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Lacunary Ideal Convergence in Probabilistic Normed Space

Bipan Hazarika, Ayhan Esi

Abstract. The aim of this paper is to study the notion of lacunary I-convergence in probabilistic normed spaces as a variant of the notion of ideal convergence. Also lacunary I-limit points and lacunary I-cluster points have been defined and the relation between them has been established. Furthermore, lacunary Cauchy and lacunary I-Cauchy sequences are introduced and studied. Finally, we provided example which shows that our method of convergence in probabilistic normed spaces is more general.

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1 Introduction

Steinhaus [45] and Fast [13] independently introduced the notion of statistical convergence for sequences of real numbers. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on it was further investigated from various points of view. For example, statistical convergence has been investigated in summability theory by (Connor [7], Fridy [15], Šalát [40]), number theory and mathematical analysis by (Buck [1], Mitrinović et al. [37]), topological groups (Çakalli [2, 3]), topological spaces (Di Maio and Kočinac [34]), function spaces (Caserta and Kočinac [5]), measure theory (Cheng et al. [6], Connor and Swardson [8], Miller [36]). Fridy and Orhan [16] introduced the concept of lacunary statistical convergence. Some work on lacunary statistical convergence can be found in [2, 17, 20, 33].

Kostyrko, et al. [28] introduced the notion of I-convergence as a generalization of statistical convergence which is based on the structure of an admissible ideal I of subset of natural numbers $\mathbb{N}$. Kostyrko et al. [29] gave some of basic properties of I-convergence and dealt with extremal I-limit points. Further details on ideal convergence can be found in [4, 11, 12, 21–25, 32, 41, 46], and many others. The notion of lacunary ideal convergence of real sequences was introduced in [47, 48], and Hazarika [18, 19] introduced the lacunary ideal convergent sequences of fuzzy real numbers and studied some properties. Debnath [10] introduced the notion of lacunary ideal convergence in intuitionistic fuzzy normed linear spaces. Recently, Yamanci and Gürdal [49] introduced the notion of lacunary ideal convergence in random n-normed space.

A family I of subsets of $\mathbb{N}$, positive integers, i.e. $I \subset 2^{\mathbb{N}}$, is an ideal on $\mathbb{N}$ if and only if

(i) $\phi \in I$,

(ii) $A \cup B \in I$ for each $A, B \in I$,

(iii) each subset of an element of I is an element of I.

A non-empty family of sets $F \subset 2^{\mathbb{N}}$ is a filter on $\mathbb{N}$ if and only if

(a) $\phi \notin F$,

(b) $A \cap B \in F$ for each $A, B \in F$,

(c) any superset of an element of F is in F.

An ideal I is called *non-trivial* if $I \neq \phi$ and $\mathbb{N} \notin I$. Clearly I is a non-trivial ideal if and only if $F = F(I) = \{\mathbb{N} - A : A \in I\}$ is a filter in $\mathbb{N}$, called the filter associated with the ideal I.

A non-trivial ideal I is called *admissible* if and only if $\{\{n\} : n \in \mathbb{N}\} \subset I$. A non-trivial ideal I is *maximal* if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset.

Recall that a sequence $x = (x_k)$ of points in $\mathbb{R}$ is said to be I-convergent to a real number ℓ if $\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\} \in I$ for every $\varepsilon > 0$ [28]. In this case we write $I - \lim x_k = \ell$.

By a lacunary sequence $\theta = (k_r)$, where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $J_r = (k_{r-1}, k_r]$ and we let $h_r = k_r - k_{r-1}$. The space of lacunary strongly convergent sequences $\mathcal{N}_\theta$ was defined by Freedman et al. [14] as follows:

$$\mathcal{N}_\theta = \left\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k \in J_r} |x_k - L| = 0, \text{ for some } L \right\}.$$

Menger [35] proposed the probabilistic concept of the distance by replacing the number $d(p, q)$ as the distance between points p, q by a probability distribution function $F_{p,q}(x)$. He interpreted $F_{p,q}(x)$ as the probability that the distance between p and q is less than x. This led to the development of the area now called probabilistic metric spaces. This is Šerstnev [44] who first used this idea of Menger to introduce the concept of a PN space. For an extensive view on this subject, we refer to [9, 26, 31, 42, 43]. Subsequently, Mursaleen and Mohiuddine [38] and Rahmat[39] studied the ideal convergence in probabilistic normed spaces and V. Kumar and K. Kumar [30] studied I-Cauchy and I^*-Cauchy sequences in probabilistic normed spaces.

The notion of statistical convergence depends on the density (asymptotic or natural) of subsets of $\mathbb{N}$. A subset E of $\mathbb{N}$ is said to have natural density $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n}|\{k \le n : k \in E\}| \text{ exists.}$$

Definition 1. A sequence $x = (x_k)$ is said to be *statistically convergent* to ℓ if for every $\varepsilon > 0$

$$\delta(\{k \in \mathbb{N} : |x_k - \ell| \ge \varepsilon\}) = 0.$$

In this case, we write $S - \lim x = \ell$ or $x_k \to \ell(S)$ and S denotes the set of all statistically convergent sequences.

Definition 2. ([47,48]) Let $I \subset 2^{\mathbb{N}}$ be a non-trivial ideal. A real sequence $x = (x_k)$ is said to be *lacunary I-convergent* or *I_θ-convergent* to $L \in \mathbb{R}$ if, for every $\varepsilon > 0$ the set

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} |x_k - L| \ge \varepsilon \right\} \in I.$$

L is called the I_θ-limit of the sequence $x = (x_k)$, and we write $I_\theta - \lim x = L$.

In this paper we study the concept of lacunary I-convergence in probabilistic normed spaces. We also define lacunary I-limit points and lacunary I-cluster points in probabilistic normed space and prove some interesting results.

2 Basic definitions and notations

Now we recall some notations and basic definitions that we are going to use in this paper.

Definition 3. A *distribution function* (briefly a d.f.) F is a function from the extended reals $(-\infty, +\infty)$ into $[0,1]$ such that

(a) it is non-decreasing;

(b) it is left-continuous on $(-\infty, +\infty)$;

(c) $F(-\infty) = 0$ and $F(+\infty) = 1$.

The set of all d.f.'s will be denoted by Δ. The subset of Δ consisting of proper d.f's, namely of those elements F such that $\ell^+ F(-\infty) = F(-\infty) = 0$ and $\ell^- F(+\infty) = F(+\infty) = 1$ will be denoted by D. A *distance distribution function* (briefly, d.d.f.) is a d.f. F such that $F(0) = 0$. The set of all d.d.f.f's will be denoted by Δ^+, while $D^+ := D \cap \Delta^+$ will denote the set of proper d.d.f.'s.

Definition 4. A *triangular norm* or, briefly, a *t-norm* is a binary operation $T : [0,1] \times [0,1] \to [0,1]$ that satisfies the following conditions (see [27]):

(T1) T is commutative, i. e., $T(s,t) = T(t,s)$ for all s and t in $[0,1]$;

(T2) T is associative, i. e., $T(T(s,t),u) = T(s,T(t,u))$ for all s,t and u in $[0,1]$;

(T3) T is nondecreasing, i. e., $T(s,t) \leq T(s',t)$ for all $t \in [0,1]$ whenever $s \leq s'$;

(T4) T satisfies the boundary condition $T(1,t) = t$ for every $t \in [0,1]$.

T^* is a continuous t-*conorm*, namely, a continuous binary operation on $[0,1]$ that is related to a continuous t-norm through $T^*(s,t) = 1 - T(1-s, 1-t)$. Notice that by virtue of its commutativity, any t-norm T is nondecreasing in each place. Some examples of t-norms T and its t-conorms T^* are: $M(x,y) = \min\{x,y\}, \Pi(x,y) = x.y$ and $M^*(x,y) = \max\{x,y\}, \Pi^*(x,y) = x + y - x.y$.

Definition 5. A Menger PN space under T is a PN space (X, ν, τ, τ^*), denoted by (X, ν, T), in which $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$, for some continuous t-norm T and its t-conorm T^*.

Definition 6. Let (X, ν, T) be a PN space and $x = (x_k)$ be a sequence in X. We say that (x_k) is *convergent* to $\ell \in X$ with respect to the probabilistic norm ν if for each $\varepsilon > 0$ and $\alpha \in (0,1)$ there exists a positive integer m such that $\nu_{x_k - \ell}(\varepsilon) > 1 - \alpha$ whenever $k \geq m$. The element ℓ is called the limit of the sequence (x_k) and we shall write $\nu - \lim x_k = \ell$ or $x_k \xrightarrow{\nu} \ell$ as $k \to \infty$.

Definition 7. A sequence (x_k) in X is said to be *Cauchy* with respect to the probabilistic norm ν if for each $\varepsilon > 0$ and $\alpha \in (0,1)$ there exists a positive integer $M = M(\varepsilon, \alpha)$ such that $\nu_{x_k - x_p}(\varepsilon) > 1 - \alpha$ whenever $k, p \geq M$.

Definition 8. Let (X, ν, T) be a probabilistic normed space, and let $r \in (0,1)$ and $x \in X$. The set
$$B\left(x, r; t\right) = \{y \in X : \nu_{y-x}(t) > 1 - r\}$$
is called the open ball with center x and radius r with respect to t.

Throughout the paper, we denote I as an admissible ideal of subsets of $\mathbb{N}$ and $\theta = (k_r)$ as a fixed lacunary sequence, respectively, unless otherwise stated.

3 Main results

We now obtain our main results.

Definition 9. Let $I \subset 2^{\mathbb{N}}$ and (X, ν, T) be a PNS. A sequence $x = (x_k)$ in X is said to be I_θ-convergent to $L \in X$ with respect to the probabilistic norm ν if, for every $\varepsilon > 0$ and $\alpha \in (0,1)$ the set
$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) \leq 1 - \alpha \right\} \in I.$$
L is called the $I_\theta-$limit of the sequence $x = (x_k)$ in X, and we write $I_\theta^\nu - \lim x = L$.

Example 1. Let $(\mathbb{R}, |.|)$ denote the space of all real numbers with the usual norm, and let $T(a,b) = ab$ for all $a, b \in [0,1]$. For all $x \in \mathbb{R}$ and every $t > 0$, consider $\nu_x(t) = \frac{t}{t+|x|}$. Then $(\mathbb{R}\nu, T)$ is a PNS. If we take $I = \{A \subset \mathbb{N} : \ \delta(A) = 0\}$, where $\delta(A)$ denotes the natural density of the set A, then I is a non-trivial admissible ideal. Define a sequence $x = (x_k)$ as follows:

$$x_k = \begin{cases} 1 & \text{if } k = i^2, i \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Then for every $\alpha \in (0,1)$ and for any $\varepsilon > 0$, the set

$$K = \left\{ r \in \mathbb{N} : \ \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k}(\varepsilon) \leq 1 - \alpha \right\}$$

will be a finite set. Hence, $\delta(K) = 0$ and consequently $K \in I$, i.e., $I_\theta^\nu - \lim x = 0$.

Lemma 1. *Let (X, ν, T) be a PNS and $x = (x_k)$ be a sequence in X. Then, for every $\varepsilon > 0$ and $\alpha \in (0,1)$ the following statements are equivalent:*

(i) $I_\theta^\nu - \lim x = L$,

(ii) $\left\{ r \in \mathbb{N} : \ \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) \leq 1 - \alpha \ \right\} \in I$,

(iii) $\left\{ \ r \in \mathbb{N} : \ \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) > 1 - \alpha \ \right\} \in F(I)$,

(iii) $I_\theta - \lim \nu_{x_k - L}(\varepsilon) = 1$.

Theorem 1. *Let (X, ν, T) be a PNS and if a sequence $x = (x_k)$ in X is I_θ-convergent to $L \in X$ with respect to the probabilistic norm ν, then $I_\theta^\nu - \lim x$ is unique.*

Proof. Suppose that $I_\theta^\nu - \lim x = L_1$ and $I_\theta^\nu - \lim x = L_2$ $(L_1 \neq L_2)$. Given $\alpha > 0$ and choose $\beta \in (0,1)$ such that

$$T(1-\beta, 1-\beta) > 1 - \alpha. \tag{1}$$

Then for $\varepsilon > 0$, define the following sets:

$$K_1 = \left\{ r \in \mathbb{N} : \ \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L_1}\left(\frac{\varepsilon}{2}\right) \leq 1 - \beta \ \right\},$$

$$K_2 = \left\{ r \in \mathbb{N} : \ \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L_2}\left(\frac{\varepsilon}{2}\right) \leq 1 - \beta \ \right\}.$$

Since $I_\theta^\nu - \lim x = L_1$, using Lemma 1, we have $K_1 \in I$. Also, using $I_\theta^\nu - \lim x = L_2$, we get $K_2 \in I$. Now let

$$K = K_1 \cup K_2.$$

Then $K \in I$. This implies that its complement K^c is a non-empty set in $F(I)$. Now if $r \in K^c$, let us consider $r \in K_1^c \cap K_2^c$. Then we have

$$\frac{1}{h_r}\sum_{k\in J_r}\nu_{x_k-L_1}\left(\frac{\varepsilon}{2}\right) > 1-\beta \text{ and } \frac{1}{h_r}\sum_{k\in J_r}\nu_{x_k-L_2}\left(\frac{\varepsilon}{2}\right) > 1-\beta.$$

Now, we choose an $s \in \mathbb{N}$ such that

$$\nu_{x_s-L_1}\left(\frac{\varepsilon}{2}\right) > \frac{1}{h_r}\sum_{k\in J_r}\nu_{x_k-L_1}\left(\frac{\varepsilon}{2}\right) > 1-\beta$$

and

$$\nu_{x_s-L_2}\left(\frac{\varepsilon}{2}\right) > \frac{1}{h_r}\sum_{k\in J_r}\nu_{x_k-L_2}\left(\frac{\varepsilon}{2}\right) > 1-\beta$$

e. g., consider $\max\left\{\nu_{x_k-L_1}\left(\frac{\varepsilon}{2}\right), \nu_{x_k-L_2}\left(\frac{\varepsilon}{2}\right) : \ k \in J_r\right\}$ and choose that k as s for which the maximum occurs. Then from (1), we have

$$\nu_{L_1-L_2}(\varepsilon) \geq T\left(\nu_{x_s-L_1}\left(\frac{\varepsilon}{2}\right), \nu_{x_s-L_2}\left(\frac{\varepsilon}{2}\right)\right) > T\left(1-\beta, 1-\beta\right) > 1-\alpha.$$

Since $\alpha > 0$ is arbitrary, we have $\nu_{L_1-L_2}(\varepsilon) = 1$ for all $\varepsilon > 0$, which implies that $L_1 = L_2$. Therefore, we conclude that $I_\theta^\nu - \lim x$ is unique. □

Here, we introduce the notion of θ-convergence in a PNS and discuss some properties.

Definition 10. Let (X, ν, T) be a PNS. A sequence $x = (x_k)$ in X is θ-convergent to $L \in X$ with respect to the probabilistic norm ν if, for $\alpha \in (0,1)$ and every $\varepsilon > 0$, there exists $r_o \in \mathbb{N}$ such that

$$\frac{1}{h_r}\sum_{k\in J_r}\nu_{x_k-L}(\varepsilon) > 1-\alpha$$

for all $r \geq r_o$. In this case, we write $\nu^\theta - \lim x = L$.

Theorem 2. *Let (X, ν, T) be a PNS and let $x = (x_k)$ in X. If $x = (x_k)$ is θ-convergent with respect to the probabilistic norm ν, then $\nu^\theta - \lim x$ is unique.*

Proof. Suppose that $\nu^\theta - \lim x = L_1$ and $\nu^\theta - \lim x = L_2 \left(L_1 \neq L_2\right)$. Given $\alpha \in (0,1)$ and choose $\beta \in (0,1)$ such that $T\left(1-\beta, 1-\beta\right) > 1-\alpha$. Then for any $\varepsilon > 0$, there exists $r_1 \in \mathbb{N}$ such that

$$\frac{1}{h_r}\sum_{k\in J_r}\nu_{x_k-L_1}\left(\varepsilon\right) > 1-\alpha$$

for all $r \geq r_1$. Also, there exists $r_2 \in \mathbb{N}$ such that

$$\frac{1}{h_r}\sum_{k\in J_r}\nu_{x_k-L_2}\left(\varepsilon\right) > 1-\alpha$$

for all $r \geq r_2$. Now, consider $r_o = \max\{r_1, r_2\}$. Then for $r \geq r_o$, we will get an $s \in \mathbb{N}$ such that

$$\nu_{x_s - L_1}\left(\frac{\varepsilon}{2}\right) > \frac{1}{h_r}\sum_{k \in J_r} \nu_{x_k - L_1}\left(\frac{\varepsilon}{2}\right) > 1 - \beta$$

and

$$\nu_{x_s - L_2}\left(\frac{\varepsilon}{2}\right) > \frac{1}{h_r}\sum_{k \in J_r} \nu_{x_k - L_2}\left(\frac{\varepsilon}{2}\right) > 1 - \beta.$$

Then, we have

$$\nu_{L_1 - L_2}(\varepsilon) \geq T\left(\nu_{x_s - L_1}\left(\frac{\varepsilon}{2}\right), \nu_{x_s - L_2}\left(\frac{\varepsilon}{2}\right)\right) > T(1-\beta, 1-\beta) > 1 - \alpha.$$

Since $\alpha > 0$ is arbitrary, we have $\nu_{L_1 - L_2}(\varepsilon) = 1$ for all $\varepsilon > 0$, which implies that $L_1 = L_2$. ☐

Theorem 3. *Let (X, ν, T) be a PNS and let $x = (x_k)$ in X. If $\nu^\theta - \lim x = L$, then $I_\theta^\nu - \lim x = L$.*

Proof. Let $\nu^\theta - \lim x = L$, then for every $\varepsilon > 0$ and given $\alpha \in (0,1)$, there exists $r_0 \in \mathbb{N}$ such that

$$\frac{1}{h_r}\sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) > 1 - \alpha$$

for all $r \geq r_0$. Therefore the set

$$B = \left\{ r \in \mathbb{N} : \ \frac{1}{h_r}\sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) \leq 1 - \alpha \right\} \subseteq \{1, 2, ..., n_0 - 1\}.$$

But, with I being admissible, we have $B \in I$. Hence $I_\theta^\nu - \lim x = L$. ☐

Theorem 4. *Let (X, ν, T) be a PNS and $x = (x_k), y = (y_k)$ be two sequence in X.*
(i) If $I_\theta^\nu - \lim x_k = L_1$ and $I_\theta^\nu - \lim y_k = L_2$, then $I_\theta^\nu - \lim(x_k \pm y_k) = L_1 \pm L_2$;
(ii) If $I_\theta^\nu - \lim x_k = L$ and a be a non-zero real number, then $I_\theta^\nu - \lim a x_k = aL$. If $a = 0$, then result is true only if I is admissible of N.

Proof. (i) We shall prove, if $I_\theta^\nu - \lim x_k = L_1$ and $I_\theta^\nu - \lim y_k = L_2$, then $I_\theta^\nu - \lim(x_k + y_k) = L_1 + L_2$, only. The proof of the other part follows similarly.

Take $\varepsilon > 0, \alpha \in (0,1)$ and choose $\beta \in (0,1)$ such that the condition (1) holds. If we define

$$A_1 = \left\{ r \in \mathbb{N} : \ \frac{1}{h_r}\sum_{k \in J_r} \nu_{x_k - L_1}\left(\frac{\varepsilon}{2}\right) \leq 1 - \beta \right\}$$

and

$$A_2 = \left\{ r \in \mathbb{N} : \ \frac{1}{h_r}\sum_{k \in J_r} \nu_{y_k - L_2}\left(\frac{\varepsilon}{2}\right) \leq 1 - \beta \right\},$$

then $A_1^c \cap A_2^c \in F(I)$. We claim that

$$A_1^c \cap A_2^c \subset \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{(x_k - L_1) + (y_k - L_2)}(\varepsilon) > 1 - \alpha \right\}.$$

Let $n \in A_1^c \cap A_2^c$. Now, using (1), we have

$$\frac{1}{h_r} \sum_{n \in J_r} \nu_{(x_n - L_1) + (y_n - L_2)}(\varepsilon) \geq T\left(\frac{1}{h_r} \sum_{n \in J_r} \nu_{x_n - L_1}\left(\frac{\varepsilon}{2}\right), \frac{1}{h_r} \sum_{n \in J_r} \nu_{y_n - L_2}\left(\frac{\varepsilon}{2}\right) \right)$$

$$> T(1-\beta, 1-\beta) > 1 - \alpha.$$

Hence

$$A_1^c \cap A_2^c \subset \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{(x_k - L_1) + (y_k - L_2)}(\varepsilon) > 1 - \alpha \right\}.$$

As $A_1^c \cap A_2^c \in F(I)$, so

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{(x_k - L_1) + (y_k - L_2)}(\varepsilon) \leq 1 - \alpha \right\} \in I.$$

Therefore $I_\theta^\nu - \lim(x_k + y_k) = L_1 + L_2$.

(ii) Suppose $a \neq 0$. Since $I_\theta^\nu - \lim x_k = L$, for each $\varepsilon > 0$ and $\alpha \in (0,1)$, the set

$$A(\varepsilon, \alpha) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) < 1 - \alpha \right\} \in F(I).$$

If $n \in A(\varepsilon, \alpha)$, then we have

$$\frac{1}{h_r} \sum_{k \in J_r} \nu_{ax_k - aL}(\varepsilon) = \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}\left(\frac{\varepsilon}{|a|}\right)$$

$$\geq T\left(\frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon), \nu_0\left(\frac{\varepsilon}{|a|} - \varepsilon\right) \right)$$

$$\geq T\left(\frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon), 1 \right) \geq \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) > 1 - \alpha.$$

Hence

$$A(\varepsilon, \alpha) \subset \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{ax_k - aL}(\varepsilon) > 1 - \alpha \right\}$$

and

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{ax_k - aL}(\varepsilon) > 1 - \alpha \right\} \in F(I).$$

It follows that

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{ax_k - aL}(\varepsilon) \leq 1 - \alpha \right\} \in I.$$

Hence $I_\theta^\nu - \lim ax_k = aL$.

Next suppose that $a = 0$. Then for each $\varepsilon > 0$ and $\alpha \in (0,1)$, we have

$$\frac{1}{h_r} \sum_{k \in J_r} \nu_{0x_k - 0L}(\varepsilon) = \frac{1}{h_r} \sum_{k \in J_r} \nu_0(\varepsilon) = 1 > 1 - \alpha,$$

it follows that $\nu^\theta - \lim x = \ell$. Hence from Theorem 3, $I_\theta^\nu - \lim x = \ell$. □

Theorem 5. *Let (X, ν, T) be a PNS and let $x = (x_k)$ in X. If $\nu^\theta - \lim x = L$, then there exists a subsequence (x_{m_k}) of $x = (x_k)$ such that $\nu - \lim x_{m_k} = L$.*

Proof. Let $\nu^\theta - \lim x = L$. Then, for every $\varepsilon > 0$ and given $\alpha \in (0,1)$, there exists $r_0 \in \mathbb{N}$ such that

$$\frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) > 1 - \alpha$$

for all $r \geq r_0$. Clearly, for each $r \geq r_0$, we can select an $m_k \in J_r$ such that

$$\nu_{x_{m_k} - L}(\varepsilon) > \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) > 1 - \alpha.$$

It follows that $\nu - \lim x_{m_k} = L$. □

Definition 11. Let (X, ν, T) be a PNS and let $x = (x_k)$ be a sequence in X. Then,

(1) An element $L \in X$ is said to be I_θ-limit point of $x = (x_k)$ if there is a set $M = \{m_1 < m_2 < ... < m_k < ...\} \subset \mathbb{N}$ such that the set $M^\imath = \{r \in \mathbb{N} : \ m_k \in J_r\} \notin I$ and $\nu^\theta - \lim x_{m_k} = L$.

(2) An element $L \in X$ is said to be I_θ-cluster point of $x = (x_k)$ if for every $\varepsilon > 0$ and $\alpha \in (0,1)$, we have

$$\left\{ r \in \mathbb{N} : \ \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) > 1 - \alpha \right\} \notin I.$$

Let $\Lambda_\nu^{I_\theta}(x)$ denote the set of all I_θ-limit points and $\Gamma_\nu^{I_\theta}(x)$ denote the set of all I_θ-cluster points in X, respectively.

Theorem 6. *Let (X,ν,T) be a PNS. For each sequence $x=(x_k)$ in X, we have $\Lambda_{\nu}^{I_\theta}(x)\subset\Gamma_{\nu}^{I_\theta}(x)$.*

Proof. Let $L\in\Lambda_{\nu}^{I_\theta}(x)$, then there exists a set $M\subset\mathbb{N}$ such that $M^\imath\notin I$, where M and $M^\imath$ are as in Definition 5, satisfies $\nu^\lambda-\lim x_{m_k}=L$. Thus, for every $\varepsilon>0$ and $\alpha\in(0,1)$, there exists $r_0\in\mathbb{N}$ such that

$$\frac{1}{h_r}\sum_{k\in J_r}\nu_{x_{m_k}-L}(\varepsilon)>1-\alpha$$

for all $r\geq r_0$. Therefore,

$$B=\left\{r\in\mathbb{N}:\ \frac{1}{h_r}\sum_{k\in J_r}\nu_{x_k-L}(\varepsilon)>1-\alpha\right\}\supseteq M^\imath\setminus\{m_1,m_2,...,m_{n_0}\}.$$

Now, with I being admissible, we must have $M^\imath\setminus\{m_1,m_2,...,m_{k_0}\}\notin I$ and as such $B\notin I$. Hence $L\in\Gamma_{\nu}^{I_\theta}(x)$. ∎

Theorem 7. *Let (X,ν,T) be a PNS. For each sequence $x=(x_k)$ in X, the set $\Gamma_{\nu}^{I_\theta}(x)$ is a closed set in X with respect to the usual topology induced by the probabilistic norm ν^θ.*

Proof. Let $y\in\overline{\Gamma_{\nu}^{I_\theta}(x)}$. Take $\varepsilon>0$ and $\alpha\in(0,1)$. Then there exists $L_0\in\Gamma_{\nu}^{I_\theta}(x)\cap B(y,\alpha,\varepsilon)$. Choose $\delta>0$ such that $B(L_0,\delta,\varepsilon)\subset B(y,\alpha,\varepsilon)$. We have

$$G=\left\{r\in\mathbb{N}:\ \frac{1}{h_r}\sum_{k\in J_r}\nu_{x_k-y}(\varepsilon)>1-\alpha\right\}$$

$$\supseteq\left\{r\in\mathbb{N}:\ \frac{1}{h_r}\sum_{k\in J_r}\nu_{x_k-L_0}(\varepsilon)>1-\delta\right\}=H.$$

Thus $H\notin I$ and so $G\notin I$. Hence $y\in\Gamma_{\nu}^{I_\theta}(x)$. ∎

Theorem 8. *Let (X,ν,T) be a PNS and let $x=(x_k)$ in X. Then the following statements are equivalent:*

(1) *L is an $I_\theta-$limit point of x,*

(2) *There exist two sequences y and z in X such that $x=y+z$ and $\nu^\theta-\lim y=L$ and $\left\{r\in\mathbb{N}:\ k\in J_r,\ z_k\neq\overline{\theta}\right\}\in I$, where $\overline{\theta}$ is the zero element of X.*

Proof. Suppose that (1) holds. Then there exist sets M and $M^\imath$ as in Definition 11 such that $M^\imath\notin I$ and $\nu^\theta-\lim x_{m_k}=L$. Define the sequences y and z as follows:

$$y_k=\begin{cases}x_k & \text{if } k\in J_r;\ r\in M^\imath,\\ L & \text{otherwise}\end{cases}$$

and

$$z_k = \begin{cases} \overline{\theta} & \text{if } k \in J_r;\, r \in M^\imath, \\ x_k - L & \text{otherwise.} \end{cases}$$

It sufficies to consider the case $k \in J_r$ such that $r \in \mathbb{N} \diagdown M^\imath$. Then for each $\alpha \in (0,1)$ and $\varepsilon > 0$, we have $\nu_{y_k - L}(\varepsilon) = 1 > 1 - \alpha$. Thus, in this case,

$$\frac{1}{h_r} \sum_{k \in J_r} \nu_{y_k - L}(\varepsilon) = 1 > 1 - \alpha.$$

Hence $\nu^\theta - \lim y = L$. Now $\{r \in \mathbb{N} :\ k \in J_r,\ z_k \neq \theta\} \subset \mathbb{N} \diagdown M^\imath$ and so $\{r \in \mathbb{N} :\ k \in J_r,\ z_k \neq \theta\} \in I$.

Now, suppose that (2) holds. Let $M^\imath = \{r \in \mathbb{N} :\ k \in J_r,\ z_k = \theta\}$. Then, clearly $M^\imath \in F(I)$ and so it is an infinite set. Construct the set $M = \{m_1 < m_2 < ... < m_k < ...\} \subset \mathbb{N}$ such that $m_k \in J_r$ and $z_{m_k} = \overline{\theta}$. Since $x_{m_k} = y_{m_k}$ and $\nu^\theta - \lim y = L$ we obtain $\nu^\theta - \lim x_{m_k} = L$. This completes the proof. □

Theorem 9. *Let (X, ν, T) be a PNS and $x = (x_k)$ be a sequence in X. Let I be an admissible ideal in N. If there is an I_θ^ν-convergent sequence $y = (y_k)$ in X such that $\{k \in N : y_k \neq x_k\} \in I$ then x is also I_θ^ν-convergent.*

Proof. Suppose that $\{k \in \mathbb{N} : y_k \neq x_k\} \in I$ and $I_\theta^\nu - \lim y = \ell$. Then for every $\alpha \in (0,1)$ and $\varepsilon > 0$, the set

$$\left\{ r \in \mathbb{N} :\ \frac{1}{h_r} \sum_{k \in J_r} \nu_{y_k - L}(\varepsilon) \leq 1 - \alpha \right\} \in I.$$

For every $0 < \alpha < 1$ and $\varepsilon > 0$, we have

$$\left\{ r \in \mathbb{N} :\ \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) \leq 1 - \alpha \right\} \tag{2}$$

$$\subseteq \{k \in \mathbb{N} : y_k \neq x_k\} \cup \left\{ r \in \mathbb{N} :\ \frac{1}{h_r} \sum_{k \in J_r} \nu_{y_k - L}(\varepsilon) \leq 1 - \alpha \right\}.$$

As the both sets of right-hand side of (2) are in I, therefore we have that

$$\left\{ r \in \mathbb{N} :\ \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) \leq 1 - \alpha \right\} \in I.$$

This completes the proof of the theorem. □

Definition 12. Let (X, ν, T) be a PNS. A sequence $x = (x_k)$ in X is said to be θ-Cauchy sequence with respect to the probabilistic norm ν if, for every $\varepsilon > 0$ and $\alpha \in (0,1)$, there exist $r_0, m \in \mathbb{N}$ satisfying

$$\frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - x_m}(\varepsilon) > 1 - \varepsilon$$

for all $r \geq r_0$.

Definition 13. Let I be an admissible ideal of $\mathbb{N}$. Let (X, ν, T) be a PNS. A sequence $x = (x_k)$ in X is said to be I_θ-Cauchy sequence with respect to the probabilistic norm ν if, for every $\varepsilon > 0$ and $\alpha \in (0,1)$, there exists $m \in \mathbb{N}$ satisfying

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - x_m}(\varepsilon) > 1 - \varepsilon \right\} \in F(I).$$

Definition 14. Let I be an admissible ideal of $\mathbb{N}$. Let (X, ν, T) be a PNS. A sequence $x = (x_k)$ in X is said to be I_θ^*-Cauchy sequence with respect to the probabilistic norm ν if there exists a set $M = \{m_1 < m_2 < ... < m_k < ...\} \subset \mathbb{N}$ such that the set $M^\imath = \{r \in \mathbb{N} : m_k \in J_r\} \in F(I)$ and the subsequence (x_{m_k}) of $x = (x_k)$ is a θ-Cauchy sequence with respect to the probabilistic norm ν.

The following theorem is an analogue of Theorem 3, so the proof is omitted.

Theorem 10. *Let I be an admissible ideal of N. Let (X, ν, T) be a PNS. If a sequence $x = (x_k)$ in X is θ-Cauchy sequence with respect to the probabilistic norm ν, then it is I_θ-Cauchy sequence with respect to the same norm.*

The proof of the following theorem is similar to that of Theorem 5.

Theorem 11. *Let (X, ν, T) be a PNS. If a sequence $x = (x_k)$ in X is θ-Cauchy sequence with respect to the probabilistic norm ν, then there is a subsequence of $x = (x_k)$ which is ordinary Cauchy sequence with respect to the same norm.*

The following theorem can be proved easily using similar techniques as in the proof of Theorem 6.

Theorem 12. *Let I be an admissible ideal of N. Let (X, ν, T) be a PNS. If a sequence $x = (x_k)$ in X is I_θ^*-Cauchy sequence with respect to the probabilistic norm ν, then it is I_θ-Cauchy sequence as well.*

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Bipan Hazarika
Department of Mathematics
Rajiv Gandhi University
Rono Hills, Doimukh-791112
Arunachal Pradesh, India
E-mail: *bh_rgu@yahoo.co.in*

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Ayhan Esi
Adiyaman University
Science and Art Faculty
Department of Mathematics
02040, Adiyaman, Turkey
E-mail: *aesi23@hotmail.com*

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The multiplicative Zagreb co-indices on two graph operators

Mansoureh Deldar, Mehdi Alaeiyan

Abstract. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The first and second multiplicative Zagreb co-indices are defined as:

$$\overline{\prod}_1(G) = \prod_{uv \notin E(G)} [d_G(u) + d_G(v)] \ , \ \overline{\prod}_2(G) = \prod_{uv \notin E(G)} [d_G(u) d_G(v)],$$

respectively, where $d_G(u)$ is the degree of the vertex u of G. The aim of this paper is to investigate the multiplicative Zagreb co-indices of the subdivision graphs of tadpole graphs and wheel graphs

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1 Introduction

Throughout the paper, we consider connected finite graphs without any loops or multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The degree of $v \in V(G)$, denoted by $d_G(v)$, is the number of vertices in G adjacent to v. A graphical invariant is a number related to a graph which is a structural invariant, in other words, it is a fixed number under graph automorphisms. In chemical graph theory, these invariants are also known as the topological indices. The Zagreb indices are among the oldest topological indices, and were introduced in 1972 [13]. Gutman and Trinajstic examined the dependence of total π-electron energy on molecular structure, elaborated in [12]. The first and second Zagreb indices of G are denoted by $M_1(G)$ and $M_2(G)$, respectively, and defined as follows:

$$M_1(G) = \sum_{v \in V(G)} d_G^2(v) \quad \text{and} \quad M_2(G) = \sum_{uv \in V(G)} d_G(u) d_G(v).$$

The first Zagreb index can be also expressed as a sum over edges of G:

$$M_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)].$$

The main properties of the Zagreb indices were summarized in [4, 5, 10]. In particular, Deng [5] gave a unified approach to determine extremal values of Zagreb indices for trees, unicyclic graphs and bicyclic graphs. Other recent results on ordinary Zagreb indices can be found in [15]. Note that the contribution of non-adjacent vertex pairs should be taken into account when computing the weighted Wiener polynomials of certain composite graphs [4]. The first and second Zagreb co-indices, as the sums involved run over the edges of the complement of G, are denoted by $\overline{M}_1(G)$ and $\overline{M}_2(G)$ and were defined in 2010 [1] as follows:

$$\overline{M}_1(G) = \sum_{uv \notin E(G)} [d_G(u) + d_G(v)] \quad \text{and} \quad \overline{M}_2 = \sum_{uv \notin E(G)} [d_G(u) d_G(v)].$$

The multiplicative versions of Zagreb indices were introduced by Gutman in 2012 [9]. The first and second multiplicative Zagreb indices of G are denoted by $\prod_1(G)$ and $\prod_2(G)$, respectively, and are defined as:

$$\prod\nolimits_1(G) = \prod\nolimits_{u \in V(G)} d_G(u)^2 \quad \text{and} \quad \prod\nolimits_2(G) = \prod\nolimits_{uv \in E(G)} [d_G(u) d_G(v)].$$

The first and second multiplicative Zagreb indices were extensively studied in [9, 18, 19]. In particular, Gutman have determined the extremal tree with respect to multiplicative Zagreb indices. In 2012 Xu and Hua [19] provided a unified approach to extremal trees, unicyclic and bicyclic graphs with respect to this multiplicative version of Zagreb indices. Xu et al. introduced the first and second multiplicative Zagreb co-indices of G [14]. The first and second multiplicative Zagreb co-indices of G are denoted by $\overline{\prod}_1(G)$ and $\overline{\prod}_2(G)$, respectively, and defined as:

$$\overline{\prod}\nolimits_1(G) = \prod\nolimits_{uv \notin E(G)} [d_G(u) + d_G(v)] \quad \text{and} \quad \overline{\prod}\nolimits_2(G) = \prod\nolimits_{uv \notin E(G)} [d_G(u) d_G(v)].$$

The subdivision graph $S(G)$ is the graph obtained from G by replacing each of its edges by a path of length 2, or equivalently, by inserting an additional vertex into each edge of G, and the operator $R(G)$ is the graph obtained from G by adding a new vertex corresponding to each edge of G and by joining each new vertex to the end vertices of the edge corresponding to it [16]. The tadpole graph, $T_{n,k}$, is the graph obtained by joining a cycle graph C_n to a path of length k [17]. The wheel graph W_{n+1} is defined as the graph $K_1 + C_n$, where K_1 is the singleton graph and C_n is the cycle graph. In this paper we will calculate the multiplicative Zagreb co-indices of $T_{n,k}$, W_{n+1} and the subdivision $S(G)$ and $R(G)$ on these graphs.

2 The multiplicative Zagreb co-indices on $S(G)$ and $R(G)$ for tadpole graph

In this section, we compute the multiplicative Zagreb co-indices on two graph operators $S(G)$ and $R(G)$ for tadpole graph $T_{n,k}$. At first we prove the following lemma, which plays an important role in the proofs.

Proposition 1. *For a connected graph G, we have*

$$\overline{\prod}\nolimits_2(G) = \prod_{v \in V(G)} d_G(v)^{(n-1-d_G(v))}.$$

Proof. By definition of complement graph of G we find that for each vertex $v \in V(G)$, the factor $d_G(v)$ occurs $n-1-d_G(v)$ times in $\overline{\prod}_2(G)$. Thus this theorem follows immediately. $\square$

Theorem 1. *For the tadpole graph, the multiplicative Zagreb co-indices satisfy the following equations:*

$$\overline{\prod}_1(T_{n,k}) = (2^{n^2+k^2+2nk-7n-7k+16})(5^{n+k-5})(3^{k+n-4})$$

and

$$\overline{\prod}_2(T_{n,k}) = (2^{n^2+k^2-5n-5k+2nk+6})(3^{n+k-4}).$$

Proof. The tadpole graph $T_{n,k}$ contains $n+k-2$ vertices of degree 2, one vertex of degree 3 and a pendent vertex. The subdivision graph $S(T_{n,k})$ contains $n+k$ additional vertices of degree 2. In $T_{n,k}$, let v_l be a vertex of degree 3 and $v_{1'}$ and $v_{2'}$ be the neighbors of v_l in the cycle C_n and v_j be the neighbor of v_l in the path P_{k+1}. Let v_1 be the pendent vertex in $T_{n,k}$. We calculate $\overline{\prod}_1[d_G(u)+d_G(v)]$:
1. Among the vertices in C_n.
2. From cycle C_n to the path P_{k+1}.
3. Among the vertices in the path P_{k+1}.

Case I. In C_n, $v_{1'}$ and $v_{2'}$ are non-adjacent with $n-3$ vertices of degree 2. Remaining $n-3$ vertices in C_n are non-adjacent with $n-4$ vertices of degree 2 and one vertex of degree 3. Also v_l is non-adjacent with $n-3$ vertices of degree 2. Hence in C_n, $\overline{\prod}_1[d_G(u)+d_G(v)] = (4^{n^2-5n+6})(5^{2n-6})$. Since one edge is shared between a pair of vertices, $\overline{\prod}_1[d_G(u)+d_G(v)]$ in C_n is

$$\overline{\prod}_1[d_G(u)+d_G(v)] = (4^{n^2-5n+6}5^{2n-6})^{\frac{1}{2}}. \tag{2.1}$$

Case II. From cycle C_n to path P_{k+1}, all the $n-1$ vertices other than v_l in C_n are non-adjacent with v_1. Also all of $n-1$ vertices except v_l in C_n are non-adjacent with $k-1$ vertices of degree 2 and one vertex of degree 1. Hence

$$\overline{\prod}_1[d_G(u)+d_G(v)] = (4^{(k-1)(n-1)})(3^{(n-1)}). \tag{2.2}$$

Case III: In the path P_{k+1}, the vertex v_l is non-adjacent with $k-2$ vertices of degree 2 and one vertex of degree 1. The neighbor of v_l in P_{k+1} is non-adjacent with $k-3$ vertices of degree 2 and one vertex of degree 1. The vertex v_j is non-adjacent with $k-4$ vertices of degree 2 and one vertex of degree 1 and one vertex of degree 3 for $3 \leq j \leq k-1$. Also the vertex v_2 has $k-3$ non-adjacent vertices of degree 2 and one vertex of degree 3. The vertex v_2 has $k-2$ non-adjacent vertices of degree 2 and one vertex of degree 3. Thus $\overline{\prod}_1[d_G(u)+d_G(v)] = (5^{2k-4})(4^{k^2-5k+8})(3^{2k-6})$. Since one edge is shared between a pair of vertices,

$$\overline{\prod}_1[d_G(u)+d_G(v)] = 5^{k-2}2^{k^2-5k+8}3^{k-3}. \tag{2.3}$$

The product of equations (2.1), (2.2) and (2.3) implies that

$$\overline{\prod}_1(T_{n,k}) = (2^{n^2+k^2+2nk-7n-7k+16})(5^{n+k-5})(3^{k+n-4}).$$

By Proposition 1, $\overline{\prod}_2(T_{n,k})$ can be easily obtained,

$$\overline{\prod}_2(T_{n,k}) = (2^{n^2+k^2-5n-5k+2nk+6})(3^{n+k-4}).$$

□

Theorem 2. *For the subdivision graph $S(G)$ of a tadpole graph, the multiplicative Zagreb co-indices are:*

$$\overline{\prod}_1(S(T_{n,k})) = (2^{4n^2+4k^2-14n-14k+8nk+16})(5^{2n+2k-5})(3^{2k+2n-5})$$

and

$$\overline{\prod}_2(S(T_{n,k})) = (2^{4n^2+4k^2-10n-10k+8nk+6})(3^{2n+2k-4}).$$

Proof. $S(T_{n,k})$ contains $2(n+k-1)$ vertices of degree 2, one vertex of degree 3 and a pendent vertex. In $S(T_{n,k})$, let v_l be the vertex of degree 3 and $v_{1'}$ and $v_{2'}$ be the neighbors of v_l in the cycle $S(C_n)$ and v_j be the neighbor of v_1 in the path $S(P_{k+1})$. Let v_1 be the pendent vertex in $S(T_{n,k})$. We calculate $\overline{\prod}_1[d_G(u)+d_G(v)]$:
1. Among the vertices in $S(C_n)$.
2. From cycle $S(C_n)$ to the path $S(P_{k+1})$.
3. Among the vertices in the path $S(P_{k+1})$.
In $S(C_n)$, $v_{1'}$ and $v_{2'}$ are non-adjacent with $2n-3$ vertices of degree 2. Remaining $2n-3$ vertices in $S(C_n)$ are non-adjacent with $2n-4$ vertices of degree 2 and one vertex of degree 3. Also v_1 is non-adjacent with $2n-3$ vertices of degree 2. Hence in $S(C_n)$, $\overline{\prod}_1[d_G(u)+d_G(v)] = (4^{(4n^2-10n+6)})(5^{4n-6})$. Since one edge is shared between a pair of vertices, $\overline{\prod}_1[d_G(u)+d_G(v)]$ in $S(C_n)$ is

$$\overline{\prod}_1[d_G(u)+d_G(v)] = 2^{4n^2-10n+6}5^{2n-3}. \tag{2.4}$$

From cycle $S(C_n)$ to path $S(P_{k+1})$, all the $2n-1$ vertices other than v_l in $S(C_n)$ are non-adjacent with v_1. Also all of $2n-1$ vertices except v_l in $S(C_n)$ are non-adjacent with $2k-1$ vertices of degree 2 and one vertex of degree 1. In the $S(P_{k+1})$, the vertex v_l is non-adjacent with $2k-2$ vertices of degree 2 and pendent vertex. Hence

$$\overline{\prod}_1[d_G(u)+d_G(v)] = (4^{4nk-2n-2k+2})(3^{(2n-1)})(5^{2k-2}). \tag{2.5}$$

In the path $S(P_{k+1})$, the neighbor of v_1 in $S(P_{k+1})$ is non-adjacent with $2k-3$ vertices of degree 2 and one vertex of degree 1. The vertex v_j is non-adjacent with $2k-4$ vertices of degree 2 and one vertex of degree 1 for $3 \leq j \leq 2k-1$. Also the vertex v_2 has $2k-3$ non-adjacent vertices of degree 2. Thus $\overline{\prod}_1[d_G(u)+d_G(v)] = (4^{4k^2-18k++18})(3^{4k-4})$. Since one edge is shared between a pair of vertices,

$$\overline{\prod}_1[d_G(u)+d_G(v)] = (2^{4k^2-18k+18})(3^{2k-2}). \tag{2.6}$$

By multiplying equations (2.4), (2.5) and (2.6) we have:

$$\overline{\prod}_1(S(T_{n,k})) = (2^{4n^2+4k^2-14n-14k+8nk+16})(5^{2n+2k-5})(3^{2k+2n-5}).$$

By Proposition 1, it can be easily obtained:

$$\overline{\prod}_2(S(T_{n,k})) = (2^{4n^2+4k^2-10n-10k+8nk+6})(3^{2n+2k-4}).$$

□

Theorem 3. *For the tadpole graph $T_{n,k}$ we have:*

$$\overline{\prod}_1(R(T_{n,k})) = (2^{\frac{1}{2}(7n^2+7k^2-23k-17n+23)})(3^{k^2+n^2-4k-3n+3})(5^{k+n-5})$$

and

$$\overline{\prod}_2(R(T_{n,k})) = (2^{6(n+k)^2-17(n+k)+17})(3^{2(n+k)-7}).$$

Proof. The vertices which are of degree l in $S(T_{n,k})$ are of degree $2l$ in $R(T_{n,k})$. All the subdivision vertices are of the same degree in both $S(T_{n,k})$ and in $R(T_{n,k})$.
In the cycle $R(C_n)$, the vertices which are adjacent to v_1 make the sum 8 with remaining $n-3$ vertices in the cycle and the remaining $n-3$ vertices make the sum 8 with $n-4$ vertices in the cycle. Also v_1 makes the sum 10 with the $n-3$ vertices. All the n subdivision vertices make the sum 4 with the remaining $n-1$ subdivision vertices. The vertex v_1 makes the sum 8 with $n-2$ subdivision vertices. The $n-1$ vertices other than v_1 make the sum 6 with the $n-2$ subdivision vertices. Therefore in $R(C_n)$,

$$\overline{\prod}_1[d_G(u)+d_G(v)] = [(2^{7n^2-15n+4})(3^{n^2-3n+2})(5^{2n-6})]^{\frac{1}{2}}. \tag{2.7}$$

To calculate $\overline{\prod}_1[d_G(u)+d_G(v)]$ from $R(C_n)$ to $R(P_{k+1})$, all the $n-1$ vertices in the cycle other than v_1 make the sum 6 with v_l and k subdivision vertices in the path. All the n subdivision vertices in the cycle make the sum 4 with v_l and k subdivision vertices in the path. Also all n subdivision vertices in $R(C_n)$ make the sum 6 with $k-1$ vertices in the path. So from cycle to path,

$$\overline{\prod}_1[d_G(u)+d_G(v)] = (2^{7nk-4k-n+2})(3^{2nk-k-1}). \tag{2.8}$$

In the path $R(P_{k+1})$, vertex v_1 makes the sum 8 with $k-1$ subdivision vertices in the path as well as with v_l. Also v_1 makes the sum 10 with $k-2$ vertices in the path. The subdivision vertex v_j in the path makes the sum 4 with the remaining $k-1$ subdivision vertices as well as with v_l. It also makes the sum 6 with $k-2$ vertices in the path. The neighbors of v_j in the path make the sum 8 with $k-3$ vertices and 6 with $k-2$ vertices and so on. Thus in the path,

$$\overline{\prod}_1 [d_G(u) + d_G(v)] = [(2^{7k^2-15k+15})(3^{2k^2-6k+4})(5^{2k-4})]^{\frac{1}{2}}. \tag{2.9}$$

By multiplying equations (2.7), (2.8) and (2.9) we have:

$$\overline{\prod}_1 (R(T_{n,k})) = (2^{\frac{1}{2}(7n^2+7k^2-23k-17n+23)})(3^{k^2+n^2-4k-3n+3})(5^{k+n-5}).$$

By Proposition 1, $\overline{\prod}_2(R(T_{n,k})) = (2^{6(n+k)^2-17(n+k)+17})(3^{2(n+k)-7})$. □

3 The multiplicative Zagreb co-indices on $S(G)$ and $R(G)$ for wheel graph

In this section we compute the multiplicative Zagreb co-indices on two graph operators $S(G)$ and $R(G)$ for wheel graph W_{n+1}.

Theorem 4. *The multiplicative Zagreb co-indices for the wheel graph W_{n+1} are*

$$\overline{\prod}_1 (W_{n+1}) = 6^{\frac{n^2-3n}{2}} \ , \ \overline{\prod}_2 (W_{n+1}) = 3^{n(n-3)}.$$

Proof. In W_{n+1}, the hub of the wheel is of degree n and the remaining vertices are of degree 3. Each vertex on C_n has $n-3$ non-adjacent vertices of degree 3. Hence $\overline{\prod}_1[d_G(u) + d_G(v)] = 6^{n^2-3n}$. Since one edge is shared between a pair of vertices, then

$$\overline{\prod}_1 (W_{n+1}) = 6^{\frac{n^2-3n}{2}}.$$

Proposition 1 implies that

$$\overline{\prod}_2 (W_{n+1}) = 3^{n(n-3)}.$$

□

Theorem 5. *For the subdivision graph $S(G)$ of a wheel graph, the multiplicative Zagreb co-indices are*

$$\overline{\prod}_1 S(W_{n+1}) = [5^{4n-6} \ 4^{4n-2} \ 6^{n-1} \ (2+n)^2 \ (n+3)^2]^{\frac{n}{2}}$$

and

$$\overline{\prod}_2 S(W_{n+1}) = (3^{3n-3} 4^{3n-2} n^3)^n.$$

Proof. $S(W_{n+1})$ contains n vertices of degree 3, $2n$ vertices of degree 2 and one vertex of degree n. Each vertex of degree 3 has $n-1$ non-adjacent vertices of degree 3, $2n-3$ non-adjacent vertices of degree 2 and one vertex of degree n. So,

$$\overline{\prod}_1 [d_G(u) + d_G(v)] = [6^{n-1}\ 5^{2n-3}\ (3+n)]^n. \tag{3.1}$$

The subdivision vertices of degree 2 on $S(C_n)$ are non-adjacent with $n-2$ vertices of degree 3, $2n-1$ vertices of degree 2 and one vertex of degree n. Hence

$$\overline{\prod}_1 [d_G(u) + d_G(v)] = [5^{n-2}\ 4^{2n-1}\ (2+n)]^n. \tag{3.2}$$

The remaining subdivision vertices of degree 2 are non-adjacent with $n-1$ vertices of degree 3 and $2n-1$ vertices of degree 2. So,

$$\overline{\prod}_1 [d_G(u) + d_G(v)] = [5^{n-1}\ 4^{2n-1}]^n. \tag{3.3}$$

The hub of the wheel has n non-adjacent vertices of degree 3 and n non-adjacent vertices of degree 2. Hence

$$\overline{\prod}_1 [d(u) + d(v)] = [n + 2(3+n)]^n. \tag{3.4}$$

The equations (3.1), (3.2), (3.3) and (3.4) make the product

$$\overline{\prod}_1 [d(u) + d(v)] = [5^{4n-6}\ 4^{4n-2}\ 6^{n-1}\ (2+n)^2\ (n+3)^2]^n.$$

Since one edge is shared between a pair of vertices, then

$$\overline{\prod}_1 S(W_{n+1}) = [5^{4n-6}\ 4^{4n-2}\ 6^{n-1}\ (2+n)^2\ (n+3)^2]^{\frac{n}{2}}.$$

Proposition 1 implies that

$$\overline{\prod}_2 S(W_{n+1}) = (3^{3n-3} 4^{3n-2} n^3)^n.$$

$\square$

Theorem 6. *For the subdivision graph $R(G)$ of a wheel graph, the multiplicative Zagreb co-indices are*

$$\overline{\prod}_1 R(W_{n+1}) = [(2n+2)^2\ 2^{24n-28}\ 3^{n-3}]^{\frac{n}{2}}$$

and

$$\overline{\prod}_2 R(W_{n+1}) = 2^{6n^2+6k^2+12nk-17n-17k+9} 3^{2n+2k-7}.$$

Proof. In $R(W_{n+1})$, n vertices are of degree 6, hub of the wheel is of degree $2n$ and all subdivision vertices are of degree 2. Hence, $\overline{\prod}_1[d(u)+d(v)]$ with respect to the hub of the wheel is

$$\overline{\prod}_1[d(u)+d(v)] = (2n+2)^n. \tag{3.5}$$

The product of [d(u)+d(v)] degrees with respect to all the n vertices of C_n is given by

$$\overline{\prod}_1[d(u)+d(v)] = [8^{2n-3}\ 12^{n-3}]^n. \tag{3.6}$$

With respect to the n subdivision vertices on the spokes of the wheel, $\overline{\prod}_1[d(u)+d(v)]$ is

$$\overline{\prod}_1[d(u)+d(v)] = [4^{2n-1}\ 8^{n-1}]^n. \tag{3.7}$$

The calculation with respect to n subdivision vertices on the edge of the cycle C_n of $R(W_{n+1})$ is

$$\overline{\prod}_1[d(u)+d(v)] = [8^{n-2}\ 4^{2n-1}\ (2n+2)]^n. \tag{3.8}$$

The equations (3.5), (3.6), (3.7) and (3.8) make the product

$$\overline{\prod}_1[d(u)+d(v)] = [(2n+2)^2\ 2^{22n-28}\ 3^{n-3}]^n.$$

Since one edge is shared by a pair of vertices, then

$$\overline{\prod}_1 R(W_{n+1}) = [(2n+2)^2\ 2^{24n-28}\ 3^{n-3}]^{\frac{n}{2}}.$$

Proposition 1 implies that

$$\overline{\prod}_2 R(W_{n+1}) = 2^{6n^2+6k^2+12nk-17n-17k+9}3^{2n+2k-7}.$$

□

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Mansoureh Deldar, Mehdi Alaeiyan
Department of Mathematics
Karaj Branch, Islamic Azad University
Karaj, Iran.
E-mail: *deldaraz@yahoo.com; Alaeiyan@iust.ac.ir.*

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On Lagrange algorithm for reduced algebraic irrationalities*

N. M. Dobrovol'skii, I. N. Balaba, I. Yu. Rebrova,
N. N. Dobrovol'skii

Abstract. In this paper the properties of Lagrange algorithm for expansion of algebraic number are refined. It has been shown that for reduced algebraic irrationalities the quantity of elementary arithmetic operations which needed for the computation of next incomplete quotient does not depend on the value of this incomplete quotient.

It is established that beginning with some index all residual fractions for an arbitrary reduced algebraic irrationality are the generalized Pisot numbers. An asymptotic formula for conjugate numbers to residual fractions is obtained.

The definition of generalized Pisot numbers differs from the definition of Pisot numbers by absence of the requirement to be integer.

Mathematics subject classification: 11J17.
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1 Introduction

The continued fraction expansion of algebraic irrationalities is one of the most difficult questions in the modern number theory. Various aspects of this theory can be seen in the papers [1–9, 11–13] Even in such developed theory as the theory of continued fractions of quadratic irrationalities one can find new interesting facts (see [10, 14]). The paper [17] describes the set of reduced algebraic irrationalities of n-th degree and asserts that this set has the property of rational convexity.

The aim of this paper is the refinement of properties of Lagrange algorithm for reduced algebraic irrationalities of n-th degree and for Pisot numbers in general case.

The case of the reduced algebraic irrationalities of n-th degree is very important for us. This case is connected with totally real algebraic fields of n-th degree which underly the construction of algebraic lattice used in quadrature formulas with weights in K. K. Frolov's method (see [5–7, 15, 16]).

2 Necessary definitions and facts

We begin with the definition of a reduced algebraic irrationality of n-th degree. Here we follow [8, 9, 17].

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Definition 1. Let

$$f(x) = \sum_{k=0}^{n} a_k x^k \in \mathbb{Z}[x], \quad a_n > 0$$

be irreducible polynomial with integer coefficients [1] such that all its roots $\alpha^{(k)}$ ($k = 1, 2, \ldots, n$) are different real numbers satisfying the following condition

$$-1 < \alpha^{(n)} < \ldots < \alpha^{(2)} < 0, \quad \alpha^{(1)} > 1.$$

The algebraic number $\alpha = \alpha^{(1)}$ is called a reduced algebraic irrationality of n-th degree.

Note that for minimal polynomial $f(x)$ that defines a reduced algebraic irrationality α of n-th degree we always have $a_0 < 0$, since $f(x)$ has only one root α belonging to $[0; \infty)$ and $f(x) > 0$ for $x > \alpha$, so $f(0) < 0$. Besides the following inequalities hold

$$a_n + a_{n-1} + \ldots + a_1 + a_0 = f(1) < 0,$$
$$a_n - a_{n-1} + \ldots + (-1)^{n-1} a_1 + (-1)^n a_0 = (-1)^n f(-1) > 0.$$

For any real number α which is a reduced algebraic irrationality of n-th degree consider infinite continued fraction expansion

$$\alpha = \alpha_0 = q_0 + \cfrac{1}{q_1 + \cfrac{1}{\ddots + \cfrac{1}{q_n + \cfrac{1}{\ddots}}}} = q_0 + \cfrac{1}{q_1 + \cfrac{1}{\ddots + \cfrac{1}{q_k + \cfrac{1}{\alpha_{k+1}}}}}.$$

As usually by P_k and Q_k we denote numerator and denominator of k-th order convergent of continued fraction and by α_k we denote its residual fraction of order k.

Thus $\alpha = \alpha_0$ and the equality

$$\alpha = \frac{\alpha_{k+1} P_k + P_{k-1}}{\alpha_{k+1} Q_k + Q_{k-1}}, \quad k \geq -1,$$

is valid if we assume as usually that $P_{-1} = 1$, $P_{-2} = 0$ and $Q_{-1} = 0$, $Q_{-2} = 1$.

It is easy to show that

$$\alpha_{k+1} = \frac{\alpha Q_{k-1} - P_{k-1}}{P_k - \alpha Q_k}, \quad k \geq -1.$$

[1] In particular, the irreducibility of a polynomial means that $(a_0, \ldots, a_n) = 1$.

Lemma 1. *For an arbitrary reduced algebraic irrationality α of n-th degree its residual fractions α_1 is a reduced algebraic irrationality of n-th degree too that satisfies the irreducible polynomial*

$$f_1(x) = \sum_{k=0}^{n} a_{k,1} x^k \in \mathbb{Z}[x], \quad a_{n,1} > 0,$$

where

$$a_{k,1} = \frac{b_k}{d},\ d = (b_0, \ldots, b_n),\ b_k = -\sum_{m=n-k}^{n} a_m C_m^{m+k-n} q_0^{m+k-n}\ (0 \le k \le n).$$

Proof. See [8].

Theorem 1. *For an arbitrary reduced algebraic irrationality α of n-th degree all its residual fractions α_m are reduced algebraic irrationalities of n-th degree, satisfying the irreducible polynomials*

$$f_m(x) = \sum_{k=0}^{n} a_{k,m} x^k \in \mathbb{Z}[x], \quad a_{n,m} > 0,$$

where

$$a_{k,m} = \frac{b_{k,m}}{d_m}, \quad d_m = (b_{0,m}, \ldots, b_{n,m}),$$
$$b_{k,m} = -\sum_{l=n-k}^{n} a_{l,m-1} C_l^{l+k-n} q_{m-1}^{l+k-n} \quad (0 \le k \le n).$$

Proof. See [8].

Theorem 2. *An incomplete quotient q_k is uniquely defined as an integer which satisfies the following condition*

$$f_k(q_k) < 0, \quad f_k(q_k + 1) > 0.$$

Proof. See [8].

It is not hard to see that to compute q_k we need to calculate $O(\ln q_k)$ values of polynomial $f_k(x)$. Indeed, consider the sequence $f_k(1)$, $f_k(2)$, …, $f_k(2^m)$, $f_k(2^{m+1})$, where $m = [\log_2(q_k)]$. It is clear that $f_k(2^j) < 0$ for all $0 \le j \le m$ and $f_k(2^{m+1}) > 0$. Further using the method of interval bisection contract the segment $[2^m; 2^{m+1}]$ to the segment $[q_k; q_k + 1]$, that will require to compute yet m values of $f_k(x)$. □

Here in fact Lagrange algorithm of expansion for algebraic irrationality of arbitrary degree $n \ge 2$ is described.

Theorem 1 is generalized to the case for continued fraction of arbitrary totally real algebraic irrationality α of degree n. First we shall show Lemma on the transformation of the roots.

Lemma 2. *Let*

$$f(x)=\sum_{k=0}^{n}a_kx^k\in\mathbb{Z}[x],\quad a_n>0$$

be irreducible polynomial with integral coefficients such that all its roots $\alpha^{(k)}$ $(k=1,2,\ldots,n)$ *are different real number satisfying the following condition*

$$\alpha^{(n)}<\ldots<\alpha^{(2)}<\alpha^{(1)},$$

and for integer number q *the following inequalities hold:*

$$\begin{cases}\alpha^{(k)}<q & \text{for } k\ge k_0,\\ q<\alpha^{(k)}<q+1 & \text{for } k_0>k\ge k_1,\\ \alpha^{(k)}>q+1 & \text{for } k_1>k\ge 1.\end{cases}$$

Then the polynomial

$$g(x)=-f\left(q+\frac{1}{x}\right)\cdot x^n=\sum_{k=0}^{n}b_kx^k.$$

has roots $\beta^{(k)}=\frac{1}{\alpha^{(k)}-q}$ $(k=1,2,\ldots,n)$ *satisfying the following inequalities*

$$\begin{cases}\beta^{(k)}<0 & \text{for } k\ge k_0,\\ 1<\beta^{(k)} & \text{for } k_0>k\ge k_1,\\ 0<\beta^{(k)}<1 & \text{for } k_1>k\ge 1.\end{cases}$$

Proof. See [8].

Theorem 3. *For an arbitrary totally real algebraic irrationality* α *of* n*-th degree all its residual fractions* α_m *are reduced algebraic irrationalities of* n*-th degree beginning with some index* m_0+1.

Proof. See [8].

3 Refinement of Lagrange algorithm for reduced algebraic irrationalities

Denote by $\mathbb{PZ}_n[x]$ the set of all irreducible polynomials with integer coefficients of n-th degree considered in Definition 1.

Lemma 3. *If polynomial*

$$f_0(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0\in\mathbb{PZ}_n[x]$$

and $\alpha^{(1)}>\alpha^{(2)}>\ldots>\alpha^{(n)}$ *are its roots, then for the continued fraction expansion*

$$\alpha^{(1)}=\alpha_0=q_0+\cfrac{1}{q_1+\cfrac{1}{\ddots+\cfrac{1}{q_n+\cfrac{1}{\ddots}}}}$$

we have

$$\left[-\frac{a_{n-1}}{a_n}\right] \le q_0 < -\frac{a_{n-1}}{a_n} + n - 1. \tag{1}$$

Proof. Indeed, using Viete's formula we have

$$-\frac{a_{n-1}}{a_n} = \alpha^{(1)} + \alpha^{(2)} + \ldots + \alpha^{(n)}.$$

Since $\alpha^{(1)}$ is a reduced algebraic irrationality of degree n, then

$$-1 < \alpha^{(n)} < \ldots < \alpha^{(2)} < 0, \quad \alpha^{(1)} > 1.$$

So

$$-n + 1 < \alpha^{(2)} + \ldots + \alpha^{(n)} < 0$$

and

$$-\frac{a_{n-1}}{a_n} < \alpha^{(1)} < -\frac{a_{n-1}}{a_n} + n - 1.$$

Since $q_0 < \alpha^{(1)} < q_0 + 1$ we get the statement of Lemma. □

Revise Lemma 1.

Lemma 4. *For a reduced algebraic irrationality α of degree n its residual fraction α_1 is a reduced algebraic irrationality of n-th degree too that satisfies the irreducible polynomial*

$$f_1(x) = \sum_{k=0}^{n} a_{k,1} x^k \in \mathbb{Z}[x], \quad a_{n,1} > 0,$$

where

$$a_{k,1} = \frac{b_k}{d_0},\ d_0 = (b_0, \ldots, b_n),\ b_k = -\sum_{m=n-k}^{n} a_m C_m^{m+k-n} q_0^{m+k-n}\ (0 \le k \le n).$$

The polynomial $f_1(x)$ has the roots

$$\alpha_1^{(j)} = \frac{1}{\alpha^{(j)} - q_0} \quad (1 \le j \le n)$$

and the following equality holds:

$$f_1(x) = \frac{-f_0(q_0)}{d_0} \prod_{j=1}^{n} \left(x - \frac{1}{\alpha^{(j)} - q_0}\right) \in \mathbb{PZ}_n[x].$$

Proof. Consider the polynomial

$$g(x) = -x^n f\left(q_0 + \frac{1}{x}\right).$$

We have:

$$g(x) = -a_n \prod_{j=1}^{n} \left(q_0 x + 1 - \alpha^{(j)} x\right) =$$
$$= -a_n \prod_{j=1}^{n} \left(q_0 - \alpha^{(j)}\right) \prod_{j=1}^{n} \left(x - \frac{1}{\alpha^{(j)} - q_0}\right) =$$
$$= -f_0(q_0) \prod_{j=1}^{n} \left(x - \frac{1}{\alpha^{(j)} - q_0}\right)$$

and $\alpha_1 = \frac{1}{\alpha^{(1)} - q_0}$.

On the other hand

$$g(x) = -\sum_{j=0}^{n} a_j (q_0 x + 1)^j x^{n-j} = -\sum_{j=0}^{n} a_j \sum_{\nu=0}^{j} C_j^{\nu} q_0^{\nu} x^{n-j+\nu} =$$
$$= -\sum_{k=0}^{n} x^k \sum_{m=n-k}^{n} a_m C_m^{k+m-n} q_0^{k+m-n} = \sum_{k=0}^{n} b_k x^k,$$

where

$$b_k = -\sum_{m=n-k}^{n} a_m C_m^{k+m-n} q_0^{k+m-n} \in \mathbb{Z} \quad (0 \leq k \leq n).$$

Since $1 \leq q_0 < \alpha^{(1)} < q_0 + 1$ we obtain

$$b_n = -\sum_{m=0}^{n} a_m q_0^m = -f_0(q_0) > 0,$$
$$\frac{1}{\alpha^{(1)} - q_0} > 1, \quad -1 < \frac{1}{\alpha^{(j)} - q_0} < 0 \quad (2 \leq j \leq n).$$

So for $d_0 = (b_0, \ldots, b_n)$ the polynomial $f_1(x) = \frac{1}{d_0} g(x) \in \mathbb{PZ}_n[x]$ and Lemma is completely proved. □

Theorem 4. *Let $\alpha = \alpha_0$ be a reduced algebraic irrationality of n-th degree satisfying the irreducible polynomial*

$$f_0(x) = \sum_{k=0}^{n} a_{k,0} x^k \in \mathbb{Z}[x], \quad a_{n,0} > 0.$$

And let a sequence of the polynomials $f_m(x)$ $(m \geq 1)$ and a sequence of natural numbers q_m $(m \geq 0)$ define the recurrence relations

$$f_{m-1}(q_{m-1}) < 0, \quad f_{m-1}(q_{m-1} + 1) > 0 \quad (m \geq 1), \tag{2}$$
$$\left[-\frac{a_{n-1,m-1}}{a_{n,m-1}}\right] \leq q_{m-1} < -\frac{a_{n-1,m-1}}{a_{n,m-1}} + n - 1 \quad (m \geq 1), \tag{3}$$

$$f_m(x) = \sum_{k=0}^{n} a_{k,m} x^k \in \mathbb{Z}[x], \quad a_{n,m} > 0,$$
$$a_{k,m} = \frac{b_{k,m}}{d_{m-1}},\ d_{m-1} = (b_{0,m}, \ldots, b_{n,m}),$$
$$b_{k,m} = -\sum_{\nu=n-k}^{n} a_{\nu,m-1} C_{\nu}^{\nu+k-n} q_{m-1}^{\nu+k-n}\ (0 \le k \le n). \tag{4}$$

Then:

(1) the polynomials $f_m(x)$ have the roots

$$\alpha_m^{(j)} = \frac{\alpha^{(j)} Q_{m-2} - P_{m-2}}{P_{m-1} - \alpha^{(j)} Q_{m-1}} \quad (1 \le j \le n); \tag{5}$$

(2)

$$f_m(x) = \frac{-f_{m-1}(q_{m-1})}{d_{m-1}} \prod_{j=1}^{n} \left(x - \alpha_m^{(j)}\right) \in \mathbb{PZ}_n[x];$$

(3) α has the following continued fraction expansion

$$\alpha = \alpha_0 = q_0 + \cfrac{1}{q_1 + \cfrac{1}{\ddots + \cfrac{1}{q_n + \cfrac{1}{\ddots}}}}.$$

Proof. The proof is by induction on m.

For $m = 1$ the statements of theorem are valid by Lemma 4 and the equalities $Q_0 = 1$, $P_0 = q_0$, $Q_{-1} = 0$ and $P_{-1} = 1$.

Suppose the statements are valid for $m \ge 1$, applying Lemma 4 to reduced algebraic irrationality $\alpha_m^{(1)}$ we get the statements (2) – (4).

Further we obtain

$$\alpha_{m+1}^{(j)} = \frac{1}{\alpha_m^{(j)} - q_m} = \frac{1}{\frac{\alpha^{(j)} Q_{m-2} - P_{m-2}}{P_{m-1} - \alpha^{(j)} Q_{m-1}} - q_m} = \frac{\alpha^{(j)} Q_{m-1} - P_{m-1}}{P_m - \alpha^{(j)} Q_m}$$

and the statement (5) holds.

By (5) numbers $\alpha_m^{(1)}$ are the residual fractions for α ($m = 0, 1, \ldots$), so a sequence $q_0, q_1, \ldots$ is a sequence of incomplete quotients for α. This completes the proof. □

It is easy to show that we need to calculate $O(\ln n)$ values of $f_m(x)$ for the computation of q_m. Indeed, for $A = \left[-\frac{a_{n-1,m}}{a_{n,m}}\right]$ consider a sequence of $f_m(A)$, $f_m(A+1)$, $\ldots$, $f_m(A+n-1)$ consisting of n members. It is clear that if $f_m(A+n-1) < 0$ then $q_m = A + n - 1$. Otherwise using the method of interval bisection contract the segment $[A; A+n-1]$ to the segment $[q_m; q_m+1]$ that will require to compute yet $O(\ln n)$ values of $f_m(x)$.

Thus the new version of Lagrange algorithm for expansion of an algebraic irrationality of arbitrary degree $n \geq 2$ in the case of reduced algebraic irrationality of n-th degree has a new property: for the computation of next incomplete quotient of continued fraction expansion of this irrationality we need to calculate at most $O(\ln n)$ values of polynomial $f_m(x)$. Since for the computation of coefficients of a polynomial $f_m(x)$ via the coefficients of a polynomial $f_{m-1}(x)$ we need at most $O\left(n^2\right)$ elementary arithmetic operations then the quantity of operations for the computation of next incomplete quotient does not depend on the value of this incomplete quotient.

Make an essential remark. If we will not use the greatest common divisor d_{m-1} in formula (4), then all coefficients will be increased and time for practical realisation using symbolic arithmetic will increase too. The calculation of d_{m-1} requires additional time, but it is compensated by range extension for calculations of incomplete quotients. On the other hand, even establishing that $d_{m-1} = 1$ requires time which depends on the polynomial coefficients, but it does not depend on the value of incomplete quotient.

4 The case of generalized Pisot numbers

Now we give the definition of generalized Pisot numbers.

Definition 2. Let

$$f(x) = \sum_{k=0}^{n} a_k x^k \in \mathbb{Z}[x], \quad a_n > 0,$$

be an arbitrary irreducible polynomial with integer coefficients such that its roots $\alpha^{(k)}$ $(k = 1, 2, \ldots, n)$ satisfy the following conditions

$$|\alpha^{(j)}| < 1 \ (2 \leq j \leq n), \quad \alpha^{(1)} > 1.$$

The algebraic number $\alpha = \alpha^{(1)}$ is called a generalized Pisot number of n-th degree.

It is easy to see that the definition of generalized Pisot numbers differs from the definition of Pisot numbers by absence of the requirement to be integer.

Theorem 5. *Let $\alpha = \alpha_0$ be a real algebraic irrationality of n-th degree satisfying the irreducible polynomial*

$$f_0(x) = \sum_{k=0}^{n} a_{k,0} x^k \in \mathbb{Z}[x], \quad a_{n,0} > 0,$$

and α have the following continued fraction expansion

$$\alpha = \alpha_0 = q_0 + \cfrac{1}{q_1 + \cfrac{1}{\ddots + \cfrac{1}{q_n + \cfrac{1}{\ddots}}}}.$$

Let a sequence of the polynomials $f_m(x)$ $(m \geq 1)$ be defined by the recurrence relations

$$\begin{aligned} f_m(x) &= \sum_{k=0}^{n} a_{k,m} x^k \in \mathbb{Z}[x], \quad a_{n,m} > 0, \\ a_{k,m} &= \frac{b_{k,m}}{d_{m-1}},\ d_{m-1} = (b_{0,m}, \ldots, b_{n,m}), \\ b_{k,m} &= -\sum_{\nu=n-k}^{n} a_{\nu,m-1} C_{\nu}^{\nu+k-n} q_{m-1}^{\nu+k-n} \quad (0 \leq k \leq n). \end{aligned} \tag{6}$$

Then:

(1) the polynomials $f_m(x)$ have the following roots

$$\alpha_m^{(j)} = \frac{\alpha^{(j)} Q_{m-2} - P_{m-2}}{P_{m-1} - \alpha^{(j)} Q_{m-1}} \quad (1 \leq j \leq n). \tag{7}$$

(2)

$$f_m(x) = \frac{-f_{m-1}(q_{m-1})}{d_{m-1}} \prod_{j=1}^{n} \left(x - \alpha_m^{(j)} \right). \tag{8}$$

(3) beginning with some index m_0 all residual fractions $\alpha_m^{(1)}$ are generalized Pisot numbers $(m \geq m_0)$.

Proof. Consider a sequence of the polynomials

$$g_m(x) = -x^n f_{m-1}\left(q_{m-1} + \frac{1}{x} \right) \quad (m \geq 1).$$

Repeating arguments of Lemma 4 and Theorem 4 we get (7) and (8).

To prove the last statement of Theorem, transforming expression (7) we obtain:

$$\alpha_m^{(j)} = \frac{Q_{m-2}}{Q_{m-1}} \frac{\alpha^{(j)} - \frac{P_{m-2}}{Q_{m-2}}}{\frac{P_{m-1}}{Q_{m-1}} - \alpha^{(j)}} \quad (1 \leq j \leq n). \tag{9}$$

For $j = 1$ we have the obvious inequality $\alpha_m^{(1)} > 1$ using the definition of a residual fraction.

Let $2 \le j \le n$, then

$$\alpha_m^{(j)} = \frac{Q_{m-2}}{Q_{m-1}}\left(-1+\frac{\frac{P_{m-1}}{Q_{m-1}}-\frac{P_{m-2}}{Q_{m-2}}}{\frac{P_{m-1}}{Q_{m-1}}-\alpha^{(j)}}\right) = \frac{Q_{m-2}}{Q_{m-1}}\left(-1+\frac{\frac{(-1)^m}{Q_{m-1}Q_{m-2}}}{\frac{P_{m-1}}{Q_{m-1}}-\alpha^{(j)}}\right) =$$

$$= \frac{Q_{m-2}}{Q_{m-1}}\left(-1+\frac{(-1)^m}{Q_{m-1}Q_{m-2}\left(\frac{P_{m-1}}{Q_{m-1}}-\alpha^{(j)}\right)}\right). \tag{10}$$

Since

$$\lim_{m\to\infty}\left|\frac{P_{m-1}}{Q_{m-1}}-\alpha^{(j)}\right| = \left|\alpha^{(1)}-\alpha^{(j)}\right|,$$

and all roots are distinct, it follows that

$$|\alpha_m^{(j)}| \le \frac{Q_{m-2}}{Q_{m-1}}\left(1+\frac{2}{Q_{m-1}Q_{m-2}\delta}\right) = \frac{Q_{m-2}}{Q_{m-1}}+\frac{2}{Q_{m-1}^2\delta} < 1, \tag{11}$$

for $m > m_0$, where

$$\delta = \min_{2\le j\le n}\left|\alpha^{(1)}-\alpha^{(j)}\right| > 0.$$

By (11) we obtain that beginning with index m_0 all residual fractions $\alpha_m^{(1)}$ are generalized Pisot numbers. This completes the proof. □

The importance of generalized Pisot numbers for Lagrange algorithm of continued fraction expansion of an algebraic number is explained by the following generalization of Lemma 3.

Lemma 5. *If*

$$f_0(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \in \mathbb{Z}[x]$$

is a minimal polynomial for generalized Pisot number $\alpha^{(1)} = \alpha_0$, then for the continued fraction expansion

$$\alpha^{(1)} = \alpha_0 = q_0 + \cfrac{1}{q_1 + \cfrac{1}{\ddots + \cfrac{1}{q_n + \cfrac{1}{\ddots}}}}$$

the following inequality holds

$$\left[-\frac{a_{n-1}}{a_n}\right] + 1 - n \le q_0 < -\frac{a_{n-1}}{a_n} + n - 1. \tag{12}$$

Proof. Indeed, using Viete's formula we have:

$$-\frac{a_{n-1}}{a_n} = \alpha^{(1)} + \alpha^{(2)} + \ldots + \alpha^{(n)}.$$

Since a minimal polynomial $f_0(x)$ is irreducible it follows that

$$\alpha^{(2)} + \alpha^{(3)} + \ldots + \alpha^{(n)} \neq 0,$$

for otherwise we have $\alpha^{(1)} = -\frac{a_{n-1}}{a_n} \in \mathbb{Q}$ and get a contradiction with the irreducibility of minimal polynomial $f_0(x)$.

As $\alpha^{(1)}$ is a generalized Pisot number then

$$|\alpha^{(j)}| < 1 \quad (2 \leq j \leq n).$$

So

$$0 < |\alpha^{(2)} + \ldots + \alpha^{(n)}| < n - 1$$

and

$$-\frac{a_{n-1}}{a_n} + 1 - n < \alpha^{(1)} < -\frac{a_{n-1}}{a_n} + n - 1.$$

Since $q_0 < \alpha^{(1)} < q_0 + 1$ we obtain the statement of Lemma. □

Thus, from Theorem 5 and Lemma 5 it follows that beginning with some index m_0 all incomplete quotients q_m $(m \geq m_0)$ require for their calculations at most $O(\ln n)$ computations of values of polynomial $f_m(x)$.

5 Conclusion

The results of this paper show that reduced algebraic irrationalities in the case of totally real algebraic fields and generalized Pisot numbers in general case play a fundamental role in the continued fraction expansion of algebraic irrationalities. Beginning with some index all residual fractions are the reduced algebraic numbers in the first case and generalized Pisot numbers in the second case.

The formulas (10) and (11) imply that beginning with some index m_0 a peculiar asymptotic formula for the conjugate to residual fractions takes place

$$\alpha_m^{(j)} = -\frac{Q_{m-2}}{Q_{m-1}} + O\left(\frac{2}{Q_{m-1}^2 \delta}\right).$$

Hence beginning with index m_0 some more powerful analog of Lemma 3 holds, which is valid for any real irrationality. The next article will be devoted to the study of this phenomenon.

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N. M. Dobrovol'skii, I. N. Balaba, I. Yu. Rebrova
Received July 20, 2015
Tula State Lev Tolstoy Pedagogical University
Lenina prospect, 125, 300026, Tula, Russia
E-mail: *dobrovol@tspu.tula.ru; ibalaba@mail.ru; i_rebrova@mail.ru*

N. N. Dobrovol'skii
E-mail: *nikolai.dobrovolsky@gmail.com*

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Asymmetric ID-Based Encryption System, Using an Explicit Pairing Function of the Reciprocity Law

S. V. Vostokov, R. P. Vostokova, I. A. Budanaev

Abstract. In this paper, we describe a new approach for building an asymmetric ID-based encryption (**IBE**) system and an authentication protocol without disclosure, using the idea of Explicit Hilbert Pairing.

Mathematics subject classification: 11A15, 11F33, 11T71.
Keywords and phrases: ID-based systems, Asymmetric Encryption, Explicit Pairing, Reciprocity Law, Hilbert's Ninth Problem, Frobenius Operator, Protocol, Explicit Hilbert Pairing.

1 Introduction

This paper proposes a new approach for creating ID-based systems, using the Explicit Pairing Reciprocity Law from works [1, 2]. The Reciprocity Law was first examined by P. Fermat, when he proved that $x^2 + 1$ is divisible by a prime number p, for some integer x, if and only if $p = 4k + 1$. The Quadratic Reciprocity Law for Legendre exponential symbols was formulated by L. Euler and proved by C. F. Gauss in the 18th century. In the 19th century attempts were made to obtain an explicit formula for the product of the symbols of power residues in an arbitrary number field, containing the necessary roots of 1. Partial results were obtained by Kummer, Dirichlet and Eisenstein. After new insight about the deep analogy of algebraic numbers and algebraic functions was proposed by L. Kroneker, Hilbert implemented this idea and devised a plan to obtain an Explicit Reciprocity Law (Hilbert's 9th problem, 1900). The first part of this plan was the construction of field theory of classes which was completed in the early 20th century by mathematicians such as W. Furtwängler, T. Takagi, E. Artin, and H. Hasse. This theory reduces the calculation of the product of global power residues to the product of local normed residue symbols (pairing or Hilbert symbol). The first explicit, but not complete formula for this pairing in the circumferential extension of the p-adic numbers of field Q_p, were found in 1928 by E. Artin and H. Hasse. In 1950 I. R. Shafarevich constructed the basis of the multiplicative group of a local field (finite extension of the p-adic Q_p numbers), on the elements of which he proposed the method for calculating the Hilbert pairing. Definitive and complete formulas for the Hilbert Pairing were obtained by S. Vostokov in 1978 [1], and later independently by H. Bruckner [3].

In this paper we use the Explicit Hilbert Pairing from [1], in the case of a circular field $Q_p(\xi)$, where ξ is a primitive root of degree p of 1 (see Section 2), for the authentication protocol without disclosure (see Section 3).

2 Explicit Hilbert Pairing

Consider the multiplicative group of power series $U = 1 + XZ_p[X]$. Let Δ be the Frobenius Operator on the ring of Laurent Series $Z_p[X]$, acting on series $f(X)$ of $Z_p[X]$ as follows:

$$\Delta f(X) = f^{\Delta}(X) = f(X^p).$$

Further, we define the function $l(f(X))$ for series $f(X)$ from the group $U(X)$:

$$l(f(X)) = \frac{1}{p}\log\frac{f(X)^p}{f(X)^{\Delta}}.$$

Lemma 1. *The function $l(f)$ has integer coefficients in Z_p. In addition, $l(f)$ has the following properties:*

1. $l(f(X)g(X)) = l(f(X)) + l(g(X))$
2. $l(f(X)^a) = al(f(X))$

for series $f(X)$ and $g(X)$ of group $U(X)$ and the integer a of Z_p.

Proof. The first property was proven in ([4], *Lemma* 2). The second property follows from the corresponding property of the logarithm and the additive property of the operator Δ. □

We now define the pairing $< *, * >$ on $U(X) \times U(X)$ by the formula

$$< f(X), g(X) >= \{res_x(l(f(X))\frac{d}{dX}\log g(X) - l(g(X))\frac{d}{dX}\frac{\Delta}{p}\log f(X))X^{-p}\} \bmod p.$$

Proposition 1. *The pairing $< *, * >$ has the following properties:*

1. *It is bilinear, i.e.*
 $< f_1f_2, g >=< f_1, g > + < f_2, g >$,
 $< f^a, g >= a < f, g >$
 for series f_1,f_2,g, of $U(X)$ and an integer a of Z_p, and similar equalities for the second argument.

2. *It is skew-symmetric, i.e.*
 $< f, g > + < g, f >= 0.$

Proof. Bilinearity of the pairing follows from the corresponding properties of the function $l(f)$ and logarithm. Let us now prove the skew-symmetry. We denote

$$\Phi(f, g) = l(f)d\log g - l(g)d\Delta\log f.$$

From the definition of the function $l(f)$ it follows that

$$\Phi(f,g) = l(g)dl(f) - l(g)d\log f + l(f)d\log g,$$

therefore

$$\Phi(f,g) + \Phi(g,f) = l(f)dl(g) + l(g)dl(f) = d(l(f)l(g)).$$

We conclude that

$$< f,g > + < g,f >= \{res_x(d(l(f)l(g))X^{-p}\} \equiv \{res_x(d(l(f)l(g))X(-p)\} \equiv 0 \bmod p$$

and skew-symmetry of the pairing is proved. □

Remark 1. The pairing $< *,* >$ has the property of independence of each of the arguments too. For that, let $Eis(X)$ be the Eisenstein irreducible polynomial of degree $p-1$,

$$Eis(X) = \frac{((1+X)^p - 1)}{X^p},$$

and let $r(X)$ be the remainder from the division of $f(X) - 1$ by polynomial $u(X)$. Then

$$< f(X), g(X) >=< r(X), g(X) > .$$

Remark 2. Properties of the pairing $< *,* >$ from Proposition 2, are similar to those of the Weil pairing and are proven in [1, 2]. For the general case see [5], Chapter VII).

3 Authentication Protocol without Disclosure

Proof of security of the protocol under discussion is determined by the properties of the proposed pairing function and non-polynomial complexity problem of the discrete logarithm in a polynomial ring with integer coefficients, which in general case is polynomially reduced to the discrete logarithm in finite fields [4].

3.1 Protocol Parameters and its Members

Let A (Alice) and V (verifier) be the members of the protocol. The secret known by Alice is some polynomial $a(X)$ of the group $U(X)$. Both parties of the protocol know the number s, the polynomial $Eis(X)$ and the polynomial $A(X) = a^s \mod Eis(X)$. According to the classical problem of authentication protocol without disclosure, Alice must prove to the verifier her knowledge of the secret polynomial $a(X)$, without disclosing it.

3.2 The Choreography of the Protocol

1. Alice selects a random polynomial $r(X)$ and determines the polynomial $R(X) = r(X)^s \mod Eis(X)$
2. Alice sends to the verifier the value of $R(X)$
3. V can request from A one of the following responses
   - the first response is the polynomial
   $$z(X) :< z(X), R(X) >= s < z(X), z(X) >$$
   - the second response is the polynomial
   $$y(X) :< y(X), R(X)A(X) >= s < y(X), y(X) > .$$
4. For the first response, A uses the known polynomial $r(X)$ and forms the polynomial $z(X) = r(X)$. For the second response, Alice uses the secret polynomial $a(X)$ to calculate the polynomial $y(X) = r(X)a(X) \mod Eis(X)$
5. V verifies the correctitude of the answers of A
   - for the first response
   $$< z(X), R(X) >=< r(X), r(X)^s >= s < r(X), r(X) >,$$
   - for the second response:
   $$\begin{aligned} < y(X), R(X)A(X) > &=< r(X)a(X), R(X)A(X) > \\ &= s < r(X)a(X), r(X), a(X) > \\ &= s < y(X), y(X) > . \end{aligned}$$

The above steps are performed until the verifier is convinced that Alice knows the secret polynomial $a(X)$. All the properties of the given protocol correspond to the properties of the classic authentication protocol without disclosure.

4 Final Remarks

In this paper, we propose a new system authentication protocol without disclosure. The system uses the idea of Explicit Hilbert Pairing of the Reciprocity Law (see [1, 2]). Explicit Hilbert pairing is used because it is bilinear and skew-symmetric (see Section 2). These properties make it interesting and paramount to building the system's protocol. The principle described in this paper can not only be used in other applications, like digital signature, but also as an extension to other security models.

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S. V. Vostokov
Sankt-Petersburg State University
E-mail: *s.vostokov@spbu.ru*

R. P. Vostokova
Baltic State Technical University "VOENMEH"
E-mail: *rvostokova@yandex.ru*

I. A. Budanaev
Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
E-mail: *ivan.budanaev@gmail.com*

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Almost periodicity of functions on universal algebras

Mitrofan M. Choban, Dorin I. Pavel

Abstract. The Bohr compactification is well known for groups and semigroups [1,4,7,11,13]. In the present paper the analogue of Bohr compactification is considered for universal algebras. Some questions posed by J. E. Hart and K. Kunen [9] are answered.

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Keywords and phrases: Universal algebra, compactification, almost periodic function, weakly almost periodic function, stable pseudometric.

1 Introduction

The aim of the present article is to study the compactifications of topological universal algebras generated by special functions. Any space is considered to be Tychonoff and non-empty. We use the terminology from [8].

The discrete sum $\Omega = \oplus\{\Omega_n : n \in N = \{0, 1, 2, ...\}\}$ of the pairwise disjoint discrete spaces $\{\Omega_n : n \in N\}$ is called a signature. A topological Ω-algebra or a topological universal algebra of the signature Ω is a family $\{G, e_{nG} : n \in N\}$, where G is a non-empty topological space and $e_{nG} : \Omega_n \times G^n \to G$ is a continuous mapping for each $n \in N$.

Subalgebras, homomorphisms, isomorphisms and Cartesian products of topological Ω-algebras are defined in traditional way [4, 5, 7, 9].

Let G be a topological space and $n \in N$. A continuous mapping $\lambda : G^n \to G$ is called an n-ary operation on G.

If G is a topological Ω-algebra and $\omega \in \Omega_n$, then $\omega : G^n \to G$, where $\omega(x) = e_{nG}(\omega, x)$ for every $x \in G^n$, is an n-ary operation on G.

A pair (Y, φ) is a generalized compactification or a g-compactification of a topological space X if Y is a compact space, $\varphi : X \to Y$ is a continuous mapping and the set $\varphi(X)$ is dense in Y. If (Z, φ) and (Y, ψ) are g-compactifications of X, then $(Z, \varphi) \leq (Y, \psi)$ if and only if there exists a continuous mapping $g : Y \to Z$ such that $\varphi = g \circ \psi$. If $\varphi : X \to Y$ is an embedding, then a pair (Y, φ) is called a compactification and we consider that $X \subseteq Y$ and $\varphi(x) = x$ for each $x \in X$.

If (Y, φ) and (Z, ψ) are g-compactifications of X, $(Y, \varphi) \leq (Z, \psi)$ and $(Z, \psi) \leq (Y, \varphi)$, then the g-compactifications (Y, φ), (Z, ψ) are called equivalent and there exists a unique homeomorphism $g : Y \to Z$ such that $\psi = g \circ \varphi$. We identify the equivalent g-compactifications. In this case the class of all g-compactifications of the space X is a set.

A pair (E, φ) is an algebraical g-compactification or an ag-compactification of a topological Ω-algebra G if E is a compact topological Ω-algebra, $\varphi : G \to E$ is a continuous homomorphism and the set $\varphi(G)$ is dense in E. If (Z, φ) and (Y, ψ) are ag-compactifications of G and $(Y, \varphi) \leq (Z, \psi)$, then the unique continuous mapping $g : Y \to Z$, for which $\psi = g \circ \varphi$, is a continuous homomorphism of Y onto Z. If $(Y, \varphi) \leq (Z, \psi)$ and $(Z, \psi) \leq (Y, \varphi)$, then the ag-compactifications (Y, φ), (Z, ψ) are called equivalent and there exists a unique topological isomorphism $g : Y \to Z$ such that $\psi = g \circ \varphi$.

If a pair (E, φ) is an ag-compactification and a compactification of a topological Ω-algebra G, then (E, φ) is called an a-compactification of G. If $\Omega = \Omega_0$, then any g-compactification of a topological Ω-algebra G is an ag-compactification.

If G is a topological Ω-algebra, then $Com_\Omega(G)$ is the set of all ag-compactifications of the topological Ω-algebra G.

The following properties are obvious.

Property 1. *The set $Com_\Omega(G)$ is a complete lattice for every topological Ω-algebra G and for every non-empty subset $L \subseteq Com_\Omega(X)$ there exist the maximal element $\vee L$ and the minimal element $\wedge L$.*

Property 2. *In the lattice of all ag-compactifications of a topological Ω-algebra G there exists the maximal a-compactification $(\beta_\Omega G, \beta_G)$, which is called the Bohr compactification of the topological Ω-algebra G.*

Property 3. *In the lattice of all ag-compactifications of a topological Ω-algebra G there exists the minimal ag-compactification $(\mu_a G, \mu_G)$, which is the singleton Ω-algebra.*

As a rule, the Bohr compactification of a topological Ω-algebra G is an ag-compactification of G.

Fix a topological space G. Let $C(G)$ be the space of real-valued continuous functions on the space G in the topology of uniform convergence. The topology on $C(G)$ is generated by the metric $d(f, g) = \sup\{|f(x) - g(x)| : x \in X\}$. Let $C^\circ(G)$ be the subspace of bounded functions. Then $C^\circ(G)$ is a Banach algebra (ring) with the norm $||f|| = \sup\{|f(x)| : x \in G\}$. For some $f, g \in C(G)$ it is possible that $d(f, g) = \infty$. We have $C(G) = C^\circ(G)$ if and only if the space G is pseudocompact. If $C(G) \neq C^\circ(G)$, then $C(G)$ is a linear space, but is not a topological linear space. The space $C(G)$ is a topological ring relative to the operations $f + g$ and $f \cdot g$. For any number $\lambda \in \mathbb{R}$ the correspondence $t_\lambda(f) = \lambda f$ is a continuous mapping of $C(G)$ into $C(G)$. For $\lambda \neq 0$, the correspondence $t_\lambda : C(G) \to C(G)$ is a homeomorphism.

Compactifications of the spaces can be produced in a variety of ways. One way is by use of subspaces of the space $C^\circ(G)$.

Let $F \subseteq C^\circ(G)$ be a non-empty subspace. Consider the mapping $e_F : G \to \mathbb{R}^F$, where $e_F(x) = (f(x) : f \in F)$. Denote by $b_F G$ the closure of the set $e_F(G)$ in $\mathbb{R}^F$. Then $(b_F G, e_F)$ is a g-compactification of G and $\beta G = \beta_{C^\circ(G)} G$ is the Stone-Čech maximal compactification of G [8]. Moreover, the family of functions $\bar{F} = \{g \in C(b_F G) : g \circ e_F \in F\}$ separates points of the space $b_F G$. Hence, if F is a ring which contains all constant functions and is closed in $C^\circ(G)$, then, by Stone-Weierstrass theorem ([8], Theorem 3.2.21), we have $C(b_F G) = \bar{F}$ and $F = \{g \circ e_F : g \in C(b_F G\}$.

Let (E, φ) be a g-compactification of a topological Ω-algebra G. If $C_E(G) = \{f \circ \varphi : f \in C(E)\}$, then $C_E(G)$ is the maximal subalgebra of the Banach algebra $C^{\circ}(G)$ such that $(E, \varphi) = (b_{C_E(G)}G, e_{C_E(G)})$. Denote $C^{\Omega}(G) = C_{\beta_\Omega G}(G)$.

Question A. Let G be a topological Ω-algebra and $F \subseteq C^{\circ}(G)$. Under which conditions $(b_F G, e_F)$ is an Ω-algebra ag-compactification of G and $e_F : G \to b_F G$ is a homomorphism?

Question B. Let G be a topological Ω-algebra and $f \in C(G)$. Under which conditions $f \in C^{\Omega}(G)$?

In [9] J. E. Hart and K. Kunen had formulated the next problems for the class E of all compact Ω-algebras:

Problem 1. To define the compactification $\beta_\Omega G$ as for groups directly with some notion of almost periodicity for functions ([9], Remark 2.4.1).

Problem 2. To give a method of construction of the Bohr compactification of an arbitrary algebra ([9], Remarks 2.4.1 and 3.1.6).

In this paper these problems are considered for arbitrary algebras.

We need the following elementary assertion.

Lemma 1. *Let (X, d) be a complete metric space. For a non-empty subset L of the space X the following assertions are equivalent:*

1. The closure $H = cl_X L$ of the set L in X is a compact subset of X.

2. For every $\epsilon > 0$ there exists a finite subset $S(\epsilon)$ of X such that $d(x, S(\epsilon)) = inf\{d((x, y) : y \in S(\epsilon)\} \leq \epsilon$ for each $x \in L$.

Proof. Follows immediately from Theorem 4.3.29 from [8], which affirms that a metrizable space Y is compact if and only if on Y there exists a metric ρ which is both totally bounded and complete. $\square$

2 Almost periodicity on topological spaces

Fix a topological space G. Denote by $\Pi(G)$ the set of all continuous mappings $\varphi : G \to G$. Relative to the operation of composition $\varphi \circ \psi$, where $(\varphi \circ \psi)(x) = \varphi(\psi(x))$ for $\psi, \psi \in \Pi(G)$ and $x \in G$, the set $\Pi(G)$ is a semigroup with identity e_G, where $e_G(x) = x$ for each $x \in G$. A semigroup with identity is called a monoid. We say that $\Pi(G)$ is the monoid of all continuous translations of G. If $f \in C(G)$ and $\varphi \in \Pi(G)$, then $f_\varphi = f \circ \varphi$ ($f_\varphi(x) = f(\varphi(x))$ for any $x \in G$). Evidently, $f_\varphi \in C(G)$.

Fix a non-empty subset $P \subseteq \Pi(G)$. We say that P is a set of continuous translations of G. The set P is called a transitive set of translations of G if for any two points $x, y \in G$ there exists $\varphi \in P$ such that $\varphi(x) = y$. Obviously, the monoid $\Pi(G)$ is transitive.

For any function $f \in C(G)$ we put $P(f) = \{f_\varphi : \varphi \in P\}$. If $f \in C^{\circ}(G)$, then $P(f) \subseteq C^{\circ}(G)$.

Definition 1. A function $f \in C(G)$ is called a P-periodic function on a space G if the closure $\bar{P}(f)$ of the set $P(f)$ in the space $C(G)$ is a compact set.

Denote by P-$ap(G)$ the subspace of all P-periodic functions of a space G and P°-$ap(G) = P$-$ap(G) \cap C^{\circ}(G)$.

If the set P is finite, then P-$ap(G) = C(G)$.

Theorem 1. *Let P be a set of continuous translations of G. Then P-$ap(G)$ has the following properties:*

1. P-$ap(G)$ is a linear subspace of the linear space $C(G)$.

2. P-$ap(G)$ is a topological subring of the topological ring $C(G)$.

3. P-$ap(G)$ is a closed subspace of the complete metric space $C(G)$. In particular, P-$ap(G)$ is a complete metric space.

4. If $f \in C(G)$ is a constant function, then $f \in P$-$ap(G)$.

5. If $f \in P$-$ap(G)$, then for any $x \in G$ there exists a number $c(f, x) > 0$ such that $|f(\varphi(x))| \leq c(f, x)$ for any $\varphi \in P$.

6. If $f \in P$-$ap(G)$, $\psi \in \Pi(G)$ and $g(x) = f(\psi(x))$ for each $x \in G$, then $g \in P$-$ap(G)$. In particular, $P(f) \subseteq P$-$ap(G)$ and $f_{\psi} \in P$-$ap(G)$ for all $f \in P$-$ap(G)$ and $\psi \in \Pi(G)$.

7. P°-$ap(G)$ is a Banach algebra of continuous functions.

Proof. Fix $f, g \in P$-$ap(G)$. Since $\bar{P}(f)$, $\bar{P}(g)$, $-\bar{P}(f)$, $\bar{P}(f) + \bar{P}(g)$ and $\bar{P}(f) \cdot \bar{P}(g)$ are compact subsets of P-$ap(G)$ and $-\bar{P}(f) = \bar{P}(-f)$, $\bar{P}(f + g) \subseteq \bar{P}(f) + \bar{P}(g)$, $\bar{P}(f \cdot g) \subseteq \bar{P}(f) \cdot \bar{P}(g)$, then $-f, f + g, f \cdot g \in P$-$ap(G)$. Hence P-$ap(G)$ is a topological subring of the topological ring $C(G)$.

If $f \in P$-$ap(G)$ and $\lambda \in \mathbb{R}$, then the correspondence $t_{\lambda}(f) = \lambda f$ is a continuous mapping of $C(G)$ into $C(G)$ and $\bar{P}(\lambda f) = t_{\lambda}(\bar{P}(f))$. Hence $\lambda f \in P$-$ap(G)$ and P-$ap(G)$ is a linear subspace of the linear space $C(G)$.

Let $\{f_n \in P\text{-}ap(G) : n \in \mathbb{N}\}$ and $f = lim_{n \to \infty} f_n$. It is well known that $f \in C(G)$. Fix $\epsilon > 0$. There exist $n \in \mathbb{N}$ and a finite subset S of $C(G)$ such that:

– $|f_n(x) - f(x)| \leq \epsilon/3$ for each $x \in G$;

– $d(g, S) \leq \epsilon/3$ for each $g \in P(f_n)$.

Fix $\varphi \in P$. For a given $\epsilon > 0$ there exists $g \in S$ such that $|g(x) - f_n(\varphi(x)) \leq \epsilon/3 + \epsilon/3$ for each $x \in G$. Then $|g(x) - f(\varphi(x))| \leq |g(x) - f_n(\varphi(x))| + |f_n(\varphi(x)) - f(\varphi(x))| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$. Hence $d(h, S) \leq \epsilon$ for each $h \in P(f)$. By virtue of Lemma 1, $f \in P$-$ap(G)$. Hence, P-$ap(G)$ is a closed subspace of the complete metric space $C(G)$.

The Assertion 4 is obvious.

Assume that $f \in C(G)$, $b \in G$ and the set $\{f(\varphi(x)) : \varphi \in P\}$ is unbounded. Then there exists a sequence $\{\varphi_n \in P : n \in \mathbb{N}\}$ such that $|f(\varphi_1(b))| \geq 2 + |f(b)|$ and $|f(\varphi_{n+1}(b))| \geq 2 + |f(\varphi_n(b))|$ for each $n \in \mathbb{N}$. We put $g_n(x) = f(\varphi_n(x))$. Then $d(f, g_n) \geq 2n$ for each $n \in \mathbb{N}$. Hence $P(f)$ is an unbounded subset of $C(G)$ and $f \notin P$-$ap(G)$. The Assertion 5 is proved.

Fix $\psi \in \Pi(G)$. Consider the mapping $\Phi : C(G) \longrightarrow C(G)$, where $\Phi(h)(x) = h(\psi(x))$ for all $h \in C(G)$ and $x \in G$. We have $d(\Phi(f), \Phi(g)) \leq d(f, g)$ for all $f, g \in C(G)$. Fix now $f \in P$-$ap(G)$ and put $g(x) = f(\psi(x))$ for each $x \in G$. Let $\epsilon > 0$. Then there exists a finite subset S of $C(G)$ such that $d(h, S) \leq \epsilon$ for each $h \in P(f)$. We have $g_{\varphi}(x) = g(\varphi(x)) = f(\varphi(\psi(x)))$ for each $x \in G$ and each $\varphi \in P$.

Assume that $\varphi \in P$, $\delta > 0$, $h \in C(G)$ and $d(f_\varphi, h) \leq \delta$. Since $|f(\varphi(x) - h(x)| \leq \delta$ for any $x \in G$, we have $|f(\varphi(\psi(x)) - h(\psi(x))| \leq \delta$ for any $x \in G$. Hence, the set $\Phi(S)$ is finite and $d(h, \Phi(S)) \leq \epsilon$ for each $h \in P(g)$. By virtue of Lemma 1, the Assertion 6 is proved. The Assertion 7 is obvious. □

Corollary 1. *If P is a transitive set of translations of G, then any function $f \in P$-$ap(G)$ is bounded and P-$ap(G)$ is a Banach algebra of continuous functions.*

Theorem 2. *Let P be a set of continuous translations of G and F be a compact subset of the complete metric space P-$ap(G)$. Then the closure H of the set $P(F)$ $= \cup\{P(f) : f \in F\}$ is a compact subset of the space P-$ap(G)$.*

Proof. Fix $\epsilon > 0$. There exists a finite subset S_1 of F such that $d(h, S_1) \leq \epsilon/2$ for each $h \in F$. For each $f \in F$ there exists a finite subset S_f of $P(f)$ such that $d(h, S_f) \leq \epsilon/2$ for each $h \in P(f)$. We put $S = \cup\{S_f : f \in S_1\}$. Fix $h \in F$ and $\varphi \in P$. There exists $f \in S_1$ such that $d(f, h) \leq \epsilon/2$. In continuation, there exists $g \in S_f$ such that $d(f_\varphi, g) \leq \epsilon/2$. Since $d(h_\varphi, f_\varphi) \leq d(h, f)$, we have $d(h_\varphi, g) \leq d(h_\varphi, f_\varphi) + d(f_\varphi, g) \leq \epsilon$. Hence $d(h, S) \leq \epsilon$ for each $h \in P(F)$. Lemma 1 completes the proof. □

Definition 2. Let G be a space and $\Gamma = \{P_\alpha : \alpha \in A\}$ be a non-empty family of non-empty subsets of the semigroup $\Pi(G)$. A function $f \in C(G)$ is called a Γ-periodic function of a space G if the function $f \in C(G)$ is P_α-periodic for any $\alpha \in A$.

Let G be a space and $\Gamma = \{P_\alpha : \alpha \in A\}$ be a non-empty family of non-empty subsets of the semigroup $\Pi(G)$. Denote by Γ-$ap(G)$ the subspace of all Γ-periodic functions of a space G. By definition, we have Γ-$ap(G) = \cap\{P_\alpha$-$ap(G) : \alpha \in A\}$.

From Theorem 1 follows

Corollary 2. *Let G be a space and $\Gamma = \{P_\alpha : \alpha \in A\}$ be a non-empty family of non-empty subsets of the semigroup $\Pi(G)$. Then Γ-$ap(G)$ has the following properties:*

1. Γ-$ap(G)$ is a linear subspace of the linear space $C(G)$.

2. Γ-$ap(G)$ is a topological subring of the topological ring $C(G)$.

3. Γ-$ap(G)$ is a closed subspace of the complete metric space $C(G)$. In particular, Γ-$ap(G)$ is a complete metric space.

4. If $f \in C(G)$ is a constant function, then $f \in \Gamma$-$ap(G)$.

5. If $f \in \Gamma$-$ap(G)$, $\psi \in \Pi(G)$ and $g(x) = f(\psi(x))$ for each $x \in G$, then $g \in \Gamma$-$ap(G)$. In particular, $f_\psi \in \Gamma$-$ap(G)$ for all $f \in \Gamma$-$ap(G)$ and $\psi \in \Pi(G)$.

6. Γ°-$ap(G)$ is a Banach algebra of continuous functions.

Let G be a space and $\Gamma = \{P_\alpha : \alpha \in A\}$ be a non-empty family of non-empty subsets of the semigroup $\Pi(G)$. A finite oriented set $(\alpha_1, \alpha_2, ..., \alpha_n)$, where $\alpha_1, \alpha_2, ..., \alpha_n \in A$ and $n \geq 1$, is called an A-cortege of the length n. For any A-cortege $\beta = (\alpha_1, \alpha_2, ..., \alpha_n)$ we put $B_\beta = \{\varphi_{\alpha_1} \circ \varphi_{\alpha_2} \circ ... \circ \varphi_{\alpha_n} : (\varphi_{\alpha_1}, \varphi_{\alpha_2}, ...\varphi_{\alpha_n}) \in P_{\alpha_1} \times P_{\alpha_2} \times ... \times P_{\alpha_n}\}$. Denote by A_∞ the set of all A-corteges and $\Gamma_\infty =$

$\{B_\beta : \beta \in A_\infty\}$. Then Γ_∞ is a non-empty family of non-empty subsets of the monoid $\Pi(G)$, $A \subseteq A_\infty$ and $\cup\{B_\beta : \beta \in A_\infty\}$ is a semigroup of the monoid $\Pi(G)$.

From Theorem 2 follows

Corollary 3. *Let G be a space and $\Gamma = \{P_\alpha : \alpha \in A\}$ be a non-empty family of non-empty subsets of the semigroup $\Pi(G)$. Then Γ_∞-ap$(G) = \Gamma$-ap(G).*

3 Almost periodicity on dynamical systems

A topological monoid is a topological space A with a continuous mapping $\cdot : A \times A \to A$ for which there exists a point $1 \in A$ such that $1 \cdot x = x \cdot 1 = x$ and $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for each $x, y, z \in X$. The element 1 is the unity of monoid A and we say that $xy = x \cdot y$ is the product of x, y.

A dynamical system is a triple (G, S, m), where S is a topological monoid, G is a Tychonoff space and $m : S \times G \to G$ is a continuous action on G, i. e. $m(s, m(t, x)) = m(st, x)$ and $m(1, x) = x$ for all $s, t \in S$ and $x \in G$. In theory of finite state machines and in automata theory the dynamical system (G, S, m) is called a semiautomaton, where S is called the input alphabet, G is called the set of states and m is the transition function.

Remark 1. Let G be a non-empty space. Then the semigroup $\Pi(G)$ is a monoid. Consider the evaluation action $e_G : \Pi(G) \times G \longrightarrow G$, where $e_G(\varphi) = \varphi(x)$ for all $x \in G$ and $\varphi \in \Pi(G)$. If S is a submonoid of the monoid $\Pi(G)$ and $m = e_G|S \times G$, then (G, S, m) is a dynamical system. In particular, $(G, \Pi(G), e_G)$ is a dynamical system.

Fix a discrete monoid S and a dynamical system (G, S, m). Then G is a topological universal algebra of the signature S. All operations from S are unary.

For any continuous real-valued function $f : G \to \mathbb{R}$ and any $s \in S$ we consider the function $f_s : G \to \mathbb{R}$, where $f_s(x) = f(m(s, x))$ for each $x \in G$, and put $S(f) = \{f_s : s \in S\}$.

A continuous function $f : X \to \mathbb{R}$ is called an almost periodic function of the dynamical system (G, S, m) if the closure $cl_{C(G)}S(f)$ is a compact subset of $C(G)$. Denote by $S(m)$-ap(G) the class of all almost periodic functions on G and $S(m)^\circ$-ap$(G) = S(m)$-ap$(G) \cap C^\circ(G)$.

Remark 2. Any element $s \in S$ generates the continuous mapping $m_s : G \longrightarrow G$, where $m_s(x) = m(s, x)$ for any point $x \in G$. We put $S_G = \{m_s : s \in G\}$. Then S_G is a submonoid of the monoid $\Pi(G)$. By construction, $f_s = f_{m_s}$ for all $s \in S$ and $f \in C(G)$. Hence $S(f) = S_G(f)$ for any function $f \in C(G)$. In particular, $S(m)$-ap$(G) = S_G$-ap(G).

The continuous action $m : S \times G \longrightarrow G$ generates the continuous action $m_C : S \times C(G) \longrightarrow C(G)$, where $m_C(s, f) = f_s$ for all $s \in S$ and $f \in C(G)$. Hence $(S, C(G), m_C)$ is a dynamical system generated by the continuous action $m : S \times G \longrightarrow G$. From Theorem 2 it follows that $m_C(S$-ap$(G) = S(m)$-ap(G). Therefore

$(S\text{-}ap(G), S, m_C)$ is a dynamical system too, generated by the continuous action $m : S \times G \longrightarrow G$.

From Theorem 1 follows

Corollary 4. *Let G be a space, S be a discrete monoid and (G, S, m) be a dynamical system. Then the space S-$ap(G)$ has the following properties:*

1. $S(m)$-$ap(G)$ is a linear subspace of the linear space $C(G)$.

2. $S(m)$-$ap(G)$ is a topological subring of the topological ring $C(G)$.

3. $S(m)$-$ap(G)$ is a closed subspace of the complete metric space $C(G)$. In particular, S-$ap(G)$ is a complete metric space.

4. If $f \in C(G)$ is a constant function, then $f \in S$-$ap(G)$.

5. If $f \in S(m)$-$ap(G)$, $\psi \in \Pi(G)$ and $g(x) = f(\psi(x))$ for each $x \in G$, then $g \in S(m)$-$ap(G)$. In particular, $S(f) \subseteq S(m)$-$ap(G)$ for any $f \in S(m)$-$ap(G)$.

6. $S(m)^\circ$-$ap(G)$ is a Banach algebra of continuous functions.

If ρ is a pseudometric on G, $x \in G$ and $r > 0$, then $B(x, \rho, r) = \{y \in G : \rho(x, y) < r\}$ is the r-ball with the center x. The pseudometric ρ is continuous if the sets $B(x, \rho, r)$ are open in G.

A pseudometric ρ on G is totally bounded if for any real number $r > 0$ there exists a finite subset F of G such that $\rho(x, F) = min\{\rho(x, y) : y \in F\} < r$ for each $x \in G$.

A pseudometric ρ on (G, S, m) is totally S-bounded if it is totally bounded and for any real number $r > 0$ there exists a finite subset L of S such that: for each $s \in S$ there exists $s_r \in L$ such that $\rho(m(s, x), m(s_r, x)) < r$ for each $x \in G$.

A pseudometric $\rho : G \times G \to \mathbb{R}$ is S-invariant on (G, S, m) if ρ is continuous, $\rho(x, y) < \infty$ and $\rho(m(s, x), m(s, y)) \leq \rho(x, y)$ for all $x, y \in G$ and $s \in S$.

If $f : G \to \mathbb{R}$ is a function, then we put $\rho_f(x, y) = sup\{|f_s(x) - f_s(y)| : s \in S\}$ for all $x, y \in G$.

Theorem 3. *Fix a dynamical system (G, S, m) and $f \in S$-$ap(G)$. Then:*

1. ρ_f is an S-invariant pseudometric.

2. ρ_f is a continuous pseudometric on G.

3. ρ_f is a totally bounded pseudometric if and only if the function f is bounded.

4. ρ_f is a totally S-bounded pseudometric provided the function f is bounded and for any real number $r > 0$ there exists a finite subset L of S such that: for each $s \in S$ there exists $s_r \in L$ such that $|f(m(ts, x) - f(m(ts_r, x))| < r$ for each $x \in G$ and every $t \in S$.

Proof. 1. Fix two points $x, y \in G$. By virtue of the Assertion 5 from Theorem 1, there exists a number $c > 0$ such that $|f_s(x))| \leq c$ and $|f_s(y))| \leq c$ for any $s \in S$. Hence $\rho_f(x, y) \leq 2c < \infty$. Let $\mu \in S$ and $g = f_\mu$. Then $g \in S$-$ap(G)$, $g_s = f_{s\mu}$ for any $s \in G$ and $\rho_f(m(s, x), m(s, y)) = sup\{|g_s(x) - g_s(y)| : s \in S\} = sup\{|f_{s\mu}(x) - f_{s\mu}(y)| : s \in S\} \leq sup\{|f_s(x) - f_s(y)| : s \in S\} = \rho_f(x, y)$. Hence the pseudometric ρ_f is S-invariant.

2. Now fix a number $r > 0$ and a point $b \in G$. Then there exists a finite subset L of S such that $1 \in L$ and for each $s \in S$ there exists $l(s) \in L$ such

that $d(f_s, f_{l(s)}) < r/3$. Since the set L is finite, the set $U(b, L, r) = \{x \in G : |f_s(x) - f_s(b)| < r/3, s \in L\}$ is open in G. Hence $|f_s(x) - f_s(b)| \leq |f_s(x) - f_{s(l)}(x)| + |f_{s(l)}(x) - f_{s(l)}(b)| + |f_{s(l)}(b) - f_s(b)| < r$ for all $s \in L$ and $x \in U(b, L, r)$. Therefore $U(b, L, r) \subseteq B(b, \rho_f, r)$ and $B(b, \rho_f, r)$ is an open subset of G. Thus ρ_f is a continuous pseudometric on G. By construction, $\rho_f(m(s, x), m(l(s), x)) = sup\{|f_t(m(s, x)) - f_t m(l(s), x)| : t \in S\} = sup\{\rho_f((m(t \cdot s, x), m(t \cdot l(s), x)) : t \in S\}$.

3. Assume that the function f is bounded and $r > 0$. There exists a finite subset L of S such that $1 \in L$ and for each $s \in S$ there exists $l(s) \in L$ such that $d(f_s, f_{l(s)}) < r/3$. Since the functions f_s are bounded and the set L is finite, there exists a finite subset F of G such that for each $x \in G$ there exists $x(f) \in F$ such that $|f_s(x) - f_s(x(f))| < r/3$ for each $s \in L$. Hence $\rho_f(x, F) < r$ for each $x \in G$ and ρ_f is a totally bounded pseudometric.

4. Fix $b \in G$. Since $|f(x) - f(b)| \leq \rho_f(b, x)$ the function f is bounded if and only if the pseudometric ρ_f is bounded (i.e. $sup\{\rho_f(x, y) : x, y \in G\} < \infty$).

5. Assume that the function f is bounded and for any real number $r > 0$ there exists a finite subset L_r of S such that: for each $s \in S$ there exists $s_r \in L_r$ such that $|f(m(ts, x) - f(m(ts_r, x))| < r$ for each $x \in G$ and every $t \in S$.

Fix $r > 0$ and $s \in S$. Then $\rho_f(m(s, x), m(s_r, x)) = sup\{|f(m(ts, x) - f(m(ts_r, x))| : t \in S\} \leq r$. The proof is complete. $\square$

If ρ is a bounded pseudometric on G and $a \in G$, then we put $f_{(\rho,a)}(x) = \rho(a, x)$ for any $x \in G$.

Theorem 4. *If ρ is a totally S-bounded S-invariant pseudometric on a dynamical system (G, S, m) and $a \in G$, then $f_{(\rho,a)} \in S(m)$-$ap(G)$ and the function $f_{(\rho,a)}$ is bounded for each $a \in G$.*

Proof. Fix $a \in G$ and $r > 0$. Let $g = f_{(\rho,a)}$. We have $g_s(x) = \rho(a, m(s, x))$ for all $x \in G$ and $s \in S$. Obviously, the function g is bounded. By assumption, there exists a finite subset L of S such that: for each $s \in S$ there exists $s_r \in L$ such that $\rho(m(s, x), m(s_r, x)) < r$ for each $x \in G$. We have $|g_s(x) - g_{s_r}(x)| = |\rho(a, m(s, x)) - \rho(a, m(s_r, x))| \leq \rho(m(s, x), m(s_r, x)) < r$. By virtue of Lemma 1, the assertion is proved. $\square$

Theorem 5. *Fix a dynamical system (G, S, m). Then there exist a dynamical system $(\beta_{ap(S,m)}G, S, m_G)$ and a continuous mapping $\varphi : G \longrightarrow ap_{(S,m)}G$ such that:*

1. $\beta_{ap(S,m)}G$ is a compact space and the set $\varphi(G)$ is dense in $b_{ap(S,m)}G$.

2. φ is a homomorphism, i.e. $\varphi(m(s, x) = m(s, \varphi(x))$ for all $s \in S$ and $x \in G$.

3. $S(m)^{\circ}$-$ap(G) = \{g \circ \varphi : g \in C(\beta_{ap(S,m)}G)\}$.

4. $C(\beta_{ap(S,m)}G)\} = S(m)$-$ap(\beta_{ap(S,m)}G)\}$.

5. The topology of the space $\beta_{ap(S,m)}G$ is induced by the family of all S-invariant pseudometrics on the dynamical system $(\beta_{ap(S,m)}G, S, m_G)$.

Proof. Let $F = S(m)^{\circ}$-$ap(G)$. Consider the mapping $e_F : G \to \mathbb{R}^F$, where $e_F(x) = (f(x) : f \in F)$. Denote by $b_F G = \beta_{ap(S,m)}G$ the closure of the set $e_F(G)$ in $\mathbb{R}^F$. We put $\varphi = e_F$. Then $(b_F G, e_F)$ is a compactification of G. For any $f \in F$ consider the pseudometric $\rho_f(x,y) = sup\{|f_s(x) - f_s(y)| : s \in S\}$ for all $x, y \in G$. By virtue of Theorem 3, the pseudometric ρ_f is continuous, stable and totally bounded on (G, S, m). Since $|f(y) - f(x)| \leq \rho_f(x,y)$ for all $x, y \in G$, there exists a continuous pseudometric $\bar{\rho_f}$ on $b_F G$ such that $\rho_f(x,y) = \bar{\rho_f}(\varphi(x), \varphi(y))$ for all $x, y \in G$. We say that $\bar{\rho_f}$ is the continuous extension of ρ_f on $b_F G$. The topology of the compact space is induced by the pseudometrics $\{\bar{\rho_f} : f \in F\}$.

For every $f \in F$ there exists a unique function $\bar{f} \in C(b_F G)$ such that $f = \bar{f} \circ \varphi$. Hence $\bar{F} = \{\bar{f} : f \in F\}$ is a closed subalgebra of the Banach algebra $C(b_F G)$.

Fix $s \in S$. The mapping $m_s : G \to G$, where $m_s(x) = m(s,x)$ for every $x \in G$ is continuous. If $x \in G$, then we put $\mu_s(\varphi(x)) = \varphi(m_s(x))$. For $x, y \in G$ with $\varphi(x) = \varphi(y)$ we have $0 \leq \rho_f(m_s(x), m_s(y)) \leq \rho_f(x,y) = 0$ for any $f \in F$ and $\varphi(m_s(x)) = \varphi(m_s(y))$. Therefore μ_s is a single-valued continuous mapping of $\varphi(G)$ into $\varphi(G)$.

We have $\rho_f(m_s(x), m_s(y)) \leq \rho_f(x,y)$ for all $x, y \in G$. Hence the mapping m_s is uniformly continuous for every pseudometric $\bar{\rho_f}$, $f \in F$. Therefore there exists a continuous extension $\nu_s : b_F G \longrightarrow b_F G$ of μ_s. By construction, $\nu_s \circ \nu_t = \nu_{s \cdot t}$. We prove that $(\beta_{ap(S,m)} G, S, m_G)$, where $m_G(s,x) = \nu_s(x)$ for each $x \in b_F G = \beta_{ap(S,m)} G$, is a dynamical system.

By construction, φ is a homomorphism.

The mapping $\psi : F \longrightarrow C(b_F G)$, where $\psi(f) = \bar{f}$ for each $f \in F$ is an isometrical embedding. Hence $\psi(F) \subseteq S(m)\text{-}ap(\beta_{ap(S,m)} G)\}$. It is obvious that $g \circ \varphi \in F$ for any $g \in S(m)\text{-}ap(\beta_{ap(S,m)} G)\}$. Therefore $S(m)^{\circ}\text{-}ap(G) = \{g \circ \varphi : g \in S(m) - ap(\beta_{ap(S,m)} G)\}$.

Since $\{\bar{f} : f \in F\} = \{g|\varphi(G) : g \in S(m)\text{-}ap(\beta_{ap(S,m)} G)\}$, by Stone-Weierstrass theorem ([8], Theorem 3.2.21), we have $S(m)\text{-}ap(\beta_{ap(S,m)} G) = C(\beta_{ap(S,m)} G)$. The topology of the space $\beta_{ap(S,m)} G$ is induced by the family of S-invariant pseudometrics $\{\rho_g : g \in C(\beta_{ap(S,m)} G)\}$ on the dynamical system $(\beta_{ap(S,m)} G, S, m_G)$. The proof is complete. □

Remark 3. We say that the dynamical system $(\beta_{ap(S,m)} G, S, m_G)$ is the maximal a-compactification of the dynamical system (G, S, m).

4 Almost periodicity on universal algebras

Fix a discrete signature $\Omega = \oplus\{\Omega_n : n \in N = \{0, 1, 2, ...\}\}$, where $\{\Omega_n : n \in N\}$ is a non-empty family of pairwise disjoint discrete spaces.

Let $P(\Omega)$ be a minimal set of operations on Ω-algebras for which:

P1. $\Omega \subseteq P(\Omega)$.

P2. If $n \geq 1$, $\omega \in \Omega_n$, $p_1, ..., p_n \in P(E)$, p_i is an m_i-ary operation and $m = m_1 + ... + m_n$, then $p = \omega(p_1, ..., p_n)$ is an m-ary operation, $p(x_1, ..., x_m) = \omega(p_1(x_1, ..., x_m), ..., p_n(x_{m-m_n+1}, ..., x_m))$.

P3. If $u_0(x) = x$ for any Ω-algebra G and every $x \in G$, then $u_0 \in P(\Omega)$.

The set $P(\Omega)$ is called the set of Ω-polynomials. If G is a topological Ω-algebra and $p \in P(\Omega)$ is an n-ary polynomial, then $p : G^n \to G$ is a continuous operation.

Let $\lambda : G^n \to G$ be an n-ary operation. If $n = 0$, then we put $\lambda(x) = \lambda(G^0)$ for each $x \in G$ and $T_\lambda(G) = \{\lambda\}$. If $n = 1$, then $T_\lambda(G) = \{\lambda\}$. Let $n \geq 2$ and $1 \leq i \leq n$. For every $a = (a_1, ..., a_n) \in G^n$ we put $t_{ia\lambda}(x) = \lambda(a_1, ..., a_{i-1}, x, a_{i+1}, ..., a_n)$ for each $x \in G$. We put $T_{i\lambda}(G) = \{t_{ia\lambda} : a \in G^n\}$ and $T_\lambda(G) = \cup\{T_{i\lambda}(G) : i \leq n\}$. Therefore $T_\lambda(G)$ is a set of translations on a space G. If λ is a continuous operation, then $T_\lambda(G) \subseteq \Pi(G)$.

Now we put $T_\Omega(G) = \cup\{T_\omega(G) : \omega \in P(\Omega)\}$ for any topological Ω-algebra G. By construction, $T_\Omega(G)$ is a monoid of continuous translations of the space G and $T_\Omega(G) \subseteq \Pi(G)$.

If G is a topological Ω-algebra and $m_\Omega = e_G|T_\Omega(G) \times G$, then $(G, T_\Omega(G), m_\Omega)$ is a dynamical system, generated by the structure of Ω-algebra on G.

Definition 3. Let G be a topological Ω-algebra. The set Ω-$AP(G) = T_\Omega(G)(m_\Omega)$-$ap(G)$ is called the algebra of almost periodic continuous functions on the topological Ω-algebra G.

All statements proved in the above two Sections are true for almost periodic continuous functions on the topological Ω-algebras. The set Ω°-$AP(G) = C^\circ(G) \cap (\Omega$-$AP(G))$ is a Banach algebra of continuous functions on G.

Definition 4. An Ω-algebra G is called Ω-finite if there exists a finite subset $F \subseteq P(\Omega)$ such that $T_\Omega(G) = \cup\{T_\omega(G) : \omega \in F\}$.

Any finite Ω-algebra is Ω-finite. If Ω is a structure of a semigrup, or of a monoid, or a group on G, then G is is Ω-finite.

Definition 5. An Ω-algebra G is called a right (left) Mal'cev algebra if there exists a ternary operation $p \in P(\Omega)$ such that $p(x, x, y) = y$ (respectively $p(y, x, x) = y$) for all $x, y \in G$. If $p(x, x, y) = p(y, x, x) = y$, then G is called a Mal'cev algebra [4, 10].

Proposition 1. *Let G be a right (left) Mal'cev topological Ω-algebra. Then the monoid $T_\Omega(G)$ is transitive on G. Moreover, any almost periodic function $f \in \Omega$-$AP(G)$ is bounded and Ω-$AP(G)$ is a Banach algebra of continuous functions on G.*

Proof. Assume that $p \in P(\Omega)$ is a ternary operation and $p(x, x, y) = y$ for all $x, y \in G$. Fix $a, b \in G$. If $\varphi(x) = p(x, a, b)$, then $\varphi \in T_\Omega(G)$ and $\varphi(a) = b$. Hence the monoid $T_\Omega(G)$ is transitive on G. Corollary 1 completes the proof. $\square$

A pseudometric $\rho : G \times G \to \mathbb{R}$ is stable on a topological Ω-algebra G if ρ is continuous, $\rho(x, y) < \infty$ and $\rho(\omega(x_1, ..., x_n), \omega(y_1, ..., y_n)) \leq \Sigma\{\rho(x_i, y_i) : i \leq n\}$ for all $x_1, y_1, ..., x_n, y_n \in G$, $n \geq 1$ and $\omega \in \Omega$.

If ρ is a stable pseudometric on a topological Ω-algebra G, $n \leq 1$ and $p \in P(\Omega)$ is an n-ary polynomial, then $\rho(p(x_1, ..., x_n), p(y_1, ..., y_n)) \leq \Sigma\{\rho(x_i, y_i) : i \leq n\}$ for all $x_1, y_1, ..., x_n, y_n \in G$.

In [5] the following theorem was proved:

Theorem 6. *Let ρ be a continuous pseudometric on a topological Ω-algebra G. The pseudometric ρ is stable if and only if it is $T_\Omega(G)$-invariant.*

From Theorems 6 and 5 follows

Corollary 5. *Fix a topological Ω-algebra G. Then there exist an Ω-algebra $\beta_{ap(\Omega)}G$ and a continuous homomorphism $\alpha_G : G \longrightarrow \beta ap_(\Omega)G$ such that:*

1. $\beta_{ap(\Omega)}G$ is a compact Ω-algebra and the set $\alpha_G(G)$ is dense in $\beta_{ap(\Omega)}G$.

2. Ω°-$AP(G) = \{g \circ \varphi : g \in C(\beta_{ap(\Omega)}G)\}$.

3. $C(\beta_{ap(\Omega)}G)\} = \Omega$-$AP(\beta_{ap(S,m)}G)$.

4. The topology of the space $\beta_{ap(\Omega)}G)$ is induced by the family of all stable pseudometrics on the topological Ω-algebra $\beta_{ap(\Omega)}G)$.

5. The a-compactification $(\beta_{ap(\Omega)}G, \alpha G) = (b_F G, e_F)$, where $F = \Omega^\circ$-$AP(G)$.

Remark 4. We say that the topological Ω-algebra $\beta_{ap(\Omega)}G$ is the maximal almost periodic a-compactification of the topological Ω-algebra G.

Lemma 2. *Let G be a topological Ω-algebra and ρ be a stable totally bounded pseudometric on G. If $\omega \in P(\Omega)$, $c \in G$ and $h(x) = \rho(c, x)$ for any $x \in G$, then the function h is bounded and the closure of the set $\{h_\varphi : \varphi \in T_\omega(G)\}$ in $C^\circ(G)$ is a compact set.*

Proof. Since ρ is totally bounded, by construction, $h \in C^\circ(G)$. Fix $\epsilon > 0$. If ω is n-ary polynomial and $n \leq 1$, then the assertion of Lemma is obvious. Assume that $n \geq 2$ and ω is an n-ary polynomial. There exists a finite subset $L \subseteq G$ such that $\rho(x, L) < \epsilon/2$ for any $x \in G$. For every $i \leq n$ we put $T_{(i,\omega,L)} = \{t_{ia\lambda} : a = (a_1, ..., a_n) \in L^n\}$ and $T_{(\omega,L)}(G) = \cup\{T_{(i,\omega,L)}(G) : i \leq n\}$. Obviously, the set $T_{(\omega,L)}(G)$ is finite. Fix $\varphi \in T_\omega(G)$. Then $\varphi(x) = \omega(x_1, ..., x_{i-1}, x, x_{i+1}, ..., x_n)$ for some $i \leq n$ and $x = (x_1, ..., x_n) \in G^n$. There exists $a = (a_1, ..., a_n) \in L^n$ such that $\rho(x_j, a_j) < \epsilon/n$ for each $j \leq n$. Let $\psi = t_{ia\omega}$. Then $\psi \in T_{(\omega,L)}(G)$ and $h_\varphi(x) - h_\psi(x) < \Sigma\{\rho(x_j, a_j) : j \leq n, j \neq i\} < \epsilon$. Lemma 1 completes the proof. □

Lemma 3. *Let G be a a compact topological Ω-algebra, $n \in \mathbb{N}$ and $\omega \in P(\Omega)$ be an n-ary polynomial. If $h \in C(G)$, then the set $\{h_\varphi : \varphi \in T_\omega(G)\}$ in $C^\circ(G)$ is a compact set.*

Proof. If $n \leq 1$, then the assertion of Lemma is obvious. Assume that $n \geq 2$. Let $h \in C(G)$. Fix $i \leq n$. Let $G_k = G$ for any k and $Z_i = \Pi\{G_j : j \leq n, j \neq i\}$. For any $z = (z_1, ..., z_{i-1}, ..., z_{i+1}, ..., z_n) \in Z_i$ we put $\Psi_i(z)(x) = h(\omega(z_1, ..., z_{i-1}, x, z_{i+1}, ..., z_n))$ for each $x \in G$. Then $\Psi_i : Z_i \longrightarrow C(G)$ is a continuous mapping. Since $\Psi_i(Z_i) = \{h_\psi : \psi \in T_{i\omega}\}$, the set $\{h_\psi : \psi \in T_{i\omega}\}$ is compact. Hence the set $\{h_\varphi : \varphi \in T_\omega(G)\}$ is compact too. □

Corollary 6. *Let G be an Ω-finite topological Ω-algebra and ρ be a stable totally bounded pseudometric on G. If $c \in G$ and $h(x) = \rho(c, x)$ for any $x \in G$, then $h \in \Omega^\circ$-$AP(G)$.*

Corollary 7. *Let G be a compact Ω-finite topological Ω-algebra. Then:*

1. *The topology of G is induced by a family of stable pseudometrics.*
2. *Ω°-$AP(G) = C(G)$.*

Corollary 8. *Let G be an Ω-finite topological Ω-algebra. For any bounded continuous pseudometric ρ on G we put $C(G, \rho) = \{a + b \cdot \rho(z, x) : z \in G, a, d \in \mathbb{R}\}$. Then the set $\cup\{C(G, \rho) : \rho$ is a totally bounded stable pseudometric on $G\}$ is a dense subset of the Banach algebra Ω°-$AP(G)$.*

5 Weakly almost periodic functions on algebras

Fix a discrete signature $\Omega = \oplus\{\Omega_n : n \in N = \{0, 1, 2, ...\}\}$.

Definition 6. Let G be a topological Ω-algebra. A function $f \in C(G)$ is called a weakly almost periodic function on G if the closure of the set $\{f_t = f \circ t : t \in T_\omega(G)\}$ in $C(G)$ is compact for every $\omega \in P(\Omega)$.

If $\Gamma(\Omega) = \{T_\omega(G) : \omega \in P(\Omega)\}$, then Ω-$wAP(G) = \Gamma(\Omega)$-$ap(G)$ is the algebra of weakly almost periodic continuous functions on the topological Ω-algebra G. Hence Corollary 2 is true for the algebra of weakly almost periodic continuous functions on the topological Ω-algebra G. Moreover, if $\Gamma_0(\Omega) = \{T_\omega(G) : \omega \in \Omega\}$, then from Corollary 3 it follows that Ω-$wAP(G) = \Gamma_0(\Omega)$-$ap(G)$. Obviously, Ω-$AP(G) \subseteq \Omega$-$wAP(G)$. Let Ω°-$wAP(G) = \Omega$-$wAP(G) \cap C^\circ(G)$.

Theorem 7. *Let G be a a compact topological Ω-algebra. Then Ω-$wAP(G) = C(G)$.*

Proof. Follows from Lemma 3. □

Example 1. Let G be the compact space of all complex numbers z with $|z| = 1$. Relatively to the multiplicative operation $\{\cdot\}$ and inverse operation $\{^{-1}\}$ the space G is a compact commutative group with the unite 1. Let $g : G \longrightarrow G$ be a homeomorphism and $\omega_g(x, y) = x \cdot y$ for all $x, y \in G$. Then (G, ω_g) is a topological quasigroup. Denote by $P(g)$ the translations of the topological quasigroup (G, ω_g). Obviously, $g \in P(g)$.

In [6] such homeomorphism g_0 was constructed for which only constant functions are continuous almost periodic on (G, ω_{g_0}) and every stable pseudometric ρ on (G, ω_{g_0}) is trivial ($\rho(x, y) = 0$ for all $x, y \in G$). Let $\Omega_1 = \{^{-1}, g_0, g_0^{-1}\}$, $\Omega_2 = \{\cdot\}$ and $\Omega = \Omega_1 \cup \Omega_2$. Then $\omega_{g_0},^{-1}, g_0, g_0^{-1} \in P(\Omega)$ and only constant functions are continuous almost periodic on the Ω-algebra G. In particular, every stable pseudometric ρ on the Ω-algebra G is trivial. Therefore the Ω-algebra G is not Ω-finite. Since G is a compact space, then, by virtue of Theorem 7, we have Ω-$wAP(G) = C(G)$.

Definition 7. Let $\{\rho_\mu : \mu \in M\}$ be a family of pseudometrics on an Ω-algebra G. The family $\{\rho_\mu : \mu \in M\}$ is called a stable set of pseudometrics if the set M is non-empty and for every $\alpha \in M$, every $n \geq 1$ and every $\lambda \in \Omega_n$ there exists $\beta = \beta(\lambda, \alpha) \in M$ such that $\rho_\alpha(x_1, y_1) \leq \rho_\beta(x_1, y_1)$ and $\rho_\alpha(\lambda(x_1, ...x_n), \lambda(y_1, ..., y_n)) \leq \sum\{\rho_\beta(x_i, y_i) : i \leq n\}$ for all $x_1, y_1, ..., x_n, y_n \in G$.

Remark 5. Let $T(R)$ be the topology induced by a stable set of pseudometrics $R = \{\rho_\mu : \mu \in M\}$ on an Ω-algebra G. Then for each $n \geq 1$ and $\omega \in \Omega_n$ the operation ω is continuous relative to the topology $T(R)$.

Lemma 4. *Let $\{\rho_\mu : \mu \in M\}$ be a stable net of pseudometrics on an Ω-algebra G. Then for every $\alpha \in M$, every $n \geq 1$ and every n-ary polynomial $\lambda \in P(\Omega_)$ there exists $\beta = \beta(\lambda, \alpha) \in M$ such that $\rho_\alpha(x_1, y_1) \leq \rho_\beta(x_1, y_1)$ and $\rho_\alpha(\lambda(x_1, ...x_n), \lambda(y_1, ..., y_n)) \leq \sum\{\rho_\beta(x_i, y_i) : i \leq n\}$ for all $x_1, y_1, ..., x_n, y_n \in G$.*

Proof. Assume that $n, m_1, m_2, ..., m_n \geq 1$, $\lambda \in \Omega_n$, $p_1, p_2, ..., p_n \in P(\Omega)$ and for each $i \leq n$ the polynomial p_i is m_i-ary and for every $\alpha \in M$ there exists $\beta_i = \beta(p_i, \alpha) \in M$ such that $\rho_\alpha(x_1, y_1) \leq \rho_{\beta_i}(x_1, y_1)$ and $\rho_\alpha(p_i(x_1, ...x_{m_i}), p_i(y_1, ..., y_{m_i})) \leq \sum\{\rho_{\beta_i}(x_i, y_i) : i \leq m_i\}$ for all $x_1, y_1, ..., x_{m_i}, y_{m_i} \in G$. Put $p = \lambda(p_1, ..., p_n)$ and $m = m_1 + ... + m_n$. Then p is m-ary polynomial.

Fix $\alpha \in M$. We put $\alpha_1 = \beta(p_1, \alpha)$, $\alpha_2 = \beta(p_2, \alpha_1)$,..., $\alpha_n = \beta(p_n, \alpha_{n-1})$ and $\beta = \beta(\lambda, \alpha_n)$. Then $\rho_\alpha(x_1, y_1) \leq \rho_\beta(x_1, y_1)$ and $\rho_\alpha(p(x_1, ...x_m), p(y_1, ..., y_m)) \leq \sum\{\rho_\beta(x_i, y_i) : i \leq m\}$ for all $x_1, y_1, ..., x_m, y_m \in G$. The proof is complete. □

Lemma 5. *Let A be a non-empty set and $\{\rho_\mu : \mu \in M_\alpha\}$ be a stable set of pseudometrics on an Ω-algebra G for each $\alpha \in A$. If $M = \cup\{M_\alpha : \alpha \in A\}$, then the family $\{\rho_\mu : \mu \in M\}$ is a stable set of pseudometrics on the Ω-algebra G.*

Proof. It is obvious. □

Proposition 2. *Let $R = \{\rho_\mu : \mu \in M\}$ be a stable set of continuous totally bounded pseudometrics on a topological Ω-algebra G. Then there exist a compact topological Ω-algebra G/R, a continuous homomorphism $p_R : G \longrightarrow G/R$ and a stable set of continuous totally bounded pseudometrics $\bar{R} = \{\bar{\rho}_\mu : \mu \in M\}$ on a topological Ω-algebra G/R such that:*

1. *The topology of the space G/R is induced by the family of pseudometrics $\bar{R}$.*
2. *$\bar{\rho}_\mu(p_R(x), p_R(y)) = \rho_\mu(x, y)$ for all $x, y \in G$ and $\mu \in M$.*
3. *$(G/R, p_R)$ is an a-compactification of the topological Ω-algebra G.*

Proof. Fix $\mu \in M$. Then there exists a metric space (Y_μ, d_μ) and a mapping $p_\mu : G \to Y_\mu$ of G onto Y_μ such that $d_\mu(p_\mu(x), p_\mu(y)) = \rho_\mu(x, y)$ for all $x, y \in G$. Denote by $(G_\mu, \bar{d_\mu})$ the completion of the metric space (Y_μ, d_μ). Since the metric d_μ is totally bounded, G_μ is a compact space.

Consider the continuous mapping $p_R : G \longrightarrow \Pi\{G_\mu : \mu \in M\}$, where $p_R(x) = (p_\mu(x) : \mu \in M\}$ for each point $x \in G$. We put $Y = p_R(G)$ and by G/R denote the closure of Y in the compact space $\Pi\{G_\mu : \mu \in M\}$. For each $\mu \in M$ on G/R there exists a continuous pseudometric $\bar{\rho_\mu}$ such that $\bar{\rho}_\mu(p_R(x), p_R(y)) = \rho_\mu(x, y)$ for all $x, y \in G$.

Fix $n \geq 1$ and $\omega \in \Omega_n$. Let $a = (a_1, ..., a_n) \in Y^n$. Fix $b = (b_1, ..., b_n) \in G^n$ such that $p_R(b_i) = a_i$ for any $i \leq n$. We put $\omega(a) = p(\omega(b))$. We affirm that the mapping $\omega : Y^n \longrightarrow Y$ is single-valued. Let $c = (c_1, ..., c_n) \in G^n$ and $p_R(c_i) = a_i$ for any $i \leq n$. Suppose that $p_R(\omega(c) \neq p_R(\omega(b))$. Then there exists $\alpha \in M$ such that $\rho_\alpha(\omega(c), \omega(b)) > 0$. Since R is a stable set of pseudometrics, there exists $\beta = \beta(\omega, \alpha) \in M$ such that $\rho_\alpha(x_1, y_1) \leq \rho_\beta(x_1, y_1)$ and $\rho_\alpha(\omega(x_1, ...x_n), \omega(y_1, ..., y_n)) \leq \sum\{\rho_\beta(x_i, y_i) : i \leq n\}$ for all $x_1, y_1, ..., x_n, y_n \in G$. In particular, $0 < \rho_\alpha(\omega(c), \omega(b)) \leq \sum\{\rho_\beta(c_i, b_i) : i \leq n\}$. Thus $\rho_\beta(c_i, b_i) > 0$ for some $i \leq n$. Since $p_R(c_i) = p_R(a_i)$, we have $\rho_\mu(c_i, b_i) = 0$, a contradiction. Thus $\omega : Y^n \to Y$ is an n-ary operation on Y and on Y there exists the structure of Ω-algebra relative to which p_R is a homomorphism.

By construction, the pseudometrics $\bar{R}$ forms a stable set of pseudometrics on Y. Hence Y is a topological algebra and p_R is a continuous homomorphism of G onto Y.

Let $U(\bar{R})$ be the uniformity generated by the pseudometrics $\bar{R}$ on G/R and $(Y, U(\bar{R})_Y)$ be the uniform subspace of the uniform space $(G/R, U(\bar{R}))$. By the definition of a stable set of pseudometrics, the operation $\omega : Y^n \longrightarrow G/R$ is a uniformly continuous mapping for each $n \geq 1$ and every $\omega \in \Omega_n$. Hence the operation ω is continuous extendable on G/R^n and on G/R there exists a structure of topological Ω-algebra such that Y is a subalgebra of the compact Ω-algebra G/R. The proof is complete. $\square$

Assume that v is a unary operation and $v(x) = x$ for each Ω-algebra G and any point $x \in G$. Let M_Ω be the family of all finite ordered subsets of $\Omega \cup \{v\}$ such that v is the first element in each $\alpha \in M_\Omega$. If $\alpha = (\alpha_1, ..., \alpha_n), \beta = (\beta_1, ..., \beta_m) \in M_\Omega$, then:

– $\alpha \leq \beta$ if and only if $n \leq m$ and $\alpha_i \beta_i$ for any $i \leq n$;

– $c(\alpha) = n$ and $c(\beta) = m$.

The set $\{v\}$ is the minimal element of the set M_Ω and $c(\{v\}) = 1$. If $\lambda \in \Omega$, then (λ), (λ, λ),..., $(\lambda, \lambda, ..., \lambda)$ are distinct elements.

Let $\alpha \in M_\Omega$ and $c(\alpha) = 1$. Then $\{v\} \subseteq \alpha \subseteq \{v\} \cup \Omega_0$. We put $P(\alpha) = \alpha \cup \{v(\omega) : \omega \in \alpha\}$.

Assume that $\alpha, \beta \in M_\Omega$, $\alpha \leq \beta$, $c(\beta) = c(\alpha) + 1$ and the polynomials $P(\alpha)$ are constructed. Then $P(\beta) = \beta \cup P(\alpha) \cup \{\omega(p_1, p_2, ..., p_n) : p_1, p_2, ..., p_n \in P(\alpha) \cup \beta, \omega \in \beta \cap \Omega_n, n \geq 1\}$. By induction, the set $P(\alpha)$ is constructed for each $\alpha \in M_\Omega$. Any set $P(\alpha)$ is finite, $P(\alpha) \subseteq P(\beta)$ for $\alpha \leq \beta$ and $P(\Omega) = \cup\{P(\beta) : \beta \in M_\Omega\}$. Let $T(\alpha) = \cup\{T(\lambda) : \lambda \in \alpha\}$ for each $\alpha \in M_\Omega$.

Assume that f is a function on an Ω-algebra G. For each $\alpha \in M_\Omega$ we put $\rho_{(f,\alpha)}(x, y) = sup\{|f_t(y) - f_t(x)| : t \in T(\alpha)\}$ for all $x, y \in G$.

Proposition 3. *Let G be a topological Ω-algebra and $f \in \Omega$-$wAP(G)$. Then:*

1. $R(f) = \{\rho_{(f,\alpha)} : \alpha \in M_\Omega\}$ is a stable set of continuous pseudometrics on G.

2. If the function f is bounded, then the pseudometrics $\{\rho_{(f,\alpha)}\}$ are totally bounded.

3. $\rho_{(f,\alpha)}(x, y) \geq |f(x) - f(y)$ for all $x, y \in G$.

Proof. 1. Since $v \in \alpha$, we have $\rho_{(f,\alpha)}(x, y) \geq |f(x) - f(y)|$ for all $x, y \in G$.

2. Since f is a weakly almost periodic continuous function and the set of polynomials $P(\alpha)$ is finite for any $\alpha \in M_\Omega$, the closure of the set $\alpha(f) = \{t_f : t \in T(\alpha\}$

in $C(G)$ is a compact set. From this fact it follows that the pseudometric $\rho_{(f,\alpha)}$ is continuous and $\rho_{(f,\alpha)}(x,y) < \infty$ for all $\alpha \in M_\Omega$ and $x, y \in G$.

3. Fix $\alpha \in M_\Omega$, $n \geq 1$ and $\omega \in \Omega_n$. Assume that $\alpha = (\alpha_1, ..., \alpha_m)$ for some $m \geq 1$. We put $\beta = (\alpha_1, ..., \alpha_n, \omega)$. Then $\alpha < \beta$ and $c(\beta) = c(\alpha) + 1$.

Since $T(\alpha) \subseteq T(\beta)$, we have $\rho_{(f,\alpha)}(x,y) \leq \rho_{(f,\beta)}(x,y)$ for all $x, y \in G$.

Fix $x_1, y_1, ..., x_n, y_n \in G$. Since $\varphi \circ \psi \in T(\beta)$ for any $\varphi \in T(\alpha)$ and each $\alpha\psi \in T(\omega)$, we have $\rho_{(f,\alpha)}(\omega(x_1, ...x_m), \omega(y_1, ..., y_m)) \leq \sum\{\rho_{(f,\beta)}(x_i, y_i) : i \leq m\}$. Hence $R(f)$ is a stable set of continuous pseudometrics on G.

4. Assume now that the function f is bounded. Fix $\epsilon > 0$ and $\alpha \in M_\Omega$.

Since the closure of the set $\alpha(f) = \{f_t : t \in T(\alpha\}$ in $C(G)$ is a compact set, there exists a finite set $L = \{t_1, t_2, ..., t_k\} \subseteq T(\alpha)$ such that for each $t \in T(\alpha)$ there exists $i \leq k$ such that $d(f_t, f_{t_i}) < \epsilon/3$. Assume that $v \in L$.

We put $g(x) = \Sigma\{|f_t(x)| : t \in L\}$. The function g is continuous and bounded. There exists a finite subset F of G such that $min\{|g(x) - g(y)| : y \in F\} < \epsilon/6$ for any $x \in G$. Hence for each $x \in G$ there exists $x(f) \in F$ such that $|f_t(x) - f_t(x(f))| < \epsilon/3$ for any $t \in L$. We affirm that $d_{(f,\alpha)}(x, x(f)) < \epsilon$. Suppose that $x \in G$ and $d_{(f,\alpha)}(x, x(f)) \geq \epsilon > 0$. Then there exist $\varphi \in T(\alpha)$ and $t \in L$ such that $|f_\varphi(x) - f_\varphi(x(f))| > \epsilon$ and $d(f_\varphi, f_t) < \epsilon/3$. By construction, we have $|f_\varphi(x) - f_\varphi(x(f))| = |f_\varphi(x) - f_t(x) + f_t(x) - f_t(x(f)) + f_t(x(f)) - f_\varphi(x(f))| \leq |f_\varphi(x) - f_t(x)| + |f_t(x) - f_t(x(f))| + |f_t(x(f)) - f_\varphi(x(f))| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$, a contradiction. Therefore the pseudometrics $\{\rho_{(f,\alpha)}\}$ are totally bounded. The proof is complete. □

Corollary 9. *Let G be a topological Ω-algebra. Then the maximal a-compactification $(\beta_\Omega G, \beta_G) = (b_F G, e_F)$, where $F = \Omega^\circ$-$wAP(G)$.*

Corollary 10. *Let G be a compact Ω-finite topological Ω-algebra. Then Ω-$wAP(G)$ $= \Omega$-$AP(G)$.*

Remark 6. Let G be a topological Ω-algebra and F be a closed subalgebra of the algebra Ω°-$wAP(G)$ with the following proprieties:

– if f is a constant function, then $f \in F$;

– if $\in F$ and $t \in T(\Omega)$, then $f_t \in F$.

Then $(b_F G, e_F)$ is an a-compactification of G. Any a-compactification can be constructed in this way.

6 Cartesian product of topological algebras

Let $\Omega = \oplus\{\Omega_n : n \in N\}$ be a discrete signature.

For any nulary polynomial $\omega \in P(\Omega)$ and any Ω-algebra G there exists a unique neutral element $\omega_G \in G$ such that $e_{0G}(\omega, G^0) = \omega_G$.

Fix a class $\mathcal{K}$ of topological Ω-algebras with the following properties:

1. If $A \in \mathcal{K}$, then A is a Tychonoff space.

2. The Cartesian product of algebras from $\mathcal{K}$ is an algebra from $\mathcal{K}$.

3. There exists a nulary polynomial $1 \in P(\Omega)$ such that for the point $1_G = e_{0G}(1, G^0)$, each $n \geq 1$ and every $\lambda \in \Omega_n$ we have $\lambda(1_G, ...1_G) = 1_G$ for every $G \in K$.

4. There exists a ternary polynomial $p \in P(\Omega)$ such that $p(x,x,y) = p(y,x,x) = y$ for all $G \in \mathcal{K}$ and $x, y \in G$.

5. There exists a binary polynomial $v \in P(\Omega)$ such that $v(1_G, x) = v(x, 1_G) = x$ for all $G \in \mathcal{K}$ and $x \in G$.

6. If G is a Tychonoff topological Ω-algebra with the properties 3-5, then $G \in \mathcal{K}$.

We may assume that $1 \in \Omega_0$, $p \in \Omega_3$ and $v \in \Omega_2$.

A mapping $\varphi : X \to Y$ is injective if $f(x) \neq f(y)$ for every two distinct points $x, y \in X$.

Lemma 6. *Let $\varphi : A \to B$ be a homomorphism of a topological Ω-algebra $A \in \mathcal{K}$ into an Ω-algebra B, A_1 be a dense subset of A and $\varphi_1 = \varphi|A_1 : A_1 \to B$ be an injective mapping. Then φ is injective too.*

Proof. We may consider that $B = \varphi(A)$. On B we consider the quotient topology $\{U \subseteq B : \varphi^{-1}(U) \textit{ is open in } A\}$. Since $A \in \mathcal{K}$, B is a topological Ω-algebra and $\varphi : A \to B$ is an open continuous mapping (see [4]). Suppose that $a, b \in A$, $a \neq b$ and $\varphi(a) = \varphi(b)$. We fix two open subsets U, V of A for which $a \in U$, $b \in V$ and $U \cap V = \emptyset$. Then the set $W = \varphi(U) \cap \varphi(V)$ is open in B, $\varphi(A_1)$ is a dense subset of B, $\varphi(a) = \varphi(b) \in W$ and $W \cap \varphi(A_1) = \emptyset$, a contradiction. The proof is complete. □

Lemma 7. *Let $A \in \mathcal{K}$ and A be a dense subalgebra of the topological Ω-algebra B. Then $B \in \mathcal{K}$.*

Proof. Is obvious. □

Theorem 8. *Let $\{G_\mu \in \mathcal{K} : \mu \in M\}$ be a non-empty family of topological Ω-algebras and $G = \Pi\{G_\mu \in \mathcal{K} : \mu \in M\}$. Then:*

1. $\beta_{ap(\Omega)} G = \Pi\{\beta_{ap(\Omega)} G_\mu : \mu \in M\}$ and $\alpha_G(x) = (\alpha_{G_\mu}(x_\mu) : \mu \in M)$ for each point $x = (x_\mu) : \mu \in M) \in G$.

2. $(\beta_\Omega G, \beta_G) = \Pi\{\beta_\Omega G_\mu : \mu \in M\}$ and $\beta_G(x) = (\beta_{G_\mu}(x_\mu) : \mu \in M)$ for each point $x = (x_\mu) : \mu \in M) \in G$.

Proof. From Lemma 7 it follows that $\beta_{ap(\Omega)} A \in \mathcal{K}$ for any $A \in \mathcal{K}$.

Let $M = \{1, 2\}$. Then $G = G_1 \times G_2$. There exists a continuous homomorphism $\psi : \beta_{ap(\Omega)} G \longrightarrow \beta_{ap(\Omega)} G_1 \times \beta_{ap(\Omega)} G_2$ such that $\psi(\alpha_G(x,y) = (\alpha_{G_1}(x), \alpha_{G_2}(y))$ for every point $(x, y) \in G$.

We can identify $x \in G_1$ with $(x, 1_{G_2}) \in G$ and $y \in G_2$ with $(1_{G_1}, y) \in G$. In this case $1_G = (1_{G_1}, 1_{G_2})$ and G_1, G_2 are subalgebras of the algebra G. If $h \in \Omega\text{-}AP(G)$, then:

– for each $y \in G_2$ there exists $h_y \in \Omega\text{-}AP(G_1)$ such that $h_y(x) = h(x, y)$ for each $x \in G_1$;

– for each $x \in G_1$ there exists $h_x \in \Omega\text{-}AP(G_2)$ such that $h_x(y) = h(x, y)$ for each $y \in G_2$.

Hence $\psi|\alpha_G(G)$ is an injective mapping. From Lemma 6 it follows that ψ is an isomorphism. Hence the assertions 1 of theorem are true for any finite set M.

Suppose that the set M is infinite. If $B \subseteq M$, then we put $G_B = \Pi\{G_\mu : \mu \in B\}$. Let $G = G_M$ and $\pi_B : G \to G_B$ be the natural projection. We identify G_B with the subalgebra $\{x = (x_\mu : \mu \in M) \in G : x_\mu = 0_{G_\mu}$ *for any* $\mu \in M \setminus B\}$. In this case $\pi_B : G \to G_B$ is the retraction.

Let $\overline{r}_E G_B = \Pi\{r_E G_\mu : \mu \in M\}$ and identity $\overline{r}_E G_B$ with the subalgebra $\{x = (x_\mu : \mu \in M) : x_\mu = 0_{r_E G_\mu}$ for every $\mu \in M \backslash B\}$ of the algebra $\overline{r}_E G = \overline{r}_E G_M$. Let $\overline{\pi}_B : \overline{r}_E G \to \overline{r}_E G_B$ be the natural projection. We put $G' = \cup\{G_B \subseteq G : B$ is a finite subset of $M\}$. Then G' is a dense subalgebra of the topological Ω-algebra G. If $B \subseteq M$, then $G''_B = r_B(G_B)$ and $G'' = \cup\{G''_B : B$ is a finite subset of $M\} = r_\mu(G')$. For every finite subset $B \subseteq M$ the mapping $\nu_M | G''_B : G''_B \to \overline{r}_E G_B$ is a topological isomorphism. Hence $\nu_M : G'' \to \overline{r}_E G_M$ is an injection. Lemma 6 completes the proof of Assertions 1. The proof of Assertions 2 is similar. The proof is complete. $\square$

Theorem 9. *Let $G \in \mathcal{K}$ be a pseudocompact topological Ω-algebra B. Then:*

1. On βG there exists a structure of topological Ω-algebra such that $\beta G \in \mathcal{K}$ and G is a dense subalgebra of the Ω-algebra βG.

2. Ω-$wAP(G) = C(G) = C^\circ(G)$.

Proof. In [12] it was proved that for any pseudocompact topological Mal'cev E-algebra G and each $n \in \mathbb{N}$ the space G^n is pseudocompact. From the I. Glicksberg's theorem ([8], Problem 3.12.20 (d), p. 299) it follows that $\beta(G^n) = (\beta G)^n$ for each $n \in \mathbb{N}$. Thus for each $n \in \mathbb{N}$ and every $\omega \in \Omega_n$ there exists a continuous extension $\omega : \beta(G^n) \longrightarrow \beta G$ of the mapping $\omega : G^n \longrightarrow G$. Therefore on βG there exists a structure of topological Ω-algebra such that G is a dense subalgebra of the Ω-algebra βG. From Lemma 7 it follows that $\beta G \in \mathcal{K}$. Theorem 7 completes the proof. $\square$

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M. M. Choban
Department of Mathematics, Tiraspol State University,
Chişinău, Republic of Moldova, MD 2069
E-mail: *mmchoban@gmail.com*

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D. I. Pavel
Department of Informatics, Tiraspol State University,
Chişinău, Republic of Moldova, MD 2069
E-mail: *dorinp@mail.md*

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Lattice of all topologies of countable module over countable rings

V. I. Arnautov, G. N. Ermakova

Abstract. For any countable ring R with discrete topology τ_0 and any countable R-module M the lattice of all (R, τ_0)-module topologies contains:
– A sublattice which is isomorphic to the lattice of all real numbers with the usual order;
– Two to the power of continuum (R, τ_0)-module topologies each of which is a coatom.

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1 Introduction

For any ring R with discrete topology τ_0 and any R-module M the question of the existence of non-discrete Hausdorff (R, τ_0)-module topologies was considered in [1] and [2]. In particular, it was proved that any infinite module over any discrete ring R admits non-discrete Hausdorff module topology and an example of a topological ring (R, τ_0) and an R-module M was constructed for which the lattice of all (R, τ_0)-module topologies does not contain Hausdorff topologies.

In fact (see below Remark 3.1) for this topological ring (R, τ_0), the lattice of all (R, τ_0)-module topologies on this R-module M contains only anti-discrete topology.

The present paper is a continuation of these works and is devoted to the study of properties of the lattice of all topologies on countable modules over discrete ring.

The main result of this article is Theorem 3.2, in which it is proved that for any countable ring R with discrete topology τ_0 and any countable R-module M, the lattice of all (R, τ_0)-module topologies contains a sublattice which is isomorphic to the lattice of real numbers with the usual order and contains two to the power of continuum coatoms.

Similar results for countable groups and countable rings were obtained in [3, 4] and [5], respectively.

Furthermore, it was shown that the condition that the ring should be countable is essential in Theorem 3.2, namely, we constructed an example of an infinite discrete ring (R, τ_0) and a countable R-module M such that every (R, τ_0)-module topology on M which has a countable or finite basis of the filter of neighbourhoods of zero is anti-discrete.

2 Preliminary results

To present the main results we recall the following two well known theorems (see, for example, [1]).

Theorem 2.1. *A set $\Omega = \{V_\gamma | \gamma \in \Gamma\}$ of subsets of a ring R is a basis of the filter of neighbourhoods of zero for some ring topology τ on the ring R if and only if the following conditions are satisfied:*

1. $0 \in \bigcap\limits_{\gamma \in \Gamma} V_\gamma$;

2. For any subsets V_1 and $V_2 \in \Omega$ there exists a subset $V_3 \in \Omega$ such that $V_3 \subseteq V_1 \cap V_2$;

3. For any subset $V_1 \in \Omega$ there exists a subset $V_2 \in \Omega$ such that $V_2 + V_2 \subseteq V_1$;

4. For any subset $V_1 \in \Omega$ there exists a subset $V_2 \in \Omega$ such that $-V_2 \subseteq V_1$;

5. For any subset $V_1 \in \Omega$ and any element $r \in R$ there exists a subset $V_2 \in \Omega$ such that $r \cdot V_2 \subseteq V_1$ and $V_2 \cdot r \subseteq V_1$;

6. For any subset $V_1 \in \Omega$ there exists a subset $V_2 \in \Omega$ such that $V_2 \cdot V_2 \subseteq V_1$.

Theorem 2.2. *If (R, τ) is a topological ring and M is an R-module, then a set $\Lambda = \{U_\delta | \delta \in \Delta\}$ of subsets of the module M is a basis of the filter of neighborhoods of zero for some (R, τ)-module topology τ_1 of the module M if and only if the following conditions are satisfied:*

1. $0 \in \bigcap\limits_{\delta \in \Delta} U_\delta$*;*

2. For any subsets U_1 and $U_2 \in \Lambda$ there exists a subset $U_3 \in \Lambda$ such that $U_3 \subseteq U_1 \cap U_2$;

3. For any subset $U_1 \in \Lambda$ there exists a subset $U_2 \in \Lambda$ such that $U_2 + U_2 \subseteq U_1$;

4. For any subset $U_1 \in \Lambda$ there exists a subset $U_2 \in \Lambda$ such that $-U_2 \subseteq U_1$;

5. For any subset $U_1 \in \Lambda$ and any element $r \in R$ there exists a subset $U_2 \in \Lambda$ such that $r \cdot U_2 \subseteq U_1$;

6. For any subset $U_1 \in \Lambda$ and any element $m \in M$ there exists a neighborhood V_2 of zero of the topological ring (R, τ) such that $V_2 \cdot m \subseteq U_1$;

7. For any subset $U_1 \in \Lambda$ there exists a neighborhood V_2 of zero of the topological ring (R, τ) and a subset $U_2 \in \Lambda$ such that $V_2 \cdot U_2 \subseteq U_1$.

Theorem 2.3. *(see the proof in [5], Theorem 3.1) If R is a countable ring and τ_0 is a non-discrete, Hausdorff ring topology such that the topological ring (R, τ_0) has a countable basis of the filter of neighborhoods of zero, then the following statements are true:*

1. For any infinite set A of natural numbers there exists a ring topology $\tau(A)$ such that the topological ring $(R, \tau(A))$ has a countable basis of the filter of neighborhoods of zero and such that $\tau_0 \leq \tau(A)$;

2. $\sup\{\tau(A), \tau(B)\}$ *is the discrete topology for any infinite sets A and B of natural numbers such that $A \cap B$ is a finite set;*

3. There exist the continuum of Hausdorff ring topologies each having a countable basis of the filter of neighbourhoods of zero and stronger than τ_0 *and such that any two of them are comparable;*

4. There are two to the power of continuum topologies such that $\sup\{\tau_1, \tau_2\}$ *is the discrete topology for any two different topologies;*

5. There are two to the power of continuum coatoms in the lattice of all ring topologies.

Remark 2.4. From the proof of Theorem 3.1 in [5] it is easy to see that all topologies which are indicated in this theorem are stronger than the topology τ_0.

Remark 2.5. As in the proof of the Statement 3.1.3 of Theorem 3.1 in [5] ring topology τ_r is defined for every real number r and $\tau_t \leq \tau_s$ if and only if $s \leq t$, then the lattice of all ring topologies contains a sublattice which is anti-isomorphic to the lattice of all real numbers with the usual order for any countable ring.

In addition, since the mapping σ such that $\sigma(r) = -r$ is an anti-isomorphism of the lattice of all real numbers on itself, then the lattice of all ring topologies contains a sublattice which is isomorphic to the lattice of all real numbers with the usual order for any countable ring.

3 Basic results

Remark 3.1. We will show that for the topological ring (R, τ_0) and for the R-module M, which are constructed in [3] and [4], any (R, τ_0)-module topology of the module M is anti-discrete.

Thus, let:

– R be the ring of polynomials of an argument x over the field of rational numbers Q;

– $M = \{r \cdot z | r \in Q\}$ be a one-dimensional vector space over the field of rational numbers Q;

– $\left(\sum\limits_{i=0}^{n} r_i \cdot x^i\right) \cdot (r \cdot z) = \left(\sum\limits_{i=0}^{n} r_i \cdot r\right) \cdot z$ for any element $\sum\limits_{i=0}^{n} r_i \cdot x^i \in R$ and any element $r \cdot z \in M$;

– The set $\Omega = \{R \cdot x^n | n = 1, 2, \ldots\}$ is a basis of the filter of neighbourhoods of zero in the topological ring (R, τ_0).

Now let τ be an (R, τ_0)-module topology of the module M and let U be an arbitrary neighbourhood of zero in the topological module (M, τ).

If $r \cdot z \in M$, then according to the condition 6 of Theorem 2.2, there exists a neighbourhood V of zero in the topological ring (R, τ_0) such that $V \cdot (r \cdot z) \subseteq U$, and hence $(R \cdot x^n) \cdot (r \cdot z) \subseteq V \cdot (r \cdot z) \subseteq U$ for some natural number n.

Then $(r\cdot z) = x^n \cdot (r\cdot z) \in (R\cdot x^n)\cdot(r\cdot z) \subseteq V\cdot(r\cdot z) \subseteq U$. From the arbitrariness of the element $r\cdot z$ it follows that $U = M$, and hence the topology τ is anti-discrete.

Theorem 3.2. *If (R, τ_0) is a countable ring with the discrete topology τ_0 and M is a countable R-module then the following statements are true:*

1. For any infinite set A of natural numbers there exists an (R, τ_0)-module topology $\tau(A)$ which has a countable basis of the filter of neighborhoods of zero and such that $\sup\{\tau(A), \tau(B)\}$ is the discrete topology for any infinite sets A and B of natural numbers such that $A \cap B$ is a finite set;

2. There exist continuum of (R, τ_0)-module topologies which have a countable basis of the filter of neighbourhoods of zero and such that any two of them are comparable;

3. There exist two to the power of continuum coatoms in the lattice of all (R, τ_0)-module topologies on the module M;

4. The lattice of all (R, τ_0)-module topologies on the module M contains a sublattice which is anti-isomorphic to the lattice of all real numbers with the usual order, and contains a sublattice which is isomorphic to the lattice of all real numbers with the usual order.

Proof. We define the operation of multiplication on the group $\hat{R}(+) = \{(r, m) | r \in R, m \in M\}$, which is the direct sum of the groups $R(+)$ and $M(+)$, as follows: $(r_1, m_1)\cdot(r_2, m_2) = (r_1\cdot r_2, r_1\cdot m_2)$ for any elements $r_1, r_2 \in R$ and any elements $m_1, m_2 \in M$.

It is easy to see that $\hat{R}(+,\cdot)$ is a ring, and the set $\hat{I} = \{(0, m) | m \in M\}$ is an ideal of the ring $\hat{R}$.

If $\psi(0, m) = m$, then $\psi : \hat{I} \to M$ is a bijective mapping. Then putting $\hat{\psi}(\hat{U}) = \{\psi(0, m) | (0, m) \in \hat{U}\}$ for each subset $\hat{U} \subseteq \hat{I}$, we define a bijective mapping $\hat{\psi}$ of the set of all subsets of the set $\hat{I}$ on the set of all subsets of the set M.

Let $\hat{\Delta}$ be the lattice of all ring topologies on the ring $\hat{R}$ such that the ideal $\hat{I}$ is open, and let Δ be the lattice of all (R, τ_0)-module topologies on the module M. We show that the lattices $\hat{\Delta}$ and Δ are isomorphic.

Let $\hat{\tau} \in \hat{\Delta}$. As $\hat{I}$ is an open ideal in the topological ring $(\hat{R}, \hat{\tau})$ then the topological ring $(\hat{R}, \hat{\tau})$ has a basis $\hat{\Omega}$ of the filter of neighborhoods of zero such that $\hat{V} \subseteq \hat{I}$ for any $\hat{V} \in \hat{\Omega}$.

Since τ_0 is the discrete topology, then from Theorems 2.1 and 2.2 it follows that the set $\{\hat{\psi}(\hat{V}) | \hat{V} \in \hat{\Omega}\}$ is a basis of the filter of neighborhoods of zero for some $(R, \tau_0)-$module topology on the module M, and any (R, τ_0)-module topology on the module M can be obtained in this way.

Since any module topology is given in a unique way by any basis of the filter of neighborhoods of zero, we have identified mapping $\widetilde{\psi} : \hat{\Omega} \to \Omega$. It is easy to see that this map is bijective, and $\hat{\tau}_1 \leq \hat{\tau}_2$ if and only if $\widetilde{\psi}(\hat{\tau}_1) \leq \widetilde{\psi}(\hat{\tau}_2)$, i.e. $\widetilde{\psi} : (\hat{\Omega}, \leq) \to (\Omega, \leq)$ is a lattice isomorphism.

As noted above (see Introduction), there exists a non-discrete Hausdorff (R, τ_0)-module topology $\bar{\tau}_0$ on the module M. If $\hat{\tau}_0 = \hat{\Psi}^{-1}(\bar{\tau}_0)$, then $\hat{I}$ is an open ideal in the topological ring $(\hat{R}, \hat{\tau}_0)$. Then the topological ring $(\hat{R}, \hat{\tau}_0)$ has a basis $\hat{B}$ of the filter of neighborhoods of zero such that $\hat{U} \subseteq \hat{I}$ for every $\hat{U} \in \hat{B}$ and $\bigcap_{\hat{U} \in \hat{B}} \hat{U} = \{0\}$.

From countability of the ring $\hat{R}$, it follows that there exists a countable subset $\hat{B}_0 \subseteq \hat{B}$ such that $\bigcap_{\hat{U} \in \hat{B}_0} \hat{U} = \{0\}$ and the conditions of Theorem 2.1 are satisfied. Hence, there is a Hausdorff topology $\tilde{\tau}_0 \in \hat{\Omega}$ such that topological ring $(\hat{R}, \tilde{\tau}_0)$ has a countable basis of the filter of neighborhoods of zero and $\hat{I}$ is an open ideal.

Then Statements 1 – 5 of Theorem 2.3 are true for the topological ring $(\hat{R}, \tilde{\tau}_0)$, and from Remark 2.4 it follows that $\hat{I}$ is an open ideal for any topology, which is obtained according of Statements 1 – 5 of Theorem 2.3, i.e. all these topologies belong to $\hat{\Omega}$. As the lattice $\hat{\Omega}$ is isomorphic to the lattice Ω, then Statements 1 – 3 of Theorem 3.2 are true.

In addition, the Statement 4 of Theorem 3.2 follows from Remark 2.5.

The theorem is proved. □

Remark 3.3. We will construct an example of a ring (R, τ_0) with discrete ring topology τ_0 and countable R-module M such that every non-discrete (R, τ_0)-module topology which has a finite or countable basis of the filter of neighborhoods of zero, is anti-discrete.

This example shows that the requirement that the ring R should be countable is essential in Statements 1 and 2 of Theorem 3.2.

As for any ring R with the discrete topology τ_0 any infinite module allows a non-discrete Hausdorff (R, τ_0)-module topology, then the lattice of all (R, τ_0)-module topologies contains coatoms.

However, the following questions remain unresolved:

– How many coatoms are in the lattice of all module topologies on any infinite module over any ring with discrete topology?

– Do there exist a ring with discrete topology and an infinite module for which the lattice of all module topologies has only one coatom?

– Do there exist a ring with discrete topology and an infinite module for which the lattice of all module topologies is a chain?

Example 3.4. Let X be a set with the cardinality of continuum and let $Y = \{y_1, y_2, \ldots\}$ be a countable set. We consider the free associative algebra R over the two-element field Z_2 which is generated by the set X and the linear space M over Z_2 for which the set Y is a basis.

We consider the set $\tilde{N}$ of all countable strictly increasing sequences of natural numbers.

If ω_0 is the smallest countable transfinite number and ω_c is the smallest transfinite number with the cardinality of continuum, then:

$\tilde{N} = \{\tilde{m}_\alpha | \omega_0 \leq \alpha < \omega_c\}$ and $X = \{x_\alpha | 1 \leq \alpha < \omega_c\}$.

We define the multiplication of elements of the set $Y \cup \{0\}$ by elements of the set X as follows:

– If $\alpha < \omega_c$, then we let $x_\alpha \cdot 0 = 0$;

– If $\alpha < \omega_0$, i.e. α is a natural number, then we let $x_\alpha \cdot y_k = y_{\alpha+k-1}$ for any natural number k;

– If $\omega_0 \leq \alpha < \omega_c$, then $\tilde{m}_\alpha$ is an increasing sequence of natural numbers, i.e. $\tilde{m}_\alpha = (m_1, m_2, \ldots)$, and then we let $x_\alpha \cdot y_k = y_1$ if $k \in \{m_1, m_2, \ldots\}$ and $x_\alpha \cdot y_k = 0$ if $k \notin \{m_1, m_2, \ldots\}$.

Then, using the associative and distributive laws, we can extend the operation of the multiplication of elements of the ring R on the elements of the group M so that the group M will be a R-module.

We show now that every non-discrete module topology on the R-module M which has a finite or countable basis of the filter of neighbourhoods of zero is anti-discrete.

Assume the contrary, i.e. that on the R-module M there exists a non-discrete module topology τ which has a finite or countable basis Ω of the filter of neighbourhoods of zero and which is not anti-discrete.

If $\{0\} \neq \bigcap\limits_{V \in \Omega} V$ and $0 \neq g \in \bigcap\limits_{V \in \Omega} V\}$, then there exists a natural number n such that $g = k_1 \cdot y_1 + k_2 \cdot y_2 + \ldots + k_n \cdot y_n$ and $k_n \neq 0$, i.e. $k_n = 1$.

Since the sequence $(n, n+1, n+2, \ldots) \in \tilde{N}$, then $(n, n+1, n+2, \ldots) = \tilde{m}_\alpha$ for some transfinite number $\omega_0 \leq \alpha < \omega_c$.

Now if $V \in \Omega$, then (see Theorem 2.2, the property 5) for the element $x_\alpha \in R$, there exists a neighbourhood $V_1 \in \Omega$ such that $x_\alpha \cdot V_1 \subseteq V$. Then (see above, the definition of multiplication of elements from M by elements from R) $y_1 = x_\alpha \cdot y_n = x_\alpha \cdot g = x_\alpha \cdot V_1 \subseteq V$.

So, we have proved that $y_1 \in V$ for every neighbourhood $V \in \Omega$.

If $V \in \Omega$ and $h = y_{k_1} + y_{k_2} + \ldots + y_{k_s} \in M$, then there exists a neighbourhood $V'_1 \in \Omega$ such that $\underbrace{V'_1 + V'_1 + \ldots V'_1}_{s\ items} \subseteq V$ and there exist neighbourhoods of $V_{k_1}, V_{k_2}, \ldots, V_{k_s} \in \Omega$ such that $x_{k_i} \cdot V_{k_i} \subseteq V'_1$ for every natural number $1 \leq i \leq s$.

Then

$$h = y_{k_1} + y_{k_2} + \ldots + y_{k_s} = x_{k_1} \cdot y_1 + x_{k_2} \cdot y_1 + \ldots + x_{k_s} \cdot y_1 \subseteq \underbrace{V'_1 + V'_1 + \ldots + V'_1}_{s\ items} \subseteq V.$$

The arbitrariness of the element $h \in M$ implies that $V = M$, and from the arbitrariness of the neighbourhood V we have that the topology τ is anti-discrete for the case when $\{0\} \neq \bigcap\limits_{V \in \Omega} V$.

Now let $\{0\} = \bigcap\limits_{V \in \Omega} V$. The further proof will be realized in several steps.

Step I. *We show that for any natural number n and any neighborhood $V_0 \in \Omega$ there exists an element $h \in V_0$ such that*

$$h = k_{n+1} \cdot y_{n+1} + k_{n+2} \cdot y_{n+2} + \ldots + k_{n+t} \cdot y_{n+t}$$

and $k_i \in \{0, 1\}$ for $n+1 \leq i \leq n+t$.

Let V_1 be a neighbourhood of zero in (M, τ) such that $V_1 - V_1 \subseteq V_0$.

As for the natural number n the set

$$M_n = \{l_1 \cdot y_1 + l_2 \cdot y_2 + \ldots + l_{n-1} \cdot y_{n-1} | l_i \in \{0, 1\}, 1 \le i \le n-1\}$$

is finite, then there exist elements $g = k_1 \cdot y_1 + k_2 \cdot y_2 + \ldots + k_m \cdot y_m \in V_1$ and $g' = k'_1 \cdot y_1 + k'_2 \cdot y_2 + \ldots + k'_m \cdot y_m \in V_1$ such that $k_i = k'_i$ for $1 \le i \le n$. Then

$$h = g - g' = (k_{n+1} - k'_{n+1}) \cdot y_{n+1} + (k_{n+2} - k'_{n+2}) \cdot y_{n+2} + \ldots + (k_m - k'_m) \cdot y_m \in V_1 - V_1 \subseteq V_0.$$

By this the statement indicated in Step I is proved.

Step II. *By induction we construct an increasing sequence $n_1, n_2, \ldots$ of natural numbers and a sequence $g_1, g_2, \ldots$ of elements of the module M.*

If $\Omega = \{V_1, V_2, \ldots\}$, then we take an element

$$g_1 = k_1 \cdot y_1 + k_2 \cdot y_2 + \ldots + k_{n_1} \cdot y_{n_1} \in V_1.$$

According to the statement indicated in Step I, for the natural number n_1 and the neighbourhood V_2 there exists an element

$$g_2 = k_{n_1+1} \cdot y_{n_1+1} + k_{n_1+2} \cdot y_{n_1+2} + \ldots + k_{n_2} \cdot y_{n_2} \in V_2.$$

Assume that for any number $2 \le i \le k$ we have constructed a natural number n_i and an element

$$g_i = k_{n_{i-1}+1} \cdot y_{n_{i-1}+1} + k_{n_{i-1}2} \cdot y_{n_{i-1}+2} + \ldots + k_{n_i} \cdot y_{n_i} \in V_i.$$

Then according to the statement indicated in Step I, for the natural number n_k and the neighbourhood V_{k+1} there exists an element

$$g_{k+1} = k_{n_k+1} \cdot y_{n_k+1} + k_{n_k+2} \cdot y_{n_k+2} + \ldots + k_{n_{k+1}} \cdot y_{n_{k+}} \in V_{k+1}.$$

So, we have identified an increasing sequence $n_1, n_2, \ldots$ of natural numbers and the sequence $g_1, g_2, \ldots$ of elements of the module M such that

$$g_i = k_{n_{i-1}1} \cdot y_{n_{i-1}+1} + k_{n_{i-1}+2} \cdot y_{n_{i-1}+2} + \ldots + k_{n_i} \cdot y_{n_i} \in V_i$$

for any natural number i.

Step III. *We verify that $y_1 \in \bigcap\limits_{i=1}^{\infty} V_i$.*

If $n_1, n_2, \ldots$ is the sequence of natural numbers which was built in the second Step, then it belongs to $\tilde{N}$, and hence, $(n_1, n_2, \ldots) = \tilde{m}_\alpha$ for some transfinite number $\omega_0 \le \alpha < \omega_c$.

If i is any natural number, then for the element x_α and the neighbourhood of zero V_i there exists a natural number j such that $x_\alpha \cdot V_j \subseteq V_i$. Then, the definition of multiplication of elements from M by elements from R implies that $y_1 = x_\alpha \cdot g_j \in x_\alpha \cdot V_j \subseteq V_i$.

The arbitrariness of the natural number i implies that $y_1 \in \bigcap\limits_{i=1}^{\infty} V_i$. This contradicts the assumption that $\{0\} = \bigcap\limits_{i=1}^{\infty} V_i$, and hence the case $\{0\} = \bigcap\limits_{i=1}^{\infty} V_i$ is impossible.

Thus, any non-discrete module topology on R-module M which has a finite or countable basis of the filter of neighbourhoods of zero is anti-discrete.

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V. I. Arnautov
Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
5 Academiei str., MD-2028, Chisinau
Moldova
E-mail: *arnautov@math.md*

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G. N. Ermacova
Transnistrian State University
25 October str., 128, Tiraspol, 278000
Moldova
E-mail: galla0808@yandex.ru

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Stationary Nash Equilibria for Average Stochastic Games with Finite State and Action Spaces

Dmitrii Lozovanu

Abstract. We study the problem of the existence of stationary Nash equilibria in infinite n-person stochastic games with limiting average payoff criteria for the players. The state and action spaces in the games are assumed to be finite. We present some results for the existence of stationary Nash equilibria in a multichain average stochastic game with n players. Based on constructive proof of these results we propose an approach for determining the optimal stationary strategies of the players in the case when stationary Nash equilibria in the game exist.

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1 Introduction

In this paper we investigate n-person average stochastic games with finite state and action spaces. The problem we are interested in is the existence of Nash equilibria in stationary strategies. This problem has been studied by many authors (see [4–6, 8, 9, 12, 13, 19–21]) however the existence of stationary Nash equilibria or ε-Nash equilibrium have been proved only for some classes of average stochastic games. Rogers [16] and Sobel [19] showed that stationary Nash equilibria exist for nonzero-sum stochastic games with average payoffs when the transition probability matrices induced by any stationary strategies of the players are unichain. Mertens and Neyman [12] proved the existence of uniform ε-optimal strategies in two-player zero-sum games, i.e. they showed that for every $\varepsilon > 0$ each of the two players has a strategy that guarantees the discounted value up to ε for every discount factor sufficiently close to 0. Important results for two-person non-zero sum games with average payoffs have been obtained by Vieille [20] where he shows the existence of ε-Nash equilibria. Flesch et al.[7] constructed a three-player average stochastic game with given starting state for which stationary Nash equilibria does not exist, however a cyclic Markov equilibrium for such a game exists. In general case the existence of Nash equilibria for an arbitrary stochastic game with average payoffs is an open problem. Here we formulate a condition for the existence of stationary Nash equilibria in n-person average stochastic games and based on constructive proof of this condition we propose a continuous model for the considered games that allows determining stationary Nash equilibria if such equilibria exist.

2 Formulation of average stochastic game

We present the general formulation of n-person average stochastic game and specify some basic notions that we shall use in the paper.

2.1 The framework of n-person stochastic game

A stochastic game with n players consists of the following elements:

– a state space X (which we assume to be finite);

– a finite set $A^i(x)$ of actions with respect to each player $i \in \{1, 2, \dots, n\}$ for an arbitrary state $x \in X$;

– a payoff $f^i(x, a)$ with respect to each player $i \in \{1, 2, \dots, n\}$ for each state $x \in X$ and for an arbitrary action vector $a \in \prod_i A^i(x)$;

– a transition probability function $p : X \times \prod_{x \in X} \prod_{i=1}^{n} A^i(x) \times X \rightarrow [0, 1]$ that gives the probability transitions $p^a_{x,y}$ from an arbitrary $x \in X$ to an arbitrary $y \in Y$ for a fixed action vector $a \in \prod_i A^i(x)$, where $\sum_{y \in X} p^a_{x,y} = 1, \ \ \forall x \in X, \ a \in \prod_i A^i(x)$;

– a starting state $x_0 \in X$.

The game starts in the state x_0 and the play proceeds in *a sequence of stages*. At *stage* t players observe state x_t and simultaneously and independently choose actions $a^i_t \in A^i(x_t), \ \ i = 1, 2, \dots, n$. Then nature selects state $y = x_{t+1}$ according to probability transitions $p^{a_t}_{x_t,y}$ for the given action vector $a_t = (a^1_t, a^2_t, \dots, a^n_t)$. Such a play of the game produces a sequence of states and actions $x_0, a_0, x_1, a_1, \dots, x_t, a_t, \dots$ that defines the corresponding stream of stage payoffs $f^1_t = f^1(x_t, a_t), \ f^2_t = f^2(x_t, a_t), \dots, \ f^n_t = f^n(x_t, a_t), \ \ t = 0, 1, 2, \dots$. *The infinite average stochastic game* is the game with payoffs of players

$$\omega^i_{x_0} = \liminf_{t \to \infty} \ \mathsf{E}\left(\frac{1}{t} \sum_{\tau=0}^{t-1} f^i_\tau\right), \quad i = 1, 2, \dots, n,$$

where $\omega^i_{x_o}$ expresses the *average payoff per transition* of player i in infinite game. In the case $n = 1$ this game becomes the average Markov decision problem with a transition probability function $p : X \times \prod_{x \in X} A(x) \times X \rightarrow [0, 1]$ and immediate rewards $f(x, a) = f^1(x, a)$ in the states $x \in X$ for given actions $a \in A(x) = A^1(x)$.

In the paper we will study the stochastic games when players use pure and mixed stationary strategies of selection of the actions in the states.

2.2 Pure and mixed stationary strategies of the players

A strategy of player $i \in \{1, 2, \dots, n\}$ in a stochastic game is a mapping s^i that for every state $x_t \in X$ provides a probability distribution over the set of actions $A^i(x_t)$. If these probabilities take only values 0 and 1, then s^i is called *a pure strategy*, otherwise s^i is called *a mixed strategy*. If these probabilities depend only on the state $x_t = x \in X$ (i. e. s^i do not depend on t), then s^i is called *a stationary strategy*. This means that *a pure stationary strategy* of player $i \in \{1, 2, \dots, n\}$ can be regarded as a map

$$s^i : x \to a^i \in A^i(x) \quad \text{for} \quad x \in X$$

that determines for each state x an action $a^i \in A^i(x)$, i.e. $s^i(x) = a^i$. Obviously, the corresponding sets of pure stationary strategies $S^1, S^2, \dots, S^n$ of the players in the game with finite state and action spaces are finite sets.

In the following we will identify a pure stationary strategy $s^i(x)$ of player i with the set of boolean variables $s^i_{x,a^i} \in \{0, 1\}$, where for a given $x \in X$ $s^i_{x,a^i} = 1$ if and only if player i fixes the action $a^i \in A^i(x)$. So, we can represent the set of pure stationary strategies S^i of player i as the set of solutions of the following system:

$$\begin{cases} \sum\limits_{a^i \in A^i(x)} s^i_{x,a^i} = 1, & \forall x \in X; \\ s^i_{x,a^i} \in \{0, 1\}, & \forall x \in X, \quad \forall a^i \in A^i(x). \end{cases}$$

If in this system we change the restriction $s^i_{x,a^i} \in \{0, 1\}$ for $x \in X$, $a^i \in A^i(x)$ by the condition $0 \le s^i_{x,a^i} \le 1$ then we obtain the set of stationary strategies in the sense of Shapley [17], where s^i_{x,a^i} is treated as the probability of choices of the action a^i by player i every time when the state x is reached by any route in the dynamic stochastic game. Thus, we can identify the set of mixed stationary strategies of the players with the set of solutions of the system

$$\begin{cases} \sum\limits_{a^i \in A^i(x)} s^i_{x,a^i} = 1, & \forall x \in X; \\ s^i_{x,a^i} \ge 0, & \forall x \in X, \quad \forall a^i \in A^i(x) \end{cases} \tag{1}$$

and for a given profile $s = (s^1, s^2, \dots, s^n)$ of mixed strategies $s^1, s^2, \dots, s^n$ of the players the probability transition $p^s_{x,y}$ from a state x to a state y can be calculated as follows

$$p^s_{x,y} = \sum_{(a^1, a^2, \dots, a^n) \in A(x)} \prod_{k=1}^{n} s^k_{x,a^k} p^{(a^1, a^2, \dots, a^n)}_{x,y}. \tag{2}$$

In the sequel we will distinguish stochastic games in pure and mixed stationary strategies.

2.3 Average stochastic games in pure stationary strategies

Let $s = (s^1, s^2, \dots, s^n)$ be a profile of pure stationary strategies of the players and denote by $a(s) = (a^1(s), a^2(s), \dots, a^n(s)) \in \prod\limits_{x \in X} \prod\limits_{i=1}^{n} A^i(x)$ the action vector that corresponds to s and determines the probability distributions $p^s_{x,y} = p^{a(s)}_{x,y}$ in the states $x \in X$. Then the average payoffs per transition $\omega^1_{x_0}(s),\ \omega^2_{x_0}(s), \dots, \omega^n_{x_0}(s)$ for the players are determined as follows

$$\omega^i_{x_0}(s) = \sum_{y \in X} q^s_{x_0,y} f^i(y, a(s)), \quad i = 1, 2, \dots, n,$$

where $q^s_{x_o,y}$ represent the limiting probabilities in the states $y \in X$ for the Markov process with probability transition matrix $P^s = (p^s_{x,y})$ when the transitions start in x_0. So, if for the Markov process with probability matrix P^s the corresponding limiting probability matrix $Q^s = (q^s_{x,y})$ is known then $\omega^1_x, \omega^2_x, \dots, \omega^n_x$ can be determined for an arbitrary starting state $x \in X$ of the game. The functions $\omega^1_{x_0}(s),\ \omega^2_{x_0}(s), \dots, \omega^n_{x_0}(s)$ on $S = S^1 \times S^2 \times \cdots \times S^n$ define a game in normal form that corresponds to an infinite *average stochastic game in pure stationary strategies*. This game is determined by the set of states X, the sets of actions of the players $\{A^i(x)\}_{i=\overline{1,n}}$, the probability function p, the set of stage payoffs $\{f^i(x,a\}_{i=\overline{1,n}}$ and the starting position of the game x_0. Therefore we denote this game by $(X, \{A^i(x)\}_{i=\overline{1,n}}, \{f^i(x,a\}_{i=\overline{1,n}}, p, x_0)$. If the starting position of the game is chosen randomly according to distribution probabilities $\{\theta_x\}$ in X then such a game we denote $(X, \{A^i(x)\}_{i=\overline{1,n}}, \{f^i(x,a\}_{i=\overline{1,n}}, p, \{\theta_x\})$.

If an arbitrary profile $s = (s^1, s^2, \dots, s^n)$ of pure stationary strategies in a stochastic game induces a probability matrix P^s that corresponds to a Markov unichain then we say that the game possesses the unichain property and shortly we call it *unichain stochastic game;* otherwise we call it *multichain stochastic game.*

For an average stochastic game in pure strategies a Nash equilibrium may not exist. Therefore in this paper we study stochastic games in the case when players use mixed stationary strategies.

2.4 Stochastic games in mixed stationary strategies

Let $s = (s^1, s^2, \dots, s^n)$ be a profile of mixed stationary strategies of the players. Then elements of probability transition matrix $P^s = (p^s_{x,y})$ in the Markov process induced by s can be calculated according to (2). Therefore if $Q^s = (q^s_{x,y})$ is the limiting probability matrix of P^s then the average payoffs per transition $\omega^1_{x_0}(s),\ \omega^2_{x_0}(s), \dots, \omega^n_{x_0}(s)$ for the players are determined as follows

$$\omega^i_{x_0}(s) = \sum_{y \in X} q^s_{x_0,y} f^i(y, s), \quad i = 1, 2, \dots, n,$$

where

$$f^i(y,s) = \sum_{(a^1,a^2,\dots,a^n) \in A(y)} \prod_{k=1}^{n} s^k_{y,a^k} f^i(y, a^1, a^2, \dots, a^n)$$

expresses the average payoff (immediate reward) in the state $y \in X$ of player i when the corresponding stationary strategies $s^1, s^2, \dots, s^n$ have been applied by players $1, 2, \dots, n$ in y.

Let $\overline{S}^1, \overline{S}^2, \dots, \overline{S}^n$ be the corresponding sets of mixed stationary strategies for the players $1, 2, \dots, n$, i.e. each $\overline{S}^i$ for $i \in \{1, 2, \dots, n\}$ represents the set of solutions of system (1). Then the functions $\omega^1_{x_0}(s)$, $\omega^2_{x_0}(s), \dots, \omega^n_{x_0}(s)$ on $\overline{S} = \overline{S}^1 \times \overline{S}^2 \times \cdots \times \overline{S}^n$, define a game in normal form. This game corresponds to an infinite *average stochastic game in mixed stationary strategies.*

3 Preliminaries

We present some results for the average Markov decision problem and for the average stochastic game with unichain property that we shall use for the multichain average stochastic games.

3.1 A continuous model for the average Markov decision problem with unichain property

In [9] it has been shown that an average Markov decision problem with unichain property can be formulated as the following optimization problem:
Maximize
$$\psi(s, q) = \sum_{x \in X} \sum_{a \in A(x)} f(x, a) s_{x,a}\, q_x \tag{3}$$
subject to
$$\begin{cases} q_y - \sum\limits_{x \in X} \sum\limits_{a \in A(x)} p^a_{x,y} s_{x,a} q_x = 0, & \forall y \in X; \\ \sum\limits_{x \in X} q_x = 1; & \\ \sum\limits_{a \in A(x)} s_{x,a} = 1, & \forall x \in X; \\ s_{x,a} \geq 0, \quad \forall x \in X,\ a \in A(x). & \end{cases} \tag{4}$$
Here $f(x, a)$ represents the immediate reward in the state $x \in X$ for a given action $a \in A(x)$ in the unichain problem and $p^a_{x,y}$ expresses the probability transition from $x \in X$ to $y \in X$ for $a \in A(x)$. The variables $s_{x,a}$ correspond to strategies of selection of the actions $a \in A(x)$ in the states $x \in X$ and q_x for $x \in X$ represent the corresponding limiting probabilities in the states $x \in X$ for the probability transition matrix $P^s = (p^s_{x,y})$ induced by stationary strategy s.

In this problem the average reward $\psi(s, q)$ is maximized under the conditions (4) that determines the set of feasible stationary strategies in the unichain problem. An optimal solution (s^*, q^*) of problem (3), (4) with $s^*_{x,a} \in \{0, 1\}$ corresponds to an optimal stationary strategy $s^* : X \to A$ where $a^* = s^*(x)$ for $x \in X$ if $s^*_{x,a} = 1$. Using the notations $\alpha_{x,a} = s_{x,a} q_x$, for $x \in X, a \in A(x)$, problem (3), (4) can be easily transformed into the following linear programming problem:
Maximize
$$\overline{\psi}(\alpha) = \sum_{x \in X} \sum_{a \in A(x)} f(x, a)\, \alpha_{x,a} \tag{5}$$

subject to

$$\begin{cases} q_y - \sum\limits_{x\in X}\sum\limits_{a\in A(x)} p^a_{x,y}\alpha_{x,a} = 0, & \forall y \in X;\\ \sum\limits_{x\in X} q_x = 1; & \\ \sum\limits_{a\in A(x)} \alpha_{x,a} - q_x = 0, & \forall x \in X;\\ \alpha_{x,a} \geq 0, \ \ \forall x \in X,\ a \in A(x). & \end{cases} \tag{6}$$

This problem can be simplified by eliminating q_x from (6) and finally we obtain the problem in which it is necessary to maximize the objective function (5) on the set of solutions of the following system:

$$\begin{cases} \sum\limits_{a\in A(y)} \alpha_{y,a} - \sum\limits_{x\in X}\sum\limits_{a\in A(x)} p^a_{x,y}\,\alpha_{x,a} = 0, \quad \forall y \in X;\\ \sum\limits_{x\in X}\sum\limits_{a\in A(x)} \alpha_{x,a} = 1;\\ \alpha_{x,a} \geq 0, \ \ \forall x \in X,\ a \in A(x). \end{cases} \tag{7}$$

Based on the mentioned above relationship between problem (3), (4) and problem (5), (7) in [9] the following lemma is proven.

Lemma 1. *Let an average Markov decision problem be given, where an arbitrary stationary strategy s generates a Markov unichain, and consider the function*

$$\psi(s) = \sum_{x\in X}\sum_{a\in A(x)} f(x,a)s_{x,a}\,q_x$$

where q_x for $x \in X$ satisfy the condition

$$\begin{cases} q_y - \sum\limits_{x\in X}\sum\limits_{a\in A(x)} p^a_{x,y}s_{x,a}q_x = 0, \quad \forall y \in X;\\ \sum\limits_{x\in X} q_x = 1. \end{cases}$$

Then the function $\psi(s)$ on the set $\overline{S}$ of solutions of the system

$$\begin{cases} \sum\limits_{a\in A(x)} s_{x,a} = 1, \quad \forall x \in X;\\ s_{x,a} \geq 0, \ \ \forall x \in X,\ a \in A(x) \end{cases}$$

depends only on $s_{x,a}$ for $x \in X, a \in A(x)$, and $\psi(s)$ is quasi-monotone on $\overline{S}$.

Thus, the average unichain decision problem can be represented as the problem of the maximization of a quasi-monotone function $\overline{\psi}(s)$ on a compact set $\overline{S}$. Using this result in [10] it has been shown that an average stochastic game with unichain property can be formulated as a continuous game with quasi-monotone payoffs.

3.2 Determining stationary Nash equilibria for average stochastic games with unichain property

An average stochastic game with unichain property can be formulated in the terms of stationary strategies as follows.

Let $\overline{S} = \overline{S}^1 \times \overline{S}^2 \times \cdots \times \overline{S}^n$, where each $\overline{S}^i$ for $i \in \{1, 2, \ldots, n\}$ represents the set of solutions of system (1), i.e. $\overline{S}^i$ represents the set of mixed stationary strategies for player i. On $\overline{S}$ we define the average payoffs for the players as follows:

$$\psi^i(s^1, s^2, \ \ldots, \ s^n) = \sum_{x \in X} \sum_{(a^1, a^2, \ldots, a^n) \in A(x)} \prod_{k=1}^{n} s^k_{x,a^k} f^i(x, a^1, a^2, \ldots, a^n) q_x,$$

$$i = 1, 2, \ldots, n,$$

where q_x for $x \in X$ are determined uniquely from the following system of linear equations

$$\begin{cases} \displaystyle\sum_{x \in X} \sum_{(a^1, a^2, \ldots, a^n) \in A(x)} \prod_{k=1}^{n} s^k_{x,a^k} p^{(a^1, a^2, \ldots, a^n)}_{x,y} q_x = q_y, \quad \forall y \in X; \\ \displaystyle\sum_{x \in X} q_x = 1, \end{cases}$$

where $s^i \in \overline{S}^i, \ i = 1, 2, \ldots, n.$

The functions $\psi^i(s^1, s^2, \ \ldots, \ s^n), \ i = 1, 2, \ldots, n$ on $\overline{S}$ define a game in normal form that corresponds to a stationary average stochastic game with unichain property. For this game in [11] the following results are proven.

Lemma 2. *For an arbitrary unichain stochastic game each payoff function $\psi^i(s^1, \ s^2, \ \ldots, \ s^n), \quad i \in \{1, \ 2, \ \ldots, \ n\}$ possesses the property that $\psi^i(\overline{s}^1, \overline{s}^2, \ \ldots, \ \overline{s}^{i-1}, s^i, \overline{s}^{i+1}, \ \ldots, \ \overline{s}^n)$ is quasi-monotone with respect to $s^i \in \overline{S}^i$ for arbitrary fixed $\overline{s}^k \in \overline{S}^k, \ k = 1, 2, \ldots, i-1, i+1, \ldots, n.$*

Based on this lemma in [11] the following theorem is proven.

Theorem 1. *Let $(X, \ A, \ \{X_i\}_{i=\overline{1,n}}, \ \{f^i(x, a)\}_{i=\overline{1,n}}, \ p, \ x)$ be a stochastic game with a given starting position $x \in X$ and average payoff functions*

$$\psi^1(s^1, \, s^2, \, \ldots, \, s^m), \ \psi^2(s^1, \, s^2, \, \ldots, \, s^n), \ \ldots, \ \psi^m(s^1, \, s^2, \, \ldots, \, s^m)$$

of players $1, 2, \ldots, n$, respectively. If for an arbitrary $s = (s^1, s^2, \ldots, s^n) \in S$ of the game the transition probability matrix $P^s = (p^s_{x,y})$ corresponds to a Markov unichain then for the continuous game on $\overline{S}$ there exists a Nash equilibrium $s^* = (s^{1^*}, s^{2^*}, \ldots, s^{n*})$ which is a Nash equilibrium for an arbitrary starting state $x \in X$ of the game and $\psi^i(s^{1^*}, s^{2^*}, \ldots, s^{m*}) = \omega^i_x(s^{1^*}, s^{2^*}, \ldots, s^{m*}), \ \forall x \in X, \ i = 1, 2, \ldots, n.$

4 Some auxiliary results for multichain average decision problem

In this section we propose a continuous model for the multichain average decision problem and extend the results from Section 3.1 for the general case of decision problem. We shall use these results in the next section for the multichain average stochastic games.

4.1 Linear programming approach for multichain decision problem

It is well-known that the optimal stationary strategies for a multichain average Markov decision problem can be found using the following linear programming problem (see [11, 14]):
Maximize

$$\overline{\psi}(\alpha,\beta)=\sum_{x\in X}\sum_{a\in A(x)}f(x,a)\alpha_{x,a} \tag{8}$$

subject to

$$\begin{cases}\sum\limits_{a\in A(y)}\alpha_{y,a}-\sum\limits_{x\in X}\sum\limits_{a\in A(x)}p_{x,y}^{a}\,\alpha_{x,a}=0, \quad \forall y\in X;\\[2mm] \sum\limits_{a\in A(y)}\alpha_{y,a}+\sum\limits_{a\in A(y)}\beta_{y,a}-\sum\limits_{x\in X}\sum\limits_{a\in A(x)}p_{x,y}^{a}\beta_{x,a}=\theta_y, \ \forall y\in X;\\[2mm] \qquad \alpha_{x,a}\geq 0, \qquad \beta_{y,a}\geq 0, \quad \forall x\in X, \ a\in A(x),\end{cases} \tag{9}$$

where θ_y for $y\in X$ represent arbitrary positive values that satisfy the condition $\sum\limits_{y\in X}\theta_y=1$. Recall that $f(x,a)$ denotes the immediate cost in a state $x\in X$ for a given action $a\in A(x)$ in the decision problem and $p_{x,y}^{a}$ represent the corresponding probability transitions from a state $x\in X$ to the states $y\in X$ for $a\in A(x)$, where $\sum\limits_{y\in X}p_{x,y}^{a}=1$.

This problem generalizes the unichain linear programming model (5), (7) from Section 3.1. In (9) the restrictions

$$\sum_{a\in A(y)}\alpha_{y,a}+\sum_{a\in A(y)}\beta_{y,a}-\sum_{x\in X}\sum_{a\in A(x)}p_{x,y}^{a}\beta_{x,a}=\theta_y, \quad \forall y\in X \tag{10}$$

with the condition $\sum\limits_{y\in X}\theta_y=1$ generalize the constraint

$$\sum_{x\in X}\sum_{a\in A(y)}\alpha_{y,a}=1 \tag{11}$$

in the unichain model. Condition (11) is obtained if we sum (10) over y.

The relationship between feasible solutions of problem (8),(9) and stationary strategies in the average Markov decision problem can be established on the basis of the following *randomized stationary decision rule* (see [14]):

Let (α, β) be a feasible solution of the linear programming problem (8), (9) and denote $X_\alpha = \{x \in X | \sum_{a\in X} \alpha_{x,a} > 0\}$. Then (α, β) possesses the properties that $\sum_{a\in A(x)} \beta_{x,a} > 0$ for $x \in X \setminus X_\alpha$ and a stationary randomized decision rule $d_{\alpha,\beta}(x)$ for a feasible solution (α, β) is defined by

$$s_{d_{\alpha,\beta}(x)}(a) = \begin{cases} \dfrac{\alpha_{x,a}}{\sum\limits_{a\in A(x)} \alpha_{x,a}} & \text{if } x \in X_\alpha; \\ \dfrac{\beta_{x,a}}{\sum\limits_{a\in A(x)} \beta_{x,a}} & \text{if } x \in X \setminus X_\alpha, \end{cases} \tag{12}$$

where $s_{d_{x,y}(x)}(a)$ expresses the probability of choosing the actions $a \in A(x)$ in the states $x \in X$ for the average decision problem under decision rule d. This means that for a given feasible solution (α, β) the decision rule d determines a stationary strategy $s_{x,a} = s_{d_{\alpha,\beta}(x)}(a)$ of choosing the actions $a \in A(x)$ in the states $x \in X$. If for each $x \in X_\alpha$ it holds $\alpha_{x,a} > 0$ for a single $a \in A(x)$ and for each $x \in X \setminus X_\alpha$ it holds $\beta_{x,a} > 0$ for a single $a \in A(x)$ then (12) generates a deterministic decision rule

$$d_{\alpha,\beta}(x) = \begin{cases} a & \text{if } \alpha_{x,a} > 0 \text{ and } x \in X_\alpha; \\ a' & \text{if } \beta_{x,a'} > 0 \text{ and } x \in X \setminus X_\alpha \end{cases}$$

that corresponds to a pure stationary strategy s, where $s_{x,a} = s_{d_{\alpha,\beta}(x)}(a)$ for $x \in X$ and $a \in A(x)$.

Remark 1. In [14] problem (8), (9) is regarded as the dual model of the following linear programming problem:
Minimize

$$\phi(\varepsilon, \omega) = \sum_{x\in X} \theta_x \omega_x \tag{13}$$

subject to

$$\begin{cases} \varepsilon_x + \omega_x \geq f(x,a) + \sum\limits_{y\in X} p^a_{x,y} \varepsilon_y, & \forall x \in X, \quad \forall a \in A(x); \\ \omega_x \geq \sum\limits_{y\in X} p^a_{x,y} \omega_y, & \forall x \in X, \quad \forall a \in A(x). \end{cases} \tag{14}$$

The optimal value of objective function in this problem as well as the optimal value of objective function in problem (8), (9) express the optimal average reward when the initial state is chosen according to distribution $\{\theta_x\}$. Solving problem (13), (14) we obtain the value ω^*_x for each $x \in X$ that represents the optimal average reward when transition starts in x with probability equal to 1. This means that if (α^*, β^*) is the optimal solution of problem (8), (9) then we can determine the optimal strategy s^* and the optimal values of object functions of problems (13), (14) and (8), (9), where $\phi(\varepsilon^*, \omega^*) = \overline{\psi}(\alpha^*, \beta^*)$.

4.2 Multichain decision model in the terms of stationary strategies

The continuous model we propose for the multichain average decision problem that generalizes the unichain continuous model (3), (4) is the following:
Maximize

$$\psi(s,q,w) = \sum_{x\in X}\sum_{a\in A(x)} f(x,a)s_{x,a}q_x \tag{15}$$

subject to

$$\begin{cases} q_y - \sum\limits_{x\in X}\sum\limits_{a\in A(x)} p^a_{x,y}\, s_{x,a}q_x = 0, & \forall y\in X;\\ q_y + w_y - \sum\limits_{x\in X}\sum\limits_{a\in A(x)} p^a_{x,y}s_{x,a}w_x = \theta_y, & \forall y\in X;\\ \sum\limits_{a\in A(y)} s_{y,a} = 1, & \forall y\in X;\\ s_{x,a}\ge 0,\ \ \forall x\in X,\ \forall a\in A(x); \quad w_x\ge 0,\ \forall x\in X, \end{cases} \tag{16}$$

where θ_y are the same values as in problem (8), (9) and $s_{x,a},\ q_x,\ w_x$ for $x\in X$, $a\in A(x)$ represent the variables that must be found.

Theorem 2. *Optimization problem (15), (16) determines the optimal stationary strategies of the multichain average Markov decision problem.*

Proof. Indeed, if we assume that each action set $A(x), x\in X$ contains a single action a' then system (9) is transformed into the following system of equations

$$\begin{cases} q_y - \sum\limits_{x\in X} p_{x,y}q_x = 0, & \forall y\in X;\\ q_y + w_y - \sum\limits_{x\in X} p_{x,y}w_x = \theta_y, & \forall y\in X \end{cases}$$

with conditions $q_y, w_y\ge 0$ for $y\in X$, where $q_y=\alpha_{y,a'},\ w_y=\beta_{y,a'},\ \forall y\in X$ and $p_{x,y}=p^{a'}_{x,y},\ \forall x,y\in X$. This system uniquely determines q_x for $x\in X$ and determines w_x for $x\in X$ up to an additive constant in each recurrent class of $P=(p_{x,y})$ (see [14]). Here q_x represents the limiting probability in the state x when transitions start in the states $y\in X$ with probabilities θ_y and therefore the condition $q_x\ge 0$ for $x\in X$ can be released. Note that w_x for some states may be negative, however always the additive constants in the corresponding recurrent classes can be chosen so that w_x became nonnegative. In general, we can observe that in (16) the condition $w_x\ge 0$ for $x\in X$ can be released and this does not influence the value of objective function of the problem. In the case $|A(x)|=1,\ \forall x\in X$ the average cost is determined as $\psi = \sum\limits_{x\in X} f(x)q_x$, where $f(x)=f(x,a), \forall x\in X$.

If the action sets $A(x),\ x\in X$ may contain more than one action then for a given stationary strategy $s\in \overline{S}$ of selection of the actions in the states we can find the average cost $\psi(s)$ in a similar way as above by considering the probability matrix $P^s = (p^s_{x,y})$, where

$$p^s_{x,y} = \sum_{a\in A(x)} p^a_{x,y} s_{x,a} \tag{17}$$

expresses the probability transition from a state $x \in X$ to a state $y \in X$ when the strategy s of selections of the actions in the states is applied. This means that we have to solve the following system of equations

$$\begin{cases} q_y - \sum\limits_{x\in X} p^s_{x,y} q_x = 0, & \forall y \in X; \\ q_y + w_y - \sum\limits_{x\in X} p^s_{x,y} w_x = \theta_y, & \forall y \in X. \end{cases}$$

If in this system we take into account (17) then this system can be written as follows

$$\begin{cases} q_y - \sum\limits_{x\in X} \sum\limits_{a\in A(x)} p^a_{x,y}\, s_{x,a} q_x = 0, & \forall y \in X; \\ q_y + w_y - \sum\limits_{x\in X} \sum\limits_{a\in A(x)} p^a_{x,y} s_{x,a} w_x = \theta_y, & \forall y \in X. \end{cases} \tag{18}$$

An arbitrary solution (q, w) of the system of equations (18) uniquely determines q_y for $y \in X$ that allows us to determine the average cost per transition

$$\psi(s) = \sum_{x\in X} \sum_{a\in X} f(x, a) s_{x,a} q_x \tag{19}$$

when the stationary strategy s is applied. If we are seeking for an optimal stationary strategy then we should add to (18) the conditions

$$\sum_{a\in A(x)} s_{x,a} = 1,\ \forall x \in X;\ s_{x,a} \geq 0,\ \forall x \in X, a \in A(x) \tag{20}$$

and to maximize (19) under the constraints (18), (20). In such a way we obtain problem (15), (16) without conditions $w_x \geq 0$ for $x \in X$. As we have noted the conditions $w_x \geq 0$ for $x \in X$ do not influence the values of the objective function (15) and therefore we can preserve such conditions that show the relationship of the problem (15), (16) with problem (8), (9). $\square$

The relationship between feasible solutions of problem (8), (9) and feasible solutions of problem (15), (16) can be established on the basis of the following lemma.

Lemma 3. *Let (s, q, w) be a feasible solution of problem (15), (16). Then*

$$\alpha_{x,a} = s_{x,a} q_x, \quad \beta_{x,a} = s_{x,a} w_x, \quad \forall x \in X, a \in A(x) \tag{21}$$

represent a feasible solution (α, β) of problem (8), (9) and $\varphi(s, q, w) = \overline{\psi}(\alpha, \beta)$. If (α, β) is a feasible solution of problem (8), (9) then a feasible solution (s, q, w) of *problem (15), (16) can be determined as follows:*

$$s_{x,a} = \begin{cases} \dfrac{\alpha_{x,a}}{\sum\limits_{a \in A(x)} \alpha_{x,a}} & \textit{for } \ x \in X_\alpha, \ a \in A(x); \\ \dfrac{\beta_{x,a}}{\sum\limits_{a \in A(x)} \beta_{x,a}} & \textit{for } \ x \in X \setminus X_\alpha, \ a \in A(x); \end{cases} \tag{22}$$

$$q_x = \sum_{a \in A(x)} \alpha_{x,a}, \quad w_x = \sum_{a \in A(x)} \beta_{x,a} \quad \textit{for } \ x \in X.$$

Proof. Assume that (s, q, w) is a feasible solution of problem (15), (16) and (α, β) is determined according to (21). Then by introducing (21) in (8),(9) we can observe that (9) is transformed in (16) and $\psi(s, q, w) = \overline{\psi}(\alpha, \beta)$, i.e. (α, β) is a feasible solution of problem (8), (9). The second part of lemma follows directly from the properties of feasible solutions of problems (8),(9) and (15),(16). □

Note that an arbitrary pure stationary strategy s of problem (15), (16) corresponds to a basic solution (α, β) of problem (8), (9) for which (22) holds, however system (9) may contain basic solutions for which stationary strategies determined through (22) do not correspond to pure stationary strategies. Moreover two different feasible solutions of problem (8), (9) may generate through (22) the same stationary strategy. Such solutions of system (9) are considered *equivalent solutions* for the decision problem.

Corollary 1. *If* $(\alpha^i, \beta^i), \ \ i = \overline{1, k}$, *represent the basic solutions of system (9) then the set of solutions*

$$M = \Big\{ (\alpha, \beta) | \ (\alpha, \beta) = \sum_{i=1}^{k} \lambda^i (\alpha^i, \beta^i), \ \ \sum_{i=1}^{k} \lambda^i = 1, \ \ \lambda^i > 0, \ i = \overline{1, k} \Big\}$$

determines all feasible stationary strategies of problem (15), (16) through (22).

An arbitrary solution (α, β) *of system (9) can be represented as follows:* $\alpha = \sum_{i=1}^{k} \lambda^i \alpha^i$, *where* $\sum_{i=1}^{k} \lambda^i = 1; \ \ \lambda^i \geq 0, \ \ i = \overline{1, k}$, *and* β *represents a solution of the system*

$$\begin{cases} \sum\limits_{a \in A(y)} \beta_{x,a} - \sum\limits_{z \in X} \sum\limits_{a \in A(z)} p_{z,x}^{a} \beta_{z,a} = \theta_x - \sum\limits_{a \in A(x)} \alpha_{x,a}, \ \forall x \in X; \\ \beta_{y,a} \geq 0, \ \ \forall x \in X, \ a \in A(x). \end{cases}$$

If (α, β) *is a feasible solution of problem (8), (9) and* $(\alpha, \beta) \notin M$ *then there exists a solution* $(\alpha', \beta') \in M$ *that is equivalent to* (α, β) *and* $\overline{\psi}(\alpha, \beta) = \overline{\psi}(\alpha', \beta')$.

4.3 The main property of the object function

Using problem (15), (16) we can now extend the results from Section 3.1 for the general case of average decision problem.

Theorem 3. *Let an average Markov decision problem be given and consider the function*

$$\psi(s)=\sum_{x\in X}\sum_{a\in A(x)} f_{(x,a)}s_{x,a}\,q_x, \tag{23}$$

where q_x for $x\in X$ satisfy the condition

$$\begin{cases} q_y-\sum\limits_{x\in X}\sum\limits_{a\in A(x)} p^a_{x,y}\,s_{x,a}q_x=0, & \forall y\in X;\\ q_y+w_y-\sum\limits_{x\in X}\sum\limits_{a\in A(x)} p^a_{x,y}s_{x,a}w_x=\theta_y, & \forall y\in X. \end{cases} \tag{24}$$

Then on the set $\overline{S}$ of solutions of the system

$$\begin{cases} \sum\limits_{a\in A(x)} s_{x,a}=1, \quad \forall x\in X;\\ \qquad s_{x,a}\geq 0, \quad \forall x\in X,\ a\in A(x) \end{cases} \tag{25}$$

the function $\psi(s)$ depends only on $s_{x,a}$ for $x\in X, a\in A(x)$ and $\psi(s)$ is quasi-monotone on $\overline{S}$.

Proof. For an arbitrary $s\in\overline{S}$ system (24) uniquely determines q_x for $x\in X$ and determines w_x for $x\in X$ up to a constant in each recurrent class of $P^s=(p^s_{x,y})$, where $p^s_{x,y}=\sum\limits_{a\in A(x)} p^a_{x,y}s_{x,a}$, $\forall x,y\in X$. This means that $\psi(s)$ is determined uniquely for an arbitrary $s\in\overline{S}$, i.e. the first part of the theorem holds.

Now let us prove the second part of the theorem.

Consider arbitrary two strategies s', $s''\in\overline{S}$ and assume that $s'\neq s''$. Then according to Lemma 3 there exist feasible solutions (α',β') and (α'',β'') of linear programming problem (8), (9) for which

$$\psi(s')=\overline{\psi}(\alpha',\beta'),\quad \psi(s'')=\overline{\psi}(\alpha'',\beta'''), \tag{26}$$

where

$$\alpha'_{x,a}=s'_{x,a}q'_x,\quad \alpha''_{x,y}=s''_{x,a}q''_x,\quad \forall x\in X,\ a\in A(x);$$

$$\beta'_{x,a}=s'_{x,a}w'_x,\quad \beta''_{x,y}=s''_{x,a}q''_x,\quad \forall x\in X,\ a\in A(x);$$

$$q'_x=\sum_{a\in A(x)}\alpha'_{x,a}\qquad w'_{x,a}=\sum_{a\in A(x)}\beta'_{x,a},\quad \forall x\in X;$$

$$q''_x=\sum_{a\in A(x)}\alpha''_{x,a}\quad w''_{x,a}=\sum_{a\in A(x)}\beta''_{x,a},\quad \forall x\in X.$$

The function $\overline{\psi}(\alpha,\beta))$ is linear and therefore for an arbitrary feasible solution $(\overline{\alpha},\overline{\beta})$ of problem (8), (9) holds

$$\overline{\psi}(\overline{\alpha},\overline{\beta}) = t\overline{\psi}(\alpha',\beta') + (1-t)\overline{\psi}(\alpha'',\beta'') \tag{27}$$

if $0 \le t \le 1$ and

$$(\overline{\alpha},\overline{\beta}) = t(\alpha',\beta') + (1-t)(\alpha'',\beta'').$$

Note that $(\overline{\alpha},\overline{\beta})$ corresponds to a stationary strategy $\overline{s}$ for which

$$\psi(\overline{s}) = \overline{\psi}(\overline{\alpha},\overline{\beta}), \tag{28}$$

where

$$\overline{s}_{x,a} = \begin{cases} \dfrac{\overline{\alpha}_{x,a}}{\overline{q}_x} & \text{if } \ x \in X_{\overline{\alpha}}; \\ \dfrac{\beta_{x,a}}{\overline{w}_x} & \text{if } \ x \in X \setminus X_{\overline{\alpha}}. \end{cases} \tag{29}$$

Here $X_{\overline{\alpha}} = \{x \in X | \sum\limits_{a\in A(x)} \overline{\alpha}_{x,a} > 0\}$ is the set of recurrent states induced by $P^{\overline{s}} = (p^{\overline{s}}_{x,y})$, where $p^{\overline{s}}_{x,y}$ are calculated according to (17) for $s = \overline{s}$ and

$$\overline{q}_x = tq'_x + (1-t)q'', \quad \overline{w}_x = tw'_x + (1-t)w''_x, \ \ \forall x \in X.$$

We can see that $X_{\overline{\alpha}} = X_{\alpha'} \cup X_{\alpha''}$, where $X_{\alpha'} = \{x \in X | \sum\limits_{a\in A(x)} \alpha'_{x,a} > 0\}$ and $X_{\alpha''} = \{x \in X | \sum\limits_{a\in A(x)} \alpha''_{x,a} > 0\}$.

The value

$$\psi(\overline{s}) = \sum_{x\in X} \sum_{a\in A(x)} f(x,a)\overline{s}_{x,a}\overline{q}_x$$

is determined by $f(x,a)$, $\overline{s}_{x,a}$ and $\overline{q}_x$ in recurrent states $x \in X_{\overline{\alpha}}$ and it is equal to $\overline{\psi}(\overline{\alpha},\overline{\beta})$. If we use (29) then for $x \in X_{\overline{\alpha}}$ and $a \in A(x)$ we have

$$\overline{s}_{x,a} = \frac{t\alpha'_{x,a} + (1-t)\alpha''_{x,a}}{tq'_x + (1-t)q''_x} = \frac{ts'_{x,a}q'_x + (1-t)s''_{x,a}q''_x}{tq'_x + (1-t)q''_x} =$$

$$= \frac{tq'_x}{tq'_x + (1-t)q''_x}s'_{x,a} + \frac{(1-t)q''_x}{tq'_x + (1-t)q''_x}s''_{x,a}$$

and for $x \in X \setminus X_{\overline{\alpha}}$ and $a \in A(x)$ we have

$$\overline{s}_{x,a} = \frac{t\beta'_{x,a} + (1-t)\beta''_{x,a}}{tw'_x + (1-t)w''_x} = \frac{ts'_{x,a}w'_x + (1-t)s''_{x,a}w''_x}{tw'_x + (1-t)w''_x} =$$

$$= \frac{tw'_x}{tw'_x + (1-t)w''_x}s'_{x,a} + \frac{(1-t)w''_x}{tw'_x + (1-t)w''_x}s''_{x,a}.$$

So, we obtain

$$\overline{s}_{x,a} = t_x s'_{x,a} + (1-t_x)s''_{x,a}, \quad \forall a \in A(x), \tag{30}$$

where

$$t_x = \begin{cases} \dfrac{tq'_x}{tq'_x + (1-t)q''_x} & \text{if } \ x \in X_{\overline{\alpha}}; \\ \dfrac{tw'_x}{tw'_x + (1-t)w''_x} & \text{if } \ x \in X \setminus X_{\overline{\alpha}}. \end{cases} \tag{31}$$

and from (26)–(28) we have

$$\psi(\overline{s}) = t\psi(s') + (1-t)\psi(s''). \tag{32}$$

This means that if we consider the set of strategies

$$S(s', s'') = \{\overline{s} |\ \overline{s}_{x,a} = t_x s'_{x,a} + (1-t_x)s''_{x,a}, \quad \forall x \in X, a \in A(x)\}$$

then for an arbitrary $\overline{s} \in S(s', s'')$ holds

$$\min\{\psi(s'), \psi(s'')\} \le \psi(\overline{s}) \le \max\{\psi(s'), \psi(s'')\}, \tag{33}$$

i.e $\psi(s)$ is monotone on $S(s', s'')$. Moreover, using (30)–(33) we obtain that $\overline{s}$ possesses the properties

$$\lim_{t\to 1} \overline{s}_{x,a} = s'_{x,a}, \forall x \in X, a \in A(x); \qquad \lim_{t\to 0} \overline{s}_{x,a} = s''_{x,a}, \forall x \in X, a \in A(x). \tag{34}$$

and respectively

$$\lim_{t\to 1} \psi(\overline{s}) = \psi(s'); \qquad \lim_{t\to 0} \psi(\overline{s}) = \psi(s'').$$

In the following we show that the function $\psi(s)$ is quasi-monotone on $\overline{S}$. To prove this it is sufficient to show that for an arbitrary $c \in R$ the *sublevel set*

$$L_c^-(\psi) = \{s \in \overline{S} |\ \psi(s) \le c\}$$

and the *superlevel set*

$$L_c^+(\psi) = \{s \in \overline{S} |\ \psi(s) \ge c\}$$

of function $\psi(s)$ are convex. These sets can be obtained respectively from the *sublevel set*

$$L_c^-(\overline{\psi}) = \{(\alpha, \beta) |\ \overline{\psi}(\alpha, \beta)) \le c\}$$

and the *superlevel set*

$$L_c^+(\overline{\psi}) = \{(\alpha, \beta) |\ \overline{\psi}(\alpha, \beta)) \ge c\}$$

of function $\overline{\psi}(\alpha, \beta)$ for linear programming problem (8), (9) using (22).

Denote by $(\alpha^i, \beta^i), \quad i = \overline{1,k}$ the basic solutions of system (9). According to Corollary 1 all feasible strategies of problem (8), (9) can be obtained trough (22) using the basic solutions (α^i, β^i), $i = \overline{1,k}$. Each (α^i, β^i), $i = \overline{1,k}$, determines a stationary strategy

$$s^i_{x,a} = \begin{cases} \dfrac{\alpha^i_{x,a}}{q^i_x}, & \text{for } \ x \in X_{\alpha^i}, \ a \in A(x); \\ \dfrac{\beta^i_{x,a}}{w^i_x}, & \text{for } \ x \in X \setminus X_{\alpha^i}, \ a \in A(x) \end{cases} \tag{35}$$

for which $\psi(s^i) = \overline{\psi}(\alpha^i, \beta^i)$ where

$$X_{\alpha^i} = \{x \in X | \sum_{a \in A(x)} \alpha^i_{x,a} > 0\}, \quad q^i_x = \sum_{a \in A(x)} \alpha^i_{x,a}, \quad w^i_x = \sum_{a \in A(x)} \beta^i_{x,a}, \ \forall x \in X. \tag{36}$$

An arbitrary feasible solution (α, β) of system (9) determines a stationary strategy

$$s_{x,a} = \begin{cases} \dfrac{\alpha_{x,a}}{q_x}, & \text{for } \ x \in X_\alpha, \ a \in A(x); \\ \dfrac{\beta_{x,a}}{w_x}, & \text{for } \ x \in X \setminus X_\alpha, \ a \in A(x), \end{cases} \tag{37}$$

for which $\psi(s) = \overline{\psi}(\alpha, \beta)$ where

$$X_\alpha = \{x \in X | \sum_{a \in A(x)} \alpha_{x,a} > 0\}, \quad q_x = \sum_{a \in A(x)} \alpha_{x,a}, \quad w_x = \sum_{a \in A(x)} \beta_{x,a}, \ \forall x \in X.$$

Taking into account that (α, β) can be represented as

$$(\alpha, \beta) = \sum_{i=1}^k \lambda^i (\alpha^i, \beta^i), \text{ where } \sum_{i=1}^k \lambda^i = 1, \quad \lambda^i \geq 0, \ i = \overline{1,k} \tag{38}$$

we have $\overline{\psi}(\alpha, \beta) = \sum\limits_{i=1}^k \overline{\psi}(\alpha^i, \beta^i)\lambda^i$ and we can consider

$$X_\alpha = \bigcup_{i=1}^k X_{\alpha^i}; \quad \alpha = \sum_{i=1}^k \lambda^i \alpha^i; \quad q = \sum_{i=1}^k \lambda^i q^i; \quad w = \sum_{i=1}^k \lambda^i w^i. \tag{39}$$

Using (35)–(39) we obtain:

$$s_{x,a} = \frac{\alpha_{x,a}}{q_x} = \frac{\sum_{i=1}^k \lambda^i \alpha^k_{x,a}}{q_x} = \frac{\sum_{i=1}^k \lambda^i s^i_{x,a} q^i_x}{q_x} = \sum_{i=1}^k \frac{\lambda^i q^i_x}{q_x} s^i_{x,a}, \quad \forall x \in X_\alpha, \ a \in A(x);$$

$$s_{x,a} = \frac{\beta_{x,a}}{w_x} = \frac{\sum_{i=1}^k \lambda^i \beta^k_{x,a}}{w_x} = \frac{\sum_{i=1}^k \lambda^i s^i_{x,a} w^i_x}{w_x} = \sum_{i=1}^k \frac{\lambda^i w^i_x}{w_x} s^i_{x,a}, \quad \forall x \in X \backslash X_\alpha, a \in A(x)$$

and

$$q_x = \sum_{i=1}^k \lambda^i q^i_x, \qquad w_x = \sum_{i=1}^k \lambda^i w^i_x \ \text{ for } \quad x \in X. \tag{40}$$

So,

$$s_{x,a} = \begin{cases} \displaystyle\sum_{i=1}^{k} \frac{\lambda^i q_x^i}{q_x} s_{x,a}^i & \text{if } \ q_x > 0; \\ \displaystyle\sum_{i=1}^{k} \frac{\lambda^i w_x^i}{w_x} s_{x,a}^i & \text{if } \ q_x = 0, \end{cases} \tag{41}$$

where q_x and w_x are determined according to (40).

We can see that if $\lambda^i, s^i, q^i,\ i = \overline{1,k}$ are given then the strategy s defined by (41) is a feasible strategy because $s_{x,a} \geq 0, \forall x \in X, a \in A(x)$ and $\sum_{a\in A(x)} s_{x,a} = 1,\ \forall x \in X$. Moreover, we can observe that $q_x = \sum\limits_{i=1}^{k} \lambda^i q_x^i,\ \ w_x = \sum\limits_{i=1}^{k} \lambda^i w_x^i$ for $x \in X$ represent a solution of system (24) for the strategy s defined by (41). This can be verified by introducing (40) and (41) in (24); after such a substitution all equations from (24) are transformed into identities. For $\psi(s)$ we have

$$\psi(s) = \sum_{x\in X} \sum_{a\in A(x)} f(x,a) s_{x,a} q_x = \sum_{x\in X_\alpha} \sum_{a\in A(x)} f(x,a) \sum_{i=1}^{k} \left(\frac{\lambda^i q_x^i}{q_x} s_{x,a}^i \right) q_x =$$

$$\sum_{i=1}^{k} \Bigg(\sum_{x\in X_{\alpha^i}} \sum_{a\in A(x)} f(x,a) s_{x,a}^i q_x^i \Bigg) \lambda^i = \sum_{i=1}^{k} \psi(s^i) \lambda^i,$$

i.e.

$$\psi(s) = \sum_{i=1}^{k} \psi(s^i) \lambda^i, \tag{42}$$

where s is the strategy that corresponds to (α, β).

Thus, assuming that the strategies $s^1, s^2, \ldots, s^k$ correspond to basic solutions $(\alpha^1, \beta^1), (\alpha^2, \beta^2), \ldots, (\alpha^k, \beta^k)$ of problem (8), (9) and $s \in \overline{S}$ corresponds to an arbitrary solution (α, β) of this problem that can be expressed as convex combination of basic solutions of problem (8), (9) with the corresponding coefficients $\lambda^1, \lambda^2, \ldots, \lambda^k$, we can express the strategy s and the corresponding value $\psi(s)$ by (40)–(42). In general the representation (40)–(42) of strategy s and of the value $\psi(s)$ is valid for an arbitrary finite set of strategies from $\overline{S}$ if (α, β) can be represented as convex combination of the finite number of feasible solutions $(\alpha^1, \beta^1), (\alpha^2, \beta^2), \ldots, (\alpha^k, \beta^k)$ that correspond to $s^1, s^2, \ldots, s^k$; in the case $k = 2$ from (40)–(42) we obtain (30)–(32). It is evident that for a feasible strategy $s \in S$ the representation (40), (41) may be not unique, i.e. two differen vectors $\overline{\Lambda} = (\overline{\lambda}^1, \overline{\lambda}^2, \ldots, \overline{\lambda}^k)$ and $\overline{\overline{\Lambda}} = \overline{\overline{\lambda}}^1, \overline{\overline{\lambda}}^2, \ldots, \overline{\overline{\lambda}}^k$ may be that determine the same strategy s via (40), (41). In the following we will assume that $s^1, s^2, \ldots, s^k$ represent the system of linear independent basic solutions of system (25), i.e. each $s^i \in \overline{S}$ corresponds to a pure stationary strategy.

Thus, an arbitrary strategy $s \in \overline{S}$ is determined according to (40), (41) where $\lambda^1, \lambda^2, \dots, \lambda^k$ correspond to a solution of the following system

$$\sum_{i=1}^{k} \lambda^i = 1;\ \ \lambda^i \geq 0,\ \ i = \overline{1,k}.$$

Consequently, the sublevel set $L_c^-(\psi)$ of function $\psi(s)$ represents the set of strategies s determined by (40), (41), where $\lambda^1, \lambda^2, \dots, \lambda^k$ satisfy the condition

$$\begin{cases} \sum\limits_{i=1}^{k} \psi(s^i)\lambda^i \leq c; \\ \sum\limits_{i=1}^{k} \lambda^i = 1; \quad \lambda^i \geq 0, \quad i = \overline{1,k} \end{cases} \tag{43}$$

and the superlevel set $L_c^+(\psi)$ of $\psi(s)$ represents the set of strategies s determined by (40),(41), where $\lambda^1, \lambda^2, \dots, \lambda^k$ satisfy the condition

$$\begin{cases} \sum\limits_{i=1}^{k} \psi(s^i)\lambda^i \geq c; \\ \sum\limits_{i=1}^{k} \lambda^i = 1; \quad \lambda^i \geq 0, \quad i = \overline{1,k}. \end{cases} \tag{44}$$

Respectively the level set $L_c(\psi) = \{s \in \overline{S} |\ \psi(s) = c\}$ of function $\psi(s)$ represents the set of strategies s determined by (40), (41), where $\lambda^1, \lambda^2, \dots, \lambda^k$ satisfy the condition

$$\begin{cases} \sum\limits_{i=1}^{k} \psi(s^i)\lambda^i = c; \\ \sum\limits_{i=1}^{k} \lambda^i = 1; \quad \lambda^i \geq 0, \quad i = \overline{1,k}. \end{cases} \tag{45}$$

Let us show that $L_c^-(\psi),\ L_c^+(\psi),\ L_c(\psi)$ are convex sets. We present the proof of convexity of sublevel set $L_c^-(\psi)$. The proof of convexity of $L_c^+(\psi)$ and $L_c(\psi)$ is similar to the proof of convexity of $L_c^-(\psi)$.

Denote by Λ the set of solutions $(\lambda^1, \lambda^2, \dots, \lambda^k)$ of system (43). Then from (40), (41), (43) we have

$$L_c^-(\psi) = \prod_{x \in X} \hat{S}_x$$

where $\hat{S}_x$ represents the set of strategies

$$s_{x,a} = \begin{cases} \dfrac{\sum_{i=1}^{k} \lambda^i q_x^i s_{x,a}^i}{\sum_{i=1}^{k} \lambda^i q_x^i} & \text{if } \sum_{i=1}^{k} \lambda^i q_x^i > 0, \\[2ex] \dfrac{\sum_{i=1}^{k} \lambda^i w_x^i s_{x,a}^i}{\sum_{i=1}^{k} \lambda^i w_x^i} & \text{if } \sum_{i=1}^{k} \lambda^i q_x^i = 0, \end{cases} \qquad a \in A(x)$$

in the state $x \in X$ determined by $(\lambda^1, \lambda^2, \dots, \lambda^k) \in \Lambda$.

For an arbitrary $x \in X$ the set Λ can be represented as follows $\Lambda = \Lambda_x^+ \cup \Lambda_x^0$, where

$$\Lambda_x^+ = \{(\lambda^1, \lambda^2, \dots, \lambda^k) \in \Lambda | \sum_{i=1}^{k} \lambda^i q_x^i > 0\},$$

$$\Lambda_x^0 = \{(\lambda^1, \lambda^2, \dots, \lambda^k) \in \Lambda | \sum_{i=1}^{k} \lambda^i q_x^i = 0\}$$

and $\sum_{i=1}^k \lambda^i w_x^i > 0$ if $\sum_{i=1}^k \lambda^i q_x^i = 0$.

Therefore $\hat{S}_x$ can be expressed as follows $\hat{S}_x = \hat{S}_x^+ \cup \hat{S}_x^0$, where $\hat{S}_x^+$ represents the set of strategies

$$s_{x,a} = \frac{\sum_{i=1}^k \lambda^i q_x^i s_{x,a}^i}{\sum_{i=1}^k \lambda^i q_x^i}, \text{ for } a \in A(x) \tag{46}$$

in the state $x \in X$ determined by $(\lambda^1, \lambda^2, \dots, \lambda^k) \in \Lambda_x^+$ and $\hat{S}_x^0$ represents the set of strategies

$$s_{x,a} = \frac{\sum_{i=1}^k \lambda^i w_x^i s_{x,a}^i}{\sum_{i=1}^k \lambda^i w_x^i}, \text{ for } a \in A(x) \tag{47}$$

in the state $x \in X$ determined by $(\lambda^1, \lambda^2, \dots, \lambda^k) \in \Lambda_x^0$.

Therefore $\hat{S}_x$ can be expressed as follows $\hat{S}_x = \hat{S}_x^+ \cup \hat{S}_x^0$, where $\hat{S}_x^+$ represents the set of strategies

$$s_{x,a} = \frac{\sum_{i=1}^k \lambda^i q_x^i s_{x,a}^i}{\sum_{i=1}^k \lambda^i q_x^i}, \text{ for } a \in A(x) \tag{48}$$

in the state $x \in X$ determined by $(\lambda^1, \lambda^2, \dots, \lambda^k) \in \Lambda_x^+$ and $\hat{S}_x^0$ represents the set of strategies

$$s_{x,a} = \frac{\sum_{i=1}^k \lambda^i w_x^i s_{x,a}^i}{\sum_{i=1}^k \lambda^i w_x^i}, \text{ for } a \in A(x) \tag{49}$$

in the state $x \in X$ determined by $(\lambda^1, \lambda^2, \dots, \lambda^k) \in \Lambda_x^0$.

Thus, if we analyze (48) then observe that $s_{x,a}$ for a given $x \in X$ represents a linear-fractional function with respect to $\lambda^1, \lambda^2, \dots, \lambda^k$ defined on convex set Λ_x^+ and $\hat{S}_x^+$ is the image of $s_{x,a}$ on Λ_x^+. Therefore $\hat{S}_x^+$ is a convex set. If we analyze (49) then observe that $s_{x,a}$ for given $x \in X$ represents a linear-fractional function with respect to $\lambda^1, \lambda^2, \dots, \lambda^k$ on convex set Λ_x^0 and $\hat{S}_x^0$ is the image of $s_{x,a}$ on Λ_x^0. Therefore $\hat{S}_x^0$ is a convex set (see [1]). Additionally we can observe that $\Lambda_x^+ \cap \Lambda_x^0 = \emptyset$ and in the case $\Lambda_x^+, \Lambda_x^0, \neq \emptyset$ the set Λ_x^0 represents the limit inferior of Λ_x^+. Using this property and taking into account (34) we can conclude that each strategy $s_x \in \hat{S}_x^0$ can be regarded as the limit of a sequence of strategies $\{s_x^t\}$ from $\hat{S}_x^+$. Therefore we obtain that $\hat{S}_x = \hat{S}_x^+ \cup \hat{S}_x^0$ is a convex set. This involves the convexity of the sublevel set $L_c^-(\psi)$. In analogues way using (44) and (45) we can show that the superlevel set $L_c^+(\psi)$ and the level set $L_c(\psi)$ a convex set. This means that the function $\psi(s)$ is quasi-monotone on $\overline{S}$. $\square$

5 Existence of stationary Nash equilibria for the multichain average stochastic game

In this section we present an result concerned with the existence of stationary Nash equilibria in a multichain average stochastic game with n players. We prove this result using a continuous model for the considered game that generalizes the continuous model from Section 3.

5.1 A continuous model for the multichain stochastic game

The continuous model for a multichain average stochastic game that generalizes the continuous model (23)–(25) is the following:

Let $\overline{S}^i,\ i \in \{1, 2, \dots n\}$ be the set of solutions of the system

$$\begin{cases} \sum\limits_{a_i \in A^i(x)} s^i_{x,a^i} = 1, \quad \forall x \in X; \\ \qquad\qquad s^i_{x,a^i} \geq 0, \quad \forall x \in X,\ a^i \in A^i(x). \end{cases} \tag{50}$$

that determines the set of stationary strategies of player i. Each $\overline{S}^i$ is a convex compact set and an arbitrary its extreme point corresponds to a basic solution s^i of system (50), where $s^i_{x,a^i} \in \{0, 1\},\ \forall x \in X,\ a^i \in A(x),$ i.e each basic solution of this system corresponds to a pure stationary strategy of player i. On the set $\overline{S} = \overline{S}^1 \times \overline{S}^2 \times \dots \times \overline{S}^n$ we define n payoff functions

$$\begin{cases} \psi^i(s^1, s^2, \ \dots, \ s^n) = \sum\limits_{x \in X} \ \sum\limits_{(a^1, a^2, \dots, a^n) \in A(x)} \ \prod\limits_{k=1}^{n} \ s^k_{x,a^k} f^i(x, a^1, a^2 \dots a^n) q_x, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, 2, \dots, n, \end{cases} \tag{51}$$

where q_x for $x \in X$ are determined uniquely from the following system of linear equations

$$\begin{cases} q_y - \sum\limits_{x \in X} \ \sum\limits_{(a^1, a^2, \dots, a^n) \in A(x)} \ \prod\limits_{k=1}^{n} \ s^k_{x,a^k} p^{(a^1, a^2, \dots, a^n)}_{x,y} q_x = 0, \quad \forall y \in X; \\ q_y + w_y - \sum\limits_{x \in X} \ \sum\limits_{(a^1, a^2, \dots, a^n) \in A(x)} \ \prod\limits_{k=1}^{n} \ s^k_{x,a^k} p^{(a^1, a^2, \dots, a^n)}_{x,y} w_x = \theta_x, \forall y \in X, \end{cases} \tag{52}$$

for an arbitrary profile $(s^1, s^2, \ \dots, \ s^m) \in \overline{S}$. Each $(s^1, s^2, \ \dots, \ s^n) \in \overline{S}$ in the considered continuous game corresponds to a profile of mixed stationary strategies of the players and $\psi^i(s^1, s^2, \ \dots, \ s^n),\ i = 1, 2, \dots, n,$ defined by (51), (52) represent the corresponding average payoffs of the players in the case when the staring state is chosen according to distribution $\{\theta_x\}$. If $\theta_x = 0,\ \forall x \in X \setminus \{x_0\}$ and $\theta_{x_0} = 1$ then we obtain the continuous game model for the average stochastic game with given starting state x_0, i.e. $\psi^i(s^1, s^2, \dots, s^n) = \omega^i_{x_0}(s^1, s^2, \dots, s^n),\ i = 1, 2, \dots, n.$

5.2 The main result

From Theorem 3 as a corollary we can obtain the following lemma.

Lemma 4. *For an arbitrary average stochastic game with* $\theta_x > 0, \forall x \in X$ *each payoff function* $\psi^i(s^1, s^2, \ldots, s^n)$, $i \in \{1, 2, \ldots, n\}$ *possesses the property that* $\psi^i(\overline{s}^1, \overline{s}^2, \ldots, \overline{s}^{i-1}, s^i, \overline{s}^{i+1}, \ldots, \overline{s}^n)$ *is quasi-monotone with respect to* $s^i \in \overline{S}^i$ *for arbitrary fixed* $\overline{s}^k \in \overline{S}^k$, $k = 1, 2, \ldots, i-1, i+1, \ldots, n$.

Proof. Indeed, if players $1, 2, \ldots, i-1, i+1, \ldots, n$ fix their stationary strategies $\overline{s}^k \in \overline{S}^k$, $k = 1, 2, \ldots, i-1, i+1, \ldots, n$, then we obtain an average decision problem with respect to $s^i \in \overline{S}^i$ and average cost function $\psi^i(\overline{s}^1, \overline{s}^2, \ldots, \overline{s}^{i-1}, s^i, \overline{s}^{i+1}, \ldots, \overline{s}^n)$. According to Theorem 3 this function possesses the property that the value of the function is uniquely determined by $s^i \in \overline{S}^i$ and it is quasi-monotone with respect to s^i on $\overline{S}^i$. $\square$

Theorem 4. *Let* $(X, A, \{X_i\}_{i=\overline{1,n}}, \{f^i(x,a)\}_{i=\overline{1,n}}, p, \{\theta_x\})$ *be an average stochastic game with given distribution* $\{\theta_x\}$ *for the initial state and consider the continuous game with average payoffs* $\psi^i(s^1, s^2, \ldots, s^n)$, $i = 1, 2, \ldots, n$ *for the players. If for an arbitrary profile* $\overline{s} = (\overline{s}^1, \overline{s}^2, \ldots, \overline{s}^n) \in \overline{S}$ *each payoff function* $\psi^i(s^1, s^2, \ldots, s^n)$, $i \in \{1, 2, \ldots, n\}$ *possesses the property that*

$$\lim_{s^i \to \overline{s}^i} \psi^i(\overline{s}^1, \overline{s}^2, \ldots, \overline{s}^{i-1}, s^i, \overline{s}^{i+1}, \ldots, \overline{s}^n) = \psi^i(\overline{s}^1, \overline{s}^2, \ldots, \overline{s}^{i-1}, \overline{s}^i, \overline{s}^{i+1}, \ldots, \overline{s}^n)$$

then for the considered continuous game there exists a Nash equilibrium $s^* = (s^{1^*}, s^{2^*}, \ldots, s^{n*}) \in \overline{S}$ *that is a stationary Nash equilibrium for the average stochastic game* $(X, A, \{X_i\}_{i=\overline{1,n}}, \{f^i(x,a)\}_{i=\overline{1,n}}, p, x\})$ *with an arbitrary initial state* $x \in X$.

Proof. According to Lemma 4 each function $\psi^i(s^1, s^2, \ldots, s^n)$, $i \in \{1, 2, \ldots, n\}$ satisfies the condition that $\psi^i(\overline{s}^1, \overline{s}^2, \ldots, \overline{s}^{i-1}, s^i, \overline{s}^{i+1}, \ldots, \overline{s}^n)$ is quasi-monotone with respect to $s^i \in \overline{S}^i$ for an arbitrary fixed $\overline{s}^k \in \overline{S}^k$, $k = 1, 2, \ldots, i-1, i+1, \ldots, n$. In the considered game each subset $\overline{S}^i$ is convex and compact and according to the conditions of the theorem each payoff function $\psi^i(s^1, s^2, \ldots, s^n)$ is continue with respect to s^i in $\overline{S}^i$. Therefore, these conditions (see [2, 3, 15, 18]) provide the existence of a Nash equilibrium $s^* = (s^{1^*}, s^{2^*}, \ldots, s^{n*})$ for the game with payoff functions $\psi^i(s^1, s^2, \ldots, s^n)$, $i \in \{i, 2, \ldots, n\}$ on $\overline{S}^1 \times \overline{S}^2 \times \cdots \times \overline{S}^n$. $\square$

6 Conclusion

The results presented in the paper show that for finite state space stochastic games with average payoffs stationary Nash equilibria exist if the conditions of Theorem 4 are satisfied. For determining stationary Nash equilibria in the considered games the continuous model from Section 5.1 can be used. For average stochastic games with unichain property the continuous model from Section 3.2 can be used.

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Dmitrii Lozovanu
Institute of Mathematics and Computer Science
5 Academiei str., Chişinău, MD−2028
Moldova

E-mail: *lozovanu@math.md*

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General form transversals in groups

Eugene Kuznetsov

Abstract. The classical notion of transversal in group to its subgroup is generalised. It is made with the help of reducing any conditions on the choice of representatives of the left (right) cosets in group to its subgroup. Obtained general form transversals are investigated and some its properties are studied.

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1 Introduction

In the theory of quasigroups and loops the following notion of left (right) transversal in group to its subgroup is well-known [1–4].

Definition 1. Let G be a group and H be its subgroup. Let $\{H_i\}_{i\in E}$ be the set of all left (right) cosets in G to H (E is a set of indexes with distinguished element 1), and we assume $H_1 = H$. A set $T = \{t_i\}_{i\in E}$ of representativities of the left (right) cosets (by one from each coset H_i and $t_1 = e \in H$) is called a **left (right) transversal** in G to H.

As is easy to see, in this definition the choice of representatives of left (right) cosets in G to H is not free – there exist two conditions: $H_1 = H$ and $t_1 = e \in H$. Let us reduce these two conditions and investigate obtained below general form transversals in group to its subgroup.

2 General form transversals in group to its subgroup

2.1 Definitions and elementary properties

Let G be a group and H be its subgroup. Below we shall use the following notations:

E is an index set (E contains a distinguished element 1);

left (right) cosets in the group G to its subgroup H are numbered by the indexes from E;

$\{H_i\}_{i\in E}$ is the set of all left (right) cosets in G to H;

e is the unit of group G;

Below all definitions and propositions will be formulated for the left cosets in G to H; for the right cosets in G to H it may be done analogously.

Definition 2. Let G be a group and H be its subgroup. Let $\{H_i\}_{i\in E}$ be the set of all left cosets in G to H. A set $T=\{t_i\}_{i\in E}$ of representativities of the left (right) cosets (by one from each coset H_i, i.e. $t_i \in H_i$) is called a **left general form transversal** in G to H (see also [6, 7]).

Remark 1. Generally speaking the numbering of left cosets $\{H_i\}_{i\in E}$ in G to H may be such that the subgroup H obtain an index $a \in E$ which is different from 1, i.e. $H = H_a \neq H_1$.

Remark 2. Generally speaking the unit e of the group G (and subgroup H) may not belong to the left general form transversal T in G to H, i.e. $e \notin T$.

Definition 3. If for left general form transversal $T=\{t_i\}_{i\in E}$ in G to H the following condition holds: $t_{i_0} = e$ for some $i_0 \in E$, then such transversal T is called a **left reduced transversal** in G to H. In opposite case T is called a **left non-reduced transversal** in G to H.

Definition 4. If for left general form transversal $T=\{t_i\}_{i\in E}$ in G to H the following condition holds: $H = H_1$ (i.e. the index of the subgroup H in the set of left cosets in G to H is equal to 1), then such transversal T is called a **left ordered transversal** in G to H. In opposite case T is called a **left non-ordered transversal** in G to H.

Definition 5. A left general form transversal $T=\{t_i\}_{i\in E}$ in G to H which is a left reduced and ordered transversal in G to H is usually called a **left transversal** in G to H.

Example 1. Let us have:

$$\begin{aligned} G &= S_3 = \{id, (12), (13), (23), (123), (132)\}, \\ H &= St_1(S_3) = \{id, (23)\}. \end{aligned}$$

Left cosets in G to H:

$$\begin{aligned} H_{i_1} &= H = \{id, (23)\}, \\ H_{i_2} &= \{(12), (123)\}, \\ H_{i_3} &= \{(13), (132)\}, \\ E &= \{i_1, i_2, i_3\} \equiv \{1, 2, 3\}. \end{aligned}$$

1. $i_1 \neq 1$ and $T = \{(23), (12), (132)\}$. Then T is a left non-reduced non-ordered general form transversal in G to H.

2. $i_1 = 1$ and $T = \{(23), (12), (132)\}$. Then T is a left non-reduced ordered general form transversal in G to H.

3. $i_1 \neq 1$ and $T = \{id, (123), (132)\}$. Then T is a left reduced non-ordered general form transversal in G to H.

4. $i_1 = 1$ and $T = \{id, (12), (13)\}$. Then T is a left (reduced and ordered) transversal in G to H.

Theorem 1. *For an arbitrary left general form transversal $T = \{t_i\}_{i\in E}$ in G to H the following statements are true:*

1. *For every $h \in H$ the set $T_h = Th = \{t_i h\}_{i\in E}$ is a left general form transversal in G to H too.*
2. *There exists an element $h_0 \in H$ such that the set $T_{h_0} = Th_0$ is a left reduced (maybe non-ordered) general form transversal in G to H.*
3. *For every $\pi \in G$ the set ${}_\pi T = \pi T = \{\pi t_i\}_{i\in E}$ is a left general form transversal in G to H too.*
4. *There exists an element $\pi_0 \in G$ such that the set ${}_{\pi_0}T = \pi_0 T = \{\pi_0 t_i\}_{i\in E}$ is a left (reduced and ordered) transversal in G to H.*

Proof. **1**. For every $i \in E$ and $h \in H$ we have

$$t_i \in H_i \quad \Longrightarrow \quad t_i h \in H_i,$$

and so

$$(Th) \cap H_i = \{t_i h\},$$

i.e. Th is a left general form transversal in G to H.

2. Let

$$T \cap H = h^*,$$

i.e. h^* is a representative of general form transversal T in the subgroup H. Then we put

$$h_0 = (h^*)^{-1}.$$

We obtain

$$h^* \in T \quad \Longrightarrow \quad e = h^* \cdot (h^*)^{-1} \in (Th_0),$$

i.e. due to item **1** general form transversal $T_1 = Th_0$ is a left reduced (maybe non-ordered) general form transversal in G to H

3. Let us take an arbitrary element $\pi \in G$ and consider the set

$${}_\pi T = \pi T = \{\pi t_i\}_{i\in E}.$$

Because T is a left general form transversal in G to H then

$$G = \bigcup_{i\in E} (t_i H).$$

So we obtain

$$G = \pi G = \pi \cdot \left(\bigcup_{i\in E} (t_i H) \right) = \bigcup_{i\in E} ((\pi t_i) H),$$

i.e. every element $g \in G$ may be presented in the form $g = t^* h$, where $h \in H$ and $t^* \in \pi T$.

Now let us show that for every $i, j \in E$, $i \neq j$, the following equality is true

$$((\pi t_i)H) \cap ((\pi t_j)H) = \varnothing.$$

Let us assume that it is not true, and so

$$\pi t_i h_1 = \pi t_j h_2 = g_0$$

for some $h_1, h_2 \in H$. Then we obtain

$$t_i h_1 = t_j h_2 \quad \Longrightarrow \quad t_i = t_j h_2 h_1^{-1} \in t_j H \quad \Longrightarrow \quad (t_i H) \cap (t_j H) \neq \varnothing,$$

that is in contradiction to the fact that T is a left general form transversal in G to H.

4. Let us consider the left coset H_1 and take the element

$$\pi^* = t_1 = H_1 \cap T.$$

Then we may take $\pi_0 = (\pi^*)^{-1}$. Really we have

$$e = (\pi^*)^{-1} \cdot \pi^* = \pi_0 t_1 \in \pi_0 T,$$

i.e. with the help of item **3** the left general form transversal $\pi_0 T$ is a left (reduced and ordered) transversal in G to H. □

2.2 A transversal operation

Definition 6. Let $T = \{t_i\}_{i \in E}$ be a left general form transversal in G to H. Define the following operation on the set E:

$$x \overset{(T)}{\cdot} y = z \quad \Leftrightarrow \quad t_x t_y = t_z h, \quad h \in H.$$

Theorem 2. *For an arbitrary left general form transversal $T = \{t_i\}_{i \in E}$ in G to H the following statements are true:*

1. *There exists an element $a_0 \in E$ such that the system $\left\langle E, \overset{(T)}{\cdot}, a_0 \right\rangle$ is a left quasigroup with right unit a_0.*
2. *If a left general form transversal $T = \{t_i\}_{i \in E}$ is a reduced (but non-ordered) transversal in G to H, then there exists an element $a_0 \in E$ such that the system $\left\langle E, \overset{(T)}{\cdot}, a_0 \right\rangle$ is a left loop with unit a_0.*
3. *If a left general form transversal $T = \{t_i\}_{i \in E}$ is an ordered (but non-reduced) transversal in G to H, then the system $\left\langle E, \overset{(T)}{\cdot}, 1 \right\rangle$ is a left quasigroup with right unit 1.*

4. If a left general form transversal $T = \{t_i\}_{i\in E}$ is an ordered and reduced transversal in G to H, then the system $\left\langle E, \overset{(T)}{\cdot}, 1 \right\rangle$ is a left loop with unit 1*.*

Proof. **1**. For any arbitrary $a, b \in E$ consider the following equivalent equations on the set E:

$$\begin{aligned} a \overset{(T)}{\cdot} x &= b, \\ t_a t_x &= t_b h, \quad h \in H, \\ t_x &= t_a^{-1} t_b h = t_c h^*, \quad h^* \in H, \\ x &= c, \end{aligned}$$

for some $c \in E$; moreover, the element $c = c(a, b)$ is uniquely determined by the elements $a, b \in E$. So the system $\left\langle E, \overset{(T)}{\cdot} \right\rangle$ is a left quasigroup. If a_0 is the index of subgroup H as a left coset in G to H, i.e. $H \equiv H_{a_0}$, then $t_{a_0} = h_0 \in H$ for some element h_0. For every $x \in E$ we have the following equivalent equations on the set E:

$$\begin{aligned} x \overset{(T)}{\cdot} a_0 &= u, \\ t_x t_{a_0} &= t_u h, \quad h \in H, \\ t_x h_0 &= t_u h, \quad h \in H, \\ t_x &= t_u h h_0^{-1} = t_u h^*, \quad h^* \in H, \\ u &= x, \end{aligned}$$

i.e. for every $x \in E$: $x \overset{(T)}{\cdot} a_0 = x$. It means that the system $\left\langle E, \overset{(T)}{\cdot}, a_0 \right\rangle$ is a left quasigroup with right unit a_0.

2. If a left general form transversal $T = \{t_i\}_{i\in E}$ is a reduced (but non-ordered) transversal in G to H, then $t_{a_0} = e \in H$. For every $x \in E$ we have the following equivalent equations on the set E:

$$\begin{aligned} a_0 \overset{(T)}{\cdot} x &= u, \\ t_{a_0} t_x &= t_u h, \quad h \in H, \\ e t_x &= t_u h, \quad h \in H, \\ t_x &= t_u h, \quad h \in H, \\ u &= x, \end{aligned}$$

i.e. for every $x \in E$: $a_0 \overset{(T)}{\cdot} x = x$. It means that the system $\left\langle E, \overset{(T)}{\cdot}, a_0 \right\rangle$ is a left loop with two-sided unit a_0.

3. If a left general form transversal $T = \{t_i\}_{i\in E}$ is an ordered (but non-reduced) transversal in G to H, then the proof is analogous to the proof of the item **1**, but we have $a_0 = 1$ too (because $H_1 \equiv H_{a_0} \equiv H$). So we obtain that the system $\left\langle E, \overset{(T)}{\cdot}, 1 \right\rangle$ is a left loop with unit 1.

4. It is an evident corollary of the items **2** and **3** $\square$

2.3 Permutation representation

Definition 7. Let G be a group and H be its subgroup. A **permutation representation** $\hat{G}$ of the group G by left cosets to its subgroup H is the following map φ:

$$\varphi : G \to S_E,$$

$$\varphi : g \to \hat{g},$$

$$\hat{g}\,(x) = y \quad \overset{def}{\Leftrightarrow} \quad g \cdot (H_x) = H_y, \qquad x, y \in E.$$

If some left general form transversal $T = \{t_i\}_{i \in E}$ in G to H is chosen, then the last formula may be rewritten in the following form:

$$\hat{g}\,(x) = y \quad \overset{def}{\Leftrightarrow} \quad g \cdot (t_x \cdot H) = t_y \cdot H.$$

The map φ is a homomorphism from the group G to the symmetric group S_E. The kernel of this homomorphism is called a *core* of G to H:

$$Core_G H = \bigcap_{\pi \in G} (\pi H \pi^{-1}).$$

If $Core_G H = \{e\}$, then the above-mentioned representation is a strict representation and φ is an isomorphism.

It is easy to show that with the help of factorisation on the core it is always possible to take into consideration the strict permutation representation $\hat{G}$ of the group G by left cosets to its subgroup H. So below we assume that the above-mentioned representation is a strict representation.

Theorem 3. *For an arbitrary left general form transversal $T = \{t_i\}_{i \in E}$ in G to H the following statements are true:*

1. *There exists an element $a_0 \in E$ such that for every $h \in H$: $\hat{h}\,(a_0) = a_0$.*
2. *The following identities are fulfilled:*
   (a) *For all $x, y \in E$:* $\qquad \widehat{t_x}(y) = x \overset{(T)}{\cdot} y;$
   (b) *For all $x, y \in E$:* $\qquad \widehat{t_x^{-1}}(y) = x \overset{(T)}{\diagdown} y,$ *where "$\overset{(T)}{\diagdown}$" is a left division for the operation $\left\langle E, \overset{(T)}{\cdot}, a_0 \right\rangle$ (i.e. $x \overset{(T)}{\diagdown} y = z \Longleftrightarrow x \overset{(T)}{\cdot} z = y$);*
   (c) *For every $x \in E$:* $\qquad \widehat{t_x}(a_0) = x.$

3. *If a left general form transversal $T = \{t_i\}_{i\in E}$ is a reduced (but non-ordered) transversal in G to H, then the following identities are fulfilled:*

   (a) *For all $x, y \in E$:* $\quad \widehat{t_x}(y) = x \overset{(T)}{\cdot} y$;

   (b) *For all $x, y \in E$:* $\quad \widehat{t_x}^{-1}(y) = x \overset{(T)}{\diagdown} y$;

   (c) *For every $x \in E$:* $\quad \widehat{t_x}(a_0) = \widehat{t_{a_0}}(x) = x$.

4. *If a left general form transversal $T = \{t_i\}_{i\in E}$ is an ordered (but non-reduced) transversal in G to H, then the following identities are fulfilled:*

   (a) *For all $x, y \in E$:* $\quad \widehat{t_x}(y) = x \overset{(T)}{\cdot} y$;

   (b) *For all $x, y \in E$:* $\quad \widehat{t_x}^{-1}(y) = x \overset{(T)}{\diagdown} y$;

   (c) *For every $x \in E$:* $\quad \widehat{t_x}(1) = x$.

5. *If a left general form transversal $T = \{t_i\}_{i\in E}$ is an ordered and reduced transversal in G to H, then the following identities are fulfilled:*

   (a) *For all $x, y \in E$:* $\quad \widehat{t_x}(y) = x \overset{(T)}{\cdot} y$;

   (b) *For all $x, y \in E$:* $\quad \widehat{t_x}^{-1}(y) = x \overset{(T)}{\diagdown} y$;

   (c) *For every $x \in E$:* $\quad \widehat{t_x}(1) = \widehat{t_1}(x) = x$.

Proof. **1**. According to item **1** of Theorem 2 there exists an element $a_0 \in E$ such that $H \equiv H_{a_0}$ (i.e. $t_{a_0} = h_0 \in H$). Then for every $h \in H$ we have the following equivalent equalities:

$$\begin{aligned}
\hat{h}\,(a_0) &= a_1, \\
ht_{a_0} &= t_{a_1}h^*, \quad h^* \in H, \\
hh_0 &= t_{a_1}h^*, \quad h^* \in H, \\
t_{a_1} &= hh_0(h^*)^{-1} \in H, \\
t_{a_1} &= t_{a_0}, \\
a_1 &= a_0.
\end{aligned}$$

So we obtain that $\hat{h}\,(a_0) = a_0$.

2. **a**. For all $x, y \in E$ we have the following equivalent equalities:

$$\begin{aligned}
x \overset{(T)}{\cdot} y &= u, \\
t_x t_y &= t_u h, \quad h \in H, \\
t_x t_y H &= t_u H, \\
\widehat{t_x}(y) &= u.
\end{aligned}$$

So we obtain that $\widehat{t}_x(y) = x \overset{(T)}{\cdot} y$.

b. For all $x, y \in E$ we have the following equivalent equalities:

$$\begin{aligned} \widehat{t}_x^{-1}(y) &= u, \\ \widehat{t}_x(u) &= y, \\ x \overset{(T)}{\cdot} u &= y, \\ u &= x \overset{(T)}{\diagdown} y, \end{aligned}$$

where "$\overset{(T)}{\diagdown}$" is a left division for the operation $\left\langle E, \overset{(T)}{\cdot}, a_0 \right\rangle$ (i.e. $x \overset{(T)}{\diagdown} y = z \Longleftrightarrow x \overset{(T)}{\cdot} z = y$). So we obtain that $\widehat{t}_x^{-1}(y) = x \overset{(T)}{\diagdown} y$.

c. According to item **1** of Theorem 2 there exists an element $a_0 \in E$ such that for every $x \in E$ $x \overset{(T)}{\cdot} a_0 = x$. Then due to item **2a** we have for every $x \in E$

$$\widehat{t}_x(a_0) = x \overset{(T)}{\cdot} a_0 = x.$$

3. Let the left general form transversal $T = \{t_i\}_{i \in E}$ be a reduced (but non-ordered) transversal in G to H. Then $t_{a_0} = e$. So all identities from the item **2** of present Theorem are true; moreover, we have for every $x \in E$

$$\widehat{t}_{a_0}(x) = \hat{e}(x) = id(x) = x.$$

4. Let the left general form transversal $T = \{t_i\}_{i \in E}$ be an ordered (but non-reduced) transversal in G to H. Then $a_0 = 1$. So all identities from the item **2** of present Theorem are true; moreover, we have for every $x \in E$

$$\widehat{t}_x(1) = x.$$

5. It is an evident corollary of the items **3** and **4**. □

Theorem 4. *For an arbitrary left general form transversal $T = \{t_i\}_{i \in E}$ in G to H the following statements are true:*

1. *If $P = \{p_i\}_{i \in E}$ is a left general form transversal in G to H such that for every $x \in E$:*

$$\begin{aligned} P &= Th_0, \\ p_{x'} &= t_x h_0, \end{aligned}$$

   *where $h_0 \in H$ is an arbitrary fixed element (see item **1** from Theorem 1), then the transversal operation $\left\langle E, \overset{(P)}{\cdot} \right\rangle$ is isotopic to the transversal operation $\left\langle E, \overset{(T)}{\cdot} \right\rangle$, and this isotopy has the form $(id, \hat{h}_0, id)$.*

2. *If $S = \{s_i\}_{i \in E}$ is a left general form transversal in G to H such that for every $x \in E$:*

$$\begin{aligned} S &= \pi T, \\ s_{x'} &= \pi t_x, \end{aligned}$$

*where $\pi \in G$ is an arbitrary fixed element (see item **3** from Theorem 1), then the transversal operation $\left\langle E, \overset{(S)}{\cdot} \right\rangle$ is isotopic to the transversal operation $\left\langle E, \overset{(T)}{\cdot} \right\rangle$, and this isotopy has the form (π^{-1}, id, π).*

Proof. **1**. Let $P = \{p_i\}_{i \in E}$ be a left general form transversal in G to H such that for every $x \in E$:

$$\begin{aligned} P &= T h_0, \\ p_{x'} &= t_x h_0, \end{aligned}$$

where $h_0 \in H$ is an arbitrary fixed element. According to items **1** and **2** from Theorem 3 there exists an element $a_0 \in E$ such that for every $h \in H$

$$\begin{aligned} \hat{h}\,(a_0) &= a_0, \\ \widehat{t_x}(a_0) &= x, \\ \hat{p}_{x'}(a_0) &= x', \end{aligned}$$

for all $x, x' \in E$. Then we have for all $x \in E$

$$x' = \hat{p}_{x'}(a_0) = \hat{t}_x \hat{h}_0(a_0) = \widehat{t_x}(a_0) = x,$$

i.e. for all $x \in E$

$$p_x = t_x h_0.$$

According to item **2** from Theorem 3 we obtain for all $x, y \in E$:

$$x \overset{(P)}{\cdot} y = \hat{p}_x(y) = \hat{t}_x \hat{h}_0(y) = x \overset{(T)}{\cdot} \hat{h}_0(y),$$

i.e. the transversal operation $\left\langle E, \overset{(P)}{\cdot} \right\rangle$ is isotopic to the transversal operation $\left\langle E, \overset{(T)}{\cdot} \right\rangle$, and this isotopy has the form $(id, \hat{h}_0, id)$.

2. Let $S = \{s_i\}_{i \in E}$ be a left general form transversal in G to H such that for every $x \in E$:

$$\begin{aligned} S &= \pi T, \\ s_{x'} &= \pi t_x, \end{aligned}$$

where $\pi \in G$ is an arbitrary fixed element. Analogously to the item **1** of this Theorem we have

$$x' = \hat{s}_{x'}(a_0) = \hat{\pi}\,\hat{t}_x(a_0) = \hat{\pi}(x),$$

i.e. for every $x \in E$

$$\begin{aligned} s_{\pi(x)} &= \pi t_x, \\ s_x &= \pi t_{\pi^{-1}(x)}. \end{aligned}$$

Then according to item **2** from Theorem 3 we obtain for all $x, y \in E$:

$$x \overset{(S)}{\cdot} y = \hat{s}_x(y) = \hat{\pi}\,\hat{t}_{\hat{\pi}^{-1}(x)}(y) = \hat{\pi}(\hat{\pi}^{-1}(x) \overset{(T)}{\cdot} y),$$

i.e. the transversal operation $\left\langle E, \overset{(S)}{\cdot} \right\rangle$ is isotopic to the transversal operation $\left\langle E, \overset{(T)}{\cdot} \right\rangle$, and this isotopy has the form (π^{-1}, id, π). □

Remark 3. The last statement allows us to see a new sense of Theorem 2. Now it is evident that the transition from a general form transversal to the reduced (or ordered) transversal is just a transition from a left quasigroup transversal operation to a left loop transversal operation (which is its isotope).

3 Quasigroup and loop general form transversals

Definition 8. Let $T = \{t_i\}_{i \in E}$ be a left general form transversal in G to H. If its transversal operation $\left\langle E, \overset{(T)}{\cdot} \right\rangle$ is a quasigroup, then the transversal T is called a left **quasigroup** general form transversal in G to H (in [5] such transversal is called a **stable** transversal in G to H).

Remark 4. According to item **1** from Theorem 2 there exists an element $a_0 \in E$ such that a_0 is a right unit in the operation $\left\langle E, \overset{(T)}{\cdot} \right\rangle$; so if T is a left quasigroup general form transversal in G to H, then the system $\left\langle E, \overset{(T)}{\cdot}, a_0 \right\rangle$ is a quasigroup with the right unit a_0.

Theorem 5. *If $T = \{t_i\}_{i \in E}$ is a left quasigroup general form transversal in G to H, then there exists an element $a_0 \in E$ such that the system $\left\langle E, \overset{(T)}{\cdot}, a_0 \right\rangle$ is a loop.*

Proof. It is an evident corollary from the item **2** of Theorem 2. □

Definition 9. A left reduced quasigroup general form transversal in G to H is usually called a left **loop** general form transversal in G to H.

Theorem 6. *The following statements are equivalent:*

1. *A set $T = \{t_x\}_{x\in E}$ is a left quasigroup general form transversal in G to H;*

2. *For every $\pi \in G$ the set $T\pi = \{t_x\pi\}_{x\in E}$ is a left general form transversal in G to H;*

3. *For all $\pi_1, \pi_2 \in G$ the set $\pi_1 T\pi_2 = \{\pi_1 t_x \pi_2\}_{x\in E}$ is a left general form transversal in G to H;*

4. *For every $\pi \in G$ the set $T = \{t_x\}_{x\in E}$ is a left general form transversal in G to $H^\pi = \pi H\pi^{-1}$.*

Proof. **1⇒2**. Let a set $T = \{t_x\}_{x\in E}$ be a left quasigroup general form transversal in G to H. Then the system $\left\langle E, \overset{(T)}{\cdot} \right\rangle$ is a quasigroup. Let an element $\pi \in G$ be an arbitrary fixed element from G. We shall consider the set $T\pi = \{t_x\pi\}_{x\in E}$ and prove that this set is a left general form transversal in G to H.

Because $T = \{t_x\}_{x\in E}$ is a left quasigroup general form transversal in G to H, then

$$\pi = t_{c_0} h_0$$

for some $t_{c_0} \in T$ and $h_0 \in H$. Because the operation $\left\langle E, \overset{(T)}{\cdot} \right\rangle$ is a quasigroup, then for every $x \in E$ we have

$$t_x\pi = t_x t_{c_0} h_0 = t_{x \overset{(T)}{\cdot} c_0} h_1 = t_{R_{c_0}(x)} h_1$$

for some $h_1 \in H$. Then every element $g \in G$ may be represented in the following form:

$$g = t_{c_1} h^* = t_{R_{c_0}(c_1/c_0)} h_1 h_1^{-1} h^* = t_{c_1/c_0} \pi h_1^{-1} h^* = (t_{c_1/c_0}\pi) h^{**}, \qquad h^{**} \in H.$$

Let us assume that this representation is not unique, i.e. there exist $a, b \in E$, $a \neq b$ and $h_1, h_2 \in H$ such that

$$t_a \pi h_1 = g = t_b \pi h_2.$$

According to item **1** of Theorem 3 there exists an element $a_0 \in E$ such that we have the following equivalent equalities

$$\begin{aligned}
\hat{t}_a \hat{\pi}\, \hat{h}_1(a_0) &= \hat{t}_b \hat{\pi}\, \hat{h}_2(a_0) \\
\hat{t}_a \hat{\pi}\, (a_0) &= \hat{t}_b \hat{\pi}\, (a_0) \\
\hat{t}_a \hat{t}_{c_0} \hat{h}_0(a_0) &= \hat{t}_b \hat{t}_{c_0} \hat{h}_0(a_0) \\
\hat{t}_a \hat{t}_{c_0}(a_0) &= \hat{t}_b \hat{t}_{c_0}(a_0) \\
\hat{t}_a(c_0) &= \hat{t}_b(c_0)
\end{aligned}$$

$$\begin{aligned} a \overset{(T)}{\cdot} c_0 &= b \overset{(T)}{\cdot} c_0 \\ a &= b \end{aligned}$$

because the operation $\left\langle E, \overset{(T)}{\cdot} \right\rangle$ is a quasigroup. We obtain a contradiction and so the above mentioned representation is unique. Then the set $T\pi = \{t_x\pi\}_{x\in E}$ is a left general form transversal in G to H.

2$\Rightarrow$**3**. It is evident due to item **3** of Theorem 1.

3$\Rightarrow$**4**. If the condition of item **3** holds then a fortiori is true that for every $\pi \in G$ the set $\pi T\pi^{-1} = \{\pi t_x \pi^{-1}\}_{x\in E}$ is a left general form transversal in G to H. So for all $a, b \in E$, $a \neq b$ we have the following equivalent statements:

$$\begin{cases} G = \bigcup\limits_{x\in E} (\pi t_x \pi^{-1})H, \\ \varnothing = (\pi t_a \pi^{-1} H) \cap (\pi t_b \pi^{-1} H), \end{cases}$$

$$\begin{cases} G = \pi^{-1} G \pi = \pi^{-1} \left(\bigcup\limits_{x\in E} (\pi t_x \pi^{-1})H \right) \pi = \bigcup\limits_{x\in E} t_x(\pi^{-1} H \pi), \\ \varnothing = \pi^{-1} \cdot \varnothing \cdot \pi = \pi^{-1}((\pi t_a \pi^{-1} H) \cap (\pi t_b \pi^{-1} H))\pi = (t_a(\pi^{-1}H\pi)) \cap (t_b(\pi^{-1}H\pi)). \end{cases}$$

Because the element $\pi \in G$ is an arbitrary element from G then the element π^{-1} will be an arbitrary element from G too. So the set $T = \{t_x\}_{x\in E}$ is a left general form transversal in G to $H^{\pi'} = \pi' H \pi'^{-1}$ for every $\pi' \in G$ (where $\pi' = \pi^{-1}$).

4$\Rightarrow$**1**. Let for every $\pi \in G$ a set T be a left general form transversal in G to $H^\pi = \pi H \pi^{-1}$. In order to prove that the set T is a left quasigroup general form transversal in G to H, it is sufficient to prove that for all arbitrary fixed elements $a, b \in E$ the equation

$$x \overset{(T)}{\cdot} a = b$$

has unique solution in the set E.

We have the following equivalent equalities:

$$\begin{aligned} x \overset{(T)}{\cdot} a &= b \\ t_x t_a &= t_b h, \quad h \in H \end{aligned}$$

$$t_x = t_b h t_a^{-1} = (t_b t_a^{-1}) \cdot (t_a h t_a^{-1}) \tag{1}$$

Because the set T is a left general form transversal in G to $H^{t_a} = t_a H t_a^{-1}$ (when $\pi = t_a$), then there exists the unique element $c = c(a, b) \in E$ such that

$$t_b t_a^{-1} \in t_c \cdot (t_a H t_a^{-1}).$$

Substituting this product in (1) we obtain:

$$t_x = t_c \cdot (t_a h' t_a^{-1}) \cdot (t_a h t_a^{-1}) = t_c \cdot (t_a h^* t_a^{-1}), \quad h^* \in H.$$

Because the set T is a left general form transversal in G to $H^{t_a} = t_a H t_a^{-1}$, then $x = c$. The proof is finished. $\square$

Corollary 1. *The following statements are equivalent:*

1. *A set $T = \{t_x\}_{x\in E}$ is a left quasigroup general form transversal in G to H;*
2. *For every $\pi \in G$ the set $T\pi = \{t_x\pi\}_{x\in E}$ is a left quasigroup general form transversal in G to H;*
3. *For all $\pi_1, \pi_2 \in G$ the set $\pi_1 T\pi_2 = \{\pi_1 t_x \pi_2\}_{x\in E}$ is a left quasigroup general form transversal in G to H.*

Theorem 7. *The following statements are equivalent:*

1. *A set $T = \{t_x\}_{x\in E}$ is a left loop general form transversal in G to H;*
2. *For every $\pi \in G$ the set $T\pi = \{t_x\pi\}_{x\in E}$ is a left general form transversal in G to H;*
3. *For all $\pi \in G$ the set $\pi T\pi^{-1} = \{\pi t_x \pi^{-1}\}_{x\in E}$ is a left reduced general form transversal in G to H;*
4. *For every $\pi \in G$ the set $T = \{t_x\}_{x\in E}$ is a left reduced general form transversal in G to $H^\pi = \pi H\pi^{-1}$.*

Proof. It is an evident corollary from Theorems 1 and 6. □

Corollary 2. *The following statements are equivalent:*

1. *A set $T = \{t_x\}_{x\in E}$ is a left loop general form transversal in G to H;*
2. *For every $\pi \in G$ the set $T\pi = \{t_x\pi\}_{x\in E}$ is a left quasigroup general form transversal in G to H;*
3. *For all $\pi \in G$ the set $\pi T\pi^{-1} = \{\pi t_x \pi^{-1}\}_{x\in E}$ is a left loop general form transversal in G to H.*

Theorem 8. *Let $T = \{t_x\}_{x\in E}$ be a left loop general form transversal in G to H. According to Definition 9 and Theorem 3 there exists an element $a_0 \in E$ such that $\widehat{t}_{a_0} = id$. Then for every $x \in E,\ \ x \neq a_0$, the permutation $\widehat{t}_x$ is a fixed-point-free permutation on the set E.*

Proof. Let the conditions of Theorem hold and assume that it is not true, i.e. there exist $c_0 \in E$ and $a_1 \in E,\ \ a_1 \neq a_0$, such that

$$\begin{cases} \widehat{t}_{a_1}(c_0) = c_0, \\ a_1 \neq a_0. \end{cases}$$

Then according to Theorem 2 we have the following equivalent equalities

$$\widehat{t}_{a_1}(c_0) \quad = \quad c_0,$$

$$\begin{aligned} a_1 \overset{(T)}{\cdot} c_0 &= c_0 = a_1 \overset{(T)}{\cdot} c_0, \\ a_1 \overset{(T)}{\cdot} c_0 &= a_1 \overset{(T)}{\cdot} c_0, \\ a_1 &= a_0, \end{aligned}$$

since the system $\left\langle E, \overset{(T)}{\cdot}, a_0 \right\rangle$ is a loop. But the last equality contradicts to the assumption that $a_1 \neq a_0$. The proof is finished. $\square$

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Eugene Kuznetsov
Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
5 Academiei str., Chishinau, MD-2028
Moldova

E-mail: *kuznet1964@mail.ru*

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Operations on level graphs of bipolar fuzzy graphs

Wieslaw A. Dudek, Ali A. Talebi

Abstract. We define the notion of level graphs of bipolar fuzzy graphs and use its to characterizations of various classical and new operations on bipolar fuzzy graphs.

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Keywords and phrases: Bipolar fuzzy graph, level graph, cross product, lexicographic product of fuzzy graphs.

1 Introduction

The theory of graphs is an extremely useful tool for solving numerous problems in different areas such as geometry, algebra, operations research, optimization, and computer science. In many cases, some aspects of a graph-theoretic problem may be uncertain. For example, the vehicle travel time or vehicle capacity on a road network may not be known exactly. In such cases, it is natural to deal with the uncertainty using the methods of fuzzy sets, and fuzzy logic. But, the using of fuzzy graphs as models of various systems (social, economics systems, communication networks and others) leads to difficulties. In many domains, we deal with bipolar information. It is noted that positive information represents what is granted to be possible, while negative information represents what is considered to be impossible. The bipolar fuzzy sets as an extension of fuzzy sets were introduced by Zhang [20, 21] in 1994. In a bipolar fuzzy set, the membership degree range is $[-1, 1]$, the member degree 0 of an element shows that the element is irrelevant to the corresponding property. If membership degree of an element is positive, it means that the element somewhat satisfies the property, and a negative membership degree shows that the element somewhat satisfies the implicit counter-property. The bipolar fuzzy graph model is more precise, flexible, and compatible as compared to the classical and fuzzy graph models. This is the motivation to generalize the notion of fuzzy graphs to the notion of bipolar fuzzy graphs. In 1965, Zadeh [19] introduced the notion of a fuzzy subset of a set as a method for representing uncertainty. Now, the theory of fuzzy sets has become a vigorous area of research in different disciplines including medical, life science, management sciences, engineering, statistics, graph theory, signal processing, pattern recognition, computer networks and expert systems. Fuzzy graphs and fuzzy analogs of several graph theoretical notions were discussed by Rosenfeld [13], whose basic idea was introduced by Kauffmann [7] in 1973. Rosenfeld considered the fuzzy relations between fuzzy sets and developed the structure of fuzzy graphs. Some operations on fuzzy graphs were introduced by Mordeson and Peng [11]. Akram and

Dudek [3] generalized some operations to interval-valued fuzzy graphs. The concept of intuitionistic fuzzy graphs was introduced by Shannon and Atanassov [16], they investigated some of their properties in [17]. Parvathi et al. defined operations on intuitionistic fuzzy graphs in [12]. Akram introduced the concept of bipolar fuzzy graphs in [1], he discussed the concept of isomorphism of these graphs, and investigated some of their important properties, also defined some operations on bipolar fuzzy graphs (see also [2, 4–6]).

In this paper, we define the notion of level graphs of a bipolar fuzzy graph and investigate some of their properties. Next we show that level graphs can be used to the characterization of various products of two bipolar fuzzy graphs.

2 Preliminaries

In this section, we review some definitions that are necessary for this paper.

Let V be a nonempty set. Denote by $\widetilde{V^2}$ the collection of all 2-element subsets of V. A pair (V, E), where $E \subseteq \widetilde{V^2}$, is called a *graph*.

Further, for simplicity, the subsets of the form $\{x, y\}$ will be denoted by xy.

Definition 1. Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two graphs and let $V = V_1 \times V_2$.

- The *union of graphs* G_1^* and G_2^* is the graph $(V_1 \cup V_2, E_1 \cup E_2)$.
- The graph $(V_1 \cup V_2, E_1 \cup E_2 \cup E')$, where E' is the set of edges joining vertices of V_1 and V_2, is denoted by $G_1^* + G_2^*$ and is called the *join of graphs* G_1^* and G_2^*.
- The *Cartesian product of graphs* G_1^* and G_2^*, denoted by $G_1^* \times G_2^*$, is the graph (V, E) with
  $E = \{(x, x_2)(x, y_2) \,|\, x \in V_1\,,\ x_2y_2 \in E_2\} \cup \{(x_1, z)(y_1, z) \,|\, z \in V_2\,,\ x_1y_1 \in E_1\}$.
- The *cross product of graphs* G_1^* and G_2^*, denoted by $G_1^* * G_2^*$, is the graph (V, E) such that $E = \{(x_1, x_2)(y_1, y_2) \,|\, x_1y_1 \in E_1\,,\ x_2y_2 \in E_2\}$.
- The *lexicographic product of graphs* G_1^* and G_2^*, denoted by $G_1^* \bullet G_2^*$, is the graph (V, E) such that
  $E = \{(x, x_2)(x, y_2) \,|\, x \in V_1,\ x_2y_2 \in E_2\} \cup \{(x_1, x_2)(y_1, y_2) \,|\, x_1y_1 \in E_1,\ x_2y_2 \in E_2\}$.
- The *strong product of graphs* G_1^* and G_2^*, denoted by $G_1^* \boxtimes G_2^*$, is the graph (V, E) such that
  $E = \{(x, x_2)(x, y_2) \,|\, x \in V_1\,,\ x_2y_2 \in E_2\} \cup \{(x_1, z)(y_1, z) \,|\, z \in V_2\,,\ x_1y_1 \in E_1\} \cup \{(x_1, x_2)(y_1, y_2) \,|\, x_1y_1 \in E_1\,,\ x_2y_2 \in E_2\}$.
- The *composition of graphs* G_1^* and G_2^*, denoted by $G_1^*[G_2^*]$, is the graph (V, E) such that
  $E = \{(x, x_2)(x, y_2) \,|\, x \in V_1\,,\ x_2y_2 \in E_2\} \cup \{(x_1, z)(y_1, z) \,|\, z \in V_2\,,\ x_1y_1 \in E_1\} \cup \{(x_1, x_2)(y_1, y_2) \,|\, x_2, y_2 \in V_2, \,,\ x_2 \neq y_2\,,\ x_1y_1 \in E_1\}$.

One can find the corresponding examples clarifying the above concepts in [9,10, 14,15,18].

Definition 2. Let X be a set, a mapping $A = (\mu_A^N, \mu_A^P) : X \to [-1, 0] \times [0, 1]$ is called a *bipolar fuzzy set* on X. For every $x \in X$, the value $A(x)$ is written as $(\mu_A^N(x), \mu_A^P(x))$.

We use the *positive membership degree* $\mu_A^P(x)$ to denote the satisfaction degree of elements x to the property corresponding to a bipolar fuzzy set A, and the *negative membership degree* $\mu_A^N(x)$ to denote the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar fuzzy set A.

Definition 3. A *fuzzy graph* of a graph $G^* = (V, E)$ is a pair $G = (\sigma, \mu)$, where σ and μ are fuzzy sets on V and $\widetilde{V^2}$, respectively, such that $\mu(x, y) \le \min(\sigma(x), \sigma(y))$ for all $xy \in E$ and $\mu(xy) = 0$ for $xy \in \widetilde{V^2} \setminus E$.

Let $G^* = (V, E)$ be a crisp graph and let A, B be bipolar fuzzy sets on V and E, respectively. The pair (A, B) is called a *bipolar fuzzy pair* of a graph G^*.

Definition 4. ([1]) A *bipolar fuzzy graph* of a graph $G^* = (V, E)$ is a bipolar fuzzy pair $G = (A, B)$ of G^*, where $A = (\mu_A^N, \mu_A^P)$ and $B = (\mu_B^N, \mu_B^P)$ are such that

$$\mu_B^P(xy) \le \min(\mu_A^P(x), \mu_A^P(y)), \quad \mu_B^N(xy) \ge \max(\mu_A^N(x), \mu_A^N(y)) \quad \text{for all } xy \in E.$$

A fuzzy graph (σ, μ) of a graph G^* can be considered as an bipolar fuzzy graph $G = (A, B)$, where $\mu_A^N(x) = 0$ for all $x \in V$, $\mu_B^N(xy) = 0$ for all $xy \in E$ and $\mu_B^P = \mu$, $\mu_A^P = \sigma$.

Definition 5. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be bipolar fuzzy pair of graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Consider two bipolar fuzzy sets $A = (\mu_A^N, \mu_A^P)$ and $B = (\mu_B^N, \mu_B^P)$.

- The *union* $G_1 \cup G_2$ is defined as the pair (A, B) of bipolar fuzzy sets determined on the union of graphs G_1^* and G_2^* such that

$$\text{(i)} \qquad \mu_A^P(x) = \begin{cases} \mu_{A_1}^P(x) & \text{if } x \in V_1 \text{ and } x \notin V_2 \\ \mu_{A_2}^P(x) & \text{if } x \in V_2 \text{ and } x \notin V_1 \\ \max(\mu_{A_1}^P(x), \mu_{A_2}^P(x)) & \text{if } x \in V_1 \cap V_2, \end{cases}$$

$$\text{(ii)} \qquad \mu_A^N(x) = \begin{cases} \mu_{A_1}^N(x) & \text{if } x \in V_1 \text{ and } x \notin V_2 \\ \mu_{A_2}^N(x) & \text{if } x \in V_2 \text{ and } x \notin V_1 \\ \min(\mu_{A_1}^N(x), \mu_{A_2}^N(x)) & \text{if } x \in V_1 \cap V_2, \end{cases}$$

$$\text{(iii)} \qquad \mu_B^P(xy) = \begin{cases} \mu_{B_1}^P(xy) & \text{if } xy \in E_1 \text{ and } xy \notin E_2 \\ \mu_{B_2}^P(xy) & \text{if } xy \in E_2 \text{ and } xy \notin E_1 \\ \max(\mu_{B_1}^P(xy), \mu_{B_2}^P(xy)) & \text{if } xy \in E_1 \cap E_2, \end{cases}$$

$$\text{(iv)} \quad \mu_B^N(xy) = \begin{cases} \mu_{B_1}^N(xy) & \text{if } xy \in E_1 \text{ and } xy \notin E_2 \\ \mu_{B_2}^N(xy) & \text{if } xy \in E_2 \text{ and } xy \notin E_1 \\ \min(\mu_{B_1}^N(xy), \mu_{B_2}^N(xy)) & \text{if } xy \in E_1 \cap E_2. \end{cases}$$

- The *join* $G_1 + G_2$ is the pair (A, B) of bipolar fuzzy sets defined on the join $G_1^* + G_2^*$ such that

$$\text{(i)} \quad \mu_A^P(x) = \begin{cases} \mu_{A_1}^P(x) & \text{if } x \in V_1 \text{ and } x \notin V_2 \\ \mu_{A_2}^P(x) & \text{if } x \in V_2 \text{ and } x \notin V_1 \\ \max(\mu_{A_1}^P(x), \mu_{A_2}^P(x)) & \text{if } x \in V_1 \cap V_2, \end{cases}$$

$$\text{(ii)} \quad \mu_A^N(x) = \begin{cases} \mu_{A_1}^N(x) & \text{if } x \in V_1 \text{ and } x \notin V_2 \\ \mu_{A_2}^N(x) & \text{if } x \in V_2 \text{ and } x \notin V_1 \\ \min(\mu_{A_1}^N(x), \mu_{A_2}^N(x)) & \text{if } x \in V_1 \cap V_2, \end{cases}$$

$$\text{(iii)} \quad \mu_B^P(xy) = \begin{cases} \mu_{B_1}^P(xy) & \text{if } xy \in E_1 \text{ and } xy \notin E_2 \\ \mu_{B_2}^P(xy) & \text{if } xy \in E_2 \text{ and } xy \notin E_1 \\ \max(\mu_{B_1}^P(xy), \mu_{B_2}^P(xy)) & \text{if } xy \in E_1 \cap E_2 \\ \min(\mu_{A_1}^P(x), \mu_{A_2}^P(x)) & \text{if } xy \in E', \end{cases}$$

$$\text{(iv)} \quad \mu_B^N(xy) = \begin{cases} \mu_{B_1}^N(xy) & \text{if } xy \in E_1 \text{ and } xy \notin E_2 \\ \mu_{B_2}^N(xy) & \text{if } xy \in E_2 \text{ and } xy \notin E_1 \\ \min(\mu_{B_1}^N(xy), \mu_{B_2}^N(xy)) & \text{if } xy \in E_1 \cap E_2 \\ \max(\mu_{A_1}^N(x), \mu_{A_2}^N(y)) & \text{if } xy \in E'. \end{cases}$$

- The *Cartesian product* $G_1 \times G_2$ is the pair (A, B) of bipolar fuzzy sets defined on the Cartesian product $G_1^* \times G_2^*$ such that

  (i) $\mu_A^P(x_1, x_2) = \min(\mu_{A_1}^P(x_1), \mu_{A_2}^P(x_2))$,
  $\mu_A^N(x_1, x_2) = \max(\mu_{A_1}^N(x_1), \mu_{A_2}^N(x_2))$ for all $(x_1, x_2) \in V_1 \times V_2$,

  (ii) $\mu_B^P((x, x_2)(x, y_2)) = \min(\mu_{A_1}^P(x), \mu_{B_2}^P(x_2y_2))$,
  $\mu_B^N((x, x_2)(x, y_2)) = \max(\mu_{A_1}^N(x), \mu_{B_2}^N(x_2y_2))$ for all $x \in V_1$ and $x_2y_2 \in E_2$,

  (iii) $\mu_B^P((x_1, z)(y_1, z)) = \min(\mu_{B_1}^P(x_1y_1), \mu_{A_2}^P(z))$,
  $\mu_B^N((x_1, z)(y_1, z)) = \max(\mu_{B_1}^N(x_1y_1), \mu_{A_2}^N(z))$ for all $z \in V_2$ and $x_1y_1 \in E_1$.

- The *cross product* $G_1 * G_2$ is the pair (A, B) of bipolar fuzzy sets defined on the cross product $G_1^* * G_2^*$ such that

  (i) $\mu_A^P(x_1, x_2) = \min(\mu_{A_1}^P(x_1), \mu_{A_2}^P(x_2))$,
  $\mu_A^N(x_1, x_2) = \max(\mu_{A_1}^N(x_1), \mu_{A_2}^N(x_2))$ for all $(x_1, x_2) \in V_1 \times V_2$,

  (ii) $\mu_B^P((x_1, x_2)(y_1, y_2)) = \min(\mu_{B_1}^P(x_1y_1), \mu_{B_2}^P(x_2y_2))$,
  $\mu_B^N((x_1, x_2)(y_1, y_2)) = \max(\mu_{B_1}^N(x_1y_1), \mu_{B_2}^N(x_2y_2))$ for all $x_1y_1 \in E_1$ and for all $x_2y_2 \in E_2$.

- The *lexicographic product* $G_1 \bullet G_2$ is the pair (A, B) of bipolar fuzzy sets defined on the lexigographic product $G_1^* \bullet G_2^*$ such that

(i) $\mu_A^P(x_1,x_2) = \min(\mu_{A_1}^P(x_1), \mu_{A_2}^P(x_2))$,
$\mu_A^N(x_1,x_2) = \max(\mu_{A_1}^N(x_1), \mu_{A_2}^N(x_2))$ for all $(x_1,x_2) \in V_1 \times v_2$,

(ii) $\mu_B^P((x,x_2)(x,y_2)) = \min(\mu_{A_1}^P(x), \mu_{B_2}^P(x_2y_2))$,
$\mu_B^N((x,x_2)(x,y_2)) = \max(\mu_{A_1}^N(x), \mu_{B_2}^N(x_2y_2))$ for all $x \in V_1$ and for all $x_2y_2 \in E_2$,

(iii) $\mu_B^P((x_1,x_2)(y_1,y_2)) = \min(\mu_{B_1}^P(x_1y_1), \mu_{B_2}^P(x_2y_2))$,
$\mu_B^N((x_1,x_2)(y_1,y_2)) = \max(\mu_{B_1}^N(x_1y_1), \mu_{B_2}^N(x_2y_2))$ for all $x_1y_1 \in E_1$ and for all $x_2y_2 \in E_2$.

- The *strong product* $G_1 \boxtimes G_2$ of G_1 is the pair (A,B) of bipolar fuzzy sets defined on the strong product $G_1^* \boxtimes G_2^*$ such that

(i) $\mu_A^P(x_1,x_2) = \min(\mu_{A_1}^P(x_1), \mu_{A_2}^P(x_2))$,
$\mu_A^N(x_1,x_2) = \max(\mu_{A_1}^N(x_1), \mu_{A_2}^N(x_2))$ for all $(x_1,x_2) \in V_1 \times V_2$,

(ii) $\mu_B^P((x,x_2)(x,y_2)) = \min(\mu_{A_1}^P(x), \mu_{B_2}^P(x_2y_2))$,
$\mu_B^N((x,x_2)(x,y_2)) = \max(\mu_{A_1}^N(x), \mu_{B_2}^N(x_2y_2))$ for all $x \in V_1$ and for all $x_2y_2 \in E_2$,

(iii) $\mu_B^P((x_1,z)(y_1,z)) = \min(\mu_{B_1}^P(x_1y_1), \mu_{A_2}^P(z))$,
$\mu_B^N((x_1,z)(y_1,z)) = \max(\mu_{B_1}^N(x_1y_1), \mu_{A_2}^N(z))$ for all $z \in V_2$ and for all $x_1y_1 \in E_1$,

(iv) $\mu_B^P((x_1,x_2)(y_1,y_2)) = \min(\mu_{B_1}^P(x_1y_1), \mu_{B_2}^P(x_2y_2))$,
$\mu_B^N((x_1,x_2)(y_1,y_2)) = \max(\mu_{B_1}^N(x_1y_1), \mu_{B_2}^N(x_2y_2))$ for all $x_1y_1 \in E_1$ and for all $x_2y_2 \in E_2$.

- The *composition* $G_1[G_2]$ is the pair (A,B) of bipolar fuzzy sets defined on the composition $G_1^*[G_2^*]$ such that

(i) $\mu_A^P(x_1,x_2) = \min(\mu_{A_1}^P(x_1), \mu_{A_2}^P(x_2))$,
$\mu_A^N(x_1,x_2) = \max(\mu_{A_1}^N(x_1), \mu_{A_2}^N(x_2))$ for all $(x_1,x_2) \in V_1 \times V_2$,

(ii) $\mu_B^P((x,x_2)(x,y_2)) = \min(\mu_{A_1}^P(x), \mu_{B_2}^P(x_2y_2))$,
$\mu_B^N((x,x_2)(x,y_2)) = \max(\mu_{A_1}^N(x), \mu_{B_2}^N(x_2y_2))$ for all $x \in V_1$ and for all $x_2y_2 \in E_2$,

(iii) $\mu_B^P((x_1,z)(y_1,z)) = \min(\mu_{B_1}^P(x_1y_1), \mu_{A_2}^P(z))$,
$\mu_B^N((x_1,z)(y_1,z)) = \max(\mu_{B_1}^N(x_1y_1), \mu_{A_2}^N(z))$ for all $z \in V_2$ and for all $x_1y_1 \in E_1$,

(iv) $\mu_B^P((x_1,x_2)(y_1,y_2)) = \min(\mu_{A_2}^P(x_2), \mu_{A_2}^P(y_2), \mu_{B_1}^P(x_1y_1))$,
$\mu_B^N((x_1,x_2)(y_1,y_2)) = \max(\mu_{A_2}^N(x_2), \mu_{A_2}^N(y_2), \mu_{B_1}^N(x_1y_1))$ for all $x_2, y_2 \in V_2$, where $x_2 \neq y_2$ and for all $x_1y_1 \in E_1$.

3 Level graphs of bipolar fuzzy graphs

In this section we define the level graph of a bipolar fuzzy graph and discuss some important operations on bipolar fuzzy graphs by characterizing these operations by their level counterparts graphs.

Definition 6. Let $A : X \to [-1,0] \times [0,1]$ be a bipolar fuzzy set on X. The set $A_{(a,b)} = \{x \in X \mid \mu_A^P(x) \ge b\,,\ \mu_A^N(x) \le a\}$, where $(a,b) \in [-1,0] \times [0,1]$, is called the (a,b)-*level set* of A.

The following theorem is important in this paper. It is substantial modification of the transfer principle for fuzzy sets described in [8].

Theorem 1. *Let V be a set, and $A = (\mu_A^N, \mu_A^P)$ and $B = (\mu_B^N, \mu_B^P)$ be bipolar fuzzy sets on V and $\widetilde{V^2}$, respectively. Then $G = (A,B)$ is a bipolar fuzzy graph if and only if $(A_{(a,b)}, B_{(a,b)})$, called the (a,b)-level graph of G, is a graph for each pair $(a,b) \in [-1,0] \times [0,1]$.*

Proof. Let $G = (A,B)$ be a bipolar fuzzy graph. For every $(a,b) \in [-1,0] \times [0,1]$, if $xy \in B_{(a,b)}$, then $\mu_B^N(xy) \le a$ and $\mu_B^P(xy) \ge b$. Since G is a bipolar fuzzy graph,

$$a \ge \mu_B^N(xy) \ge \max(\mu_A^N(x), \mu_A^N(y))$$

and

$$b \le \mu_B^P(xy) \le \min(\mu_A^P(x), \mu_A^P(y)),$$

and so $a \ge \mu_A^N(x)$, $a \ge \mu_A^N(y)$, $b \le \mu_A^P(x)$, $b \le \mu_A^P(y)$, that is, $x, y \in A_{(a,b)}$. Therefore, $(A_{(a,b)}, B_{(a,b)})$ is a graph for each $(a,b) \in [-1,0] \times [0,1]$.

Conversely, let $(A_{(a,b)}, B_{(a,b)})$ be a graph for all $(a,b) \in [-1,0] \times [0,1]$. For every $xy \in \widetilde{V^2}$, let $\mu_B^N(xy) = a$ and $\mu_B^P(xy) = b$. Then $xy \in B_{(a,b)}$. Since $(A_{(a,b)}, B_{(a,b)})$ is a graph, we have $x, y \in A_{(a,b)}$; hence $\mu_A^N(x) \le a$, $\mu_A^P(x) \ge b$, $\mu_A^N(y) \le a$ and $\mu_A^P(x) \ge b$. Therefore,

$$\mu_B^N(xy) = a \ge \max(\mu_A^N(x), \mu_A^N(y))$$

and

$$\mu_B^P(xy) = b \le \min(\mu_A^P(x), \mu_A^P(y)),$$

that is $G = (A,B)$ is a bipolar fuzzy graph. □

Theorem 2. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be bipolar fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then $G = (A,B)$ is the Cartesian product of G_1 and G_2 if and only if for each pair $(a,b) \in [-1,0] \times [0,1]$ the (a,b)-level graph $(A_{(a,b)}, B_{(a,b)})$ is the Cartesian product of $((A_1)_{(a,b)}, (B_1)_{(a,b)})$ and $((A_2)_{(a,b)}, (B_2)_{(a,b)})$.*

Proof. Let $G = (A, B)$ be the Cartesian product of bipolar fuzzy graphs G_1 and G_2. For every $(a, b) \in [-1, 0] \times [0, 1]$, if $(x, y) \in A_{(a,b)}$, then

$$\min(\mu^P_{A_1}(x), \mu^P_{A_2}(y)) = \mu^P_A(x, y) \geq b$$

and

$$\max(\mu^N_{A_1}(x), \mu^N_{A_2}(y)) = \mu^N_A(x, y) \leq a,$$

hence $x \in (A_1)_{(a,b)}$ and $y \in (A_2)_{(a,b)}$; that is $(x, y) \in (A_1)_{(a,b)} \times (A_2)_{(a,b)}$. Therefore, $A_{(a,b)} \subseteq (A_1)_{(a,b)} \times (A_2)_{(a,b)}$. Now if $(x, y) \in (A_1)_{(a,b)} \times (A_2)_{(a,b)}$, then $x \in (A_1)_{(a,b)}$ and $y \in (A_2)_{(a,b)}$. It follows that $\min(\mu^P_{A_1}(x), \mu^P_{A_2}(y)) \geq b$ and $\max(\mu^N_{A_1}(x), \mu^N_{A_2}(y)) \leq a$. Since (A, B) is the Cartesian product of G_1 and G_2, $\mu^P_A(x, y) \geq b$ and $\mu^N_A(x, y) \leq a$; that is $(x, y) \in A_{(a,b)}$. Therefore, $(A_1)_{(a,b)} \times (A_2)_{(a,b)} \subseteq A_{(a,b)}$ and so $(A_1)_{(a,b)} \times (A_2)_{(a,b)} = A_{(a,b)}$.

We now prove $B_{(a,b)} = E$, where E is the edge set of the Cartesian product $(G_1)_{(a,b)} \times (G_2)_{(a,b)}$ for all $(a, b) \in [-1, 0] \times [0, 1]$. Let $(x_1, x_2)(y_1, y_2) \in B_{(a,b)}$. Then, $\mu^P_B((x_1, x_2)(y_1, y_2)) \geq b$ and $\mu^N_B((x_1, x_2)(y_1, y_2)) \leq a$. Since (A, B) is the Cartesian product of G_1 and G_2, one of the following cases holds:

(i) $x_1 = y_1$ and $x_2y_2 \in E_2$,

(ii) $x_2 = y_2$ and $x_1y_1 \in E_1$.

For the case (i), we have

$$\mu^P_B((x_1, x_2)(y_1, y_2)) = \min(\mu^P_{A_1}(x_1), \mu^P_{B_2}(x_2y_2)) \geq b,$$

$$\mu^N_B((x_1, x_2)(y_1, y_2)) = \max(\mu^N_{A_1}(x_1), \mu^N_{B_2}(x_2y_2)) \leq a,$$

and so $\mu^P_{A_1}(x_1) \geq b$, $\mu^N_{A_1}(x_1) \leq a$, $\mu^P_{B_2}(x_2y_2) \geq b$ and $\mu^N_{B_2}(x_2y_2) \leq a$. It follows that $x_1 = y_1 \in (A_1)_{(a,b)}$, $x_2y_2 \in (B_2)_{(a,b)}$; that is $(x_1, x_2)(y_1, y_2) \in E$. Similarly, for the case (ii), we conclude that $(x_1, x_2)(y_1, y_2) \in E$. Therefore, $B_{(a,b)} \subseteq E$. For every $(x, x_2)(x, y_2) \in E$, $\mu^P_{A_1}(x) \geq b$, $\mu^N_{A_1}(x) \leq a$, $\mu^P_{B_2}(x_2y_2) \geq b$ and $\mu^N_{B_2}(x_2y_2) \leq a$. Since (A, B) is the Cartesian product of G_1 and G_2, we have

$$\mu^P_B((x, x_2)(x, y_2)) = \min(\mu^P_{A_1}(x), \mu^P_{B_2}(x_2y_2)) \geq b,$$

$$\mu^N_B((x, x_2)(x, y_2)) = \max(\mu^N_{A_1}(x), \mu^N_{B_2}(x_2y_2)) \leq a.$$

Therefore, $(x, x_2)(x, y_2) \in B_{(a,b)}$. Similarly, for every $(x_1, z)(y_1, z) \in E$, we have $(x_1, z)(y_1, z) \in B_{(a,b)}$. Therefore, $E \subseteq B_{(a,b)}$, and so $B_{(a,b)} = E$.

Conversely, suppose that the (a, b)-level graph $(A_{(a,b)}, B_{(a,b)})$ is the Cartesian product of $((A_1)_{(a,b)}, (B_1)_{(a,b)})$ and $((A_2)_{(a,b)}, (B_2)_{(a,b)})$ for all $(a, b) \in [-1, 0] \times [0, 1]$. Let $\min(\mu^P_{A_1}(x_1), \mu^P_{A_2}(x_2)) = b$ and $\max(\mu^N_{A_1}(x_1), \mu^N_{A_2}(x_2)) = a$ for some $(x_1, x_2) \in V_1 \times V_2$. Then $x_1 \in (A_1)_{(a,b)}$ and $x_2 \in (A_2)_{(a,b)}$. By the hypothesis, $(x_1, x_2) \in A_{(a,b)}$, hence

$$\mu^P_A((x_1, x_2)) \geq b = \min(\mu^P_{A_1}(x_1), \mu^P_{A_2}(x_2))$$

and

$$\mu_A^N((x_1,x_2)) \le a = \max(\mu_{A_1}^N(x_1), \mu_{A_2}^N(x_2)).$$

Now let $\mu_A^N(x_1,x_2) = c$ and $\mu_A^P(x_1,x_2) = d$, then we have $(x_1,x_2) \in A_{(c,d)}$. Since $(A_{(c,d)}, B_{(c,d)})$ is the Cartesian product of levels $((A_1)_{(c,d)}, (B_1)_{(c,d)})$ and $((A_2)_{(c,d)}, (B_2)_{(c,d)})$, then $x_1 \in (A_1)_{(c,d)}$ and $x_2 \in (A_2)_{(c,d)}$. Hence,

$$\mu_{A_1}^P(x_1) \ge d = \mu_A^P(x_1,x_2), \qquad \mu_{A_1}^N(x_1) \le c = \mu_A^N(x_1,x_2),$$

$$\mu_{A_2}^P(x_2) \ge d = \mu_A^P(x_1,x_2) \quad \text{and} \quad \mu_{A_2}^N(x_2) \le c = \mu_A^N(x_1,x_2).$$

It follows that

$$\min(\mu_{A_1}^P(x_1), \mu_{A_2}^P(x_2)) \ge \mu_A^P(x_1,x_2)$$

and

$$\max(\mu_{A_1}^N(x_1), \mu_{A_2}^N(x_2)) \le \mu_A^N(x_1,x_2).$$

Therefore,

$$\mu_A^P(x_1,x_2) = \min(\mu_{A_1}^P(x_1), \mu_{A_2}^P(x_2))$$

and

$$\mu_A^N(x_1,x_2) = \max(\mu_{A_1}^N(x_1), \mu_{A_2}^N(x_2)) \quad \text{for all } (x_1,x_2) \in V_1 \times V_2.$$

Similarly, for every $x \in V_1$ and every $x_2y_2 \in E_2$, let

$$\min(\mu_{A_1}^P(x), \mu_{B_2}^P(x_2y_2)) = b, \qquad \max(\mu_{A_1}^N(x), \mu_{B_2}^N(x_2y_2)) = a,$$

$$\mu_B^P((x,x_2)(x,y_2)) = d \quad \text{and} \quad \mu_B^N((x,x_2)(x,y_2)) = c.$$

Then we have $\mu_{A_1}^P(x) \ge b$, $\mu_{B_2}^P(x_2y_2) \ge b$, $\mu_{A_1}^N(x) \le a$, $\mu_{B_2}^N(x_2y_2) \le a$ and $(x,x_2)(x,y_2) \in B_{(c,d)}$, i.e., $x \in (A_1)_{(a,b)}$, $x_2y_2 \in (B_2)_{(a,b)}$ and $(x,x_2)(x,y_2) \in B_{(c,d)}$. Since $(A_{(a,b)}, B_{(a,b)})$ (respectively, $(A_{(c,d)}, B_{(c,d)})$) is the Cartesian product of levels $((A_1)_{(a,b)}, (B_1)_{(a,b)})$ and $((A_2)_{(a,b)}, (B_2)_{(a,b)})$ (respectively, $((A_1)_{(c,d)}, (B_1)_{(c,d)})$ and $((A_2)_{(c,d)}, (B_2)_{(c,d)})$), we have $(x,x_2)(x,y_2) \in B_{(a,b)}$, $x \in (A_1)_{(c,d)}$, and $x_2y_2 \in (B_2)_{(c,d)}$, which implies $(x,x_2)(x,y_2) \in B_{(a,b)}$, $\mu_{A_1}^P(x) \ge d$, $\mu_{A_1}^N(x) \le c$, $\mu_{B_2}^P(x_2y_2) \ge d$ and $\mu_{B_2}^N(x_2y_2) \le c$. It follows that

$$\mu_B^N((x,x_2)(x,y_2)) \le a = \max(\mu_{A_1}^N(x), \mu_{B_2}^N(x_2y_2)),$$

$$\mu_B^P((x,x_2)(x,y_2)) \ge b = \min(\mu_{A_1}^P(x), \mu_{B_2}^P(x_2y_2)),$$

$$\min(\mu_{A_1}^P(x), \mu_{B_2}^P(x_2y_2)) \ge d = \mu_B^P((x,x_2)(x,y_2)),$$

and

$$\max(\mu_{A_1}^N(x), \mu_{B_2}^N(x_2y_2)) \le c = \mu_B^N((x,x_2)(x,y_2)).$$

Therefore,

$$\mu_B^P((x,x_2)(x,y_2)) = \min(\mu_{A_1}^P(x), \mu_{B_2}^P(x_2y_2)),$$

$$\mu_B^N((x,x_2)(x,y_2)) = \max(\mu_{A_1}^N(x), \mu_{B_2}^N(x_2y_2))$$

for all $x \in V_1$ and $x_2y_2 \in E_2$.

As above we can show that

$$\mu_B^P((x_1,z)(y_1,z)) = \min(\mu_{B_1}^P(x_1y_1), \mu_{A_2}^P(z)),$$

$$\mu_B^N((x_1,z)(y_1,z)) = \max(\mu_{B_1}^N(x_1y_1), \mu_{A_2}^N(z))$$

for all $z \in V_2$ and for all $x_1y_1 \in E_1$. This completes the proof. $\square$

Now by Theorem 1 and Theorem 2 we have the following corollary.

Corollary 1. *If $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are bipolar fuzzy graphs, then the Cartesian product $G_1 \times G_2$ is a bipolar fuzzy graph.*

Theorem 3. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be bipolar fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then $G = (A, B)$ is the composition of G_1 and G_2 if and only if for each $(a, b) \in [-1, 0] \times [0, 1]$ the (a, b)-level graph $(A_{(a,b)}, B_{(a,b)})$ is the composition of $((A_1)_{(a,b)}, (B_1)_{(a,b)})$ and $((A_2)_{(a,b)}, (B_2)_{(a,b)})$.*

Proof. Let $G = (A, B)$ be the composition of bipolar fuzzy graphs G_1 and G_2. By the definition of $G_1[G_2]$ and the same argument as in the proof of Theorem 2, we have $A_{(a,b)} = (A_1)_{(a,b)} \times (A_2)_{(a,b)}$. Now we prove $B_{(a,b)} = E$, where E is the edge set of the composition $(G_1)_{(a,b)}[(G_2)_{(a,b)}]$ for all $(a, b) \in [-1, 0] \times [0, 1]$. Let $(x_1, x_2)(y_1, y_2) \in B_{(a,b)}$. Then $\mu_B^P((x_1, x_2)(y_1, y_2)) \geq b$ and $\mu_B^N((x_1, x_2)(y_1, y_2)) \leq a$. Since $G = (A, B)$ is the composition $G_1[G_2]$, one of the following cases holds:

(i) $x_1 = y_1$ and $x_2y_2 \in E_2$,

(ii) $x_2 = y_2$ and $x_1y_1 \in E_1$,

(iii) $x_2 \neq y_2$ and $x_1y_1 \in E_1$.

For the cases (i) and (ii), similarly as in the cases of (i) and (ii) in the proof of Theorem 2, we obtain $(x_1, x_2)(y_1, y_2) \in E$. For the case (iii), we have

$$\mu_B^P((x_1,x_2)(y_1,y_2)) = \min(\mu_{A_2}^P(x_2), \mu_{A_2}^P(y_2), \mu_{B_1}^P(x_1y_1)) \geq b,$$

$$\mu_B^N((x_1,x_2)(y_1,y_2)) = \max(\mu_{A_2}^N(x_2), \mu_{A_2}^N(y_2), \mu_{B_1}^N(x_1y_1)) \leq a.$$

Thus, $\mu_{A_2}^P(x_2) \geq b$, $\mu_{A_2}^P(y_2) \geq b$, $\mu_{B_1}^P(x_1y_1) \geq b$, $\mu_{A_2}^N(x_2) \leq a$, $\mu_{A_2}^N(y_2) \leq a$ and $\mu_{B_1}^N(x_1y_1) \leq a$. It follows that $x_2, y_2 \in (A_2)_{(a,b)}$ and $x_1y_1 \in (B_1)_{(a,b)}$; that is $(x_1, x_2)(y_1, y_2) \in E$. Therefore, $B_{(a,b)} \subseteq E$.

For every $(x, x_2)(x, y_2) \in E$, $\mu_{A_1}^P(x) \geq b$, $\mu_{A_1}^N(x) \leq a$, $\mu_{B_2}^P(x_2y_2)) \geq b$ and $\mu_{B_2}^N(x_2y_2) \leq a$. Since $G = (A, B)$ is the composition $G_1[G_2]$, we have

$$\mu_B^P((x,x_2)(x,y_2)) = \min(\mu_{A_1}^P(x), \mu_{B_2}^P(x_2y_2)) \geq b,$$

$$\mu_B^N((x,x_2)(x,y_2)) = \max(\mu_{A_1}^N(x), \mu_{B_2}^N(x_2y_2)) \leq a.$$

Therefore, $(x, x_1)(x, y_2) \in B_{(a,b)}$. Similarly, for every $(x_1, z)(y_1, z) \in E$, we have $(x, x_2)(x, y_2) \in B_{(a,b)}$. For every $(x_1, x_2)(y_1, y_2) \in E$, where $x_2 \neq y_2$, is $x_1 \neq y_1$, $\mu^P_{B_1}(x_1y_1) \geq b$, $\mu^N_{B_1}(x_1y_1) \leq a$, $\mu^P_{A_2}(y_2) \geq b$, $\mu^N_{A_2}(y_2) \leq a$, $\mu^P_{A_2}(x_2) \geq b$ and $\mu^N_{A_2}(x_2) \leq a$. Since $G = (A, B)$ is the composition $G_1[G_2]$, we have

$$\mu^P_B((x_1, x_2)(y_1, y_2)) = \min(\mu^P_{A_2}(x_2), \mu^P_{A_2}(y_2), \mu^P_{B_1}(x_1y_1)) \geq b,$$

$$\mu^N_B((x_1, x_2)(y_1, y_2)) = \max(\mu^N_{A_2}(x_2), \mu^N_{A_2}(y_2), \mu^N_{B_1}(x_1y_1)) \leq a,$$

hence $(x_1, x_2)(y_1, y_2) \in B_{(a,b)}$. Therefore $E \subseteq B_{(a,b)}$, and so $E = B_{(a,b)}$.

Conversely, suppose that $(A_{(a,b)}, B_{(a,b)})$, where $(a, b) \in [-1, 0] \times [0, 1]$, is the composition of $((A_1)_{(a,b)}, (B_1)_{(a,b)})$ and $((A_2)_{(a,b)}, (B_2)_{(a,b)})$. By the definition of the composition and the proof of Theorem 2, we have

(i) $\mu^P_A(x_1, x_2) = \min(\mu^P_{A_1}(x_1), \mu^P_{A_2}(x_2))$,
$\mu^N_A(x_1, x_2) = \max(\mu^N_{A_1}(x_1), \mu^N_{A_2}(x_2))$ for all $(x_1, x_2) \in V_1 \times V_2$,

(ii) $\mu^P_B((x, x_2)(x, y_2)) = \min(\mu^P_{A_1}(x), \mu^P_{B_2}(x_2y_2))$,
$\mu^N_B((x, x_2)(x, y_2)) = \max(\mu^N_{A_1}(x), \mu^N_{B_2}(x_2y_2))$ for all $x \in V_1$ and $x_2y_2 \in E_2$,

(iii) $\mu^P_B((x_1, z)(y_1, z)) = \min(\mu^P_{B_1}(x_1y_1), \mu^P_{A_2}(z))$,
$\mu^N_B((x_1, z)(y_1, z)) = \max(\mu^N_{B_1}(x_1y_1), \mu^N_{A_2}(z))$ for all $z \in V_2$ and $x_1y_1 \in E_1$.

Similarly, by the same argumentation as in the proof of Theorem 2, we obtain

$$\mu^P_B((x_1, x_2)(y_1, y_2)) = \min(\mu^P_{A_2}(x_2), \mu^P_{A_2}(y_2), \mu^P_{B_1}(x_1y_1)),$$

$$\mu^N_B((x_1, x_2)(y_1, y_2)) = \max(\mu^N_{A_2}(x_2), \mu^N_{A_2}(y_2), \mu^N_{B_1}(x_1y_1))$$

for all $x_2, y_2 \in V_2\, (x_2 \neq y_2)$ and for all $x_1y_1 \in E_1$. This completes the proof. □

Corollary 2. *If $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are bipolar fuzzy graphs, then their composition $G_1[G_2]$ is a bipolar fuzzy graph.*

Theorem 4. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be bipolar fuzzy graphs of $G^*_1 = (V_1, E_1)$ and $G^*_2 = (V_2, E_2)$, respectively, and $V_1 \cap V_2 = \varnothing$. Then $G = (A, B)$ is the union of G_1 and G_2 if and only if each (a, b)-level graph $(A_{(a,b)}, B_{(a,b)})$ is the union of $((A_1)_{(a,b)}, (B_1)_{(a,b)})$ and $((A_2)_{(a,b)}, (B_2)_{(a,b)})$.*

Proof. Let $G = (A, B)$ be the union of bipolar fuzzy graphs G_1 and G_2. We show that $A_{(a,b)} = (A_1)_{(a,b)} \cup (A_2)_{(a,b)}$ for each $(a, b) \in [-1, 0] \times [0, 1]$. Let $x \in A_{(a,b)}$, then $x \in V_1 \setminus V_2$ or $x \in V_2 \setminus V_1$. If $x \in V_1 \setminus V_2$, then $\mu^P_{A_1}(x) = \mu^P_A(x) \geq b$ and $\mu^N_{A_1}(x) = \mu^N_A(x) \leq a$, which implies $x \in (A_1)_{(a,b)}$. Analogously $x \in V_2 \setminus V_1$ implies $x \in (A_2)_{(a,b)}$. Therefore, $x \in (A_1)_{(a,b)} \cup (A_2)_{(a,b)}$, and so $A_{(a,b)} \subseteq (A_1)_{(a,b)} \cup (A_2)_{(a,b)}$.

Now let $x \in (A_1)_{(a,b)} \cup (A_2)_{(a,b)}$. Then we have $x \in (A_1)_{(a,b)}$ and $x \notin (A_2)_{(a,b)}$ or $x \in (A_2)_{(a,b)}$ and $x \notin (A_1)_{(a,b)}$. For the first case, we have $\mu^P_A(x) = \mu^P_{A_1}(x) \geq b$ and $\mu^N_A(x) = \mu^N_{A_1}(x) \leq a$, which implies $x \in A_{(a,b)}$. For the second case, we have

$\mu^P_A(x) = \mu^P_{A_2}(x) \geq b$ and $\mu^N_A(x) = \mu^N_{A_2}(x) \leq a$. Hence $x \in A_{(a,b)}$. Consequently, $(A_1)_{(a,b)} \cup (A_2)_{(a,b)} \subseteq A_{(a,b)}$. To prove that $B_{(a,b)} = (B_1)_{(a,b)} \cup (B_2)_{(a,b)}$ for all $(a,b) \in [-1,0] \times [0,1]$ consider $xy \in B_{(a,b)}$. Then $xy \in E_1 \setminus E_2$ or $xy \in E_2 \setminus E_1$. For $xy \in E_1 \setminus E_2$ we have $\mu^P_{B_1}(xy) = \mu^P_B(xy) \geq b$ and $\mu^N_{B_1}(xy) = \mu^N_B(xy) \leq a$. Thus $xy \in (B_1)_{(a,b)}$. Similarly $xy \in E_2 \setminus E_1$ gives $xy \in (B_2)_{(a,b)}$. Therefore $B_{(a,b)} \subseteq (B_1)_{(a,b)} \cup (B_2)_{(a,b)}$. If $xy \in (B_1)_{(a,b)} \cup (B_2)_{(a,b)}$, then $xy \in (B_1)_{(a,b)} \setminus (B_2)_{(a,b)}$ or $xy \in (B_2)_{(a,b)} \setminus (B_1)_{(a,b)}$. For the first case, $\mu^P_B(xy) = \mu^P_{B_1}(xy) \geq b$ and $\mu^N_B(xy) = \mu^N_{B_1}(xy) \leq a$, hence $xy \in B_{(a,b)}$. In the second case we obtain $xy \in B_{(a,b)}$. Therefore, $(B_1)_{(a,b)} \cup (B_2)_{(a,b)} \subseteq B_{(a,b)}$.

Conversely, let for all $(a,b) \in [-1,0] \times [0,1]$ the level graph $(A_{(a,b)}, B_{(a,b)})$ be the union of $((A_1)_{(a,b)}, (B_1)_{(a,b)})$ and $((A_2)_{(a,b)}, (B_2)_{(a,b)})$. Let $x \in V_1$, $\mu^P_{A_1}(x) = b$, $\mu^N_{A_1}(x) = a$, $\mu^P_A(x) = d$ and $\mu^N_A(x) = c$. Then $x \in (A_1)_{(a,b)}$ and $x \in A_{(c,d)}$. But by the hypothesis $x \in A_{(a,b)}$ and $x \in (A_1)_{(c,d)}$. Thus, $\mu^P_A(x) \geq b$, $\mu^N_A(x) \leq a$, $\mu^P_{A_1}(x) \geq d$ and $\mu^N_{A_1}(x) \leq c$. Therefore, $\mu^P_{A_1}(x) \leq \mu^P_A(x)$, $\mu^N_A(x) \geq \mu^N_{A_1}(x)$, $\mu^P_{A_1}(x) \geq \mu^P_A(x)$ and $\mu^N_{A_1}(x) \leq \mu^N_A(x)$. Hence $\mu^P_{A_1}(x) = \mu^P_A(x)$ and $\mu^N_A(x) = \mu^N_{A_1}(x)$. Similarly, for every $x \in V_2$, we get $\mu^P_{A_2}(x) = \mu^P_A(x)$ and $\mu^N_A(x) = \mu^N_{A_2}(x)$. Thus, we conclude that

$$\text{(i)} \begin{cases} \mu^P_A(x) = \mu^P_{A_1}(x) & \text{if } x \in V_1 \\ \mu^P_A(x) = \mu^P_{A_2}(x) & \text{if } x \in V_2, \end{cases}$$

$$\text{(ii)} \begin{cases} \mu^N_A(x) = \mu^N_{A_1}(x) & \text{if } x \in V_1 \\ \mu^N_A(x) = \mu^N_{A_2}(x) & \text{if } x \in V_2. \end{cases}$$

By a similar method as above, we obtain

$$\text{(iii)} \begin{cases} \mu^P_B(xy) = \mu^P_{B_1}(xy) & \text{if } xy \in E_1 \\ \mu^P_B(xy) = \mu^P_{B_2}(xy) & \text{if } xy \in E_2, \end{cases}$$

$$\text{(iv)} \begin{cases} \mu^N_B(xy) = \mu^N_{B_1}(xy) & \text{if } xy \in E_1 \\ \mu^N_B(xy) = \mu^N_{B_2}(xy) & \text{if } xy \in E_2. \end{cases}$$

This completes the proof. □

Corollary 3. *If G_1 and G_2 are bipolar fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, in which $V_1 \cap V_2 = \varnothing$, then $G_1 \cup G_2$ is a bipolar fuzzy graph.*

Theorem 5. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be bipolar fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, and $V_1 \cap V_2 = \varnothing$. Then $G = (A, B)$ is the join of G_1 and G_2 if and only if each (a,b)-level graph $(A_{(a,b)}, B_{(a,b)})$ is the join of $((A_1)_{(a,b)}, (B_1)_{(a,b)})$ and $((A_2)_{(a,b)}, (B_2)_{(a,b)})$.*

Proof. Let $G = (A, B)$ be the join of bipolar fuzzy graphs G_1 and G_2. Then by the definition and the proof of Theorem 4, $A_{(a,b)} = (A_1)_{(a,b)} \cup (A_2)_{(a,b)}$ for all $(a,b) \in [-1,0] \times [0,1]$. We show that $B_{(a,b)} = (B_1)_{(a,b)} \cup (B_2)_{(a,b)} \cup E'_{(a,b)}$ for all $(a,b) \in [-1,0] \times [0,1]$, where $E'_{(a,b)}$ is the set of all edges joining the vertices $(A_1)_{(a,b)}$ and $(A_2)_{(a,b)}$.

From the proof of Theorem 4 it follows that $(B_1)_{(a,b)} \cup (B_2)_{(a,b)} \subseteq B_{(a,b)}$. If $xy \in E'_{(a,b)}$, then $\mu^P_{A_1}(x) \geq b$, $\mu^N_{A_1}(x) \leq a$, $\mu^P_{A_2}(y) \geq b$ and $\mu^N_{A_2}(y) \leq a$. Hence

$$\mu^P_B(xy) = \min(\mu^P_{A_1}(x), \mu^P_{A_2}(y)) \geq b$$

and

$$\mu^N_B(xy) = \max(\mu^N_{A_1}(x), \mu^N_{A_2}(y)) \leq a.$$

It follows that $xy \in B_{(a,b)}$. Therefore, $(B_1)_{(a,b)} \cup (B_2)_{(a,b)} \cup E'_{(a,b)} \subseteq B_{(a,b)}$. For every $xy \in B_{(a,b)}$, if $xy \in E_1 \cup E_2$, then $xy \in (B_1)_{(a,b)} \cup (B_2)_{(a,b)}$, by the proof of Theorem 4. If $x \in V_1$ and $y \in V_2$, then

$$\min(\mu^P_{A_1}(x), \mu^P_{A_2}(y)) = \mu^P_B(xy) \geq b$$

and

$$\max(\mu^N_{A_1}(x), \mu^N_{A_2}(y)) = \mu^N_B(xy) \leq a,$$

hence $x \in (A_1)_{(a,b)}$ and $y \in (A_2)_{(a,b)}$. So, $xy \in E'_{(a,b)}$. Therefore, $B_{(a,b)} \subseteq (B_1)_{(a,b)} \cup (B_2)_{(a,b)} \cup E'_{(a,b)}$.

Conversely, let each level graph $(A_{(a,b)}, B_{(a,b)})$ be the join of $((A_1)_{(a,b)}, (B_1)_{(a,b)})$ and $((A_2)_{(a,b)}, (B_2)_{(a,b)})$. From the proof of Theorem 4, we have

$$\text{(i)} \begin{cases} \mu^P_A(x) = \mu^P_{A_1}(x) & \text{if } x \in V_1 \\ \mu^P_A(x) = \mu^P_{A_2}(x) & \text{if } x \in V_2, \end{cases}$$

$$\text{(ii)} \begin{cases} \mu^N_A(x) = \mu^N_{A_1}(x) & \text{if } x \in V_1 \\ \mu^N_A(x) = \mu^N_{A_2}(x) & \text{if } x \in V_2, \end{cases}$$

$$\text{(iii)} \begin{cases} \mu^P_B(xy) = \mu^P_{B_1}(xy) & \text{if } xy \in E_1 \\ \mu^P_B(xy) = \mu^P_{B_2}(xy) & \text{if } xy \in E_2, \end{cases}$$

$$\text{(iv)} \begin{cases} \mu^N_B(xy) = \mu^N_{B_1}(xy) & \text{if } xy \in E_1 \\ \mu^N_B(xy) = \mu^N_{B_2}(xy) & \text{if } xy \in E_2. \end{cases}$$

Let $x \in V_1$, $y \in V_2$, $\min(\mu^P_{A_1}(x), \mu^P_{A_2}(y)) = b$, $\max(\mu^N_{A_1}(x), \mu^N_{A_2}(y)) = a$, $\mu^P_B(xy) = d$ and $\mu^N_B(xy) = c$. Then $x \in (A_1)_{(a,b)}$, $y \in (A_2)_{(a,b)}$ and $xy \in B_{(c,d)}$. It follows that $xy \in B_{(a,b)}$, $x \in (A_1)_{(c,d)}$ and $y \in (A_2)_{(c,d)}$. So, $\mu^P_B(xy) \geq b$, $\mu^N_B(xy) \leq a$, $\mu^P_{A_1}(x) \geq d$, $\mu^N_{A_1}(x) \leq c$, $\mu^P_{A_2}(y) \geq d$ and $\mu^N_{A_2}(y) \leq c$. Therefore,

$$\mu^P_B(xy) \geq b = \min(\mu^P_{A_1}(x), \mu^P_{A_2}(y)) \geq d = \mu^P_B(xy),$$

$$\mu^N_B(xy) \leq a = \max(\mu^N_{A_1}(x), \mu^P_{A_2}(y)) \leq c = \mu^N_B(xy).$$

Thus,

$$\mu^P_B(xy) = \min(\mu^P_{A_1}(x), \mu^P_{A_2}(y)), \qquad \mu^N_B(xy) = \max(\mu^N_{A_1}(x), \mu^N_{A_2}(y)),$$

as desired. $\square$

Theorem 6. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be bipolar fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then $G = (A, B)$ is the cross product of G_1 and G_2 if and only if each (a, b)-level graph $(A_{(a,b)}, B_{(a,b)})$ is the cross product of $((A_1)_{(a,b)}, (B_1)_{(a,b)}$ and $((A_2)_{(a,b)}, (B_2)_{(a,b)})$.*

Proof. Let $G = (A, B)$ be the cross product of G_1 and G_2. By the definition of the Cartesian product $G_1 \times G_2$ and the proof of Theorem 2, we have $A_{(a,b)} = (A_1)_{(a,b)} \times (A_2)_{(a,b)}$ for all $(a, b) \in [-1, 0] \times [0, 1]$. We show that

$$B_{(a,b)} = \{(x_1, x_2)(y_1, y_2) \,|\, x_1y_1 \in (B_1)_{(a,b)}\,,\ x_2y_2 \in (B_2)_{(a,b)}\}$$

for all $(a, b) \in [-1, 0] \times [0, 1]$. Indeed, if $(x_1, x_2)(y_1, y_2) \in B_{(a,b)}$, then

$$\mu_B^P((x_1, x_2)(y_1, y_2)) = \min(\mu_{B_1}^P(x_1y_1), \mu_{B_2}^P(x_2y_2)) \geq b,$$

$$\mu_B^N((x_1, x_2)(y_1, y_2)) = \max(\mu_{B_1}^N(x_1y_1), \mu_{B_2}^P(x_2y_2)) \leq a,$$

hence $\mu_{B_1}^P(x_1y_1) \geq b$, $\mu_{B_2}^P(x_2y_2) \geq b$, $\mu_{B_1}^N(x_1y_1) \leq a$ and $\mu_{B_2}^P(x_2y_2) \leq a$. So, $x_1y_1 \in (B_1)_{(a,b)}$ and $x_2y_2 \in (B_2)_{(a,b)}$. Now if $x_1y_1 \in (B_1)_{(a,b)}$ and $x_2y_2 \in (B_2)_{(a,b)}$, then $\mu_{B_1}^P(x_1y_1) \geq b$, $\mu_{B_1}^N(x_1y_1) \leq a$, $\mu_{B_2}^P(x_2y_2) \geq b$ and $\mu_{B_2}^N(x_2y_2) \leq a$. It follows that

$$\mu_B^P((x_1, x_2)(y_1, y_2)) = \min(\mu_{B_1}^P(x_1y_1), \mu_{B_2}^P(x_2y_2)) \geq b,$$

$$\mu_B^N((x_1, x_2)(y_1, y_2)) = \max(\mu_{B_1}^N(x_1y_1), \mu_{B_2}^N(x_2y_2)) \leq a,$$

because $G = (A, B)$ is the cross product $G_1 * G_2$. Therefore, $(x_1, x_2)(y_1, y_2) \in B_{(a,b)}$.

Conversely, let each (a, b)-level graph $(A_{(a,b)}, B_{(a,b)})$ be the cross product of $((A_1)_{(a,b)}, (B_1)_{(a,b)})$ and $((A_2)_{(a,b)}, (B_2)_{(a,b)})$. In view of the fact that the cross product $(A_{(a,b)}, B_{(a,b)})$ has the same vertex set as the cartesian product of $((A_1)_{(a,b)}, (B_1)_{(a,b)})$ and $((A_2)_{(a,b)}, (B_2)_{(a,b)})$, and by the proof of Theorem 2, we have

$$\mu_A^P((x_1, x_2)) = \min(\mu_{A_1}^P(x_1), \mu_{A_2}^P(x_2)),$$

$$\mu_A^N((x_1, x_2)) = \max(\mu_{A_1}^N(x_1), \mu_{A_2}^N(x_2)),$$

for all $(x_1, x_2) \in V_1 \times V_2$.

Let $\min(\mu_{B_1}^P(x_1y_1), \mu_{B_2}^P(x_2y_2)) = b$, $\max(\mu_{B_1}^N(x_1y_1), \mu_{B_2}^N(x_2y_2)) = a$, $\mu_B^P((x_1, x_2)(y_1, y_2)) = d$ and $\mu_B^N((x_1, x_2)(y_1, y_2)) = c$ for $x_1y_1 \in E_1$, $x_2y_2 \in E_2$. Then $\mu_{B_1}^P(x_1y_1) \geq b$, $\mu_{B_2}^P(x_2y_2) \geq b$, $\mu_{B_1}^N(x_1y_1) \leq a$, $\mu_{B_2}^N(x_2y_2) \leq a$ and $(x_1, x_2)(y_1, y_2) \in B_{(c,d)}$, hence $x_1y_1 \in (B_1)_{(a,b)}$, $x_2y_2 \in (B_2)_{(a,b)}$, and consequently, $x_1y_1 \in (B_1)_{(c,d)}$, $x_2y_2 \in (B_2)_{(c,d)}$ since $B_{(c,d)} = \{(x_1, x_2)(y_1, y_2) \,|\, x_1y_1 \in (B_1)_{(c,d)}\,,\ x_2y_2 \in (B_2)_{(c,d)}\}$. It follows that $(x_1, x_2)(y_1, y_2) \in B_{(a,b)}$, $\mu_{B_1}^P(x_1y_1) \geq d$, $\mu_{B_1}^N(x_1y_1) \leq c$, $\mu_{B_2}^P(x_2y_2) \geq d$ and $\mu_{B_2}^N(x_2y_2) \leq c$. Therefore,

$$\mu_B^P((x_1, x_2)(y_1, y_2)) = d \leq \min(\mu_{B_1}^P(x_1y_1), \mu_{B_2}^P(x_2y_2)) = b \leq \mu_B^P((x_1, x_2)(y_1, y_2)),$$

$$\mu_B^N((x_1, x_2)(y_1, y_2)) = c \geq \max(\mu_{B_1}^N(x_1y_1), \mu_{B_2}^N(x_2y_2)) = a \geq \mu_B^N((x_1, x_2)(y_1, y_2)).$$

Hence

$$\mu_B^P((x_1,x_2)(y_1,y_2)) = \min(\mu_{B_1}^P(x_1y_1), \mu_{B_2}^P(x_2y_2)),$$

$$\mu_B^N((x_1,x_2)(y_1,y_2)) = \max(\mu_{B_1}^N(x_1y_1), \mu_{B_2}^N(x_2y_2)),$$

which completes our proof. □

Corollary 4. *The cross product of two bipolar fuzzy graphs is a bipolar fuzzy graph.*

Theorem 7. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be bipolar fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then $G = (A, B)$ is the lexicographic product of G_1 and G_2 if and only if $G_{(a,b)} = (G_1)_{(a,b)} \bullet (G_2)_{(a,b)}$ for all $(a, b) \in [-1, 0] \times [0, 1]$.*

Proof. Let $G = (A, B) = G_1 \bullet G_2$. By the definition of the Cartesian product $G_1 \times G_2$ and the proof of Theorem 2, we have $A_{(a,b)} = (A_1)_{(a,b)} \times (A_2)_{(a,b)}$ for all $(a, b) \in [-1, 0] \times [0, 1]$. We show that $B_{(a,b)} = E_{(a,b)} \cup E'_{(a,b)}$ for all $(a, b) \in [-1, 0] \times [0, 1]$, where $E_{(a,b)} = \{(x, x_2)(x, y_2) \,|\, x \in V_1 \,,\, x_2y_2 \in (B_2)_{(a,b)}\}$ is the subset the edge set of the direct product $(G_1)_{(a,b)} \times (G_2)_{(a,b)}$, and $E'_{(a,b)} = \{(x_1, x_2)(y_1, y_2) \,|\, x_1y_1 \in (B_1)_{(a,b)} \,,\, x_2y_2 \in (B_2)_{(a,b)}\}$ is the edge set of the cross product $(G_1)_{(a,b)} * (G_2)_{(a,b)}$. For every $(x_1, x_2)(y_1, y_2) \in B_{(a,b)}$, $x_1 = y_1$, $x_2y_2 \in E_2$ or $x_1y_1 \in E_1$, $x_2y_2 \in E_2$. If $x_1 = y_1$, $x_2y_2 \in E_2$, then $(x_1, x_2)(y_1, y_2) \in E_{(a,b)}$, by the definition of the Cartesian product and the proof of Theorem 2. If $x_1y_1 \in E_1$, $x_2y_2 \in E_2$, then $(x_1, x_2)(y_1, y_2) \in E'_{(a,b)}$, by the definition of the cross product and the proof of Theorem 6. Therefore, $B_{(a,b)} \subseteq E_{(a,b)} \cup E'_{(a,b)}$. From the definition of the Cartesian product and the proof of Theorem 2, we conclude that $E_{(a,b)} \subseteq B_{(a,b)}$, and also from the definition of the cross product and the proof of Theorem 6, we obtain $E'_{(a,b)} \subseteq B_{(a,b)}$. Therefore, $E_{(a,b)} \cup E'_{(a,b)} \subseteq B_{(a,b)}$.

Conversely, let $G_{(a,b)} = (A_{(a,b)}, B_{(a,b)}) = (G_1)_{(a,b)} \bullet (G_2)_{(a,b)}$ for all $(a, b) \in [-1, 0] \times [0, 1]$. We know that $(G_1)_{(a,b)} \bullet (G_2)_{(a,b)}$ has the same vertex set as the Cartesian product $(G_1)_{(a,b)} \times (G_2)_{(a,b)}$. Now by the proof of Theorem 2, we have

$$\mu_A^P((x_1, x_2)) = \min(\mu_{A_1}^P(x_1), \mu_{A_2}^P(x_2)),$$

$$\mu_A^N((x_1, x_2)) = \max(\mu_{A_1}^N(x_1), \mu_{A_2}^N(x_2))$$

for all $(x_1, x_2) \in V_1 \times V_2$.

Assume that for some $x \in V_1$ and $x_2y_2 \in E_2$ is $\min(\mu_{A_1}^P(x), \mu_{B_2}^P(x_2y_2)) = b$, $\max(\mu_{A_1}^N(x), \mu_{B_2}^N(x_2y_2)) = a$, $\mu_B^P((x, x_2)(x, y_2)) = d$ and $\mu_B^N((x, x_2)(x, y_2)) = c$. Then, in view of the definitions of the Cartesian and lexicographic products, we have

$$(x, x_2)(x, y_2) \in (B_1)_{(a,b)} \bullet (B_2)_{(a,b)} \Leftrightarrow (x, x_2)(x, y_2) \in (B_1)_{(a,b)} \times (B_2)_{(a,b)},$$

$$(x, x_2)(x, y_2) \in (B_1)_{(c,d)} \bullet (B_2)_{(c,d)} \Leftrightarrow (x, x_2)(x, y_2) \in (B_1)_{(c,d)} \times (B_2)_{(c,d)}.$$

From this, by the same argument as in the proof of Theorem 2, we can conclude

$$\mu_B^P((x,x_2)(x,y_2)) = \min(\mu_A^P(x), \mu_{B_2}^P(x_2y_2)),$$

$$\mu_B^N((x,x_2)(x,y_2)) = \max(\mu_A^N(x), \mu_{B_2}^N(x_2y_2)).$$

Suppose now that we have $\mu_B^P((x_1,x_2)(y_1,y_2)) = d$, $\mu_B^N((x_1,x_2)(y_1,y_2)) = c$, $\min(\mu_{B_1}^P(x_1y_1), \mu_{B_2}^P(x_2y_2)) = b$, $\max(\mu_{B_1}^N(x_1y_1), \mu_{B_2}^N(x_2y_2)) = a$ for $x_1y_1 \in E_1$ and $x_2y_2 \in E_2$. Then, in view of the definitions of the cross product and the lexicographic product, we have

$$(x_1,x_2)(y_1,y_2) \in (B_1)_{(a,b)} \bullet (B_2)_{(a,b)} \Leftrightarrow (x_1,x_2)(y_1,y_2) \in (B_1)_{(a,b)} * (B_2)_{(a,b)},$$

$$(x_1,x_2)(y_1,y_2) \in (B_1)_{(c,d)} \bullet (B_2)_{(c,d)} \Leftrightarrow (x_1,x_2)(y_1,y_2) \in (B_1)_{(c,d)} * (B_2)_{(c,d)}.$$

By the same argument as in the proof of Theorem 6, we can conclude

$$\mu_B^P((x_1,x_2)(y_1,y_2)) = \min(\mu_{B_1}^P(x_1y_1), \mu_{B_2}^P(x_2y_2)),$$

$$\mu_B^N((x_1,x_2)(y_1,y_2)) = \max(\mu_{B_1}^N(x_1y_1), \mu_{B_2}^N(x_2y_2)),$$

which completes the proof. □

Corollary 5. *The lexicographic product of two bipolar fuzzy graphs is a bipolar fuzzy graph.*

Lemma 1. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be bipolar fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, such that $V_1 = V_2$, $A_1 = A_2$ and $E_1 \cap E_2 = \varnothing$. Then $G = (A, B)$ is the union of G_1 and G_2 if and only if $(A_{(a,b)}, B_{(a,b)})$ is the union of $((A_1)_{(a,b)}, (B_1)_{(a,b)})$ and $((A_2)_{(a,b)}, (B_2)_{(a,b)})$ for all $(a,b) \in [-1,0] \times [0,1]$.*

Proof. Let $G = (A, B)$ be the union of bipolar fuzzy graphs G_1 and G_2. Then by the definition of the union and the fact that $V_1 = V_2$, $A_1 = A_2$, we have $A = A_1 = A_2$, hence $A_{(a,b)} = (A_1)_{(a,b)} \cup (A_2)_{(a,b)}$. We now show that $B_{(a,b)} = (B_1)_{(a,b)} \cup (B_2)_{(a,b)}$ for all $(a,b) \in [-1,0] \times [0,1]$. For every $xy \in (B_1)_{(a,b)}$ we have $\mu_B^P(xy) = \mu_{B_1}^P(xy) \geq b$ and $\mu_B^N(xy) = \mu_{B_1}^N(xy) \leq a$, hence $xy \in B_{(a,b)}$. Therefore, $(B_1)_{(a,b)} \subseteq B_{(a,b)}$. Similarly, we obtain $(B_2)_{(a,b)} \subseteq B_{(a,b)}$. Thus, $(B_1)_{(a,b)} \cup (B_2)_{(a,b)} \subseteq B_{(a,b)}$. For every $xy \in B_{(a,b)}$ either $xy \in E_1$ or $xy \in E_2$. If $xy \in E_1$, $\mu_{B_1}^P(xy) = \mu_B^P(xy) \geq b$ and $\mu_{B_1}^N(xy) = \mu_B^N(xy) \leq a$ and hence $xy \in (B_1)_{(a,b)}$. If $xy \in E_2$, we have $xy \in (B_2)_{(a,b)}$. Therefore, $B_{(a,b)} \subseteq (B_1)_{(a,b)} \cup (B_2)_{(a,b)}$.

Conversely, suppose that the (a,b)-level graph $(A_{(a,b)}, B_{(a,b)})$ be the union of $((A_1)_{(a,b)}, (B_1)_{(a,b)})$ and $((A_2)_{(a,b)}, (B_2)_{(a,b)})$. Let $\mu_A^P(x) = b$, $\mu_A^N(x) = a$, $\mu_{A_1}^P(x) = d$ and $\mu_{A_1}^N(x) = c$ for some $x \in V_1 = V_2$. Then $x \in A_{(a,b)}$ and $x \in (A_1)_{(c,d)}$, so $x \in (A_1)_{(a,b)}$ and $x \in A_{(c,d)}$, because $A_{(a,b)} = (A_1)_{(a,b)}$ and $A_{(c,d)} = (A_1)_{(c,d)}$. It follows that $\mu_{A_1}^P(x) \geq b$, $\mu_{A_1}^N(x) \leq a$, $\mu_A^P(x) \geq d$ and $\mu_A^N(x) \leq c$. Therefore, $\mu_{A_1}^P(x) \geq \mu_A^P(x)$, $\mu_{A_1}^N(x) \leq \mu_A^N(x)$, $\mu_A^P(x) \geq \mu_{A_1}^P(x)$ and $\mu_A^N(x) \leq \mu_{A_1}^N(x)$. So, $\mu_A^P(x) = \mu_{A_1}^P(x)$ and $\mu_A^N(x) = \mu_{A_1}^N(x)$. Since $A_1 = A_2$, $V_1 = V_2$, then $A = A_1 = A_1 \cup A_2$.

By a similar method, we conclude that

(i) $\begin{cases} \mu_B^P(xy) = \mu_{B_1}^P(xy) & \text{if } xy \in E_1 \\ \mu_B^P(xy) = \mu_{B_2}^P(xy) & \text{if } xy \in E_2, \end{cases}$

(ii) $\begin{cases} \mu_B^N(xy) = \mu_{B_1}^N(xy) & \text{if } xy \in E_1 \\ \mu_B^N(xy) = \mu_{B_2}^N(xy) & \text{if } xy \in E_2. \end{cases}$

This completes the proof. □

Theorem 8. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be bipolar fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then $G = (A, B)$ is the strong product of G_1 and G_2 if and only if each $G_{(a,b)}$, where $(a, b) \in [-1, 0] \times [0, 1]$, is the strong product of $(G_1)_{(a,b)}$ and $(G_2)_{(a,b)}$.*

Proof. According to the definitions of the strong product, the cross product and the Cartesian product, we obtain $G_1 \boxtimes G_2 = (G_1 \times G_2) \cup (G_1 * G_2)$ and

$$(G_1)_{(a,b)} \boxtimes (G_2)_{(a,b)} = ((G_1)_{(a,b)} \times (G_2)_{(a,b)}) \cup ((G_1)_{(a,b)} * (G_2)_{(a,b)})$$

for all $(a, b) \in [-1, 0] \times [0, 1]$. Now by Theorem 6, Theorem 2 and Lemma 1, we see that

$$\begin{aligned} G = G_1 \boxtimes G_2 &\Longleftrightarrow G = (G_1 \times G_2) \cup (G_1 * G_2) \\ &\Longleftrightarrow G_{(a,b)} = (G_1 \times G_2)_{(a,b)} \cup (G_1 * G_2)_{(a,b)} \\ &\Longleftrightarrow G_{(a,b)} = ((G_1)_{(a,b)} \times (G_2)_{(a,b)}) \cup ((G_1)_{(a,b)} * (G_2)_{(a,b)}) \\ &\Longleftrightarrow G_{(a,b)} = (G_1)_{(a,b)} \boxtimes (G_2)_{(a,b)} \end{aligned}$$

for all $(a, b) \in [-1, 0] \times [0, 1]$. □

Corollary 6. *The strong product of two bipolar fuzzy graphs is a bipolar fuzzy graph.*

4 Conclusion

Graph theory is one of the branches of modern mathematics applied to many areas of mathematics, science, and technology. In computer science, graphs are used to represent networks of communication, computational devices, image segmentation, clustering and the flow of computation. In many cases, some aspects of a graph theoretic problem may be uncertain, and we deal with bipolar information. Bipolarity is met in many areas such as knowledge representation, reasoning with conditions, inconsistency handling, constraint satisfaction problem, decision, learning, etc. In this paper, we define the notion of level graph of a bipolar fuzzy graph and investigate some of their properties. We define three kinds of new operations of bipolar fuzzy graphs and discuss these operations and some defined important operations on bipolar fuzzy graphs by characterizing these operations by their level counterparts graphs.

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W. A. Dudek
Faculty of Pure and Applied Mathematics
Wroclaw University of Science and Technology
50-370 Wroclaw, Poland
E-mail: *wieslaw.dudek@pwr.edu.pl*

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A. A. Talebi
Department of Mathematics
Faculty of Mathematical Sciences
University of Mazandaran, Babolsar, Iran
E-mail: *a.talebi@umz.ac.ir*

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On pseudoisomorphy and distributivity of quasigroups

Fedir M. Sokhatsky

Abstract. A repeated bijection in an isotopism of quasigroups is called a *companion* of the third component. The last is called a *pseudoisomorphism* with the companion. Isotopy coincides with pseudoisomorphy* in the class of inverse property loops and with isomorphy in the class of commutative inverse property loops. This result is a generalization of the corresponding theorem for commutative Moufang loops. A notion of middle distributivity is introduced: a quasigroup is *middle distributive* if all its middle translations are automorphisms. In every quasigroup two identities of distributivity (left, right and middle) imply the third. This fact and some others help us to find a short proof of a theorem which gives necessary and sufficient conditions for a quasigroup to be distributive. There is but a slight difference between this theorem and the well-known Belousov's theorem.

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This article is dedicated to the memory of my dear teacher professor Valentin Danilovich Belousov

Introduction

V. D. Belousov's monograph [1] was published almost 50 years ago and became very popular among mathematicians. It is still a desk book for many algebraists.

The growth of applications of the quasigroup theory in information processing, and expansion of research methods by computer tools and nascence of computer algebra have increased the need to form a coherent quasigroup theory. The author hopes that the proposed article will promote the development of this theory.

Here, a different approach to the proof of Belousov's theorem is suggested. Due to this approach, it became possible to significantly simplify the proof of the theorem and all related statements. The article is self-contained, i. e., it includes all the necessary properties with proofs despite the fact that some of them are well known and can be found in [1], [2]. A historical overview of the results of distributive quasigroups is not discussed here because it has already been done in [4].

In the first part of the paper, some properties of loop isotopy are established and they are applied in the second part. The importance of study of isotopy relation in quasigroup theory is explained by the following fact: each homotopism of

*isotopy, pseudoisomorphy, isomorphy denote relation among groupoids and isotopism, psuedoisomorphism, isomorphism are the corresponding sequence of bijections quasigroups can be represented as a composition of isotopisms from quasigroups to loops and homomorphisms of loops. V. D. Belousov [1] has proposed a programm of development of the quasigroup theory in problems, which are mainly related to the study of isotopy.

Isotopisms with two coinciding components are proposed to be considered. The repeated bijection is called a *companion* of the third component. The third component is called a *pseudoisomorphism*. This notion is a generalization of the notion of pseudoautomorphism, its companions are bijections, but not elements. The following fact shows the importance of the concept: isotopy coincides with pseudoisomorphy for inverse property loops (Corollary 3). It is easy to deduce that isotopy coincides with isomorphy for commutative inverse property loops (Corollary 5). This result is a generalization of the corresponding theorem for commutative Moufang loops [1, Theorem 6.7], [2, Theorem IV.5.6].

Questions about the relations between different types of isotopy arise. For example, when are pseudoisomorphic quasigroups isomorphic? A partial answer is given in Theorem 1: pseudoisomorphic commutative loops with coinciding nuclei are isomorphic. Or what properties are invariant under pseudoisomorphy? Etc.

It is suggested to consider also the middle distributivity identity, defining it in the similar way as the identities of the left and right distributivity: a quasigroup is *middle distributive* if all its middle translations are automorphisms of the quasigroup. It is proved that in every quasigroup two identities of distributivity imply the third (Theorem 9). Therefore, any distributive quasigroup satisfies left, right and middle distributive identities. This fact and some others help us to give a short proof of Theorem 3, which gives necessary and sufficient conditions for a quasigroup to be distributive. There is but a slight difference between this theorem and the well-known Belousov's theorem (Corollary 11).

The theorem implies that every distributive quasigroup is defined over some commutative Moufang loop by an automorphism of the loop which satisfies (16). This identity is equivalent to all identities of distributivity in the loop. Finally, it is proved that any two automorphisms defining distributive quasigroups over the same commutative Moufang loop 1) differ in a central endomorphism of the loop (Corollary 13); 2) define isomorphic distributive quasigroups if and only if they are conjugate by an automorphism of the loop (Corollary 14).

1 Preliminaries

Let Q be an arbitrary set and $(\cdot)$ be an invertible operation defined on Q, then the pair $(Q;\cdot)$ is called a *quasigroup. Invertibility* means that for arbitrary $a, b \in Q$ each of the equations $x \cdot a = b$ and $a \cdot y = b$ is uniquely solvable in Q.

A τ-parastrophe $(Q;\overset{\tau}{\cdot})$ of a quasigroup $(Q;\cdot)$ is defined by

$$x_{1\tau} \overset{\tau}{\cdot} x_{2\tau} = x_{3\tau} :\Leftrightarrow\ x_1 \cdot x_2 = x_3$$

for every $\tau \in S_3 := \{\iota, \ell, r, s, s\ell, sr\}$, where $s := (12)$, $\ell := (13)$, $r := (23)$. Special notation: $(*) := (\overset{s}{\cdot})$, $(\backslash) := (\overset{r}{\cdot})$, $(/) := (\overset{\ell}{\cdot})$. All parastrophes can be defined by identities. Some of them are the following

$$(x \cdot y)/y = x, \quad (x/y) \cdot y = x, \quad x\backslash(x \cdot y) = y, \quad x \cdot (x\backslash y) = y. \tag{1}$$

A *left* $L_{a,\tau}$, *right* $R_{a,\tau}$ and *middle* $M_{a,\tau}$ translations of the quasigroup $(Q; \overset{\tau}{\cdot})$ are defined by

$$L_{a,\tau}(x) := a \overset{\tau}{\cdot} x, \quad R_{a,\tau}(x) := x \overset{\tau}{\cdot} a, \quad M_{a,\tau}(x) = y :\Leftrightarrow x \overset{\tau}{\cdot} y = a \tag{2}$$

for any $a \in Q$ and $\tau \in S_3$. As usual, the translations $L_{a,\iota}$, $R_{a,\iota}$, $M_{a,\iota}$ are denoted by L_a, R_a, M_a respectively. In general, there are six parastrophes of a quasigroup. The set of all their translations consists of the following six transformations:

$$\begin{aligned} &L_a(x) = a \cdot x = a \overset{\iota}{\cdot} x, \quad R_a(x) = x \cdot a = x \overset{\iota}{\cdot} a, \quad M_a(x) = x\backslash a = x \overset{r}{\cdot} a, \\ &L_a^{-1}(x) = a\backslash x = a \overset{r}{\cdot} x, \quad R_a^{-1}(x) = x/a = x \overset{\ell}{\cdot} a, \quad M_a^{-1}(x) = a/x = a \overset{\ell}{\cdot} x. \end{aligned} \tag{3}$$

The relations among translations of parastrophic operations are easily verifiable (see, for example [3]) and can be expressed in the following table:

$\ddots\ \tau$	ι	s	ℓ	r	$s\ell$	sr
$L_{a,\tau}$	L_a	R_a	M_a^{-1}	L_a^{-1}	R_a^{-1}	M_a
$R_{a,\tau}$	R_a	L_a	R_a^{-1}	M_a	M_a^{-1}	L_a^{-1}
$M_{a,\tau}$	M_a	M_a^{-1}	L_a^{-1}	R_a	L_a	R_a^{-1}

(4)

A triplet (α, β, γ) of mappings from a set Q_o into a set Q is called a *homotopism* of a groupoid $(Q_o; \circ)$ into a groupoid $(Q; \cdot)$ if

$$\gamma(x \circ y) = \alpha x \cdot \beta y$$

holds for all $x, y \in Q_o$. A homotopism (α, β, γ) is called an *isotopism* if α, β, γ are bijections. If in addition $Q_o = Q$ and $(\cdot) = (\circ)$, then it is an *autotopism* of $(Q; \cdot)$.

A triplet $(Q; \cdot, e)$ is called a *loop* if $(Q; \cdot)$ is a quasigroup and e is its *neutral element*, i.e., $e \cdot x = x \cdot e = x$ holds for all $x \in Q$.

Left, *right* and *middle nuclei* of a loop $(Q; \cdot, e)$ are defined by

$$\begin{aligned} N_\ell^{(\cdot)} &:= \{a \mid ax \cdot y = a \cdot xy\} = \{a \mid (L_a, \iota, L_a) \text{ is an autotopism of } (Q; \cdot, e)\}, \\ N_r^{(\cdot)} &:= \{a \mid x \cdot ya = xy \cdot a\} = \{a \mid (\iota, R_a, R_a) \text{ is an autotopism of } (Q; \cdot, e)\}, \\ N_m^{(\cdot)} &:= \{a \mid xa \cdot y = x \cdot ay\} = \{a \mid (R_a^{-1}, L_a, \iota) \text{ is an autotopism of } (Q; \cdot, e)\}. \end{aligned} \tag{5}$$

An element of a loop is called *central* if it commutes and associates with all elements of the loop. In other words, c is central if

$$c \in N_\ell^{(\cdot)} \cap N_r^{(\cdot)} \cap N_m^{(\cdot)} \cap \{a \mid ax = xa\}.$$

An element a of a loop $(Q;\cdot,e)$ is called a *Moufang* element if there exists a bijection λ of Q such that $(L_a;R_a;\lambda)$ is an autotopism of the loop, i.e.,

$$ay\cdot za=\lambda(y\cdot z) \tag{6}$$

for all $y,z\in Q$. Remark that if we put $y=e$, thereafter $z=e$, we obtain $\lambda=L_aR_a=R_aL_a$. A loop is called a *Moufang* loop if its every element is Moufang, i.e. if one of the identities

$$xy\cdot zx=x(y\cdot z)\cdot x,\qquad xy\cdot zx=x\cdot(y\cdot z)x \tag{7}$$

hold.

2 Pseudoisomorphy

Let $(Q_o;\circ)$ and $(Q;\cdot)$ be groupoids, $\alpha,\beta:\ Q_o\to Q$ be bijections, then α will be called

- a *left pseudoisomorphism* if (β,α,β) is an isotopism of the groupoids;
- a *right pseudoisomorphism* if (α,β,β) is an isotopism of the groupoids;
- a *middle pseudoisomorphism* if (β,β,α) is an isotopism of the groupoids;
- a *pseudoisomorphism* if it is both left and right pseudoisomorphism.

In these cases, the bijection β will be called a *companion* of the corresponding pseudoisomorphism. If $\alpha=\beta$ the pseudoisomorphism is an isomorphism.

It is easy to see that the set of all left (right and middle) pseudoautomorphisms of a quasigroup as well as their corresponding companions forms groups Ψ_ℓ, Ψ_ℓ^* (Ψ_r, Ψ_r^* and Ψ_m, Ψ_m^* respectively).

Relationships between pseudoisomorphy and neutrality are given in the following proposition.

Proposition 1. *Let $(Q;\cdot)$ be a quasigroup and θ be its*

1) *left pseudoautomorphism with a companion β, then $(Q;\cdot)$ has a left neutral element if and only if $\beta=L_a\theta$ for some element $a\in Q$;*

2) *right pseudoautomorphism with a companion β, then $(Q;\cdot)$ has a right neutral element if and only if $\beta=R_b\theta$ for some element $b\in Q$;*

3) *middle pseudoautomorphism with a companion β, then $(Q;\cdot)$ has a neutral element if and only if $\beta=L_c^{-1}\theta$ for some element $c\in Q$ such that $xc=cx$ for all $x\in Q$.*

Proof. Let θ be a left pseudoautomorphism of a quasigroup $(Q;\cdot)$ with a companion β, then $(\beta;\theta;\beta)$ is an autotopism of $(Q;\cdot)$, i.e.,

$$\beta x\cdot\theta y=\beta(x\cdot y)$$

Putting $x:=e$ and $a:=\beta e$, where e denotes the left neutral element of $(Q;\cdot)$, we obtain $\beta=L_a\theta$. Conversely, let the previous equality be true for some $a\in Q$, i.e.,

$$(a\cdot\theta x)\cdot\theta y=a\cdot\theta(x\cdot y).$$

Substituting $x=e:=\theta^{-1}L_a^{-1}a$, we obtain

$$(a\cdot L_a^{-1}a)\cdot\theta y=a\cdot\theta(e\cdot y),\quad \text{i.e.}\quad a\cdot\theta y=a\cdot\theta(e\cdot y).$$

Cancelling out, we have $y=e\cdot y$. The item 2) can be proved analogously.

To prove 3) suppose that $(\beta;\beta;\theta)$ is an autotopism of $(Q;\cdot)$ and let e denote its neutral element, i.e.,

$$\beta x\cdot\beta y=\theta(x\cdot y)$$

for all $x,y\in Q$. When $x=e$ and $y=e$ the equality implies $L_c\beta=\theta$ and $R_c\beta=\theta$ respectively, where $c:=\beta e$, so $xc=cx$ for all $x\in Q$. Conversely, since $(L_c^{-1}\theta,L_c^{-1}\theta,\theta)$ is an autotopism of $(Q;\cdot)$, then

$$L_c^{-1}\theta x\cdot L_c^{-1}\theta y=\theta(x\cdot y)$$

holds. As c commutes with all elements of Q, i.e. $L_c=R_c$, it is easy to verify that $e:=\theta^{-1}L_c(c)$ is a neutral element in $(Q;\cdot)$ replacing successively x and y with e in the centralized formula. □

Note. Proposition 1 implies that for loops the introduced concept of pseudoautomorphism coincides with the well-known notion, except the notion of companion. A companion is a bijection in the definition given here, and an element in the well-known notion, but both of them uniquely define each other. Indeed, let a bijection β be a companion of θ, then

$$\beta x\cdot\theta y=\beta(x\cdot y)\quad\text{or}\quad\theta y\cdot\beta x=\beta(y\cdot y)$$

holds. Let e denote the neutral element of the loop and let $x:=e$, we obtain $L_{\beta e}\theta=\beta$ or $R_{\beta e}\theta=\beta$. In both cases βe is a companion element of the pseudoautomorphism θ. Conversely, if an element c is a companion of θ, then the bijection $L_a\theta$ is its companion, in the case when θ is a left pseudoautomorphism; and $R_a\theta$ is its companion if θ is a right pseudoautomorphism. We will use both companions: an element and a bijection, but companion-element does not exist in the case when the quasigroup has no left and no right neutral elements.

2.1 Isotopism of loops

Some relations between isotopy and pseudoisomorphy for loops are given in the following lemma.

Lemma 1. *Let $(\alpha;\beta;\gamma)$ be an arbitrary isotopism of a loop $(Q_o;\circ,e)$ on a quasigroup $(Q;\cdot)$ and let $a:=\alpha e$, $b:=\beta e$. Then the following statements are true.*

1. $\alpha=R_b^{-1}\gamma$, $\beta=L_a^{-1}\gamma$;
2. *β is a left pseudoisomorphism, i.e. $\alpha=\gamma$ if and only if b is a right neutral element in $(Q;\cdot)$;*
3. *If β is a left pseudoisomorphism, then*
   (a) *the loops $(Q;\circ)$ and $(Q;\odot)$ are isomorphic, where $x\odot y:=L_a^{-1}(ax\cdot y)$, i.e., $(\alpha;\beta;\gamma)=(L_a\beta;\beta;L_a\beta)$,*
   (b) *β is an isomorphism of $(Q;\circ)$ and $(Q;\cdot)$ if and only if $a\in N_\ell^{(\cdot)}$;*
4. *α is a right pseudoisomorphism, i.e. $\beta=\gamma$ if and only if a is a left neutral element in $(Q;\cdot)$;*
5. *If α is a right pseudoisomorphism, then*
   (a) *the loops $(Q;\circ)$ and $(Q;\bullet)$ are isomorphic, where $x\bullet y:=R_b^{-1}(x\cdot yb)$, i.e., $(\alpha;\beta;\gamma)=(\alpha;R_b\alpha;R_b\alpha)$;*
   (b) *α is an isomorphism of the quasigroups $(Q;\circ)$ and $(Q;\cdot)$ if and only if $b\in N_r^{(\cdot)}$;*
6. *γ is a middle pseudoisomorphism, i.e. $\alpha=\beta$, if and only if $a:=\alpha e=\beta e$ and $a\cdot x=x\cdot a$ for all $x\in Q$.*
7. *If γ is a middle pseudoisomorphism, then*
   (a) *the loops $(Q;\circ)$ and $(Q;\star)$ are isomorphic, where $x\star y:=L_a^{-1}x\cdot L_a^{-1}y$, i.e., $(\alpha;\beta;\gamma)=(L_a^{-1}\gamma;L_a^{-1}\gamma;\gamma)$;*
   (b) *γ is an isomorphism between $(Q;\circ)$ and $(Q;\cdot)$ if and only if $a\in N_m^{(\cdot)}$ and $a\cdot a$ is a neutral element of the quasigroup $(Q;\cdot)$;*
8. *$\alpha=\beta=\gamma$ is an isomorphism if and only if $\alpha e=\beta e$ is a neutral element of the quasigroup $(Q;\cdot)$.*

Proof. The condition of the lemma means the truth of the equality

$$\gamma(x\circ y)=\alpha x\cdot\beta y \tag{8}$$

for all $x,y\in Q$. We successively put $x:=e$, $y:=e$ and obtain

$$\gamma y=\alpha(e)\cdot\beta y=a\cdot\beta y,\qquad \gamma x=\alpha x\cdot\beta(e)=\alpha x\cdot b.$$

Herefrom $\beta = L_a^{-1}\gamma$ and $\alpha = R_b^{-1}\gamma$, that is why the items 1, 2 are obvious.

Now suppose that β is a left pseudoisomorphism, i.e., $\alpha = \gamma$. But $\alpha = R_b^{-1}\gamma$, so $R_b = \iota$, then the equality (8) can be written as follows

$$L_a\beta x \cdot \beta y = L_a\beta(x \circ y). \tag{9}$$

Applying L_a^{-1} to the equality and replacing x with $\beta^{-1}x$, y with $\beta^{-1}y$, we obtain

$$L_a^{-1}(L_a x \cdot y) = \beta(\beta^{-1}x \circ \beta^{-1}y). \tag{10}$$

So, β is an isomorphism between $(Q;\circ)$ and $(Q;\odot)$.

If β is an isomorphism of $(Q;\circ)$ and $(Q;\cdot)$, then (10) implies

$$L_a^{-1}(L_a x \cdot y) = x \cdot y.$$

It means that $a \in N_\ell^{(\cdot)}$.

Thus, items 3a, 3b have been proved. The other statements of the lemma can be proved in the same way. $\square$

This lemma immediately implies the following corollary.

Corollary 1. *Let $(\alpha;\beta;\gamma)$ be an isotopism of a loop $(Q;\circ;e_1)$ on a loop $(Q;\cdot,e)$, then*

- *β is a left pseudoisomorphism if and only if $\beta e_1 = e$;*
- *α is a right pseudoisomorphism if and only if $\alpha e_1 = e$;*
- *γ is a middle pseudoisomorphism if and only if $a := \alpha e_1 = \beta e_1$ and $ax = xa$ for all $x \in Q$;*
- *γ is an isomorphism if and only if $\alpha e_1 = \beta e_1 = e$.*

Lemma 2. *Let θ be a left (or right) pseudoisomorphism with a companion c of a commutative loop $(Q;\oplus)$ on a commutative loop $(Q;+)$ with coinciding neclei, then θ is an isomorphism and c is a central element in the loop $(Q;+)$.*

Proof. Conditions of the lemma imply that

$$(c + \theta x) + \theta y = c + \theta(x \oplus y) \tag{11}$$

is true for all $x, y \in Q$. Using commutativity of both operations, we obtain

$$\theta y + (c + \theta x) = c + \theta(y \oplus x).$$

Mutually relabeling x and y, we have

$$\theta x + (c + \theta y) = c + \theta(x \oplus y).$$

So, the left sides of this equality and (11) are equal:

$$(c+\theta x)+\theta y=\theta x+(c+\theta y).$$

It means that c belongs to the middle nucleus of $(Q;+)$. But, according to the lemma's condition, the middle nucleus coincides with the center of the loop. Therefore, we can cancel out c in (11) and conclude that the pseudoisomorphism θ is an isomorphism of these loops. $\square$

This lemma immediately implies the following theorem.

Theorem 1. *Pseudoisomorphic commutative loops with coinciding nuclei are isomorphic.*

3 Inverse property loops

Inverse property loop (briefly IP-loop) is a loop $(Q;\cdot,e)$ that has a transformation I of Q such that

$$Ix\cdot(x\cdot y)=y,\qquad (x\cdot y)\cdot Iy=x$$

for all x, $y\in Q$. It is easy to verify that $Ix=x^{-1}$, $I^{-1}=I$ and $x\cdot x^{-1}=x^{-1}\cdot x=e$.

IP-loop $(Q;\cdot)$ with a neutral element e and unary operation $I(x):=x^{-1}$ will be denoted by $(Q;\cdot,I,e)$.

Lemma 3. *Let $(\alpha;\beta;\gamma)$ be an isotopism of an IP-loop $(Q;\circ,I_1,e_1)$ on an IP-loop $(Q;\cdot,I,e)$, then both the triplets $(I\alpha I_1;\gamma;\beta)$ and $(\gamma;I\beta I_1;\alpha)$ are isotopisms of the same loops.*

Proof. The conditions of the lemma imply the equality $\alpha x\cdot\beta y=\gamma(x\circ y)$. We put here successively $y:=I_1x\circ u$ and $x=v\circ I_1y$:

$$\alpha x\cdot\beta(I_1x\circ u)=\gamma u,\qquad \alpha(v\circ I_1y)\cdot\beta y=\gamma v.$$

In the first equality, we replace x with I_1t, in the second one y with I_1z:

$$\beta(t\circ u)=I\alpha I_1t\cdot\gamma u,\quad \alpha(v\circ z)=\gamma v\cdot I\beta I_1z.$$

Thus, $(I\alpha I_1;\gamma;\beta)$ and $(\gamma;I\beta I_1;\alpha)$ are isotopisms of $(Q;\circ,I_1,e_1)$ on $(Q;\cdot,I,e)$. $\square$

Corollary 2. *Nuclei of an inverse property loop coincide.*

Proof. Let $(Q;\cdot,I,e)$ be an IP-loop. Belonging of an element a to the left nucleus $N_\ell^{(\cdot)}$ of the loop means that the triplet $(L_a;L_a;\iota)$ is an autotopism of $(Q;\cdot,I,e)$. Lemma 3 implies that both

$$(IL_aI;L_a;\iota)\quad\text{and}\quad(\iota;IL_aI;IL_aI)^{-1}$$

are its autotopisms. Using the equality $IL_aI = R_a^{-1}$, we conclude that both

$$(R_a^{-1}; L_a; \iota) \quad \text{and} \quad (\iota; R_a; R_a)$$

are autotopisms. So, an arbitrary element $a \in Q$ belongs to the left and middle nucleus as well as to the left and right nucleus simultaneously, i.e., the nuclei coincide. □

Lemma 4. *The sets of all left and right pseudoisomorphisms between inverse property loops coincide. If α is a pseudoisomorphism of an inverse property loop $(Q; \circ, I_1, e_1)$ on an inverse property loop $(Q; \cdot, I, e)$, then $\alpha e_1 = e$; $I\alpha = \alpha I_1$.*

Proof. Let $(\alpha; \beta; \beta)$ be an isotopism of an IP-loop $(Q; \circ)$ on an IP-loop $(Q; \cdot)$. Applying Lemma 3, we conclude that $(I\beta I_1; \alpha; I\beta I_1)$ and $(I\alpha I_1, \beta, \beta)$ are isotopisms of these loops. So, α is a left pseudoisomorphism of these loops. Since any two components of an isotopism of quasigroups uniquely define the third, then $I\alpha I_1 = \alpha$, i.e., $I\alpha = \alpha I_1$. □

Theorem 2. *Let $T := (\alpha; \beta; \gamma)$ be an isotopism of an inverse property loop $(Q_1; \circ, e_1)$ on an inverse property loop $(Q; \cdot; e)$ and let $a := \alpha(e_1)$, $b := \beta(e_1)$, then:*

1. *$\theta := L_a^{-1}\alpha$ is a pseudoisomorphism of $(Q_1; \circ, I_1, e_1)$ on $(Q; \cdot; I, e)$ with the right companion $c := b \cdot a^{-1}$;*

2. *the elements a, b, $a \cdot b$ are Moufang;*

3. *$(\alpha; \beta; \gamma) = (L_a; R_a; L_aR_a)(\theta; R_c\theta; R_c\theta)$.*

Proof. Lemma 1 and Lemma 3 imply that $\alpha = R_b^{-1}\gamma$, $\beta = L_a^{-1}\gamma$ and the triplet $T_1 := (I\alpha I_1; \gamma; \beta)$ is an isotopism of these loops. Hence, the triplet

$$TT_1^{-1} = (R_b^{-1}\gamma; L_a^{-1}\gamma; \gamma)(I_1\gamma^{-1}R_bI; \gamma^{-1}; \gamma^{-1}L_a) = (\lambda; L_a^{-1}; L_a)$$

is an autotopism of $(Q; \cdot, e)$ for some bijection λ of the set Q. According to Lemma 3,

$$T_2 := (L_a; IL_a^{-1}I; \lambda) = (L_a; R_a; \lambda)$$

is an autotopism of $(Q; \cdot, e)$. So, a is Moufang in $(Q; \cdot, e)$ and $\lambda = L_aR_a = R_aL_a$.

Lemma 3 implies that $(\gamma; I\beta I_1; \alpha)$ and $(\beta; I\gamma I_1; I\alpha I_1)$ are autotopisms, consequently, the elements $ab = \alpha(e_1) \cdot \beta(e_1) = \gamma(e_1 \circ e_1) = \gamma(e_1)$ and $b = \beta(e_1)$ are Moufang too. Hence, the item 2. has been proved.

Then $T_2^{-1}T$ is an isotopism of $(Q_1; \circ, I_1, e_1)$ on $(Q; \cdot, I, e)$ and

$$T_2^{-1}T = (L_a^{-1}\alpha; R_a^{-1}\beta; L_a^{-1}R_a^{-1}\gamma).$$

As $L_a^{-1}\alpha(e_1) = L_a^{-1}a = e$, by virtue of Corollary 1 and Proposition 1, $L_a^{-1}\alpha =: \theta$ is a pseudoisomorphism with the right companion $c := R_a^{-1}\beta(e_1) = b \cdot a^{-1}$. This proves the item 1). Thus, $T_2^{-1}T = (\theta; R_c\theta; R_c\theta)$. Therefrom, we obtain the item 3. □

Corollary 3. *Isotopic inverse property loops are pseudoisomorphic.*

Proof. It follows from the item 1 of Theorem 2. □

Corollary 4. *Let (α, β, γ) be an isotopism of a commutative inverse property loop $(Q_o; \circ, e)$ on a commutative inverse property loop $(Q; +, 0)$, then there exists an isomorphism θ of $(Q_o; \circ, e)$ on $(Q; +, 0)$, a central element c in $(Q; +, 0)$ and a Moufang element $a \in Q$ such that $\alpha = L_a\theta$, $\beta = L_aL_c\theta$, $\gamma = L_a^2L_c\theta$.*

Proof. According to Theorem 2 there exists a pseudoisomorphism θ of $(Q_o; \circ, e)$ on $(Q; +, 0)$ with a companion c and a Moufang element a such that

$$\alpha = L_a\theta, \qquad \beta = R_aR_c\theta, \qquad \gamma = L_aR_aR_c\theta.$$

Since the nuclei coincide in these loops (Corollary 2), then by virtue of Lemma 2 θ is an isomorphism of these loops and c is a central element in the loop $(Q; +, 0)$. Commutativity means $L_x = R_x$ for all x. □

Corollary 5. *Isotopic commutative inverse property loops are isomorphic.*

Proof. The proof follows from Corollary 4. □

Since every Moufang loop has the inverse property, then the following statement is true.

Corollary 6. *Isotopic commutative Moufang loops are isomorphic.*

Corollary 7. *In an arbitrary inverse property loop the set of all Moufang elements form a subloop, which is a Moufang loop.*

Proof. Let a, b be Moufang elements of an IP-loop $(Q; \cdot, I, e)$, i.e.

$$(L_a, R_a, L_aR_a) \quad \text{and} \quad (L_b, R_b, L_bR_b)$$

are autotopisms. Then their inverses and composition are autotopisms too. By virtue of the item 2 of Theorem 2, the elements $a^{-1} = L_a^{-1}(e)$ and $a \cdot b = L_aL_b(e)$ are Moufang. Consequently, Moufang elements form a subloop. □

4 Distributive quasigroups

A quasigroup is called *left* (*right*, *middle*) *distributive* if every its left (right, middle) translations is its automorphism.

In other words, such quasigroups are defined by the identity of *left*, *right*, *middle distributivity*:

$$x \cdot yz = xy \cdot xz, \tag{12}$$

$$yz \cdot x = yx \cdot zx, \tag{13}$$

$$yz\backslash x = (y\backslash x) \cdot (z\backslash x) \tag{14}$$

respectively.

A quasigroup is called *distributive* if it is both left and right distributive.

Lemma 5. *For any element $a \in Q$ of a distributive quasigroup $(Q;\cdot)$, the translations L_a, R_a, M_a are pairwise commuting automorphisms of every parastrophe of the quasigroup.*

Proof. The left and right distributivity mean that L_a and R_a are automorphisms of $(Q;\cdot)$. Since automorphism groups of all parastrophes coincide, then L_a, R_a as well as L_a^{-1}, R_a^{-1} are automorphisms of all parastrophes of the quasigroup.

Multiply the equality $z \cdot (z\backslash y) = y$ (see (1)) by $z\backslash u$ from the right and use (13):

$$z(z\backslash u) \cdot (z\backslash y)(z\backslash u) = y(z\backslash u).$$

As $z(z\backslash u) = u$ and L_z^{-1}, L_y are automorphisms of $(Q;\cdot)$ (see (3)), then

$$u \cdot z\backslash (yu) = yz\backslash yu.$$

Let $yu = a$, i.e., $y\backslash a = u$, then

$$(y\backslash a)(z\backslash a) = yz\backslash a.$$

It means that for arbitrary $a \in Q$ the middle translation M_a is an automorphism of $(Q;\cdot)$, and, consequently, of every its parastrophe.

Every of the identities (12), (13), (14) implies idempotency $xx = x$ (when $x = y = z$). The previous identity implies the equalities $L_a(a) = R_a(a) = M_a(a) = a$, that is why

$$L_aR_a(x) = L_a(xa) = L_a(x) \cdot L_a(a) = L_a(x) \cdot a = R_aL_a(x),$$

$$M_aL_a(x) = M_a(ax) = M_a(a) \cdot M_a(x) = a \cdot M_a(x) = L_aM_a(x).$$

Analogously, $M_aR_a = R_aM_a$. □

Corollary 8. *All parastrophes of a distributive quasigroup are distributive and pairwise distributive.*

In other words, for every $\sigma, \tau \in S_3$ the follow identities are true

$$x \overset{\sigma}{\cdot} (y \overset{\tau}{\cdot} z) = (z \overset{\sigma}{\cdot} y) \overset{\tau}{\cdot} (x \overset{\sigma}{\cdot} z), \qquad (y \overset{\tau}{\cdot} z) \overset{\sigma}{\cdot} x = (y \overset{\sigma}{\cdot} x) \overset{\tau}{\cdot} (z \overset{\sigma}{\cdot} x).$$

Proof. From the table (4), we conclude that L_x, R_x, M_x, L_x^{-1}, R_x^{-1}, M_x^{-1}, where $x \in Q$, are all translations of all parastrophes of a quasigroup $(Q;\cdot)$. That is why Lemma 5 implies this corollary. □

Corollary 9. *Every two of the identities (12), (13), (14) imply the third.*

Proof. If a quasigroup $(Q;\cdot)$ satisfies (12) and (13), then Lemma 5 implies (14). If (12) and (14) hold in the quasigroup, then the table (4) implies that $(Q;\backslash)$ is left and right distributive and, according to Lemma 5, it is middle distributive. Relations between translations (the table (4)) induce right distributivity of $(Q;\cdot)$, i.e., (13) holds.

The implication (13) & (14) $\Rightarrow$ (12) can be proved in the same way. □

Corollary 10. *A quasigroup is distributive if and only if all its translations are its automorphisms.*

The following theorem is a specification of the corresponding Belousov's result.

Theorem 3. *A quasigroup $(Q;\cdot)$ is distributive if and only if there exists a commutative Moufang loop $(Q;+)$ and its automorphism φ such that $\psi := \iota - \varphi$ is an automorphism of $(Q;+)$ and*

$$x \cdot y = \varphi x + \psi y, \tag{15}$$

$$x + (y + z) = (\varphi x + y) + (\psi x + z). \tag{16}$$

Proof. Let $(Q;\cdot)$ be an arbitrary distributive quasigroup and 0 be an arbitrary fixed element from Q. In this proof, we will write L, R, M instead of L_0, R_0, M_0. We define an operation $(+)$ on the set Q putting

$$x + y := R^{-1}(x) \cdot L^{-1}(y). \tag{17}$$

Herefrom

$$x \cdot y = R(x) + L(y). \tag{18}$$

Idempotency of $(Q;\cdot)$ implies that 0 is a neutral element in $(Q;+)$.

Since L and R are commuting automorphisms of $(Q;\cdot)$, then they are automorphisms of the loop $(Q;+)$. For example,

$$\begin{aligned} L(x+y) \quad &\overset{(17)}{=} L(R^{-1}(x) \cdot L^{-1}(y)) \overset{\textit{Lemma } 5}{=} LR^{-1}(x) \cdot LL^{-1}(y) = \\ &\overset{\textit{Lemma } 5}{=} R^{-1}L(x) \cdot L^{-1}L(y) \overset{(17)}{=} L(x) + L(y). \end{aligned}$$

We show that $(Q;+)$ is a right IP-loop, i.e., for some mapping I the identity

$$(y + x) + I(x) = y \tag{19}$$

holds. Put $I := LMR^{-1}$ and, for brevity, we denote $u := R^{-2}(y)$, $t := R^{-1}L^{-1}(x)$. Hence, we have

$$\begin{aligned} (y+x) + I(x) &\overset{(17)}{=} R^{-1}\big(R^{-1}(y) \cdot L^{-1}(x)\big) \cdot L^{-1}LMR^{-1}(x) = \\ &\overset{\text{Lemma } 5}{=} \big(R^{-2}(y) \cdot R^{-1}L^{-1}(x)\big) \cdot LMR^{-1}L^{-1}(x) = ut \cdot \big(0 \cdot M(t)\big) = \\ &\overset{(12)}{=} (ut \cdot 0)\big(ut \cdot M(t)\big) \overset{(13)}{=} (ut \cdot 0)\big(uM(t) \cdot tM(t)\big) = (ut \cdot 0)\big(uM(t) \cdot 0\big) = \\ &\overset{(13)}{=} \big(ut \cdot uM(t)\big) \cdot 0 \overset{(12)}{=} R\big(u \cdot tM(t)\big) = R(u \cdot 0) = R^2R^{-2}(y) = y. \end{aligned}$$

To prove commutativity of $(+)$, we note that for all $x, y \in Q$ the equality

$$(x + y) + I(x) = y \tag{20}$$

holds. Denote $z := R^{-2}(x)$, $v := R^{-1}L^{-1}(y)$, then

$$\begin{aligned}(x+y)+I(x) &\overset{(17)}{=} R^{-1}\big(R^{-1}(x)\cdot L^{-1}(y)\big)\cdot L^{-1}LMR^{-1}(x) =\\ &= \Big(R^{-2}(x)\cdot R^{-1}L^{-1}(y)\Big)\cdot M\Big(R^{-2}(x)\cdot 0\Big) = zv\cdot M(z0) =\\ &= L_{z0}^{-1}\Big(z0\cdot\big(zv\cdot M(z0)\big)\Big) \overset{(12)}{=} L_{z0}^{-1}\Big((z0\cdot zv)\cdot\big(z0\cdot M(z0)\big)\Big) =\\ &= L_{z0}^{-1}\Big((z0\cdot zv)\cdot 0\Big) \overset{(12)}{=} L_{z0}^{-1}\Big((z\cdot 0v)\cdot 0\Big) =\\ &\overset{(13)}{=} L_{z0}^{-1}\Big(z0\cdot(0v\cdot 0)\Big) = 0v\cdot 0 = RLR^{-1}L^{-1}(y) = y.\end{aligned}$$

The equality of the right sides of (19) and (20) implies the equality of their left sides: $(y+x)+I(x) = (x+y)+I(x)$, that is why $y+x = x+y$. Hence, $(Q;+)$ is a commutative IP-loop.

Using (18), we replace the second and the forth appearances of the operation $(\cdot)$ with $(+)$ in (12):

$$x\cdot(Ry+Lz) = R(xy)+L(xz).$$

Replacing Ry with y and Lz with z, we obtain:

$$L_x(y+z) = RL_xR^{-1}(y) + LL_xL^{-1}(z).$$

It means that the triplet $(RL_xR^{-1}; LL_xL^{-1}; L_x)$ is an autotopism of the IP-loop $(Q;+)$ for all $x\in Q$. Theorem 2 implies that the element $L_x(0) = x\cdot 0 = R(x)$ is a Moufang element in $(Q;+)$. As R is a bijection of Q, then an arbitrary element from Q is Moufang, so $(Q;+)$ is a commutative Moufang loop.

Idempotency $x\cdot x = x$ of $(\cdot)$ means that $\varphi x + \psi x = x$, i.e., $\psi = \iota - \varphi$.

It remains to prove that in a commutative Moufang loop $(Q;+)$ which has two commuting automorphisms φ and ψ such that the equality (15) holds, two identities of distributivity (12) and (13) are equivalent to the identity (16). For this purpose, we replace $(\cdot)$ with $(+)$ in (12) and (13):

$$\varphi x + (\psi\varphi y + \psi^2 z) = (\varphi^2 x + \varphi\psi y) + (\psi\varphi x + \psi^2 z),$$

$$(\varphi^2 y + \varphi\psi z) + \psi x = (\varphi^2 y + \varphi\psi x) + (\psi\varphi z + \psi^2 x).$$

In the first identity, we replace φx with x, $\psi\varphi y$ with y and $\psi^2 z$ with z, and in the second one $\varphi^2 y$ with y, $\varphi\psi z$ with z and ψx with x. Since $\varphi\psi = \psi\varphi$, then we obtain identities being equivalent to above mentioned:

$$x + (y+z) = (\varphi x + y) + (\psi x + z),$$

$$(y+z) + x = (y + \varphi x) + (z + \psi x).$$

Commutativity of $(+)$ implies coincidence of both of them with (16). $\square$

Remark that it is easy to verify that middle distributivity (14) coincides with (16) if we replace $(\cdot)$ with $(+)$.

Corollary 11 (V. D. Belousov [1]). *Every distributive quasigroup is isotopic to a commutative Moufang loop.*

Note that Theorem 3 implies that any distributive quasigroup can be considered as a *corresponding algebra* $(Q;+,\varphi)$ which satisfies the conditions:

1) $(Q;+)$ is a commutative Moufang loop;
2) φ and $\iota-\varphi:=\psi$ are automorphisms of $(Q;+)$;
3) the identity (16) holds.

(Compare with Belousov-Onoi module [4].) We will also say that "*the automorphism φ defines a distributive quasigroup $(Q;\cdot)$ on the commutative Moufang loop $(Q;+)$*".

Theorem 3 creates a possibility for studying distributive quasigroups via commutative Moufang loops. For example, we have to answer questions like "When distributive quasigroups are isotopic? isomorphic?" and so on. The next three propositions give answers to some of such questions.

Corollary 12. *Distributive quasigroups are isotopic if and only if the corresponding commutative Moufang loops are isomorphic.*

Proof. The truth of the corollary follows from Corollary 5. □

Taking into account Corollary 12, we may restrict our attention to distributive quasigroups defined on the same commutative Moufang loop and the first question that arises is the following: "What relation between automorphisms of the same commutative Moufang loop which define distributive quasigroups?"

Corollary 13. *Let an automorphism φ of a commutative Moufang loop $(Q;+)$ define a distributive quasigroup on $(Q;+)$. Then a bijection φ_o defines a distributive quasigroup on $(Q;+)$ if and only if there exists a homomorphism ν from $(Q;+)$ into its center such that $\varphi_o=\varphi+\nu$ and $\psi_o=\iota-\varphi-\nu$ are bijections of Q.*

Proof. Let automorphisms φ and φ_o define distributive quasigroups on a commutative Moufang loop $(Q;+)$. It implies that (16) and

$$x+(y+z)=(\varphi_o x+y)+(\psi_o x+z)$$

hold. Consequently, the right sides of these identities are equal:

$$(\varphi x+y)+(\psi x+z)=(\varphi_o x+y)+(\psi_o x+z).$$

Replace z with $-\psi_o x+z$ and y with $-\varphi x+y$:

$$y+(\psi x+(-\psi_o x+z))=(\varphi_o x+(-\varphi x+y))+z. \tag{21}$$

Let $\nu := \varphi_o - \varphi$, then $\psi - \psi_o = (\iota - \varphi) - (\iota - \varphi_o) = \varphi_o - \varphi = \nu$. When $y = 0$ and when $z = 0$ the equality (21) implies

$$\psi x + (-\psi_o x + z) = \nu x + z \qquad \text{and} \qquad y + \nu x = \varphi_o x + (-\varphi x + y).$$

So, (21) can be written as follows

$$y + (\nu x + z) = (y + \nu x) + z.$$

So, ν is a mapping from the loop $(Q;+)$ into its center and $\varphi_o = \varphi + \nu$.

Since φ_o is an automorphism of the loop $(Q;+)$, then

$$(\varphi + \nu)x + (\varphi + \nu)y = (\varphi + \nu)(x + y),$$

i.e.,

$$(\varphi x + \nu x) + (\varphi y + \nu y) = (\varphi x + \varphi y) + \nu(x + y).$$

As νx is a central element for all $x \in Q$, then we can change the left side of the equality:

$$(\varphi x + \varphi y) + \nu x + \nu y = (\varphi x + \varphi y) + \nu(x + y).$$

Cancelling out $\varphi x + \varphi y$, we obtain a homomorphic property for ν.

Vice versa, let ν be an arbitrary homomorphism from a commutative Moufang loop $(Q;+)$ into its center and let $\nu + \varphi$ and $\iota - \varphi - \nu$ be bijections of Q. Define transformations

$$\varphi_o := \varphi + \nu \quad \text{and} \quad \psi_o := \iota - \varphi - \nu = \iota - \varphi_o = \psi - \nu.$$

Both of them are automorphisms of $(Q;+)$. Indeed, they are bijections according to the assumption. In the following proof of the homomorphic property of φ_o we are using the fact that νx is a central element of $(Q;+)$ for arbitrary $x \in Q$:

$$\begin{aligned}
\varphi_o(x + y) &= (\varphi + \nu)(x + y) = \varphi(x + y) + \nu(x + y) = (\varphi x + \varphi y) + (\nu x + \nu y) = \\
&= (\varphi x + \nu x) + (\varphi y + \nu y) = (\varphi + \nu)x + (\varphi + \nu)y = \varphi_o x + \varphi_o y.
\end{aligned}$$

As $\psi_o = \psi - \nu$, we have

$$\begin{aligned}
\psi_o x + \psi_o y &= (\psi - \nu)x + (\psi - \nu)y = (\psi x - \nu x) + (\psi y - \nu y) = \\
&= (\psi x + \psi y) - (\nu x + \nu y) = \psi(x + y) - \nu(x + y) = (\psi - \nu)(x + y) = \\
&= \psi_o(x + y).
\end{aligned}$$

It remains to prove that (16) is true for φ_o. For this purpose, we add the neutral element 0 in the form $0 = \nu x + (-\nu x)$ to the right side of (16):

$$x + (y + z) = (\varphi x + \nu x + y) + (\psi x - \nu x + z) = (\varphi_o x + y) + (\psi_o x + z).$$

Thus, according to Theorem 3, the automorphism φ_o defines a distributive quasigroup on the commutative Moufang loop $(Q;+)$. $\square$

The next theorem gives a isomorphy criterion of distributive quasigroups (it is close to [5, Lemma 12.3]).

Theorem 4. *Distributive quasigroups are isomorphic if and only if their corresponding algebras are isomorphic.*

Proof. Let $(Q;\circ)$ and $(Q;\cdot)$ be distributive quasigroups, which are defined on commutative Moufang loops $(Q_o;\oplus,0')$ and $(Q;+,0)$ by their automorphisms φ_o and φ respectively, that is $(Q_o;\oplus,\varphi_o)$ and $(Q;+,\varphi)$ are corresponding algebras. According to Theorem 3, the mappings $\psi:=\iota-\varphi$ and $\psi_o:=\iota\ominus\varphi_o$ are automorphisms of $(Q;+,0)$ and $(Q_o;\oplus,0')$ respectively, besides (15) and

$$x\circ y=\varphi_o x\oplus\psi_o y$$

hold.

Let α be an isomorphism from $(Q_o;\circ)$ onto $(Q;\cdot)$, i.e.,

$$\alpha x\cdot\alpha y=\alpha(x\circ y)$$

for all $x,y\in Q$. This equality can be written as follows

$$\varphi\alpha x+\psi\alpha y=\alpha(\varphi_o x\oplus\psi_o y).$$

Replace x with $\varphi_o^{-1}(x)$ and y with $\psi_o^{-1}(y)$:

$$\varphi\alpha\varphi_o^{-1}(x)+\psi\alpha\psi_o^{-1}(y)=\alpha(x\oplus y).$$

The obtained equality means that the triplet $(\varphi\alpha\varphi_o^{-1},\psi\alpha\psi_o^{-1},\alpha)$ is an isotopism from the Moufang loop $(Q_o;\oplus)$ onto the Moufang loop $(Q;+)$. According to Corollary 4, there exists an isomorphism θ from $(Q_o;\oplus,0')$ onto $(Q;+,0)$, a central element c of $(Q;+)$ and an element $a\in Q$ such that the equalities

$$\varphi\alpha\varphi_o^{-1}=L_a\theta,\qquad \psi\alpha\psi_o^{-1}=L_aL_c\theta,\qquad \alpha=L_a^2L_c\theta$$

are true. Using the third equality, we substitute $L_a^2L_c\theta$ for α in the first one:

$$\varphi L_a^2L_c\theta\varphi_o^{-1}=L_a\theta.$$

Using Moufang identity (7), centrality of c and diassociativity of $(Q;+)$, we have

$$\begin{aligned}L_a^2L_c(x)&=a+(a+(c+x))=(a{+}c)+(a{+}x)=L_c^{-1}((a{+}c)+((a{+}c)+x))=\\&=L_c^{-1}(((a{+}c)+(a{+}c))+x)=L_c^{-1}L_{2(a+c)}(x).\end{aligned}$$

Consequently,

$$\varphi L_c^{-1}L_{2(a+c)}\theta\varphi_o^{-1}=L_a\theta.$$

As φ is an automorphism of $(Q;+)$, then

$$L_{\varphi c}^{-1}L_{\varphi(2(c+a))}\varphi\theta\varphi_o^{-1}=L_a\theta.$$

Therefrom

$$L_{\varphi(2(c+a))}\varphi\theta\varphi_o^{-1} = L_{\varphi c+a}\theta.$$

Since $\varphi\theta\varphi_o^{-1}(0') = 0$ and $\theta(0') = 0$, the previous equality implies $\varphi(2(c+a)) = \varphi c+a$. Therefore, $\varphi\theta\varphi_o^{-1} = \theta$, i.e., $\varphi\theta = \theta\varphi_o$. Thus, θ is an isomorphism from the algebra $(Q_o; \oplus, \varphi_o)$ onto the algebra $(Q; +, \varphi)$.

Vice versa, let θ be an isomorphism from $(Q_o; \oplus, \varphi_o)$ onto $(Q; +, \varphi)$. It means, that θ is an bijection from Q_o onto Q and the following relations hold:

$$\theta(x) + \theta(y) = \theta(x \oplus y), \qquad \varphi\theta = \theta\varphi_o$$

for all $x, y \in Q_o$. These equalities imply $\psi\theta = \theta\psi_o$. Indeed,

$$\begin{aligned} \psi\theta(x) &= (\iota - \varphi)\theta(x) = \theta(x) - \varphi\theta(x) = \\ &= \theta(x) \ominus \theta\varphi_o(x) = \theta(x \ominus \varphi_o(x)) = \theta(\iota \ominus \varphi_o)(x) = \theta\psi_o(x). \end{aligned}$$

That is why, we have

$$\theta x \cdot \theta y = \varphi\theta x + \psi\theta y = \theta\varphi_o x + \theta\psi_o y = \theta(\varphi x \oplus \psi y) = \theta(x \circ y).$$

Hence, θ is an isomorphism from $(Q; \circ)$ onto $(Q; \cdot)$. $\square$

Corollary 14. *Let distributive quasigroups $(Q; \circ)$ and $(Q; \cdot)$ be defined on a commutative Moufang loop $(Q; +)$ by its automorphisms φ_o and φ respectively. Then the quasigroups are isomorphic if and only if there exists an automorphism θ of the loop $(Q; +)$ such that $\varphi_o = \theta^{-1}\varphi\theta$.*

This corollary immediately implies that there exist exactly $p-3$ non-isomorphic distributive quasigroups of a prime power $p \geqslant 3$.

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Fedir M. Sokhatsky
Vasyl Poryk, 5, ap.37, Vinnytsia, 21021
Ukraine

E-mail: *fmsokha@ukr.net*

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On spectrum of medial T_2-quasigroups

A. V. Scerbacova, V. A. Shcherbacov

Abstract. There exist medial T_2-quasigroups of any order of the form

$$2^{k_1}3^{k_2}5^{k_3}11^{k_4}17^{k_5}23^{k_6}53^{k_7}59^{k_8}83^{k_9}101^{k_{10}}p_1^{\alpha_1}p_2^{\alpha_2}\dots p_m^{\alpha_m},$$

where $k_1 \geq 2$, $k_2, \dots, k_{10} \geq 1$, p_i are prime numbers of the form $6t+1$, $\alpha_i \in \mathbb{N}$, $i \in \{1, \dots, m\}$. Some other results on T_2-quasigroups are given.

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1 Introduction

Definitions and elementary properties of quasigroups can be found in [1, 2, 18]. Most of presented here results are given in [20]. Quasigroups have some applications in cryptology [24]. The most usable in cryptology quasigroup property is the property of orthogonality of quasigroups [9].

V. D. Belousov [3, 4] (see also [10]) by the study of orthogonality of quasigroup parastrophes proved that there exist exactly seven parastrophically non-equivalent identities which guarantee that a quasigroup is orthogonal to at least one its parastrophe: s

$$x(x \cdot xy) = y \qquad (C_3\ \text{law}) \qquad (1)$$
$$x(y \cdot yx) = y \qquad \text{of type}\ T_2\ [3] \qquad (2)$$
$$x \cdot xy = yx \qquad (\text{Stein's 1st law}) \qquad (3)$$
$$xy \cdot x = y \cdot xy \qquad (\text{Stein's 2nd law}) \qquad (4)$$
$$xy \cdot yx = y \qquad (\text{Stein's 3rd law}) \qquad (5)$$
$$xy \cdot y = x \cdot xy \qquad (\text{Schroder's 1st law}) \qquad (6)$$
$$yx \cdot xy = y \qquad (\text{Schroder's 2nd law}). \qquad (7)$$

The names of identities (3)–(7) originate from Sade's paper [19]. We follow [6] in the name of identity (1).

All these identities can be obtained in a unified way using criteria of orthogonality and quasigroup translations [15]. For example, identity (2), which guarantees orthogonality of a quasigroup $(Q, \cdot)$ and its (2 3)-parastrophe, can be obtained from the following translation identity

$$L_y^2 x = P_y x. \tag{8}$$

Using table of translations of quasigroup parastrophes [23] we can rewrite identity (8) in the following parastrophically equivalent [4] forms:

$$\begin{aligned} &R_y^2 x = P_y^{-1} x, \\ &P_y^{-2} x = L_y^{-1} x, \\ &L_y^{-2} x = R_y x, \\ &R_y^{-2} x = L_y x, \\ &P_y^2 x = R_y^{-1} x. \end{aligned} \tag{9}$$

Passing to "standard" identities we obtain from the identities (9) the following identities that are parastrophically equivalent to the identity (2):

$$\begin{aligned} &(xy \cdot y)x = y, \\ &(y\backslash x)(y/x) = y, \\ &y(y \cdot xy) = x, \\ &(yx \cdot y)y = x, \\ &x(y/(x/y)) = y. \end{aligned} \tag{10}$$

A quasigroup $(Q, \cdot)$ with the identity $x \cdot x = x$ is called idempotent. The set $\mathfrak{Q}$ of natural numbers for which there exist quasigroups with a property T, for example, the property of idempotency, is called the spectrum of the property T in the class of quasigroups. Often the following phrase is used: spectrum of quasigroups with a property T. Therefore we can say that spectra of quasigroups with identities (3)–(7) were studied in [5, 6, 8, 12, 17, 25].

It is clear that the identity (2) and any from identities (10) have the same spectrum because order of any parastroph of a quasigroup $(Q, \cdot)$ is equal to the order of quasigroup $(Q, \cdot)$.

Idempotent models of the identity $(yx \cdot y)y = x$ can be associated with a class of resolvable Mendelsohn designs [5]. In [5] "it is shown that the spectrum of $(yx \cdot y)y = x$ contains all integers $n \geq 1$ with the exception of $n = 2, 6$ and the possible exception of $n \in \{10, 14, 18, 26, 30, 38, 42, 158\}$. It is also shown that idempotent models of $(yx \cdot y)y = x$ exist for all orders $n > 174$".

Here we study in the main the spectrum of medial T_2-quasigroups. Such quasigroups can be easy constructed and they can be used in cryptology.

2 Medial T_2-quasigroups

The problem of the study of T_2-quasigroups is posed in [3,4]. In [26] the following proposition (Proposition 7) is proved. We formulate this proposition in a slightly changed form.

Theorem 1. *If a T_2-quasigroup $(Q, \cdot)$ is isotopic to an abelian group $(Q, \oplus)$, then for every element $b \in Q$ there exists an isomorphic copy $(Q, +) \cong (Q, \oplus)$ such that $x \cdot y = IL_b^3(x) + L_b(y) + b$, for all $x, y \in Q$, where $x + Ix = 0$ for all $x \in Q$.*

Definition 1. A quasigroup $(Q, \cdot)$ of the form $x \cdot y = \varphi x + \psi y + b$, where $(Q, +)$ is an abelian group, φ, ψ are automorphisms of the group $(Q, +)$, b is a fixed element of the set Q is called T-quasigroup. If, additionally, $\varphi\psi = \psi\varphi$, then $(Q, \cdot)$ is called medial quasigroup [1, 2, 16, 18].

Theorem 2. *A T-quasigroup $(Q, \cdot)$ of the form*

$$x \cdot y = \varphi x + \psi y + b \tag{11}$$

satisfies T_2-identity if and only if $\varphi = I\psi^3$, $\psi^5 + \psi^4 + 1 = (\psi^2 + \psi + 1)(\psi^3 - \psi + 1) = 0$, where 1 is identity automorphism of the group $(Q, +)$ and 0 is zero endomorphism of this group, $\psi^2 b + \psi b + b = 0$.

Proof. We rewrite T_2-identity using the right part of the form (11) as follows:

$$\varphi x + \psi(\varphi y + \psi(\varphi y + \psi x + b) + b) + b = y \tag{12}$$

or, taking into consideration that $(Q, +)$ is an abelian group, φ, ψ are its automorphisms, after simplification of equality (12) we have

$$\varphi x + \psi\varphi y + \psi^2\varphi y + \psi^3 x + \psi^2 b + \psi b + b = y. \tag{13}$$

If we put in the equality (13) $x = y = 0$, then we obtain

$$\psi^2 b + \psi b + b = 0, \tag{14}$$

where 0 is the identity (neutral) element of the group $(Q, +)$.

Therefore we can rewrite equality (13) in the following form

$$\varphi x + \psi\varphi y + \psi^2\varphi y + \psi^3 x = y. \tag{15}$$

If we put in the equality (15) $y = 0$, then we obtain that $\varphi x + \psi^3 x = 0$. Therefore $\varphi = I\psi^3$, where, as above, $x + Ix = 0$ for all $x \in Q$.

Notice in any abelian group $(Q, +)$ the map I is an automorphism of this group. Really, $I(x + y) = Iy + Ix = Ix + Iy$.

Moreover, $I\alpha = \alpha I$ for any automorphism of the group $(Q, +)$. Indeed, $\alpha x + I\alpha x = 0$. On the other hand $\alpha x + \alpha I x = \alpha(x + Ix) = \alpha 0 = 0$. Comparing the left sides we have $\alpha x + I\alpha x = \alpha x + \alpha I x$, $I\alpha x = \alpha I x$, $\alpha I = I\alpha$.

It is well known that $I^2 = \varepsilon$, i.e., $-(-x) = x$. Indeed, from the equality $x + Ix = 0$ using commutativity we have $Ix + x = 0$. On the other hand $I(x + Ix) = 0$, $Ix + I^2 x = 0$. Then $Ix + x = Ix + I^2 x$, $x = I^2 x$ for all $x \in Q$.

If we put in the equality (15) $x = 0$, then we obtain that

$$\psi\varphi y + \psi^2\varphi y = y. \tag{16}$$

If we substitute in the equality (16) the expression $I\psi^3$ for φ, then we have $I\psi^5 y + I\psi^4 y = y$, $\psi^5 y + \psi^4 y = Iy$, $\psi^5 y + \psi^4 y + y = 0$. The last condition can be written in the form $\psi^5 + \psi^4 + 1 = 0$, where 1 is identity automorphism of the group $(Q, +)$ and 0 is zero endomorphism of this group.

It is easy to check that $\psi^5 + \psi^4 + 1 = (\psi^2 + \psi + 1)(\psi^3 - \psi + 1)$.

Converse. If we take into consideration that $\psi^2 b + \psi b + b = 0$, then from equality (13) we obtain equality (15). If we substitute in equality (15) the following equality $\varphi = I\psi^3$, then we obtain $\psi I\psi^3 y + \psi^2 I\psi^3 y = y$, $\psi^4 Iy + \psi^5 Iy = y$ which is equivalent to the equality $\psi^5 y + \psi^4 y + y = 0$. Therefore T-quasigroup $(Q, \cdot)$ is T_2-quasigroup. □

Remark 1. Proposition 6 in [8] states almost the same as Theorem 2.

Corollary 1. *Any T_2-T-quasigroup is medial.*

Proof. The proof follows from the equality $\varphi = I\psi^3$ (see Theorem 2). □

Corollary 2. *A T-quasigroup $(Q, \cdot)$ of the form $x \cdot y = \varphi x + \psi y$ satisfies T_2-identity if and only if $\varphi = I\psi^3$, $\psi^5 + \psi^4 + 1 = 0$.*

Proof. It is easy to see. □

Corollary 3. *A T-quasigroup $(Q, \cdot)$ of the form $x \cdot y = \varphi x + \psi y + b$ satisfies T_2-identity if $\varphi = I\psi^3$, $\psi^2 + \psi + 1 = 0$.*

Proof. The proof follows from Theorem 2 and the following fact: if $\psi^2 + \psi + 1 = 0$, then $\psi^5 + \psi^4 + 1 = 0$. In this case the following equality $\psi^2 b + \psi b + b = 0$ is also true. □

Corollary 4. *A T-quasigroup $(Q, \cdot)$ of the form $x \cdot y = \varphi x + \psi y + b$ satisfies T_2-identity if $\varphi = I\psi^3$, $\psi^3 - \psi + 1 = 0$, $\psi^2 b + \psi b + b = 0$.*

Proof. The proof follows from Theorem 2 and the following fact: if $\psi^3 - \psi + 1 = 0$, then $\psi^5 + \psi^4 + 1 = 0$. □

Lemma 1. *Any T-quasigroup of the form $x \cdot y = \varphi x + \psi y + b$ is idempotent if and only if $\varphi + \psi = \varepsilon$, $b = 0$.*

Proof. It is easy to see. See also [16]. □

Corollary 5. *Any T_2-T-quasigroup of the form $x \cdot y = \varphi x + \psi y + b$ is idempotent if and only if $\varphi = I\psi^3$, $\psi^3 - \psi + 1 = 0$, $b = 0$.*

Proof. We can use Theorem 2 and Lemma 1. Indeed, from the equality $I\psi^3 = \varepsilon - \psi$ we have that $\psi^3 = I + \psi$, $\psi^3 - \psi + 1 = 0$. □

Example 1. The following T_2-quasigroup is non-medial and therefore it is not a T-quasigroup (see Corollary 1). It is clear that this quasigroup is not idempotent.

$*$	0	1	2	3	4	5	6	7	8
0	0	1	3	4	2	5	6	7	8
1	2	0	1	6	7	3	5	8	4
2	1	4	5	8	0	6	2	3	7
3	7	3	0	5	8	1	4	2	6
4	6	2	8	0	5	7	3	4	1
5	8	7	2	3	4	0	1	6	5
6	4	8	7	1	6	2	0	5	3
7	3	5	6	7	1	4	8	0	2
8	5	6	4	2	3	8	7	1	0

3 T_2-quasigroups from the rings of residues

We use rings of residues modulo n, say $(R, +, \cdot, 1)$, and Theorem 2 to construct T_2-quasigroups. Here $(R, +)$ is cyclic group of order n, i.e., it is the group $(Z_n, +)$ with the generator element 1. It is clear that in many cases the element 1 is not a unique generator element, $(R, \cdot)$ is a commutative semigroup [13].

Multiplication of an element $b \in R$ by all elements of the group $(R, +)$ induces an endomorphism of the group $(R, +)$, i.e., $b \cdot (x + y) = b \cdot x + b \cdot y$. If $g.c.d.(b, n) = 1$, then the element b induces an automorphism of the group $(R, +)$ and it is called an invertible element of the ring $(R, +, \cdot, 1)$.

Next theorem is a specification of Theorem 2 on medial T_2-quasigroups defined using rings of residues modulo n. We denote by the symbol $\mathbb{Z}$ the set of integers, we denote by $|n|$ module of the number n.

Theorem 3. *Let $(Z_r, +, \cdot, 1)$ be a ring of residues modulo r such that $f(k) = (k^5 + k^4 + 1) \equiv 0 \pmod r$ for some $k \in \mathbb{Z}$. If $g.c.d.(|k|, r) = 1$, $k^2 \cdot b + k \cdot b + b \equiv 0 \pmod r$ for some $b \in Z_r$, then there exists T_2-quasigroup $(Z_r, \circ)$ of the form $x \circ y = -k^3 \cdot x + k \cdot y + b$ and of order r.*

Proof. We can use Theorem 2. The fact that $g.c.d.(|k|, r) = 1$ guarantees that the multiplication by the number k induces an automorphism of the group $(Z_r, +)$. In this case the map $-k^3$ is also a permutation as a product of permutations. □

Example 2. Let $k = -3$. Then $f(-3) = (-3)^5 + (-3)^4 + 1 = -161 = -(7) \cdot (23)$. Therefore $-161 \equiv 0 \pmod 7$ and $-161 \equiv 0 \pmod{23}$ and we have theoretical possibility to construct T_2 quasigroups of order $7, 23, 161$.

Case 1. Let $r = 7$. Then $k = -3 = 4 \pmod 7$. In this case $-(k^3) = -(-3)^3 = 27 = 6 \pmod 7$. It is clear that the elements 6 and 4 are invertible elements of the ring $(Z_7, +, \cdot, 1)$. Therefore the quasigroup $(Z_7, *)$ with the form $x * y = 6 \cdot x + 4 \cdot y$ is T_2-quasigroup of order 7.

Check. We have $6x + 4(6y + 4(6y + 4x)) = y$, $70x + 24y + 96y = y$, $y = y$, since $70 \equiv 0 \pmod 7, 120 \equiv 1 \pmod 7$.

In order to construct T_2-quasigroups over the ring $(Z_7, +, \cdot, 1)$ with non-zero element b we must solve congruence $(-3)^2 \cdot b + (-3) \cdot b + b \equiv 0 \pmod 7$. We have $7 \cdot b \equiv 0 \pmod 7$. The last equation is true for any possible value of the element b. Therefore the following quasigroups are T_2-quasigroups of order 7: $x \circ y = 6 \cdot x + 4 \cdot y + i$, for any $i \in \{1, 2, \dots, 5, 6\}$.

Case 2. Let $r = 23$. Then $k = -3 = 20 \pmod{23}$. In this case $-(k^3) = -(-3)^3 = 27 = 4 \pmod{23}$. It is clear that the elements 20 and 4 are invertible elements of the ring $(Z_{23}, +, \cdot, 1)$. Therefore quasigroup $(Z_{23}, *)$ with the form $x * y = 4 \cdot x + 20 \cdot y$ is T_2-quasigroup of order 23.

Check. We have $4x + 20(4y + 20(4y + 20x)) = y$, $4x + 80y + 1600y + 8000x = y$, $y = y$, since $8004 \equiv 0 \pmod{23}, 1680 \equiv 1 \pmod{23}$. This quasigroup is idempotent. Indeed, $4 + 20 = 24 \equiv 1 \bmod 23$.

In order to construct T_2-quasigroups over the ring $(Z_{23}, +, \cdot, 1)$ with non-zero element b we must solve congruence $(-3)^2 \cdot b + (-3) \cdot b + b \equiv 0 \pmod{23}$. We have $7 \cdot b \equiv 0 \pmod{23}$. This congruence modulo has unique solution $b \equiv 0 \bmod 23$, since $g.c.d.(7, 23) = 1$.

Case 3. Let $r = 161$. Then $k = -3 = 158 \pmod{161}$. Recall the number 161 is not prime. In this case $-(k)^3 = -(-3)^3 = 27 \pmod{161}$, $g.c.d.(27, 161) = 1$, the elements 158 and 27 are invertible elements of the ring $(Z_{161}, +, \cdot, 1)$. Therefore quasigroup $(Z_{161}, \circ)$ with the form $x \circ y = 27 \cdot x + 158 \cdot y$ is medial T_2-quasigroup of order 161.

Check. $27x + 4266y + 674028y + 3944312x = y$, $y = y$, since $3944339 \equiv 0 \pmod{161}$, $678294 \equiv 1 \pmod{161}$.

In order to construct T_2-quasigroups over the ring $(Z_7, +, \cdot, 1)$ with non-zero element b we must solve congruence $7 \cdot b \equiv 0 \pmod{161}$. It is clear that $g.c.d.(7, 161) = 7$. Therefore this congruence has 6 non-zero solutions, namely, $b \in \{23, 46, 69, 92, 115, 138\} = D$.

The following quasigroups are T_2-quasigroups of order 161: $x \circ y = 27 \cdot x + 158 \cdot y + i$, for any $i \in D$.

Example 3. We list some values of the polynomial f:

$$
\begin{gathered}
f(-20) = -3039999, f(-19) = -2345777, f(-18) = -1784591, \\
f(-17) = -1336335, f(-16) = -983039, f(-15) = -708749, \\
f(-14) = -499407, f(-13) = -342731, f(-12) = -228095, \\
f(-11) = -146409, f(-10) = -89999, f(-9) = -52487, \\
f(-8) = -28671, f(-7) = -14405, f(-6) = -6479, f(-5) = -2499, \\
f(-4) = -767, f(-3) = -161, f(-2) = -15, f(-1) = 1, f(1) = 3, \\
f(2) = 49, f(3) = 325, f(4) = 1281, f(5) = 3751, \\
f(6) = 9073, f(7) = 19209, f(8) = 36865, f(9) = 65611, \\
f(10) = 110001, f(11) = 175693, f(12) = 269569, f(13) = 399855,
\end{gathered}
$$

$$f(14) = 576241, f(15) = 810001, f(12) = 269569, f(17) = 1503379,$$
$$f(18) = 1994545, f(19) = 2606421, f(20) = 3360001.$$

The set of prime divisors of the numbers of the set $\{f(-20), f(-19), \dots, f(-1), f(1), \dots, f(20)\}$ contains the following primes:

$$\{3, 5, 7, 13, 19, 23, 37, 43, 59, 61, 73, 101, 157, 211, 241, 307, 347,$$
$$421, 503, 719, 833, 977, 991, 1163, 1319, 2729, 3359, 5813, 6841\}.$$

It is possible to use presented numbers for the construction of T_2-quasigroups over the rings of residues.

Theorem 4. *There exist medial T_2-quasigroups of any prime order p such that $p = 6m + 1$, where $m \in \mathbb{N}$.*

Proof. We use Corollary 3. Let $(Z_p, +, \cdot, 1)$ be a ring (a Galois field) of residues modulo p, where p is prime of the form $6t+1$, $t \in \mathbb{N}$. Quadratic equation $\psi^2+\psi+1 = 0$ has two roots $h_1 = (-1-\sqrt{-3})/2$ and $h_2 = (-1+\sqrt{-3})/2$. Since p is prime, then $g.c.d(h_1, p) = g.c.d(h_2, p) = 1$.

It is known [11] that the number -3 is a quadratic residue modulo any prime p such that $p = 6m + 1$. Finally, if the number $(-1 - \sqrt{-3})$ is odd, then the number $(-1 - \sqrt{-3} + p)$ is even. □

We prove the fact that the number -3 is a quadratic residue modulo any prime p such that $p = 6m + 1$ additionally in the following

Lemma 2. *The number -3 is quadratic residue modulo of odd prime p if p can be presented in the form $6t + 1$, where $t \in \mathbb{N}$.*

Proof. We use for proving this fact information from [7, p. 187-188]. We represent prime p, $p > 2$, in the following form: $p = 4qt+r$, where $1 \leq r < 4q$, $g.c.d.(r, 4q) = 1$, q or $-q$ is a prime. The number q or $-q$ is a quadratic residue modulo p if and only if

$$(-1)^{\frac{r-1}{2}\cdot\frac{q-1}{2}}\left(\frac{r}{q}\right) = 1,$$

where $\left(\frac{r}{q}\right)$ is Legendre symbol, or, speaking more formally, Legendre-Jacobi-Kronecker symbol.

If $r = 1$, then $(-1)^{\frac{1-1}{2}\cdot\frac{-3-1}{2}}\left(\frac{1}{-3}\right) = \left(\frac{1}{-3}\right) = 1$.

If $r = 5$, then $(-1)^{\frac{5-1}{2}\cdot\frac{-3-1}{2}}\left(\frac{5}{-3}\right) = \left(\frac{5}{-3}\right) = -1$.

If $r = 7$, then $(-1)^{\frac{7-1}{2}\cdot\frac{-3-1}{2}}\left(\frac{7}{-3}\right) = \left(\frac{7}{-3}\right) = 1$.

If $r = 11$, then $(-1)^{\frac{11-1}{2}\cdot\frac{-3-1}{2}}\left(\frac{11}{-3}\right) = \left(\frac{11}{-3}\right) = -1$.

Therefore prime p has the form $p = 12t + 1$ or $p = 12t + 7$. Combining the last equalities we have that $p = 6t + 1$. □

In order to construct T_2-quasigroups it is possible to use direct products of T_2-quasigroups. It is clear that direct product of T_2-quasigroups is a T_2-quasigroup.

It is possible to use also the following arguments. The class of T_2 quasigroups is defined using T_2-identity, and it forms a variety in signature with three binary operations, namely, with the operations $\cdot$, $/$, and $\backslash$ [13]. It is known that any variety is closed relative to the operator of direct product [13].

Therefore we can formulate the following

Theorem 5. *There exist medial T_2-quasigroups of any order of the form $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$, where p_i are prime numbers of the form $6t+1$, $\alpha_i \in \mathbb{N}$, $i \in \{1, \dots, m\}$.*

Notice that in this section and in the next section examples of medial quasigroups of prime order of the form $6 \cdot t + 5$ (for example, $5, 11, 23, 59$) are given.

Example 4. Using Corollary 5 and ideas of Example 2 we construct medial idempotent T_2-quasigroups over some cyclic groups Z_r ($r < 174$). Notice that such quasigroups are distributive [1, 16]. We have:

$x \cdot y = -2x + 3y \bmod 5$; $x \cdot y = -x + 2y \bmod 7$;
$x \cdot y = -4x + 5y \bmod 11$; $x \cdot y = -11x + 12y \bmod 17$;
$x \cdot y = -12x + 13y \bmod 19$; $x \cdot y = -19x + 20y \bmod 23$;
$x \cdot y = -2x + 3y \bmod 25$; $x \cdot y = -22x + 23y \bmod 35$;
$x \cdot y = -23x + 24y \bmod 37$; $x \cdot y = -32x + 33y \bmod 43$;
$x \cdot y = -36x + 37y \bmod 49$; $x \cdot y = -15x + 16y \bmod 53$;
$x \cdot y = -37x + 38y \bmod 55$; $x \cdot y = -16x + 17y \bmod 59$;
$x \cdot y = -45x + 46y \bmod 59$; $x \cdot y = -3x + 4y \bmod 61$;
$x \cdot y = -59x + 60y \bmod 67$; $x \cdot y = -15x + 16y \bmod 77$;
$x \cdot y = -58x + 59y \bmod 79$; $x \cdot y = -16x + 17y \bmod 83$;
$x \cdot y = -62x + 63y \bmod 85$; $x \cdot y = -71x + 72y \bmod 89$;
$x \cdot y = -12x + 13y \bmod 95$; $x \cdot y = -45x + 46y \bmod 97$;
$x \cdot y = -7x + 8y \bmod 101$; $x \cdot y = -11x + 12y \bmod 101$;
$x \cdot y = -8x + 9y \bmod 103$; $x \cdot y = -72x + 73y \bmod 107$;
$x \cdot y = -82x + 83y \bmod 109$; $x \cdot y = -58x + 59y \bmod 113$;
$x \cdot y = -12x + 13y \bmod 115$; $x \cdot y = -113x + 114y \bmod 119$;
$x \cdot y = -4x + 5y \bmod 121$; $x \cdot y = -102x + 103y \bmod 125$;
$x \cdot y = -50x + 51y \bmod 133$; $x \cdot y = -63x + 64y \bmod 137$;
$x \cdot y = -118x + 119y \bmod 149$; $x \cdot y = -46x + 47y \bmod 157$;
$x \cdot y = -127x + 128y \bmod 161$; $x \cdot y = -32x + 33y \bmod 167$;
$x \cdot y = -33x + 34y \bmod 173$; $x \cdot y = -75x + 76y \bmod 173$.

Using Mace 4 [14] we construct the following examples of medial T_2-quasigroups.

$*$	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

$\boxtimes$	0	1	2	3
0	0	2	3	1
1	1	3	2	0
2	2	0	1	3
3	3	1	0	2

$\circ$	0	1	2	3	4
0	0	2	4	1	3
1	2	1	3	4	0
2	4	3	2	0	1
3	1	4	0	3	2
4	3	0	1	2	4

$\diamond$	0	1	2	3	4	5	6	7
0	0	2	4	1	6	3	7	5
1	6	1	5	2	0	7	3	4
2	7	4	2	5	3	6	0	1
3	4	7	0	3	5	1	2	6
4	5	3	6	7	4	2	1	0
5	2	0	7	6	1	5	4	3
6	3	5	1	4	7	0	6	2
7	1	6	3	0	2	4	5	7

We recall (see Section 1) that in [5] it is proved that idempotent models of identity $(yx \cdot y)y = x$ (therefore also idempotent models of T_2-quasigroups) exist for all orders $n > 174$.

Remark 2. From Example 4 and the example of medial idempotent T_2-quasigroup of order 8 we obtain partial spectrum of idempotent medial T_2-quasigroups of order less than 174.

Lemma 3. *There exist medial T_2-quasigroups of order 2^k for any $k \geq 2$.*

Proof. It follows since T_2-quasigroup with the operation $\boxtimes$ is medial quasigroup of order 2^2 and T_2-quasigroup with the operation $\diamond$ is medial quasigroup of order 2^3 and $g.c.d.(2, 3) = 1$. $\square$

Example 5. There exists medial T_2-quasigroup of order 2^{11} since $11 = 2 \cdot 1 + 3 \cdot 3$.

Example 6. Quasigroup $(Z_{341}, \circ)$, $x \circ y = -125x + 5y$, is an example of medial non-idempotent T_2-quasigroup. Notice, in this example $5^2 + 5 + 1 = 31$, $5^3 - 5 + 1 = 121$, but $31 \cdot 121 \equiv 0 \bmod 341$, i.e. $5^5 + 5^4 + 1 \equiv 0 \bmod 341$.

It is possible to check that quasigroup $(Z_{341}, \circ)$ is isomorphic to the direct product of quasigroup $(Z_{31}, *)$, where $x * y = -x + 5y$, and quasigroup $(Z_{11}, \star)$, where $x \star y = -4x + 5y$.

Quasigroup with operation $x \cdot y = 13x + 18y \bmod 35$ is isomorphic to the direct product of quasigroup of order five with the operation $x * y = -2x + 3y \bmod 5$ and quasigroup of order seven with the operation $x \star y = -x + 4y \bmod 7$.

See [21, 22] about direct products of medial quasigroups.

Combining Lemma 3, Theorem 5, and constructed examples we formulate the following

Theorem 6. *There exist medial T_2-quasigroups of any order of the form*

$$2^{k_1}3^{k_2}5^{k_3}11^{k_4}17^{k_5}23^{k_6}53^{k_7}59^{k_8}83^{k_9}101^{k_{10}}p_1^{\alpha_1}p_2^{\alpha_2}\dots p_m^{\alpha_m},$$

where $k_1 \geq 2$, $k_2, \dots, k_{10} \geq 1$, p_i are prime numbers of the form $6t+1$, $\alpha_i \in \mathbb{N}$, $i \in \{1, \dots, m\}$.

Notice that direct calculation demonstrates that no solution of the equations $x^2+x+1=0$, $x^3-x+1=0$, $x^5+x^4+1=0$ exists in the field $GF(29)$.

4 Annex

Computer calculations show that there exist the following medial idempotent T_2-quasigroups of order r of the form $r=6t+5$. Such quasigroups of orders less than 174 are given in Example 4 and we omit them here. We give such quasigroups up to $r=1155$. We present triplets in which the permutations φ, ψ and the order r of quasigroup $(Z_r, \varphi, \psi, 0)$ are given:

$(-97, 98, 185)$;	$(-153, 154, 191)$;	$(-202, 203, 209)$;	$(-32, 33, 215)$;
$(-33, 34, 227)$;	$(-232, 233, 245)$;	$(-208, 209, 251)$;	$(-118, 119, 263)$;
$(-202, 203, 275)$;	$(-151, 152, 281)$;	$(-59, 60, 293)$;	$(-247, 248, 305)$;
$(-170, 171, 317)$;	$(-164, 165, 323)$;	$(-327, 328, 335)$;	$(-22, 23, 347)$;
$(-312, 313, 359)$;	$(-15, 16, 371)$;	$(-39, 40, 383)$;	$(-66, 67, 389)$;
$(-137, 138, 395)$;	$(-309, 310, 401)$;	$(-356, 357, 407)$;	$(-113, 114, 413)$;
$(-55, 56, 419)$;	$(-402, 403, 425)$;	$(-310, 311, 431)$;	$(-12, 13, 437)$;
$(-249, 250, 449)$;	$(-313, 314, 467)$;	$(-290, 291, 473)$;	$(-197, 198, 479)$;
$(-142, 143, 485)$;	$(-494, 495, 503)$;	$(-317, 318, 515)$;	$(-127, 128, 521)$;
$(-477, 478, 539)$;	$(-82, 83, 545)$;	$(-233, 234, 557)$;	$(-237, 238, 563)$;
$(-109, 110, 569)$;	$(-127, 128, 575)$;	$(-99, 100, 581)$;	$(-111, 112, 593)$;
$(-71, 72, 599)$;	$(-367, 368, 605)$;	$(-538, 539, 617)$;	$(-71, 72, 623)$;
$(-504, 505, 629)$;	$(-552, 553, 641)$;	$(-266, 267, 659)$;	$(-582, 583, 665)$;
$(-125, 126, 671)$;	$(-591, 592, 677)$;	$(-354, 355, 701)$;	$(-484, 485, 707)$;
$(-117, 118, 719)$;	$(-419, 420, 731)$;	$(-59, 60, 737)$;	$(-436, 437, 743)$;
$(-393, 394, 749)$;	$(-66, 67, 773)$;	$(-517, 518, 785)$;	$(-736, 737, 791)$;
$(-225, 226, 797)$;	$(-424, 425, 809)$;	$(-322, 323, 821)$;	$(-150, 151, 827)$;
$(-232, 233, 833)$;	$(-541, 542, 839)$;	$(-134, 135, 851)$;	$(-532, 533, 869)$;
$(-477, 478, 875)$;	$(-389, 390, 881)$;	$(-512, 513, 905)$;	$(-165, 166, 911)$;
$(-147, 148, 935)$;	$(-709, 710, 941)$;	$(-210, 211, 953)$;	$(-337, 338, 959)$;
$(-706, 707, 971)$;	$(-957, 958, 977)$;	$(-208, 209, 983)$;	$(-548, 549, 989)$;
$(-542, 543, 995)$;	$(-810, 811, 1007)$;	$(-180, 181, 1019)$;	$(-637, 638, 1031)$;

$$\begin{array}{llll}
(-674, 675, 1037); & (-267, 268, 1043); & (-82, 83, 1049); & (-427, 428, 1055); \\
(-433, 434, 1067); & (-269, 270, 1091); & (-536, 537, 1097); & (-889, 890, 1103); \\
(-761, 762, 1109); & (-382, 383, 1115); & (-753, 754, 1121); & (-134, 135, 1127); \\
(-1038, 1039, 1133); & (-997, 998, 1139); & (-872, 873, 1145); & (-561, 562, 1151).
\end{array}$$

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A. V. Scerbacova
Gubkin Russian State Oil and Gas University
Leninsky Prospect, 65, Moscow 119991
Russia
E-mail: *scerbik33@yandex.ru*

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V.A. Shcherbacov
Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
Academiei str. 5, MD−2028 Chişinău
Moldova
E-mail: *scerb@math.md*