_0>0$$

which contradicts (44). The obtained contradiction proves our statement. Theorem is completely proved. □

Theorem 7. *Let φ be an r homogeneous cocycle over $(Y,\mathbb{T},\sigma)$ of the degree zero and Y be a compact metric space.*

Then the trivial motion $u=0$ of the cocycle φ is asymptotically stable if and only if it is attracting.

Proof. To prove this statement it is sufficient to show that under the conditions of Theorem if the trivial motion $u=0$ of the cocycle φ is attracting, then it is asymptotically stable. If we suppose that it is not true, then there are $\varepsilon_0>0$, $\delta_k\to0$ ($\delta_k>0$) and $t_k\to+\infty$ as $k\to\infty$, $u_k\in\mathbb{R}^n$ and $y_k\in Y$ such that

$$|u_k|\le\delta_k\ \text{ and }\ |\varphi(t_k,u_k,y_k)|\ge\varepsilon_0. \tag{45}$$

Reasoning as in the proof of Theorem 6 we can suppose that the sequence $\{\varphi(t_k,u_k,y_k)\}$ converges. Denote its limit by $\bar{u}=\lim\limits_{k\to\infty}\varphi(t_k,u_k,y_k)$. Passing to the limit in (45) as $k\to\infty$ we obtain $\bar{u}\neq 0$. On the other hand $(\bar{u},\bar{y})\in J\subseteq\Theta=\{0\}\times Y$ (see the proof of Theorem 6) and, consequently, $\bar{u}=0$. The obtained contradiction completes the proof of Theorem. □

Corollary 5. *Let $r\in(0,+\infty)^n$ and φ be an r homogeneous cocycle over $(Y,\mathbb{T},\sigma)$ with the fibre $\mathbb{R}^n$. If the space is compact, then the following statements are equivalent:*

1. *the trivial motion $u=0$ of the cocycle φ is asymptotically stable;*
2. *the skew-product dynamical system $(X,\mathbb{T}_+,\pi)$ generated by φ is pointwise dissipative.*

Proof. This statement follows from Theorems 4 and 7. □

Lemma 9. *Let $\langle\mathbb{R}^n,\varphi,(Y,\mathbb{T},\sigma)\rangle$ be a cocycle over $(Y,\mathbb{T},\sigma)$ with the fibre $\mathbb{R}^n$, then the following statements hold:*

1. *the trivial motion $u=0$ of the cocycle φ is positively uniformly stable if and only if for any $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that $\rho(u)<\delta$ implies $\rho(\varphi(t,u,y))<\varepsilon$ for any $(t,u)\in\mathbb{T}_+\times Y$;*
2. $\lim\limits_{t\to+\infty}|\varphi(t,u,y)|=0$ *if and only if* $\lim\limits_{t\to+\infty}\rho(\varphi(t,u,y))=0$.

Proof. Assume that the trivial motion of the cocycle φ is positively uniformly stable, then for arbitrary $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that $\rho(u)<\delta$ implies $\rho(\varphi(t,u,y))<\varepsilon$ for any $(t,u)\in\mathbb{T}_+\times Y$. If we suppose that it is not true, then there exist $\varepsilon_0>0$, $\delta_k\to 0$ $(\delta_k>0)$, $\rho(u_k)<\delta_k$ $(u_k\in\mathbb{R}^n)$, $(t_k,y_k)\in\mathbb{T}_+\times Y$ such that

$$\rho(\varphi(t_k,u_k,y_k))\geq\varepsilon_0 \tag{46}$$

for any $k\in\mathbb{N}$. Let $a,b\in\mathcal{K}_\infty$ be the functions figuring in (3), then from (3) and (46) we obtain

$$0<a(\varepsilon_0)\leq a(\rho(\varphi(t_k,u_k,y_k)))\leq|\varphi(t_k,u_k,y_k)|. \tag{47}$$

On the other hand by positively uniform stability of trivial motion for φ we can choose a positive number $\delta(\varepsilon_0)$ such that

$$|\varphi(t,u,y)|<a(\varepsilon_0)$$

for any $|u|<\delta(\varepsilon_0)$ and $(t,y)\in\mathbb{T}_+\times Y$. Note that $|u_k|\leq b(\rho(u_k))<b(\delta_k)\to 0$ as $k\to\infty$ and, consequently, there exists a number $k_0\in\mathbb{N}$ such that $|u_k|<\delta(\varepsilon_0)$ for any $k\geq k_0$. Thus we have

$$|\varphi(t,u_k,y)|<a(\varepsilon_0) \tag{48}$$

for any $k \geq k_0$ and $(t, y) \in \mathbb{T}_+ \times Y$. In particular, from (48) we receive

$$|\varphi(t_k, u_k, y_k)| < a(\varepsilon_0) \tag{49}$$

for any $k \geq k_0$. The inequalities (47) and (49) are contradictory. The obtained contradiction proves our statement. The converse statement can be proved using absolutely the same arguments as above.

Let $(u, y) \in \mathbb{R}^n \times Y$ be so that

$$\lim_{t \to +\infty} |\varphi(t, u, y)| = 0. \tag{50}$$

Since $a(\rho(\varphi(t, u, y))) \leq |\varphi(t, u, y)|$, then

$$\rho(\varphi(t, u, y)) \leq a^{-1}(|\varphi(t, u, y)|) \tag{51}$$

for any $(t, u, y) \in \mathbb{T}_+ \times \mathbb{R}^n \times Y$. Passing to the limit in (51) as $t \to +\infty$ and taking into account (50) we obtain $\lim_{t \to +\infty} \rho(\varphi(t, u, y)) = 0$. Then we have $|\varphi(t, u, y)| \leq b(\rho(\varphi(t, u, y)))$ and, consequently, $\lim_{t \to +\infty} |\varphi(t, u, y)| = 0$. Lemma is completely proved. □

Theorem 8. *Assume that the following conditions are fulfilled:*

1. *the cocycle $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{T}, \sigma) \rangle$ is r-homogeneous of the degree zero;*
2. *the space Y is compact.*

Then the following statements are equivalent:

a. *the trivial motion of the cocycle φ is asymptotically stable;*

b. *there exit positive numbers $\mathcal{N}$ and ν such that*

$$\rho(\varphi(t, u, y)) \leq \mathcal{N} e^{-\nu t} \rho(u)$$

for any $u \in \mathbb{R}^n$, $y \in Y$ and $t \geq 0$.

Proof. To prove the theorem it is sufficient to establish the implication $a. \Rightarrow b.$, since the converse statement is obvious.

Since the cocycle φ is r homogeneous of the degree zero and the trivial motion $u = 0$ is attracting, then from Lemmas 9 and 6 we have $W^s_y(0) = \mathbb{R}^n$ for any $y \in Y$. Consider the skew-product dynamical system $(X, \mathbb{T}_+, \pi)$ generated by the cocycle φ ($X := \mathbb{R}^n \times Y$ and $\pi := (\varphi, \sigma)$). Taking into account that Y is a compact space and $W^s_y(0) = \mathbb{R}^n$ (for any $y \in Y$) according to Theorem 5 we conclude that the dynamical system $(X, \mathbb{T}_+, \pi)$ is compactly dissipative and its Levinson center $J \subseteq \Theta := \{0\} \times Y$. This means that for any compact subset $K \subset X = \mathbb{R}^n \times Y$ the following statements hold:

1.
$$M(K) := \sup_{(t,u,y)\in \mathbb{T}_+\times K} |\varphi(t,u,y)| < +\infty;$$

2.
$$m_K(t) := \sup_{(u,y)\in K} |\varphi(t,u,y)| \to 0$$

as $t\to +\infty$.

Note that $S_{r,p}\times Y$ is a compact subset of $X=\mathbb{R}^n\times Y$, because $S_{r,p}$ is a compact subset of $\mathbb{R}^n$. Denote by

$$m(t) := \sup_{(u,y)\in S_{r,p}\times Y} \rho(\varphi(t,u,y)) \tag{52}$$

and

$$M := \sup_{(t,u,y)\in \mathbb{T}_+\times S_{r,p}\times Y} \rho(\varphi(t,u,y)). \tag{53}$$

Let a,b be the functions from $\mathcal{K}_\infty$ figuring in (3), then we obtain

$$\rho(\varphi(t,u,y)) \le a^{-1}(|\varphi(t,u,y)|) \le a^{-1}(M(S_{r,p})) \tag{54}$$

and

$$\rho(\varphi(t,u,y)) \le a^{-1}(|\varphi(t,u,y)|) \le a^{-1}(m_{S_{r,p}}(t)) \tag{55}$$

for any $t\in\mathbb{T}_+$, $u\in S_{r,p}$ and $y\in Y$. From (52)-(55) we have the following statements:

1. $0<m(t)\le M$ for any $t\in\mathbb{T}_+$;
2. $m(t)\to 0$ as $t\to+\infty$.

From Lemma 6 (item (ii)) we obtain

$$\rho(\varphi(t,u,y)) \le m(t)\rho(u)$$

for any $t\in\mathbb{T}_+$ and $u\ne 0$, where

$$m(t) := \sup\{\rho(\varphi(t,u,y))|\ (u,y)\in S_{r,p}\times Y\}.$$

Indeed, $\Lambda^r_{\rho(u)^{-1}}u\in S_{r,p}$ for any $u\ne 0$ and, consequently,

$$\rho(\varphi(t,\Lambda^r_{\rho(u)^{-1}}u,y)) \le \sup_{(v,y)\in S_{r,p}\times Y} \rho(\varphi(t,v,y)) = m(t) \tag{56}$$

for any $u\ne 0$ and $(t,y)\in\mathbb{T}_+\times Y$. In particular from (56) we obtain

$$\rho(\varphi(t,\Lambda^r_{\rho(\varphi(\tau,u,y))^{-1}}\varphi(\tau,u,y),\sigma(\tau,y))) \le \sup_{(\tilde u,\tilde y)\in S_{r,p}\times Y} \rho(\varphi(t,\tilde u,\tilde y) = m(t)$$

for any $t,\tau\in\mathbb{T}_+$ and $(u,y)\in S_{r,p}\times Y$.

Finally, by the equality (13) we have

$$m(t+\tau) = \sup_{(u,y)\in S_{r,p}\times Y} \rho(\varphi(t+\tau,u,y)) =$$
$$\sup_{(u,y)\in S_{r,p}\times Y} \rho(\varphi(\tau,u,y))\rho(\varphi(t,\Lambda^r_{\mu^{-1}}\varphi(\tau,u,y),\sigma(\tau,y))) \le$$
$$\sup_{(u,y)\in S_{r,p}\times Y} \rho(\varphi(\tau,u,y)) \times \sup_{(u,y)\in S_{r,p}\times Y} \rho(\varphi(t,\Lambda^r_{\mu^{-1}}\varphi(\tau,u,y),\sigma(\tau,y))) \le m(\tau)m(t)$$

because

$$\Lambda^r_{\mu^{-1}}\varphi(\tau,u,y) \in S_{r,p}$$

if $\mu = \rho(\varphi(\tau,u,y))$ and

$$\sup_{(u,y)\in S_{r,p}\times Y} \rho(\varphi(t,\Lambda^r_{\mu^{-1}}\varphi(\tau,u,y),\sigma(\tau,y))) \le \sup_{(\tilde{u},\tilde{y})\in S_{r,p}\times Y} \rho(\varphi(t,\tilde{u},\tilde{y})) = m(t).$$

By Lemma 3 there exist positive numbers $\mathcal{N}$ and ν such that $m(t) \le \mathcal{N}e^{-\nu t}$ for any $t \in \mathbb{T}_+$. $\square$

5 Asymptotic Stability of Nonautonomous Generalized Homogeneous Dynamical Systems: The Case of the Compact and Minimal Phase Space of Driving System

In this Section we suppose that the complete metric space Y is compact and the dynamical system $(Y,\mathbb{T},\sigma)$ is minimal, i.e., every trajectory $\Sigma_y := \{\sigma(t,y):\ t\in\mathbb{T}\}$ is dense in Y (this means that $H(y) = Y$ for all $y \in Y$, where $H(y) := \overline{\Sigma}_y$).

Theorem 9. *[6, Ch.II, pp.94-95] Let $\langle \mathbb{R}^n,\varphi,(Y,\mathbb{S},\sigma)\rangle$ be a cocycle over two-sided dynamical system $(Y,\mathbb{S},\sigma)$ with the fibre $\mathbb{R}^n$. Assume that the following conditions are fulfilled:*

1. *the trivial motion $u = 0$ of the cocycle φ is uniformly stable;*
2. *there exist positive number δ_0 and point $y_0 \in Y$ such that $B(0,\delta_0) \subset W^s_{y_0}$, where $B(0,r) := \{u \in \mathbb{R}^n |\ |u| < r\}$.*

Then the trivial motion $u = 0$ of the cocycle φ is asymptotically stable, i.e., there exists a positive number β such that $B(0,\beta) \subset W^s_y(0)$ for any $y \in Y$.

Theorem 10. *Let $\langle \mathbb{R}^n,\varphi,(Y,\mathbb{S},\sigma)\rangle$ be a cocycle over two-sided dynamical system $(Y,\mathbb{S},\sigma)$ with the fibre $\mathbb{R}^n$. Assume that the following conditions are fulfilled:*

1. *the cocycle $\langle \mathbb{R}^n,\varphi,(Y,\mathbb{T},\sigma)\rangle$ is r-homogeneous of the degree zero;*
2. *the trivial motion $u = 0$ of the cocycle φ is stable;*
3. *there exit a point $y_0 \in Y$ and positive number δ_{y_0} such that $B(0,\delta_{y_0}) \subset W^s_{y_0}(0)$.*

Then the trivial motion $u = 0$ of the cocycle φ is globally uniformly asymptotically stable, i.e., $W^s_y(0) = \mathbb{R}^n$ for any $y \in Y$.

Proof. By Theorem 9 there exists a positive number δ_0 such that $B(0, \delta_0) \subset W^s_y(0)$ for any $y \in Y$. According to Lemma 8 we have $W^s_y(0) = \mathbb{R}^n$ for any $y \in Y$. Theorem is proved. □

Theorem 11. *Let $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be an r-homogeneous cocycle of the degree zero over two-sided dynamical system $(Y, \mathbb{S}, \sigma)$.*

Then the following statements are equivalent:

1. *the trivial motion $u = 0$ of the cocycle φ is uniformly stable and there exists a point $y_0 \in Y$ and positive number δ_{y_0} such that $B(0, \delta_{y_0}) \subset W^s_{y_0}(0)$;*
2. *there exist positive numbers $\mathcal{N}$ and ν such that $\rho(\varphi(t, u, y)) \le \mathcal{N} e^{-\nu t} \rho(u)$ for any $u \in \mathbb{R}^n, y \in Y$ and $t \ge 0$.*

Proof. According to Theorem 10 under the conditions of Theorem 11 the trivial motion $u = 0$ of the cocycle φ is (globally) uniformly asymptotically stable. To finish tha proof of Theorem it is sufficient to Apply Theorem 8. □

6 Applications

6.1 Ordinary Differential Equations

Let $\mathbb{R}^n$ be n-dimensional real or complex Euclidean space. Let us consider a differential equation

$$u' = f(t, u), \tag{57}$$

where $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$. Along with the equation (57) we consider its *H-class* [4, 15, 21, 22], i.e., the family of the equations

$$v' = g(t, v), \tag{58}$$

where $g \in H(f) := \overline{\{f^\tau |\ \tau \in \mathbb{R}\}}$, $f^\tau(t, u) = f(t + \tau, u)$ for any $(t, u) \in \mathbb{R} \times \mathbb{R}^n$ and by bar we denote the closure in $C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$. We will suppose also that the function f is *regular* [20, ChIV], i.e., for every equation (58) the conditions of existence, uniqueness (on the maximal interval of definition of the solutions) and extendability on $\mathbb{R}_+$ are fulfilled. Denote by $\varphi(t, v, g)$ the solution of equation (58), passing through the point $v \in \mathbb{R}^n$ at the initial moment $t = 0$. Then from the general properties of solutions of *ordinary differential equations (ODEs)* it follows that the mapping $\varphi : \mathbb{R}_+ \times \mathbb{R}^n \times H(f) \to \mathbb{R}^n$ is well defined and it satisfies the following conditions (see for example [4, ChIV] and [20, ChIV]):

1) $\varphi(0, v, g) = v$ for any $v \in \mathbb{R}^n$ and $g \in H(f)$;

2) $\varphi(t, \varphi(\tau, v, g), g^\tau) = \varphi(t + \tau, v, g)$ for every $v \in \mathbb{R}^n$, $g \in H(f)$ and $t, \tau \in \mathbb{R}_+$;

3) the mapping $\varphi : \mathbb{R}_+ \times \mathbb{R}^n \times H(f) \to \mathbb{R}^n$ is continuous.

Denote by $Y := H(f)$ and $(Y, \mathbb{R}, \sigma)$ the dynamical system of translations on Y, induced by the dynamical system of translations $(C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n), \mathbb{R}, \sigma)$. The triplet $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{R}, \sigma) \rangle$ is a cocycle over $(Y, \mathbb{R}_+, \sigma)$ with the fibre $\mathbb{R}^n$. Thus the equation (57) generates a cocycle $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{R}, \sigma) \rangle$ satisfying Condition (C).

Note that under the conditions listed above the equation (57) (respectively, H-class (58)) can be written in the form (2). Indeed, let $Y := H(f)$ and $(Y, \mathbb{R}, \sigma)$ be the dynamical system of translations on Y. Denote by F the mapping from $Y \times \mathbb{R}^n$ into $\mathbb{R}^n$ defined by the equality

$$F(g, u) := g(0, u). \tag{59}$$

It is not difficult to check that the mapping $F : H(f) \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous. Finally, note that we can rewrite the equation (58) as follows

$$u' = F(\sigma(t, g), u) \quad (g \in H(f)). \tag{60}$$

Definition 13. A function $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is said to be r homogeneous ($r \in (0, +\infty)^n$) of degree $m \in \mathbb{R}$ if $f(t, \Lambda_\varepsilon^r u) = \lambda^m \Lambda_\varepsilon^r f(t, u)$ for any $(\varepsilon, t, u) \in (0, +\infty) \times \mathbb{R} \times \mathbb{R}^n$.

Remark 5. If the function $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is r homogeneous of a degree $m \geq 0$, then $f(t, 0) = 0$ for any $t \in \mathbb{R}$.

Lemma 10. *If the function $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is r homogeneous of a degree m, then the mapping $F : Y \times \mathbb{R}^n \to \mathbb{R}^n$ ($Y = H(f)$) defined by the equality (59) is r homogeneous of a degree m with respect to $u \in \mathbb{R}^n$ uniformly in $y \in Y$.*

Proof. Let $g \in H(f)$, then there exists a sequence $\{t_k\} \subset \mathbb{R}$ such that

$$g(t, u) = \lim_{k \to \infty} f(t + t_k, u)$$

uniformly with respect to (t, u) on every compact subset from $\mathbb{R} \times \mathbb{R}^n$. Notice that

$$F(g, \Lambda_\varepsilon^r u) = \lim_{k \to \infty} f(t + t_k, \Lambda_\varepsilon^r u) = \lambda^m \Lambda_\varepsilon^r \lim_{k \to \infty} f(t + t_k, u) = \lambda^m \Lambda_\varepsilon^r F(g, u)$$

for any $(\varepsilon, g, u) \in (0, +\infty) \times H(f) \times \mathbb{R}^n$. Lemma is proved. □

Corollary 6. *Assume that the function $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is r homogeneous of the degree zero, then the cocycle $\langle \mathbb{R}^n, \varphi, (H(f), \mathbb{R}, \sigma) \rangle$ generated by the equation (57) is r homogeneous of the degree zero.*

Proof. This statement follows from Lemmas 2 and 10. □

Let $f(t, 0) \equiv 0$ and the function $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ be regular.

Definition 14. The trivial solution of the equation (57) is said to be:

1. *uniformly stable* if for any positive number ε there exists a number $\delta = \delta(\varepsilon)$ ($\delta \in (0, \varepsilon)$) such that $|x| < \delta$ implies $|\varphi(t, x, f^\tau)| < \varepsilon$ for any $t \in \mathbb{R}_+$ and $\tau \in \mathbb{R}$;

2. *attracting (respectively, uniformly attracting)* if there exists a positive number a such that

$$\lim_{t\to+\infty} |\varphi(t,x,f^{\tau})| = 0$$

for any $|x| \le a$ and $\tau \in \mathbb{R}$ (respectively, uniformly with respect to $|x| \le a$ and $t \in \mathbb{R}$);

3. *asymptotically stable (respectively, uniformly asymptotically stable*, if it is uniformly stable and attracting (respectively, uniformly attracting).

Remark 6. If the function $f \in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$ is regular and $f(t,0) = 0$ for any $t \in \mathbb{R}$, then it is easy to show that the trivial solution of equation (57) is uniformly attracting if and only if there exists a positive number a such that

$$\lim_{t\to+\infty} \sup_{|x|\le a,\ g\in H(f)} |\varphi(t,u,g)| = 0. \tag{61}$$

Remark 7. 1. Note that from the results given in the works [1, 19] it follows the equivalence of standard definition (see, for example,[12, Ch.V]) of the uniform stability (respectively, global uniform asymptotically stability) and of the one given above for the equation (57) with regular right hand side.

2. From the results of G. Sell [19, 20] it follows that for the differential equations (57) with the regular and Lagrange stable right hand side f the following statements are equivalent:

1. the trivial solution of equation (57) is uniformly asymptotically stable;
2. the trivial motion of the cocycle $\langle\mathbb{R}^n,\varphi,(H(f),\mathbb{R},\sigma)\rangle$ generated by (57) is uniformly asymptotically stable.

Theorem 12. *Let $f \in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$. Assume that the following conditions are fulfilled:*

1. *the function f is regular and $f(t,0) = 0$ for any $t \in \mathbb{R}$;*
2. *the function f is r homogeneous of the degree zero.*

Then the following statements are equivalent:

1. *the trivial solution of the equation (57) is uniformly asymptotically stable;*
2. *there exit positive numbers $\mathcal{N}$ and ν such that*

$$\rho(\varphi(t,u,g)) \le \mathcal{N}e^{-\nu t}\rho(u)$$

for any $u \in \mathbb{R}^n$, $g \in H(f)$ and $t \ge 0$, where $\rho(u) = |u|_{r,p}$.

Proof. Let $Y := H(f)$ and $(Y, \mathbb{R}, \sigma)$ be the shift dynamical system on $Y = H(f)$. Denote by $\langle \mathbb{R}^n, \varphi, (H(f), \mathbb{R}, \sigma) \rangle$ (shortly φ) the cocycle generated by the differential equation (57). Since the function f is r homogeneous of the degree zero, then by Corollary 6 the cocycle φ generated by the equation (57) is r homogeneous of the degree zero. To finish the proof of Theorem 12 it is sufficient to take into account Remarks 6–7 and apply Theorem 3. $\square$

Theorem 13. *Let $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ be a regular function. Assume that the following conditions are fulfilled:*

1. *$f(t, 0) = 0$ for any $t \in \mathbb{R}$;*
2. *the function f is r homogeneous of the degree zero and Lagrange stable.*

Then the following statements are equivalent:

1. *the trivial solution of the equation (57) is asymptotically stable;*
2. *there exit positive numbers $\mathcal{N}$ and ν such that*
$$\rho(\varphi(t, u, g)) \leq \mathcal{N} e^{-\nu t} \rho(u)$$
for any $u \in \mathbb{R}^n$, $g \in H(f)$ and $t \geq 0$.

Proof. Let $Y := H(f)$ and $(Y, \mathbb{R}, \sigma)$ be the shift dynamical system on $Y = H(f)$. Note that the space Y is compact because the function f is Lagrange stable. Since the function f is r homogeneous of the degree zero, then by Corollary 6 the cocycle $\langle \mathbb{R}^n, \varphi, (H(f), \mathbb{R}, \sigma) \rangle$ generated by the equation (57) is r homogeneous of the degree zero. To finish the proof of Theorem 13 it suffices to take into account Remark 7 and apply Theorem 8. $\square$

Remark 8. 1. If the function f is τ-periodic, then the equivalence of the conditions (i) and (ii) was established in the work [17].

2. If the function f is homogeneous of the degree zero (in the classical sense, i.e., $f(t, \varepsilon x) = \varepsilon f(t, x)$ for any $\varepsilon > 0$ and $(t, x) \in \mathbb{R} \times \mathbb{R}^n$), then the equivalence of the uniform asymptotically stability and exponential stability was established in the work [12, Ch.VII]. If the function f is r homogeneous of the degree zero the equivalence of the uniform asymptotic stability and exponential stability was established in the work [9]

Recall that the function $f \in C(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n)$ is said to be recurrent in time if the motion $\sigma(t, f)$ generated by f in the shift dynamical system $(C(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n), \mathbb{T}, \sigma)$ is recurrent.

Remark 9. Note that the function f is recurrent in time if and only if its hull $H(f)$ is a compact and minimal set of the shift dynamical system $(C(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n), \mathbb{T}, \sigma)$ (see for example [8, Ch.I]).

Theorem 14. *Let $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ be a regular function. Assume that the following conditions are fulfilled:*

1. *the function f is recurrent in time and $f(t,0)=0$ for any $t\in\mathbb{R}$;*
2. *the function f is r homogeneous of the degree zero.*

Then the following statements are equivalent:

1. *the trivial solution of equation (57) is uniformly stable and there exists a positive number a such that*
$$\lim_{t\to+\infty}|\varphi(t,u,f)|=0 \tag{62}$$
for any $u\in B[0,a]:=\{u\in\mathbb{R}^n|\ |u|\le a\}$;
2. *there exit positive numbers $\mathcal{N}$ and ν such that*
$$\rho(\varphi(t,u,g))\le\mathcal{N}e^{-\nu t}\rho(u)$$
for any $u\in\mathbb{R}^n$, $g\in H(f)$ and $t\ge 0$.

Proof. Let $Y:=H(f)$ and $(Y,\mathbb{R},\sigma)$ be the shift dynamical system on $Y=H(f)$. Note that the space Y is a compact and minimal set because the function f is recurrent in time (see Remark 9). Since the function f is r homogeneous of the degree zero, then by Corollary 6 the cocycle $\langle\mathbb{R}^n,\varphi,(H(f),\mathbb{R},\sigma)\rangle$ generated by the equation (57) is r homogeneous of the degree zero. To finish the proof of Theorem 14 it is sufficient to take into account Remark 7 and to apply Theorem 11. □

Here is an example illustrating the theorems proved in this subsection.

Example 2. Denote by $C(\mathbb{R},\mathbb{R})$ the space of all continuous functions $\psi:\mathbb{R}\to\mathbb{R}$ equipped with the compact-open topology and $(C(\mathbb{R},\mathbb{R}),\mathbb{R},\sigma)$ the shift dynamical system on $C(\mathbb{R},\mathbb{R})$. Consider the system of differential equations
$$\begin{cases}\dot{x}_1=-x_1+p(t)\sqrt{|x_2|}\\ \dot{x}_2=-x_2\end{cases}, \tag{63}$$
where $p\in C(\mathbb{R},\mathbb{R})$.

Note that the function $F\in C(\mathbb{R}\times\mathbb{R}^2,\mathbb{R}^2)$, where $F(t,x):=(-x_1+p(t)\sqrt{|x_2|},-x_2)$ and $x:=(x_1,x_2)$, is $r=(1,2)$ homogeneous. This means that $F(t,\Lambda_\mu x))=\Lambda_\mu F(t,x)$ for any $(t,\mu,x)\in\mathbb{R}\times(0,+\infty)\times\mathbb{R}^n$, where $\Lambda_\mu x=(\mu x_1,\mu^2x_2)$.

Recall that the function p is called Lagrange stable if the set $H(p):=\overline{\{p^h|\ h\in\mathbb{R}\}}$ ($p^h(t):=p(t+h)$ for any $t\in\mathbb{R}$) is a compact subset of $C(\mathbb{R},\mathbb{R})$.

Along this the system (63) we consider its H-class, i.e., the family of systems of differential equations
$$\begin{cases}\dot{x}_1=-x_1+q(t)\sqrt{|x_2|}\\ \dot{x}_2=-x_2\end{cases}\qquad(q\in H(p)). \tag{64}$$

Denote by $Y:=H(p)$, $(Y,\mathbb{R},\sigma)$ the shift dynamical system on $Y=H(p)$ and $\varphi(t,u,q)$ the unique solution of the system (64) passing through the point $u\in\mathbb{R}^2$ at the initial moment $t=0$. Then $\langle\mathbb{R}^2,\varphi,(Y,\mathbb{R},\sigma)$ is a cocycle over $(Y,\mathbb{R},\sigma)$ with the fibre $\mathbb{R}^2$.

Lemma 11. *Assume that the function p is Lagrange stable, then the skew-product dynamical system $(X,\mathbb{R}_+,\pi)$ generated by the cocycle φ ($X := \mathbb{R}^2\times Y$ and $\pi = (\varphi,\sigma)$) is pontwise dissipative.*

Proof. Consider a function $V : \mathbb{R}^2\times Y \to \mathbb{R}_+$ defined by the equality

$$V(u_1,u_2,q) := u_1^2 + u_2^2$$

for any $(u_1,u_2,q)\in\mathbb{R}^2\times H(p)$. Note that

$$\frac{dV}{dt}\Big|_{t=0} := \lim_{t\to 0^+}\frac{V(\pi(t,x)) - V(x)}{t} = -2(u_1^2+u_2^2) + 2q(0)u_1\sqrt{|u_2|}. \qquad (65)$$

Since the function p is bounded, then there exists a positive number R_0 such that

$$-2(u_1^2+u_2^2) + 2q(0)u_1\sqrt{|u_2|} \le -{u_1}^2 - {u_2}^2 \qquad (66)$$

for any $|u| := (u_1^2+u_2^2)^{1/2}\ge R_0$. From (65) and (66) we obtain

$$\frac{dV}{dt}\Big|_{t=0} \le -u_1^2 - u_2^2$$

for any $|u| := (u_1^2+u_2^2)^{1/2}\ge R_0$. According to Theorem 5.3 from [7, Ch.V] the skew-product dynamical system $(X,\mathbb{R}_+,\pi)$ generated by the cocycle φ is pointwise dissipative. Lemma is proved. □

Corollary 7. *The trivial motion $u = 0$ of the cocycle φ generated by the system (63) is attracting.*

Proof. This statement follows from Lemma 11 and Theorem 4. □

Corollary 8. *If the function p is Lagrange stable, then there are positive numbers $\mathcal{N}$ and ν such that*

$$\rho(\varphi(t,u,q)) \le \mathcal{N}e^{-\nu t}\rho(u)$$

for any $(t,u,q)\in\mathbb{R}_+\times\mathbb{R}^2\times H(p)$, where $\rho(u) := (u_1^4+u_2^2)^{1/4}$.

Proof. This statement follows from Corollary 7 and Theorems 13 and 7. □

6.2 Difference Equations

6.2.1 Discrete Nonautonomous Dynamical Systems

Definition 15. Let $\mathbb{T}\subseteq\mathbb{Z}$ and $(\mathbb{R}^n,\mathbb{T},\lambda)$ be a discrete linear dynamical system on $\mathbb{R}^n$. A function $F\in C(Y\times\mathbb{R}^n,\mathbb{R}^n)$ is said to be λ-homogeneous if

$$F(y,\lambda(\tau,w)) = \lambda(\tau,F(y,w)) \text{ (or equivalently } F(y,\lambda^\tau w) = \lambda^\tau F(y,w))$$

for any $(y,\tau,w)\in Y\times\mathbb{T}\times\mathbb{R}^n$.

Consider the difference equation

$$u(t+1) = F(\sigma(t,y), u(t)),\ \ (y \in Y) \tag{67}$$

where $F \in C(Y \times \mathbb{R}^n, \mathbb{R}^n)$. We will suppose also that the function F is *regular*, i.e., for every equation (67) the conditions of existence and uniqueness (on the maximal interval of definition of solutions) are fulfilled. Denote by $\varphi(t,u,y)$ the unique solution of the equation (67) with the initial data $\varphi(0,u,y) = u$, then the continuous mapping $\varphi : \mathbb{Z}_+ \times \mathbb{R}^n \times Y \to \mathbb{R}^n$ satisfying the condition $\varphi(t+\tau,u,y) = \varphi(t,\varphi(\tau,u,y),\sigma(\tau,y))$ for any $t,\tau \in \mathbb{Z}_+$ and $(u,y) \in \mathbb{R}^n \times Y$ is well defined. Then the triplet $\langle \mathbb{R}^n, \varphi, (Y,\mathbb{Z},\sigma)\rangle$ is a cocycle generated by (67) and satisfying Condition (C).

Lemma 12. *Assume that the function $F \in C(Y \times \mathbb{R}^n, \mathbb{R}^n)$ is λ-homogeneous, then the cocycle $\langle \mathbb{R}^n, \varphi, (Y,\mathbb{Z},\sigma)\rangle$ generated by the equation (67) is λ-homogeneous.*

Proof. To prove this statement we consider the function $\psi(t) := \lambda^\tau \varphi(t,u,y)$. It is easy to check that

$$\begin{gathered}\psi(t+1) = \lambda^\tau \varphi(t+1,u,y) = \lambda^\tau F(\sigma(t,y), \varphi(t,u,y)) = \\ F(\sigma(t,y), \lambda^\tau \varphi(t,u,y)) = F(\sigma(t,y), \psi(t))\end{gathered}$$

for any $t \in \mathbb{Z}_+$. Since $\psi(0) = \lambda^\tau u$, then we obtain $\psi(t) = \varphi(t, \lambda^\tau u, y)$, i.e., $\lambda^\tau \varphi(t,u,y) = \varphi(t,\lambda^\tau u, y)$ for any $t,\tau \in \mathbb{Z}_+$ and $(u,y) \in \mathbb{R}^n \times Y$. Lemma is proved. ☐

6.2.2 Homogeneous Difference Equations

Let us consider a difference equation

$$u(t+1) = f(t, u(t)), \tag{68}$$

where $f \in C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n)$. Along with equation (68) we consider its *H-class* [4, 15, 21, 22], i.e., the family of equations

$$v(t+1) = g(t, v(t)), \tag{69}$$

where $g \in H(f) := \overline{\{f^\tau |\ \tau \in \mathbb{Z}\}}$, $f^\tau(t,u) = f(t+\tau,u)$ for any $(t,u) \in \mathbb{Z} \times \mathbb{R}^n$ and by bar we denote the closure in $C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n)$. Assume that the function $f \in C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n)$ is regular, that is, for any $u \in \mathbb{R}^n$ and $g \in H(f)$ the equation (69) has a unique (on the maximal domain of definition) solution $\varphi(t,v,g)$ passing through the point $v \in \mathbb{R}^n$ at the initial moment $t = 0$. Then from the general properties of solutions of *difference equations (DEs)* it follows that the mapping $\varphi : \mathbb{Z}_+ \times \mathbb{R}^n \times H(f) \to \mathbb{R}^n$ is well defined and it satisfies the following conditions (see for example [4, ChIV] and [20, ChIV]):

1) $\varphi(0,v,g) = v$ for any $v \in \mathbb{R}^n$ and $g \in H(f)$;

2) $\varphi(t, \varphi(\tau, v, g), g^{\tau}) = \varphi(t + \tau, v, g)$ for every $v \in \mathbb{R}^n$, $g \in H(f)$ and $t, \tau \in \mathbb{Z}_+$;

3) the mapping $\varphi : \mathbb{Z}_+ \times \mathbb{R}^n \times H(f) \to \mathbb{R}^n$ is continuous.

Denote by $Y := H(f)$ and $(Y, \mathbb{Z}, \sigma)$ the dynamical system of translations on Y induced by the dynamical system of translations $(C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n), \mathbb{Z}, \sigma)$. The triplet $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{Z}, \sigma)\rangle$ is a cocycle over $(Y, \mathbb{Z}, \sigma)$ with the fibre $\mathbb{R}^n$. Thus equation (68) generates a cocycle $\langle \mathbb{R}^n, \varphi, (Y, \mathbb{Z}, \sigma)\rangle$. Note that under the conditions listed above the equation (68) (respectively, H-class (69)) can be written in the form

$$u(t + 1) = F(\sigma(t, y), u(t)) \quad (y \in Y = H(f)). \tag{70}$$

Indeed, let $Y := H(f)$ and $(Y, \mathbb{Z}, \sigma)$ be the dynamical system of translations on Y. Denote by F the mapping from $Y \times \mathbb{R}^n$ into $\mathbb{R}^n$ defined by the equality

$$F(g, u) := g(0, u). \tag{71}$$

It is easy to check that the mapping $F : H(f) \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

Definition 16. A function $f \in C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n)$ is said to be r homogeneous ($r \in (0, +\infty)^n$) of the degree zero if $f(t, \Lambda^r_\mu u) = \Lambda^r_\mu f(t, u)$ for any $(\mu, t, u) \in (0, +\infty) \times \mathbb{Z} \times \mathbb{R}^n$.

Remark 10. If the function $f \in C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n)$ is r homogeneous of the degree zero, then $f(t, 0) = 0$ for any $t \in \mathbb{Z}$.

Lemma 13. *If the function $f \in C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n)$ is r homogeneous of the degree zero, then the mapping $F : Y \times \mathbb{R}^n \to \mathbb{R}^n$ ($Y = H(f)$) defined by the equality (70) is r homogeneous of the degree zero with respect to $u \in \mathbb{R}^n$ uniformly in $y \in Y$.*

Proof. This statement can be proved using the same arguments as in the proof of Lemma 10. □

Corollary 9. *Assume that the function $f \in C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n)$ is r homogeneous of the degree zero, then the cocycle $\langle \mathbb{R}^n, \varphi, (H(f), \mathbb{Z}, \sigma)\rangle$ generated by the equation (68) is r homogeneous of the degree zero.*

Proof. This statement follows from Lemmas 12 and 13. □

6.2.3 Asymptotic Stability of Nonautonomous Difference Equations

Let $f \in C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n)$ and $f(t, 0) \equiv 0$ for any $t \in \mathbb{Z}$.

Definition 17. The trivial solution of equation (68) is said to be:

1. *uniformly stable* if for any positive number ε there exists a number $\delta = \delta(\varepsilon)$ ($\delta \in (0, \varepsilon)$) such that $|x| < \delta$ implies $|\varphi(t, x, f_\tau)| < \varepsilon$ for any $(t, \tau) \in \mathbb{Z}_+ \times \mathbb{Z}$;

2. *attracting* (respectively, *uniformly attracting*) if there exists a positive number a such that
$$\lim_{t\to+\infty}|\varphi(t,x,f_\tau)|=0 \tag{72}$$
for any $|x|\le a$ and $\tau\in\mathbb{Z}$;

3. *asymptotically stable* if it is uniformly stable and attracting (respectively, the equality (72) holds uniformly with respect to $|u|\le a$ and $\tau\in\mathbb{Z}$).

Remark 11. If the function $f\in C(\mathbb{Z}\times\mathbb{R}^n,\mathbb{R}^n)$ is regular and $f(t,0)=0$ for any $t\in\mathbb{Z}$, then it is easy to show that the trivial solution of the equation (68) is uniformly attracting if and only if there exists a positive number a such that
$$\lim_{t\to+\infty}\sup_{|x|\le a,\ g\in H(f)}|\varphi(t,u,g)|=0. \tag{73}$$

Remark 12. 1. By slight modifications of the reasoning from the works [1,19] we can establish the equivalence of the standard definition (see for example [11, Ch.V] and [14, Ch.IV]) of uniform stability (respectively, global uniform asymptotic stability) and of the one given above for the difference equation (68).

2. Using the same ideas as in the works of G. Sell [19, 20] we can prove that for the difference equations (68) with the Lagrange stable right hand side f the following statements are equivalent:

1. the trivial solution of the equation (68) is uniformly asymptotically stable;
2. the trivial motion of the cocycle $\langle\mathbb{R}^n,\varphi,(H(f),\mathbb{Z},\sigma)\rangle$ generated by (68) is uniformly asymptotically stable.

Theorem 15. *Let $f\in C(\mathbb{Z}\times\mathbb{R}^n,\mathbb{R}^n)$. Assume that the following conditions are fulfilled:*

1. *the function f is regular and $f(t,0)=0$ for any $t\in\mathbb{Z}$;*
2. *the function f is r homogeneous of the degree zero.*

Then the following statements are equivalent:

1. *the trivial solution of the equation (68) is uniformly asymptotically stable;*
2. *there exit positive numbers $\mathcal{N}$ and ν such that*
$$\rho(\varphi(t,u,g))\le\mathcal{N}e^{-\nu t}\rho(u)$$
for any $u\in\mathbb{R}^n$, $g\in H(f)$ and $t\in\mathbb{Z}_+$.

Proof. Let $Y:=H(f)$ and $(Y,\mathbb{Z},\sigma)$ be the shift dynamical system on $Y=H(f)$. Denote by $\langle\mathbb{R}^n,\varphi,(H(f),\mathbb{Z},\sigma)\rangle$ the cocycle generated by the difference equation (68). Since the function f is r homogeneous of the degree zero, then by Corollary 6 the cocycle φ generated by the equation (68) is r homogeneous of the degree zero. To finish the proof of Theorem 15 it suffices to take into account Remarks 11 – 12 and apply Theorem 3. ∎

Theorem 16. *Let $f \in C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n)$. Assume that the following conditions are fulfilled:*

1. *$f(t, 0) = 0$ for any $t \in \mathbb{Z}$;*
2. *the function f is r homogeneous of the degree zero and Lagrange stable.*

Then the following statements are equivalent:

1. *the trivial solution of the equation (68) is uniformly asymptotically stable;*
2. *the trivial solution of the equation (68) is globally uniformly asymptotically stable;*
3. *there exit positive numbers $\mathcal{N}$ and ν such that*

$$\rho(\varphi(t, u, g)) \leq \mathcal{N} e^{-\nu t} \rho(u) \tag{74}$$

for any $u \in \mathbb{R}^n$, $g \in H(f)$ and $t \geq 0$, where $\rho(u) = |u|_{r,p}$.

Proof. Let $Y := H(f)$ and $(Y, \mathbb{Z}, \sigma)$ be the shift dynamical system on $Y = H(f)$. Since the function f is Lagrange stable, then the set Y is compact. Denote by $\langle \mathbb{R}^n, \varphi, (H(f), \mathbb{Z}, \sigma) \rangle$ the cocycle generated by the difference equation (68). Since the function f is r homogeneous of the degree zero, then by Corollary 9 the cocycle φ generated by equation (68) is r homogeneous of the degree zero. To finish the proof of Theorem 16 it suffices to take into account Remark 12 and apply Theorem 8. □

Theorem 17. *Let $f \in C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n)$ be a regular function. Assume that the following conditions are fulfilled:*

1. *the function f is recurrent in time and $f(t, 0) = 0$ for any $t \in \mathbb{Z}$;*
2. *the function f is r homogeneous of the degree zero.*

Then the following statements are equivalent:

1. *the trivial solution of the equation (68) is uniformly stable and there exists a positive number a such that*

$$\lim_{t \to +\infty} |\varphi(t, u, f)| = 0 \tag{75}$$

for any $u \in B[0, a]$;

2. *there exit positive numbers $\mathcal{N}$ and ν such that*

$$\rho(\varphi(t, u, g)) \leq \mathcal{N} e^{-\nu t} \rho(u)$$

for any $u \in \mathbb{R}^n$, $g \in H(f)$ and $t \geq 0$.

Proof. Let $Y := H(f)$ and $(Y, \mathbb{Z}, \sigma)$ be the shift dynamical system on $Y = H(f)$. Note that the space Y is a compact and minimal set because the function f is recurrent in time (see Remark 9). Since the function f is r homogeneous of the degree zero, then by Corollary 6 the cocycle $\langle \mathbb{R}^n, \varphi, (H(f), \mathbb{Z}, \sigma) \rangle$ generated by the equation (68) is r homogeneous of the degree zero. To finish the proof of Theorem 17 it suffices to take into account Remark 12 and apply Theorem 8. □

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8 Conflict of Interests

The author declare that he does not have conflict of interests.

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The linear Fredholm integral equations with functionals and parameters

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Abstract. The theory of linear Fredholm integral-functional equations of the second kind with linear functionals and a parameter is considered. The necessary and sufficient conditions are obtained for the coefficients of the equation and those parameter values in the neighbourhood of which the equation has solutions. The leading terms of the asymptotics of the solutions are constructed. The constructive method is proposed for constructing a solution both in the regular case and in the irregular one. In the regular case, the solution is constructed as a Taylor series in powers of the parameter. In the irregular case, the solution is constructed as a Laurent series in powers of the parameter. The example is used to illustrate the proposed constructive theory and method.

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1 Introduction

This paper deals with some issues in the theory of linear integral equations with linear functionals. Modern views on the fundamental laws of nature are often stated in terms of integral equations [1–5]. Many inverse problems in mathematical physics can be formulated or reduced to nonclassical integral equations. In [6] the problem for identification of external force and heat source density dynamics was reduced to solution of Volterra integral equations of the first kind. The analysis of integral operators includes questions of finding eigenvalues and adjoint functions [7], studying the convergence of their asymptotics, existence and convergence theorems of approximate methods [4,5]. At the end of 20th century, A. P. Khromov found a new class of integral operators with discontinuous kernels and began a systematic study of them [2]. Under very general assumptions, he derived the conditions under which eigenfunction expansions of these operators behave like trigonometric Fourier series. However, these conditions as well as the construction of the classical discontinuous Fredholm resolvent in the form of the ratio of two integer analytic expansions over a parameter are difficult to verify. In the works [5, 8, 9] a class of equations with discontinuous kernels was distinguished and studied.

In [10] the branching solutions of the Cauchy problem for nonlinear loaded differential equations with bifurcation parameters were studied. The purpose of this

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study is to prove the properties of the resolvent integral operator as applied to the second kind Fredholm integral equations with local and integral loads, and to formulate and prove constructive theorems of existence and convergence to the desired solution of successive approximations.

Let us consider the equation

$$x - \mathcal{L}x - \lambda \mathcal{K}x = f, \tag{1}$$

where linear operators $\mathcal{L}$ and $\mathcal{K}$ are given as follows

$$\mathcal{L}x := \sum_{k=1}^{n} a_k(t)\langle \gamma_k, x\rangle,$$

$$\mathcal{K}x := \int_a^b K(t,s)x(s)\,ds,$$

λ is a parameter. All the functions in equation (1) are assumed to be continuous. Kernel $K(t,s)$ can be symmetric and it is also continuous both in t and s. The desired solution $x(t)$ is constructed in $\mathcal{C}_{[a,b]}$.

Linear functionals $\langle \gamma_k, x\rangle$ in applications corresponds to the *loads* imposed on the desired solution. The loads can be local ($\langle \gamma_k, x\rangle = x(t_k)$, $t_k \in [a,b]$) or integral such as $\langle \gamma_k, x\rangle = \int_a^b \gamma_k(t)x(t)\,dt$, where $\gamma_k(t)$ are piecewise continuous functions for $t \in [a,b]$ or $\langle \gamma_k, x\rangle = \int_a^b x(t)\,d\gamma_k(t)$, $\gamma_k(t)$ is a given function of limited variation.

The objective is to construct the solution $x(t,\lambda)$ for $\lambda \in \mathbb{R}^1$ of equation (1). For operator $\mathcal{L}x$ below the following brief notation

$$\mathcal{L}x := \sum_{k=1}^{n} a_k(t)\langle \gamma_k, x\rangle \equiv (\vec{a}(t), \langle \vec{\gamma}, x\rangle)$$

is used, where conventional notation $(\cdot,\cdot)$ for scalar product is used. Here $\vec{a}(t) = (a_1(t), \cdots, a_n(t))^T$, $a_i(y) \in \mathcal{C}_{[a,b]}$, $\langle \vec{\gamma}, x\rangle = (\langle \gamma_1, x\rangle, \ldots, \langle \gamma_n, x\rangle)^T$.

Loaded differential equations have been intensively studied during the last decades. The term "loaded equation" was first used in the works of A. M. Nakhushev, here readers may refer to his monograph [3]. Loaded equations appear in many applications, see e.g.[11, 12]. But theory and numerical methods for the loaded integral equations remained less developed. In paper [13] the problem statement for the integral equation with single load is given. Then, in [14, 15] theory of the Hammerstein integral equations with loads and bifurcation parameters was proposed. In [16] the Fredholm resolvent was employed for computing H_2-norm for linear periodic systems.

The similar statement is addressed in the present paper and analytical method is described which makes it possible to consider integral equations with arbitrary finite number of local and integral loads. An example of functionals that generate local and integral loads in the space $\mathcal{C}_{[a,b]}$ is the functional

$$\langle \gamma, x\rangle := \sum_{i=1}^{m} \alpha_i x(t_i) + \sum_{i=1}^{n} \int_{a_i}^{b_i} m_i(s)x(s)\,ds,$$

where $\alpha_i \in \mathbb{R}^1$, $[a_i, b_i] \subset [a, b]$, $m_i(s) \in \mathcal{C}_{[a_i,b_i]}$, $t_i \in [a, b]$.

2 System of equations to determine the load

Let us introduce the following condition

I. $\langle \gamma_k, K(t,s)\rangle = 0$, $k = 1, \ldots, n$, $s \in [a, b]$ and vectors $\vec{x}_\gamma = (\langle \gamma_1, x\rangle, \ldots, \langle \gamma_n, x\rangle)^T$, $\vec{f}_\gamma = (\langle \gamma_1, f\rangle, \ldots, \langle \gamma_n, f\rangle)^T$.

Lemma 1. *Let condition* **I** *be fuilfilled. Then load vector $\vec{x}_j$ necesserily satisfies system*

$$(E - A_0)\vec{c} = \vec{f}_\gamma, \tag{2}$$

where $A_0 = [\langle \gamma_i, a_k\rangle]_{i,k=1}^n$, E is $(n \times n)$ identity matrix.

Proof. Let us apply the functionals $\langle \gamma_i, \cdot\rangle$, $i = 1, \ldots, n$, to both parts of equation (1). Using **I**, the following system can be derived

$$\langle \gamma_i, x\rangle - \sum_{k=1}^{n} \langle \gamma_i, a_k\rangle \langle \gamma_k, x\rangle = \langle \gamma_i, f\rangle,\ i = 1, \ldots, n. \tag{3}$$

System of linear algebraic equations (3) is, in fact, system (2) presented in coordinate system. Lemma is proved. □

From this Lemma follows:

Corollary 1. *Let condition* **I** *be fuilfilled and system (2) has no solution. Then equation (1) has no solution in class of continuous functions.*

Let condition **I** be fuilfilled and vector $\vec{c}^* \in \mathbb{R}^n$ satisfies system (2). Then solution $x(t, \lambda)$ of equation (1) depends on vector $\vec{c}^*$ and satisfies the following Fredholm integral equation of the 2nd kind

$$x(t, \lambda) - \lambda \int_a^b K(t, s)x(s, \lambda)\, ds = f(t) + (\vec{a}(t), \vec{c}^*).$$

Lemma 2. *Solution of equation (1) for arbitrary λ, except the characteristic numbers λ_i of kernel $K(t, s)$, is defined by the following formula*

$$x(t, \lambda) = (\vec{a}(t), \vec{x}_\gamma(\lambda)) + \int_a^b \Gamma(t, s, \lambda)(\vec{a}(s), \vec{x}_\gamma(\lambda))\, ds + \int_a^b \Gamma(t, s, \lambda) f(s)\, ds + f(t). \tag{4}$$

Here $\Gamma(t, s, \lambda) = \frac{D(t,s,\lambda)}{D(\lambda)}$, $D(t, s, \lambda)$ and $D(\lambda)$ are entire analytic functions of parameter λ, $D(\lambda_i) = 0$. Load vector $\vec{x}_\gamma(\lambda)$ necesserily must satisfy the following system of n linear algebraic equations

$$(E - A_0 - A(\lambda))\vec{x}_\gamma(\lambda) = \vec{b}(\lambda) \tag{5}$$

with matrix

$$A(\lambda) = \left\langle \gamma_i, \int_a^b \Gamma(t,s,\lambda) a_k(s)\, ds \right\rangle_{i,k=1}^{n} \tag{6}$$

and vector

$$\vec{b}(\lambda) = \left\langle \gamma_i, f(t) + \int_a^b \Gamma(t,s,\lambda) f(s)\, ds \right\rangle_{i=1}^{n}.$$

The set of characteristic numbers $\{\lambda_i\}$ is a finite and countable set .

Proof. It is known (see sec. 9 (3) in book [17]) that an inverse operator $(I - \lambda K)^{-1}$ is defined by Fredholm formula [18]:

$$(I - \lambda K)^{-1} = I + \lambda \int_a^b \frac{D(t,s,\lambda)}{D(\lambda)} [\cdot]\, ds.$$

Functions $D(t,s,\lambda)$ and $D(\lambda)$ are entire analytical funcations with respect to λ, defined for $\lambda \in \mathbb{R}^1$. Moreover, the characteristic numbers of kernel $K(t,s)$ of operator $\mathcal{K}$ are zeros of denominator $D(\lambda)$. Thus, the inverse operator $(I - \lambda K)^{-1}$ can be called discontinuous operator. Indeed, the function $\Gamma(t,s,\lambda)$ in solution (4) has the 2nd kind discontinuities at points $\{\lambda_i\}$. By solving system (5) and substituting its solution into (4), we find the solution of the original problem (1). The lamma is proved.

□

Remark 1. In system (5) in general case the matrix $A(\lambda)$ and vector $\vec{b}(\lambda)$ will have 2nd kind discontinuities at points λ.

Let us distinguish the class of kernels $K(t,s)$ when matrix A_0 and vector $\vec{b}(\lambda)$ can be specified. Let the kernel $K(t,s)$ generate the nilpotency of the operator $\mathcal{K}$.

Let $|\lambda| < \frac{1}{||\mathcal{K}||}$. In that case the solution of equation $x - \lambda \mathcal{K} x = f$ for arbitrary source function f is defined uniquely as follows

$$x = f + \lambda \mathcal{K} f + \lambda^2 \mathcal{K}^2 f + \cdots + \lambda^p \mathcal{K}^p f.$$

Here

$$\mathcal{K}^n f = \int_a^b K_n(t,s) f(s)\, ds,$$

where

$$K_n(t,s) = \int_a^b K(t,z) K_{n-1}(z,s)\, dz.$$

Here $K_1(t,s) := K(t,s)$, $K_{p+1}(t,s) = 0$ due to the nilpotency of the operator $\mathcal{K}$ for some $p \geq 1$. Therefore, formula (4) can be presented in the following constructive form

$$x(t,\lambda,\vec{x}_\gamma) = f(t) + (\vec{a}(t), \vec{x}_\gamma) + \int_a^b \big(\lambda K(t,s) + \lambda^2 K_2(t,s) + \cdots \tag{7}$$

$$\cdots + \lambda^p K_p(t,s)\big)(f(s) + (\vec{a}(s), \vec{x}_\gamma))\,ds.$$

Correspondingly, we derive the refined system of linear algebraic equations (5) with respect to the load vector because

$$A(\lambda) = \langle \gamma_i, \int_a^b (\lambda K(t,s) + \lambda^2 K_2(t,s) + \cdots + \lambda^p K_p(t,s)) a_k(s)\,ds \rangle_{i,k=1}^n, \tag{8}$$

$$\vec{b}(\lambda) = \langle \gamma_i,\ f(t) + \int_a^b (\lambda K(t,s) + \lambda^2 K_2(t,s) + \cdots + \lambda^p K_p(t,s)) f(s)\,ds \rangle_{i=1}^n. \tag{9}$$

Thus, $A(\lambda)$ and $\vec{b}(\lambda)$ are continuous in λ. It is to be noted that if $\langle \gamma_i, K(t,s)\rangle = 0$, $i = 1,\dots,n$, then $A(\lambda) = 0$, and system (5) degenerates into system (2) introduced in Lemma 1. Therefore, in this case vector $\vec{x}_\gamma$ from solution (7) to given problem (1) can be determined. Then the following theorem can be formulated.

Theorem 1. *Let operator $\mathcal{K}$ be nilpotent and $\langle \gamma_i, K(t,s)\rangle = 0$, $i = 1,\dots,n$, $\forall s \in [a,b]$. Then solution of equation (1) exists as functional polynomial (7) of p-th order in parameter λ. Coefficients of polynomial (7) depend on selection of the load vector $\vec{x}_\gamma$ in $\mathbb{R}^n$.*

If operator $\mathcal{K}$ is not nilpotent and the identity $\langle \gamma_i, K(t,s)\rangle = 0$ is not satisfied, then the solution $x(t,\lambda)$ of equation (1) can be found in the class of continuous in t functions. This solution can be represented in the punctured neighbourhood $0 < |\lambda| < \rho$ in the form of Laurent series with pole at point $\lambda = 0$.

3 Successive approximations

Let $\det(E - A_0) \neq 0$. Then there exists a neighbourhood of λ $|\lambda| < \rho$ such that system (5) has a solution $\vec{x}_\rho(\lambda) \to (E - A_0)^{-1}\vec{f}_\gamma$ as $\lambda \to 0$. Positive ρ exists since $||(E - A_0)^{-1}A(\lambda)|| \to 0$ as $\lambda \to 0$.

Let us call the case of $\det(E - A_0) \neq 0$ regular.

Theorem 2. *In the regular case $\det(E - A_0) \neq 0$ there exists a neighbourhood $|\lambda| < \rho$ in which equation (1) has the unique solution continuous in t and holomorphic in λ.*

Corollary 2. *Let $\det(E - A_0) \neq 0$, $||(I-L)^{-1}\mathcal{K}|| \leq l$. Fix the scalar $q < 1$. Then for $|\lambda| \leq \frac{q}{l}$ equation (1) has the unique solution. Moreover, solution is holomorphic in λ. The sequence $\{x_n\}$, where $x_n = \lambda(I-L)^{-1}\mathcal{K}x_{n-1} + (I-L)^{-1f}$, $x_0 = 0$, uniformly converges to the desired solution $x(t,\lambda)$ of equation (1) at the rate of a geometric progression with the denominator $q < 1$.*

Let us focus now on the irregular case of $\det(E - A_0) = 0$. Let $A_0 = E$. Then $\det(E - A_0) = 0$ and we have irregular case. Let $\frac{d^i}{d\lambda^i} A(\lambda)\big|_{\lambda=0}$ for $i = 0, 1, \ldots, p-1$ be zero matrices and $\frac{d^p}{d\lambda^p} A(\lambda)\big|_{\lambda=0} \neq 0$. Then the load vector $\vec{x}_\gamma$ satisfies the following system

$$\big(-E - A_p^{-1} \sum_{m=p+1}^{\infty} \lambda^{m-p} A_m\big)\vec{x}_\gamma = \lambda^{-p} A^{-1} \vec{b}(\lambda),$$

where

$$A_p = \frac{1}{p!}\left(\frac{d^p}{d\lambda^p} A(\lambda)\right)\Big|_{\lambda=0}.$$

Let's select neighbourhood $|\lambda| < \rho$ such that

$$||A_p^{-1} \sum_{m=p+1}^{\infty} \lambda^{m-p} A_m|| \leq q < 1.$$

Then

$$\lambda^p \vec{x}_\gamma = -\sum_{n=0}^{\infty} (-A_p^{-1} \sum_{m=p}^{\infty} \lambda^{m-p} A_m)^n A_p^{-1} \vec{b}(\lambda),$$

which series converges to holomorphic function

$$\vec{\nu}(\lambda) = -\sum_{n=0}^{\infty} (-A_p^{-1} \sum_{m=p}^{\infty} \lambda^{m-p} A_m)^n A_p^{-1} \vec{b}(\lambda)$$

at the rate of a geometric sequence with the denominator $q < 1$ for $|\lambda| \leq \rho$. Therefore, the load $\vec{x}_\gamma(\lambda) = \lambda^{-p} \vec{\nu}(\lambda)$ is a Laurent series with pth order pole. Then the following theorem is true.

Theorem 3. *Let $A_0 = E$, $A(\lambda) = \sum_{m=p}^{\infty} A_m \lambda^m$, $p \geq 1$. Let the matrix A_p be not singular. Then there exists punctured neighbourhood $0 < |\lambda| \leq p$ such that the equation (1) has a solution $x(t, \lambda)$ with pole at point $\lambda = 0$ of order less than or equal to p.*

Example 1. Let us consider the equation

$$x(t, \lambda) - a(t)x(0, \lambda) = \lambda \int_0^1 b(t)m(s)x(s, \lambda)\, ds + f(t), \ \ t \in [0, 1].$$

Let us have irregular case of $a(0) = 1$. Let $b(0) \neq 0$, i.e. condition **I** is not fulfilled,

$$\frac{d}{d\lambda} A(\lambda)\big|_{\lambda=0} = b(0) \int_0^1 m(s)a(s)\, ds.$$

Let

$$\int_0^1 m(s)a(s)\, ds \neq 0.$$

Then all the conditions of Theorem 3 are fulfilled for $p = 1$. Then equation has solution $x(\lambda)$ for $|\lambda| > 0$ with 1st order pole at point $\lambda = 0$. The desired solution is the following

$$x(t,\lambda) = f(t,\lambda) - \frac{b(t)}{b(0)} f(0) + a(t)x(0,\lambda),$$

where the load $x(0,\lambda)$ is constructed as follows

$$x(0,\lambda) \equiv \frac{1}{(a,m)}\left[-\frac{f(0)}{\lambda b(0)} - (f,m) + \frac{f(0)}{b(0)}(b,m)\right],$$

where $(a,m) = \int_0^1 a(t)m(t)dt$, $(f,m) = \int_0^1 f(t)m(t)dt$, $(b,m) = \int_0^1 b(t)m(t)dt$. In this example the solution is constructed in an explicit form.

4 Conclusion and generalizations

The linear Fredholm integral functional equations of the second kind with linear functionals are studied. Necessary and sufficient conditions are formulated. Constructive methods are proposed for both regular and irregular cases. The solution in form of a Taylor series is constructed in terms of powers of the parameters. In the irregular case, the solution is constructed as a Laurent series of powers of the parameters. The constructive theory and methods are demonstrated using a model example. The case of $A_0 \neq E$ remained not addressed in this paper. The most complete results can be derived for the case of symmetric matrix A_0. In that case solution of equation (1) can be also presented as Laurent series with pole at point $\lambda = 0$. The corresponding sufficient condition can be derived based on generalized Jordan chains of the theory of perturbed nonlinear operators [7]. The bifurcation theory of nonlinear loaded integral equations, using the approach of this article in combination with representation theory and group symmetry [19], will also be addressed in future works. Some results in this direction are published in [4, 5, 14]. The numerical solution of Fredholm integral-functional equations of the second kind with linear functionals and parameter will be also addressed in future works.

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B-spline collocation method for solving Fredholm integral equations with discontinuous right-hand side

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Abstract. In this paper, we propose a method for approximating the solution of the linear Fredholm integral equation of the second kind which is defined on a closed contour Γ in the complex plane. The right-hand side of the equation is a piecewise continuous function that is numerically defined on a finite set of points on Γ. To approximate the solution, we use a linear combination of B-spline functions and Heaviside step functions defined on Γ. We discuss both theoretical and practical aspects of the pointwise convergence of the method, including its performance in the vicinity of the points where discontinuities occur.

Mathematics subject classification: 65R20, 65D07.
Keywords and phrases: Fredholm integral equation, piecewise continuous function, closed contour, complex plane, numerical approximation, B-spline, step function, convergence.

1 Introduction and problem formulation

Let a closed and piecewise smooth contour Γ be the boundary of the simply connected domain $\Omega^+ \subset \mathbb{C}$, and let the point $z = 0 \in \Omega^+$. Consider the Riemann function $z = \psi(w)$ that performs the conformal map of the domain D^- from the outside of the circle $\Gamma_0 := \{w \in \mathbb{C} : \ |w| = 1\}$ onto the domain Ω^- from the outside of the contour Γ, such that $\psi(\infty) = \infty, \ \psi'(\infty) > 0$. The function $\psi(w)$ transforms the circle Γ_0 onto the contour Γ. Next, we consider that the points of the contour Γ are defined by means of the function $\psi(w)$.

Let $f : \Gamma \to \mathbb{C}$ be a continuous or piecewise continuous function on Γ, and in this context, we will use the notation $f \in PC(\Gamma)$. If the function $f \in PC(\Gamma)$ is discontinuous on Γ, we consider that it has finite jump discontinuities, being left-continuous at the discontinuity points.

Let's consider the linear Fredholm integral equation of the second kind

$$\varphi(t) - \lambda \int_\Gamma K(t,s)\,\varphi(s)\,ds = f(t)\,, \ \ t \in \Gamma, \tag{1}$$

which is defined on the contour Γ described above. The kernel function is continuous in both variables, $K \in C(\Gamma \times \Gamma)$. The right-hand side function $f \in PC(\Gamma)$, and the constant $\lambda \in \mathbb{C}$ satisfies the sufficient condition for equation (1) to have a unique solution $\varphi \in PC(\Gamma)$.

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Considering that the right-hand side f is numerically defined on the set of points $\{t_j\}$ on the contour Γ, we aim to develop an efficient method for computing a sequence of approximations φ_n to the solution φ which converges pointwise to φ on Γ as $n \to \infty$.

Global or piecewise polynomial approximation is generally ineffective for approximating piecewise continuous functions, except when studying convergence in the norm of Lebesgue spaces $L_p,\ 1 < p < \infty$. In such cases, it is shown that the sequence of interpolation polynomials converges to the solution φ, with the exception of a countable set of points [1].

It is known that if algebraic polynomials or spline functions of order $m \geq 2$ are used to approximate the piecewise continuous function φ, then in the vicinity of the discontinuity points, the approximation error does not tend to zero, no matter how much we increase the amount of informations required for constructing the approximation.

For applications, it is of interest to define analytically a sequence of approximation functions φ_n that converge pointwise to the piecewise continuous function φ, including in the vicinity of the discontinuity points.

Linear spline functions can be employed as an approximation technique, but in this case, the convergence rate of the approximation process can be exceedingly slow [2]. Some numerical results show that the oscillatory effect disappears and pointwise convergence of the approximations is attained, even in the vicinity of discontinuity points, when the approximation φ_n is constructed as a linear combination of B-spline functions of order $m \geq 2$ [2,3]. However, in the vicinity of discontinuity points, the convergence rate of the approximations is exceedingly slow. Furthermore, it should be noted that continuous curves in the complex plane frequently lead to a heavily distorted approximation of discontinuous curves.

The proposed approximation method entails constructing a sequence of piecewise continuous approximations for the function φ, with the objective of incorporating the convergence properties of B-spline functions. Specifically, we define the sequence of approximations φ_n as a linear combination of B-spline functions and Heaviside step functions. Previous studies have examined these approximations on intervals of the real axis [4]. In this paper, we investigate the case where the approximations are defined on the contour Γ in the complex plane.

Let $\{t_j\}_{j=1}^{n_B}$ be the set of distinct points on the contour Γ where the values of the function $f \in PC(\Gamma)$ are defined. We consider that the points t_j are generated based on the relation

$$t_j = \psi(w_j),\ w_j = e^{i\theta_j},\ \theta_j = 2\pi(j-1)/n_B,\ j = 1, ..., n_B.$$

We denote by $\Gamma_j := arc[t_j, t_{j+1}]$ the set of points of the contour Γ, located between the points t_j and t_{j+1} (see Figure 1).

We admit that the values $f\left(t_r^d\right)$ of the function f are known at the discontinuity points $t_r^d,\ r = 1, ..., n_{pd}$, on the contour Γ. For the function f, defined numerically, in [5] and [6] several algorithms have been proposed for establishing the locations of the discontinuity points on Γ.

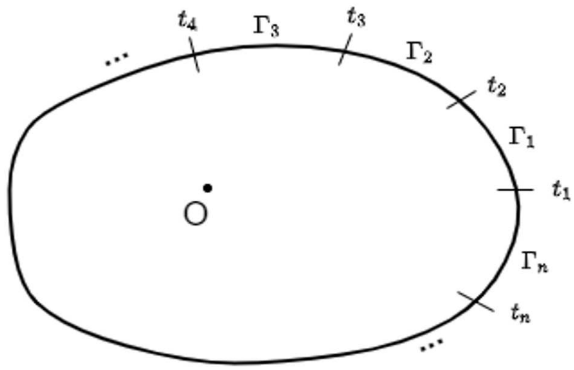

Figure 1: The contour and notations used

2 The computational scheme for approximating the solution of the integral equation

The algorithm we propose for approximating the solution φ of equation (1) is based on the concept of B-spline functions of order $m \geq 2$ which are defined at the points t_j of the contour Γ. These B-spline functions are defined using the recursive formula

$$B_{m,j}(t) := \frac{m}{m-1}\left(\frac{t-t_j^B}{t_{j+m}^B - t_j^B} B_{m-1,j}(t) + \frac{t_{j+m}^B - t}{t_{j+m}^B - t_j^B} B_{m-1,j+1}(t)\right), \ j = 1, ..., n_B, \tag{2}$$

where $B_{1,j}(t) = \begin{cases} \frac{1}{t_{j+1}^B - t_j^B} & \text{if } t \in arc\left[t_j^B, t_{j+1}^B\right) \\ 0 & \text{otherwise} \end{cases}$. The set of nodes $\left\{t_j^B\right\}_{j=1}^{n_B+m}$ satisfies the condition $t_j^B = t_j, \ j = 1, ..., n_B, \ t_{n_B+1}^B = t_1^B, t_{n_B+2}^B = t_2^B, ..., t_{n_B+m}^B = t_m^B$ (see [3]). For a fixed $m \geq 2$, the B-spline functions (2) have an explicit representation [3].

We define the Heaviside step function H on the contour Γ, constructed using the discontinuity points $t_r^d, \ r = 1, ..., n_{pd}$:

$$H\left(t - t_r^d\right) := \begin{cases} 0 & \text{if } t \in \Gamma_1 \cup ... \cup \Gamma_{s-1} \cup arc\left[t_s^B, t_r^d\right) \\ 1 & \text{if } t \in arc\left[t_r^d, t_{s+1}^B\right) \cup \Gamma_{s+1} \cup ... \cup \Gamma_{n_B} \end{cases},$$

where $\Gamma_s = arc\left[t_s^B, t_{s+1}^B\right], \ t_r^d \in \Gamma_s$.

Taking into account that the solution φ of equation (1) is a function with jump discontinuities on the contour Γ, and the linear combination of B-spline functions generates a continuous curve, we will seek the approximation of the solution φ of equation (1) in the form

$$\varphi_{n_B}^H(t) := \sum_{k=1}^{n_B} \alpha_k B_{m,k}(t) + \sum_{r=1}^{n_{pd}} \beta_r H\left(t - t_r^d\right), \tag{3}$$

where the coefficients $\alpha_k \in \mathbb{C},\ k = 1, ..., n_B$, and $\beta_r \in \mathbb{C},\ r = 1, ..., n_{pd}$, are determined by imposing the interpolation conditions

$$\varphi^H_{n_B}\left(t^C_j\right) - \lambda \int_\Gamma K\left(t^C_j, s\right)\varphi^H_{n_B}(s)\,ds = f\left(t^C_j\right),\ \ j = 1, ..., n. \tag{4}$$

In relation (4), where $n := n_B + n_{pd}$, the following elements of the B-spline knot set are selected as interpolation points $t^C_j,\ j = 1, ..., n$:

1. the first n_B interpolation points $t^C_j,\ j = 1, ..., n_B$, are the nodes $t^B_j = t_j,\ j = 1, ..., n_B$;
2. the remaining n_{pd} interpolation points $t^C_j,\ j = n_B + 1, ..., n$, are the discontinuity points $t^d_r,\ r = 1, ..., n_{pd}$, of the function f.

If among the interpolation points $t^C_j,\ j = 1, ..., n_B$, there are discontinuity points $t^d_j = \psi\left(e^{i\theta^d_j}\right)$ of the function f on Γ, then instead of them we consider the points $\tilde{t}^d_j = \psi\left(e^{i\left(\theta^d_j - \varepsilon_2\right)}\right)$, where $\varepsilon_2 > 0$ is a small value, for example, $\varepsilon_2 = 0.01$. Since the function is left continuous, for a sufficiently small ε_2, it can be considered that the value of the function f at point $\tilde{t}^d_j$ coincides with its value at point t^d_j.

Taking into account the representation

$$\int_\Gamma K(t,s)\,\varphi^H_{n_B}(s)\,ds = \int_\Gamma K(t,s)\left(\sum_{k=1}^{n_B}\alpha_k B_{m,k}(s) + \sum_{r=1}^{n_{pd}}\beta_r H\left(s - t^d_r\right)\right)ds =$$

$$= \sum_{k=1}^{n_B}\alpha_k I^{1,m}_k(t) + \sum_{r=1}^{n_{pd}}\beta_r I^2_r(t),$$

where $I^{1,m}_k(t) := \int_\Gamma K(t,s)\,B_{m,k}(s)\,ds,\ \ I^2_r(t) := \int_\Gamma K(t,s)\,H\left(s - t^d_r\right)ds,$ we can write the interpolation conditions (4) in the form

$$\sum_{k=1}^{n_B}\left(B_{m,k}\left(t^C_j\right) - \lambda I^{1,m}_k\left(t^C_j\right)\right)\alpha_k + \sum_{r=1}^{n_{pd}}\left(H\left(t^C_j - t^d_r\right) - \lambda I^2_r\left(t^C_j\right)\right)\beta_r =$$

$$= f\left(t^C_j\right),\ \ j = 1, ..., n. \tag{5}$$

The relation (5) can be written in matrix form as $B\bar{x} = \bar{f}$, where

$$B = \{m_{j,k}\}^n_{j,k=1},\ \ m_{j,k} := B_{m,k}\left(t^C_j\right) - \lambda I^{1,m}_k\left(t^C_j\right),\ \ j = 1, ..., n,\ \ k = 1, ..., n_B,$$

$$m_{j,k} := H\left(t^C_j - t^d_r\right) - \lambda I^2_r\left(t^C_j\right),\ \ j = 1, ..., n,\ \ k = n_B + 1, ..., n,$$

$$\bar{x} = \left(\alpha_1, ..., \alpha_{n_B}, \beta_1, ..., \beta_{n_{pd}}\right)^T,\ \ \bar{f} = \left(f\left(t^C_1\right), ..., f\left(t^C_n\right)\right)^T.$$

If we consider $t_j^C = t_{j+1}^B,\ \ j = 1, ..., n_B$, for $m = 2$, and $t_j^C = t_{j+2}^B,\ \ j = 1, ..., n_B$, for $m = 3$ and $m = 4$, then the elements of the matrix B can be calculated as follows:

$$B = B_1 - \lambda B_2. \tag{6}$$

The properties of matrix

$$B_1 = \begin{pmatrix} B_{m,1}\left(t_1^C\right) & \cdots & B_{m,n_B}\left(t_1^C\right) & H\left(t_1^C - t_1^d\right) & \cdots & H\left(t_1^C - t_{n_{pd}}^d\right) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ B_{m,1}\left(t_n^C\right) & \cdots & B_{m,n_B}\left(t_n^C\right) & H\left(t_n^C - t_1^d\right) & \cdots & H\left(t_n^C - t_{n_{pd}}^d\right) \end{pmatrix}$$

have been examined in [3], and the elements of matrix

$$B_2 = \begin{pmatrix} I_1^{1,m}\left(t_1^C\right) & \cdots & I_{n_B}^{1,m}\left(t_1^C\right) & I_1^2\left(t_1^C\right) & \cdots & I_{n_{pd}}^2\left(t_1^C\right) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ I_1^{1,m}\left(t_n^C\right) & \cdots & I_{n_B}^{1,m}\left(t_n^C\right) & I_1^2\left(t_n^C\right) & \cdots & I_{n_{pd}}^2\left(t_n^C\right) \end{pmatrix}$$

can be determined as follows:

For the elements $I_k^{1,m}\left(t_j^C\right),\ j = 1, ..., n,\ k = 1, ..., n_B$, the following relations hold:

$$I_k^{1,m}\left(t_j^C\right) = \int_\Gamma K\left(t_j^C, s\right) B_{m,k}(s)\, ds = \int_{arc\left[t_k^B, t_{k+m}^B\right]} K\left(t_j^C, s\right) B_{m,k}(s)\, ds =$$

$$= \sum_{r=1}^{m} \int_{arc\left[t_{k+r-1}^B, t_{k+r}^B\right]} K\left(t_j^C, s\right) p_k^{(r)}(s)\, ds = \sum_{r=1}^{m} \int_{\theta_{k+r-1}^B}^{\theta_{k+r}^B} g_{j,k}^r(\theta)\, d\theta,$$

where $g_{j,k}^r := K\left(t_j^C, \psi\left(e^{i\theta}\right)\right) p_k^{(r)}\left(\psi\left(e^{i\theta}\right)\right) \psi'\left(e^{i\theta}\right) ie^{i\theta}$, and $p_k^{(r)}(s)$ represents the components of the B-spline function $B_{m,k}(s)$ of the corresponding order m. For example, for $m = 4$, we have:

$$p_k^{(1)}(s) = \frac{4\left(s - t_k^B\right)^3}{\left(t_{k+4}^B - t_k^B\right)\left(t_{k+3}^B - t_k^B\right)\left(t_{k+2}^B - t_k^B\right)\left(t_{k+1}^B - t_k^B\right)},$$

$$p_k^{(2)}(s) = 4\left(I_1 + I_2\right),$$

where

$$I_1 := \frac{s - t_k^B}{t_{k+4}^B - t_k^B}\left(I_1^1 + I_1^2\right),$$

$$I_1^1 = \frac{\left(s - t_k^B\right)\left(t_{k+2}^B - s\right)}{\left(t_{k+3}^B - t_k^B\right)\left(t_{k+2}^B - t_k^B\right)\left(t_{k+2}^B - t_{k+1}^B\right)},$$

$$I_1^2 = \frac{\left(s - t_{k+1}^B\right)\left(t_{k+3}^B - s\right)}{\left(t_{k+3}^B - t_k^B\right)\left(t_{k+3}^B - t_{k+1}^B\right)\left(t_{k+2}^B - t_{k+1}^B\right)},$$

$$I_2 := \frac{\left(t_{k+4}^B - s\right)\left(s - t_{k+1}^B\right)^2}{\left(t_{k+4}^B - t_k^B\right)\left(t_{k+4}^B - t_{k+1}^B\right)\left(t_{k+3}^B - t_{k+1}^B\right)\left(t_{k+2}^B - t_{k+1}^B\right)},$$

$$p_k^{(3)}(s) = 4\left(I_3 + I_4\right),$$

and

$$I_3 := \frac{\left(t_{k+3}^B - s\right)^2\left(s - t_k^B\right)}{\left(t_{k+4}^B - t_k^B\right)\left(t_{k+3}^B - t_k^B\right)\left(t_{k+3}^B - t_{k+1}^B\right)\left(t_{k+3}^B - t_{k+2}^B\right)},$$

$$I_4 := \frac{t_{k+4}^B - s}{t_{k+4}^B - t_k^B}\left(I_4^1 + I_4^2\right),$$

$$I_4^1 = \frac{\left(s - t_{k+1}^B\right)\left(t_{k+3}^B - s\right)}{\left(t_{k+4}^B - t_{k+1}^B\right)\left(t_{k+3}^B - t_{k+1}^B\right)\left(t_{k+3}^B - t_{k+2}^B\right)}$$

$$I_4^2 = \frac{\left(s - t_{k+2}^B\right)\left(t_{k+4}^B - s\right)}{\left(t_{k+4}^B - t_{k+1}^B\right)\left(t_{k+4}^B - t_{k+2}^B\right)\left(t_{k+3}^B - t_{k+2}^B\right)},$$

$$p_k^{(4)}(s) = \frac{4\left(t_{k+4}^B - s\right)^3}{\left(t_{k+4}^B - t_k^B\right)\left(t_{k+4}^B - t_{k+1}^B\right)\left(t_{k+4}^B - t_{k+2}^B\right)\left(t_{k+4}^B - t_{k+3}^B\right)}.$$

The integrals $\int_{\theta_{k+r-1}^B}^{\theta_{k+r}^B} g_{j,k}^r(\theta)\,d\theta$ are approximated using the generalized trapezoidal rule, which is also applicable to functions with complex values [7]:

$$I := \int_{\theta_{in}}^{\theta_f} g(\theta)\,d\theta \approx I_N := h\left(0.5\left(g(\theta_{in}) + g(\theta_f)\right) + \sum_{j=1}^{N-1} g(\theta_{in} + jh)\right), \qquad (7)$$

$$h := \left(\theta_f - \theta_{in}\right)/N.$$

As $N \to \infty$, it has been shown in [7] that $I_N \to I$ at the rate of a geometric progression.

For the elements $I_r^2\left(t_j^C\right),\ j = 1, ..., n,\ r = 1, ..., n_{pd}$, the relations

$$I_r^2\left(t_j^C\right) = \int_\Gamma K\left(t_j^C, s\right) H\left(s - t_r^d\right) ds = \int_{arc\left[t_r^d, \psi(1)\right]} K\left(t_j^C, s\right) ds = \int_{\theta_r^d}^{2\pi} q_j(\theta)\,d\theta,$$

hold true, where $q_j(\theta) := K\left(t_j^C, \psi\left(e^{i\theta}\right)\right)\psi'\left(e^{i\theta}\right) ie^{i\theta}$. Similarly, the integrals $\int_{\theta_r^d}^{2\pi} q_j(\theta)\,d\theta$ will be approximated using the generalized trapezoidal rule (7).

It should be noted that the functions $g_{j,k}^r(\theta)$ and $q_j(\theta)$ do not depend on the function $f(t)$. Therefore, they can be evaluated at any point $\theta \in [0, 2\pi]$, allowing for the approximation of the integrals $I_k^{1,m}\left(t_j^C\right)$ and $I_r^2\left(t_j^C\right)$, respectively.

3 About the convergence of the method and a numerical example

After determining the solution $\alpha_k,\ k = 1, ..., n_B,\ \beta_r,\ r = 1, ..., n_{pd}$, of the system (5), we construct the approximation (3) of the function $\varphi(t)$ and calculate its values at the points $t \in \Gamma$. The convergence of the approximation sequence $\varphi^H_{n_B}$, defined by (3), to the function $\varphi \in PC(\Gamma)$ as $n_B \to \infty$ has been established in [3].

We exhibit the convergence of the proposed method through a numerical example. Consider the Riemann function $z = \psi(w)$ that performs the conformal transformation of the set $\{w \in \mathbb{C} : |w| > 1\}$ on the domain Ω^- from the outside of the contour Γ as $\psi(w) = w + 1/\left(3w^3\right)$. Thus, $\psi(w)$ transforms the unit circle Γ_0 onto the astroid Γ (see Figure 2).

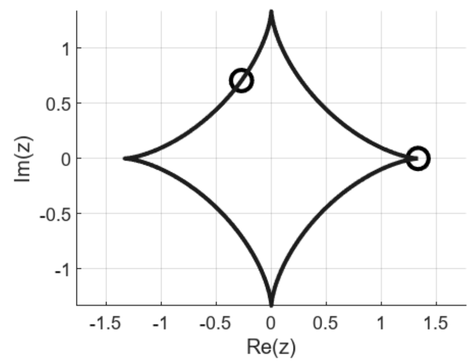

Figure 2: The contour and discontinuity points

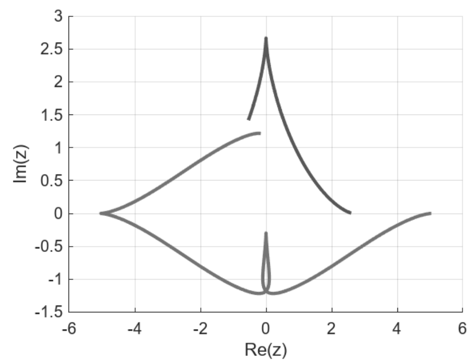

Figure 3: Graph of the solution

For testing purposes, we consider in the integral equation (1) the kernel function $K(t, s) = t^2 + s^2$, the constant $\lambda = 0.5$, and the right-hand side $f(t)$ given analytically on Γ:

$$f(t) = \begin{cases} 2t - \lambda u \ \text{ if } \ \theta \in \left(0, \theta^d_1\right] \\ t^3 + 2t - \lambda u \ \text{ if } \ \theta \in \left(\theta^d_1, \theta^d_2\right] \\ t^3 + 2t - \lambda u \ \text{ if } \ \theta = 0 \end{cases} .$$

We have $\theta^d_1 = 0.7\pi,\ \ \theta^d_2 = 2\pi$ and $u := (0.78148 - 0.081271i)\, t^2 + 0.91818 + 0.025237i$. The function f has $n_{pd} = 2$ jump discontinuity points on Γ, $t^d_j = \psi\left(e^{i\theta^d_j}\right),\ j = 1, 2$ (see Figure 2 and Figure 3).

Likewise, the exact solution $\varphi \in PC(\Gamma)$ for the given test problem is known to be

$$\varphi(t) = \begin{cases} 2t \ \text{ if } \ \theta \in \left(0, \theta^d_1\right] \\ t^3 + 2t \ \text{ if } \ \theta \in \left(\theta^d_1, \theta^d_2\right] \\ t^3 + 2t \ \text{ if } \ \theta = 0 \end{cases} .$$

It has two discontinuity points, the same as the right-hand side f.

The approximation algorithm for the solution of equation (1) takes as initial data the values f_j of the function f at the points

$$t_j = \psi\left(e^{i\theta_j}\right) \in \Gamma,\ \theta_j = 2\pi\,(j-1)\,/n_B,\ n_B \in \mathbb{N},\ k = 1, ..., n_B.$$

The coefficients of the approximation for the solution of equation (1) are determined as a linear combination according to (3), where B-spline functions of order $m = 4$ are considered. The number of points where the value of the function f is given on Γ is $n_B = 320$. Consequently, the solution to the system of equations $B\bar{x} = \bar{f}$ is determined, where $\bar{x} = \left(\alpha_1, ..., \alpha_{n_B}, \beta_1, ..., \beta_{n_{pd}}\right)^T$, $\bar{f} = \left(f\left(t_1^c\right), ..., f\left(t_n^c\right)\right)^T$, $n = n_B + n_{pd}$, and the matrix B has the form specified in (6).

The integrals $I_k^{1,m}\left(t_j^C\right)$ and $I_r^2\left(t_j^C\right)$, which define the components of the matrix B, are approximated using the generalized trapezoidal rule (7), with the parameter $N = 200$.

For values $n_B = 160$ and $n_B = 320$ in Figure 4 and Figure 5 the error obtained at the approximation of the solution φ by $\varphi_{n_B}^H$ is presented. It can be seen that the maximum error decreases significantly for $n_B = 320$.

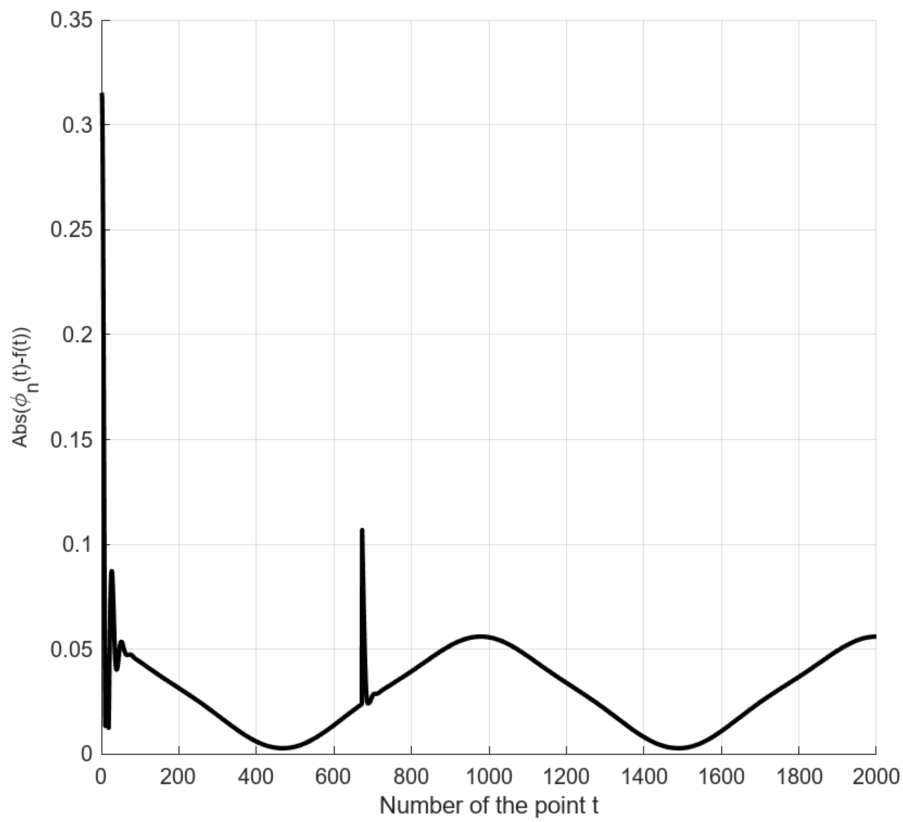

Figure 4: The approximation error for nB=160

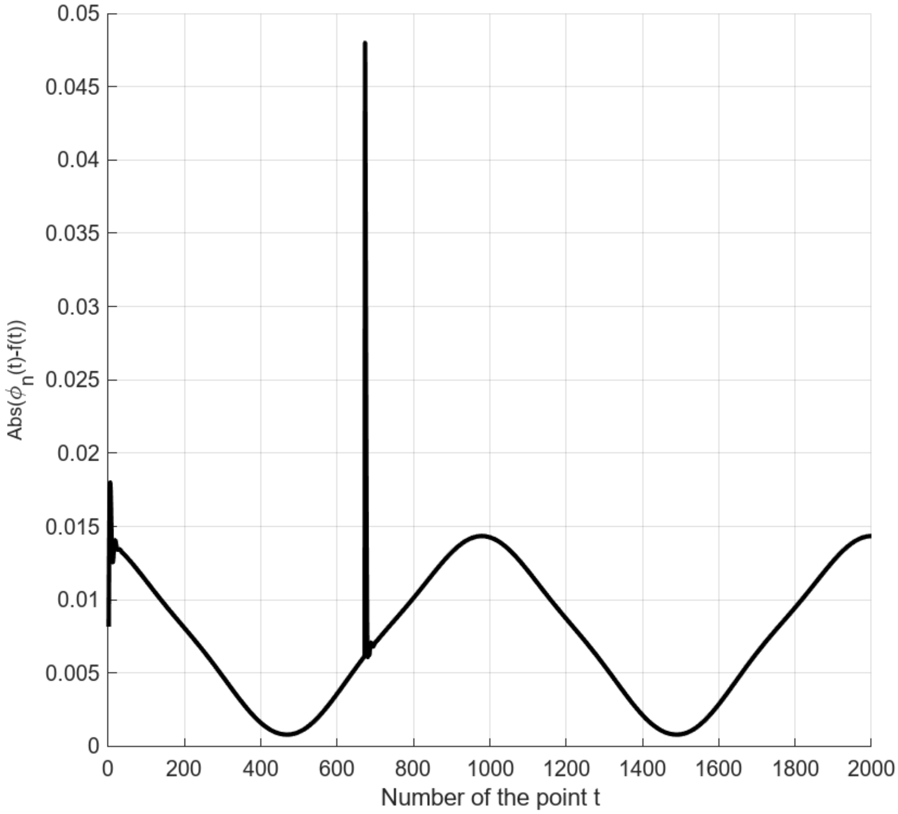

Figure 5: The approximation error for nB=320

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On recursive 1-differentiability of the quasigroup prolongations

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Abstract. The recursive differentiability of finite binary quasigroups is investigated. We consider the Bruck and Belousov constructions of prolongation of finite quasigroups and give necessary and sufficient conditions when such prolongations are recursively 1-differentiable.

Mathematics subject classification: 20N05.
Keywords and phrases: recursive derivative, recursively differentiable quasigroup, complete mapping, transversal, prolongations of quasigroups.

The first construction of a quasigroup prolongation was proposed by Bruck (see [1]) for the case of idempotent quasigroups in 1944. However, the notion of prolongation was introduced by Belousov (see [2]) in 1967. Constructions of prolongations of finite quasigroups have been given by Osborn (1961), Yamamoto (1961), Denes and Pasztor (1963), Belousov and Belyavskaya (1968), Belyavskaya (1969), Deriyenko and Dudek (2008, 2013) and others (see [7]).

Belousov considered a construction of prolongations based on complete mappings [2]. Recall that a complete mapping of a quasigroup $Q, \cdot)$ is a bijection $x \mapsto \theta(x)$ of Q onto Q such that $x \cdot \theta(x) = \theta_1(x)$ is also a bijective mapping of Q onto Q. The determination of all quasigroups, in particular groups, which possess a complete mapping remains at present an open problem [7]. In finite case, the complete mappings of quasigroups define transversals of their Cayley tables. A transversal of a latin square of order q is a set of q cells, taken by one from each row and each column, such that the elements in these cells are pairwise different.

Let $(Q, \cdot)$ be a finite quasigroup of order q, and let $\sigma : Q \mapsto Q$ be a complete mapping. Then $\{(x, \sigma(x)) | x \in Q\}$ is a transversal of the latin square given by the Cayley table of $(Q, \cdot)$. The prolongation $(Q', \circ)$ of $(Q, \cdot)$, where $Q' = Q \cup \{\xi\}$ and $\xi \notin Q$, considered by Belousov, is defined as follows:

$$x \circ y = \begin{cases} x \cdot y \text{ if } y \neq \sigma(x) \text{ and } x,\ y \in Q; \\ \xi \text{ if } y = \sigma(x) \text{ and } x,\ y \in Q; \\ x \cdot \sigma(x) \text{ if } y = \xi \text{ and } x \in Q; \\ \sigma^{-1}(y) \cdot y \text{ if } x = \xi \text{ and } y \in Q; \\ \xi \text{ if } x = y = \xi. \end{cases}$$

Analogously, we may construct prolongations of order $q + k$ if $(Q, \cdot)$ has k pairwise distinct transversals.

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In the present work we study the recursive differentiability of Bruck and Belousov prolongations, obtained by adding one element to a finite quasigroup.

The notions of recursive derivative and recursively r-differentiable k-quasigroup ($r \geq 0, k \geq 2$) have been introduced in [3] in connection with complete k-recursive codes.

Let Q be a finite set of q elements. Any nonempty subset C of Q^n is called an n-code (or a code of length n) over the alphabet Q. An n-code $C \subseteq Q^n$, where $|Q| = q$, with the minimum Hamming distance d, is called an $[n, k, d]_q$-code if $|C| = q^k$. It is known that the parameters of an $[n, k, d]_q$-code satisfy the inequality $d \leq n - k + 1$ [7]. An $[n, k, d]_q$-code with $d = n - k + 1$, i.e. which attains the Singleton bound, is called an MDS-code. At present it is an open problem to determine all values of the parameters q, n and d (for a fixed $k \geq 2$) such that there exist $[n, k, d]_q$-codes meeting the Singleton bound.

A code C of length n over an alphabet Q is called a complete k-recursive code, where $1 \leq k \leq n$, if there exists a mapping $f : Q^k \mapsto Q$ such that the components of every code word $u = (u_0, u_1, ..., u_{n-1}) \in C$ satisfy the conditions:

$$u_{i+k} = f(u_i, u_{i+1}, ..., u_{i+k-1}),$$

for every $i = 0, 1, ..., n - k$. So, if C is a complete k-recursive code of length n, over an alphabet Q, then there exist the mappings $f^{(0)}, f^{(1)}, ..., f^{(n-k-1)} : Q^k \mapsto Q$ such that $C = \{(x_1, ..., x_k, f^{(0)}(x_1^k), ..., f^{(n-k-1)}(x_1^k)) \mid x_1, ..., x_k \in Q\}$, where

$f^{(0)}(x_1^k) = f(x_1^k)$,
$f^{(1)}(x_1^k) = f(x_2, ..., x_k, f^{(0)}(x_1^k))$,
....
$f^{(t)}(x_1^k) = f(x_{t+1}, ..., x_k, f^{(0)}(x_1^k), ..., f^{(t-1)}(x_1^k))$, for $t < k$, and
$f^{(t)}(x_1^k) = f(f^{(t-k)}(x_1^k), ..., f^{(t-1)}(x_1^k))$, for $t \geq k$.

The mapping $f^{(t)}(x_1^k)$, where $t \geq 0$, is called the recursive derivative of order t of f. We say that a k-ary quasigroup (Q, f) is recursively s-differentiable if its recursive derivatives $f^{(1)}, ..., f^{(s)}$ are quasigroup operations. A complete k-recursive code $C = \{(x_1, ..., x_k, f^{(0)}(x_1^k), ..., f^{(n-k-1)}(x_1^k)) \mid x_1, ..., x_k \in Q\}$ is an MDS-code if and only if the system of k-recursive derivatives $\{f^{(0)}, ..., f^{(n-k-1)}\}$ is strongly orthogonal [3, 6]. As a corollary from this result we get that if the given above code C attains the Singleton bound then the k-ary operation f is recursively $(n-k-1)$-differentiable.

As orthogonal systems of binary quasigroups are strongly orthogonal, we obtain the following statement.

Theorem 1 [3] *A complete 2-recursive code of length n*

$$C = \{(x, y, f^{(0)}(x, y), ..., f^{(n-3)}(x, y)) \mid x, y \in Q\}$$

attains the Singleton bound if and only if (Q, f) is a recursively $(n-3)$-differentiable quasigroup. In this case, $\{f^{(0)}, ..., f^{(n-3)}\}$ is an orthogonal system of quasigroups.

It follows from Theorem 1 that:
1) a binary finite quasigroup (Q, f) is recursively r-differentiable if and only if the complete 2-recursive code

$$C = \{(x, y, f^{(0)}(x, y), ..., f^{(r)}(x, y)) \mid x, y \in Q\}$$

is an MDS-code;
2) the maximum order r of recursive differentiability of a finite binary quasigroup of order q satisfies the inequality $r \leq q - 2$ (see [5]).

Various methods of construction of binary recursively differentiable quasigroups are given in [3-6]. In particular, it is proved in [3] that, for every positive integer q, excepting 1, 2, 6, and possibly 14, 18, 24 and 42, there exist recursively 1-differentiable binary quasigroups of order q. Later, in 2009, it was shown that there exist recursively 1-differentiable quasigroups of order 42 (see [4]), but the question is still opened for 14, 18 and 24.

Another open problem is to determine the maximum order r of the recursive differentiability of a finite k-quasigroup. As it was mentioned above, in the binary case we have $r \leq q - 2$ and there exist recursively $(q-2)$-differentiable binary quasigroups of every primary order $q \geq 3$ [3]. Necessary and sufficient conditions when a binary finite abelian group is recursively r-differentiable, for $r \geq 1$, are given in [6]. A generalization of this result for a class of n-ary groups is considered in [5]. Also a table with maximum known values of r for binary finite quasigroups of order up to 200 is given in [5], where it is shown, in particular, that there exist finite recursively 1-differentiable n-quasigroups of every odd order $q \geq 3$, for every $n \geq 2$.

Our aim in the present paper is to find necessary and sufficient conditions when the prolongations of finite binary quasigroups, obtained using Bruck and Belousov constructions, are recursively 1-differentiable. Let $(Q, \cdot)$ be a finite quasigroup of order n and $Q = \{1, 2, ..., n\}$ such that the mapping $x \mapsto x \cdot x$ is a bijection. Then the main diagonal of the Cayley table of $(Q, \cdot)$ is a transversal, which entries are given by the mapping $\theta : Q \mapsto Q, \theta(x) = x \cdot x$. As it was mentioned above, Bruck considered such prolongations for idempotent quasigroups, i.e. in the case $\theta = \epsilon$ be the identical mapping on Q.

Following Bruck's idea, the operation of the prolongation $(Q', \circ)$ of a quasigroup $(Q, \cdot)$, where $Q = \{1, ..., n\}$ and $Q' = Q \cup \{\xi\}, \xi \notin Q$, is defined as follows:

$$x \circ y = \begin{cases} x \cdot y \text{ if } x \neq y \text{ and } x,\ y \in Q; \\ \xi \text{ if } x = y \text{ and } x \in Q; \\ \theta(x) \text{ if } y = \xi \text{ and } x \in Q; \\ \theta(y) \text{ if } x = \xi \text{ and } y \in Q; \\ \xi \text{ if } x = y = \xi. \end{cases} \tag{1}$$

So, the prolongation $(Q', \circ)$ is a quasigroup with the Cayley table:

$\circ$	1	...	n	ξ
1	ξ	...	...	$\theta(1)$
...	...	...	...	...
n	...	...	ξ	$\theta(n)$
ξ	$\theta(1)$	...	$\theta(n)$	ξ

Table 1

where $x \circ y = x \cdot y$, for every $x \neq y$ from Q.

Remark that not every transversal on the main diagonal gives 1-differentiable prolongations as it is shown in the following statement.

Proposition 1. *Let $(Q, \cdot)$ be a finite quasigroup such that the mapping $\theta : Q \mapsto Q$, $\theta(x) = x \cdot x$ is a bijection. If the prolongation $(Q', \circ)$, given by (1), where $Q' = Q \cup \{\xi\}$, $\xi \notin Q$, is a quasigroup, then $\theta(x) \neq x, \forall x \in Q$.*

Proof. Indeed, if there exists an element $a \in Q$ such that $a = \theta(a) = a \cdot a$, then using (1) we get: $a \overset{1}{\circ} a = a \cdot (a \cdot a) = a \cdot a = a$ and $\xi \overset{1}{\circ} a = a \cdot (\xi \cdot a) = a \cdot \theta(a) = a \cdot a = a$, so $(Q', \circ)$ can not be a quasigroup. $\square$

Lemma 1. *Let $(Q, \cdot)$ be a finite quasigroup of order n, $Q = \{1, ..., n\}$ and $Q' = Q \cup \{\xi\}$ where $\xi \notin Q$. If the mapping $\theta : Q \mapsto Q$, $\theta(x) = x \cdot x$ is a bijection and $\theta(x) \neq x$, $\forall x \in Q$, then the recursive derivative of order 1 of the operation " $\circ$ ", given in (1), is the following:*

$$x \overset{1}{\circ} y = \begin{cases} y \cdot (x \cdot y) \text{ if } y \neq x \cdot y,\ x \neq y,\ x, y \in Q; \\ \xi \text{ if } y = x \cdot y\ x \neq y,\ x, y \in Q; \\ \theta(y) \text{ if } x = y,\ x \in Q; \\ \theta^2(x) \text{ if } y = \xi,\ x \in Q; \\ y \cdot \theta(y) \text{ if } x = \xi,\ y \in Q; \\ \xi \text{ if } x = y = \xi. \end{cases} \tag{2}$$

Proof. Using (1) and the fact that $x \overset{1}{\circ} y = y \circ (x \circ y), \forall x, y \in Q'$, we have:

$$x \overset{1}{\circ} y = \begin{cases} y \circ (x \cdot y) \text{ if } x \neq y \text{ and } x, y \in Q; \\ y \circ \xi \text{ if } x = y,\ y \in Q; \\ \xi \circ \theta(x) \text{ if } y = \xi,\ x \in Q; \\ y \circ \theta(y) \text{ if } x = \xi,\ y \in Q; \\ \xi \text{ if } x = y = \xi. \end{cases}$$

Now, using (1) for " $\circ$ " in the previous formulas, we get:

$$x \overset{1}{\circ} y = \begin{cases} y \cdot (x \cdot y) \text{ if } y \neq x \cdot y,\ x \neq y \text{ and } x, y \in Q; \\ \xi \text{ if } y = x \cdot y,\ x \neq y \text{ and } x, y \in Q; \\ \theta(y) \text{ if } x = y,\ y \in Q; \\ \theta^2(x) \text{ if } y = \xi,\ x \in Q; \\ y \cdot \theta(y) \text{ if } x = \xi,\ y \neq \theta(y),\ y \in Q; \\ \xi \text{ if } x = \xi,\ y = \theta(y),\ y \in Q; \\ \xi \text{ if } x = y = \xi. \end{cases}$$

If the mapping $\theta : Q \mapsto Q$, $\theta(x) = x \cdot x$ is a bijection and $\theta(x) \neq x$, $\forall x \in Q$, then the prolongation $(Q', \circ)$ is a quasigroup and its recursive derivative $(\overset{1}{\circ})$ is defined as it is shown in (2). $\square$

Remark 1. According to Lemma 1, the Cayley table of the recursive derivative $(Q', \overset{1}{\circ})$ is the following:

$\overset{1}{\circ}$	...	x	...	y	...	ξ
...	...	...	...	...	...	...
x	...	$\theta(x)$	...	z	...	$\theta^2(x)$
...	...	...	...	...	...	...
ξ	...	$x \cdot \theta(x)$	...	...	...	ξ

Table 2

where

$$z = \begin{cases} y \cdot (x \cdot y) \text{ if } y \neq x \cdot y; \\ \xi \text{ if } y = x \cdot y. \end{cases}$$

Theorem 2. *Let $(Q, \cdot)$ be a finite quasigroup such that the mapping $\theta : Q \mapsto Q$, $\theta(x) = x \cdot x$ is a bijection and $\theta(x) \neq x, \forall x \in Q$. Then the prolongation $(Q', \circ)$ obtained using Bruck's construction, where $Q' = Q \cup \{\xi\}$, $\xi \notin Q$, is recursively 1-differentiable if and only if the following conditions are satisfied:*
1. $\{f_x \mid x \in Q\} = Q$, where $f_x \cdot x = x, \forall x \in Q$;
2. θ is a complete mapping of $(Q, \cdot)$;
3. for each $x \in Q, \{\theta(x), y \cdot (x \cdot y), \theta^2(x) \mid y \in Q,\ x \neq y,\ y \neq x \cdot y\} = Q$.

Proof. According to Proposition 1, the condition $\theta(x) \neq x, \forall x \in Q$, implies the fact that the prolongation $(Q', \circ)$ is a quasigroup, so the equation $x \overset{1}{\circ} a = b \Leftrightarrow a \circ (x \circ a) = b$ has a unique solution in $(Q', \overset{1}{\circ})$ and consequently, the rows in Table 2 are permutations of Q'. For $x, y \in Q$, the entry of the cell (x, y) is ξ if and only if $y = x \cdot y$, i.e. if and only if $x = f_y$ is the left local unit of y. Thus ξ will appear exactly once in each row and each column of Table 2 if and only if $\{f_y \mid y \in Q\} = Q$. The row of the element ξ in Table 2 is a permutation of Q' if and only if $x \mapsto x \cdot \theta(x)$ is a bijection on Q, i.e. if and only if θ is a complete mapping of $(Q, \cdot)$.

Finally, the row of $x \in Q$ is a permutation of Q' if and only if
$$\{\theta(x), y \cdot (x \cdot y), \theta^2(x) \mid x \neq y, y \neq x \cdot y, y \in Q\} = Q.$$
$\square$

Example 1. The prolongation of the quasigroup $(Q, \cdot)$, obtained using the transversal T={(1,1),(2,2),(3,3)}, is recursively 1-differentiable.

$\cdot$	1	2	3
1	**2**	1	3
2	1	**3**	2
3	3	2	**1**

$\circ$	1	2	3	ξ
1	ξ	1	3	2
2	1	ξ	2	3
3	3	2	ξ	1
ξ	2	3	1	ξ

$\overset{1}{\circ}$	1	2	3	ξ
1	2	1	ξ	3
2	ξ	3	2	1
3	3	ξ	1	2
ξ	1	2	3	ξ

As it was mentioned above, the Belousov's idea of prolongation uses an arbitrary transversal of the Cayley table, not necessarily one on the main diagonal.

Let $\{(x, \theta(x)) \mid x \in Q\}$, where $\theta \in S_Q$, be a transversal of a finite quasigroup $(Q, \cdot)$. Then the mapping $\theta' : Q \to Q, \theta'(x) = x \cdot \theta(x)$ is a bijection. Following the Bruck's idea, Belousov considered the prolongation $(Q', \circ)$, where $Q' = Q \cup \{\xi\}, \xi \notin Q$ and

$$x \circ y = \begin{cases} x \cdot y \text{ if } y \neq \theta(x) \text{ and } x,\ y \in Q; \\ \xi \text{ if } y = \theta(x) \text{ and } x,\ y \in Q; \\ \theta'(\theta^{-1}(y)) \text{ if } x = \xi \text{ and } y \in Q; \\ \theta'(x) \text{ if } y = \xi \text{ and } x \in Q; \\ \xi \text{ if } x = y = \xi. \end{cases} \tag{3}$$

Remark 2. If θ' is a bijection then $(Q', \circ)$ is a quasigroup with the following Cayley table:

$\circ$	...	$\theta(x)$	...	y	...	ξ
...	...	...	...	...	...	...
x	...	$\theta'(x)$	...	$x \cdot y$	...	$\theta'(x)$
...	...	...	...	...	...	...
ξ	...	...	...	$\theta'(\theta^{-1}(y))$	...	ξ

Table 3

Let $(Q, \cdot)$ be a finite quasigroup and $\theta \in S_Q$ such that $\theta' : Q \to Q, \theta'(x) = x \cdot \theta(x)$ is a bijection. Then the recursive derivative $(Q', \overset{1}{\circ})$ of the prolongation $(Q', \circ)$ given in (3) is the following:

$$x \overset{1}{\circ} y = \begin{cases} y \cdot (x \cdot y) \text{ if } y \neq \theta(x \cdot y),\ y \neq \theta(x) \text{ and } x,\ y \in Q; \\ \xi \text{ if } y = \theta(x \cdot y),\ y \neq \theta(x) \text{ and } x,\ y \in Q; \\ \theta'(\theta(x)) \text{ if } y = \theta(x) \text{ and } x, y \in Q; \\ y \cdot \theta'(\theta^{-1}(y)) \text{ if } y \neq \theta(\theta'(\theta^{-1}(y))),\ x = \xi \text{ and } y \in Q; \\ \xi \text{ if } y = \theta(\theta'(\theta^{-1}(y))),\ x = \xi \text{ and } y \in Q; \\ \theta'(\theta^{-1}(\theta'(x))) \text{ if } y = \xi \text{ and } x \in Q; \\ \xi \text{ if } x = y = \xi. \end{cases} \tag{4}$$

Proof. Indeed, (4) follows from (3), using the definition of the recursive derivative $x \overset{1}{\circ} y = y \circ (x \circ y), \forall x, y \in Q$. □

Remark 3. If $y = \theta(\theta'(\theta^{-1}(y)))$, where $y \in Q$, then $\xi \overset{1}{\circ} y = \xi = \xi \overset{1}{\circ} \xi$, so $(Q', \overset{1}{\circ})$ is not a quasigroup.

Now, using (4) and Remark 3, we get the following statement.

Lemma 2. *Let $(Q, \cdot)$ be a finite quasigroup, $\theta \in S_Q$ such that $\theta' : Q \mapsto Q$, $\theta'(x) = x \cdot \theta(x)$ is a bijection and $y \neq \theta(\theta'(\theta^{-1}(y))), \forall y \in Q$. Then the recursive derivative*

$(Q', \overset{1}{\circ})$ *of the Belousov's prolongation* $(Q', \circ)$ *is:*

$$x \overset{1}{\circ} y = \begin{cases} y \cdot (x \cdot y) \text{ if } y \neq \theta(x \cdot y),\ y \neq \theta(x) \text{ and } x,\ y \in Q; \\ \xi \text{ if } y = \theta(x \cdot y),\ y \neq \theta(x) \text{ and } x,\ y \in Q; \\ \theta'(\theta(x)) \text{ if } y = \theta(x) \text{ and } x, y \in Q; \\ y \cdot \theta'(\theta^{-1}(y)) \text{ if } x = \xi \text{ and } y \in Q; \\ \theta'(\theta^{-1}(\theta'(x))) \text{ if } y = \xi \text{ and } x \in Q; \\ \xi \textit{ if } x = y = \xi. \end{cases} \tag{5}$$

Proof. The proof follows from (4) and the condition $y \neq \theta(\theta'(\theta^{-1}(y))), \forall y \in Q$. □

Remark 4. The Cayley table of $(Q', \overset{1}{\circ})$, given in (5) is the following:

$\overset{1}{\circ}$	...	$\theta(x)$	...	y	...	ξ
...	...	...	...	...	...	...
x	...	$\theta'(\theta(x))$	...	w	...	$\theta'(\theta^{-1}(\theta'(x))$
...	...	...	...	...	...	...
ξ	...	...	...	$y \cdot \theta'(\theta^{-1}(y))$	...	ξ

Table 4

where

$$w = \begin{cases} y \cdot xy \text{ if } y \neq \theta(x \cdot y),\ y \neq \theta(x); \\ \xi \text{ if } y = \theta(x \cdot y),\ y \neq \theta(x). \end{cases}$$

Theorem 3. *Let* $(Q, \cdot)$ *be a finite quasigroup,* $\theta \in S_Q$ *such that the mapping* $\theta' : Q \mapsto Q$, $\theta'(x) = x \cdot \theta(x)$ *is a bijection and* $\theta^{-1}(y) \neq \theta'(\theta^{-1}(y)), \forall y \in Q$. *Then the Belousov's prolongation* $(Q', \circ)$ *is recursively 1-differentiable if and only if the following conditions hold:*
1. $\{\theta^{-1}(y)/y \mid y \in Q\} = Q$*;*
2. the mapping $y \mapsto y \cdot \theta'(\theta^{-1}(y))$ *is a bijection on* Q*;*
3. for each $x \in Q$, $\{\theta'(\theta(x)), y \cdot xy, \theta'(\theta^{-1}(\theta'(x)))| y \neq \theta(x \cdot y), y \neq \theta(x), y \in Q\} = Q$.

Proof. According to Belousov's construction, $(Q', \circ)$ is a quasigroup, so the equation $x \overset{1}{\circ} a = b \Leftrightarrow a \circ (x \circ a) = b$ has a unique solution in Q', for every $a, b \in Q'$. Thus the rows in the Cayley table (5) are permutations of Q'. The element ξ appears in a cell (x, y) with $x, y \in Q$ if $y = \theta(x \cdot y)$, $y \neq \theta(x)$, i.e. if $x = \theta^{-1}(y)/y$. If $(Q', \overset{1}{\circ})$ is a quasigroup, then $\{\theta^{-1}(y)/y \mid y \in Q\} = Q$.

According to Table 4, the row of ξ is a permutation of Q' if and only if the mapping $y \mapsto y \cdot \theta'(\theta^{-1}(y))$ is a bijection on Q.

Finally, the row of $x \in Q$ in Table 4, is a permutation of Q' if and only if the third condition is fulfilled. □

Example 2. The prolongation of the quasigroup $(Q, \cdot)$, obtained using the transversal T=$\{$(1,2),(2,1),(3,3)$\}$, is recursively 1-differentiable.

$\cdot$	1	2	3
1	2	**3**	1
2	**1**	2	3
3	3	1	**2**

$\circ$	1	2	3	ξ
1	2	ξ	1	3
2	ξ	2	3	1
3	3	1	ξ	2
ξ	1	3	2	ξ

$\overset{1}{\circ}$	1	2	3	ξ
1	ξ	1	3	2
2	3	2	ξ	1
3	1	ξ	2	3
ξ	2	3	1	ξ

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Zero-Order Markov Processes with Multiple Final Sequences of States

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Abstract. A zero-order Markov process with multiple final sequences of states represents a stochastic system with independent transitions that stops its evolution as soon as one of the given final sequences of states is reached. The transition time of the system is unitary and the transition probability depends only on the destination state. It is proved that the distribution of the evolution time is a homogeneous linear recurrent sequence and a polynomial algorithm to determine the initial state and the generating vector of this recurrence is developed. Using the generating function, the main probabilistic characteristics are determined.

Mathematics subject classification: 65C40, 60J22, 90C39, 90C40.
Keywords and phrases: zero-order Markov process, final sequence of states, evolution time, homogeneous linear recurrence, generating function.

1 Introduction and Problem Formulation

Let L be a discrete stochastic system with finite set of states V, $|V| = \omega$. At every discrete moment of time $t \in \mathbb{N}$ the state of the system is $v(t) \in V$. The system L starts its evolution from the state v with the probability $p^*(v)$, for all $v \in V$, where $\sum_{v \in V} p^*(v) = 1$.

Also, the transition from one state u to another state v is performed according to the same probability $p^*(v)$ that depends only on the destination state v, for every $u \in V$ and $v \in V$. Additionally, we assume that r different sequences of states $X^{(\ell)} = (x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_m^{(\ell)}) \in V^m, \ell = \overline{1, r}$, are given and the stochastic system stops transitions as soon as the states $x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_m^{(\ell)}$ are reached consecutively in given order for an arbitrary $\ell \in \{1, 2, \ldots, r\}$. The time T, when the system stops, is called evolution time of the stochastic system L with given final sequences of states $X = \{X^{(1)}, X^{(2)}, \ldots, X^{(r)}\}$.

The stochastic system L, described above, represents a zero-order Markov process with final sequences of states $X = \{X^{(1)}, X^{(2)}, \ldots, X^{(r)}\}$. For the particular case $r = 1$, several interpretations of these Markov processes were analyzed in [8] and [9]. Using these concepts, the zero-order Markov processes with final sequence of state continued to be deeply studied in [3], with several further generalizations for the games, compositions and optimization problems in [2], [4] and [6]. Also, the obtained results were extended for stochastic systems with final sequence of states

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and interdependent transitions in [1], [5] and [7]. Based on polynomial algorithms proposed in [3], the main probabilistic characteristics (expectation, variance, mean square deviation, n-order moments) of evolution time and game duration were efficiently determined.

Next, in this paper, the generalization of this problem for any $r \geq 1$ is considered. This generalized problem is a bit different than the parallel compositions, studied in [2], because the dynamics of the systems are performed in a mixed one and they are interdependent.

Our goal is to analyze the evolution time T of the stochastic system L. We prove that the distribution of the evolution time T is a homogeneous linear recurrent sequence, and a polynomial algorithm to determine the initial state and the generating vector of this recurrence is developed. Having the generating vector and the initial state of the recurrence, we can use the related algorithm from [3], which was mentioned above, for determining the main probabilistic characteristics of the evolution time.

2 Determining the Distribution of the Evolution Time

In this section we will determine the distribution law of the evolution time T. Initially, we consider the notations

$$\begin{aligned} &X_k^{(\ell)} = \{x_k^{(\ell)}\},\ \pi_k^{(\ell)} = p^*(x_k^{(\ell)}),\ w_k^{(\ell)} = \prod_{j=2}^{k} \pi_j^{(\ell)}, \\ &Y_k^{(\ell)} = (x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_k^{(\ell)}),\ Y_k = \{Y_k^{(1)}, Y_k^{(2)}, \ldots, Y_k^{(r)}\}, \end{aligned} \tag{1}$$

for each $k = \overline{1, m}$ and $\ell = \overline{1, r}$.

Let $a = (a_n)_{n=0}^{\infty}$ be the distribution of the evolution time T, i.e. $a_n = \mathbb{P}(T = n)$, $n = \overline{0, \infty}$. Since $T \geq m-1$, we have $a_n = 0$, $n = \overline{0, m-2}$. If $T = m-1$, then $\exists \ell \in \{1, 2, \ldots, r\}$ such that $v(j) = x_{j+1}^{(\ell)}$, $j = \overline{0, m-1}$, that implies

$$a_{m-1} = \mathbb{P}(T = m-1) = \sum_{\ell=1}^{r} \prod_{j=1}^{m} p^*(x_j^{(\ell)}) =$$

$$= \sum_{\ell=1}^{r} \left(\pi_1^{(\ell)} \pi_2^{(\ell)} \ldots \pi_m^{(\ell)} \right) = \sum_{\ell=1}^{r} \left(\pi_1^{(\ell)} w_m^{(\ell)} \right). \tag{2}$$

We consider $\forall n \in \mathbb{Z}$. Let be $S(V) = \{A \mid A \subseteq V\}$. Denote by $P_{\Phi}^{(\ell)}(n)$ the probability that $T = n$ and $v(j) \in \Phi_j$, $j = \overline{0, t-1}$, for all $\Phi = (\Phi_j)_{j=0}^{t-1} \in (S(V))^t$, $t \in \mathbb{N}$ and $\ell = \overline{1, r}$. We introduce the following functions on $\mathbb{Z}$, $k = \overline{0, m}$, $\ell = \overline{1, r}$:

$$\begin{aligned} \beta_k^{(\ell)}(n) &= P_{(X_1^{(\ell)}, X_2^{(\ell)}, \ldots, X_k^{(\ell)})}(n), \\ \gamma_k^{(\ell)}(n) &= P_{(X_2^{(\ell)}, X_3^{(\ell)}, \ldots, X_k^{(\ell)})}(n). \end{aligned} \tag{3}$$

For $\forall n \geq m$, we have

$$\beta_k^{(\ell)}(n) = P_{(X_1^{(\ell)},X_2^{(\ell)},\ldots,X_k^{(\ell)})}(n) =$$

$$= \pi_1^{(\ell)} P_{(X_2^{(\ell)},\ldots,X_k^{(\ell)})}(n-1) - \pi_1^{(\ell)} \sum_{j=1}^{r} u_{j,k}^{(\ell)} P_{(X_2^{(j)},\ldots,X_m^{(j)})}(n-1) =$$

$$= \pi_1^{(\ell)} \left(\gamma_k^{(\ell)}(n-1) - \sum_{j=1}^{r} u_{j,k}^{(\ell)} \gamma_m^{(j)}(n-1) \right),\ k = \overline{0,m}, \ell = \overline{1,r}, \tag{4}$$

where

$$u_{j,k}^{(\ell)} = \begin{cases} 1, & k = 0 \text{ or } Y_k^{(j)} = Y_k^{(\ell)} \\ 0, & k \neq 0 \text{ and } Y_k^{(j)} \neq Y_k^{(\ell)} \end{cases}. \tag{5}$$

We consider the sets

$$T_s^{(\ell)} = \{s+1\} \cup \{t \in \{2,3,\ldots,s\} \mid (x_t^{(\ell)}, x_{t+1}^{(\ell)}, \ldots, x_s^{(\ell)}) \in Y_{s+1-t}\},$$

for each $s = \overline{1,m}$ and $\ell = \overline{1,r}$. The minimal elements from these sets are

$$t_s^{(\ell)} = \min_{k \in T_s^{(\ell)}} k,\ s = \overline{1,m},\ \ell = \overline{1,r}. \tag{6}$$

The value $t_s^{(\ell)}$ represents the position in the sequence $(x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_s^{(\ell)})$ starting with which, if we overlap a final sequence of states $X^{(\tau_s^{(\ell)})} \in X$, the superposed elements are equal. Here by $\tau_s^{(\ell)}$ we denote the minimal index from the set $\{1,2,\ldots,r\}$ that satisfies given condition.

Next, for $s = \overline{1,m}$ and $\ell = \overline{1,r}$, we obtain

$$\gamma_s^{(\ell)}(n) = P_{(X_2^{(\ell)},X_3^{(\ell)},\ldots,X_s^{(\ell)})}(n) =$$

$$= \pi_2^{(\ell)} \pi_3^{(\ell)} \ldots \pi_{t_s^{(\ell)}-1}^{(\ell)} P_{(X_{t_s^{(\ell)}}, X_{t_s^{(\ell)}+1}, \ldots, X_s)}(n - t_s^{(\ell)} + 2) =$$

$$= w_{t_s^{(\ell)}-1}^{(\ell)} P_{(X_1^{(\tau_s^{(\ell)})}, X_2^{(\tau_s^{(\ell)})}, \ldots, X_{s+1-t_s^{(\ell)}}^{(\tau_s^{(\ell)})})}(n - t_s^{(\ell)} + 2) =$$

$$= w_{t_s^{(\ell)}-1}^{(\ell)} \beta_{s+1-t_s^{(\ell)}}^{(\tau_s^{(\ell)})}(n - t_s^{(\ell)} + 2). \tag{7}$$

Particularly, for $s = 0$, we have

$$\gamma_0^{(\ell)}(n) = a_n = \gamma_1^{(\ell)}(n) = w_1^{(\ell)} \beta_0^{(\tau_1^{(\ell)})}(n) = \beta_0^{(\ell)}(n),$$

which implies

$$\beta_0^{(\ell)}(n) = \pi_1^{(\ell)} \left(\gamma_0^{(\ell)}(n-1) - \sum_{j=1}^{r} u_{j,0}^{(\ell)} \gamma_m^{(j)}(n-1) \right)$$

$$= \pi_1^{(\ell)} \left(\beta_0^{(\ell)}(n-1) - \sum_{j=1}^{r} u_{j,0}^{(\ell)} w_{t_m^{(j)}-1}^{(j)} \beta_{m+1-t_m^{(j)}}^{(\tau_m^{(j)})}(n - t_m^{(j)} + 1) \right) \tag{8}$$

and, for $k = \overline{1,m}$,

$$\beta_k^{(\ell)}(n) = \pi_1^{(\ell)} \left(\gamma_k^{(\ell)}(n-1) - \sum_{j=1}^{r} u_{j,k}^{(\ell)} \gamma_m^{(j)}(n-1) \right) =$$

$$= \pi_1^{(\ell)} \left(w_{t_k^{(\ell)}-1}^{(\ell)} \beta_{k+1-t_k^{(\ell)}}^{(\tau_k^{(\ell)})}(n - t_k^{(\ell)} + 1) - \right.$$

$$\left. - \sum_{j=1}^{r} u_{j,k}^{(\ell)} w_{t_m^{(j)}-1}^{(j)} \beta_{m+1-t_m^{(j)}}^{(\tau_m^{(j)})}(n - t_m^{(j)} + 1) \right). \tag{9}$$

Since $2 \le t_s^{(\ell)} \le s+1 \le m+1,\ s = \overline{1,m},\ \ell = \overline{1,r}$, there exist some real coefficients $v_{jks\ell}^{(i)}$, $k, j, s = \overline{0, m-1}$, $i, \ell = \overline{1,r}$, such that

$$\beta_k^{(\ell)}(n) = \sum_{i=1}^{r} \sum_{j=0}^{m-1} \sum_{s=0}^{m-1} v_{jks\ell}^{(i)}\ \beta_s^{(i)}(n-1-j),\ k = \overline{0,m-1},\ \ell = \overline{1,r},\ \forall n \ge m.$$

So, we have

$$\beta_k(n) = \sum_{j=0}^{m-1} \sum_{s=0}^{m-1} V_{jks}\ \beta_s(n-1-j),\ k = \overline{0,m-1},\ \forall n \ge m,$$

where $V_{jks} = (v_{jks\ell}^{(i)})_{\ell,\ i=\overline{1,r}}$, $\beta_k(n) = (\beta_k^{(\ell)}(n))_{\ell=\overline{1,r}}$, $k, j, s = \overline{0, m-1}$. This recurrence relation can be written in the form

$$\beta(n) = \sum_{j=0}^{m-1} V_j\ \beta(n-1-j),\ \forall n \ge m,$$

where $V_j = (V_{jks})_{k,s=\overline{0,m-1}}$ and $\beta(n) = ((\beta_k(n))_{k=0}^{m-1})^T$, $j = \overline{0,m-1}$, $\forall n \in \mathbb{Z}$. From this relation, we obtain that $\beta = (\beta(n))_{n=0}^{\infty} \in Rol^*[\mathcal{M}_m(\mathcal{M}_r(\mathbb{R}))][m]$ with generating vector $V = (V_j)_{j=0}^{m-1} \in G^*[\mathcal{M}_m(\mathcal{M}_r(\mathbb{R}))][m](\beta)$. Using the results from [1], we have $\beta \in Rol^*[\mathbb{R}][m^2 r]$, which implies that also

$$(\beta_k^{(\ell)}(n))_{n=0}^{\infty} \in Rol^*[\mathbb{R}][m^2 r],\ k = \overline{0,m-1},\ \ell = \overline{1,r},$$

with the same generating vector. Since

$$a_n = \beta_0^{(1)}(n),\ \forall n \ge 0, \tag{10}$$

we have

$$a = (a_n)_{n=0}^{\infty} \in Rol^*[\mathbb{R}][m^2 r].$$

Next, we will use only the relation $a \in Rol^*[\mathbb{C}][m^2r]$, the minimal generating vector being determined using the minimization method based on the matrix rank, described in [3]. So, according to this method, we have that the minimal generating vector $q = (q_0, q_1, \dots, q_{R-1}) \in G^*[\mathbb{C}][R](a)$ is obtained from the unique solution $x = (q_{R-1}, q_{R-2}, \dots, q_0)$ of the system

$$A_R^{[a]} x^T = (f_R^{[a]})^T, \tag{11}$$

where

$$f_R^{[a]} = (a_R, a_{R+1}, \dots, a_{2R-1}),\ A_n^{[a]} = (a_{i+j})_{i,j=\overline{0,n-1}},\ \forall n \in \mathbb{N}^* \tag{12}$$

and R is the rank of the matrix $A_{m^2r}^{[a]}$.

In order to apply this minimization method, we need to have only the values a_k, $k = \overline{0, 2m^2r-1}$. These values can be determined using the recurrences (8) and (9) and the relations (1), (2), (5), (6) and (10).

3 Describing the developed algorithm

In previous section we theoretically grounded the following algorithm for determining the main probabilistic characteristics of the evolution time T: the distribution $(\mathbb{P}(T = n))_{n=0}^{\infty}$, the expectation $\mathbb{E}(T)$, the variance $\mathbb{V}(T)$, the mean square deviation $\sigma(T)$ and the k-order moments $\nu_k(T)$, $k = 1, 2, \dots$.

Algorithm 1.

Input: $X^{(\ell)} = (x_1^{(\ell)}, x_2^{(\ell)}, \dots, x_m^{(\ell)}) \in V^m$, $\pi_k^{(\ell)}$, $k = \overline{1,m}$, $\ell = \overline{1,r}$;
Output: $\mathbb{E}(T)$, $\mathbb{V}(T)$, $\sigma(T)$, $\nu_k(T)$, $k = \overline{1,t}$, $t \geq 2$.

1. Determine the values a_k, $k = \overline{0, 2m^2r-1}$, using the recurrences (8) and (9) and the relations (1), (2), (5), (6) and (10);

2. Find the minimal generating vector $q = (q_0, q_1, \dots, q_{R-1}) \in G^*[\mathbb{R}][R](a)$ by solving the system (11), taking into account the relation (12);

3. Consider the distribution $a = (a_n)_{n=0}^{\infty} = (\mathbb{P}(T = n))_{n=0}^{\infty}$ of the evolution time T as a homogeneous linear recurrence with the initial state $I_R^{[a]} = (a_n)_{n=0}^{R-1}$ and the minimal generating vector $q = (q_k)_{k=0}^{R-1}$, determined at the steps 1 and 2;

4. Determine the expectation $\mathbb{E}(T)$, the variance $\mathbb{V}(T)$, the mean square deviation $\sigma(T)$ and the k-order moments $\nu_k(T)$, $k = \overline{1,t}$, of the evolution time T by using the corresponding algorithm from [3].

4 Conclusions

In this paper the zero-order Markov processes with multiple final sequences of states were studied and the evolution time of these stochastic systems was analyzed.

It was proved that the evolution time is a discrete random variable with homogeneous linear recurrent distribution. Based on this fact, the generating function is applied for determining the main probabilistic characteristics of the evolution time. The developed algorithm has polynomial complexity.

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