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Regular, Intra-regular and Duo Γ-Semirings

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Abstract. In this paper we give several characterizations of a regular Γ-semiring, intra-regular Γ-semiring and a duo Γ-semiring by using ideals, interior-ideals, quasi-ideals and bi-ideals of a Γ-semiring.

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1 Introduction

The notion of a quasi-ideal was firstly introduced for semigroups in [15] and for rings in [16] by Steinfeld. Iseki in [6] discussed some characterizations of quasi-ideals for a semiring without zero. Using quasi-ideals, Shabir, Ali, Batool in [14] characterize a class of semirings. Chinram in [2] generalizes the concept of a quasi-ideal to a Γ-semigroup and discussed some of its properties. Also in [1] Chinram gave some different characterizations of quasi-ideals in a Γ-semiring while the concept of a Γ-semiring was coined by Rao in [13]. The authors studied quasi-ideals and minimal quasi-ideals in Γ-semirings in [7] and quasi-ideals in regular Γ-semirings in [8].

The notion of a bi-ideal was first introduced for semigroups by Good and Hughes in [4]. The concept of a bi-ideal for a ring was given by Lajos [9] . Also in [10,11] Lajos discussed some characterizations of bi-ideals in semigroups. Shabir, Ali, Batool in [14] gave some properties of bi-ideals in a semiring.

The concept of a regular ring was introduced by J. von Neumann in [12] and he gave the definition of a regular ring as follows: a ring R is regular if for any $b \in R$ there exists $x \in R$ such that $b = bxb$. Analogously the concept of a regular semigroup was introduced by Green in [5] and a regular semiring was introduced by Zelznikov [17]. This concept of regularity was extended to a Γ-semiring by Rao [13] and wos studied by Dutta and Sardar in [3].

In this paper efforts are made to prove various characterizations of a regular Γ-semiring, intra-regular Γ-semiring and a duo Γ-semiring by using ideals, interior-ideals, quasi-ideals and bi-ideals of a Γ-semiring.

2 Preliminaries

First we recall some definitions of the basic concepts of Γ-semirings that we need in sequel. For this we follow Dutta and Sardar [3].

Definition 1. Let S and Γ be two additive commutative semigroups. S is called a Γ-semiring if there exists a mapping $S \times \Gamma \times S \longrightarrow S$ denoted by $a\alpha b$ for all $a, b \in S$ and $\alpha \in \Gamma$ satisfying the following conditions:
(i) $a\alpha\,(b + c) = (a\alpha b) + (a\alpha c)$,
(ii) $(b + c)\,\alpha a = (b\alpha a) + (c\alpha a)$,
(iii) $a(\alpha + \beta)c = (a\alpha c) + (a\beta c)$,
(iv) $a\alpha\,(b\beta c) = (a\alpha b)\,\beta c$; for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

Definition 2. An element $0 \in S$ is said to be an absorbing zero if $0\alpha a = 0 = a\alpha 0$, and $a + 0 = 0 + a = a$ for all $a \in S$ and $\alpha \in \Gamma$.

Definition 3. A non-empty subset T of a Γ-semiring S is said to be a sub-Γ- semiring of S if (T,+) is a subsemigroup of (S,+) and $a\alpha b \in T$ for all $a, b \in T$and $\alpha \in \Gamma$.

Definition 4. A non-empty subset T of a Γ-semiring S is called a left (respectively right) ideal of S if T is a subsemigroup of (S,+) and $x\alpha a \in T$ (respectively $a\alpha x \in T$) for all $a \in T,\ x \in S$ and $\alpha \in \Gamma$.

Definition 5. If T is both left and right ideal of a Γ-semiring S, then T is known as an ideal of S.

A quasi-ideal Q in a Γ-semiring S is defined as follows.

Definition 6. A subsemigroup Q of $(S, +)$ is a quasi-ideal of S if $(S\Gamma Q) \cap (Q\Gamma S) \subseteq Q$.

Example. Consider a Γ-semiring $S = M_{2\times 2}(N_0)$, where N_0 denotes the set of natural numbers with zero and Γ= S. Define $A\alpha B$= usual matrix product of A, α and B; for all $A, \alpha\ , B\ \in S$. Then
$Q = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in N_0 \right\}$ is a quasi-ideal of a Γ-semiring S.

Definition 7. A non-empty subset B of a Γ- semiring S is a bi-ideal of a Γ-semiring S if B is a sub-Γ-semiring of S and $B\Gamma S\Gamma B \subseteq B$.

Example. Let N be the set of natural numbers and $\Gamma = 2N$. Then N and Γ both are additive commutative semigroups. An image of a mapping $N \times \Gamma \times N \longrightarrow N$ is denoted by $a\alpha b$ and defined as $a\alpha b$ = product of a, α, b, for all $a, b \in S$ and $\alpha \in \Gamma$. Then N forms a Γ-semiring. $B = 3N$ is a bi-ideal of N.

Now we define a generalized bi-ideal and an interior-ideal of a Γ- semiring S.

Definition 8. A non-empty subset B of a Γ- semiring S is a generalized bi-ideal of a Γ- semiring S if $B\Gamma S\Gamma B \subseteq B$.

Definition 9. A non-empty subset I of a Γ- semiring S is an interior-ideal of a Γ-semiring S if I is a subsemigroup of S and $S\Gamma I\Gamma S \subseteq I$.

Proposition 1. *For each non-empty subset X of a Γ- semiring S the following statements hold.*
(i) $S\Gamma X$ is a left ideal of S.
(ii) $X\Gamma S$ is a right ideal of S.
(iii) $S\Gamma X\Gamma S$ is an ideal of S.

Proposition 2. *If S is a Γ- semiring S and $a \in S$, then the following statements hold.*
(i) $S\Gamma a$ is a left ideal of S.
(ii) $a\Gamma S$ is a right ideal of S.
(iii) $S\Gamma a\Gamma S$ is an ideal of S.

Now onwards S denotes a Γ-semiring with absorbing zero unless otherwise stated.

3 Regular Γ-Semiring

An element a of a Γ-semiring S is said to be regular if $a \in a\Gamma S\Gamma a$.
If all elements of a Γ-semiring S are regular, then S is known as a regular Γ-semiring. The following theorem was proved in [8] by the authors.

Theorem 1. *In S the following statements are equivalent.*
(1) S is regular.
(2) For every left ideal L and a right ideal R of S, $R\Gamma L = R \cap L$.
(3) For every left ideal L and a right ideal R of S,
(i) $R^2 = R\Gamma R = R$,
(ii) $L^2 = L\Gamma L = L$,
(iii) $R\Gamma L = R \cap L$ is a quasi-ideal of S.
(4) The set of all quasi-ideals of S is a regular Γ-semigroup.
(5) Every quasi-ideal of S is of the form $Q\Gamma S\Gamma Q = Q$.

Theorem 2. *The following statements are equivalent in S.*
(1) S is regular.
(2) For any bi-ideal B of S, $B\Gamma S\Gamma B = B$.
(3) For any quasi-ideal Q of S, $Q\Gamma S\Gamma Q = Q$.

Proof. (1) $\Rightarrow$ (2) Let B be a bi-ideal of S and $b \in B$. As S is regular, $b \in b\Gamma S\Gamma b \subseteq B\Gamma S\Gamma B$. Therefore $B \subseteq B\Gamma S\Gamma B$. Hence $B = B\Gamma S\Gamma B$.
(2) $\Rightarrow$ (3) As every quasi-ideal is a bi-ideal, implication (2) $\Rightarrow$ (3) holds.
(3) $\Rightarrow$ (1) Let R be a right ideal and L be a left ideal of S. Then $R \cap L$ is a quasi-ideal of S. Hence by assumption $R\cap L = (R \cap L)\,\Gamma S\Gamma\,(R \cap L) \subseteq (R\Gamma S)\,\Gamma L \subseteq R\Gamma L$. Therefore $R \cap L = R\Gamma L$. Thus S is a regular Γ-semiring by Theorem 1. □

Theorem 3. *In S the following statements are equivalent.*
(1) S is regular.
(2) For every bi-ideal B and an ideal I of S, $B \cap I = B\Gamma I\Gamma B$.
(3) For every quasi-ideal Q and an ideal I of S, $Q \cap I = Q\Gamma I\Gamma Q$.

Proof. (1) $\Rightarrow$ (2) Let B be a bi-ideal and I be an ideal of S. Now $B\Gamma I\Gamma B \subseteq B\Gamma S\Gamma B \subseteq B$ and $B\Gamma I\Gamma B \subseteq I$. Therefore $B\Gamma I\Gamma B \subseteq B\cap I$. For the reverse inclusion, let $a \in B \cap I$. As S is regular, $a \in a\Gamma S\Gamma a$. Then $a\Gamma S\Gamma a \subseteq (a\Gamma S\Gamma a)\Gamma S\Gamma a\Gamma S\Gamma a) \subseteq (B\Gamma S\Gamma B)\,\Gamma(S\Gamma I\Gamma S)\Gamma B \subseteq B\Gamma I\Gamma B$. Therefore $a \in Q\Gamma I\Gamma Q$. Hence we have $B \cap I \subseteq B\Gamma I\Gamma B$. Thus we get $B\Gamma I\Gamma B = B \cap I$.
(2) $\Rightarrow$ (3) Implication follows as every quasi-ideal of S is a bi-ideal.
(3) $\Rightarrow$ (1) Let R be a right ideal and L be a left ideal of S. Then by assumption we have, $R = R\cap S = R\Gamma S\Gamma R \subseteq R\Gamma R$ and $L\cap S = L\Gamma S\Gamma L \subseteq L\Gamma L$. Also $R\cap L = R\Gamma L$ is a quasi-ideal of S. Hence by Theorem 1, S is a regular Γ-semiring. □

Proof of the following theorem is straightforward.

Theorem 4. *In S the following statements are equivalent.*
(1) S is regular.
(2) For every bi-ideal B and a left ideal L of S, $B \cap L \subseteq B\Gamma L$.
(3) For every quasi-ideal Q and a left ideal L of S, $Q \cap L \subseteq Q\Gamma$L.
(4) For every bi-ideal B and a right ideal R of S, $B \cap R \subseteq R\Gamma B$.
(5) For every right ideal R and a quasi-ideal Q of S, $R \cap Q \subseteq R\Gamma Q$.
(6) For every left ideal L, every right ideal R and every bi-ideal B of S, $L \cap R \cap B \subseteq R\Gamma B\Gamma L$.
(7) For every left ideal, every right ideal R and every quasi-ideal Q of S, $L\cap R\cap Q \subseteq R\Gamma Q\Gamma L$.

Theorem 5. *In S the following conditions are equivalent.*
(1) S is regular.
(2) $I \cap Q = Q\Gamma I\Gamma Q$, for an ideal I and a quasi-ideal Q of S.
(3) $I \cap Q = Q\Gamma I\Gamma Q$, for an interior ideal I and a quasi-ideal Q of S.

Proof. (1) $\Rightarrow$ (2) Let Q be a quasi-ideal and I be an ideal of S. Now $Q\Gamma I\Gamma Q \subseteq Q\Gamma S\Gamma Q \subseteq Q\Gamma S$ by Proposition 1. Similarly we get $Q\Gamma I\Gamma Q \subseteq S\Gamma Q$. Therefore $Q\Gamma I\Gamma Q \subseteq (S\Gamma Q)\cap(Q\Gamma S) \subseteq Q$, since Q is a quasi-ideal. Also $Q\Gamma I\Gamma Q \subseteq I$ as I is an ideal. Therefore $Q\Gamma I\Gamma Q \subseteq Q\cap I$. For the reverse inclusion, let $a \in Q\cap I$. As S is regular, $a \in a\Gamma S\Gamma a$. We have $a \in (a\Gamma S\Gamma a)\,\Gamma S\Gamma\,(a\Gamma S\Gamma a) \subseteq (Q\Gamma S\Gamma Q)\,\Gamma(S\Gamma I\Gamma S)\Gamma Q \subseteq Q\Gamma I\Gamma Q$. Hence $Q \cap I \subseteq Q\Gamma I\Gamma Q$. Therefore $Q\Gamma I\Gamma Q = Q \cap I$.
(2) $\Rightarrow$ (1) Let Q be a quasi-ideal of S. By (2), $Q\Gamma S\Gamma Q = Q\cap S$. Hence $Q\Gamma S\Gamma Q = Q$. Therefore S is regular by Theorem 2.
(1) $\Rightarrow$ (3) Let Q be a quasi-ideal and I be an interior ideal of S. Now $Q\Gamma I\Gamma Q \subseteq Q\Gamma S\Gamma Q \subseteq Q\Gamma S$ by Proposition 1. Similarly we get $Q\Gamma I\Gamma Q \subseteq S\Gamma Q$. Therefore $Q\Gamma I\Gamma Q \subseteq (S\Gamma Q)\cap(Q\Gamma S) \subseteq Q$. Also $Q\Gamma I\Gamma Q \subseteq I$ as I is an interior ideal. Therefore $Q\Gamma I\Gamma Q \subseteq Q\cap I$. For the reverse inclusion, let $a \in Q\cap I$. As S is regular, $a \in a\Gamma S\Gamma a$. Therefore $a \in (a\Gamma S\Gamma a)\,\Gamma S\Gamma\,(a\Gamma S\Gamma a) \subseteq (Q\Gamma S\Gamma Q)\,\Gamma(S\Gamma I\Gamma S)\Gamma Q \subseteq Q\Gamma I\Gamma Q$. Therefore $Q \cap I \subseteq Q\Gamma I\Gamma Q$. Hence $Q\Gamma I\Gamma Q = Q \cap I$.
(3) $\Rightarrow$ (1) Let Q be a quasi-ideal of S. By (3), $Q\Gamma S\Gamma Q = Q\cap S$. Hence $Q\Gamma S\Gamma Q = Q$. Hence by Theorem 2, S is regular. □

Theorem 6. *In S the following statements are equivalent.*
(1) S is regular.
(2) $Q \cap L \subseteq Q\Gamma L$, for a quasi-ideal Q and a left ideal L of S.
(3) $Q \cap R \subseteq R\Gamma Q$, for a quasi-ideal Q and a right ideal R of S.

Theorem 7. *S is regular if and only if $R \cap Q \cap L \subseteq R\Gamma Q\Gamma L$, for a right ideal R, quasi-ideal Q and a left ideal L of S.*

Proof. Suppose that S is a regular Γ-semiring. Let R be a right ideal, Q be a quasi-ideal and L be a left ideal of S. Let $a \in R \cap Q \cap L$. As S is regular, $a \in a\Gamma S\Gamma a$. Therefore $a \in (a\Gamma S\Gamma a)\,\Gamma S\Gamma a \subseteq (R\Gamma S)\,\Gamma Q\Gamma\,(S\Gamma L) \subseteq R\Gamma Q\Gamma L$. Hence $R \cap Q \cap L \subseteq R\Gamma Q\Gamma L$. Conversely, let R be a right ideal and L be a left ideal of S. By assumption $R \cap S \cap L \subseteq R\Gamma S\Gamma L$. Therefore $R \cap L \subseteq R\Gamma L$. Thus we have $R \cap L = R\Gamma L$. Hence S is regular by Theorem 1. □

4 Intra-regular Γ-semiring

Now we give the definition of an intra-regular Γ-semiring.

Definition 10. A Γ-semiring S is said to be an intra-regular Γ-semiring if for any $x \in S$, $x \in S\Gamma x\Gamma x\Gamma S$.

Theorem 8. *S is intra-regular if and only if each right ideal R and left ideal L of S satisfy $R \cap L \subseteq L\Gamma R$.*

Proof. Suppose that S is an intra-regular Γ-semiring and R and L be a right ideal and a left ideal of S respectively. Let $a \in R \cap L$. As S is intra-regular, $a \in S\Gamma a\Gamma a\Gamma S$. Now $S\Gamma a\Gamma a\Gamma S = (S\Gamma a)\,\Gamma(a\Gamma \mathrm{S}) \subseteq (S\Gamma L)\,\Gamma(R\Gamma S) \subseteq L\Gamma R$. Therefore $R \cap L \subseteq L\Gamma R$. Conversely, for $a \in S$, $(a)_l = N_0 a + S\Gamma a$, $(a)_r = N_0 a + a\Gamma S$. By assumption $(a)_r \cap (a)_l \subseteq (a)_l\Gamma(a)_r$. Then $(a)_r \cap (a)_l \subseteq (a)_l\Gamma(a)_r = (N_0 a + S\Gamma a)\,\Gamma\,(N_0 a + a\Gamma S)$. Also by assumption we have $(a)_r \subseteq S\Gamma a + S\Gamma a\Gamma S$ and $(a)_l \subseteq a\Gamma S + S\Gamma a\Gamma S$. Hence we have $(a)_r \subseteq S\Gamma a + S\Gamma a\Gamma S \subseteq S\Gamma a\Gamma a\Gamma S$. Therefore we get $a \in S\Gamma a\Gamma a\Gamma S$. Thus any $a \in S$ is an intra-regular element of S. Therefore S is an intra-regular Γ-semiring. □

Theorem 9. *In S the following statements are equivalent.*
(1) S is intra-regular.
(2) For bi-ideals B_1 and B_2 of S, $B_1 \cap B_2 \subseteq S\Gamma B_1\Gamma B_2\Gamma S$.
(3) For every bi-ideal B and a quasi-ideal Q of S, $B \cap Q \subseteq (S\Gamma Q\Gamma B\Gamma S) \cap (S\Gamma B\Gamma Q\Gamma S)$.
(4) For every quasi-ideals Q_1 and Q_2 of S, $Q_1 \cap Q_2 \subseteq S\Gamma Q_1\Gamma Q_2\Gamma S$.

Proof. (1) ⇒ (2) Suppose that S is intra-regular. Let B_1 and B_2 be bi-ideals of S. Let $a \in B_1 \cap B_2$. As S is intra-regular, $a \in S\Gamma a\Gamma a\Gamma S$. $a \in S\Gamma a\Gamma a\Gamma S \subseteq S\Gamma B_1\Gamma B_2\Gamma S$. Therefore $B_1 \cap B_2 \subseteq S\Gamma B_1\Gamma B_2\Gamma S$.
(2) ⇒ (3), (3) ⇒ (4) Implications follow as every quasi-ideal is a bi-ideal.

(4) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S. Then R and L both are quasi-ideals of S. By (4), $R \cap L \subseteq S\Gamma L\Gamma R\Gamma S = (S\Gamma L)\,\Gamma\,(R\Gamma S) \subseteq L\Gamma R$. Therefore we get $R \cap L \subseteq L\Gamma R$. Thus by Theorem 8, S is an intra-regular Γ-semiring.
Thus we have proved (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$(4) $\Rightarrow$ (1). □

Theorem 10. *In S the following statements are equivalent.*
(1) S is intra-regular.
(2) For a left ideal L and a bi-ideal B of S, $L \cap B \subseteq L\Gamma B\Gamma S$.
(3) For a left ideal L and a quasi-ideal Q of S, $L \cap Q \subseteq L\Gamma Q\Gamma S$.
(4) For a right ideal R and a bi-ideal B of S, $R \cap B \subseteq S\Gamma B\Gamma R$.
(5) For a right ideal R and a quasi-ideal Q of S, $R \cap Q \subseteq S\Gamma Q\Gamma R$.

Proof. (1) $\Rightarrow$ (2) Suppose that S is intra-regular. Let L be a left ideal and B be a bi-ideal of S. Let $a \in B \cap L$. As S is intra-regular, $a \in S\Gamma a\Gamma a\Gamma S$. $a \in S\Gamma a\Gamma a\Gamma S \subseteq S\Gamma L\Gamma B\Gamma S \subseteq L\Gamma B\Gamma S$. Hence $B \cap L \subseteq L\Gamma B\Gamma S$.
(2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) As every quasi-ideal is a bi-ideal, implications follow.
(3) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S. Then R is a quasi-ideal of S. By (3), $R \cap L \subseteq L\Gamma R\Gamma S \subseteq L\Gamma R$. Therefore we get $R \cap L \subseteq L\Gamma R$. Thus by Theorem 8, S is an intra-regular Γ-semiring.
(1) $\Rightarrow$ (4) Suppose that S is intra-regular. Let R be a right ideal and B be a bi-ideal of S. Let $a \in B \cap R$. As S is intra-regular, $a \in S\Gamma a\Gamma a\Gamma S$. Hence $a \in S\Gamma a\Gamma a\Gamma S \subseteq S\Gamma B\Gamma R\Gamma S \subseteq S\Gamma B\Gamma R$. This shows that $B \cap R \subseteq S\Gamma B\Gamma R$.
(5) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S. By (5), $R\cap L \subseteq S\Gamma L\Gamma R \subseteq L\Gamma R$, since L is a quasi-ideal of S. Therefore we get $R \cap L \subseteq L\Gamma R$. This shows that S is an intra-regular Γ-semiring by Theorem 8. □

Theorem 11. *In S the following statements are equivalent.*
(1) S is intra-regular.
(2) $K \cap B \cap R \subseteq K\Gamma B\Gamma R$, for a bi-ideal B, a right ideal R and an interior ideal K of S.
(3) $I \cap B \cap R \subseteq I\Gamma B\Gamma R$, for a bi-ideal B, a right ideal R and an ideal I of S.
(4) $K \cap Q \cap R \subseteq K\Gamma Q\Gamma R$, for a quasi-ideal Q, a right ideal R and an interior ideal K of S.
(5) $I \cap Q \cap R \subseteq I\Gamma Q\Gamma R$, for a quasi-ideal Q, a right ideal R and an ideal I of S.

Proof. (1) $\Rightarrow$ (2) Suppose that S is intra-regular. Let R be a right ideal, K be an interior ideal and B be a bi-ideal of S. Let $a \in K \cap B \cap R$. As S is intra-regular, $a \in S\Gamma a\Gamma a\Gamma S$. Therefore $a \in S\Gamma a\Gamma a\Gamma S \subseteq (S\Gamma K\Gamma S)\,\Gamma B\Gamma\,(R\Gamma S\Gamma S) \subseteq K\Gamma B\Gamma R$. Thus we have $K \cap B \cap R \subseteq K\Gamma B\Gamma R$.
(2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) As every ideal is an interior ideal, implications follow.
(2) $\Rightarrow$ (4), (3) $\Rightarrow$ (5) Clearly implications follow, since quasi-ideal is a bi-ideal.
(5)$\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S. As L is a quasi-ideal of S, by (5) we have $S \cap L \cap R \subseteq S\Gamma L\Gamma R \subseteq L\Gamma R$. Therefore we have $R \cap L \subseteq L\Gamma R$. Hence by Theorem 8, S is an intra-regular Γ-semiring. □

Theorem 12. *In S the following statements are equivalent.*
(1) S is intra-regular.
(2) $I \cap B \cap L \subseteq L\Gamma B\Gamma I$, for a bi-ideal B, a left ideal L and an interior ideal I of S.
(3) $I \cap B \cap L \subseteq L\Gamma B\Gamma I$, for a bi-ideal B, a left ideal L and an ideal I of S.
(4) $I \cap Q \cap L \subseteq L\Gamma Q\Gamma I$, for a quasi-ideal Q, a left ideal L and an interior ideal I of S.
(5) $I \cap Q \cap L \subseteq L\Gamma Q\Gamma I$, for a quasi-ideal Q, a left ideal L and an ideal I of S.

Proof. (1) $\Rightarrow$ (2) Suppose that S is intra-regular. Let L be a left ideal, I be an interior ideal and B be a bi-ideal of S. Let $a \in I \cap B \cap L$. As S is intra-regular, $a \in S\Gamma a\Gamma a\Gamma S$. $a \in S\Gamma a\Gamma a\Gamma S \subseteq (S\Gamma S\Gamma L)\,\Gamma B\Gamma\,(S\Gamma I\Gamma S) \subseteq L\Gamma B\Gamma I$. Thus we have $I \cap B \cap L \subseteq L\Gamma B\Gamma I$.
(2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) Clearly implications follow, since an ideal is an interior ideal.
(2) $\Rightarrow$ (4), (4) $\Rightarrow$ (5) As every quasi-ideal is a bi-ideal, implications follow.
(5) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S. As right ideal R is a quasi-ideal, and S itself is an ideal of S, $S \cap R \cap L \subseteq L\Gamma R\Gamma S$ by (5). Therefore $L\Gamma R\Gamma S \subseteq L\Gamma R$. Thus we get $R \cap L \subseteq L\Gamma R$. Therefore S is an intra-regular Γ-semiring by Theorem 8. $\square$

5 Regular and Intra-regular Γ-semiring

Theorem 13. *For S the following statements are equivalent.*
(1) S is regular and intra-regular.
(2) Each right ideal R and left ideal L of S satisfy $R \cap L = R\Gamma L \subseteq L\Gamma R$.
(3) Each bi-ideal B of S satisfies $B = B^2 = B\Gamma B$.
(4) Each quasi-ideal Q of S satisfies $Q = Q^2 = Q\Gamma Q$.

Proof. (1) $\Leftrightarrow$ (2) Proof follows from Theorems 1 and 8.
(1) $\Rightarrow$ (3) Suppose that S is regular and intra-regular. Let B be a bi-ideal of S. Then $B^2 = B\Gamma B \subseteq B$. For the reverse inclusion, let $a \in B$. As S is regular and intra-regular, we have $a \in a\Gamma S\Gamma a$ and $a \in S\Gamma a\Gamma a\Gamma S$. Hence $a \in a\Gamma S\Gamma a \subseteq a\Gamma S\Gamma\,(a\Gamma S\Gamma a) \subseteq a\Gamma S\Gamma\,(S\Gamma a\Gamma a\Gamma S)\,\Gamma S\Gamma a \subseteq.\ (B\Gamma S\Gamma B)\Gamma(B\Gamma S\Gamma B) \subseteq B\Gamma B$. Therefore $B \subseteq B\Gamma B$. Thus we get $B = B\Gamma B = B^2$.
(3) $\Rightarrow$ (4) As every quasi-ideal is a bi-ideal, implication follows.
(4) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S. Then $R \cap L$ is a quasi-ideal of S. By (4), $R \cap L = (R \cap L)^2 = (R \cap L)\,\Gamma(R \cap L) \subseteq L\Gamma R$. This shows that S is an intra-regular Γ-semiring by Theorem 8. Similarly $R \cap L = (R \cap L)^2 = (R \cap L)\,\Gamma(R \cap L) \subseteq R\Gamma L$. Hence we get $R \cap L = R\Gamma L$. Therefore S is a regular Γ-semiring by Theorem 1. $\square$

Theorem 14. *In S the following statements are equivalent.*
(1) S is regular and intra-regular.
(2) For bi-ideals B_1 and B_2 of S, $B_1 \cap B_2 \subseteq (B_1\Gamma B_2) \cap (B_2\Gamma B_1)$.
(3) For every bi-ideal B and a quasi-ideal Q of S, $B \cap Q \subseteq (Q\Gamma B) \cap (B\Gamma Q)$.
(4) For quasi-ideals Q_1 and Q_2 of S, $Q_1 \cap Q_2 \subseteq (Q_1\Gamma Q_2) \cap (Q_2\Gamma Q_1)$.

(5) For every quasi-ideal Q and a generalized bi-ideal G of S, $G \cap Q \subseteq (G\Gamma Q) \cap (Q\Gamma G)$.
(6) For every left ideal L and a bi-ideal B of S, $B \cap L \subseteq (B\Gamma L) \cap (L\Gamma B)$.
(7) For every left ideal L and a quasi-ideal Q of S, $Q \cap L \subseteq (Q\Gamma L) \cap (L\Gamma Q)$.
(8) For every right ideal R and a bi-ideal B of S, $B \cap R \subseteq (B\Gamma R) \cap (R\Gamma B)$.
(9) For every quasi-ideal Q and a right ideal R of S, $R \cap Q \subseteq (R\Gamma Q) \cap (Q\Gamma R)$.
(10) For every left ideal L and a right ideal R of S, $R \cap L \subseteq (R\Gamma L) \cap (L\Gamma R)$.

Proof. (1) $\Rightarrow$ (2) Suppose that S is regular and intra-regular. Let B_1 and B_2 be bi-ideals of S. Let $a \in B_1 \cap B_2$. As S is regular and intra-regular, $a \in a\Gamma S\Gamma a$ and $a \in S\Gamma a\Gamma a\Gamma S$. Hence $a \in a\Gamma S\Gamma a \subseteq (a\Gamma S\Gamma S\Gamma a)\Gamma(a\Gamma S\Gamma S\Gamma a) \subseteq (B_1\Gamma S\Gamma B_1)\Gamma(B_2\Gamma S\Gamma B_2) \subseteq B_1\Gamma B_2$.

Similarly we can show that $a \in B_2\Gamma B_1$. Therefore $a \in B_1 \cap B_2$ implies $a \in B_1\Gamma B_2$ and $a \in B_2\Gamma B_1$. This gives $B_1 \cap B_2 \subseteq (B_1\Gamma B_2) \cap (B_2\Gamma B_1)$.

(2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4) Implications follow as every quasi-ideal is a bi-ideal.
(4) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S. Then R and L both are quasi-ideals of S. By (4) , $R \cap L \subseteq (R\Gamma L) \cap (L\Gamma R)$. $R \cap L \subseteq L\Gamma R$ implies S is an intra-regular Γ-semiring by Theorem 8. Also $R \cap L \subseteq R\Gamma L$. Therefore we get $R \cap L = R\Gamma L$. Hence by Theorem 1, S is a regular Γ-semiring.
(1) $\Rightarrow$ (5) Suppose that S is regular and intra-regular. Let G be a generalized bi-ideal and Q be quasi-ideal of S. Let $a \in G \cap Q$. As S is regular and intra-regular, $a \in a\Gamma S\Gamma a$ and $a \in S\Gamma a\Gamma a\Gamma S$. Therefore $a \in a\Gamma S\Gamma a \subseteq a\Gamma S\Gamma(a\Gamma S\Gamma a) \subseteq (a\Gamma S\Gamma S\Gamma a)\Gamma(a\Gamma S\Gamma S\Gamma a) \subseteq (G\Gamma S\Gamma G)\Gamma(Q\Gamma S\Gamma Q) \subseteq G\Gamma Q$. Hence $a \in G\Gamma Q$. Similarly we can show that $a \in Q\Gamma G$.Therefore $a \in G\cap Q$ implies $a \in G\Gamma Q$ and $a \in Q\Gamma G$, which gives $G \cap Q \subseteq (G\Gamma Q) \cap (Q\Gamma G)$.
(5) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S respectively. As R is a generalized bi-ideal and L is a quasi-ideal of S, proof follows from (4) $\Rightarrow$(1).
(1) $\Rightarrow$ (6) Suppose that S is regular and intra-regular. Let B be a bi-ideal and L be a left ideal of S. Let $a \in B \cap L$. As S is regular and intra-regular, $a \in a\Gamma S\Gamma a$ and $a \in S\Gamma a\Gamma a\Gamma S$. $a \in a\Gamma S\Gamma a \subseteq a\Gamma S\Gamma(a\Gamma S\Gamma a) \subseteq (a\Gamma S\Gamma S\Gamma a)\Gamma(a\Gamma S\Gamma S\Gamma a) \subseteq (B\Gamma S\Gamma B)\Gamma(S\Gamma S\Gamma S\Gamma L) \subseteq B\Gamma L$. Therefore we get $a \in B\Gamma L$. Similarly we can show that $a \in L\Gamma B$. Therefore $a \in B \cap L$ implies $a \in B\Gamma L$ and $a \in L\Gamma B$. Hence $B \cap L \subseteq (B\Gamma L) \cap (L\Gamma B)$.
(6) $\Rightarrow$ (7) As every quasi-ideal is a bi-ideal, implication follows.
(7) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S respectively. As R is a quasi-ideal of S, proof follows from (4) $\Rightarrow$ (1).
(1) $\Rightarrow$ (8) Suppose that S is regular and intra-regular. Let R be right ideal and B be a bi-ideals of S. Let $a \in B \cap R$. As S is regular and intra-regular, $a \in a\Gamma S\Gamma a$ and $a \in S\Gamma a\Gamma a\Gamma S$. Therefore $a \in a\Gamma S\Gamma a \subseteq a\Gamma S\Gamma(a\Gamma S\Gamma a) \subseteq (a\Gamma S\Gamma S\Gamma a)\Gamma(a\Gamma S\Gamma S\Gamma a) \subseteq (B\Gamma S\Gamma B)\Gamma(R\Gamma S\Gamma S\Gamma S) \subseteq B\Gamma R$. Therefore we get $a \in B\Gamma R$. Similarly we can show that $a \in R\Gamma B$. Therefore $a \in B \cap R$ implies $a \in B\Gamma R$ and $a \in R\Gamma B$, which gives $B \cap R \subseteq (B\Gamma R) \cap (R\Gamma B)$.
(8) $\Rightarrow$ (9), (9) $\Rightarrow$ (10) Implications follow as every left ideal is a quasi-ideal.

(10) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S. Proof follows from (4) $\Rightarrow$ (1). □

Theorem 15. *In S the following statements are equivalent.*
(1) S is regular and intra-regular.
(2) For bi-ideals B_1 and B_2 of S, $B_1 \cap B_2 \subseteq (B_1\Gamma B_2\Gamma B_1) \cap (B_2\Gamma B_1\Gamma B_2)$.
(3) For a quasi-ideal Q and a bi-ideal B of S, $Q \cap B \subseteq (B\Gamma Q\Gamma B) \cap (Q\Gamma B\Gamma Q)$.
(4) For quasi-ideals Q_1 and Q_2 of S, $Q_1 \cap Q_2 \subseteq (Q_1\Gamma Q_2\Gamma Q_1) \cap (Q_2\Gamma Q_1\Gamma Q_2)$.
(5) For a bi-ideal B and a left ideal L of S, $B \cap L \subseteq B\Gamma L\Gamma B$.
(6) For a quasi-ideal Q and a left ideal L of S, $Q \cap L \subseteq Q\Gamma L\Gamma Q$.
(7) For a bi-ideal B and a right ideal R of S, $B \cap R \subseteq B\Gamma R\Gamma B$.
(8) For a quasi-ideal Q and a right ideal R of S, $Q \cap R \subseteq Q\Gamma R\Gamma Q$.
(9) For a quasi-ideal Q and a generalized bi-ideal G of S, $Q \cap G \subseteq (Q\Gamma G\Gamma Q) \cap (G\Gamma Q\Gamma G)$.

Proof. (1)$\Rightarrow$ (2) Suppose that S is regular and intra-regular. Let B_1 and B_2 be bi-ideals of S. Let $a \in B_1 \cap B_2$. As S is regular and intra-regular, $a \in a\Gamma S\Gamma a$ and $a \in S\Gamma a\Gamma a\Gamma S$. Hence $a \in a\Gamma S\Gamma a \subseteq a\Gamma S\Gamma(a\Gamma S\Gamma a) \subseteq (a\Gamma S\Gamma a)\,\Gamma\,(a\Gamma S\Gamma a)\,\Gamma\,(a\Gamma S\Gamma a) \subseteq (B_1\Gamma S\Gamma B_1)\,\Gamma\,(B_2\Gamma S\Gamma B_2)\,\Gamma\,(B_1\Gamma S\Gamma B_1) \subseteq B_1\Gamma B_2\Gamma B_1$. Therefore $B_1 \cap B_2 \subseteq B_1\Gamma B_2\Gamma B_1$. In the same manner we can show that $B_1 \cap B_2 \subseteq B_2\Gamma B_1\Gamma B_2$. Thus we get $B_1 \cap B_2 \subseteq (B_1\Gamma B_2\Gamma B_1) \cap (B_2\Gamma B_1\Gamma B_2)$.
(2)$\Rightarrow$ (3), (3)$\Rightarrow$ (4) Implications follow as every quasi-ideal is a bi-ideal.
(4) $\Rightarrow$ (1) Let L be a left ideal and R be a right ideal of S. Then $R \cap L$ is a quasi-ideal of S. By (4), $(R \cap L)\cap(R \cap L) \subseteq ((R \cap L)\,\Gamma\,(R \cap L)\,\Gamma\,(R \cap L)) \subseteq L\Gamma R\Gamma R \subseteq L\Gamma R$. Hence $R \cap L \subseteq L\Gamma R$. This shows that S is an intra-regular Γ-semiring by Theorem 8. Also $R \cap L \subseteq ((R \cap L)\,\Gamma\,(R \cap L)\,\Gamma\,(R \cap L))$ implies $R \cap L \subseteq R\Gamma L$. Therefore $R \cap L = R\Gamma L$. Thus S is a regular Γ-semiring by Theorem 1.
(1)$\Rightarrow$ (5) Suppose that S is regular and intra-regular. Let B be a bi-ideal and L be a left ideal of S. Let $a \in B \cap L$. As S is regular and intra-regular, $a \in a\Gamma S\Gamma a$ and $a \in S\Gamma a\Gamma a\Gamma S$. Therefore $a \in a\Gamma S\Gamma a \subseteq a\Gamma S\Gamma(a\Gamma S\Gamma a) \subseteq (a\Gamma S\Gamma a)\,\Gamma\,(S\Gamma a)\,\Gamma\,(a\Gamma S\Gamma a) \subseteq (B\Gamma S\Gamma B)\,\Gamma\,(S\Gamma L)\,\Gamma\,(B\Gamma S\Gamma B) \subseteq B\Gamma L\Gamma B$. Hence we have $B \cap L \subseteq B\Gamma L\Gamma B$.
(5) $\Rightarrow$ (6) As every quasi-ideal is a bi-ideal, implication follows.
(6) $\Rightarrow$ (7) Proof is similar to (4)$\Rightarrow$ (1).
(1) $\Rightarrow$(7) $\Rightarrow$ (8) $\Rightarrow$ (1) can be proved similarly to (1) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$(1). Proof of (1) $\Rightarrow$ (9) is similar to (1) $\Rightarrow$ (2) and proof of (9) $\Rightarrow$ (1) is parallel to (1) $\Rightarrow$ (4) $\Rightarrow$ (1).
Thus we have shown that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$(4) $\Rightarrow$ (1), (1) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$(1) and (1) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (1) and (1) $\Rightarrow$ (9) $\Rightarrow$(1). □

Theorem 16. *In S the following statements are equivalent.*
(1) S is regular and intra-regular.
(2) $B \cap R \cap L \subseteq B\Gamma R\Gamma L$, for a bi-ideal B, right ideal R and a left ideal L of S.
(3) $Q \cap R \cap L \subseteq Q\Gamma R\Gamma L$, for a quasi-ideal Q, right ideal R and left ideal R of S.

Proof. (1) ⇒ (2) Suppose that S is regular and intra-regular. Let B be a bi-ideal, R be a right ideal and L be a left ideal of S. Let $a \in B \cap R \cap L$. As S is regular and intra-regular, $a \in a\Gamma S\Gamma a$ and $a \in S\Gamma a\Gamma a\Gamma S$. Hence $a \in a\Gamma S\Gamma a \subseteq a\Gamma S\Gamma a\Gamma S\Gamma a) \subseteq (a\Gamma S\Gamma S\Gamma a)\,\Gamma\,(a\Gamma S)\Gamma S\Gamma a) \subseteq (B\Gamma S\Gamma B)\,\Gamma\,(R\Gamma S)\,\Gamma S\Gamma L) \subseteq B\Gamma R\Gamma L$. Therefore $B \cap R \cap L \subseteq B\Gamma R\Gamma L$.
(2) ⇒ (3) As every quasi-ideal is a bi-ideal, implication follows.
(3) ⇒ (1) Let L be a left ideal and R be a right ideal of S. As R is a quasi-ideal and S itself a right ideal of S, by (3) we have $R \cap S \cap L \subseteq R\Gamma S\Gamma L \subseteq R\Gamma L$. Therefore $R \cap L \subseteq R\Gamma L$. Thus we get $R \cap L = R\Gamma L$. Hence S is a regular Γ-semiring by Theorem 1. Similarly L is a quasi-ideal and S itself a left ideal of S gives $L \cap R \cap S \subseteq L\Gamma R\Gamma S \subseteq L\Gamma R$ by (3). Thus $R \cap L \subseteq L\Gamma R$. This shows that S is an intra-regular Γ-semiring by Theorem 8. □

6 Duo Γ-semiring

Now we define the notion of a duo Γ-semiring as follows.

Definition 11. A Γ - semiring S is said to be a left (right) duo Γ - semiring if every left (right) ideal of S is a right (left) ideal.

A Γ -semiring S is said to be a duo Γ - semiring if every one-sided ideal of S is a two-sided ideal.
That is a Γ-semiring S is said to be a duo Γ-semiring if it is both left duo and right duo.

Theorem 17. *If S is regular, then S is left duo if and only if for any two left ideals A and B of S, $A \cap B = A\Gamma B$.*

Proof. Let S be a regular Γ-semiring . Assume that S is left duo. Let A and B be any two left ideals of S. As S is left duo, A is a right ideal of S. Then by Theorem 1, $A \cap B = A\Gamma B$. Conversely, suppose that the given condition holds. Let L be a left ideal of S. Then by assumption $L\Gamma S = L \cap S \subseteq L$. This shows that L is a right ideal of S. Therefore S is a left duo Γ-semiring. □

Proof of the following theorem is analogous to proof of Theorem 17.

Theorem 18. *If S is regular, then S is right duo if and only if for any two right ideals A and B of S, $A \cap B = A\Gamma B$*

Theorem 19. *If S is regular, then S is left duo if and only if every quasi-ideal of S is a right ideal of S.*

Proof. Let S be a regular Γ-semiring. Suppose that S is left duo. Let Q be any quasi-ideal of S. Then there exists a right ideal R and a left ideal L of S such that $Q = R \cap L$. Therefore $Q = R \cap L$ is a right ideal of S. Conversely, let L be a left ideal of S. Then L is a quasi-ideal of S. Hence by assumption L is a right ideal of S. Therefore S is a left duo Γ-semiring. □

Proofs of the following theorems are similar to proof of Theorem 19.

Theorem 20. *If S is regular, then S is right duo if and only if every quasi-ideal of S is a left ideal of S.*

Theorem 21. *If S is regular, then S is duo if and only if every quasi-ideal of S is an ideal of S.*

Theorem 22. *If S is regular, then S is duo if and only if every bi-ideal of S is a ideal of S.*

Theorem 23. *In S the following conditions are equivalent.*
(1) S is regular duo.
(2) $I \cap B = I\Gamma B\Gamma I$, for every ideal I and a bi-ideal B of S.
(3) $I \cap Q = I\Gamma Q\Gamma I$, for every ideal I and a quasi-ideal Q of S.

Proof. (1) ⇒ (2) Suppose that S is a regular duo Γ-semiring. Let I be an ideal and B be a bi-ideal of S. Then by Theorem 22, B is an ideal of S. Therefore $I\Gamma B\Gamma I \subseteq I$ and $I\Gamma B\Gamma I \subseteq B$, since I and B are ideals of S. Hence $I\Gamma B\Gamma I \subseteq I \cap B$. For the reverse inclusion, let $a \in I \cap B$. S is regular implies $a \in a\Gamma S\Gamma a$. $a \in a\Gamma S\Gamma a \subseteq a\Gamma S\Gamma a\Gamma S\Gamma a) \subseteq (I\Gamma S)\,\Gamma B\Gamma\,(S\Gamma I) \subseteq I\Gamma B\Gamma I$. Therefore $I \cap B \subseteq I\Gamma B\Gamma I$. Hence $I \cap B = I\Gamma B\Gamma I$.
(2) ⇒ (3) As every quasi-ideal of S is a bi-ideal of S, implication follows.
(3) ⇒ (1) Let L be a left ideal and R be a right ideal of S. Hence $S \cap L = S\Gamma L\Gamma S$ and $S \cap R = S\Gamma R\Gamma S$ by (3). Therefore $L = S\Gamma L\Gamma S$ and $R = S\Gamma R\Gamma S$. Now $L\Gamma S = S\Gamma L\Gamma S\Gamma S \subseteq S\Gamma L\Gamma S = L$ and $S\Gamma R = S\Gamma S\Gamma R\Gamma S \subseteq S\Gamma R\Gamma S = R$. Hence $L\Gamma S \subseteq L$ and $S\Gamma R \subseteq R$. This shows that L is a right ideal and R is a left ideal of S. Therefore S is a duo Γ-semiring by Definition 11. As S is a duo Γ-semiring, $R \cap L = R\Gamma L\Gamma R$ by (3). $R \cap L = R\Gamma L\Gamma R \subseteq R\Gamma L$. This shows that $R \cap L = R\Gamma L$. Hence by Theorem 1, S is regular. □

Theorem 24. *If S is a Γ- semiring then the following statements are equivalents.*
(1) S is regular duo.
(2) For every bi-ideals A and B of S, $A \cap B = A\Gamma B$.
(3) For every bi-ideal B and a quasi-ideal Q of S, $B \cap Q = B\Gamma Q$.
(4) For every bi-ideal B and a right ideal R of S, $B \cap R = B\Gamma R$.
(5) For every quasi-ideal Q and a bi-ideal B of S, $Q \cap B = Q\Gamma \mathrm{B}$.
(6) For every quasi-ideals Q_1 and Q_2 of S, $Q_1 \cap Q_2 = Q_1\Gamma Q_2$.
(7) For every quasi-ideal Q and a right ideal R of S, $Q \cap R = Q\Gamma R$.
(8) For every left ideal L and a bi-ideal B of S, $L \cap B = L\Gamma B$.
(9) For every left ideal L and a right ideal R of S, $L \cap R = L\Gamma R$.

Proof. We can prove the equivalence of statements such as (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (1), (1) ⇒ (5) ⇒ (6) ⇒ (7) ⇒ (1) and (1) ⇒ (8) ⇒ (9) ⇒ (1) . Proof of each implication is straightforward so omitted. □

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Some Properties of Meromorphic Solutions of Logarithmic Order to Higher Order Linear Difference Equations

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Abstract. This paper is devoted to the study of the growth of solutions of the linear difference equation

$$A_n(z) f(z+n) + A_{n-1}(z) f(z+n-1)$$
$$+ \cdots + A_1(z) f(z+1) + A_0(z) f(z) = 0,$$

where $A_n(z), \cdots, A_0(z)$ are entire or meromorphic functions of finite logarithmic order. We extend some precedent results due to Liu and Mao, Zheng and Tu, Chen and Shon and others.

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1 Introduction and main results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions [9, 16]. We use the notations $\mu(f)$, $\rho(f)$ to denote the lower order and the order of a meromorphic function f. Since Halburd–Korhonen [7] and Chiang–Feng [5], independently, have given a difference version of the logarithmic derivative lemma, and Halburd–Korhonen [8] subsequently showed how all key results of the Nevanlinna theory have corresponding difference variants as well, some interest appeared to investigate solutions of difference equations in the complex domain by making use of this variant of the value distribution theory, see [1, 3, 12–15].

Definition 1 (see [9])**.** Let f be an entire function of order ρ $(0 < \rho < \infty)$, the type of f is defined as

$$\tau(f) = \limsup_{r \to +\infty} \frac{\log M(r, f)}{r^{\rho}}.$$

Similarly the lower type of an entire function f of lower order μ $(0 < \mu < \infty)$ is defined by

$$\underline{\tau}(f) = \liminf_{r \to +\infty} \frac{\log M(r, f)}{r^{\mu}}.$$

We recall the following definitions. The linear measure of a set $E \subset (0,+\infty)$ is defined as $m(E) = \int_0^{+\infty} \chi_E(t)\,dt$ and the logarithmic measure of a set $F \subset (1,+\infty)$ is defined by $lm(F) = \int_1^{+\infty} \frac{\chi_F(t)}{t}dt$, where $\chi_H(t)$ is the characteristic function of a set H. The upper density of a set $E \subset (0,+\infty)$ is defined by

$$\overline{dens}E = \limsup_{r \longrightarrow +\infty} \frac{m(E \cap [0,r])}{r}.$$

The upper logarithmic density of a set $F \subset (1,+\infty)$ is defined by

$$\overline{\log dens}(F) = \limsup_{r \longrightarrow +\infty} \frac{lm(F \cap [1,r])}{\log r}.$$

Proposition 1. *For all $H \subset [1,+\infty)$ the following statements hold :*
i) If $lm(H) = \infty$, then $m(H) = \infty$;
ii) If $\overline{dens}H > 0$, then $m(H) = \infty$;
iii) If $\overline{\log dens}H > 0$, then $lm(H) = \infty$.

Proof. i) Since we have $\frac{\chi_H(t)}{t} \leq \chi_H(t)$ for all $t \in H \subset [1,+\infty)$, then

$$m(H) \geq lm(H).$$

So, if $lm(H) = \infty$, then $m(H) = \infty$. We can easily prove the results ii) and iii) by applying the definition of the limit and the properties $m(H \cap [0,r]) \leq m(H)$ and $lm(H \cap [1,r]) \leq lm(H)$. □

Definition 2 (see [9,16])**.** For $a \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the deficiency of a with respect to a meromorphic function f is defined as

$$\delta(a,f) = \liminf_{r \to +\infty} \frac{m\left(r, \frac{1}{f-a}\right)}{T(r,f)} = 1 - \limsup_{r \to +\infty} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r,f)}.$$

In recent paper [5], Chiang and Feng investigated meromorphic solutions of the linear difference equation

$$A_n(z) f(z+n) + A_{n-1}(z) f(z+n-1)$$
$$+ \cdots + A_1(z) f(z+1) + A_0(z) f(z) = 0, \tag{1}$$

where $A_n(z), \cdots, A_0(z)$ are entire functions such that $A_n(z) A_0(z) \not\equiv 0$, and proved the following result.

Theorem 1 (see [5])**.** *Let $A_0(z), A_1(z), \cdots, A_n(z)$ be entire functions such that there exists an integer l, $0 \leq l \leq n$ such that*

$$\rho(A_l) > \max_{0 \leq j \leq n, j \neq l} \{\rho(A_j)\}.$$

If $f(z)$ is a meromorphic solution of (1), then $\rho(f) \geq \rho(A_l) + 1$.

Note that in Theorem 1, equation (1) has only one dominating coefficient A_l. For the case when there is more than one coefficients which have the maximal order, Laine and Yang [12] obtained the following result.

Theorem 2 (see [12])**.** *Let $A_0(z), A_1(z), \cdots, A_n(z)$ be entire functions of finite order such that among those having the maximal order $\rho = \max_{0\le j\le n}\{\rho(A_j)\}$, one has exactly its type strictly greater than the others. Then for any meromorphic solution of (1), we have $\rho(f) \ge \rho + 1$.*

Recently, Liu and Mao [13], Zheng and Tu [15] investigated the growth of solutions of equation (1) and proved the following results.

Theorem 3 (see [15])**.** *Let $A_0(z), \cdots, A_n(z)$ be entire functions such that there exists an integer l $(0 \le l \le n)$ satisfying*

$$\max\{\rho(A_j) : j = 0, 1, \cdots, n, j \ne l\} \le \mu(A_l) < \infty$$

and

$$\max\{\tau(A_j) : \rho(A_j) = \mu(A_l) : j = 0, 1, \cdots, n, j \ne l\} < \underline{\tau}(A_l).$$

Then every meromorphic solution f of equation (1) satisfies $\mu(f) \ge \mu(A_l) + 1$.

Theorem 4 (see [13])**.** *Let H be a set of complex numbers satisfying $\overline{dens}\{|z| : z \in H\} > 0$, and let $A_0(z), \cdots, A_n(z)$ be entire functions satisfying $\max\{\rho(A_j) : j = 0, 1, \cdots, n\} \le \rho$. If there exists an integer l $(0 \le l \le n)$ such that for some constants $0 \le \beta < \alpha$ and $\varepsilon > 0$ sufficiently small, we have*

$$|A_l(z)| \ge \exp\left\{\alpha r^{\rho-\varepsilon}\right\}$$

and

$$|A_j(z)| \le \exp\left\{\beta r^{\rho-\varepsilon}\right\} \quad (j \ne l)$$

as $|z| = r \to +\infty$ for $z \in H$, then every meromorphic solution $f \not\equiv 0$ of equation (1) satisfies $\rho(f) \ge \rho(A_l) + 1$.

When the coefficients $A_0(z), A_1(z), \cdots, A_n(z)$ are meromorphic, Chen and Shon extended the result of Theorem 1 and obtained.

Theorem 5 (see [3])**.** *Let $A_0(z), \cdots, A_n(z)$ be meromorphic functions such that there exists an integer l $(0 \le l \le n)$ such that $\rho(A_l) > \max\{\rho(A_j) : j = 0, 1, \cdots, n, j \ne l\}$, $\delta(\infty, A_l) > 0$. Then every meromorphic solution $f \not\equiv 0$ of equation (1) satisfies $\rho(f) \ge \rho(A_l) + 1$.*

Obviously, we have $\rho(A_l) > 0$ and $\rho > 0$ in Theorems 1, 2 and 5. Thus, a natural question arises : How to express the growth of solutions of (1) when all coefficients $A_0(z), A_1(z), \cdots, A_n(z)$ are entire or meromorphic functions and of order zero in $\mathbb{C}$?

The main purpose of this paper is to make use of the concept of finite logarithmic order due to Chern [4] to extend previous results for meromorphic solutions to equation (1) of zero order in $\mathbb{C}$.

Definition 3 (see [4])**.** The logarithmic order of a meromorphic function f is defined as
$$\rho_{\log}(f) = \limsup_{r\to+\infty} \frac{\log T(r,f)}{\log\log r}.$$
If f is an entire function, then
$$\rho_{\log}(f) = \limsup_{r\to+\infty} \frac{\log\log M(r,f)}{\log\log r}.$$

Remark 1. Obviously, the logarithmic order of any non-constant rational function f is one, and thus, any transcendental meromorphic function in the plane has logarithmic order no less than one. However, a function of logarithmic order one is not necessarily a rational function. Constant functions have zero logarithmic order, while there are no meromorphic functions of logarithmic order between zero and one. Moreover, any meromorphic function with finite logarithmic order in the plane is of order zero.

Definition 4. The logarithmic lower order of a meromorphic function f is defined as
$$\mu_{\log}(f) = \liminf_{r\to+\infty} \frac{\log T(r,f)}{\log\log r}.$$
If f is an entire function, then
$$\mu_{\log}(f) = \liminf_{r\to+\infty} \frac{\log\log M(r,f)}{\log\log r}.$$

Definition 5 (see [2])**.** The logarithmic type of an entire function f with $1 \leq \rho_{\log}(f) < +\infty$ is defined by
$$\tau_{\log}(f) = \limsup_{r\to+\infty} \frac{\log M(r,f)}{(\log r)^{\rho_{\log}(f)}}.$$
Similarly the logarithmic lower type of an entire function f with $1 \leq \mu_{\log}(f) < +\infty$ is defined by
$$\underline{\tau}_{\log}(f) = \liminf_{r\to+\infty} \frac{\log M(r,f)}{(\log r)^{\mu_{\log}(f)}}.$$

Remark 2. It is evident that the logarithmic type of any non-constant polynomial P equals its degree $\deg(P)$; that any non-constant rational function is of finite logarithmic type, and that any transcendental meromorphic function whose logarithmic order equals one in the plane must be of infinite logarithmic type.

Recently, the concept of logarithmic order has been used to investigate the growth and the oscillation of solutions of linear differential equations in the complex plane [2] and complex linear difference and q-difference equations in the complex plane and in the unit disc [1,10,11,14]. In the following, we continue to consider growth estimates of meromorphic solutions to higher order linear difference equations, and we obtain the following results.

Theorem 6. *Let* $A_0(z), \cdots, A_n(z)$ *be entire functions such that there exists an integer* l $(0 \leq l \leq n)$ *satisfying*

$$\max\{\rho_{\log}(A_j) : j = 0, 1, \cdots, n, j \neq l\} \leq \mu_{\log}(A_l) < \infty \tag{2}$$

and

$$\max\{\tau_{\log}(A_j) : \rho_{\log}(A_j) = \mu_{\log}(A_l) : j = 0, 1, \cdots, n, j \neq l\} < \underline{\tau}_{\log}(A_l). \tag{3}$$

Then every meromorphic solution $f \not\equiv 0$ *of equation* (1) *satisfies* $\mu_{\log}(f) \geq \mu_{\log}(A_l) + 1$.

Theorem 7. *Let* H *be a set of complex numbers satisfying* $\overline{\log dens}\{|z| : z \in H\} > 0$, *and let* $A_0(z), \cdots, A_n(z)$ *be entire functions satisfying* $\max\{\rho_{\log}(A_j) : j = 0, 1, \cdots, n\} \leq \rho$ *with* $\rho > 1$. *If there exists an integer* l $(0 \leq l \leq n)$ *such that for some constants* $0 \leq \beta < \alpha$ *and* ε $(0 < \varepsilon < \rho)$ *sufficiently small, we have*

$$|A_l(z)| \geq \exp\left\{\alpha [\log r]^{\rho-\varepsilon}\right\} \tag{4}$$

and

$$|A_j(z)| \leq \exp\left\{\beta [\log r]^{\rho-\varepsilon}\right\} \quad (j \neq l) \tag{5}$$

as $|z| = r \to +\infty$ *for* $z \in H$, *then every meromorphic solution* $f \not\equiv 0$ *of equation* (1) *satisfies* $\rho_{\log}(f) \geq \rho_{\log}(A_l) + 1$.

Remark 3. By the assumptions of Theorem 7, we obtain $\rho_{\log}(A_l) = \rho$. Indeed, we have $\rho_{\log}(A_l) \leq \rho$. Suppose that $\rho_{\log}(A_l) = \eta < \rho$. Then, by Definition 3 and (4), we have for any given ε $\left(0 < \varepsilon < \frac{\rho-\eta}{2}\right)$

$$\exp\left\{\alpha [\log r]^{\rho-\varepsilon}\right\} \leq |A_l(z)| \leq \exp\left\{[\log r]^{\eta+\varepsilon}\right\}$$

as $|z| = r \to +\infty$ for $z \in H$. By ε $\left(0 < \varepsilon < \frac{\rho-\eta}{2}\right)$ this is a contradiction as $r \to +\infty$. Hence $\rho_{\log}(A_l) = \rho$.

Theorem 8. *Let* $A_0(z), \cdots, A_n(z)$ *be entire functions of finite logarithmic order such that there exists an integer* l $(0 \leq l \leq n)$ *satisfying*

$$\limsup_{r \to +\infty} \sum_{\substack{j=0 \\ j \neq l}}^{n} \frac{m(r, A_j)}{m(r, A_l)} < 1. \tag{6}$$

Then every meromorphic solution $f \not\equiv 0$ *of equation* (1) *satisfies* $\rho_{\log}(f) \geq \rho_{\log}(A_l) + 1$.

The following theorems investigate the logarithmic order of meromorphic solutions of (1) in the case when the coefficients are meromorphic functions.

Theorem 9. *Let* $A_0(z), \cdots, A_n(z)$ *be meromorphic functions such that there exists an integer* l $(0 \leq l \leq n)$ *satisfying* $\rho_{\log}(A_l) > \max\{\rho_{\log}(A_j) : j = 0, 1, \cdots, n, j \neq l\}$, $\delta(\infty, A_l) > 0$. *Then every meromorphic solution* $f \not\equiv 0$ *of equation* (1) *satisfies* $\rho_{\log}(f) \geq \rho_{\log}(A_l) + 1$.

Theorem 10. *Let $A_0(z), \cdots, A_n(z)$ be meromorphic functions of finite logarithmic order such that there exists an integer l $(0 \le l \le n)$ satisfying $\limsup_{r\to+\infty} \sum_{\substack{j=0\\ j\neq l}}^{n} \frac{m(r,A_j)}{m(r,A_l)} < 1$, $\delta(\infty, A_l) > 0$. Then every meromorphic solution $f \not\equiv 0$ of equation (1) satisfies $\rho_{\log}(f) \ge \rho_{\log}(A_l) + 1$.*

2 Some lemmas

Lemma 1 (see [1]). *Let η_1, η_2 be two arbitrary complex numbers such that $\eta_1 \neq \eta_2$ and let $f(z)$ be a finite logarithmic order meromorphic function. Let ρ be the logarithmic order of $f(z)$. Then for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O\left((\log r)^{\rho-1+\varepsilon}\right).$$

Lemma 2 (see [5]). *Let f be a meromorphic function, η a non-zero complex number, and let $\gamma > 1$, and $\varepsilon > 0$ be given real constants. Then there exists a subset $E_1 \subset (1,\infty)$ of finite logarithmic measure, and a constant A depending only on γ and η, such that for all $|z| = r \notin [0,1] \cup E_1$, we have*

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \le A\left(\frac{T(\gamma r, f)}{r} + \frac{n(\gamma r)}{r}\log^{\gamma} r \log^{+} n(\gamma r)\right), \tag{7}$$

where $n(t) = n(t,\infty,f) + n(t,\infty,1/f)$

Lemma 3 (see [6]). *Let f be a transcendental meromorphic function, let j be non-negative integer, let a be a value in the extended complex plane, and let $\alpha > 1$ be a real constant. Then there exists a constant $R > 0$ such that for all $r > R$, we have*

$$n\left(r, a, f^{(j)}\right) \le \frac{2j+6}{\log\alpha} T(\alpha r, f). \tag{8}$$

Lemma 4. *Let f be a meromorphic function with $1 \le \mu_{\log}(f) < +\infty$. Then there exists a set $E_2 \subset (1,+\infty)$ with infinite logarithmic measure such that for all $r \in E_2 \subset (1,+\infty)$, we have*

$$T(r,f) < (\log r)^{\mu_{\log}(f)+\varepsilon}. \tag{9}$$

Proof. By definition of logarithmic lower order, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ satisfying $\left(1+\frac{1}{n}\right) r_n < r_{n+1}$ and

$$\lim_{r_n\to+\infty} \frac{\log T(r_n, f)}{\log\log r_n} = \mu_{\log}(f).$$

Then for any given $\varepsilon > 0$, there exists an integer n_1 such that for all $n \ge n_1$,

$$T(r_n, f) < (\log r_n)^{\mu_{\log}(f)+\frac{\varepsilon}{2}}.$$

Set $E_2 = \bigcup\limits_{n=n_1}^{\infty} \left[\frac{n}{n+1} r_n, r_n\right]$. Then for $r \in E_2 \subset (1, +\infty)$, we obtain

$$T(r, f) \leq T(r_n, f) < (\log r_n)^{\mu_{\log}(f) + \frac{\varepsilon}{2}} \leq \left(\log \frac{n+1}{n} r\right)^{\mu_{\log}(f) + \frac{\varepsilon}{2}} < (\log r)^{\mu_{\log}(f) + \varepsilon},$$

and $lm(E_2) = \sum\limits_{n=n_1}^{\infty} \int\limits_{\frac{n}{n+1} r_n}^{r_n} \frac{dt}{t} = \sum\limits_{n=n_1}^{\infty} \log\left(1 + \frac{1}{n}\right) = \infty$. Thus, Lemma 4 is proved. □

Lemma 5. *Let f be a meromorphic function, η a non-zero complex number, and $\varepsilon > 0$ be given real constants. Then there exists a subset $E_3 \subset (1, \infty)$ of finite logarithmic measure, such that if f has finite logarithmic order ρ, then for all $|z| = r \notin [0, 1] \cup E_3$, we have*

$$\exp\left\{-\frac{(\log r)^{\rho+\varepsilon}}{r}\right\} \leq \left|\frac{f(z+\eta)}{f(z)}\right| \leq \exp\left\{\frac{(\log r)^{\rho+\varepsilon}}{r}\right\}. \tag{10}$$

Proof. By using (7) and (8), we obtain

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \leq A\left(\frac{T(\gamma r, f)}{r}\right.$$

$$\left.+\frac{12}{\log\alpha}\frac{T(\alpha\gamma r, f)}{r}\log^{\gamma} r \log^{+}\left(\frac{12}{\log\alpha} T(\alpha\gamma r, f)\right)\right)$$

$$\leq B\left(\frac{T(\beta r, f)}{r} + \frac{\log^{\beta} r}{r} T(\beta r, f) \log T(\beta r, f)\right), \tag{11}$$

for all $|z| = r \notin [0, 1] \cup E_3$ with $lm(E_3) < +\infty$, where $B > 0$ is some constant and $\beta = \alpha\gamma > 1$. Since $f(z)$ has finite logarithmic order $\rho_{\log}(f) = \rho < +\infty$, so given ε, $0 < \varepsilon < 2$, we have for sufficiently large r

$$T(r, f) < (\log r)^{\rho + \frac{\varepsilon}{2}}. \tag{12}$$

Then by using (11) and (12), we obtain

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \leq B\left(\frac{T(\beta r, f)}{r} + \frac{\log^{\beta} r}{r} T(\beta r, f) \log T(\beta r, f)\right)$$

$$\leq B\left(\frac{(\log \beta r)^{\rho+\frac{\varepsilon}{2}}}{r} + \frac{\log^{\beta} r}{r} (\log \beta r)^{\rho+\frac{\varepsilon}{2}} \log (\log \beta r)^{\rho+\frac{\varepsilon}{2}}\right) \leq \frac{(\log r)^{\rho+\varepsilon}}{r}. \tag{13}$$

From (13), we easily obtain (10). □

Lemma 6. *Let η_1, η_2 be two arbitrary complex numbers such that $\eta_1 \neq \eta_2$ and let $f(z)$ be a meromorphic function of finite logarithmic order ρ. Let $\varepsilon > 0$ be given, then there exists a subset $E_4 \subset (1, \infty)$ with finite logarithmic measure such that for all $|z| = r \notin [0,1] \cup E_4$, we have*

$$\exp\left\{-\frac{(\log r)^{\rho+\varepsilon}}{r}\right\} \leq \left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| \leq \exp\left\{\frac{(\log r)^{\rho+\varepsilon}}{r}\right\}. \tag{14}$$

Proof. We can write

$$\left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| = \left|\frac{f(z+\eta_2+\eta_1-\eta_2)}{f(z+\eta_2)}\right| \quad (\eta_1 \neq \eta_2).$$

Then by using Lemma 5, we obtain for any given $\varepsilon > 0$ and all $|z+\eta_2| = R \notin [0,1] \cup E_3$, such that $lm(E_3) < \infty$

$$\exp\left\{-\frac{(\log r)^{\rho+\varepsilon}}{r}\right\} \leq \exp\left\{-\frac{(\log(|z|+|\eta_2|))^{\rho+\frac{\varepsilon}{2}}}{|z+\eta_2|}\right\}$$

$$= \exp\left\{-\frac{(\log R)^{\rho+\frac{\varepsilon}{2}}}{R}\right\} \leq \left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right|$$

$$= \left|\frac{f(z+\eta_2+\eta_1-\eta_2)}{f(z+\eta_2)}\right| \leq \exp\left\{\frac{(\log R)^{\rho+\frac{\varepsilon}{2}}}{R}\right\}$$

$$\leq \exp\left\{\frac{(\log(|z|+|\eta_2|))^{\rho+\frac{\varepsilon}{2}}}{|z+\eta_2|}\right\} \leq \exp\left\{\frac{(\log r)^{\rho+\varepsilon}}{r}\right\},$$

where $|z| = r \notin [0,1] \cup E_4$ and E_4 is a set of finite logarithmic measure. □

By using Lemmas 2–4, we can generalize Lemma 6 into finite logarithmic lower order case as following.

Lemma 7. *Let η_1, η_2 be two arbitrary complex numbers such that $\eta_1 \neq \eta_2$ and let $f(z)$ be a meromorphic function of finite logarithmic lower order μ. Let $\varepsilon > 0$ be given, then there exists a subset $E_5 \subset (1, \infty)$ with infinite logarithmic measure such that for all $|z| = r \in E_5$, we have*

$$\exp\left\{-\frac{(\log r)^{\mu+\varepsilon}}{r}\right\} \leq \left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| \leq \exp\left\{\frac{(\log r)^{\mu+\varepsilon}}{r}\right\}.$$

Lemma 8 (see [1])**.** *Let f be a meromorphic function with $\rho_{\log}(f) \geq 1$. Then there exists a set $E_6 \subset (1, +\infty)$ with infinite logarithmic measure such that*

$$\lim_{\substack{r\to+\infty \\ r\in E_6}} \frac{\log T(r,f)}{\log\log r} = \rho.$$

Lemma 9 (see [1]). *Let f_1, f_2 be meromorphic functions satisfying $\rho_{\log}(f_1) > \rho_{\log}(f_2)$. Then there exists a set $E_7 \subset (1, +\infty)$ having infinite logarithmic measure such that for all $r \in E_7$, we have*

$$\lim_{r \to +\infty} \frac{T(r, f_2)}{T(r, f_1)} = 0.$$

Lemma 10. *Let f be an entire function with $1 \leq \mu_{\log}(f) < +\infty$. Then there exists a set $E_8 \subset (1, +\infty)$ with infinite logarithmic measure such that*

$$\underline{\tau}_{\log}(f) = \lim_{\substack{r \to +\infty \\ r \in E_8}} \frac{\log M(r, f)}{(\log r)^{\mu_{\log}(f)}}.$$

Proof. By the definition of the logarithmic lower type, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ satisfying $\left(1 + \frac{1}{n}\right) r_n < r_{n+1}$, and

$$\underline{\tau}_{\log}(f) = \lim_{r_n \to +\infty} \frac{\log M(r_n, f)}{(\log r_n)^{\mu_{\log}(f)}}.$$

Then for any given $\varepsilon > 0$, there exists an n_1 such that for $n \geq n_1$ and any $r \in \left[\frac{n}{n+1} r_n, r_n\right]$, we have

$$\frac{\log M(\frac{n}{n+1} r_n, f)}{(\log r_n)^{\mu_{\log}(f)}} \leq \frac{\log M(r, f)}{(\log r)^{\mu_{\log}(f)}} \leq \frac{\log M(r_n, f)}{(\log \frac{n}{n+1} r_n)^{\mu_{\log}(f)}}.$$

It follows that

$$\left(\frac{\log \frac{n}{n+1} r_n}{\log r_n}\right)^{\mu_{\log}(f)} \frac{\log M(\frac{n}{n+1} r_n, f)}{(\log \frac{n}{n+1} r_n)^{\mu_{\log}(f)}} \leq \frac{\log M(r, f)}{(\log r)^{\mu_{\log}(f)}}$$

$$\leq \frac{\log M(r_n, f)}{(\log r_n)^{\mu_{\log}(f)}} \left(\frac{\log r_n}{\log \frac{n}{n+1} r_n}\right)^{\mu_{\log}(f)}.$$

Set

$$E_8 = \bigcup_{n=n_1}^{\infty} \left[\frac{n}{n+1} r_n, r_n\right].$$

Then, we have

$$\lim_{\substack{r \to +\infty \\ r \in E_8}} \frac{\log M(r, f)}{(\log r)^{\mu_{\log}(f)}} = \lim_{r_n \to +\infty} \frac{\log M(r_n, f)}{(\log r_n)^{\mu_{\log}(f)}} = \underline{\tau}_{\log}(f)$$

and $lm(E_8) = \int\limits_{E_8} \frac{dr}{r} = \sum\limits_{n=n_1}^{\infty} \int\limits_{\frac{n}{n+1} r_n}^{r_n} \frac{dt}{t} = \sum\limits_{n=n_1}^{\infty} \log\left(1 + \frac{1}{n}\right) = \infty.$ □

3 Proofs of Theorems

3.1 Proof of Theorem 6

Let $f \not\equiv 0$ be a meromorphic solution of (1). We suppose $\mu_{\log}(f) < \mu_{\log}(A_l)+1 < \infty$. We divide through equation (1) by $f(z+l)$ to get

$$|A_l(z)| \le |A_n(z)| \left|\frac{f(z+n)}{f(z+l)}\right| + \cdots + |A_{l-1}(z)| \left|\frac{f(z+l-1)}{f(z+l)}\right|$$

$$+ |A_{l+1}(z)| \left|\frac{f(z+l+1)}{f(z+l)}\right| + \cdots + |A_1(z)| \left|\frac{f(z+1)}{f(z+l)}\right| + |A_0(z)| \left|\frac{f(z)}{f(z+l)}\right|. \quad (15)$$

In relation to (2) and (3), we set

$$\rho = \max\{\rho_{\log}(A_j) : j = 0, 1, \cdots, n, j \neq l\},$$

and

$$\tau = \max\{\tau_{\log}(A_j) : \rho_{\log}(A_j) = \mu_{\log}(A_l) : j = 0, 1, \cdots, n, j \neq l\}.$$

Then for sufficiently large r, we have

$$|A_j(z)| \le \exp\left\{(\log r)^{\rho+\varepsilon}\right\} \quad (j \neq l) \quad (16)$$

if $\rho_{\log}(A_j) < \mu_{\log}(A_l)$, and

$$|A_j(z)| \le \exp\left\{(\tau+\varepsilon)(\log r)^{\mu_{\log}(A_l)}\right\} \quad (j \neq l) \quad (17)$$

if $\rho_{\log}(A_j) = \mu_{\log}(A_l)$. By Lemma 7, for any given $\varepsilon > 0$, there exists a set $E_5 \subset (1, \infty)$ with infinite logarithmic measure such that for all $|z| = r \in E_5$, we have

$$\left|\frac{f(z+j)}{f(z+l)}\right| \le \exp\left\{\frac{(\log r)^{\mu_{\log}(f)+\varepsilon}}{r}\right\} \quad (j = 0, 1, \cdots, n, j \neq l). \quad (18)$$

Then we can choose ε $(0 < \varepsilon < 1)$ sufficiently small to satisfy

$$\tau + 2\varepsilon < \underline{\tau}_{\log}(A_l), \ \ \max\{\rho, \mu_{\log}(f) - 1\} + 2\varepsilon < \mu_{\log}(A_l). \quad (19)$$

Substituting (16), (17) and (18) into (15), we get for $|z| = r \in E_5$,

$$M(r, A_l) \le \exp\left\{\frac{(\log r)^{\mu_{\log}(f)+\varepsilon}}{r}\right\} O\left(\exp\left\{(\tau+\varepsilon)(\log r)^{\mu_{\log}(A_l)}\right\}\right.$$

$$\left. + \exp\left\{(\log r)^{\rho+\varepsilon}\right\}\right). \quad (20)$$

By (19) and (20) and Lemma 10, we get

$$\underline{\tau}_{\log}(A_l) = \liminf_{\substack{r\to+\infty \\ r\in E_5}} \frac{\log M(r, A_l)}{(\log r)^{\mu_{\log}(A_l)}} \le \tau + \varepsilon < \underline{\tau}_{\log}(A_l) - \varepsilon$$

which is a contradiction. Hence $\mu_{\log}(f) \ge \mu_{\log}(A_l) + 1$. □

3.2 Proof of Theorem 7

By Remark 3, we know that $\rho_{\log}(A_l) = \rho$. Let $f \not\equiv 0$ be a meromorphic solution of (1). Next we suppose $\rho_{\log}(f) < \rho_{\log}(A_l)+1 = \rho+1 < +\infty$. From the conditions of Theorem 7, there is a set H of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$ such that for $z \in H$, we have (4) and (5) as $|z| = r \to +\infty$. Set $H_1 = \{r = |z| : z \in H\}$, since $\overline{\log dens}\{|z| : z \in H\} > 0$, then by Proposition 1, H_1 is a set with $\int_{H_1} \frac{dr}{r} = \infty$. By Lemma 6, for any given ε $\left(0 < \varepsilon < \frac{\rho - \rho_{\log}(f)+1}{2}\right)$, there exists a set $E_4 \subset (1, \infty)$ with finite logarithmic measure such that for all $|z| = r \notin [0,1] \cup E_4$, we have

$$\left|\frac{f(z+j)}{f(z+l)}\right| \leq \exp\left\{\frac{(\log r)^{\rho_{\log}(f)+\varepsilon}}{r}\right\} \quad (j = 0, 1, \cdots, n, j \neq l). \tag{21}$$

Substituting (4), (5) and (21) into (15), we get for $|z| = r \in H_1 \backslash ([0,1] \cup E_4)$,

$$\exp\left\{\alpha [\log r]^{\rho-\varepsilon}\right\} \leq n \exp\left\{\beta [\log r]^{\rho-\varepsilon}\right\} \exp\left\{\frac{(\log r)^{\rho_{\log}(f)+\varepsilon}}{r}\right\},$$

it follows that

$$\exp\left\{(\alpha - \beta) [\log r]^{\rho-\varepsilon}\right\} \leq n \exp\left\{\frac{(\log r)^{\rho_{\log}(f)+\varepsilon}}{r}\right\}. \tag{22}$$

By ε $\left(0 < \varepsilon < \frac{\rho - \rho_{\log}(f)+1}{2}\right)$ and $\alpha - \beta > 0$, we obtain a contradiction from (22). Hence, we get $\rho_{\log}(f) \geq \rho + 1 = \rho_{\log}(A_l) + 1$. □

3.3 Proof of Theorem 8

Let $f \not\equiv 0$ be a meromorphic solution of (1). If $\rho_{\log}(f) = \infty$, then the result is trivial. Next we suppose $\rho_{\log}(f) = \rho < \infty$. We divide through equation (1) by $f(z+l)$ to get

$$A_l(z) = -\left(A_n(z)\frac{f(z+n)}{f(z+l)} + \cdots + A_{l-1}(z)\frac{f(z+l-1)}{f(z+l)}\right.$$
$$\left.+A_{l+1}(z)\frac{f(z+l+1)}{f(z+l)} + \cdots + A_1(z)\frac{f(z+1)}{f(z+l)} + A_0(z)\frac{f(z)}{f(z+l)}\right). \tag{23}$$

It follows

$$m(r, A_l) \leq \sum_{\substack{j=0 \\ j\neq l}}^{n} m(r, A_j) + \sum_{\substack{j=0 \\ j\neq l}}^{n} m\left(r, \frac{f(z+j)}{f(z+l)}\right) + O(1). \tag{24}$$

Suppose that

$$\limsup_{r\to+\infty} \sum_{\substack{j=0 \\ j\neq l}}^{n} \frac{m(r, A_j)}{m(r, A_l)} = \mu < \lambda < 1. \tag{25}$$

Then for sufficiently large r, we have

$$\sum_{\substack{j=0\\j\neq l}}^{n} m\left(r, A_j\right) < \lambda m\left(r, A_l\right). \tag{26}$$

By Lemma 1, we have for sufficiently large r and any given $\varepsilon > 0$

$$m\left(r, \frac{f\left(z+j\right)}{f\left(z+l\right)}\right) = O\left(\left(\log r\right)^{\rho_{\log}(f)-1+\varepsilon}\right),\ j = 0, \cdots, n,\ j \neq l. \tag{27}$$

Thus, by substituting (26) and (27) into (24), we obtain for sufficiently large r and any given $\varepsilon > 0$

$$\begin{aligned} m\left(r, A_l\right) &\leq \sum_{\substack{j=0\\j\neq l}}^{n} m\left(r, A_j\right) + \sum_{\substack{j=0\\j\neq l}}^{n} m\left(r, \frac{f\left(z+j\right)}{f\left(z+l\right)}\right) + O\left(1\right) \\ &\leq \lambda m\left(r, A_l\right) + O\left(\left(\log r\right)^{\rho_{\log}(f)-1+\varepsilon}\right). \end{aligned} \tag{28}$$

From (28), it follows that

$$\left(1-\lambda\right) m\left(r, A_l\right) \leq O\left(\left(\log r\right)^{\rho_{\log}(f)-1+\varepsilon}\right). \tag{29}$$

By (29), we obtain $\rho_{\log}\left(f\right) \geq \rho_{\log}\left(A_l\right) + 1$. Thus, Theorem 8 is proved. □

3.4 Proof of Theorem 9

Clearly, (1) has no nonzero rational solution. If $\rho_{\log}\left(f\right) = \infty$, then the result is trivial. Now suppose that f is a transcendental meromorphic solution of (1) with $\rho_{\log}\left(f\right) < \infty$. Set

$$\delta\left(\infty, A_l\right) = \liminf_{r\to+\infty} \frac{m\left(r, A_l\right)}{T\left(r, A_l\right)} = \delta > 0. \tag{30}$$

Thus from (30), we have for sufficiently large r

$$m\left(r, A_l\right) > \frac{1}{2}\delta T\left(r, A_l\right). \tag{31}$$

Thus, by substituting (27) and (31) into (24), we obtain for sufficiently large r and any given $\varepsilon > 0$

$$\begin{aligned} \frac{\delta}{2} T\left(r, A_l\right) < m\left(r, A_l\right) &\leq \sum_{\substack{j=0\\j\neq l}}^{n} m\left(r, A_j\right) + \sum_{\substack{j=0\\j\neq l}}^{n} m\left(r, \frac{f\left(z+j\right)}{f\left(z+l\right)}\right) + O\left(1\right) \\ &\leq \sum_{\substack{j=0\\j\neq l}}^{n} T\left(r, A_j\right) + O\left(\left(\log r\right)^{\rho_{\log}(f)-1+\varepsilon}\right). \end{aligned} \tag{32}$$

Since $\max\{\rho_{\log}(A_j)\ \ (j=0,\cdots,n), j\neq l\} < \rho_{\log}(A_l)$, then by Lemma 9, there exists a set $E_7\subset(1,+\infty)$ with infinite logarithmic measure such that

$$\max\left\{\frac{T(r,A_j)}{T(r,A_l)}\ \ (j=0,\cdots,n), j\neq l\right\}\to 0,\ r\to+\infty,\ r\in E_7. \tag{33}$$

Thus, by (32) and (33), we have for all $r\in E_7$, $r\to+\infty$

$$\left(\frac{\delta}{2}-o(1)\right)T(r,A_l)\leq O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right). \tag{34}$$

So that, it follows from (34) and Lemma 8 that $\rho_{\log}(f)\geq\rho_{\log}(A_l)+1$. Thus, Theorem 9 is proved. $\square$

3.5 Proof of Theorem 10

Let $f\not\equiv 0$ be a meromorphic solution of (1). If $\rho_{\log}(f)=\infty$, then the result is trivial. Next we suppose $\rho_{\log}(f)=\rho<\infty$. By substituting (26) and (27) into (24), we have for sufficiently large r and any given $\varepsilon>0$

$$(1-\lambda)\,m(r,A_l)\leq O\left((\log r)^{\rho_{\log}(f)-1+\varepsilon}\right). \tag{35}$$

By Lemma 8, we have

$$\lim_{\substack{r\to+\infty\\ r\in E_6}}\frac{\log T(r,A_l)}{\log\log r}=\rho_{\log}(A_l), \tag{36}$$

where E_6 is a set of r of infinite logarithmic linear measure. Since $\delta(\infty,A_l)=\liminf\limits_{r\to+\infty}\frac{m(r,A_l)}{T(r,A_l)}>0$, then we obtain

$$\lim_{\substack{r\to+\infty\\ r\in E_6}}\frac{\log m(r,A_l)}{\log\log r}=\rho_{\log}(A_l). \tag{37}$$

Thus, by (35) and (37), we obtain $\rho_{\log}(f)\geq\rho_{\log}(A_l)+1$. Thus, Theorem 10 is proved. $\square$

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Forbidden Set of the Rational Difference Equation $x_{n+1} = x_n x_{n-k}/(a x_{n-k+1} + x_n x_{n-k+1} x_{n-k})$

Julius Fergy T. Rabago

Abstract. This short note aims to answer one of the open problems raised by F. Balibrea and A. Cascales in [2]. In particular, the forbidden set of the nonlinear difference equation $x_{n+1} = x_n x_{n-k}/(a x_{n-k+1} + x_n x_{n-k+1} x_{n-k})$, where k is a positive integer and a is a positive constant, is found by first computing the closed form solution of the given equation. Additional results regarding the limiting properties and periodicity of its solutions are also discussed. Numerical examples are also provided to illustrate the exhibited results. Lastly, a possible generalization of the present work is offered as an open problem.

Mathematics subject classification: 39A10.
Keywords and phrases: Forbidden set, closed form solution, difference equation, open problem.

1 Introduction

Recently, various types of difference equations have been considered and examined (see, e.g., [2] and the papers cited therein). These types of equations are of great importance in various fields of mathematics and areas of pure and applied sciences. In fact, they frequently appear as discrete mathematical models of many biological and environmental phenomena, such as population growth and predator-prey interactions [6–8]. They are also extensively used in deterministic formulations of dynamical phenomena in economics and social sciences [16]. One of the reasons why these equations are being studied is because they possess rich and complex dynamics (see, e.g., [6]). Some researchers, however, focus on the problem of finding closed form solutions of some solvable systems of nonlinear difference equations. As a matter of fact, this line of research has become a growing interest in recent literature (see, e.g., [3, 5, 11–14, 17, 20, 21], as well as the references therein). In this work, we are also interested in finding a closed form solution of a certain class of difference equations, but, only as a way to solve a related problem. To be more precise, we are interested in addressing the solution to one of the open problems posted by Balibrea and Cascales in [2, Open Problem 3, Eq. 17] concerning the *forbidden set* of a certain class of rational difference equations. Specifically, given fixed constants $k \in \mathbb{N}$ and $a > 0$, we would like to find the forbidden set of the rational difference equation

$$x_{n+1} = \frac{x_n x_{n-k}}{a x_{n-k+1} + x_n x_{n-k+1} x_{n-k}}, \tag{1}$$

with real initial conditions $\{x_n\}_{n=-k}^{0}$. We shall determine the forbidden set of equation (1) by first providing its closed form solution.

Definition 1. Given a rational difference equation $x_{n+1} = \frac{P(x_n,\ldots,x_{n-k})}{Q(x_n,\ldots,x_{n-k})}$ of order $k+1$, where P and Q are two polynomials, there exists a subset $\mathcal{F}$ of $\mathbb{R}^{k+1}$ such that every initial condition of (1) lying on $\mathcal{F}$ generates a finite solution $x_{-k},\ldots,x_m$ for which it is impossible to construct x_{m+1} because its $k+1$ latest terms form a root of polynomial Q. The set $\mathcal{F}$ is known as the *forbidden set* of the equation.

Consequently, the forbidden set $\mathcal{F}$ of a rational difference equation is the set of initial conditions which eventually map to a singularity, or more intuitively, $\mathcal{F}$ is the set of initial conditions for which after a finite number of iterates we reach a value outside the domain of definition of the iteration function [2]. For some papers related to this topic, see [2] and [10], and the references cited therein.

Our main result, which answers the open problem [2, Open Problem 3, Eq. 17], is stated in Corollary 1. The term solution and sequence of iterates will be used interchangeably throughout the rest of the paper.

2 Closed Form Solution and Forbidden Set of Equation (1)

In this section, we derive the closed form solution of (1), and then deduce from the computed formula the forbidden set of the given equation. To begin with, we provide some preliminary observations regarding the right side of equation (1). First, notice that the equation can be written as

$$x_{n+1} = \frac{x_n x_{n-k}}{x_{n-k+1}(a + x_n x_{n-k})}.$$

Clearly, this form suggests that the quantities $x_n x_{n-k} + a$ and x_{n-k} should not be both zero, for all $n \in \mathbb{N}_0$, so that the sequence of iterates $\{x_n\}_{n=1}^{\infty}$ is well-defined. Hence, we assumed that $x_n x_{n-k} \neq -a$ and $x_{n-k} \neq 0$, for all $n \in \mathbb{N}_0$. These conditions shall be refined later on in the discussion by expressing them in terms of just the initial conditions $\{x_n\}_{n=-k}^{0}$ and the parameter a. Meanwhile, if $x_n x_{n-k} = 1-a$, for all $n \in \mathbb{N}_0$, and $a \neq 1$, then equation (1) reduces to

$$x_n = \frac{1-a}{x_{n-k}}, \quad n \in \mathbb{N}_0. \tag{2}$$

This implies that the solution sequence $\{x_n\}_{n=1}^{\infty}$ to (1) is *periodic* (see Definition 2). Indeed, substituting $x_{n-k} = (1-a)/x_{n-2k}$ in equation (2) yields the equation (after an adjustment in the index) $x_{n+2k} = x_n$ ($n \in \mathbb{N}_0$). Clearly, this equation shows that the sequence of iterates $\{x_n\}_{n=1}^{\infty}$ is periodic with period $2k$.

Now, in the sequel, we shall assume that $x_n x_{n-k} \neq 1-a$ for all $n \in \mathbb{N}_0$ and $a \neq 1$. Consider the transformation

$$v_{n+1} = (a+1)v_n - a v_{n-1}, \qquad n \in \mathbb{N}_0, \tag{3}$$

of equation (1) obtained through the change of variable $v_{n+1}/v_n := a + x_{n+1}x_{n-k+1}$. We determine the solution form of equation (3) through a classical method in solving linear (homogenous) recurrences. That is, we use a discrete function λ^n where $\lambda \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}_0 \cup \{-1\}$ to obtain a Binet-like form of the n-th term v_n. The inclusion of the index -1 is crucial (as we are using the ansatz $v_n = \lambda^n$) in this approach, and this we shall see as we proceed in our discussion. It is worth noting that there are many other techniques in solving linear recurrences with constant coefficients (see, e.g., [1] and [9]), and here we shall apply the method of using a discrete function as our main approach. However, we shall also remark that the method of differences, much known as *telescoping sums*, can be effectively used to derive the solution form of equation (1). For an interesting application of this method to a class of difference equations, we refer the readers to [15].

Now, to begin the computation, we let $v_n = \lambda^n$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}_0 \cup \{-1\}$. From equation (3), we get $\lambda^{n+1} = (a+1)\lambda^n - a\lambda^{n-1}$ or equivalently, $\lambda^2 - (a+1)\lambda + a = 0$ whose roots are given by $\lambda_1 = a$ and $\lambda_2 = 1$. Since $a \neq 1$, then it is evident that λ_1 and λ_2 are distinct. Therefore, by a standard result in difference equations, we can write v_n as $v_n = c_1 a^n + c_2 1^n$ for some computable constants c_1 and c_2. These coefficients are easily determined by computing the solution pair (c_1, c_2) of the system

$$\begin{cases} c_1 + c_2 = v_0 \\ c_1 + ac_2 = av_{-1}. \end{cases}$$

Thus,

$$c_1 = \frac{av_0 - av_{-1}}{a-1} \qquad \text{and} \qquad c_2 = -\frac{v_0 - av_{-1}}{a-1},$$

from which it follows that

$$v_n = \left(\frac{a^{n+1}-1}{a-1}\right) v_0 - a\left(\frac{a^n - 1}{a-1}\right) v_{-1}.$$

Form the relation $x_n x_{n-k} = v_n/v_{n-1} - a$, we obtain

$$\begin{aligned} x_n x_{n-k} &= \frac{\left(\frac{a^{n+1}-1}{a-1}\right)\frac{v_0}{v_{-1}} - a\left(\frac{a^n-1}{a-1}\right)}{\left(\frac{a^n-1}{a-1}\right)\frac{v_0}{v_{-1}} - a\left(\frac{a^{n-1}-1}{a-1}\right)} - a \\ &= \frac{\left(\frac{a^{n+1}-1}{a-1}\right)(a + x_0x_{-k}) - a\left(\frac{a^n-1}{a-1}\right)}{\left(\frac{a^n-1}{a-1}\right)(a + x_0x_{-k}) - a\left(\frac{a^{n-1}-1}{a-1}\right)} - a \\ &= \frac{a^{n+1} + \left(\frac{a^{n+1}-1}{a-1}\right)x_0x_{-k}}{a^n + \left(\frac{a^n-1}{a-1}\right)x_0x_{-k}} - a \\ &= \frac{x_0x_{-k}}{a^n + \left(\frac{a^n-1}{a-1}\right)x_0x_{-k}}. \end{aligned} \tag{4}$$

Now, replacing n by $2kj+i$ (resp., $2kj-k+i$) for $i \in I := \{-k, -k+1, \ldots, 0\}$, we get

$$x_{2kj+i}x_{2kj-k+i} = \frac{x_0x_{-k}}{a^{2kj+i} + \left(\frac{a^{2kj+i}-1}{a-1}\right)x_0x_{-k}},$$
$$x_{2kj-k+i}x_{2kj-2k+i} = \frac{x_0x_{-k}}{a^{2kj-k+i} + \left(\frac{a^{2kj-k+i}-1}{a-1}\right)x_0x_{-k}},$$

respectively. Taking the ratio of the corresponding sides of the above equations, and then taking the product of the resulting expression from $j=1$ to $j=n$, we get

$$\begin{aligned} x_{2kn+i} &= x_i \prod_{j=1}^{n} \left\{ \frac{x_{2kj+i}x_{2kj-k+i}}{x_{2kj-k+i}x_{2kj-2k+i}} \right\} \\ &= x_i \prod_{j=1}^{n} \left\{ \frac{a^{2kj-k+i} + \left(\frac{a^{2kj-k+i}-1}{a-1}\right)x_0x_{-k}}{a^{2kj+i} + \left(\frac{a^{2kj+i}-1}{a-1}\right)x_0x_{-k}} \right\}, \end{aligned} \tag{5}$$

for all $n \in \mathbb{N}_0$ and $i \in I$, with the usual convention that $\prod_{j=1}^{0}(\cdot) = 1$. Notice that, with the above indices of the iterate, we were not able to describe the form of the first $k-1$ iterates $\{x_n\}_{n=1}^{k-1}$. However, these iterates can be obtained easily by replacing i by $-i$ in (5) and let i run from 1 to $k-1$. Alternatively, we can utilize equation (4) and let n assumes the value from 1 to $k-1$. More precisely, we have

$$x_n = \frac{x_0x_{-k}}{x_{n-k}}\left(a^n + \left(\frac{a^n-1}{a-1}\right)x_0x_{-k}\right)^{-1},$$

for all $n = 1, 2, \ldots, k-1$.

Remark 1. We note that we can determine the solution form of equation (3) via telescoping sums. To do this, we transform equation (3) to the equivalent form $v_{n+1} - v_n = a(v_n - v_{n-1})$. Letting $w_{n+1} := v_{n+1} - v_n$, we can write equation (3) as $w_{n+1} = aw_n$ which, upon iterating the right-hand side, leads to $w_{n+1} = a^{n+1}w_0$. This equation, in turn, yields the relation $v_{n+1} - v_n = a^{n+1}(v_0 - v_{-1})$, and by telescoping sums we easily obtain the identity

$$v_n = v_{-1} + (v_0 - v_{-1})\sum_{j=-1}^{n-1} a^{j+1} = \left(\frac{a^{n+1}-1}{a-1}\right)v_0 - a\left(\frac{a^n-1}{a-1}\right)v_{-1}.$$

To this end, one can follow the same inductive lines as above to get the desired result. Referring to the form of v_n computed above, it is clear that a must not equate to unity since, if it is so, the quantity will be undefined.

In concluding, we have just proved the following result.

Theorem 1. *Let $k \in \mathbb{N}$ and $a > 1$ $(a \neq 1)$ be fixed. Then, every well-defined solution $\{x_n\}_{n=-k}^{\infty}$ of equation (1) takes the form*

$$x_n = \frac{x_0 x_{-k}}{x_{n-k}} \left[a^n + \left(\frac{a^n - 1}{a - 1} \right) x_0 x_{-k} \right]^{-1}, \tag{6}$$

for all $n = 1, 2, \ldots, k-1$, and

$$x_{2kn+i} = x_i \prod_{j=1}^{n} \left\{ \frac{a^{2kj-k+i} + \left(\frac{a^{2kj-k+i}-1}{a-1} \right) x_0 x_{-k}}{a^{2kj+i} + \left(\frac{a^{2kj+i}-1}{a-1} \right) x_0 x_{-k}} \right\}, \tag{7}$$

for all $n \in \mathbb{N}_0$ and $i \in I$. If, in addition, $x_0 x_{-k} = 1 - a$, $a \neq 1$, then the solution forms (6) and (7) can be simplified as

$$x_n = \frac{x_0 x_{-k}}{x_{n-k}},$$

for all $n = 1, 2, \ldots, k-1$, and

$$x_{2kn+i} = x_i,$$

for all $n \in \mathbb{N}_0$ and $i \in I$, respectively.

By a well-defined solution of (1), we mean a solution sequence $\{x_n\}_{n=-k}^{\infty}$ with real initial conditions $\{x_n\}_{n=-k}^{0}$ such that $x_i \neq 0$ for all $i \in I$, and

$$x_0 x_{-k} \notin A \cup B := \left\{ -a^j \left(\frac{a^j - 1}{a - 1} \right)^{-1} \right\}_{j=1}^{k-1} \bigcup \left\{ -a^{2kj+i} \left(\frac{a^{2kj+i} - 1}{a - 1} \right)^{-1} \right\}_{j=1}^{\infty},$$

for all $i \in I$.

As an immediate consequence of Theorem 1, we finally obtain the forbidden set for the difference equation (1) given in the following corollary.

Corollary 1. *Let $\boldsymbol{x}_0 := (x_{-k}, x_{k+1}, \ldots, x_0) \in \mathbb{R}^{k+1}$, $k \in \mathbb{N}$, and $a > 1$ $(a \neq 1)$ be fixed. Then, the forbidden set $\mathcal{F}$ of the difference equation (1) is given by*

$$\mathcal{F} = \{ \boldsymbol{x}_0 \ : \ x_i \neq 0 \textit{ for all } i \in I, \textit{ and } x_0 x_{-k} \notin A \cup B \}.$$

Remark 2. We observe that the computation of the closed form solution of equation (1) does not require the positivity of a. This suggests that, following the same line of arguments, the results can easily be extended to the case when a is an arbitrary real number not equal to zero, or possibly when $a \in \mathbb{C} \setminus \{0\}$ in general.

3 Some Results on the Behavior of Solutions of Equation (1)

In this section we examine the case when a is the unity, and present some results regarding the qualitative behavior of the solution of equation (1). Also, we provide some numerical illustrations depicting the long-time behavior of solutions of equation (1) for some given fixed constants $k \in \mathbb{N}$ and $a > 0$.

Before we proceed further, we need to recall what we mean by an *eventually periodic solution.*

Definition 2 ([8])**.** Let $k \in \mathbb{N}$. A sequence $\{x_n\}_{n=-k}$ is said to be periodic with period p if $x_{n+p} = x_n$, for all $n \geq -k$. Moreover, a solution $\{x_n\}_{n=-k}$ of (1) is called eventually periodic with period p if there exists an integer $N \geq -k$ such that $\{x_n\}_{n=-k}$ is periodic with period p; that is, $x_{n+p} = x_n$, for all $n \geq N$.

Hereinafter, we assume that $\{x_n\}_{n=-k}^{\infty}$ is a well-defined solution of (1).

3.1 Form and Periodicity of Solutions for the Case $a = 1$

For the case when $a = 1$, we have the following corollary of Theorem 1.

Corollary 2. *If $a = 1$, then every solution $\{x_n\}_{n=-k}^{\infty}$ of equation* (1) *takes the form*

$$x_n = \frac{x_0 x_{-k}}{x_{n-k}} \left(1 + n x_0 x_{-k}\right)^{-1}, \tag{8}$$

for all $n = 1, 2, \ldots, k-1$, and

$$x_{2kn+i} = x_i \prod_{j=1}^{n} \left\{ \frac{1 + (2kj - k + i)\, x_0 x_{-k}}{1 + (2kj + i)\, x_0 x_{-k}} \right\}, \tag{9}$$

for all $n \in \mathbb{N}_0$ and $i \in I$. Furthermore, the solution sequence $\{x_n\}_{n=-k}^{\infty}$ is eventually periodic with period $2k$. In addition, the forbidden set $\mathcal{F}_1$ of equation (1) *for $a = 1$ is given by*

$$\mathcal{F}_1 = \left\{ \boldsymbol{x}_0 \ : \ x_i \neq 0 \text{ for all } i \in I, \text{ and } x_0 x_{-k} \notin \left\{ -\frac{1}{j} \right\}_{j=1}^{k-1} \bigcup \left\{ -\frac{1}{2kj+1} \right\}_{j=1}^{\infty} \right\},$$

where $\boldsymbol{x}_0 := (x_{-k}, x_{k+1}, \ldots, x_0) \in \mathbb{R}^{k+1}$.

Proof. Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of (1) with $a = 1$, and $\mathcal{F}_1$ denotes its forbidden set. Formulas (8) and (9) follow directly by taking the limit of equations (6) and (7), respectively, as a approaches the unity. Meanwhile the periodicity of $\{x_n\}_{n=-k}^{\infty}$ is a consequence of the fact that

$$\frac{1 + (2kj - k + i)\, x_0 x_{-k}}{1 + (2kj + i)\, x_0 x_{-k}} \longrightarrow 1$$

as $j \to \infty$, and of course, as long as $\boldsymbol{x}_0 \notin \mathcal{F}_1$. Finally, the forbidden set $\mathcal{F}_1$ is specified by finding the values of $x_0 x_{-k}$ for which formulas (8) and (9) are undefined. $\square$

3.2 Limiting Properties of Solutions for $a \in \mathbb{R}^+ \setminus \{1\}$

Now, we examine the limiting properties of solutions of equation (1) for $a > 0$ not equal to the unity. First, we investigate the possibility that a solution to (1) is convergent to zero. To see this possibility, it suffices to determine when the subsequence $\{x_{2kn+i}\}_{n=1}^{\infty}$, for all $i \in I$, converges to zero. This situation would only be possible when $|x_{2kn+i}| < |x_{2k(n-1)+i}|$, for all $n \geq 2$ and $i \in I$. In view of equation (7), this condition is equivalent to

$$(0 <) \left| \frac{a^{2kn-k+i} + \left(\frac{a^{2kn-k+i}-1}{a-1}\right) x_0 x_{-k}}{a^{2kn+i} + \left(\frac{a^{2kn+i}-1}{a-1}\right) x_0 x_{-k}} \right| < 1,$$

for all $n \in \mathbb{N}$ and $i \in I$. Without-loss-of-generality, suppose that the numerator and the denominator are both positive. Then, after some rearrangement, the above inequality condition can be expressed as

$$0 < a^{2kn-k+i}(a^k - 1)\left(1 + \frac{x_0 x_{-k}}{a-1}\right).$$

Since this inequality must hold true for all $n \in \mathbb{N}$ and $i \in I$, then a must be greater than the unity and $x_0 x_{-k} \neq 1 - a$. Given these conditions, we conclude that every solution $\{x_n\}_{n=-k}^{\infty}$ of equation (1) will converge to zero for $a > 1$.

Similarly, we can show, without any difficulty, that every solution $\{x_n\}_{n=-k}^{\infty}$ of equation (1) will eventually be periodic whenever $a \in (0, 1)$ or $x_0 x_{-k} = 1 - a$. In either of these situations, the periodicity is given by $2k$. Indeed, for $a \in (0, 1)$, the quantity $a^{2kj-k+i}$ vanishes as j goes to infinity, for all $i \in I$. In addition, the ratio $(a^{2kj+i} - 1)/(a - 1)$ will converge to $1/(a - 1)$ as j goes to infinity, for all $i \in I$. These results imply that

$$\frac{a^{2kj-k+i} + \left(\frac{a^{2kj-k+i}-1}{a-1}\right) x_0 x_{-k}}{a^{2kj+i} + \left(\frac{a^{2kj+i}-1}{a-1}\right) x_0 x_{-k}} \longrightarrow 1,$$

as $j \to \infty$ (and of course, given that $\boldsymbol{x}_0 \notin \mathcal{F}$). Meanwhile, the case when $x_0 x_{-k} = 1 - a$ $(a \neq 1)$, has already been discussed in Section **??**, and so we shall not repeat it here. In summary, we see that following result holds.

Theorem 2. *Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of equation* (1)*. If $a > 1$ and $x_0 x_{-k} \neq 1-a$, then $\{x_n\}_{n=-k}^{\infty}$ converges to zero. If, however, $a \in (0, 1)$ or $x_0 x_{-k} = 1 - a$ $(a \neq 1)$, then $\{x_n\}_{n=-k}^{\infty}$ is eventually periodic with period $2k$.*

3.3 Numerical Examples

Finally, in this section, we provide some numerical examples that illustrate our results in the previous two sections. In these examples (see Figure 1), the initial conditions $\{x_n\}_{n=-k}^{0}$ are chosen randomly on the interval $(-1, 1)$. The results verify Corollary 2 and Theorem 2.

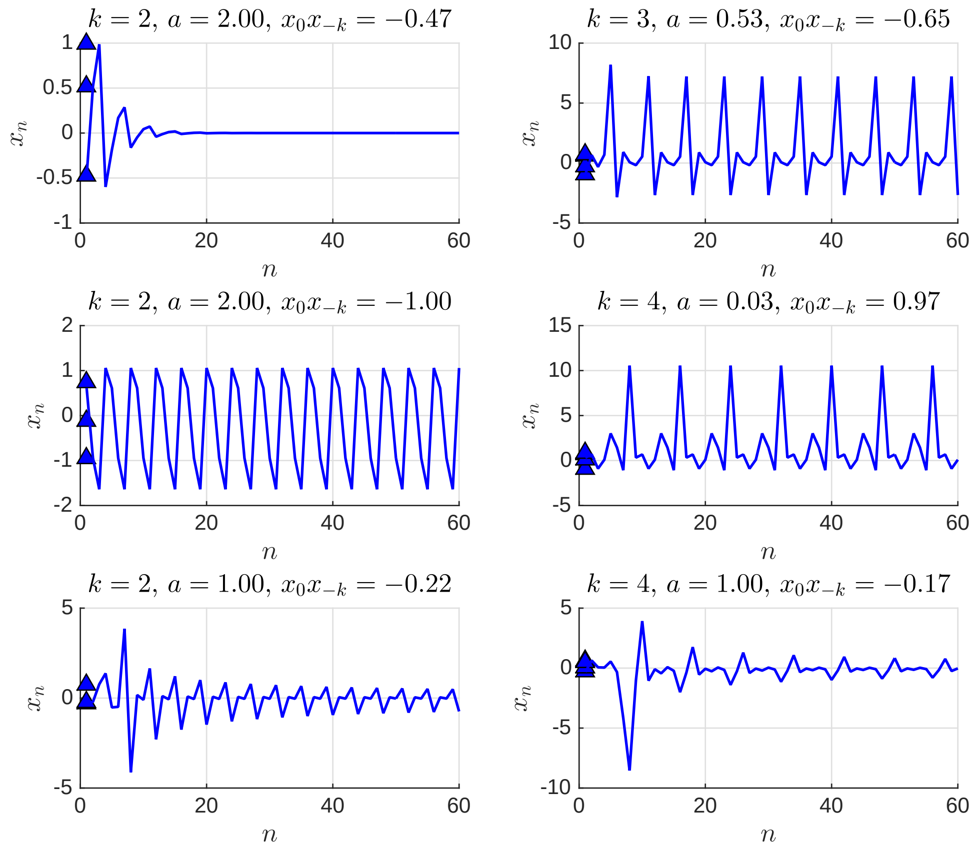

Figure 1. The uppermost plots corroborate the results in Theorem 2 for the case $a > 1$ (left plot) and the case $a \in (0, 1)$ (right plot). Meanwhile, the middle plots illustrate the case when $a > 1$ and $x_0x_{-k} \neq 1$ (right plot), and $a \in (0, 1)$ (left plot) given that $x_0x_{-k} = 1 - a$. Clearly, the solution sequences are both periodic with period four and eight, respectively. Finally, the last two (lower) plots illustrate two particular situations when $a = 1$. Evidently, the figures show that, in these situations, the solutions to (1) are eventually periodic. The solution sequences (left and right) have periods four and eight, respectively. This verifies the results stated in Corollary 2.

4 Summary and a Possible Generalization

We have successfully settled one of the open problems raised by Balibrea and Cascales in [2]. The solution form to the given rational difference equation was established by reducing the equation to a linear type difference equation. The resulting equation was then solved through a classical method in solving linear homogenous recurrence equation with constant coefficients. We emphasize that the method used here can obviously be applied to other problems offered in [2], especially to those nonlinear difference equations whose solution forms are, in structure, similar to the ones obtained here. In fact, we believe that the method employed here can be used effectively in examining the case when a is replaced by a $2k$-periodic sequence of real or complex numbers. Consequently, we believe that the discussion delivered here provides a better understanding of the forbidden set problem in the frame of rational difference equations, and had provided considerable interest in examining other classes of nonlinear difference equations.

As a possible generalization of the open problem addressed in this work, we mention that the case when a is replaced by a general number sequence is also an interesting problem to investigate. So, we ask, given fixed constants $k \in \mathbb{N}$ and $a \in \mathbb{C} \setminus \{0\}$, and a general number sequence $\{a_n\}_{n=0}^{\infty}$, what is the corresponding forbidden set for the rational difference equation

$$x_{n+1} = \frac{x_n x_{n-k}}{a_n x_{n-k+1} + x_n x_{n-k+1} x_{n-k}},$$

with real (or complex) initial conditions $\{x_n\}_{n=-k}^{0}$. Finally, we announce that the other open problems presented in [2] shall be the subject of our future investigations elsewhere.

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A New Characterization of Curves in Euclidean 4-Space $\mathbb{E}^4$

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Abstract. In the present study, we characterize a regular curve whose position vector can be written as a linear combination of its Serret-Frenet vectors in Euclidean 4-space $\mathbb{E}^4$. We investigate such curves in terms of their curvature functions. Further, we obtain some results of T-constant, N-constant and constant ratio curves in $\mathbb{E}^4$.

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1 Introduction

Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in Euclidean 4-space $\mathbb{E}^4$. Let us denote $T(s) = x'(s)$ and call as a unit tangent vector of x at s. We denote the first Serret-Frenet curvature of x by $\kappa_1(s) = \|x''(s)\|$. If $\kappa_1(s) \neq 0$, then the unit principal normal vector $N_1(s)$ of the curve x at s is given by $N_1'(s) + \kappa_1(s)T(s) = \kappa_2(s)N_2(s)$, where κ_2 is the second Serret-Frenet curvature of x. If $\kappa_2(s) \neq 0$, then the unit second principal normal vector $N_2(s)$ of the curve x at s is given by $N_2'(s) + \kappa_2(s)N_1(s) = \kappa_3(s)N_3(s)$, where κ_3 is the third Serret-Frenet curvature of x. Then we have the Serret-Frenet formulae (see [12]):

$$\begin{aligned} T'(s) &= \kappa_1(s)N_1(s), \\ N_1'(s) &= -\kappa_1(s)T(s) + \kappa_2(s)N_2(s), \\ N_2'(s) &= -\kappa_2(s)N_1(s) + \kappa_3(s)N_3(s), \\ N_3'(s) &= -\kappa_3(s)N_2(s). \end{aligned} \tag{1}$$

If the Serret-Frenet curvatures $\kappa_1(s), \kappa_2(s)$ and $\kappa_3(s)$ of x are constant functions then x is called a screw line or a helix [11]. Since these curves are the traces of 1-parameter family of the groups of Euclidean transformations, F. Klein and S. Lie called them *W-curves* [20]. If the tangent vector T of the curve x makes a constant angle with a unit vector U of $\mathbb{E}^4$ then this curve is called a general helix (or inclined curve) in $\mathbb{E}^4$ [22]. It is known that a regular curve in $\mathbb{E}^n$ is said to have constant curvature ratios if the ratios of the consecutive curvatures are constant [21]. The Frenet curves with constant curvature ratios are called ccr-curves [22]. We remark that a regular curve in $\mathbb{E}^4$ is a ccr-curve if $H_1(s) = \frac{\kappa_1}{\kappa_2}(s)$ and $H_2(s) = \frac{\kappa_3}{\kappa_2}(s)$ are constant functions.

Rectifying curves in Euclidean 3-space $\mathbb{E}^3$ are introduced by B. Y. Chen in [4] as space curves whose position vector (denoted also by x) lies in its rectifying plane, spanned by the tangent and the binormal normal vector fields $T(s)$ and $N_2(s)$ of the curve. In the same paper, B. Y. Chen gave a simple characterization of rectifying curves. In particular, it is shown in [8] that there exists a simple relation between rectifying curves and centrodes, which play an important role in mechanics kinematics as well as in differential geometry in defining the curves of constant procession. It is also provided that a twisted curve is congruent to a non-constant linear function of s [4]. Further, in the Minkowski 3-space $\mathbb{E}_1^3$, the rectifying curves are investigated in [10, 15, 16]. In [16] a characterization of the spacelike, the timelike and the null rectifying curves in the Minkowski 3-space in terms of centrodes is given.

For a unit speed regular curve $x : I \subset \mathbb{R} \to \mathbb{E}^4$, the hyperplanes at each point of $x(s)$ which are spanned by $\{T, N_1, N_3\}$, $\{T, N_2, N_3\}$ are known as the *first osculating hyperplane* and the *second osculating hyperplane,* respectively. If the position vector x lies on its first (resp. second) *osculating hyperplane* then $x(s)$ is called *osculating curve of first (resp. second)* kind. In [17], Ilarslan and Nesovic considered the rectifying curve in Euclidean 4-space $\mathbb{E}^4$. They characterized the rectifying curves given by the equation

$$x(s) = \lambda(s)T(s) + \mu(s)N_2(s) + \upsilon(s)N_3(s), \tag{2}$$

for some differentiable functions $\lambda(s)$, $\mu(s)$ and $\upsilon(s)$. Actually, these curves are *osculating curves of* second kind. Further, in the Minkowski 4-space $\mathbb{E}_1^4$, the rectifying curves are investigated in [1, 18, 19]. Recently, quaternionic rectifying curves in the semi-Euclidean space $\mathbb{E}_2^4$ have been considered in [9].

For a regular curve $x(s)$, the position vector x can be decomposed into its tangential and normal components at each point, i.e., $x = x^T + x^N$. A curve $x(s)$ with $\kappa_1(s) > 0$ is said to be of *constant ratio* if the ratio $\|x^T\| : \|x^N\|$ is constant on $x(I)$ where $\|x^T\|$ and $\|x^N\|$ denote the length of x^T and x^N, respectively [2].

Clearly a curve x in $\mathbb{E}^n$ is of constant ratio if and only if $x^T = 0$ or $\|x^T\| : \|x\|$ is constant [2]. The distance function $\rho = \|x\|$ satisfies $\|grad\rho\| = c$ for some constant c if and only if we have $\|x^T\| = c\,\|x\|$. In particular, if $\|grad\rho\| = c$ then $c \in [0, 1]$. In [4], B. Y. Chen gave a classification of constant ratio curves in Euclidean space. A curve in $\mathbb{E}^n$ is called T-*constant* (resp. N-*constant*) if the tangential component x^T (resp. the normal component x^N) of its position vector x is of constant length [3, 6]. Recently the present authors have studied curves with constant ratio in Euclidean 3-space $\mathbb{E}^3$ in [13]. For more details see also [5, 7].

In the present study, we give a generalization of rectifying curves in Euclidean 4-space $\mathbb{E}^4$. First of all, we consider a regular curve in Euclidean 4-space $\mathbb{E}^4$ as a curve whose position vector satisfies the parametric equation

$$x(s) = m_0(s)T(s) + m_1(s)N_1(s) + m_2(s)N_2(s) + m_3(s)N_3(s), \tag{3}$$

for some differentiable functions $m_i(s)$, $0 \leq i \leq 3$. Next, we characterize osculating curves of first and second kind in terms of their curvature functions $\kappa_1(s), \kappa_2(s)$ and

$\kappa_3(s)$. We give necessary and sufficient conditions for the curves given with the parametrization (3) to become W-curves. Furthermore, we obtain some results for these types of curves to become ccr-curves. Finally, we consider T-constant and N-constant curves in $\mathbb{E}^4$. Moreover, we obtain some explicit equations of constant-ratio curves in $\mathbb{E}^4$.

2 Characterization of Curves in $\mathbb{E}^4$

In the present section, we consider unit speed curves with Serret-Frenet curvatures $\kappa_1(s) > 0$, $\kappa_2(s)$, and $\kappa_3(s)$. By definition of the position vector of the curve (also defined by x), it satisfies the vectorial equation (3) for some differentiable functions $m_i(s)$, $0 \leq i \leq 3$. By taking the derivative of (3) with respect to arclength parameter s and using the Serret-Frenet equations (1), we obtain

$$\begin{aligned} x'(s) &= (m_0'(s) - \kappa_1(s)m_1(s))T(s) \\ &+(m_1'(s) + \kappa_1(s)m_0(s) - \kappa_2(s)m_2(s))N_1(s) \\ &+(m_2'(s) + \kappa_2(s)m_1(s) - \kappa_3(s)m_3(s))N_2(s) \\ &+(m_3'(s) + \kappa_3(s)m_2(s))N_3(s). \end{aligned} \tag{4}$$

It follows that

$$\begin{aligned} m_0' - \kappa_1 m_1 &= 1, \\ m_1' + \kappa_1 m_0 - \kappa_2 m_2 &= 0, \\ m_2' + \kappa_2 m_1 - \kappa_3 m_3 &= 0, \\ m_3' + \kappa_3 m_2 &= 0. \end{aligned} \tag{5}$$

The following result explicitly determines the W-curves in $\mathbb{E}^4$.

Theorem 1. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a regular curve given with the parametrization (3). If x is a W-curve of $\mathbb{E}^4$ then the position vector x is given by the curvature functions*

$$\begin{aligned} m_0(s) &= \kappa_1\left(\frac{-c_1e^{-\lambda s} + c_2e^{\lambda s}}{\lambda} + \frac{-c_3e^{-\mu s} + c_4e^{\mu s}}{\mu}\right) + c_0, \\ m_1(s) &= c_1e^{-\lambda s} + c_2e^{\lambda s} + c_3e^{-\mu s} + c_4e^{\mu s} - \frac{1}{\kappa_1}, \\ m_2(s) &= \frac{1}{\kappa_2}\left(\left(\frac{\lambda^2 + \kappa_1^2}{\lambda}\right)\left(-c_1e^{-\lambda s} + c_2e^{\lambda s}\right) + \left(\frac{\mu^2 + \kappa_1^2}{\mu}\right)\left(-c_3e^{-\mu s} + c_4e^{\mu s}\right)\right) \\ &+\frac{\kappa_1}{\kappa_2}c_0 \\ m_3(s) &= -\kappa_3\int m_2(s)ds \end{aligned} \tag{6}$$

where c_i $(0 \leq i \leq 4)$ are integral constants and

$$\lambda = \frac{\sqrt{-2a - 2\sqrt{a^2 - 4b}}}{2},$$

$$\begin{aligned} \mu &= \frac{\sqrt{-2a+2\sqrt{a^2-4b}}}{2}, \\ a &= \kappa_1^2+\kappa_2^2+\kappa_3^2, \\ b &= \kappa_1^2\kappa_3^2, \end{aligned} \tag{7}$$

are real constants.

Proof. Let x be a regular W-curve in $\mathbb{E}^4$, then by the use of the equations (5) we get

$$\begin{aligned} m_0' &= \kappa_1 m_1 + 1, \\ m_1'' &= \kappa_2 m_2' - \kappa_1 (\kappa_1 m_1 + 1), \\ m_2'' &= -\kappa_3^2 m_2 - \kappa_2 m_1', \end{aligned} \tag{8}$$

In particular, one can show that the system of equations (8) has a non-trivial solution (6). Thus, the theorem is proved. □

2.1 Osculating curve of first kind in $\mathbb{E}^4$

Definition 1. Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a regular curve in $\mathbb{E}^4$ given with the arclength parameter s. If the position vector x lies in the hyperplane spanned by $\{T, N_1, N_3\}$ then x is called an osculating curve of first kind in $\mathbb{E}^4$.

Assume that $x : I \subset \mathbb{R} \to \mathbb{E}^4$ is an osculating curve of first kind in $\mathbb{E}^4$ given with the arclength parameter s. By definition the curvature function m_2 vanishes identically. So, from (5) we get

$$\begin{aligned} m_0' - \kappa_1 m_1 &= 1, \\ m_1' + \kappa_1 m_0 &= 0, \\ \kappa_2 m_1 - \kappa_3 m_3 &= 0, \\ m_3 &= c, \end{aligned} \tag{9}$$

and therefore

$$\begin{aligned} m_0 &= \frac{-cH_2'}{\kappa_1}, \\ m_1 &= cH_2, \\ m_3 &= c, \end{aligned} \tag{10}$$

where $H_2(s) = \frac{\kappa_3}{\kappa_2}(s)$ and $c \in \mathbb{R}$ is a real constant. So, the position vector of x is given by

$$x(s) = c\left\{\frac{-H_2'}{\kappa_1}T(s) + H_2 N_1(s) + N_3(s)\right\}. \tag{11}$$

By the use of (9) with (10) we obtain the following result.

Lemma 1. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed in $\mathbb{E}^4$. Then, x is congruent to an osculating curve of first kind if and only if*

$$\left(\frac{cH_2'}{\kappa_1}\right)' + c\kappa_1 H_2 + 1 = 0 \tag{12}$$

holds, where $H_2(s) = \frac{\kappa_3}{\kappa_2}(s)$ and $c \in \mathbb{R}$.

As a consequence of (12), we obtain the following result.

Theorem 2. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a regular curve congruent to an osculating curve of first kind. If x is a ccr-curve then*

$$H_2 = \frac{-1}{c\kappa_1},$$

vhere $c = m_3$ is a real constant.

Moreover, if two of the curvature functions are constant, we may consider the following cases.

Suppose that $\kappa_1(s) = \text{constant} > 0$, $\kappa_2(s) = \text{constant} \neq 0$, and $\kappa_3(s)$ is a non-constant function. By the use of (12), we obtain the differential equation

$$c\kappa_3''(s) + c\kappa_1^2\kappa_3(s) + \kappa_1\kappa_2 = 0, \tag{13}$$

which has a non-trivial solution

$$\kappa_3(s) = -\frac{\kappa_2}{c\kappa_1} + c_1 \cos(\kappa_1 s) + c_2 \sin(\kappa_1 s).$$

Similarly, assume that $\kappa_1(s) = \text{constant} > 0$, $\kappa_3(s) = \text{constant} \neq 0$, and $\kappa_2(s)$ is a non-constant function. Then the equation (12) implies the differential equation

$$\frac{c\kappa_3}{\kappa_1}\left(\frac{1}{\kappa_2(s)}\right)'' + \frac{c\kappa_1\kappa_3}{\kappa_2(s)} + 1 = 0. \tag{14}$$

Thus, the differential equation (14) has a non-trivial solution of the form

$$\kappa_2(s) = \frac{c\kappa_1\kappa_3}{c_1\kappa_3\cos(\kappa_1 s) - c_2\kappa_3\sin(\kappa_1 s) - 1}.$$

Summing up these calculations, we obtain the following result.

Theorem 3. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in $\mathbb{E}^4$. Then x is congruent to an osculating curve of first kind if*

i) $\kappa_1(s) = \text{constant} > 0$, $\kappa_2(s) = \text{constant} \neq 0$, and

$$\kappa_3(s) = -\frac{\kappa_2}{c\kappa_1} + c_1\cos(\kappa_1 s) + c_2\sin(\kappa_1 s),$$

ii) $\kappa_1(s) = \text{constant} > 0$, $\kappa_3(s) = \text{constant} \neq 0$, and

$$\kappa_2(s) = \frac{c\kappa_1\kappa_3}{c_1\kappa_3\cos(\kappa_1 s) - c_2\kappa_3\sin(\kappa_1 s) - 1}$$

where $H_2(s) = \frac{\kappa_3}{\kappa_2}(s)$ and c, c_1 and $c_2 \in \mathbb{R}$.

2.2 Osculating curve of second kind in $\mathbb{E}^4$

Definition 2. Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a regular curve in $\mathbb{E}^4$ given with the arclength parameter s. If the position vector x lies in the hyperplane spanned by $\{T, N_2, N_3\}$ then x is called an osculating curve of second kind in $\mathbb{E}^4$.

In [17] K. Ilarslan and E. Nesovic considered the osculating curves of second kind in $\mathbb{E}^4$. Observe that they called them rectifying curves in $\mathbb{E}^4$. It means that the curvature function m_1 vanishes identically. So, from (5) we get

$$\begin{aligned} m_0' &= 1, \\ \kappa_2 m_2 - \kappa_1 m_0 &= 0, \\ m_2' - \kappa_3 m_3 &= 0, \\ m_3' + \kappa_3 m_2 &= 0, \end{aligned} \tag{15}$$

and therefore

$$\begin{aligned} m_0 &= s + b, \\ m_2 &= (s+b)H_1, \\ m_3 &= \frac{1}{\kappa_3}\left\{(s+b)H_1' + H_1\right\}, \end{aligned} \tag{16}$$

where $H_1(s) = \frac{\kappa_1}{\kappa_2}(s)$ is the first harmonic curvature of x and $b \in \mathbb{R}$. So, the position vector of x is given by

$$x(s) = (s+b)T(s) + (s+b)H_1 N_1(s) + \frac{(s+b)H_1' + H_1}{\kappa_3} N_3(s). \tag{17}$$

By the use of (9) with (10) we obtain the following result.

Theorem 4. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in $\mathbb{E}^4$. Then, x is congruent to an osculating curve of second kind if and only if*

$$\left\{\frac{(s+b)H_1' + H_1}{\kappa_3}\right\}' + \kappa_3 (s+b) H_1 = 0 \tag{18}$$

holds, where $H_1(s) = \frac{\kappa_1}{\kappa_2}(s)$, $b \in \mathbb{R}$.

In [17] K. Ilarslan and E. Nesovic gave the following result.

Theorem 5. *[17] There is no osculating curve of second kind with non-zero constant curvatures $\kappa_1(s), \kappa_2(s)$ and $\kappa_3(s)$.*

As a consequence of (18) we obtain the following result.

Theorem 6. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a regular curve congruent to an osculating curve of second kind. If x is a ccr-curve then*

$$\kappa_3(s) = \frac{\mp 1}{\sqrt{c - 2bs - s^2}}, \tag{19}$$

where $b, c \in \mathbb{R}$.

Proof. Let x be an osculating curve of second kind. If x is a ccr-curve then by definition, the curvature functions $H_1(s) = \frac{\kappa_1}{\kappa_2}$ and $H_2(s) = \frac{\kappa_3}{\kappa_2}$ are constant. So, by the use of (18) one can get

$$\kappa_3'(s) + (s+b)\kappa_3^3(s) = 0 \tag{20}$$

which has a nontrivial solution (19). □

As a consequence of differential equation (18) one can get the following solutions as in the previous section.

Corollary 1. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in $\mathbb{E}^4$. Then x is congruent to an osculating curve of second kind if*

i) $\kappa_1(s) =$ constant > 0, $\kappa_2(s) =$ constant $\neq 0$, and $\kappa_3(s) = \frac{1}{\left|\sqrt{c_1 - s^2 - 2bs}\right|}$ (see [17]),

ii) $\kappa_2(s) =$ constant $\neq 0$, $\kappa_3(s) =$ constant $\neq 0$, and

$$\kappa_1(s) = \frac{1}{s+b}\left(c_2 \sin(\kappa_3 s) + c_1 \cos(\kappa_3 s)\right),$$

iii) $\kappa_1(s) =$ constant > 0, $\kappa_3(s) =$ constant $\neq 0$, and

$$\kappa_2(s) = \frac{(s+b)\,\kappa_1}{c_1 \cos(\kappa_1 s) - c_2 \sin(\kappa_1 s)},$$

where c_1, c_2 and $b \in \mathbb{R}$.

2.3 T-constant curves in $\mathbb{E}^4$

Definition 3. Let $x : I \subset \mathbb{R} \to \mathbb{E}^n$ be a unit speed curve in $\mathbb{E}^n$. If $\left\|x^T\right\|$ is constant then x is called a *T-constant curve.* For a T-constant curve x, either $\left\|x^T\right\| = 0$ or $\left\|x^T\right\| = \lambda$ for some non-zero smooth function λ (see [3, 6]). Further, a T-constant curve x is called of first kind if $\left\|x^T\right\| = 0$, otherwise of second kind.

As a consequence of (5), we get the following results.

Theorem 7. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in $\mathbb{E}^4$ given with the parametrization (5). Then x is a T-constant curve of first kind if and only if*

$$H_2 R' + \left(\frac{\left(\frac{R'}{\kappa_2}\right)'}{\kappa_3} + \frac{R}{H_2}\right)' = 0. \tag{21}$$

where $H_2(s) = \frac{\kappa_3}{\kappa_2}(s)$ and $-m_1(s) = R(s) = \frac{1}{\kappa_1(s)}$ is the radius of the curvature of the curve x.

Proof. Let x be a T-constant curve of first kind, then from (5) we get

$$m_1 = -\frac{1}{\kappa_1},\ m_2 = \frac{m_1'}{\kappa_2},\ m_3 = \frac{m_2' + m_1\kappa_2}{\kappa_3}.$$

Further, substituting these values into $m_3' + \kappa_3 m_2 = 0$ we get the result. □

Remark 1. Any unit speed regular curve in $\mathbb{E}^4$ satisfying the equality (21) is a spherical curve lying on a sphere $S^3(r)$ of $\mathbb{E}^4$. Thus every T-constant curves of first kind are spherical.

The following theorem characterizes T-constant curve of second kind in $\mathbb{E}^4$.

Theorem 8. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in $\mathbb{E}^4$ given with the parametrization (5).Then x is a T-constant curve of second kind if and only if*

$$H_2\left(\kappa_1 m_0 - R'\right) + \left(\frac{\left(H_1 m_0 - \frac{R'}{\kappa_2}\right)'}{\kappa_3} - \frac{R}{H_2}\right)' = 0, \tag{22}$$

where $m_0 \in \mathbb{R}$, $H_1(s) = \frac{\kappa_1}{\kappa_2}(s)$, $H_2(s) = \frac{\kappa_3}{\kappa_2}(s)$ and $-m_1(s) = R(s) = \frac{1}{\kappa_1(s)}$ is the radius of the curvature of the curve x.

Proof. Let x be a T-constant curve of second kind, then from (5) we get

$$m_1 = -\frac{1}{\kappa_1},\ m_2 = \frac{m_1' + \kappa_1 m_0}{\kappa_2},\ m_3 = \frac{m_2' + m_1\kappa_2}{\kappa_3}.$$

Further, substituting these values into $m_3' + \kappa_3 m_2 = 0$, we get the result. □

The following result explicitly determines the T-constant W-curves of second kind in $\mathbb{E}^4$.

Corollary 2. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a regular T-constant curve of second kind in $\mathbb{E}^4$. If x is a W-curve of $\mathbb{E}^4$, then the position vector x has the parametrization*

$$x(s) = \lambda T - RN_1 + H_1\lambda N_2 + (bs + c)\, N_3,$$

where $R = \frac{1}{\kappa_1}$, $H_1 = \frac{\kappa_1}{\kappa_2}$, c is integral constant, $b = -H_1\kappa_3\lambda$ and $\lambda \in \mathbb{R}$.

The following result provides a simple characterization of T-constant curve of second kind in $\mathbb{E}^4$.

Theorem 9. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a T-constant curve of second kind. Then the distance function $\rho = \|x\|$ satisfies*

$$\rho = \pm\sqrt{2\lambda s + c}. \tag{23}$$

for some real constants c and $\lambda = m_0$.

Proof. Differentiating the squared distance function $\rho^2 = \langle x(s), x(s)\rangle$ and using (3) we get $\rho\rho' = m_0$. If x is a T-constant curve of second kind then by definition, the curvature function $m_0(s)$ of x is constant. It is easy to show that this differential equation has a nontrivial solution (23). □

2.4 N-constant curves in $\mathbb{E}^4$

Definition 4. Let $x : I \subset \mathbb{R} \to \mathbb{E}^n$ be a unit speed curve in $\mathbb{E}^n$. If $\left\|x^N\right\|$ is constant then x is called an N-constant curve. For an N-constant curve x, either $\left\|x^N\right\| = 0$ or $\left\|x^N\right\| = \mu$ for some non-zero smooth function μ (see [3,6]). Further, an N-constant curve x is called of first kind if $\left\|x^N\right\| = 0$, otherwise of second kind.

So, for an N-constant curve x in $\mathbb{E}^4$

$$\left\|x^N(s)\right\|^2 = m_1^2(s) + m_2^2(s) + m_3^2(s) \tag{24}$$

becomes a constant function. Therefore, by differentiation

$$m_1 m_1' + m_2 m_2' + m_3 m_3' = 0. \tag{25}$$

For the N-constant curves of first kind we give the following result.

Proposition 1. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in $\mathbb{E}^4$. Then x is an N-constant curve of first kind if and only if $x(I)$ is an open portion of a straight line through the origin.*

Proof. Suppose that x is an N-constant curve of first kind in $\mathbb{E}^4$, then by definition $\left\|x^N(s)\right\| = \mu = 0$. Further, differentiating $x(s) = m_0(s)T(s)$ and using the Frenet equation (1) we get $\kappa_1 = 0$. □

Further, for the N-constant curves of second kind, we obtain the following results.

Theorem 10. *Let $x(s) \in \mathbb{E}^4$ be a unit speed regular curve that fully lies in $\mathbb{E}^4$. If x is an N-constant curve of second kind, then the position vector x of the curve has the parametrization*

$$x(s) = (s+b)\,T(s) + (s+b)\,H_1 N_2(s) + \frac{(s+b)\,H_1' + H_1}{\kappa_3} N_3(s), \tag{26}$$

where $H_1(s) = \frac{\kappa_1}{\kappa_2}(s)$, $b \in \mathbb{R}$.

Proof. Suppose that x is an N-constant curve of second kind in $\mathbb{E}^4$, then from the equations in (5) and (25) we get $m_1 = 0$, $m_0(s) = s + b$, $m_2(s) = \frac{\kappa_1}{\kappa_2}(s) m_0$ and $m_3(s) = \frac{m_2'(s)}{\kappa_3(s)}$ for some constant function b. This completes the proof of the theorem. □

Corollary 3. *Every N-constant curve of second kind in $\mathbb{E}^4$ is an osculating curve of second kind.*

The following result provides a simple characterization of N-constant curve of second kind in $\mathbb{E}^4$.

Theorem 11. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be an N-constant curve of second kind. Then the distance function $\rho = \|x\|$ satisfies*

$$\rho = \mp\sqrt{s^2 + 2bs + d} \tag{27}$$

for some constant functions b, d.

Proof. Differentiating the squared distance function $\rho^2 = \langle x(s), x(s)\rangle$ and using (3) we get $\rho\rho' = m_0$. If x is an N-constant curve of second kind then from the previous theorem $m_0(s) = s + b$. It is easy to show that this differential equation has a nontrivial solution (27). □

Definition 5. Let $x : I \subset \mathbb{R} \to \mathbb{E}^n$ be a unit speed regular curve in $\mathbb{E}^n$. Then the position vector x can be decomposed into its tangential and normal components at each point:

$$x = x^T + x^N.$$

If the ratio $\left\|x^T\right\| : \left\|x^N\right\|$ is constant on $x(I)$ then x is said to be of *constant ratio, or equivalently* $\left\|x^T\right\| : \|x\| = c =$constant [2].

For a unit speed regular curve x in $\mathbb{E}^n$, the gradient of the distance function $\rho = \|x(s)\|$ is given by

$$grad\rho = \frac{d\rho}{ds}x'(s) = \frac{< x(s), x'(s) >}{\|x(s)\|}T(s), \tag{28}$$

where T is the tangent vector field of x.

The following results characterize constant-ratio curves.

Theorem 12. *[7] Let $x : I \subset \mathbb{R} \to \mathbb{E}^n$ be a unit speed regular curve in $\mathbb{E}^n$. Then x is of constant ratio with $\left\|x^T\right\| : \|x\| = c$ if and only if $\|grad\rho\| = c$ which is constant.*

In particular, for a curve of constant ratio we have $\|grad\rho\| = c \leq 1$.

As a consequence of (28) we obtain the following result.

Corollary 4. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^n$ be a unit speed regular curve in $\mathbb{E}^n$. If x is of constant ratio then the distance function $\rho = \frac{m_0}{c}$, where $\|grad\rho\| = c$ and $m_0 =< x(s), x'(s) >$.*

Theorem 13. *[7] Let $x : I \subset \mathbb{R} \to \mathbb{E}^n$ be a unit speed regular curve in $\mathbb{E}^n$. Then $\|grad\rho\| = c$ holds for a constant c if and only if one of the following three cases occurs:*

(i) $\|grad\rho\| = 0 \iff x(I)$ is contained in a hypersphere centered at the origin.
(ii) $\|grad\rho\| = 1 \iff x(I)$ is an open portion of a line through the origin.
(iii) $\|grad\rho\| = c \iff \rho = \|x(s)\| = cs$, for $c \in (0, 1)$.

The following result provides some simple characterization of T-constant and N-constant curves in $\mathbb{E}^4$. Observe that this result is also valid in 3-dimensional case (see [13]).

Corollary 5. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed regular curve in $\mathbb{E}^4$. Then up to a translation of the arc length function s, we have*

i) If x is a T-constant curve of first kind then $\|grad\rho\| = 0$,

ii) If x is an N-constant curve of first kind then $\|grad\rho\| = 1$,

iii) If x is a T-constant curve of second kind then $\rho^2 = m_0 s + b$,

iv) If x is an N-constant curve of second kind then $\rho^2 = (s+a)^2 + m_1$,

where m_0, m_1, a, b are real constants.

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Solvability of the boundary value problem for the equation of transition processes in semiconductors with a fractional time derivative

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Abstract. Necessary and sufficient conditions are established for the unique solvability of the initial boundary value problem for the equation describing the transition processes in semiconductors. The method of studying is the reducing to the Cauchy problem for a degenerate evolution equation of fractional order in a Banach space. Using the functional calculus in the Banach algebra of bounded linear operators a form of the considered problem solution is performed.

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Introduction

Let $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $\Delta = \sum\limits_{k=1}^{3} \frac{\partial^2}{\partial x_k^2}$ is the Laplace operator. For $\alpha > 0, \lambda, \beta, \theta \in \mathbb{R}$ consider the initial boundary value problem

$$D_t^\alpha(\lambda - \Delta)w(x,t) = \beta w(x,t) + f(x,t), \quad (x,t) \in \Omega \times \overline{\mathbb{R}}_+, \tag{1}$$

$$(1-\theta)w(x) + \theta \frac{\partial}{\partial n} w(x) = 0, \quad (x,t) \in \partial\Omega \times \overline{\mathbb{R}}_+, \tag{2}$$

$$\frac{\partial^k w}{\partial t^k}(x,0) = w_k(x), \ x \in \Omega, \ k = 0, 1, \ldots, m-1, \tag{3}$$

Here D_t^α is a fractional Caputo derivative, m is a smallest integer not exceeding or equal to α. It is worth noting that the fractional derivatives play an increasingly important role in mathematical modeling, partially for describing various physical processes [3–6].

In the case of $\alpha = 1$ equation (1) describes the transition processes in semiconductors [2]. Function $w(x,t)$ has a physical sense of the electric field potential. The unique solvability of problem (1)–(3) with $\alpha = 1$ was studied in [2]. A mixed–type optimal control problem for the corresponding distributed control system was researched in [7].

In this paper by means of solution operators theory for fractional differential equations in Banach spaces the conditions of problem (1)–(3) unique solvability in the fractional case are found and the form of solution is performed in the present work.

1 Cauchy problem for abstract fractional order equation

Let $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$, $\overline{\mathbb{R}}_+ = \{0\} \cup \mathbb{R}_+$, for $\delta > 0$ $g_\delta(t) = t^{\delta-1}/\Gamma(\delta),\ t > 0,$

$$J_t^\delta h(t) = (g_\delta * h)(t) = \int_0^t g_\delta(t-s)h(s)ds = \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}h(s)ds.$$

Let $\alpha > 0$, m is the smallest integer that is greater than or equal to α, D_t^m is a usual derivative of order m for $m \in \mathbb{N}$, D_t^α is Caputo derivative [1], i. e.

$$D_t^\alpha h(t) = D_t^m J_t^{m-\alpha}\left(h(t) - \sum_{k=0}^{m-1} h^{(k)}(0)g_{k+1}(t)\right) = J_t^{m-\alpha}D_t^m h(t),$$

when the expression on the right side is defined.

Let $\mathfrak{U}$ and $\mathfrak{V}$ be Banach spaces, $L \in \mathcal{L}(\mathfrak{U};\mathfrak{V})$ (linear and continuous operator), $M \in \mathcal{C}l(\mathfrak{U};\mathfrak{V})$ (linear, closed and densely defined operator), D_M is a domain of the operator M, $f : [0,T] \to \mathfrak{V}$ is a given function. Consider the Cauchy problem

$$u^{(k)}(0) = u_k,\ k = 0, 1, \ldots, m-1, \tag{4}$$

for the fractional differential equation

$$D_t^\alpha Lu(t) = Mu(t) + f(t). \tag{5}$$

Various initial–boundary value problems for partial differential equations or systems of equations not solved with respect to the time–fractional derivatives can be reduced to the Cauchy problem (4), (5). Such equations arise in mathematical modeling of various processes in natural and technical sciences [2, 8–10]. Partially it concerns problem (1)–(3).

The theory of fractional differential equations has been intensively developed in the last decades [1, 3–6], but a few articles concern the fractional differential equations of the form (1), not solved with respect to the fractional derivative. See [9–11].

Define L–resolvent set of operator M $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{V};\mathfrak{U})\}$. Operator M is called (L,σ)–bounded if the complement to the set $\rho^L(M)$ is bounded in $\mathbb{C}$. Define $R_\mu^L(M) = (\mu L - M)^{-1}L$, $L_\mu^L(M) = L(\mu L - M)^{-1}$,

$$P = \frac{1}{2\pi i}\int_\gamma R_\mu^L(M)d\mu \in \mathcal{L}(\mathfrak{U};\mathfrak{U}), \quad Q = \frac{1}{2\pi i}\int_\gamma L_\mu^L(M)d\mu \in \mathcal{L}(\mathfrak{V};\mathfrak{V}), \tag{6}$$

where the integrals are taken along a circle γ with a radius a, enclosing the complement to $\rho^L(M)$ in the complex plane $\mathbb{C}$. It is easy to check that operators P and Q are projectors [8]. Denote $\mathfrak{U}^0 = \ker P$, $\mathfrak{V}^0 = \ker Q$, $\mathfrak{U}^1 = \mathrm{im} P$, $\mathfrak{V}^1 = \mathrm{im} Q$. Let $L_k(M_k)$ be the restrictions of operator $L(M)$ to the subspace $\mathfrak{U}^k (D_{M_k} = D_M \cap \mathfrak{U}^k)$, $k = 0, 1$.

Theorem 1. *[8] Let operator M be (L,σ)–bounded. Then*
(i) $M_1 \in \mathcal{L}(\mathfrak{U}^1;\mathfrak{V}^1)$, $M_0 \in \mathcal{C}l(\mathfrak{U}^0;\mathfrak{V}^0)$, $L_k \in \mathcal{L}(\mathfrak{U}^k;\mathfrak{V}^k)$, $k = 0,1$;
(ii) *there exist operators* $M_0^{-1} \in \mathcal{L}(\mathfrak{V}^0;\mathfrak{U}^0)$, $L_1^{-1} \in \mathcal{L}(\mathfrak{V}^1;\mathfrak{U}^1)$.

Let us denote $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $H = M_0^{-1}L_0$. For $p \in \mathbb{N}_0$ operator M is called (L,p)–bounded if it is (L,σ)–bounded, $H^p \neq \mathbb{O}$, $H^{p+1} = \mathbb{O}$.

The solution of Cauchy problem (4), (5) is a function $u \in C^{m-1}(\overline{\mathbb{R}}_+;\mathfrak{U}) \cap C(\overline{\mathbb{R}}_+;D_M)$ such that $Lu \in C^{m-1}(\overline{\mathbb{R}}_+;\mathfrak{V})$, $g_{m-\alpha} * \left(Lu - \sum\limits_{k=0}^{m-1}(Lu)^{(k)}(0)g_{k+1}\right) \in C^m(\overline{\mathbb{R}}_+;\mathfrak{V})$, equalities (4) and (5) are valid for all $t \in \overline{\mathbb{R}}_+$.

The unique solvability of problem (4), (5) was investigated in [10]. Formulate the theorem on the existence and uniqueness of problem (4), (5) solution.

Theorem 2. *[10] Let operator M be (L,p)–bounded, $\gamma = \{\mu \in \mathbb{C} : |\mu| = r > a\}$,*

$$U(t) = \frac{1}{2\pi i}\int\limits_{\gamma} R_\mu^L(M)E_{\alpha,\beta}(\mu t^\alpha)d\mu, \ t \in \overline{\mathbb{R}}_+,$$

where $E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}$ is Mittag–Leffler function. Then for all $u_k \in \mathfrak{U}^1$, $k = 0,1,\dots m-1$, there exists a unique solution of problem (4), (5), and it has the form

$$u(t) = \sum_{k=0}^{m-1} J_t^k U(t)u_k. \tag{7}$$

If for some $l \in \{0,1,\dots m-1\}$ $u_l \notin \mathfrak{U}^1$, then problem (4), (5) has no solutions.

2 Solvability of the equation of transition processes in semiconductors

Let us return to problem (1)–(3) and reduce it to Cauchy problem (4), (5). Define the formal differential operator

$$B_\theta = (1-\theta) + \theta\frac{\partial}{\partial n}, \quad \theta \in \mathbb{R}.$$

Operator $A \in \mathcal{C}l(L_2(\Omega))$ is defined as acting on its domain

$$D_A = H_\theta^2(\Omega) = \left\{u \in H^2(\Omega) : B_\theta u(x) = 0, \ x \in \partial\Omega\right\}$$

by $Au = \Delta u$. Denote by $\{\varphi_k : \ k \in \mathbb{N}\}$ the orthonormal in the sense of the scalar product $\langle\cdot,\cdot\rangle$ in $L_2(\Omega)$ eigenfunctions of operator A, numbered in the non–increasing order with respect to the corresponding eigenvalues $\{\lambda_k : \ k \in \mathbb{N}\}$, counting their multiplicities. Note that the spectrum of operator A is real, discrete and condensed to $-\infty$ [12].

Let $\mathfrak{U} = \left\{u \in H^2(\Omega) : B_\theta u(x) = 0, x \in \partial\Omega\right\}$ (the Sobolev space), $\mathfrak{V} = L_2(\Omega)$ (the Lebesgue space), $L = \lambda - A$, $M = \beta I \in \mathcal{L}(\mathfrak{U};\mathfrak{V})$.

Theorem 3. *Let $\beta \neq 0$ or the spectrum $\sigma(A)$ do not contain λ. Then operator M is $(L,0)$–bounded.*

Proof. In conditions of the theorem consider the operator

$$\mu L - M = \sum_{k=1}^{\infty} (\mu(\lambda - \lambda_k) - \beta) \langle \cdot, \varphi_k \rangle \varphi_k.$$

Show that for

$$|\mu| > \sup_{\lambda \neq \lambda_k} \left| \frac{\beta}{\lambda - \lambda_k} \right|$$

the operator

$$(\mu L - M)^{-1} = \sum_{k=1}^{\infty} \frac{\langle \cdot, \varphi_k \rangle \varphi_k}{\mu(\lambda - \lambda_k) - \beta} : L_2(\Omega) \to \mathfrak{U}$$

exists and is continuous. For $f \in L_2(\Omega)$

$$||(\mu L - M)^{-1} f||^2_{H^2(\Omega)} = \sum_{k=1}^{\infty} \frac{(1+\lambda_k^2) |\langle f, \varphi_k \rangle|^2}{|\mu(\lambda - \lambda_k) - \beta|^2} =$$

$$= \sum_{\lambda_k = \lambda} \frac{(1+\lambda_k^2) |\langle f, \varphi_k \rangle|^2}{\beta^2} + \sum_{\lambda_k \neq \lambda} \frac{(1+\lambda_k^2) |\langle f, \varphi_k \rangle|^2}{|\lambda - \lambda_k|^2 \left| \mu - \frac{\beta}{\lambda - \lambda_k} \right|^2} \leq C ||f||^2_{L_2(\Omega)}$$

because of finitness of the first sum in the last line. Indeed,

$$\lim_{k \to \infty} \frac{1 + \lambda_k^2}{|\lambda - \lambda_k|^2} = 1,$$

so the corresponding sequence is bounded. Furthermore, the inequalities

$$\left| \mu - \frac{\beta}{\lambda - \lambda_k} \right| \geq |\mu| - \left| \frac{\beta}{\lambda - \lambda_k} \right| \geq d > 0$$

are true. Thus, the operator M is (L, σ)–bounded with a constant

$$a = \sup_{\lambda \neq \lambda_k} \left| \frac{\beta}{\lambda - \lambda_k} \right|.$$

Construct the projector

$$P = \frac{1}{2\pi i} \int\limits_{|\mu| = a+1} \sum_{\lambda_k \neq \lambda} \frac{\langle \cdot, \varphi_k \rangle \varphi_k}{\mu - \frac{\beta}{\lambda - \lambda_k}} d\mu = \sum_{\lambda_k \neq \lambda} \langle \cdot, \varphi_k \rangle \varphi_k \in \mathcal{L}(\mathfrak{U}).$$

It is obvious that the projector Q has the same form but is defined in $L_2(\Omega)$. Consequently $\mathfrak{U}^0 = \mathfrak{V}^0 = \operatorname{span}\{\varphi_k : \lambda_k = \lambda\}$, $\mathfrak{U}^1$ and $\mathfrak{V}^1$ are the closures of span $\{\varphi_k : \lambda_k \neq \lambda\}$ in the norm of spaces $\mathfrak{U}$ and $\mathfrak{V}$ respectively. Note that $\ker L = \ker P$, hence

$$0 = Lu = (\lambda I - A)u = \sum_{k=1}^{\infty}(\lambda - \lambda_k)\langle u, \varphi_k\rangle = \sum_{\lambda\neq\lambda_k}(\lambda - \lambda_k)\langle u, \varphi_k\rangle,$$

then $u = \sum_{\lambda_k=\lambda} c_k\varphi_k$ for some $c_k \in \mathbb{R}$, therefore $u \in \mathfrak{U}^0$. Inversely, if $u = \sum_{\lambda_k=\lambda} c_k\varphi_k$, then $(\lambda I - A)u = \sum_{\lambda_k=\lambda} c_k(\lambda - \lambda_k)\varphi_k = 0$ and $u \in \ker L$. Therefore $H = \mathbb{O}$ and the operator M is $(L, 0)$–bounded. □

Theorem 4. *Let $\beta \neq 0$ or the spectrum $\sigma(A)$ do not contain λ, for all $k \in \mathbb{N}$ such that $\lambda_k = \lambda$ the equalities $\langle u_l, \varphi_k\rangle = 0$, $l = 0, 1, \ldots, m-1$, are true. Then there exists a unique solution of problem (1)–(3), and it has the form*

$$u(x,t) = \sum_{\lambda_k\neq\lambda}\sum_{l=0}^{m-1} t^l E_{\alpha,\beta+l}\left(\frac{\beta t^\alpha}{\lambda - \lambda_k}\right)\langle u_l, \varphi_k\rangle\,\varphi_k(x).$$

Proof. Reduce the problem (1)–(3) to problem (4), (5). By Theorems 2, 3 obtain the required assertion. The solution is calculated using the formula (7) and the residue theorem in the same way as the projector is calculated in the previous theorem. Note that the properties of Mittag–Leffler functions imply the equality

$$u(x,t) = \sum_{\lambda_k\neq\lambda}\sum_{l=0}^{m-1} J_t^l E_{\alpha,\beta}\left(\frac{\beta t^\alpha}{\lambda - \lambda_k}\right)\langle u_l, \varphi_k\rangle\,\varphi_k(x) =$$

$$= \sum_{\lambda_k\neq\lambda}\sum_{l=0}^{m-1} t^l E_{\alpha,\beta+l}\left(\frac{\beta t^\alpha}{\lambda - \lambda_k}\right)\langle u_l, \varphi_k\rangle\,\varphi_k(x).$$

□

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On the inverse operations in the class of preradicals of a module category, I

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Abstract. In the class $\mathbb{PR}$ of preradicals of the category of left R-modules R-Mod a new operation is defined and studied, namely the left quotient with respect to join. Some properties of this operation are shown, its compatibility with the lattice operations of $\mathbb{PR}$ (meet and join of preradicals), as well as the relations with some constructions in the "big" lattice $\mathbb{PR}$. Also some particular cases are examined.

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1 Introduction and preliminary facts

The work is concerned with the theory of radicals of modules ([1], [2], [3]) and is devoted to investigation of a new operation in the class of preradicals of a module category.

Let R be a ring with unity and R-Mod be the category of unitary left R-modules. A *preradical* r of R-Mod is a subfunctor of identity functor of R-Mod, i.e. r associates to every module $M \in R$-Mod a submodule $r(M) \subseteq M$ such that $f(r(M)) \subseteq r(M')$ for every R-morphism $f: M \to M'$.

We denote by $\mathbb{PR}$ the class of all preradicals of the category R-Mod, where the partial order relation is defined as follows:

$$r_1 \leq r_2 \overset{def}{\Leftrightarrow} r_1(M) \subseteq r_2(M) \text{ for every } M \in R\text{-Mod}.$$

In the class $\mathbb{PR}$ the following operations are defined ([1]):

1) the *meet* $\underset{\alpha \in \mathfrak{A}}{\wedge} r_\alpha$ of the family of preradicals $\{r_\alpha\}_{\alpha \in \mathfrak{A}} \subseteq \mathbb{PR}$:

$$\left(\underset{\alpha \in \mathfrak{A}}{\wedge} r_\alpha\right)(M) \overset{def}{=} \bigcap_{\alpha \in \mathfrak{A}} r_\alpha(M),\ M \in R\text{-Mod};$$

2) the *join* $\underset{\alpha \in \mathfrak{A}}{\vee} r_\alpha$ of the family of preradicals $\{r_\alpha\}_{\alpha \in \mathfrak{A}} \subseteq \mathbb{PR}$:

$$\left(\underset{\alpha \in \mathfrak{A}}{\vee} r_\alpha\right)(M) \overset{def}{=} \sum_{\alpha \in \mathfrak{A}} r_\alpha(M),\ M \in R\text{-Mod};$$

3) the *product* $r \cdot s$ of preradicals $r, s \in \mathbb{PR}$:

$$(r \cdot s)(M) \overset{def}{=} r(s(M)),\ M \in R\text{-Mod};$$

4) the *coproduct* $r \# s$ of preradicals $r, s \in \mathbb{PR}$:

$$[(r \# s)(M)]/s(M) \overset{def}{=} r(M/s(M)), \; M \in R\text{-Mod}.$$

The class $\mathbb{PR}$ is a "big" complete lattice with respect to the operations meet and join.

We remark that in the book [1] the coproduct is denoted by $(r : s)$ and is defined by the rule $[(r : s)(M)]/r(M) = s(M/r(M))$, so $(r \# s) = (s : r)$.

The following properties of distributivity hold ([1]):

(1) $(\wedge r_\alpha) \cdot s = \wedge (r_\alpha \cdot s)$; (2) $(\vee r_\alpha) \cdot s = \vee (r_\alpha \cdot s)$;

(3) $(\wedge r_\alpha) \# s = \wedge (r_\alpha \# s)$; (4) $(\vee r_\alpha) \# s = \vee (r_\alpha \# s)$,

for every family of preradicals $\{r_\alpha\}_{\alpha \in \mathfrak{A}} \subseteq \mathbb{PR}$ and $s \in \mathbb{PR}$.

These relations permit to define some new operations in the class $\mathbb{PR}$. In the present work it is introduced and studied one of these operations, namely the left quotient with respect to join. The similar questions are discussed in [2], [6], [7] and [8].

Now we remind the principal types of preradicals. A preradical $r \in \mathbb{PR}$ is called:

- *an idempotent preradical*, if $r(r(M)) = r(M)$ for every $M \in R$-Mod (or if $r \cdot r = r$);
- *a radical*, if $r(M/r(M)) = 0$ for every $M \in R$-Mod (or if $r \# r = r$);
- *an idempotent radical*, if both previous conditions are fulfilled;
- *a pretorsion*, if $r(N) = N \bigcap r(M)$ for every $N \subseteq M \in R$-Mod;
- *a torsion*, if r is a radical and a pretorsion;
- *prime*, if $r \neq 1$ and for any $t_1, t_2 \in \mathbb{PR}$, $t_1 \cdot t_2 \leq r$ implies either $t_1 \leq r$ or $t_2 \leq r$;
- $\wedge$-*prime*, if for any $t_1, t_2 \in \mathbb{PR}$, $t_1 \wedge t_2 \leq r$ implies either $t_1 \leq r$ or $t_2 \leq r$;
- *irreducible*, if for any $t_1, t_2 \in \mathbb{PR}$, $t_1 \wedge t_2 = r$ implies $t_1 = r$ or $t_2 = r$.

The operations meet and join are commutative and associative, but the product and coproduct are only associative. By means of these operations four preradicals are obtained which are arranged in the following order:

$$r \cdot s \leq r \wedge s \leq r \vee s \leq r \# s$$

for every $r, s \in \mathbb{PR}$.

In the course of this work we will need the following facts and notions from general theory of preradicals (see [1]$-$[5]).

Lemma 1.1. *(Monotony of the product) For any $s_1, s_2 \in \mathbb{PR}$, $s_1 \leq s_2$ implies that $r \cdot s_1 \leq r \cdot s_2$ and $s_1 \cdot r \leq s_2 \cdot r$ for every $r \in \mathbb{PR}$.* □

Lemma 1.2. *(Monotony of the coproduct) For any $s_1, s_2 \in \mathbb{PR}$, $s_1 \leq s_2$ implies that $r \# s_1 \leq r \# s_2$ and $s_1 \# r \leq s_2 \# r$ for every $r \in \mathbb{PR}$.* □

Lemma 1.3. *If r is a pretorsion, then $r \cdot s = r \wedge s$ for every $s \in \mathbb{PR}$.* □

Lemma 1.4. *For every* $r, s, t \in \mathbb{PR}$ *we have:*

1) $(r \cdot s) \# t \geq (r \# t) \cdot (s \# t)$;

2) $(r \# s) \cdot t \leq (r \cdot t) \# (s \cdot t)$. □

Definition 1.1. The *annihilator* of preradical r is the preradical

$$a(r) = \vee \{ r_\alpha \in \mathbb{PR} \,|\, r_\alpha \cdot r = 0 \} \,.$$

Definition 1.2. The *pseudocomplement* of r in $\mathbb{PR}$ is a preradical $r^\perp \in \mathbb{PR}$ with the properties:

1) $r \wedge r^\perp = 0$;

2) If $s \in \mathbb{PR}$ is such that $s > r^\perp$, then $r \wedge s \neq 0$.

Lemma 1.5. *Each* $r \in \mathbb{PR}$ *has a unique pseudocomplement* $r^\perp$ *such that if* $s \in \mathbb{PR}$ *and* $r \wedge s = 0$, *then* $s \leq r^\perp$. □

Definition 1.3. The *supplement* of r in $\mathbb{PR}$ is a preradical $r^* \in \mathbb{PR}$ with the properties:

1) $r \vee r^* = 1$;

2) If $s \in \mathbb{PR}$ is such that $s < r^*$, then $r \vee s \neq 1$.

Lemma 1.6. *Let* $r \in \mathbb{PR}$ *and* r *possesses the supplement* r^*. *If* $s \in \mathbb{PR}$ *and* $r \vee s = 1$, *then* $s \geq r^*$. □

2 Left quotient with respect to join

We investigate the class of preradicals $\mathbb{PR}(\wedge, \vee, \cdot, \#)$ of category R-Mod provided with four operations defined above. Using these operations and the aforementioned properties of distributivity, some new inverse operations can be defined. One of them is defined and studied further.

Definition 2.1. Let $r, s \in \mathbb{PR}$. The *left quotient with respect to join of* r *by* s is defined as the greatest preradical among $r_\alpha \in \mathbb{PR}$ with the property $r_\alpha \cdot s \leq r$. We denote this preradical by $r \overset{\vee}{/} s$.

We say that r is the *numerator* and s is the *denominator* of the quotient $r \overset{\vee}{/} s$.

Now we mention the existence of the left quotient for every pair of preradicals.

Lemma 2.1. *For every* $r, s \in \mathbb{PR}$ *there exists the left quotient* $r \overset{\vee}{/} s$ *with respect to join, and it can be presented in the form* $r \overset{\vee}{/} s = \vee \{ r_\alpha \in \mathbb{PR} \,|\, r_\alpha \cdot s \leq r \}$.

Proof. The family of preradicals $\{r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s \leq r\}$ is not empty, because $0 \cdot s \leq$ $\leq r$. By the distributivity of the product with respect to the join of preradicals we obtain $\left(\bigvee\limits_{r_\alpha \cdot s \leq r} r_\alpha\right) \cdot s = \bigvee\limits_{r_\alpha \cdot s \leq r} (r_\alpha \cdot s)$. Since $r_\alpha \cdot s \leq r$ for preradicals r_α, we have $\bigvee\limits_{r_\alpha \cdot s \leq r} (r_\alpha \cdot s) \leq r$, therefore $\left(\bigvee\limits_{r_\alpha \cdot s \leq r} r_\alpha\right) \cdot s \leq r$. So the preradical $\bigvee\limits_{r_\alpha \cdot s \leq r} r_\alpha$ is one of r_α, moreover it is the greatest among r_α with the property $r_\alpha \cdot s \leq r$. Therefore we have $\vee \{r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s \leq r\} = r \mathbin{\vee\!\!/\!.} s$. □

From the proof of Lemma 2.1 it follows that $(r \mathbin{\vee\!\!/\!.} s) \cdot s \leq r$ and we will use this relation often in continuation.

Lemma 2.2. *For every* $r, s \in \mathbb{PR}$ *we have* $r \mathbin{\vee\!\!/\!.} s \geq r$.

Proof. Since $r \cdot s \leq r$ and $r \mathbin{\vee\!\!/\!.} s = \vee \{r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s \leq r\}$, it is clear that r is one of preradicals r_α. Therefore $r \leq \vee \{r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s \leq r\}$, so $r \leq r \mathbin{\vee\!\!/\!.} s$. □

The next two statements show the connection between the left quotient $r \mathbin{\vee\!\!/\!.} s$ and the partial order $(\leq)$ in $\mathbb{PR}$.

Proposition 2.3. *(Monotony in the numerator) If* $r_1, r_2 \in \mathbb{PR}$ *and* $r_1 \leq r_2$*, then* $r_1 \mathbin{\vee\!\!/\!.} s \;\leq\; r_2 \mathbin{\vee\!\!/\!.} s$ *for every* $s \in \mathbb{PR}$.

Proof. From Lemma 2.1 we have: $r_1 \mathbin{\vee\!\!/\!.} s = \vee \{r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s \leq r_1\}$ and $r_2 \mathbin{\vee\!\!/\!.} s =$ $\vee \{r'_\alpha \in \mathbb{PR} \mid r'_\alpha \cdot s \leq r_2\}$. The relations $r_1 \leq r_2$ and $r_\alpha \cdot s \leq r_1$ imply $r_\alpha \cdot s \leq r_2$, so each r_α is one of the preradicals r'_α. This proves that $\vee \{r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s \leq r_1\} \leq$ $\vee \{r'_\alpha \in \mathbb{PR} \mid r'_\alpha \cdot s \leq r_2\}$, so $r_1 \mathbin{\vee\!\!/\!.} s \;\leq\; r_2 \mathbin{\vee\!\!/\!.} s$. □

Proposition 2.4. *(Antimonotony in the denominator) If* $s_1, s_2 \in \mathbb{PR}$ *and* $s_1 \leq s_2$*, then* $r \mathbin{\vee\!\!/\!.} s_1 \geq r \mathbin{\vee\!\!/\!.} s_2$ *for every* $s \in \mathbb{PR}$.

Proof. From Lemma 2.1 we have:

$$r \mathbin{\vee\!\!/\!.} s_1 = \vee \{r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s_1 \leq r\}, \quad r \mathbin{\vee\!\!/\!.} s_2 = \vee \{r'_\alpha \in \mathbb{PR} \mid r'_\alpha \cdot s_2 \leq r\}.$$

If $s_1 \leq s_2$, then $r'_\alpha \cdot s_1 \leq r'_\alpha \cdot s_2$, but $r'_\alpha \cdot s_2 \leq r$, therefore $r'_\alpha \cdot s_1 \leq r$. So each preradical r'_α is one of the preradicals r_α and we obtain

$$\vee \{r'_\alpha \in \mathbb{PR} \mid r'_\alpha \cdot s_2 \leq r\} \leq \vee \{r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s_1 \leq r\} ,$$

i.e. $r \mathbin{\vee\!\!/\!.} s_1 \geq r \mathbin{\vee\!\!/\!.} s_2$. □

The following result is particulary useful in the further studies.

Proposition 2.5. *For every* $r, s, t \in \mathbb{PR}$ *we have:*

$$r \geq t \cdot s \Leftrightarrow r \mathbin{\vee\!\!/\!.} s \geq t \qquad .$$

Proof. $(\Rightarrow)$ Let $t \cdot s \leq r$. By Lemma 2.1 we have $r \mathbin{\vee\!\!/\!.} s = \vee \{r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s \leq r\}$. Then t is one of the preradicals r_α, therefore $t \leq \vee \{r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s \leq r\} = r \mathbin{\vee\!\!/\!.} s$.

$(\Leftarrow)$ Let $t \leq r \mathbin{\vee\!\!/\!.} s$. Then $t \cdot s \leq (r \mathbin{\vee\!\!/\!.} s) \cdot s$ and by definition $(r \mathbin{\vee\!\!/\!.} s) \cdot s \leq r$, therefore $t \cdot s \leq r$. □

Proposition 2.6. $(r \cdot s) \vee\!/. \, s \geq r$ *for every preradicals* $r, s \in \mathbb{PR}$.

Proof. From Lemma 2.1 we have $(r \cdot s) \vee\!/. \, s = \vee \{ t_\alpha \in \mathbb{PR} \mid t_\alpha \cdot s \leq r \cdot s \}$. Since $r \cdot s \leq r \cdot s$ we have that r is one of the preradicals t_α, therefore $r \leq \vee \{ t_\alpha \in \mathbb{PR} \mid t_\alpha \cdot s \leq r \cdot s \}$, i.e. $r \leq (r \cdot s) \vee\!/. \, s$. □

Proposition 2.7. *For every* $r, s, t \in \mathbb{PR}$ *the following relations are true:*

1) $(r \vee\!/. \, s) \vee\!/. \, t = r \vee\!/. \, (t \cdot s)$;

2) $(r \cdot s) \vee\!/. \, t \geq r \cdot (s \vee\!/. \, t)$.

Proof. 1) From Lemma 2.1 we have $r \vee\!/. \, s = \vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s \leq r \}$, $(r \vee\!/. \, s) \vee\!/. \, t = \vee \{ t_\beta \in \mathbb{PR} \mid t_\beta \cdot t \leq r \vee\!/. \, s \}$ and $r \vee\!/. \, (t \cdot s) = \vee \{ r'_\gamma \in \mathbb{PR} \mid r'_\gamma \cdot (t \cdot s) \leq r \}$.

($\leq$) If $t_\beta \cdot t \leq r \vee\!/. \, s$, then from the monotony of the product $(t_\beta \cdot t) \cdot s \leq (r \vee\!/. \, s) \cdot s$. By definition of the left quotient $(r \vee\!/. \, s) \cdot s \leq r$, so $t_\beta \cdot (t \cdot s) = (t_\beta \cdot t) \cdot s \leq r$. This shows that each t_β is one of the preradicals r'_γ. Therefore $\vee \{ t_\beta \in \mathbb{PR} \mid t_\beta \cdot t \leq r \vee\!/. \, s \} \leq \leq \vee \{ r'_\gamma \in \mathbb{PR} \mid r'_\gamma \cdot (t \cdot s) \leq r \}$, i.e $(r \vee\!/. \, s) \vee\!/. \, t \leq r \vee\!/. \, (t \cdot s)$.

($\geq$) Let $r'_\gamma \cdot (t \cdot s) \leq r$. Then from the associativity of the product $(r'_\gamma \cdot t) \cdot s \leq r$, therefore any preradical of the form $(r'_\gamma \cdot t)$ is one of the preradicals r_α. This implies the following relation $(r'_\gamma \cdot t) \leq \vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s \leq r \} = r \vee\!/. \, s$, which shows that each preradical r'_γ is one of the preradicals t_β. Therefore $\vee \{ r'_\gamma \in \mathbb{PR} \mid r'_\gamma \cdot (t \cdot s) \leq r \} \leq \leq \vee \{ t_\beta \in \mathbb{PR} \mid t_\beta \cdot s \leq r \vee\!/. \, s \}$, i.e. $r \vee\!/. \, (t \cdot s) \leq (r \vee\!/. \, s) \vee\!/. \, t$.

2) By the definition of left the quotient $s \geq (s \vee\!/. \, t) \cdot t$. Then $r \cdot s \geq r \cdot [(s \vee\!/. \, t) \cdot t] = [r \cdot (s \vee\!/. \, t)] \cdot t$, and from Proposition 2.5 we obtain $(r \cdot s) \vee\!/. \, t \geq r \cdot (s \vee\!/. \, t)$. □

Proposition 2.8. *For every* $r, s, t \in \mathbb{PR}$ *the following relations hold:*

1) $(r \vee\!/. \, t) \vee\!/. \, (s \vee\!/. \, t) \geq r \vee\!/. \, s$;

2) $(r \cdot t) \vee\!/. \, (s \cdot t) \geq r \vee\!/. \, s$.

Proof. 1) From Proposition 2.5 the relation of this statement is equivalent to the relation $r \vee\!/. \, t \geq (r \vee\!/. \, s) \cdot (s \vee\!/. \, t)$.

By definition of the left quotient we have $r \geq (r \vee\!/. \, s) \cdot s$ and $s \geq (s \vee\!/. \, t) \cdot t$, therefore $r \geq (r \vee\!/. \, s) \cdot s \geq (r \vee\!/. \, s) \cdot [(s \vee\!/. \, t) \cdot t] = [(r \vee\!/. \, s) \cdot (s \vee\!/. \, t)] \cdot t$. Applying Proposition 2.5 we obtain $r \vee\!/. \, t \geq (r \vee\!/. \, s) \cdot (s \vee\!/. \, t)$.

2) From Proposition 2.5 follows that the relation of this statement is equivalent to $r \cdot t \geq (r \vee\!/. \, s) \cdot (s \cdot t)$. By definition of the left quotient we have $r \geq (r \vee\!/. \, s) \cdot s$, therefore $r \cdot t \geq [(r \vee\!/. \, s) \cdot s] \cdot t = (r \vee\!/. \, s) \cdot (s \cdot t)$. □

Now we will indicate some relations between the left quotient with respect to join and the lattice operations of $\mathbb{PR}$.

Proposition 2.9. *(The distributivity of the left quotient* $r \mathbin{\vee\!/\!.} s$ *with respect to meet) Let* $s \in \mathbb{PR}$. *Then for every family of preradicals* $\{ r_\alpha | \, \alpha \in \mathfrak{A}\}$ *the following relation holds:*

$$\left(\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha\right) \mathbin{\vee\!/\!.} s = \bigwedge_{\alpha\in\mathfrak{A}} (r_\alpha \mathbin{\vee\!/\!.} s).$$

Proof. ($\geq$) By definition $r_\alpha \geq (r_\alpha \mathbin{\vee\!/\!.} s) \cdot s$, for every $\alpha \in \mathfrak{A}$. Then $\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha \geq$ $\geq \bigwedge_{\alpha\in\mathfrak{A}} [(r_\alpha \mathbin{\vee\!/\!.} s) \cdot s]$. From the distributivity of the product of preradicals relative to meet it follows that $\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha \geq \left[\bigwedge_{\alpha\in\mathfrak{A}} (r_\alpha \mathbin{\vee\!/\!.} s)\right] \cdot s$. Using Proposition 2.5 we obtain $\left(\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha\right) \mathbin{\vee\!/\!.} s \geq \bigwedge_{\alpha\in\mathfrak{A}} (r_\alpha \mathbin{\vee\!/\!.} s)$.

($\leq$) From Lemma 2.1 we have $\left(\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha\right) \mathbin{\vee\!/\!.} s = \vee \left\{ t_\beta \in \mathbb{PR} \,|\, t_\beta \cdot s \leq \bigwedge_{\alpha\in\mathfrak{A}} r_\alpha \right\}$ and $\bigwedge_{\alpha\in\mathfrak{A}} (r_\alpha \mathbin{\vee\!/\!.} s) = \bigwedge_{\alpha\in\mathfrak{A}} \left(\bigvee_{r'_\gamma \cdot s \leq r_\alpha} r'_\gamma \right)$. Since $t_\beta \cdot s \leq \bigwedge_{\alpha\in\mathfrak{A}} r_\alpha \leq r_\alpha$ for every $\alpha \in \mathfrak{A}$, we have $t_\beta \cdot s \leq r_\alpha$, so each preradical t_β is one of the preradicals r'_γ. This implies the relation $\vee \left\{ t_\beta \in \mathbb{PR} \,|\, t_\beta \cdot s \leq \bigwedge_{\alpha\in\mathfrak{A}} r_\alpha \right\} \leq \vee \{ r'_\gamma \in \mathbb{PR} \,|\, r'_\gamma \cdot s \leq r_\alpha \}$ for every $\alpha \in \mathfrak{A}$, therefore $\vee \left\{ t_\beta \in \mathbb{PR} \,|\, t_\beta \cdot s \leq \bigwedge_{\alpha\in\mathfrak{A}} r_\alpha \right\} \leq \bigwedge_{\alpha\in\mathfrak{A}} (\vee \{ r'_\gamma \in \mathbb{PR} \,|\, r'_\gamma \cdot s \leq r_\alpha \})$, which means that $\left(\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha\right) \mathbin{\vee\!/\!.} s \leq \bigwedge_{\alpha\in\mathfrak{A}} (r_\alpha \mathbin{\vee\!/\!.} s)$. □

Proposition 2.10. *In the class* $\mathbb{PR}$ *the following relations are true:*

1) $\left(\bigvee_{\alpha\in\mathfrak{A}} r_\alpha\right) \mathbin{\vee\!/\!.} s \geq \bigvee_{\alpha\in\mathfrak{A}} (r_\alpha \mathbin{\vee\!/\!.} s)$;

2) $r \mathbin{\vee\!/\!.} \left(\bigwedge_{\alpha\in\mathfrak{A}} s_\alpha\right) \geq \bigvee_{\alpha\in\mathfrak{A}} (r \mathbin{\vee\!/\!.} s_\alpha)$;

3) $r \mathbin{\vee\!/\!.} \left(\bigvee_{\alpha\in\mathfrak{A}} s_\alpha\right) \leq \bigwedge_{\alpha\in\mathfrak{A}} (r \mathbin{\vee\!/\!.} s_\alpha)$.

Proof. 1) By the definition of the left quotient we have $r_\alpha \geq (r_\alpha \mathbin{\vee\!/\!.} s) \cdot s$ for every $\alpha \in \mathfrak{A}$, therefore $\bigvee_{\alpha\in\mathfrak{A}} r_\alpha \geq \bigvee_{\alpha\in\mathfrak{A}} [(r_\alpha \mathbin{\vee\!/\!.} s) \cdot s]$. From the distributivity of the product of preradicals relative to join it follows that $\bigvee_{\alpha\in\mathfrak{A}} r_\alpha \geq \left[\bigvee_{\alpha\in\mathfrak{A}} (r_\alpha \mathbin{\vee\!/\!.} s)\right] \cdot s$ and using Proposition 2.5 we obtain $\left(\bigvee_{\alpha\in\mathfrak{A}} r_\alpha\right) \mathbin{\vee\!/\!.} s \geq \bigvee_{\alpha\in\mathfrak{A}} (r_\alpha \mathbin{\vee\!/\!.} s)$.

2) For every $\alpha \in \mathfrak{A}$ we have $\bigwedge_{\alpha\in\mathfrak{A}} s_\alpha \leq s_\alpha$. From the antimonotony in the denominator it follows that $r \mathbin{\vee\!/\!.} \left(\bigwedge_{\alpha\in\mathfrak{A}} s_\alpha\right) \geq r \mathbin{\vee\!/\!.} s_\alpha$ for all $\alpha \in \mathfrak{A}$, therefore $r \mathbin{\vee\!/\!.} \left(\bigwedge_{\alpha\in\mathfrak{A}} s_\alpha\right) \geq \bigvee_{\alpha\in\mathfrak{A}} (r \mathbin{\vee\!/\!.} s_\alpha)$.

3) For every $\alpha \in \mathfrak{A}$ we have $\bigvee\limits_{\alpha \in \mathfrak{A}} s_\alpha \geq s_\alpha$. From the antimonotony in the denominator it follows that $r \overset{\vee}{/} \left(\bigvee\limits_{\alpha \in \mathfrak{A}} s_\alpha \right) \leq r \overset{\vee}{/} s_\alpha$ for all $\alpha \in \mathfrak{A}$, therefore $r \overset{\vee}{/} \left(\bigvee\limits_{\alpha \in \mathfrak{A}} s_\alpha \right) \leq \bigwedge\limits_{\alpha \in \mathfrak{A}} (r \overset{\vee}{/} s_\alpha)$. □

3 The left quotient $r \overset{\vee}{/} s$ in particular cases

In this section we will show some particular cases of left quotient with respect to join, its relations with some constructions in the "big" lattice $\mathbb{PR}$ and its connection with certain types of preradicals (prime, $\wedge$-prime, irreducible), as well as the arrangement (relative position) of preradicals obtained by left quotient.

Proposition 3.1. *For every preradicals $r, s \in \mathbb{PR}$ the following conditions are equivalent:*

1) $r \geq s$;

2) $r \overset{\vee}{/} s = 1$.

Proof. 1) $\Rightarrow$ 2) Let $r \geq s$, then $r \geq 1 \cdot s$ and from Proposition 2.5 we obtain $r \overset{\vee}{/} s \geq 1$, therefore $r \overset{\vee}{/} s = 1$.

2) $\Rightarrow$ 1) Let $r \overset{\vee}{/} s = 1$. By the definition of the left quotient we have $(r \overset{\vee}{/} s) \cdot s \leq r$, so $1 \cdot s \leq r$, i.e $s \leq r$. □

Proposition 3.2. *Let $r, s \in \mathbb{PR}$. Then:*

1) $0 \overset{\vee}{/} s = a\,(s)$ *(see Definition 1.1);*

2) $r \overset{\vee}{/} 1 = r$.

Proof. From the definition of left quotient we have:

1) $0 \overset{\vee}{/} s = \vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s \leq 0 \} = \vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s = 0 \} = a\,(s)$;

2) $r \overset{\vee}{/} 1 = \vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot 1 \leq r \} = \vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \leq r \} = r$. □

From Propositions 3.1 and 3.2 it follows the following particular cases:

(1) $0 \overset{\vee}{/} 0 = a\,(0) = 1$;	(2) $r \overset{\vee}{/} r = 1$, $\forall r \in \mathbb{PR}$;
(3) $1 \overset{\vee}{/} s = 1$, $\forall s \in \mathbb{PR}$;	(4) $1 \overset{\vee}{/} 1 = 1$.

As in Proposition 3.1 $(r \overset{\vee}{/} r) \cdot r = 1 \cdot r = r$, for every $r \in \mathbb{PR}$.

Moreover, the distributivity of product of preradicals relative to the join implies $a\,(s) \cdot s = \left(\bigvee\limits_{r_\alpha \cdot s = 0} r_\alpha \right) \cdot s = \bigvee\limits_{r_\alpha \cdot s = 0} (r_\alpha \cdot s) = 0$ for every $s \in \mathbb{PR}$.

In continuation we will discuss the question of the relations between the annihilator $a\,(r)$ and some constructions in the "big" lattice $\mathbb{PR}$ such as pseudocomplement and supplement.

Proposition 3.3. *For every preradical $s \in \mathbb{PR}$ we have $a\,(s) \geq s^\perp$.*

Proof. By the definition of the annihilator $a\,(s) = \vee\,\{\,r_\alpha \mid r_\alpha \cdot s = 0\}$. The pseudo-complement $s^\perp$ of the preradical s, by the definition, has the property $s^\perp \wedge s = 0$. Since $s^\perp \cdot s \leq s^\perp \wedge s = 0$, we obtain $s^\perp \cdot s = 0$. So $s^\perp$ is one of the preradicals r_α, therefore $s^\perp \leq \vee\,\{\,r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s = 0\}$, i.e. $s^\perp \leq a\,(s)$. $\square$

Moreover, from Proposition 2.3 we have $r \vee\!\!/.\, s \geq 0 \vee\!\!/.\, s = a\,(s)$, therefore $r \vee\!\!/.\, s \geq s^\perp$.

Proposition 3.4. *Let* $s \in \mathbb{PR}$ *and* s *has the supplement* s^*. *Then* $a\,(s) \leq s^*$.

Proof. By definition $a\,(s) = \vee\,\{\,r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s = 0\}$. The supplement s^* of s from the definition has the property $s^* \vee s = 1$. Since $s \,\#\, s^* \geq s \vee s^* = 1$, we obtain $s \,\#\, s^* = 1$. We have that $a\,(s) \cdot s = 0$, so $s^* = 0 \,\#\, s^* = (a\,(s) \cdot s) \,\#\, s^*$. From Lemma 1.4 $(a\,(s) \cdot s) \,\#\, s^* \geq (a\,(s) \,\#\, s^*) \cdot (s \,\#\, s^*) = (a\,(s) \,\#\, s^*) \cdot 1 = a\,(s) \,\#\, s^*$, therefore $s^* \geq a\,(s) \,\#\, s^*$. But $a\,(s) \,\#\, s^* \geq a\,(s)$ and so $s^* \geq a\,(s)$. $\square$

Furtheremore, we have $s^* \geq a\,(s) \,\#\, s^*$ and $a\,(s) \,\#\, s^* \geq s^*$, so $s^* = a\,(s) \,\#\, s^*$.

In the next two statements it is shown when the cancellation properties hold (see Proposition 2.6).

Proposition 3.5. *Let* $r, s \in \mathbb{PR}$. *The following conditions are equivalent:*

1) $r = (r \cdot s) \vee\!\!/.\, s$.
2) $r = t \vee\!\!/.\, s$ *for some preradical* $t \in \mathbb{PR}$.

Proof. 1) $\Rightarrow$ 2) Let $r = (r \cdot s) \vee\!\!/.\, s$. Then $r = t \vee\!\!/.\, s$ with $t = r \cdot s$.

2) $\Rightarrow$ 1) Let $r = t \vee\!\!/.\, s$ for some preradical t. By the definition of the left quotient we have $(t \vee\!\!/.\, s) \cdot s \leq t$. From Proposition 2.3 we obtain $[(t \vee\!\!/.\, s) \cdot s] \vee\!\!/.\, s \leq t \vee\!\!/.\, s$. But from Proposition 2.6 $[(t \vee\!\!/.\, s) \cdot s] \vee\!\!/.\, s \geq t \vee\!\!/.\, s$, therefore $[(t \vee\!\!/.\, s) \cdot s] \vee\!\!/.\, s = t \vee\!\!/.\, s$. Since $t \vee\!\!/.\, s = r$, we have $(r \cdot s) \vee\!\!/.\, s = r$. $\square$

Proposition 3.6. *Let* $r, s \in \mathbb{PR}$. *The following conditions are equivalent:*

1) $r = (r \vee\!\!/.\, s) \cdot s$.
2) $r = t \cdot s$ *for some preradical* $t \in \mathbb{PR}$.

Proof. 1) $\Rightarrow$ 2) Let $r = (r \vee\!\!/.\, s) \cdot s$. Then $r = t \cdot s$ with $t = r \vee\!\!/.\, s$.

2) $\Rightarrow$ 1) Let $r = t \cdot s$ for some preradical t. By Proposition 2.6 we have $(t \cdot s) \vee\!\!/.\, s \geq t$. From the monotony of the product it follows that $[(t \cdot s) \vee\!\!/.\, s] \cdot s \geq$ $\geq t \cdot s$. But from the definition of the left quotient $[(t \cdot s) \vee\!\!/.\, s] \cdot s \leq t \cdot s$, therefore $[(t \cdot s) \vee\!\!/.\, s] \cdot s = t \cdot s$. Since $t \cdot s = r$, we have $(r \vee\!\!/.\, s) \cdot s = r$. $\square$

Now we will show the behaviour of the left quotient $r \vee\!\!/.\, s$ in the cases of some types of preradicals (prime, $\wedge$-prime, irreducible).

Proposition 3.7. *The preradical* r *is prime if and only if for every preradical* s *we have* $r \vee\!\!/.\, s = 1$ *or* $r \vee\!\!/.\, s = r$.

Proof. ($\Rightarrow$) Let $r \neq 1$. By definition $(r \mathbin{\vee\!\!/\!.} s) \cdot s \leq r$ and if r is prime, then we have $r \mathbin{\vee\!\!/\!.} s \leq r$ or $s \leq r$. If $r \mathbin{\vee\!\!/\!.} s \leq r$, then by Lemma 2.2 $r \mathbin{\vee\!\!/\!.} s \geq r$, therefore $r \mathbin{\vee\!\!/\!.} s = r$. If $s \leq r$, then from Proposition 3.1 we have $r \mathbin{\vee\!\!/\!.} s = 1$.

($\Leftarrow$) Let $t_1 \cdot t_2 \leq r$ for some preradicals $t_1, t_2 \in \mathbb{PR}$. From Proposition 2.5 we obtain $t_1 \leq r \mathbin{\vee\!\!/\!.} t_2$. For the preradical t_2 from the condition of this proposition we have $r \mathbin{\vee\!\!/\!.} t_2 = 1$ or $r \mathbin{\vee\!\!/\!.} t_2 = r$. If $r \mathbin{\vee\!\!/\!.} t_2 = 1$, then from Proposition 3.1 it follows that $t_2 \leq r$. If $r \mathbin{\vee\!\!/\!.} t_2 = r$, then $t_1 \leq r \mathbin{\vee\!\!/\!.} t_2 = r$. So for every $t_1, t_2 \in \mathbb{PR}$ with $t_1 \cdot t_2 \leq r$ we have $t_1 \leq r$ or $t_2 \leq r$, which means that the preradical r is prime. □

Proposition 3.8. *If the preradical r is $\wedge$-prime, then the quotient $r \mathbin{\vee\!\!/\!.} s$ is $\wedge$-prime for every $s \in \mathbb{PR}$.*

Proof. Suppose that $t_1 \wedge t_2 \leq r \mathbin{\vee\!\!/\!.} s$. Then from Proposition 2.5 we obtain $(t_1 \wedge t_2) \cdot s \leq r$. From the distributivity of the product of preradicals relative to meet we have $(t_1 \cdot s) \wedge (t_2 \cdot s) \leq r$. If r is $\wedge$-prime, then $t_1 \cdot s \leq r$ or $t_2 \cdot s \leq r$. From Proposition 2.5 we obtain that $t_1 \leq r \mathbin{\vee\!\!/\!.} s$ or $t_2 \leq r \mathbin{\vee\!\!/\!.} s$. So for every preradicals $t_1, t_2 \in \mathbb{PR}$ with $t_1 \wedge t_2 \leq r \mathbin{\vee\!\!/\!.} s$ we have $t_1 \leq r \mathbin{\vee\!\!/\!.} s$ or $t_2 \leq r \mathbin{\vee\!\!/\!.} s$, which means that the preradical $r \mathbin{\vee\!\!/\!.} s$ is $\wedge$-prime. □

Proposition 3.9. *Let $r, s \in \mathbb{PR}$ and $r = t \cdot s$ for some preradical $t \in \mathbb{PR}$. If the preradical r is irreducible, then the preradical $r \mathbin{\vee\!\!/\!.} s$ is irreducible.*

Proof. Let for some preradicals $t_1, t_2 \in \mathbb{PR}$ we have $t_1 \wedge t_2 = r \mathbin{\vee\!\!/\!.} s$. If $r = t \cdot s$ for some preradical t, then by Proposition 3.5 $r = (r \mathbin{\vee\!\!/\!.} s) \cdot s$, so $r = (t_1 \wedge t_2) \cdot s$. From the distributivity of the product of preradicals with respect to meet we obtain $r = (t_1 \cdot s) \wedge (t_2 \cdot s)$. If r is irreducible, then $t_1 \cdot s = r$ or $t_2 \cdot s = r$.

If $t_1 \cdot s = r$, then from Proposition 2.5 we have $t_1 \leq r \mathbin{\vee\!\!/\!.} s$. But $t_1 \geq r \mathbin{\vee\!\!/\!.} s$, because $t_1 \wedge t_2 = r \mathbin{\vee\!\!/\!.} s$, therefore $t_1 = r \mathbin{\vee\!\!/\!.} s$.

If $t_2 \cdot s = r$, then similarly we obtain $t_2 = r \mathbin{\vee\!\!/\!.} s$.

So for every preradicals $t_1, t_2 \in \mathbb{PR}$ with $t_1 \wedge t_2 = r \mathbin{\vee\!\!/\!.} s$ we have $t_1 = r \mathbin{\vee\!\!/\!.} s$ or $t_2 = r \mathbin{\vee\!\!/\!.} s$, which means that the preradical $r \mathbin{\vee\!\!/\!.} s$ is irreducible. □

The operation of the left quotient $r \mathbin{\vee\!\!/\!.} s$ implies the following arrangement of associated preradicals.

Proposition 3.10. *For every $r, s, t \in \mathbb{PR}$ the following relations are true:*

1) $r \mathbin{\vee\!\!/\!.} s = (r \wedge s) \mathbin{\vee\!\!/\!.} s$;

2) $(r \mathbin{\vee\!\!/\!.} s) \cdot s \leq r \wedge s$.

Proof. 1) From Proposition 2.9 we have $(r \wedge s) \mathbin{\vee\!\!/\!.} s = (r \mathbin{\vee\!\!/\!.} s) \wedge (s \mathbin{\vee\!\!/\!.} s)$, but $s \mathbin{\vee\!\!/\!.} s = 1$, so $(r \wedge s) \mathbin{\vee\!\!/\!.} s = (r \mathbin{\vee\!\!/\!.} s) \wedge 1 = r \mathbin{\vee\!\!/\!.} s$.

Moreover, since $r \cdot s \leq r \wedge s$, from Proposition 2.3 we obtain

$$(r \cdot s) \mathbin{\vee\!\!/\!.} s \leq (r \wedge s) \mathbin{\vee\!\!/\!.} s = r \mathbin{\vee\!\!/\!.} s.$$

2) By 1) we have $r \mathbin{\vee\!\!/\!.} s = (r \wedge s) \mathbin{\vee\!\!/\!.} s$ and from the monotony of the product of preradicals we obtain $(r \mathbin{\vee\!\!/\!.} s) \cdot s = ((r \wedge s) \mathbin{\vee\!\!/\!.} s) \cdot s$. From the definition of the left quotient we have $((r \wedge s) \mathbin{\vee\!\!/\!.} s) \cdot s \leq r \wedge s$, therefore $(r \mathbin{\vee\!\!/\!.} s) \cdot s \leq r \wedge s$. □

Corollary 3.11. *1) For every preradicals* $r, s \in \mathbb{PR}$ *the following relations hold:*

$$r \cdot s \leq (r \stackrel{\vee}{/}. s) \cdot s \leq r \wedge s \leq r \leq (r \cdot s) \stackrel{\vee}{/}. s \leq r \stackrel{\vee}{/}. s;$$

2) If r *is a pretorsion, then*

$$r \cdot s = (r \stackrel{\vee}{/}. s) \cdot s = r \wedge s \leq r \leq (r \cdot s) \stackrel{\vee}{/}. s = r \stackrel{\vee}{/}. s$$

for every $s \in \mathbb{PR}$. □

In conclusion we can say that in the class $\mathbb{PR}$ of the category R-Mod there is defined a new operation - left quotient with respect to join, which possesses a series of properties connected with the four operations of the class $\mathbb{PR}$. This new operation is concordant with a series of notions from the theory of radicals.

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Some Homomorphic Properties of Multigroups

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Abstract. Multigroup is an algebraic structure of multiset that generalized crisp group theory. In this paper, we study the concept of homomorphism and its properties in multigroups context. Some related results are established.

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1 Introduction

The idea of multigroup was proposed in [5] as an algebraic structure of multiset that generalized the concept of group. The notion is consistent with other non-classical groups in [4]. Although other researchers in [2, 3, 6, 7, 10, 11] earlier used the term multigroup as an extension of group theory (with each of them having a divergent view), the notion of multigroup in [5] is quite acceptable because it is in consonant with other non-classical groups and defined over multiset (see [9] for multisets details).

Some new results on multigroups following [5] were presented in [1]. In this paper, we study the notion of homomorphism in multigroups context, present some of its properties and obtain some results.

2 Preliminaries

Definition 1 (see [8])**.** Let $X = \{x_1, x_2, ..., x_n, ...\}$ be a set. A multiset A over X is a cardinal-valued function, that is, $C_A : X \to \mathbb{N}$ such that $x \in Dom(A)$ implies $A(x)$ is a cardinal and $A(x) = C_A(x) > 0$, where $C_A(x)$, denotes the number of times an object x occur in A. Whenever $C_A(x) = 0$, implies $x \notin Dom(A)$. The set X is called the ground or generic set of the class of all multisets (for short, msets) containing objects from X.

A multiset $A = [a, a, b, b, c, c, c]$ can be represented as $A = [a, b, c]_{2,2,3}$. Different forms of representing multiset exist other than this. See [8, 9, 12] for details.

We denote the set of all multisets by $MS(X)$.

Definition 2 (see [9])**.** Let A and B be two multisets over X, A is called a submultiset of B written as $A \subseteq B$ if $C_A(x) \leq C_B(x)$ $\forall x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper submultiset of B and denoted as $A \subset B$. A multiset is called the parent in relation to its submultiset.

Definition 3. Two multisets A and B over X are comparable to each other if $A \subseteq B$ or $B \subseteq A$.

Definition 4 (see [12])**.** Let A and B be two multisets over X. Then the intersection and union of A and B, denoted by $A \cap B$ and $A \cup B$ respectively, are defined by the rules that for any object $x \in X$,

(i) $C_{A\cap B}(x) = C_A(x) \wedge C_B(x)$,

(ii) $C_{A\cup B}(x) = C_A(x) \vee C_B(x)$,

where $\wedge$ and $\vee$ denote minimum and maximum.

Definition 5. Let $\{A_i\}_{i\in I}$ be a family of multisets over X. Then

(i) $C_{\bigcap_{i\in I} A_i}(x) = \bigwedge_{i\in I} C_{A_i}(x) \forall x \in X$,

(ii) $C_{\bigcup_{i\in I} A_i}(x) = \bigvee_{i\in I} C_{A_i}(x) \forall x \in X$.

Definition 6 (see [5])**.** Let X be a group. A multiset G is called a multigroup of X if it satisfies the following conditions:

(i) $C_G(xy) \geq C_G(x) \wedge C_G(y) \forall x, y \in X$,

(ii) $C_G(x^{-1}) \geq C_G(x) \forall x \in X$,

where C_G denotes the count function of G from X into a natural number $\mathbb{N}$.

By implication, a multiset G is called a multigroup of a group X if

$$C_G(xy^{-1}) \geq C_G(x) \wedge C_G(y), \forall x, y \in X.$$

It follows immediately from the definition that $C_G(e) \geq C_G(x)$ $\forall x \in X$, where e is the identity element of X. A multigroup G of X is complete if $G_* = X$, where $G_* = \{x \in X \mid C_A(x) > 0\}$. Also, the set G^* is defined by

$$G^* = \{x \in X \mid C_A(x) = C_A(e)\},$$

where e is the identity of X. We denote the set of all multigroups of X by $MG(X)$.

Example 1. The following are examples of multigroups.

(i) Let $Z_4 = \{0, 1, 2, 3\}$ be a group with respect to addition. Then $G = [0, 1, 2, 3]_{4,3,4,3}$ is a multigroup of Z_4.

(ii) The zeros of $f(x) = x^8 - 2x^4 + 1$ form a multigroup of a group $X = \{1, -1, i, -i\}$.

(iii) Let $X = \{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$ be a permutation group on a set $S = \{1, 2, 3\}$ such that

$$\rho_0 = (1),\ \rho_1 = (123),\ \rho_2 = (132),\ \rho_3 = (23), \rho_4 = (13),\ \rho_5 = (12).$$

Then $A = [\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5]_{7,4,4,3,3,3}$ is a multigroup of X.

Definition 7 (see [5])**.** Let $A, B \in MG(X)$. Then the product of A and B denoted as $A \circ B$, is governed by

$$C_{A\circ B}(x) = \bigvee_{x=yz} (C_A(y) \wedge C_B(z)), \forall y, z \in X.$$

Definition 8 (see [5])**.** For any multigroup $A \in MG(X)$, there exists its inverse, A^{-1} defined by

$$C_{A^{-1}}(x) = C_A(x^{-1}) \ \forall x \in X.$$

For example, let $X = \{0, 1, 2, 3\}$ be a group of $(Z_4, +)$. Let $A = [0, 1, 2, 3]_{4,3,2,3}$ be a multigroup of X, then $A^{-1} = [0, 3, 2, 1]_{4,3,2,3}$. From Definition 6 (ii), $C_A(x^{-1}) \geq C_A(x) \forall x \in X$ and also, $C_A(x) = C_A((x^{-1})^{-1}) \geq C_A(x^{-1})$. Hence, $C_A(x) = C_A(x^{-1})$. Since $C_{A^{-1}}(x) = C_A(x^{-1})$, we have $C_A(x) = C_{A^{-1}}(x)$. Therefore, $A = A^{-1}$ for every $A \in MG(X)$.

Proposition 1 (see [5])**.** *Let $A \in MS(X)$. Then $A \in MG(X)$ if and only if A satisfies the following conditions;*

(i) $A \circ A \subseteq A$,

(ii) $A^{-1} \subseteq A$ *or* $A \subseteq A^{-1}$ *or* $A^{-1} = A$,

(iii) $A \circ A^{-1} \subseteq A$.

Proposition 2 (see [5])**.** *Let $A, B \in MG(X)$, then the following hold.*

(i) $A \circ A = A$,

(ii) $A \circ B = B \circ A$,

(iii) $(A \circ B)^{-1} = B^{-1} \circ A^{-1}$,

(iv) $(A \circ B) \circ C = A \circ (B \circ C)$.

Proposition 3 (see [5])**.** *Let $A, B \in MG(X)$. Then $A \circ B \in MG(X)$ if and only if $A \circ B = B \circ A$.*

Definition 9. Let $\{A_i\}_{i\in I}, I = 1, ..., n$ be an arbitrary family of multigroups of X. Then $\{A_i\}_{i\in I} \in X$ is said to have descending or ascending chain if either $A_1 \subseteq A_2 \subseteq ... \subseteq A_n$ or $A_1 \supseteq A_2 \supseteq ... \supseteq A_n$, respectively.

3 Main Results

Throughout this section, we assume that multigroups are completely defined over the underlying groups.

Definition 10. Let X and Y be groups and let $f : X \to Y$ be a homomorphism. Let A and B be multisets over X and Y respectively. Then

(i) the image of A under f, denoted by $f(A)$, is a multigroup of Y defined by

$$C_{f(A)}(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} C_A(x), & f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

for each $y \in Y$.

(ii) the inverse image of B under f, denoted by $f^{-1}(B)$, is a multigroup of X defined by

$$C_{f^{-1}(B)}(x) = C_B(f(x)) \; \forall x \in X.$$

Definition 11. Let X and Y be groups and let $A \in MG(X)$ and $B \in MG(Y)$, respectively.

(i) A homomorphism f from X to Y is called a weak homomorphism from A to B if $f(A) \subseteq B$. If f is a weak homomorphism of A into B, then we say that A is weakly homomorphic to B denoted by $A \sim B$.

(ii) An isomorphism f from X to Y is called a weak isomorphism from A to B if $f(A) \subseteq B$. If f is a weak isomorphism of A into B, then we say that A is weakly isomorphic to B denoted by $A \simeq B$.

(iii) A homomorphism f from X to Y is called a homomorphism from A to B if $f(A) = B$. If f is a homomorphism of A onto B, then A is homomorphic to B denoted by $A \approx B$.

(iv) An isomorphism f from X to Y is called an isomorphism from A to B if $f(A) = B$. If f is an isomorphism of A onto B, then A is isomorphic to B denoted by $A \cong B$.

Definition 12. Let $f : X \to Y$ be a homomorphism. Suppose A and B are multigroups of X and Y, respectively and A is homomorphic to B. Then the kernel of f from A to B is defined by

$$kerf = \{x \in X \mid C_A(x) = C_B(e'), f(e) = e'\},$$

where e and e' are the identities of X and Y, respectively.

Proposition 4. *Let $f : X \to Y$ be a homomorphism. For $A, B \in MG(X)$, if $A \subseteq B$, then $f(A) \subseteq f(B)$.*

Proof. Straightforward. □

Proposition 5. *Let X, Y be groups and f be a homomorphism of X into Y. For $A, B \in MG(Y)$, if $A \subseteq B$, then $f^{-1}(A) \subseteq f^{-1}(B)$.*

Proof. Straightforward. □

Definition 13. Let f be a homomorphism of a group X into a group Y, and $A \in MG(X)$. If for all $x, y \in X$, $f(x) = f(y)$ implies $C_A(x) = C_A(y)$, then A is f-invariant.

Lemma 1. *Let $f : X \to Y$ be groups homomorphism and $A \in MG(X)$. If $\forall x, y \in X$, $f(x) = f(y)$, then A is f-invariant.*

Proof. Suppose $f(x) = f(y)$ $\forall x, y \in X$. Then $C_{f(A)}(f(x)) = C_{f(A)}(f(y))$ implies $C_A(x) = C_A(y)$. Hence, A is f-invariant. □

Lemma 2. *If $f : X \to Y$ is a homomorphism and $A \in MG(X)$, then*

(i) $f(A^{-1}) = (f(A))^{-1}$,

(ii) $f^{-1}(f(A^{-1})) = f((f(A))^{-1})$.

Proof. (i) Let $y \in Y$. Then we get

$$\begin{aligned} C_{f(A^{-1})}(y) &= C_{A^{-1}}(f^{-1}(y)) = C_A(f^{-1}(y)) \\ &= C_{f(A)}(y) = C_{(f(A))^{-1}}(y) \forall y \in Y. \end{aligned}$$

Hence, $f(A^{-1}) = (f(A))^{-1}$.

(ii) Similar to (i). □

Proposition 6. *Let X and Y be groups such that $f : X \to Y$ is an isomorphic mapping. If $A \in MG(X)$ and $B \in MG(Y)$, respectively, then*

(i) $(f^{-1}(B))^{-1} = f^{-1}(B^{-1})$,

(ii) $f^{-1}(f(A)) = f^{-1}(f(f^{-1}(B)))$.

Proof. Recall that, if f is an isomorphism, then $f(x) = y$ $\forall x \in X$, $\forall y \in Y$. Consequently, $f(A) = B$.
(i)

$$\begin{aligned} C_{(f^{-1}(B))^{-1}}(x) &= C_{f^{-1}(B)}(x^{-1}) = C_{f^{-1}(B)}(x) \\ &= C_B(f(x)) = C_{B^{-1}}((f(x))^{-1}) \\ &= C_{B^{-1}}(f(x)) = C_{f^{-1}(B^{-1})}(x). \end{aligned}$$

Hence, $(f^{-1}(B))^{-1} = f^{-1}(B^{-1})$.

(ii) Similar to (i). □

Proposition 7. *Let $f : X \to Y$ be a homomorphism of groups. If $\{A_i\}_{i \in I} \in MG(X)$ and $\{B_i\}_{i \in I} \in MG(Y)$, respectively, then*

(i) $f(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} f(A_i)$,

(ii) $f(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} f(A_i)$,

(iii) $f^{-1}(\bigcap_{i\in I} B_i) = \bigcap_{i\in I} f^{-1}(B_i)$,

(iv) $f^{-1}(\bigcup_{i\in I} B_i) = \bigcup_{i\in I} f^{-1}(B_i)$.

Proof. (i) Let $x \in X$ and $y \in Y$. Since f is a homomorphism, so $f(x) = y$. Then we have,

$$\begin{aligned} C_{f(\bigcup_{i\in I} A_i)}(y) &= C_{\bigcup_{i\in I} A_i}(f^{-1}(y)) \\ &= \bigvee_{i\in I} C_{A_i}(f^{-1}(y)) \\ &= \bigvee_{i\in I} C_{f(A_i)}(y) \\ &= C_{\bigcup_{i\in I} f(A_i)}(y), \forall y \in Y. \end{aligned}$$

Hence, $f(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f(A_i)$.

The proofs of (ii)-(iv) are similar to (i). □

Theorem 1. *Let X be a group and $f : X \to X$ be an automorphism. If $A \in MG(X)$, then $f(A) = A \Leftrightarrow f^{-1}(A) = A$, consequently, $f(A) = f^{-1}(A)$.*

Proof. Let $x \in X$, and suppose $f(A) = A$, we get

$$\begin{aligned} C_{f(A)}(x) &= C_A(f^{-1}(x)) = C_A(x) \\ &= C_A(f(x)) = C_{f^{-1}(A)}(x) \end{aligned}$$

implies that $f^{-1}(A) = A$.

Conversely, let $f^{-1}(A) = A$, we have

$$\begin{aligned} C_{f^{-1}(A)}(x) &= C_A(f(x)) = C_A(x) \\ &= C_A(f^{-1}(x)) = C_{f(A)}(x). \end{aligned}$$

Hence, $f(A) = A$.
Therefore, $f(A) = A \Leftrightarrow f^{-1}(A) = A$. □

Theorem 2. *Let $f : X \to Y$ be a homomorphism. If $A \in MG(X)$, then $f^{-1}(f(A)) = A$, whenever f is injective.*

Proof. Suppose f is injective, then $f(x) = y \, \forall x \in X$ and $\forall y \in Y$. Now

$$\begin{aligned} C_{f^{-1}(f(A))}(x) &= C_{f(A)}(f(x)) = C_{f(A)}(y) \\ &= C_A(f^{-1}(y)) = C_A(x). \end{aligned}$$

Hence, $f^{-1}(f(A)) = A$. □

Corollary 1. *Let $f : X \to Y$ be a homomorphism. If $B \in MG(Y)$, then $f(f^{-1}(B)) = B$, whenever f is surjective.*

Proof. Similar to Theorem 2. □

Remark. Let $f : X \to Y$ be a homomorphism, $A \in MG(X)$ and $B \in MG(Y)$, respectively. If $kerf = \{e\}$ that is, $kerf \subseteq A^*$, then $f^{-1}(f(A)) = A$ since f is one-to-one.

Proposition 8. *Let X, Y and Z be groups and $f : X \to Y, f : Y \to Z$ be homomorphisms. If $\{A_i\}_{i \in I} \in MG(X)$ and $\{B_i\}_{i \in I} \in MG(Y)$ for each $i \in I$, then*

(i) $f(A_i) \subseteq B_i \Rightarrow A_i \subseteq f^{-1}(B_i)$,

(ii) $g[f(A_i)] = [gf](A_i)$,

(iii) $f^{-1}[g^{-1}(B_i)] = [gf]^{-1}(B_i)$.

Proof. The proof of (i) is trivial.

(ii) Since f and g are homomorphisms, then $f(x) = y$ and $g(y) = z$ $\forall x \in X, \forall y \in Y$ and $\forall z \in Z$ respectively. Now

$$\begin{aligned} C_{g[f(A_i)]}(z) &= C_{f(A_i)}(g^{-1}(z)) = C_{f(A_i)}(y) \\ &= C_{A_i}(f^{-1}(y)) = C_{A_i}(x), \end{aligned}$$

and

$$\begin{aligned} C_{[gf](A_i)}(z) &= C_{g(f(A_i))}(z) = C_{f(A_i)}(g^{-1}(z)) \\ &= C_{f(A_i)}(y) = C_{A_i}(f^{-1}(y)) \\ &= C_{A_i}(x) \forall x \in X. \end{aligned}$$

Hence, $g[f(A_i)] = [gf](A_i)$.

(iii) Similar to (ii). □

Theorem 3. *Let X and Y be groups and $f : X \to Y$ be an isomorphism. Then the following statements hold.*

(i) $A \in MG(X)$ *if and only if* $f(A) \in MG(Y)$.

(ii) $B \in MG(Y)$ *if and only if* $f^{-1}(B) \in MG(X)$.

Proof. (i) Suppose $A \in MG(X)$. Let $x, y \in Y$, then $\exists f(a) = x$ and $f(b) = y$ since f is an isomorphism for all $a, b \in X$. We know that

$$C_B(x) = C_A(f^{-1}(x)) = \bigvee_{a \in f^{-1}(x)} C_A(a)$$

and

$$C_B(y) = C_A(f^{-1}(y)) = \bigvee_{b \in f^{-1}(y)} C_A(b).$$

Clearly, $a \in f^{-1}(x) \neq \emptyset$ and $b \in f^{-1}(y) \neq \emptyset$. For $a \in f^{-1}(x)$ and $b \in f^{-1}(y) \Rightarrow x = f(a)$ and $y = f(b)$. Thus $f(ab^{-1}) = f(a)f(b^{-1}) = f(a)(f(b))^{-1} = xy^{-1}$. Let $c = ab^{-1} \Rightarrow c \in f^{-1}(xy^{-1})$. Now,

$$\begin{aligned} C_B(xy^{-1}) &= \bigvee_{c \in f^{-1}(xy^{-1})} C_A(c) \\ &= C_A(ab^{-1}) \\ &\geq C_A(a) \wedge C_A(b) \\ &= C_{f^{-1}(B)}(a) \wedge C_{f^{-1}(B)}(b) \\ &= C_B(f(a)) \wedge C_B(f(b)) \\ &= C_B(x) \wedge C_B(y) \forall x, y \in Y. \end{aligned}$$

Hence, $f(A) \in MG(Y)$.

Conversely, let $a, b \in X$ and suppose $f(A) \in MG(Y)$. Then

$$\begin{aligned} C_A(ab^{-1}) &= C_{f^{-1}(B)}(ab^{-1}) \\ &= C_B(f(ab^{-1})) \\ &= C_B(f(a)f(b^{-1})) \\ &= C_B(f(a)(f(b))^{-1}) \\ &\geq C_B(f(a)) \wedge C_B(f(b)) \\ &= C_{f^{-1}(B)}(a) \wedge C_{f^{-1}(B)}(b) \\ &= C_A(a) \wedge C_A(b) \end{aligned}$$

$\forall a, b \in X$. Hence, $A \in MG(X)$.

(ii) Similar to (i). □

Corollary 2. *Let X and Y be groups and $f : X \to Y$ be an isomorphism. Then the following statements hold.*

(i) $A^{-1} \in MG(X)$ *if and only if* $f(A^{-1}) \in MG(Y)$,

(ii) $B^{-1} \in MG(Y)$ *if and only if* $f^{-1}(B^{-1}) \in MG(X)$.

Proof. By combining Definition 8 and Theorem 3, the result follows. □

Corollary 3. *Let X and Y be groups and $f : X \to Y$ be homomorphism. If $\bigcap_{i \in I} A_i \in MG(X)$ and $\bigcap_{i \in I} B_i \in MG(Y)$, then*

(i) $f(\bigcap_{i \in I} A_i) \in MG(Y)$,

(ii) $f^{-1}(\bigcap_{i \in I} B_i) \in MG(X)$.

Proof. Straightforward from Theorem 3. □

Corollary 4. *Let $f : X \to Y$ be groups homomorphism. If $\bigcup_{i\in I} A_i \in MG(X)$ and $\bigcup_{i\in I} B_i \in MG(Y)$, whenever $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$ have sup/inf assuming chain, then*

(i) $f(\bigcup_{i\in I} A_i) \in MG(Y)$,

(ii) $f^{-1}(\bigcup_{i\in I} B_i) \in MG(X)$.

Proof. Straightforward from Theorem 3. □

Theorem 4. *Let $f : X \to Y$ be an isomorphism. If $A \in MG(X)$ and $B \in MG(Y)$, then*

(i) $f(A) \circ B \in MG(Y)$ *if and only if* $f(A) \circ B = B \circ f(A)$,

(ii) $f^{-1}(B) \circ A \in MG(X)$ *if and only if* $f^{-1}(B) \circ A = A \circ f^{-1}(B)$.

Proof. (i) By Theorem 3, it follows that $f(A) \in MG(Y)$. So, $f(A), B \in MG(Y)$. Suppose $f(A) \circ B \in MG(Y)$. Then

$$\begin{aligned} C_{f(A)\circ B}(y) &= C_{(f(A))^{-1}\circ B^{-1}}(y) \\ &= C_{(B\circ f(A))^{-1}}(y) \\ &= C_{B\circ f(A)}(y) \forall y \in Y. \end{aligned}$$

Conversely, suppose $f(A) \circ B = B \circ f(A)$. Then

$$\begin{aligned} C_{(f(A)\circ B)^{-1}}(y) &= C_{(B\circ f(A))^{-1}}(y) \\ &= C_{(f(A))^{-1}\circ B^{-1}}(y) \\ &= C_{f(A)\circ B}(y) \forall y \in Y, \end{aligned}$$

and

$$\begin{aligned} C_{(f(A)\circ B)\circ(f(A)\circ B)}(y) &= C_{f(A)\circ(B\circ f(A))\circ B}(y) \\ &= C_{f(A)\circ(f(A)\circ B)\circ B}(y) \\ &= C_{(f(A)\circ f(A))\circ(B\circ B)}(y) \\ &= C_{f(A)\circ B}(x) \forall y \in Y. \end{aligned}$$

Hence, $f(A) \circ B \in MG(Y)$ by Propositions 1, 2 and 3.

(ii) Combining Propositions 1, 2 and 3, Definition 10, Theorem 3 and (i), the proof follows. □

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Stochastic Games on Markov Processes with Final Sequence of States

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Abstract. In this paper a class of stochastic games, defined on Markov processes with final sequence of states, is investigated. In these games each player, knowing the initial distribution of the states, defines his stationary strategy, represented by one proper transition matrix. The game is started by first player and, at every discrete moment of time, the stochastic system passes to the next state according to the strategy of the current player. After the last player, the first player acts on the system evolution and the game continues in this way until, for the first time, the given final sequence of states is achieved. The player who acts the last on the system evolution is considered the winner of the game. In this paper we prove that the distribution of the game duration is a homogeneous linear recurrence and we determine the initial state and the generating vector of this recurrence. Based on these results, we develop polynomial algorithms for determining the main probabilistic characteristics of the game duration and the win probabilities of players. Also, using the signomial and geometric programming approaches, the optimal cooperative strategies that minimize the expectation of the game duration are determined.

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1 Introduction and Problem Formulation

Let L be a discrete stochastic system with finite set of states V, $|V| = \omega$. At every discrete moment of time $t \in \mathbb{N}$, the state of the system is $v(t) \in V$. The system L starts its evolution from the state v with the probability $p^*(v)$, for all $v \in V$, where $\sum\limits_{v \in V} p^*(v) = 1$.

Also, the transition from one state $u \in V$ to another state $v \in V$ is performed according to the probability $p(u, v) \in [0, 1]$ such that $\sum\limits_{v \in V} p(u, v) = 1$, $\forall u \in V$. Additionally we assume that a sequence of states $X = (x_1, x_2, \ldots, x_m) \in V^m$ is given and the stochastic system stops transitions as soon as the states $x_1, x_2, \ldots, x_m$ are reached consecutively in given order. The time T when the system stops is called evolution time of the system L with given final sequence of states X.

The stochastic system L, described above, represents a Markov process with final sequence of states X. Several interpretations of these Markov processes were analyzed in 1981 by Leo J. Guibas and Andrew M. Odlyzko in [10] and by G. Zbaganu in 1992 in [9]. Various problems, related to such systems, have been studied in [1]–[6].

Also, in these papers, polynomial algorithms for determining the main probabilistic characteristics (expectation, variance, mean square deviation, n-order moments) of evolution time of the given stochastic system L were proposed.

Next, in this paper, a generalization of this problem is studied. The following game is considered. Initially, each player $\mathcal{P}_\ell$ defines his stationary strategy, represented by one transition matrix $(p^{(\ell)}(u,v))_{u,v\in V}$, $\ell = \overline{0, r-1}$. The initial distribution of states is established according to the given distribution $(p^*(v))_{v\in V}$.

The game is started by first player $\mathcal{P}_0$. At every moment of time, the stochastic system passes consecutively to the next state according to the strategy of the current player. After the last player $\mathcal{P}_{r-1}$, the first player $\mathcal{P}_0$ acts on the system evolution and the game continues in this way until the given final sequence of states X is achieved. The player $\mathcal{P}_{T mod\ r}$ who acts the last on the system evolution is considered the winner of the game.

Our goal is to study the duration T of this game, knowing the initial distribution of states $p^{*(\ell)} = (p^{*(\ell)}(v))_{v\in V}$, the stationary strategy $P^{(\ell)} = (p^{(\ell)}(u,v))_{u,v\in V}$ of each player $\mathcal{P}_\ell$, $\ell = \overline{0, r-1}$, and the final sequence of states X of the stochastic system L. We will prove that the distribution of the game duration T is a homogeneous linear recurrence ([2], [7]) and we will develop a polynomial algorithm to determine the initial state and the generating vector of this recurrence. Having the generating vector and the initial state of the recurrence, we can use the related algorithm from [2], which was mentioned above, for determining the main probabilistic characteristics of the game duration. Also, based on these results, we will show how to determine the win probabilities of players.

2 Scientific Prerequisites

The developed algorithms for probabilistic characterization of the game duration and for determining the win probabilities of players are based on the theory of homogeneous linear recurrences.

2.1 Main Properties of Homogeneous Linear Recurrences

In this section we remind several properties of these recurrences, proved and described in [1], [2] and [6], that will be helpful in the following analysis from this paper.

The sequence $a = \{a_n\}_{n=0}^{\infty}$ represents a homogeneous linear m-recurrence on the set K if $\exists q = (q_k)_{k=0}^{m-1} \in K^m$ such that $a_n = \sum\limits_{k=0}^{m-1} q_k a_{n-1-k}$, $\forall n \geq m$, where q is the generating vector and $I_m^{[a]} = (a_n)_{n=0}^{m-1}$ is the initial state of the sequence a. The recurrence a is called non-degenerate when $|q_{m-1}| \neq 0$ and degenerate otherwise. Also, a is a homogeneous linear recurrence on the set K if $\exists m \in \mathbb{N}^*$ such that a is a homogeneous linear m-recurrence on the set K.

We denote by $Rol[K]$ (respectively $Rol[K][m]$) the set of non-degenerate homogeneous linear (m-)recurrences on the set K. The set $G[K](a)$ (respectively

$G[K][m](a)$) represents the set of generating vectors (of length m) of the sequence $a \in Rol[K]$ (respectively $a \in Rol[K][m]$).

The function $G^{[a]}(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ represents the generating function of the sequence $a = (a_n)_{n=0}^{\infty}$ and the function $G_t^{[a]}(z) = \sum\limits_{n=0}^{t-1} a_n z^n$ represents the partial generating function of order t of the sequence a. We consider the unit characteristic polynomial $H_m^{[q]}(z) = 1 - zG_m^{[q]}(z)$. For an arbitrary non-zero α, the polynomial $H_{m,\alpha}^{[q]}(z) = \alpha H_m^{[q]}(z)$ represents a characteristic polynomial of the sequence a of order m. We denote by $H[K](a)$ (respectively $H[K][m](a)$) the set of characteristic polynomials (of order m) of the sequence $a \in Rol[K]$ (respectively $a \in Rol[K][m]$).

In the case when we will operate with arbitrary recurrence (not obligatory non-degenerate) for the corresponding set we will use the similar notation and will specify it with the mark "*", i.e. we will denote respectively sets by $Rol^*[K][m]$, $Rol^*[K]$, $G^*[K][m](a)$, $G^*[K](a)$, $H^*[K][m](a)$ and $H^*[K](a)$.

The sequence $a \in Rol^*[K]$ is called m-minimal on the set K if $a \in Rol^*[K][m]$ and $a \notin Rol^*[K][t]$, for all $t < m$. The number m is called the dimension of sequence a on the set K (denoted $dim[K](a) = m$).

Next, we will consider a subfield K of the field of complex numbers $\mathbb{C}$ and $a = \{a_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$. The following Theorem allows us to determine the generating function $G^{[a]}(z)$ of an arbitrary homogeneous linear recurrence a on the set $\mathbb{C}$.

Theorem 1. *If $a \in Rol^*[\mathbb{C}][m]$ and $q \in G^*[\mathbb{C}][m](a)$, then*

$$G^{[a]}(z) = \frac{G_m^{[a]}(z) - \sum\limits_{k=0}^{m-1} q_k z^{k+1} G_{m-1-k}^{[a]}(z)}{H_m^{[q]}(z)}.$$

Also the inverse theorem is true:

Theorem 2. *If $G^{[a]}(z) = \dfrac{A(z)}{B(z)}$ is a rational fraction, $B(z) = 1 - z\sum\limits_{k=0}^{m-1} q_k z^k$ and $q_k \in K$, $k = \overline{0, m-1}$, then $a \in Rol^*[K][t+1]$ and $B(z) \in H^*[K][t+1](a)$, where $t = deg(A(z))$.*

The function $L.C.M.$ means the least common multiple of respective polynomials. An algebraic property of a linear combination is:

Theorem 3. *Let $a^{(j)} \in Rol[K]$, $P_j(z) \in H[K](a^{(j)})$, $\alpha_j \in \mathbb{C}$, $j = \overline{1, t}$. Then $a = \sum\limits_{k=1}^{t} \alpha_k a^{(k)} \in Rol[K]$ and $P(z) = L.C.M.(P_1(z), P_2(z), \ldots, P_t(z)) \in H[K](a)$.*

A homogeneous linear recurrence property of polynomials is:

Theorem 4. *For each polynomial $P(X) \in \mathbb{C}[X]$ of degree $deg(P(X)) = m$, $c = (P(n))_{n=0}^{\infty} \in Rol[\mathbb{R}][m+1]$ and $Q(z) = (1-z)^{m+1} \in H[\mathbb{R}][m+1](c)$.*

The following Theorem shows us that the product of a homogeneous linear m-recurrence and a geometric progression is also a homogeneous linear m-recurrence:

Theorem 5. *We consider $a \in Rol[K][m]$, $b \in Rol[K][1]$, $(q_0) \in G[K][1](b)$ and $P(z) \in H[K][m](a)$. Then $ab = (a_n b_n)_{n=0}^{\infty} \in Rol[K][m]$ and $P(q_0 z) \in H[K][m](ab)$.*

The direct formula for homogeneous linear recurrences is given by the following theorem:

Theorem 6. *Let $a \in Rol[K][m]$, $q \in G[K][m](a)$, $H_{m,\alpha}^{[q]}(z) = \prod_{k=0}^{p-1}(z - z_k)^{s_k}$, $z_i \neq z_j$, $\forall i \neq j$. Then $a_n = I_m^{[a]} \cdot ((B^{[a]})^T)^{-1} \cdot (\beta_n^{[a]})^T$, $\forall n \in \mathbb{N}$, where $\beta_i^{[a]} = \left(\dfrac{\tau_{ij}}{z_k^i}\right)_{k=\overline{0,p-1},\ j=\overline{0,s_k-1}}$, $\tau_{ij} = \begin{cases} i^j, & if\ i^2 + j^2 \neq 0 \\ 1, & if\ i = j = 0 \end{cases}$, $i \in \mathbb{N}$, $B^{[a]} = (\beta_i^{[a]})_{i=0}^{m-1}$.*

The dimension and the unique minimal generating vector of the sequence $a \in Rol^*[\mathbb{C}][m]$ can be determined by using the following minimization method:

Theorem 7. *If $a \in Rol^*[\mathbb{C}][m]$ is a sequence with at least one non-zero element, then $dim[\mathbb{C}](a) = R$ and $q = (q_0, q_1, \ldots, q_{R-1}) \in G^*[\mathbb{C}][R](a)$, where*

$$R = rank(A_m^{[a]}),\ A_n^{[a]} = (a_{i+j})_{i,j=\overline{0,n-1}},\ f_n^{[a]} = (a_k)_{k=\overline{n,2n-1}}, \forall n \geq 1$$

and the vector $x = (q_{R-1}, q_{R-2}, \ldots, q_0)$ represents the unique solution of the system $A_R^{[a]} x^T = (f_R^{[a]})^T$.

2.2 Subsequences of Homogeneous Linear Recurrences

Next, we will extend these properties with the following new results related to homogeneous linear recurrences. These results will be very important in the process of probabilistic characterization of the game duration and determination of the win probabilities of players.

The following two theorems analyze subsequences of degenerate and non-degenerate homogeneous linear recurrences.

Theorem 8. *If $a \in Rol[\mathbb{C}][m]$, then $b = (a_{cn+t})_{n=0}^{\infty} \in Rol[\mathbb{C}][m]$, $\forall c, t \in \mathbb{N}$, with a generating vector that does not depend on t.*

Proof. Let $a \in Rol[\mathbb{C}][m]$ with generating vector $u \in G[\mathbb{C}][m](a)$. We consider all distinct roots $z_0, z_1, \ldots, z_{p-1}$ (of corresponding multiplicity $s_0, s_1, \ldots, s_{p-1}$) of the characteristic polynomial $H_m^{[u]}(z)$. Let $b = (a_{cn+t})_{n=0}^{\infty}$, where c and t are two fixed nonnegative integers.

We consider the decomposition $x^{[a]} = I_m^{[a]}((B^{[a]})^T)^{-1} = (A_{k,j})_{k=\overline{0,p-1},\ j=\overline{0,s_k-1}}$. Using Theorem 6, we have

$$a_n = x^{[a]}(\beta_n^{[a]})^T = \sum_{k=0}^{p-1}\sum_{j=0}^{s_k-1} A_{k,j}\frac{n^j}{z_k^n},\ n = \overline{0,\infty},$$

that implies

$$b_n = a_{cn+t} = \sum_{k=0}^{p-1} \sum_{j=0}^{s_k-1} A_{k,j} \frac{(cn+t)^j}{z_k^{cn+t}} = \sum_{k=0}^{p-1} \sum_{j=0}^{s_k-1} \alpha_{kjt} h_{kjtc}(n),$$

where $\alpha_{kjt} = \dfrac{A_{k,j}}{z_k^t}$ and $h_{kjtc}(n) = \dfrac{(cn+t)^j}{(z_k^c)^n}$, $k = \overline{0, p-1}$, $j = \overline{0, s_k - 1}$, $n \in \mathbb{N}$.

Since $h_{jtc} = ((cn+t)^j)_{n=0}^{\infty}$ is a sequence of polynomials of degree j, applying Theorem 4, we have $h_{jtc} \in Rol[\mathbb{C}][j+1]$ with characteristic polynomial $(1-z)^{j+1} \in H[\mathbb{C}](h_{jtc})$, $j = \overline{0, s_k - 1}$. Also, because $g_{kc} = \left(\dfrac{1}{(z_k^c)^n}\right)_{n=\overline{0,\infty}}$ is a geometric progression with common ratio $\dfrac{1}{z_k^c}$, we have $g_{kc} \in Rol[\mathbb{C}][1]$ with generating vector $\left(\dfrac{1}{z_k^c}\right) \in G[\mathbb{C}](g_{kc})$, $k = \overline{0, p-1}$. From these relations, applying Theorem 5, we obtain $h_{kjtc} = (h_{kjtc}(n))_{n=\overline{0,\infty}} = h_{jtc} \cdot g_{kc} \in Rol[\mathbb{C}][j+1]$ with characteristic polynomial $\left(1 - \dfrac{z}{z_k^c}\right)^{j+1} \in H[\mathbb{C}](h_{kjtc})$, $k = \overline{0, p-1}$, $j = \overline{0, s_k - 1}$.

Next, using Theorem 3, for $k = \overline{0, p-1}$ we have

$$f_{ktc} = \sum_{j=0}^{s_k-1} \alpha_{kjt} h_{kjtc} \in Rol[\mathbb{C}][s_k]$$

with characteristic polynomial

$$L.C.M.\left(\left\{\left(1 - \frac{z}{z_k^c}\right)^{j+1} \mid j = \overline{0, s_k - 1}\right\}\right) = \left(1 - \frac{z}{z_k^c}\right)^{s_k} \in H[\mathbb{C}](f_{ktc}).$$

Since $b_n = \sum\limits_{k=0}^{p-1} f_{ktc}(n)$, where $f_{ktc} = (f_{ktc}(n))_{n=0}^{\infty}$, applying Theorem 3, we obtain $b \in Rol[\mathbb{C}][m]$ with characteristic polynomial

$$L.C.M.\left(\left\{\left(1 - \frac{z}{z_k^c}\right)^{s_k} \mid k = \overline{0, p-1}\right\}\right) = \prod_{k=0}^{p-1} \left(1 - \frac{z}{z_k^c}\right)^{s_k} \in H[\mathbb{C}][m](b).$$

It is easy to observe that this characteristic polynomial does not depend on t. So, also the corresponding generating vector does not depend on t. In conclusion, $\forall c, t \in \mathbb{N}$, we have $b = (a_{cn+t})_{n=0}^{\infty} \in Rol[\mathbb{C}][m]$ with a generating vector that does not depend on t. □

Theorem 9. *If $a \in Rol^*[\mathbb{C}][m]$, then $b = (a_{cn+t})_{n=0}^{\infty} \in Rol^*[\mathbb{C}][m]$, $\forall c, t \in \mathbb{N}$, with a generating vector that does not depend on t.*

Proof. Let $a \in Rol^*[\mathbb{C}][m]$ with generating vector $u = (u_k)_{k=0}^{m-1} \in G^*[\mathbb{C}][m](a)$. Let s be the degree of the characteristic polynomial $H_m^{[u]}(z)$.

We consider the subsequence $\alpha = (\alpha_n)_{n=0}^{\infty}$, where $\alpha_n = a_{n+m-s}$, $\forall n \geq 0$. It is easy to observe that $\alpha \in Rol[\mathbb{C}][s]$ with generating vector $u^{(s)} \in G[\mathbb{C}][s](\alpha)$, where $u^{(s)} = (u_k)_{k=0}^{s-1}$. Applying Theorem 8, we have $\beta = (\beta_n)_{n=0}^{\infty} \in Rol[\mathbb{C}][s]$, $\forall c, t \in \mathbb{N}$, with a generating vector $v^{(s)} = (v_k)_{k=0}^{s-1} \in G[\mathbb{C}][s](\beta)$ that does not depend on t, where $\beta_n = \alpha_{cn+t} = a_{cn+t+m-s}$, $\forall n \geq 0$. From this relation, we obtain that $b = (b_n)_{n=0}^{\infty} \in Rol^*[\mathbb{C}][m]$ with a generating vector $v = (v_k)_{k=0}^{m-1} \in G^*[\mathbb{C}][m](b)$ that does not depend on t, where $b_n = a_{cn+t}$, $\forall n \geq 0$, $\forall c, t \in \mathbb{N}$ and $v_k = 0$, $k = \overline{s, m-1}$. □

2.3 Five-Dimensional Homogeneous Linear Recurrences

The following theorem analyzes homogeneous linear recurrences on the set of squared matrices with squared matrices as components:

Theorem 10. *If $a \subseteq (\mathbb{C}^r)^t$ and $a \in Rol^*[\mathcal{M}_t(\mathcal{M}_r(K))][m]$, then $a \in Rol^*[K][mtr]$.*

Proof. Let $a \subseteq (\mathbb{C}^r)^t$, $a \in Rol^*[\mathcal{M}_t(\mathcal{M}_r(K))][m]$ and $q \in G^*[\mathcal{M}_t(\mathcal{M}_r(K))][m]$. We have $a_n = \sum\limits_{k=0}^{m-1} q^{(k)} a_{n-1-k}$, $\forall n \geq m$, where $a = (a_n)_{n=0}^{\infty}$ and $q = (q^{(k)})_{k=0}^{m-1}$.

We consider the set $\Lambda(n) = \{0, 1, \ldots, n-1\}$, $\forall n \in \mathbb{N}$. Let $a_n = (a_{ni})_{i \in \Lambda(t)}$, where $a_{ni} = (a_{nij})_{j \in \Lambda(r)}$, $i \in \Lambda(t)$, $\forall n \in \mathbb{N}$. We obtain the recurrence relation

$$a_{ni} = \sum_{k=0}^{m-1} \sum_{s=0}^{t-1} q_{is}^{(k)} a_{n-1-k,s}, \ i \in \Lambda(t), \ \forall n \geq m,$$

where $q^{(k)} = (q_{is}^{(k)})_{i,s \in \Lambda(t)}$, $k \in \Lambda(m)$. This formula implies the recurrence relation

$$a_{nij} = \sum_{k=0}^{m-1} \sum_{s=0}^{t-1} \sum_{\ell=0}^{r-1} q_{isj\ell}^{(k)} a_{n-1-k,s,\ell}, \ i \in \Lambda(t), \ j \in \Lambda(r),$$

where $q_{is}^{(k)} = (q_{isj\ell}^{(k)})_{j,\ell \in \Lambda(r)}$, $i, s \in \Lambda(t)$.

Let $a^{(i,j)} = (a_{nij})_{n=0}^{\infty}$ and $q_{isj\ell} = (q_{isj\ell}^{(k)})_{k=0}^{m-1}$, $i, s \in \Lambda(t)$, $j, \ell \in \Lambda(r)$. We will determine the generating function of the sequence $a^{(i,j)}$, $i \in \Lambda(t)$, $j \in \Lambda(r)$.

$$G^{[a^{(i,j)}]}(z) = \sum_{n=0}^{\infty} a_{nij} z^n = \sum_{n=m}^{\infty} z^n \sum_{k=0}^{m-1} \sum_{s=0}^{t-1} \sum_{\ell=0}^{r-1} q_{isj\ell}^{(k)} a_{n-1-k,s,\ell} + \sum_{n=0}^{m-1} a_{nij} z^n =$$

$$= G_m^{[a^{(i,j)}]}(z) + \sum_{k=0}^{m-1} \sum_{s=0}^{t-1} \sum_{\ell=0}^{r-1} q_{isj\ell}^{(k)} z^{k+1} \sum_{n=m}^{\infty} a_{n-1-k,s,\ell} z^{n-1-k} =$$

$$= G_m^{[a^{(i,j)}]}(z) + \sum_{k=0}^{m-1} z^{k+1} \sum_{s=0}^{t-1} \sum_{\ell=0}^{r-1} q_{isj\ell}^{(k)} \sum_{n=m-1-k}^{\infty} a_{ns\ell} z^n =$$

$$= G_m^{[a^{(i,j)}]}(z) + \sum_{k=0}^{m-1} z^{k+1} \sum_{s=0}^{t-1}\sum_{\ell=0}^{r-1} q_{isj\ell}^{(k)} \left(G^{[a^{(s,\ell)}]}(z) - G_{m-1-k}^{[a^{(s,\ell)}]}(z) \right) =$$

$$= \left(G_m^{[a^{(i,j)}]}(z) - \sum_{k=0}^{m-1} z^{k+1} \sum_{s=0}^{t-1}\sum_{\ell=0}^{r-1} q_{isj\ell}^{(k)} G_{m-1-k}^{[a^{(s,\ell)}]}(z) \right) +$$

$$+ z \sum_{s=0}^{t-1}\sum_{\ell=0}^{r-1} G^{[a^{(s,\ell)}]}(z) \sum_{k=0}^{m-1} q_{isj\ell}^{(k)} z^k =$$

$$= F_{ij}(z) + z \sum_{s=0}^{t-1}\sum_{\ell=0}^{r-1} G_m^{[q_{isj\ell}]} G^{[a^{(s,\ell)}]}(z),$$

where $F_{ij}(z) = G_m^{[a^{(i,j)}]}(z) - \sum\limits_{k=0}^{m-1} z^{k+1} \sum\limits_{s=0}^{t-1}\sum\limits_{\ell=0}^{r-1} q_{isj\ell}^{(k)} G_{m-1-k}^{[a^{(s,\ell)}]}(z)$.

So, for $i \in \Lambda(t)$ and $j \in \Lambda(r)$, we have

$$G^{[a^{(i,j)}]}(z) - z \sum_{s\in\Lambda(t),\ \ell\in\Lambda(r)} G_m^{[q_{isj\ell}]} G^{[a^{(s,\ell)}]}(z) = F_{ij}(z).$$

If we denote $x_{ij} = G^{[a^{(i,j)}]}(z)$, $i \in \Lambda(t)$, $j \in \Lambda(r)$, we obtain the following system of tr linear equations with tr unknown variables:

$$x_{ij}(z) - z \sum_{s\in\Lambda(t),\ \ell\in\Lambda(r)} G_m^{[q_{isj\ell}]} x_{s,\ell}(z) = F_{ij}(z),\ i \in \Lambda(t),\ j \in \Lambda(r).$$

In matrix form, this system can be written as follows:

$$W(z)x(z) = F(z),$$

where

$$x(z) = (x_{ij}(z))_{(i,j)\in\Lambda(t)\times\Lambda(r)},$$

$$F(z) = (F_{ij}(z))_{(i,j)\in\Lambda(t)\times\Lambda(r)},$$

$$Q = ((q_{(i,j),(s,\ell)}^{(k)})_{(i,j),(s,\ell)\in\Lambda(t)\times\Lambda(r)})_{k=0}^{m-1},$$

$$q_{(i,j),(s,\ell)}^{(k)} = q_{isj\ell}^{(k)},\ i, s \in \Lambda_t,\ j, \ell \in \Lambda_r,$$

$$W(z) = I - zG_m^{[Q]}(z).$$

So, we have $x(z) = W^{-1}(z)F(z)$, $\forall z \in D\backslash F$, where D is the domain of convergence of $G^{[a]}(z)$ and F is the set of roots of the polynomial $|W(z)|$. From this relation, we can conclude that $x_{ij}(z)$ are rational fractions, $\forall i \in \Lambda(t)$, $\forall j \in \Lambda(r)$. Using Theorem 2, we have that $a^{(i,j)} \in Rol^*[K][mtr]$, $\forall i \in \Lambda(t)$, $\forall j \in \Lambda(r)$, which implies also $a \in Rol^*[K][mtr]$. $\square$

3 Game Duration

In this section we will determine the distribution law of the game duration T. We will prove that this distribution is a homogeneous linear recurrence.

3.1 Determining the Distribution of the Game Duration

Initially, we consider the sets $X_j = \{x_j\}$ and $\overline{X}_j = V \backslash X_j$, $j = \overline{1, m}$. Also, we consider the notations $\pi_j = p^*(x_j)$, $\pi_{ij}^{(\ell)} = p^{(\ell)}(x_i, x_j)$ and $\omega_j^{(\ell)} = \prod_{k=3}^{j} \pi_{k-1,\,k}^{(\ell\,\oplus(k-3))}$, for each $i, j = \overline{1, m}$ and $\ell = \overline{0, r-1}$, where $c \oplus d = (c + d) \mod r$, $\forall c, d \in \mathbb{Z}$.

If for each $\ell = \overline{0, r-1}$ there exists an index $j_\ell \in \{2, \ldots, m\}$ such that $\pi_{j_\ell - 1,\, j_\ell}^{(\ell \oplus (j_\ell - 2))} = 0$, then the evolution of the stochastic system is not finite, i.e. the game duration is unlimited. In other words, in this case we have $a_n = 0$, $\forall n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} a_n = 1$. In conclusion, also the n-order moments of the game duration are infinite. Next, we investigate the rest of cases, when the game duration is finite.

We consider $\forall n \in \mathbb{Z}$. Let be $S(V) = \{A \mid A \subseteq V\}$. Denote by $P_\Phi^{(\ell)}(n)$ the probability that $T = n$, $v(j) \in \Phi_j$, $j = \overline{0, t-1}$ and the player $\mathcal{P}_\ell$ acts first, supposing that the initial state of the system is known, for all $\Phi = (\Phi_j)_{j=0}^{t-1} \in (S(V))^t$, $t \in \mathbb{N}$ and $\ell = \overline{0, r-1}$. We introduce the following functions on $\mathbb{Z}$, $k = \overline{0, m}$, $\ell = \overline{0, r-1}$:

$$\begin{aligned}
\alpha_k^{(\ell)}(n) &= P_{(X_1, X_2, \ldots, X_{k-1}, \overline{X}_k)}^{(\ell)}(n),\\
\beta_k^{(\ell)}(n) &= P_{(X_1, X_2, \ldots, X_k)}^{(\ell)}(n),\\
\gamma_k^{(\ell)}(n) &= P_{(X_2, X_3, \ldots, X_k)}^{(\ell)}(n).
\end{aligned} \tag{1}$$

Also, we consider the sets

$$T_s = \{s+1\} \cup \{t \in \{2, 3, \ldots, s\} \mid x_{t-1+j} = x_j,\ j = \overline{1, s+1-t}\},\ s = \overline{1, m}.$$

The minimal elements from these sets are

$$t_s = \min_{k \in T_s} k,\ s = \overline{1, m}. \tag{2}$$

The value t_s represents the auto superposition level of the sequence $(x_1, x_2, \ldots, x_s)$, i.e. t_s is the position in the sequence $(x_1, x_2, \ldots, x_s)$ starting with which, if we overlap the same sequence, the superposed elements are equal.

Initially, we study the case $m \geq 2$. We have

$$\beta_k^{(\ell)}(n) = P_{(X_1, X_2, \ldots, X_k)}^{(\ell)}(n) = P_n^{(\ell)} - \sum_{j=1}^{k} \alpha_j^{(\ell)}(n),\ k = \overline{0, m}, \ell = \overline{0, r-1}, \tag{3}$$

where $P_n^{(\ell)} = P_{(\,)}^{(\ell)}(n)$, $\ell = \overline{0, r-1}$.

Directly from definition we obtain

$$\gamma_1^{(\ell)}(n) = P^{(\ell)}(n),\ \ell = \overline{0, r-1},\ \forall n \in \mathbb{Z}. \tag{4}$$

Let be $s \geq 2$. For $t_s \leq s$ and $\ell = \overline{0, r-1}$ we have

$$\gamma_s^{(\ell)}(n) = P^{(\ell)}_{(X_2, X_3, \dots, X_s)}(n) = \pi_{2,3}^{(\ell)} \pi_{3,4}^{(\ell\ \oplus 1)} \dots \pi_{t_s-1,\ t_s}^{(\ell\ \oplus (t_s-3))} P^{(\ell\ \oplus (t_s-2))}_{(X_{t_s}, \dots X_s)}(n - t_s + 2) =$$

$$= \omega_{t_s}^{(\ell)} P^{(\ell\ \oplus (t_s-2))}_{(X_1, \dots, X_{s+1-t_s})}(n - t_s + 2) = \omega_{t_s}^{(\ell)} \beta^{(\ell\ \oplus (t_s-2))}_{s+1-t_s}(n - t_s + 2) =$$

$$= \omega_{t_s}^{(\ell)} \left(P^{(\ell\ \oplus (t_s-2))}(n - t_s + 2) - \sum_{j=1}^{s+1-t_s} \alpha_j^{(\ell\ \oplus (t_s-2))}(n - t_s + 2) \right) \tag{5}$$

and in the case $t_s = s+1$, for $\ell = \overline{0, r-1}$, we obtain

$$\gamma_s^{(\ell)}(n) = P^{(\ell)}_{(X_2, X_3, \dots, X_s)}(n) = \omega_s^{(\ell)} \sum_{y \in V} p^{(\ell\ \oplus (s-2))}(x_s, y) P^{(\ell\ \oplus (s-1))}_{(\{y\})}(n - s + 1) =$$

$$= \sum_{y \in V} \omega_s^{(\ell)} p^{(\ell\ \oplus (s-2))}(x_s, y) P^{(\ell\ \oplus (s-1))}_{(\{y\})}(n - t_s + 2). \tag{6}$$

Next, we determine the values $\alpha_k^{(\ell)}(n)$, $k = \overline{1, m}$, $\ell = \overline{0, r-1}$. We have

$$\alpha_1^{(\ell)}(n) = P^{(\ell)}_{(\overline{X}_1)}(n) = \sum_{x \in V \setminus \{x_1\}} P^{(\ell)}_{(\{x\})}(n) =$$

$$= \sum_{x \in V \setminus \{x_1\}} \sum_{y \in V} P^{(\ell)}_{(\{x\},\ \{y\})}(n) = \sum_{x \in V \setminus \{x_1\}} \sum_{y \in V} p^{(\ell)}(x, y) P^{(\ell\ \oplus 1)}_{\{y\})}(n-1) =$$

$$= \sum_{y \in V} P^{(\ell\ \oplus 1)}_{\{y\})}(n-1) \sum_{x \in V \setminus \{x_1\}} p^{(\ell)}(x, y) = \sum_{y \in V} \psi_1^{(\ell)}(y) P^{(\ell\ \oplus 1)}_{(\{y\})}(n-1), \tag{7}$$

where

$$\psi_1^{(\ell)}(y) = \sum_{x \in V \setminus \{x_1\}} p^{(\ell)}(x, y),\ \forall y \in V. \tag{8}$$

For $k = 2$ we obtain

$$\alpha_2^{(\ell)}(n) = P^{(\ell)}_{(X_1,\ \overline{X}_2)}(n) = \sum_{y \neq x_2} P^{(\ell)}_{(X_1,\ \{y\})}(n) = \sum_{y \neq x_2} p^{(\ell)}(x_1, y) P^{(\ell\ \oplus 1)}_{(\{y\})}(n-1) \tag{9}$$

and for $k \geq 3$ we have

$$\alpha_k^{(\ell)}(n) = P^{(\ell)}_{(X_1, X_2, \dots, X_{k-1}, \overline{X}_k)}(n) = \pi_{1,2}^{(\ell)} P^{(\ell\ \oplus 1)}_{(X_2, X_3, \dots, X_{k-1}, \overline{X}_k)}(n-1) =$$

$$= \pi_{1,2}^{(\ell)} \left(P^{(\ell\ \oplus 1)}_{(X_2, X_3, \dots, X_{k-1})}(n-1) - P^{(\ell\ \oplus 1)}_{(X_2, X_3, \dots, X_k)}(n-1) \right) =$$

$$= \pi_{1,2}^{(\ell)} \left(\gamma_{k-1}^{(\ell\ \oplus 1)}(n-1) - \gamma_k^{(\ell\ \oplus 1)}(n-1) \right). \tag{10}$$

From the following equality

$$P^{(\ell)}(n) = \sum_{k=1}^{m} \alpha_k^{(\ell)}(n) = \alpha_1^{(\ell)}(n) + \alpha_2^{(\ell)}(n) + \sum_{k=3}^{m} \alpha_k^{(\ell)}(n),\ \forall n \geq m, \tag{11}$$

using the relations (3), (9) and (10), we obtain the formula

$$P_{X_1}^{(\ell)}(n) = \beta_1^{(\ell)}(n) = P^{(\ell)}(n) - \alpha_1^{(\ell)}(n) = \alpha_2^{(\ell)}(n) + \sum_{k=3}^{m} \alpha_k^{(\ell)}(n) =$$

$$= \sum_{y \neq x_2} p^{(\ell)}(x_1, y) P_{(\{y\})}^{(\ell \oplus 1)}(n-1) + \sum_{k=3}^{m} \pi_{1,2}^{(\ell)} \left(\gamma_{k-1}^{(\ell \oplus 1)}(n-1) - \gamma_k^{(\ell \oplus 1)}(n-1) \right) =$$

$$= \sum_{y \neq x_2} p^{(\ell)}(x_1, y) P_{(\{y\})}^{(\ell \oplus 1)}(n-1) + \pi_{1,2}^{(\ell)} \left(\gamma_2^{(\ell \oplus 1)}(n-1) - \gamma_m^{(\ell \oplus 1)}(n-1) \right) =$$

$$= \sum_{y \in V} p^{(\ell)}(x_1, y) P_{(\{y\})}^{(\ell \oplus 1)}(n-1) - \pi_{1,2}^{(\ell)} \gamma_m^{(\ell \oplus 1)}(n-1), \ \forall n \geq m. \tag{12}$$

For $x \neq x_1$, we have

$$P_{(\{x\})}^{(\ell)}(n) = \sum_{y \in V} p^{(\ell)}(x, y) P_{(\{y\})}^{(\ell \oplus 1)}(n-1). \tag{13}$$

According to the relations (4) – (10), using the mathematical induction, we can prove that there exist real coefficients $u_{jk\ell}^{(i)}(y)$ and $v_{jk\ell}^{(i)}(y)$, $j = \overline{1,m}$, $k = \overline{0, j-1}$, $y \in V$, $\ell = \overline{0, r-1}$, $i = \overline{0, r-1}$ such that, for all $n \in \mathbb{Z}$, the following relations hold:

$$\begin{cases} \alpha_j^{(\ell)}(n) = \sum\limits_{i=0}^{r-1} \sum\limits_{k=0}^{j-1} \sum\limits_{y \in V} u_{jk\ell}^{(i)}(y) \ P_{(\{y\})}^{(i)}(n-1-k), \\ \gamma_j^{(\ell)}(n-1) = \sum\limits_{i=0}^{r-1} \sum\limits_{k=0}^{j-1} \sum\limits_{y \in V} v_{jk\ell}^{(i)}(y) \ P_{(\{y\})}^{(i)}(n-1-k). \end{cases} \tag{14}$$

For $n < m - 1$ these relations are obvious and are true for all reals $u_{jk\ell}^{(i)}(y)$ and $v_{jk\ell}^{(i)}(y)$. We should prove these relations for $n \geq m$, using mathematical induction method on parameter j.

For $j = 1$ we have

$$\alpha_1^{(\ell)}(n) = \sum_{y \in V} \psi_1^{(\ell)}(y) P_{(\{y\})}^{(\ell \oplus 1)}(n-1) =$$

$$= \sum_{i=0}^{r-1} \sum_{k=0}^{0} \sum_{y \in V} u_{1k\ell}^{(i)}(y) P_{(\{y\})}^{(i)}(n-1-k) = \sum_{i=0}^{r-1} \sum_{y \in V} u_{10\ell}^{(i)}(y) P_{(\{y\})}^{(i)}(n-1),$$

where

$$u_{10\ell}^{(i)}(y) = \begin{cases} \psi_1^{(\ell)}(y), & if \ i = \ell \oplus 1 \\ 0, & if \ i \neq \ell \oplus 1 \end{cases} \tag{15}$$

and

$$\gamma_1^{(\ell)}(n-1) = P^{(\ell)}(n-1) = \sum_{y \in V} P_{(\{y\})}^{(\ell)}(n-1) = \sum_{i=0}^{r-1} \sum_{y \in V} v_{10\ell}^{(i)}(y) P_{(\{y\})}^{(i)}(n-1),$$

where
$$v_{10\ell}^{(i)}(y) = \begin{cases} 1, & if\ i = \ell \\ 0, & if\ i \neq \ell. \end{cases} \tag{16}$$

For $j = 2$ we obtain

$$\alpha_2^{(\ell)}(n) = \sum_{y \neq x_2} p^{(\ell)}(x_1, y) P_{(\{y\})}^{(\ell\ \oplus 1)}(n-1) = \sum_{i=0}^{r-1} \sum_{k=0}^{1} \sum_{y \in V} u_{2k\ell}^{(i)}(y) P_{(\{y\})}^{(i)}(n-1-k),$$

where
$$u_{20\ell}^{(i)}(y) = \begin{cases} 0, & if\ y = x_2 \\ 0, & if\ y \neq x_2\ and\ i \neq \ell\ \oplus 1 \\ p^{(\ell)}(x_1, y), & if\ y \neq x_2\ and\ i = \ell\ \oplus 1 \end{cases} \tag{17}$$

and
$$u_{21\ell}^{(i)}(y) = 0,\ \forall y \in V. \tag{18}$$

Also, we have

$$\gamma_2^{(\ell)}(n-1) = P_{(X_2)}^{(\ell)}(n-1) = \sum_{i=0}^{r-1} \sum_{k=0}^{1} \sum_{y \in V} v_{2k\ell}^{(i)}(y) P_{(\{y\})}^{(i)}(n-1-k),$$

where
$$v_{20\ell}^{(i)}(y) = \begin{cases} 0, & if\ y \neq x_2 \\ 0, & if\ y = x_2\ and\ i \neq \ell \\ 1, & if\ y = x_2\ and\ i = \ell \end{cases} \tag{19}$$

and
$$v_{21\ell}^{(i)}(y) = 0,\ \forall y \in V. \tag{20}$$

So, the relations are true for $\forall j \in \{1, 2\}$. Let these relations be true for $j = \overline{1, s-1}$, $s \geq 3$, $\forall n < \tau$ and $\forall y \in V$. We have

$$\alpha_s^{(\ell)}(\tau) = \pi_{1,2}^{(\ell)} \left(\gamma_{s-1}^{(\ell\ \oplus 1)}(\tau - 1) - \gamma_s^{(\ell\ \oplus 1)}(\tau - 1) \right) =$$

$$= \pi_{1,2}^{(\ell)} \left(\sum_{i=0}^{r-1} \sum_{k=0}^{s-2} \sum_{y \in V} v_{s-1,k,\ell}^{(i)}(y) P_{(\{y\})}^{(i)}(\tau - 1 - k) - \right.$$

$$\left. - \sum_{i=0}^{r-1} \sum_{k=0}^{s-1} \sum_{y \in V} v_{s,k,\ell}^{(i)}(y) P_{(\{y\})}^{(i)}(\tau - 1 - k) \right) =$$

$$= \sum_{i=0}^{r-1} \sum_{k=0}^{s-1} \sum_{y \in V} u_{sk\ell}^{(i)}(y) P_{(\{y\})}^{(i)}(\tau - 1 - k),$$

where
$$u_{sk\ell}^{(i)}(y) = \begin{cases} \pi_{1,2}^{(\ell)}(v_{s-1,k,\ell}^{(i)}(y) - v_{sk\ell}^{(i)}(y)), & if\ 0 \leq k \leq s-2 \\ -\pi_{1,2}^{(\ell)} v_{s,s-1,\ell}^{(i)}(y), & if\ k = s-1. \end{cases} \tag{21}$$

For $t_s \le s$ we obtain

$$\gamma_s^{(\ell)}(\tau-1)=\omega_{t_s}^{(\ell)}\left(\sum_{y\in V}P_{(\{y\})}^{(\ell\ \oplus(t_s-2))}(\tau-t_s+1)-\sum_{j=1}^{s+1-t_s}\alpha_j^{(\ell\ \oplus(t_s-2))}(\tau-t_s+1)\right)=$$

$$=\omega_{t_s}^{(\ell)}\left(\sum_{y\in V}P_{(\{y\})}^{(\ell\ \oplus(t_s-2))}(\tau-t_s+1)-\right.$$

$$\left.-\sum_{j=1}^{s+1-t_s}\sum_{i=0}^{r-1}\sum_{k=0}^{j-1}\sum_{y\in V}u_{j,k,\ell\ \oplus(t_s-2)}^{(i)}(y)P_{(\{y\})}^{(i)}(\tau-t_s-k)\right)=$$

$$=\omega_{t_s}^{(\ell)}\left(\sum_{y\in V}P_{(\{y\})}^{(\ell\ \oplus(t_s-2))}(\tau-1-(t_s-2))-\right.$$

$$\left.-\sum_{i=0}^{r-1}\sum_{k=t_s-1}^{s-1}\sum_{y\in V}P_{(\{y\})}^{(i)}(\tau-1-k)\sum_{j=k-t_s+2}^{s+1-t_s}u_{j,k-t_s+1,\ell\ \oplus(t_s-2)}^{(i)}(y)\right)=$$

$$=\sum_{i=0}^{r-1}\sum_{k=0}^{s-1}\sum_{y\in V}v_{sk\ell}^{(i)}(y)P_{(\{y\})}^{(i)}(\tau-1-k),$$

where

$$v_{sk\ell}^{(i)}(y)=\begin{cases}0, & if\ 0\le k\le t_s-3\\ 0, & \begin{array}{l}if\ k=t_s-2\ and\\ i\ne\ell\ \oplus(t_s-2)\end{array}\\ \omega_{t_s}^{(\ell)}, & \begin{array}{l}if\ k=t_s-2,\ and\\ i=\ell\ \oplus(t_s-2)\end{array}\\ -\omega_{t_s}^{(\ell)}\sum\limits_{j=k-t_s+2}^{s+1-t_s}u_{j,\ k-t_s+1,\ \ell\ \oplus(t_s-2)}^{(i)}(y), & if\ t_s-1\le k\le s-1\end{cases}\tag{22}$$

and for $t_s=s+1$ we have

$$\gamma_s^{(\ell)}(\tau-1)=\sum_{y\in V}\omega_s^{(\ell)}p^{(\ell\ \oplus(s-2))}(x_s,y)P_{(\{y\})}^{(\ell\ \oplus(s-1))}(\tau-1-(t_s-2))=$$

$$=\sum_{i=0}^{r-1}\sum_{k=0}^{j-1}\sum_{y\in V}v_{sk\ell}^{(i)}(y)P_{(\{y\})}^{(i)}(\tau-1-k),$$

where

$$v_{sk\ell}^{(i)}(y)=\begin{cases}0, & if\ 0\le k\le s-2\\ 0, & if\ k=s-1\ and\ i\ne\ell\ \oplus(s-1)\\ \omega_s^{(\ell)}p^{(\ell\ \oplus(s-2))}(x_s,y), & if\ k=s-1\ and\ i=\ell\ \oplus(s-1).\end{cases}\tag{23}$$

So, we proved the truth of the relations (14), obtaining the formulas (15) – (23) for determining coefficients of the decompositions. Substituting the decompositions (14) in the relations (12) and (13), we have

$$P^{(\ell)}_{(X_1)}(n)=\sum_{y\in V}p^{(\ell)}(x_1,y)P^{(\ell\,\oplus 1)}_{(\{y\})}(n-1)-\pi^{(\ell)}_{1,2}\gamma_m^{(\ell\,\oplus 1)}(n-1)=$$

$$=\sum_{y\in V}p^{(\ell)}(x_1,y)P^{(\ell\,\oplus 1)}_{(\{y\})}(n-1)-\pi^{(\ell)}_{1,2}\sum_{i=0}^{r-1}\sum_{k=0}^{m-1}\sum_{y\in V}v^{(i)}_{m,\;k,\;\ell\,\oplus 1}(y)P^{(i)}_{(\{y\})}(n-1-k)=$$

$$=\sum_{i=0}^{r-1}\sum_{k=0}^{m-1}\sum_{y\in V}w^{(i)}_{k,\ell}(x_1,y)P^{(i)}_{(\{y\})}(n-1-k)$$

and, for all $x\neq x_1$,

$$P^{(\ell)}_{(\{x\})}(n)=\sum_{y\in V}p^{(\ell)}(x,y)P^{(\ell\,\oplus 1)}_{(\{y\})}(n-1)=\sum_{i=0}^{r-1}\sum_{k=0}^{m-1}\sum_{y\in V}w^{(i)}_{k,\ell}(x,y)P^{(i)}_{(\{y\})}(n-1-k),$$

where

$$w^{(i)}_{k\ell}(x,y)=\begin{cases} p^{(\ell)}(x_1,y)-\pi^{(\ell)}_{1,2}v^{(\ell\,\oplus 1)}_{m,0,\ell\,\oplus 1}(y), & if\ x=x_1,\ k=0,\ i=\ell\,\oplus 1\\ -\pi^{(\ell)}_{1,2}v^{(i)}_{m,0,\ell\,\oplus 1}(y), & if\ x=x_1,\ k=0,\ i\neq\ell\,\oplus 1\\ -\pi^{(\ell)}_{1,2}v^{(i)}_{m,k,\ell\,\oplus 1}(y), & if\ x=x_1,\ 1\le k\le m-1\\ p^{(\ell)}(x,y), & if\ x\neq x_1,\ k=0,\ i=\ell\,\oplus 1\\ 0, & if\ x\neq x_1,\ k=0,\ i\neq\ell\,\oplus 1\\ 0, & if\ x\neq x_1,\ 1\le k\le m-1\end{cases}\qquad(24)$$

Thus, we obtained the recurrence relation

$$P^{(\ell)}_{(\{x\})}(n)=\sum_{i=0}^{r-1}\sum_{k=0}^{m-1}\sum_{y\in V}w^{(i)}_{k\ell}(x,y)P^{(i)}_{(\{y\})}(n-1-k),\ \forall x\in V,\ \forall n\ge m,\ \ell=\overline{0,r-1}.$$

So, we have

$$P_{(\{x\})}(n)=\sum_{k=0}^{m-1}\sum_{y\in V}W_k(x,y)P_{(\{y\})}(n-1-k),\ \forall x\in V,\ \forall n\ge m,$$

where $W_k(x,y)=(w^{(i)}_{k\ell}(x,y))_{\ell,\ i=\overline{0,\ r-1}}$, $P_{(\{x\})}(n)=(P^{(\ell)}_{(\{x\})}(n))_{\ell\ =\overline{0,\ r-1}}$, $\forall x,y\in V$, $k=\overline{0,m-1}$. This recurrence relation can be written in the form

$$h_n=\sum_{k=0}^{m-1}W_kh_{n-1-k},\ \forall n\ge m,$$

where $W_k = (W_k(x,y))_{x,y\in V}$ and $h_n = ((P_{(\{x\})}(n))_{x\in V})^T$, $k=\overline{1,m}$, $\forall n\in\mathbb{Z}$. From this relation, we obtain that $h=(h_n)_{n=0}^{\infty}\in Rol^*[\mathcal{M}_\omega(\mathcal{M}_r(\mathbb{R}))][m]$ with generating vector $W=(W_k)_{k=0}^{m-1}\in G^*[\mathcal{M}_\omega(\mathcal{M}_r(\mathbb{R}))][m](h)$. Using Theorem 10, we have $h\in Rol^*[\mathbb{R}][mr\omega]$, which implies that also

$$\left(P^{(\ell)}_{(\{x\})}(n)\right)_{n=0}^{\infty}\in Rol^*[\mathbb{R}][mr\omega],\ \forall x\in V,\ \ell=\overline{0,r-1},$$

with the same generating vector. Since

$$a^{(\ell)}(n)=\sum_{x\in V}p^*(x)P^{(\ell)}_{(\{x\})}(n),\ \forall n\in\mathbb{N},$$

we have $a^{(\ell)}=(a^{(\ell)}(n))_{n=0}^{\infty}\in Rol^*[\mathbb{R}][mr\omega]$, $\ell=\overline{0,r-1}$, with the same generating vector. Because the game is started by player $\mathcal{P}^{(0)}$, then the distribution a of the game duration T coincides with $a^{(0)}$, i.e. $a=(a_n)_{n=0}^{\infty}\in Rol^*[\mathbb{R}][mr\omega]$ with the same generating vector.

Next, we will use only the relation $a\in Rol^*[\mathbb{C}][mr\omega]$, the minimal generating vector being determined using the minimization method based on the matrix rank, given by Theorem 7. So, according to this method, we have that the minimal generating vector $q=(q_0,q_1,\dots,q_{R-1})\in G^*[\mathbb{C}][R](a)$ is obtained from the unique solution $x=(q_{R-1},q_{R-2},\dots,q_0)$ of the system

$$A_R^{[a]}x^T=(f_R^{[a]})^T, \tag{25}$$

where

$$f_R^{[a]}=(a_R,a_{R+1},\dots,a_{2R-1}),\ A_n^{[a]}=(a_{i+j})_{i,j=\overline{0,n-1}},\ \forall n\in\mathbb{N}^* \tag{26}$$

and R is the rank of the matrix $A^{[a]}_{mr\omega}$.

For this, we need to have only the values a_k, $k=\overline{0,2mr\omega-1}$. These values are determined from the formula

$$a_k=a_k^{(0)},\ k=\overline{0,2mr\omega-1}, \tag{27}$$

using the relations (3) – (13) and the initial conditions

$$a_n=a_n^{(\ell)}=P^{(\ell)}(n)=P^{(\ell)}_{(\{x\})}(n)=0,\ \forall x\in V,\ \ell=\overline{0,r-1},\ n=\overline{0,m-2},$$

$$\alpha_k^{(\ell)}(n)=0,\ k=\overline{1,m},\ n=\overline{0,m-1},\ \ell=\overline{0,r-1},$$

$$P^{(\ell)}(m-1)=\pi_{1,2}^{(\ell)}w_m^{(\ell+1)},\ a_{m-1}^{(\ell)}=\pi_1P^{(\ell)}(m-1),\ \ell=\overline{0,r-1},$$

$$P^{(\ell)}_{(\{x_1\})}(m-1)=P^{(\ell)}(m-1),\ \ell=\overline{0,r-1},$$

$$P^{(\ell)}_{(\{x\})}(m-1)=0,\ \forall x\in V\backslash\{x_1\},\ \ell=\overline{0,r-1}. \tag{28}$$

For the case $m = 1$ we have other formulas for determining the values of conditional probabilities $P^{(\ell)}_{(\{x\})}(n)$, $\ell = \overline{0, r-1}$, $\forall x \in V$, $\forall n \in \mathbb{N}$. It is easy to observe that these values can be obtained using the following formulas:

$$P^{(\ell)}_{(X_1)}(0) = 1,\ P^{(\ell)}_{(X_1)}(n) = 0,\ \forall n \in \mathbb{N}^*,\ \ell = \overline{0, r-1},$$

$$P^{(\ell)}_{(\{x\})}(0) = 0,\ P^{(\ell)}_{(\{x\})}(n) = \sum_{y \in V} p^{(\ell)}(x,y) P^{(\ell\ \oplus 1)}_{(\{y\})}(n-1),\ \forall n \in \mathbb{N}^*,\ \forall x \in V \backslash \{x_1\}. \quad (29)$$

3.2 Describing the developed algorithm

In the previous subsection we theoretically grounded the following algorithm for determining the main probabilistic characteristics (the distribution $(\mathbb{P}(T = n))_{n=0}^{\infty}$, the expectation $\mathbb{E}(T)$, the variance $\mathbb{V}(T)$, the mean square deviation $\sigma(T)$ and the k-order moments $\nu_k(T)$, $k = 1, 2, \ldots$) of the game duration T.

Algorithm 1.

Input: $X = (x_1, x_2, \ldots, x_m) \in V^m$, π_j, $\pi^{(\ell)}_{i,j}$, $i, j = \overline{1, m}$, $\ell = \overline{0, r-1}$;
Output: $\mathbb{E}(T),\ \mathbb{V}(T),\ \sigma(T),\ \nu_k(T),\ k = \overline{1, t},\ t \geq 2.$

1. Determine the values a_k, $k = \overline{0, 2mr\omega - 1}$, using the formula (27), the relations (3) – (13) and the initial conditions (28) – (29);

2. Find the minimal generating vector $q = (q_0, q_1, \ldots, q_{R-1}) \in G^*[\mathbb{R}][R](a)$ by solving the system (25), taking into account the relation (26);

3. Consider the distribution $a = (a_n)_{n=0}^{\infty} = (\mathbb{P}(T = n))_{n=0}^{\infty}$ of the game duration T as a homogeneous linear recurrence with the initial state $I^{[a]}_R = (a_n)_{n=0}^{R-1}$ and the minimal generating vector $q = (q_k)_{k=0}^{R-1}$, determined at the steps 1 and 2;

4. Determine the expectation $\mathbb{E}(T)$, the variance $\mathbb{V}(T)$, the mean square deviation $\sigma(T)$ and the k-order moments $\nu_k(T)$, $k = \overline{1, t}$, of the game duration T by using the corresponding algorithm from [2].

4 Win Probabilities

Another problem that is interesting for us is the determination of the win probabilities of the players. For solving this problem, we will consider the given game in more general case.

We consider a finite game Γ with r players $\mathcal{P}_\ell$, $\ell = \overline{0, r-1}$, who apply their own stochastic strategy S_ℓ, $0 \leq \ell < r$, in a given cyclic order $(S_0, S_1, \ldots, S_{r-1}, S_0, S_1, \ldots)$. Let T be the duration of the game Γ. The player $\mathcal{P}_{T mod\ r}$, who applies the last strategy, is considered the winner of the game.

Suppose that the distribution $d = (d_n)_{n=0}^{\infty} = (\mathbb{P}(T = n))_{n=0}^{\infty}$ of the game duration T is a homogeneous linear recurrence, i.e. there exist $m \in \mathbb{N}^*$ and the generating vector $q = (q_k)_{k=0}^{m-1} \in \mathbb{C}^m$, such that $d_n = \sum_{k=0}^{m-1} q_k d_{n-1-k}$, $\forall n \geq m$. We have $d \in Rol^*[\mathbb{C}][m]$ and $q \in G^*[\mathbb{C}][m](d)$. Next, we will show how to determine the win probability ω_ℓ for each player $\mathcal{P}_\ell$, $\ell = \overline{0, r-1}$.

If we consider the subsequence $d^{(\ell)} = (d_{rn+\ell})_{n=0}^{\infty}$ of the sequence d, then we have $\omega_\ell = G^{[d^{(\ell)}]}(1)$, $\ell = \overline{0, r-1}$. Using Theorem 8, we obtain $d^{(\ell)} \in Rol^*[\mathbb{C}][m]$, $\ell = \overline{0, r-1}$, with a common generating vector.

The minimal generating vector of these sequences can be determined using the minimization method based on matrix rank, given by Theorem 7, and the initial states of these sequences can be obtained using the initial state and the generating vector of the duration distribution d. Finally, the win probability ω_ℓ is obtained applying the formula, given by Theorem 1, for $z := 1$ and $a := d^{(\ell)}$, $\ell = \overline{0, r-1}$.

5 Optimal Cooperative Strategies of the Players

Next, we consider that the distributions p^* and $p^{(\ell)}$, $\ell = \overline{0, r-1}$, are not fixed. So, we have the game $\Gamma(p^*, p^{(0)}, p^{(1)}, \ldots, p^{(r-1)})$ with final sequence of states X, initial distribution of the states p^* and strategies of players $p^{(\ell)}$, $\ell = \overline{0, r-1}$, for every parameters p^* and $p^{(\ell)}$, $\ell = \overline{0, r-1}$. The problem is to determine the optimal distribution $\overline{p}^* = p^*$ and optimal strategies $\overline{p}^{(\ell)} = p^{(\ell)}$, $\ell = \overline{0, r-1}$, that minimize the expectation of the game duration $T(p^*, p^{(0)}, p^{(1)}, \ldots, p^{(r-1)})$ for the game $\Gamma(p^*, p^{(0)}, p^{(1)}, \ldots, p^{(r-1)})$.

Similar with results obtained in [3], the following theorems hold:

Theorem 11. *The optimal initial distribution of the states is* $\overline{p}^*$, *where* $\overline{p}^*(x_1) = 1$ *and* $\overline{p}^*(x) = 0$, $\forall x \in V \backslash \{x_1\}$.

Theorem 12. *We consider the set of active final states* $\overline{X} = \{x_1, x_2, \ldots, x_{m-1}\}$, *the set of final transitions* $\overline{Y} = \{(x_1, x_2), (x_2, x_3), \ldots, (x_{m-1}, x_m)\}$ *and the set of branch states* $\overline{Z} = \{y \in \overline{X} \backslash \{x_1\} \mid \exists x \in \overline{X}, \exists z \in \overline{X} \cup \{x_m\}, z \neq y : \ (x, y) \in \overline{Y}, (x, z) \in \overline{Y}\}$. *The optimal strategies* $\overline{p}^{(\ell)}$, $\ell = \overline{0, r-1}$, *have the following properties:*

1. $\overline{p}^{(\ell)}(x, x_1) = 1$, *if* $(x, x_1) \in \overline{Y}$ *and* $(x, z) \notin \overline{Y}$, $\forall z \neq x_1$;

2. $\overline{p}^{(\ell)}(x, x_1) = 1$, $\forall x \notin \overline{X}$;

3. $\overline{p}^{(\ell)}(x, x_1) > 0$, $\forall x \in \overline{Z}$ *and* $\overline{p}^{(\ell)}(x, x_1) = 0$ *if* $(x, x_1) \notin \overline{Y}$, $x \in \overline{X} \backslash \overline{Z}$;

4. $\overline{p}^{(\ell)}(x, y) = 0$, *if* $(x, y) \notin \overline{Y}$ *and* $y \neq x_1$;

5. $\overline{p}^{(\ell)}(x, y) > 0$, $\forall (x, y) \in \overline{Y}$;

6. $\sum_{(x,y) \in \overline{Y} \cup \{(x, x_1)\}} \overline{p}^{(\ell)}(x, y) = 1, \ \forall x \in \overline{X}$.

Theorem 13. *Let $p = (p^{(0)}, p^{(1)}, \ldots, p^{(r-1)})$. If $\delta_{i,j}(p) \not\equiv 0$, $i, j = \overline{1,2}$, then the optimal transition matrix can be determined solving the following geometric programs with posynomial equality constraints:*

$$\mathbb{E}(T(p^*, p)) = d_1 d_2^{-1} \to \min, \tag{30}$$

subject to

$$\begin{cases} \sum\limits_{(x,y)\in\overline{Y}\cup\{(x,x_1)\}} p^{(\ell)}(x,y) = 1, \ \forall x \in \overline{X}, \ \ell = \overline{0, r-1} \\ d_{1,1}^{-1} d_1 + d_{1,1}^{-1} d_{1,2} = 1 \\ d_{2,1}^{-1} d_2 + d_{2,1}^{-1} d_{2,2} = 1 \\ d_{1,1}^{-1} \delta_{1,1}(p) = 1 \\ d_{1,2}^{-1} \delta_{1,2}(p) = 1 \\ d_{2,1}^{-1} \delta_{2,1}(p) = 1 \\ d_{2,2}^{-1} \delta_{2,2}(p) = 1 \\ d_1, d_2, d_{1,1}, d_{1,2}, d_{2,1}, d_{2,2} > 0 \\ p^{(\ell)}(x,y) > 0, \ \forall (x,y) \in \overline{Y}, \ \ell = \overline{0, r-1} \\ p^{(\ell)}(x,x_1) > 0, \ \forall x \in \overline{Z}, \ \ell = \overline{0, r-1} \end{cases} \tag{31}$$

and (30) subject to

$$\begin{cases} \sum\limits_{(x,y)\in\overline{Y}\cup\{(x,x_1)\}} p^{(\ell)}(x,y) = 1, \ \forall x \in \overline{X}, \ \ell = \overline{0, r-1} \\ d_{1,1}^{-1} d_1 + d_{1,1}^{-1} d_{1,2} = 1 \\ d_{2,1}^{-1} d_2 + d_{2,1}^{-1} d_{2,2} = 1 \\ d_{1,1}^{-1} \delta_{1,2}(p) = 1 \\ d_{1,2}^{-1} \delta_{1,1}(p) = 1 \\ d_{2,1}^{-1} \delta_{2,2}(p) = 1 \\ d_{2,2}^{-1} \delta_{2,1}(p) = 1 \\ d_1, d_2, d_{1,1}, d_{1,2}, d_{2,1}, d_{2,2} > 0 \\ p^{(\ell)}(x,y) > 0, \ \forall (x,y) \in \overline{Y}, \ \ell = \overline{0, r-1} \\ p^{(\ell)}(x,x_1) > 0, \ \forall x \in \overline{Z}, \ \ell = \overline{0, r-1} \end{cases} \tag{32}$$

according to the properties described by Theorems 11 and 12, where $\delta_{i,j}(p)$, $i, j = \overline{1,2}$, are the posynomials from the decomposition

$$\mathbb{E}(T(p^*, p)) = (\delta_{1,1}(p) - \delta_{1,2}(p))(\delta_{2,1}(p) - \delta_{2,2}(p))^{-1} \tag{33}$$

that follows from the algorithm developed in [2]. The signomial programs $(30) - (31)$ *and* $(30) - (32)$ *can be handled as geometric programs using the way followed in [8]. If $\overline{p}^1$ is the optimal solution of the problem* $(30) - (31)$ *and $\overline{p}^2$ is the optimal solution of the problem* $(30) - (32)$*, then the optimal transition matrix is $\overline{p} \in \{\overline{p}^1, \overline{p}^2\}$ for which $\mathbb{E}(T(\overline{p^*}, \overline{p}))$ is minimal. If there exists at least one $\delta_{i^*,j^*}(p) \equiv 0$, then in* (31) *and* (32) *the corresponding posynomial equality constraints just disappear and the related substitution $d_{i^*,j^*} = 0$ is performed in* (31) *and substitution $d_{i^*,3-j^*} = 0$ is performed in* (32).

So, Theorem 13 shows us how to determine the optimal cooperative strategies of the players using signomial and geometric programming approaches. These methods were described in details in [8].

6 Conclusions

In this paper stationary games defined on Markov processes with final sequence of states were studied and the duration and win probabilities of these games were analyzed. It was proved that the game duration is a discrete random variable with homogeneous linear recurrence distribution. Based on this fact, the generating function is applied for determining the win probabilities and the main probabilistic characteristics of the game duration. Also, using the signomial and geometric programming approaches, the optimal cooperative strategies that minimize the expectation of the game duration are determined.

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Application of the Fast Automatic Differentiation for Calculation of Gradients of Material's Bulk Modulus and Shear Modulus

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Abstract. In computer modeling of crystal structures the gradient optimization methods are often used. This raises the need to calculate the exact gradients of the Bulk modulus and the Shear modulus of materials. With help of the Fast Automatic Differentiation the formulas that allow the calculation of the exact above-mentioned gradients were derived in the case where the total interatomic energy of the system is determined by Tersoff's Potential.

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1 Introduction and problem formulation

When modeling many solid atomic structures, such as carbon, silicon, germanium, and their compounds, the Tersoff's Potential is often used (see [1]). It is an example of the multiparticle potential based on concepts of link order: the interaction between two atoms depends on the local surrounding. The Tersoff Potential consists of ten parameters specific to the modeling material.

Various mathematical models are used to study materials of atomic structures. Some parameters of these models are unknown. They should be identified from the condition that the calculated properties of the modeled material are close to its properties, which were found experimentally. In [2] was considered an optimization problem of minimizing the following cost function

$$f(\xi) = \sum_{i=1}^{m} \omega_i (y_i(\xi) - \widetilde{y_i})^2 \tag{1}$$

where ω_i is the weight factor; $\widetilde{y_i}$ is the value of the i-th material characteristic obtained experimentally, and $y_i(\xi)$ is the value of the same material characteristic calculated using Tersoff Potential with ξ parameters ($\xi \in R^m$ are vector parameters to be identified). The solution of the problem is looked for on the set $X \subseteq R^m$, which is a parallelepiped. Its boundaries are chosen so that it obviously contained the admissible range of parameters. The quantity of items in formula (1) varies depending on the studied material. A required set of parameters has to provide the minimum deviation of the calculated characteristics of material from the known experimental values, thereby most precisely describing the modeled properties of a crystal. For numerical solution of this problem the gradient minimization methods are often used. There exists the need to calculate the exact value of the objective function gradient efficiently.

These derivatives are often calculated (in particular, see [2]) using the finite difference method. Studies have shown that finite difference method does not allow to calculate the gradient of the cost function with acceptable accuracy and requires $(m+1)$ times to calculate the value of the function.

One of the terms in formula (1) is the total energy of the system of atoms. As the interatomic potential energy, the Tersoff Potential was choosen. In [3], using the Fast Automatic Differentiation (see [4]), formulas to calculate the exact gradient of the total energy with respect to parameters of Tersoff Potential (specific for modeled substance) were received.

The other two terms in formula (1) are the Bulk modulus of elasticity (it relates to how the volume of a piece of material changes when exposed to a uniform change in pressure) and the Shear modulus. They are proportional to $B(E)$ — the second derivative of the total energy with respect to length of crystal lattice. Note that in [2] $B(E)$ is also calculated using the finite difference method.

In this paper, we build a multistep algorithm to calculate the exact value of $B(E)$ in the case where the total energy of the system is determined by Tersoff Potential. With the help of Fast Automatic Differentiation we derived formulas to calculate the gradient of $B(E)$ with respect to Tersoff parameters with machine precision.

2 Calculation of second derivative of total energy with respect to atomic lattice coefficient

Let a be the initial length of the edges of the lattice of atoms; $\widetilde{a} = \alpha a$ $(\alpha \in R)$ — length of the edges of the lattice of atoms after deformation; $\rho = \widetilde{a} - a$ — deformation parameter. Then $\dfrac{a+\rho}{a} = \left(1 + \dfrac{\rho}{a}\right) a$. If $\overline{r}_k = (x_{k1}, x_{k2}, x_{k3})$ are the coordinates of some lattice atom before deformation and $\widetilde{\overline{r}}_k = (\widetilde{x}_{k1}, \widetilde{x}_{k2}, \widetilde{x}_{k3})$ are its coordinates after deformation, then $\widetilde{x}_{k1} = \left(1 + \dfrac{\rho}{a}\right) x_{k1}$, $\widetilde{x}_{k2} = \left(1 + \dfrac{\rho}{a}\right) x_{k2}$, $\widetilde{x}_{k3} = \left(1 + \dfrac{\rho}{a}\right) x_{k3}$.

Let $E(\overline{r}_1, \overline{r}_2, ..., \overline{r}_I)$ be the total energy of atoms' system before deformation. Then $E(\widetilde{\overline{r}}_1, \widetilde{\overline{r}}_2, ..., \widetilde{\overline{r}}_I) = E\left[\left(1 + \dfrac{\rho}{a}\right)\overline{r}_1, \left(1 + \dfrac{\rho}{a}\right)\overline{r}_2, ..., \left(1 + \dfrac{\rho}{a}\right)\overline{r}_I\right]$ is the total energy of atoms' system after deformation. The Bulk modulus and the Shear modulus of the material are proportional to $B(E)$, that can be calculated by the formula:

$$B(E) = \frac{\partial^2}{\partial \rho^2} E\left[\left(1 + \frac{\rho}{a}\right)\overline{r}_1, \left(1 + \frac{\rho}{a}\right)\overline{r}_2, ..., \left(1 + \frac{\rho}{a}\right)\overline{r}_I\right]\Big|_{\rho=0}.$$

As to total energy $E(\overline{r}_1, \overline{r}_2, ..., \overline{r}_I)$ it is calculated with the help of expression

$$E(\overline{r}_1, \overline{r}_2, ..., \overline{r}_I) = \sum_{i=1}^{I} \sum_{j=1; j\neq i}^{I} V_{ij},$$

where V_{ij} is the interaction potential between atoms marked i and j (i-atom and j-atom). In present paper the Tersoff Potential is used as interaction potential:

$$V_{ij} = f_c(r_{ij})\left(V_R(r_{ij}) - b_{ij}V_A(r_{ij})\right),$$

$$f_c(r) = \begin{cases} 1, & r < R - R_{cut}, \\ \frac{1}{2}\left(1 - \sin\left(\frac{\pi(r-R)}{2R_{cut}}\right)\right), & R - R_{cut} < r < R + R_{cut}, \\ 0, & r > R + R_{cut}, \end{cases}$$

$$V_{ij}^R = V_R(r_{ij}) = \frac{D_e}{S-1}\exp\left(-\beta\sqrt{2S}(r_{ij} - r_e)\right),$$

$$V_{ij}^A = V_A(r_{ij}) = \frac{SD_e}{S-1}\exp\left(-\beta\sqrt{\frac{2}{S}}(r_{ij} - r_e)\right),$$

$$b_{ij} = \left(1 + (\gamma\zeta_{ij})^{\eta}\right)^{-\frac{1}{2\eta}}, \qquad \zeta_{ij} = \sum_{k=1;k\neq i,j}^{I} f_c(r_{ik})g_{ijk}\omega_{ijk}, \qquad \omega_{ijk} = \exp(\lambda^3\tau_{ijk}),$$

$$\tau_{ijk} = (r_{ij} - r_{ik})^3, \qquad g_{ijk} = 1 + \left(\frac{c}{d}\right)^2 - \frac{c^2}{d^2 + (h - \cos\Theta_{ijk})^2}.$$

Here I is the number of atoms in considered system; r_{ij} is the distance between i-atom and j-atom; Θ_{ijk} is the angle between two vectors, first vector begins at i-atom and finishes at j-atom, second vector begins at i-atom and finishes at k-atom; R and R_{cut} are known parameters, identified from experimental geometric properties of substance. Tersoff Potential depends on ten parameters ($m = 10$), specific to modeled substances: $D_e, r_e, \beta, S, \eta, \gamma, \lambda, c, d, h$.

Let us construct the multistep algorithm to calculate the total energy E of atoms' system (interaction potential is Tersoff Potential). The distance between i-atom and j-atom is determined by the formula:

$$r_{ij} = \sqrt{(x_{1i} - x_{1j})^2 + (x_{2i} - x_{2j})^2 + (x_{3i} - x_{3j})^2},$$

where x_{1i}, x_{2i}, x_{3i} are the Cartesian coordinates of i-atom. If Θ_{ijk} is the angle between two vectors, connecting i-atom with j-atom and k-atom respectively, then $\cos\Theta_{ijk} = q_{ijk} = \dfrac{r_{ij}^2 + r_{ik}^2 - r_{jk}^2}{2r_{ij}r_{ik}}$. For compactness further in the study we introduce vectors $\overline{u}$ and $\overline{z}$ having the following coordinates: $\overline{u}^T = [u_1, u_2, ..., u_{10}]^T$, $\overline{z}^T = [z_1, z_2, ..., z_{10}]^T$, where $u_1 = D_e$, $u_2 = r_e$, $u_3 = \beta$, $u_4 = S$, $u_5 =$

$\eta, \quad u_6 = \gamma, \quad u_7 = \lambda, \quad u_8 = c, \quad u_9 = d, \quad u_{10} = h;$

$$z_1 = \left\{ z_1^{ijk} = \sqrt{(x_{1i} - x_{1k})^2 + (x_{2i} - x_{2k})^2 + (x_{3i} - x_{3k})^2} \right\} \equiv F(1, Z_1, U_1),$$
$$z_2 = \left\{ z_2^{ijk} = \sqrt{(x_{1j} - x_{1k})^2 + (x_{2j} - x_{2k})^2 + (x_{3j} - x_{3k})^2} \right\} \equiv F(2, Z_2, U_2),$$
$$z_3 = \left\{ z_3^{ijk} = q_{ijk} = \frac{(z_{13}^{ij})^2 + (z_1^{ijk})^2 - (z_2^{ijk})^2}{2 z_1^{ijk} z_{13}^{ij}} \right\} \equiv F(3, Z_3, U_3),$$
$$z_4 = \left\{ z_4^{ijk} = f_c(z_1^{ijk}) \right\} \equiv F(4, Z_4, U_4),$$
$$z_5 = \left\{ z_5^{ijk} = g_{ijk} = 1 + \left(\frac{u_8}{u_9} \right)^2 - \frac{(u_8)^2}{(u_9)^2 + (u_{10} - z_3^{ijk})^2} \right\} \equiv F(5, Z_5, U_5),$$
$$z_6 = \left\{ z_6^{ijk} = \tau_{ijk} = (z_{13}^{ij} - z_1^{ijk})^3 \right\} \equiv F(6, Z_6, U_6),$$
$$z_7 = \left\{ z_7^{ijk} = \omega_{ijk} = \exp((u_7)^3 z_6^{ijk}) \right\} \equiv F(7, Z_7, U_7),$$
$$z_8 = \left\{ z_8^{ijk} = f_c(r_{ik}) g_{ijk} \omega_{ijk} = z_4^{ijk} z_5^{ijk} z_7^{ijk}) \right\} \equiv F(8, Z_8, U_8),$$
$$z_9 = \left\{ z_9^{ij} = \zeta_{ij} = \textstyle\sum_{k=1; k \neq i,j}^{I} z_8^{ijk} \right\} \equiv F(9, Z_9, U_9),$$
$$z_{10} = \left\{ z_{10}^{ij} = \gamma \zeta_{ij} = u_6 z_9^{ij} \right\} \equiv F(10, Z_{10}, U_{10}),$$
$$z_{11} = \left\{ z_{11}^{ij} = (\gamma \zeta_{ij})^\eta = (z_{10})^{u_5} \right\} \equiv F(11, Z_{11}, U_{11}),$$
$$z_{12} = \left\{ z_{12}^{ij} = b_{ij} = (1 + z_{11}^{ij})^{-\frac{1}{2u_5}} \right\} \equiv F(12, Z_{12}, U_{12}),$$
$$z_{13} = \left\{ z_{13}^{ij} = \sqrt{(x_{1i} - x_{1j})^2 + (x_{2i} - x_{2j})^2 + (x_{3i} - x_{3j})^2} \right\} \equiv F(13, Z_{13}, U_{13}),$$
$$z_{14} = \left\{ z_{14}^{ij} = V_{ij}^R = \tfrac{u_1}{u_4 - 1} \exp\left(-u_3 \sqrt{2u_4} (z_{13}^{ij} - u_2) \right) \right\} \equiv F(14, Z_{14}, U_{14}),$$
$$z_{15} = \left\{ z_{15}^{ij} = V_{ij}^A = \tfrac{u_1 u_4}{u_4 - 1} \exp\left(-u_3 \sqrt{\tfrac{2}{u_4}} (z_{13}^{ij} - u_2) \right) \right\} \equiv F(15, Z_{15}, U_{15}),$$
$$z_{16} = \left\{ z_{16}^{ij} = f_c(z_{13}^{ij}) \right\} \equiv F(16, Z_{16}, U_{16}),$$
$$z_{17} = \left\{ z_{17}^{ij} = V_{ij} = z_{16}^{ij} (z_{14}^{ij} - z_{12}^{ij} z_{15}^{ij}) \right\} \equiv F(17, Z_{17}, U_{17}),$$
$$\left(i = \overline{1, I}, \quad j = \overline{1, I}, \quad j \neq i, \quad k = \overline{1, I}, \quad k \neq i, j \right).$$

Note that $\quad z_3^{ijk} \left(\left(1 + \frac{\rho}{a} \right) r_{ij}, \left(1 + \frac{\rho}{a} \right) r_{ik}, \left(1 + \frac{\rho}{a} \right) r_{jk} \right) = z_3^{ijk} (r_{ij}, r_{ik}, r_{jk});$

$$z_5^{ijk} \left(\left(1 + \frac{\rho}{a} \right) r_{ij}, \left(1 + \frac{\rho}{a} \right) r_{ik}, \left(1 + \frac{\rho}{a} \right) r_{jk} \right) = z_5^{ijk} (r_{ij}, r_{ik}, r_{jk}).$$

The energy E of the atoms in the system with the help of new variables may be rewritten as follows:

$$E(z(u)) = \sum_{i=1}^{I} \sum_{j=1; j \neq i}^{I} z_{17}^{ij}.$$

Variables $z_1, z_2, ..., z_{17}$ (the phase variables) are determined by the specified above multistep algorithm $z_l = F(l, Z_l, U_l)$, $(l = 17)$, where Z_l is the set of elements z_n in the right part of the equation $z_l = F(l, Z_l, U_l)$, and U_l is the set of elements u_n that appear in the right side of this equation. Note that each component z_l depends on a number of other components (z_l^{ij} or z_l^{ijk}).

Let us introduce also the following designations: $\widetilde{z}_1, \widetilde{z}_2, ..., \widetilde{z}_{17}$ and $\widetilde{\widetilde{z}}_1, \widetilde{\widetilde{z}}_2, ..., \widetilde{\widetilde{z}}_{17}$, where

$$\widetilde{z}_n = \left\{ \widetilde{z}_n^{ijk} : \widetilde{z}_n^{ijk} = \left.\frac{\partial z_n^{ijk}(\widetilde{r}_{ij}, \widetilde{r}_{ik}, \widetilde{r}_{jk})}{\partial \rho}\right|_{\rho=0} = \right.$$

$$\left. = \left.\frac{\partial z_n^{ijk}\left(\left(1+\frac{\rho}{a}\right) r_{ij}, \left(1+\frac{\rho}{a}\right) r_{ik}, \left(1+\frac{\rho}{a}\right) r_{jk}\right)}{\partial \rho}\right|_{\rho=0} \right\}, \qquad n = \overline{1,8},$$

$$\widetilde{z}_n = \left\{ \widetilde{z}_n^{ij} : \widetilde{z}_n^{ij} = \left.\frac{\partial z_n^{ij}(\widetilde{r}_{ij})}{\partial \rho}\right|_{\rho=0} = \left.\frac{\partial z_n^{ij}\left(\left(1+\frac{\rho}{a}\right) r_{ij}\right)}{\partial \rho}\right|_{\rho=0} \right\}, \qquad n = \overline{9,17},$$

$$\widetilde{\widetilde{z}}_n = \left\{ \widetilde{\widetilde{z}}_n^{ijk} : \widetilde{\widetilde{z}}_n^{ijk} = \left.\frac{\partial^2 z_n^{ijk}(\widetilde{r}_{ij}, \widetilde{r}_{ik}, \widetilde{r}_{jk})}{\partial \rho^2}\right|_{\rho=0} = \right.$$

$$\left. = \left.\frac{\partial^2 z_n^{ijk}\left(\left(1+\frac{\rho}{a}\right) r_{ij}, \left(1+\frac{\rho}{a}\right) r_{ik}, \left(1+\frac{\rho}{a}\right) r_{jk}\right)}{\partial \rho^2}\right|_{\rho=0} \right\}, \qquad n = \overline{1,8},$$

$$\widetilde{\widetilde{z}}_n = \left\{ \widetilde{\widetilde{z}}_n^{ij} : \widetilde{\widetilde{z}}_n^{ij} = \left.\frac{\partial^2 z_n^{ij}(\widetilde{r}_{ij})}{\partial \rho^2}\right|_{\rho=0} = \left.\frac{\partial^2 z_n^{ij}\left(\left(1+\frac{\rho}{a}\right) r_{ij}\right)}{\partial \rho^2}\right|_{\rho=0} \right\}, \qquad n = \overline{9,17},$$

$$\left(i = \overline{1,I}, \quad j = \overline{1,I}, \quad j \neq i, \quad k = \overline{1,I}, \quad k \neq i,j\right).$$

The above values are calculated by the formulas:

$$\widetilde{z}_1^{ijk} = r_{ik}/a; \qquad \widetilde{z}_2^{ijk} = r_{jk}/a; \qquad \widetilde{z}_3^{ijk} = 0; \qquad \widetilde{z}_4^{ijk} = \left.\frac{\partial f_c\left(\left(1+\frac{\rho}{a}\right) r_{ik}\right)}{\partial \rho}\right|_{\rho=0};$$

$$\widetilde{z}_5^{ijk} = 0; \qquad \widetilde{z}_6^{ijk} = 3 z_6^{ijk}/a; \qquad \widetilde{z}_7^{ijk} = 3 z_6^{ijk} z_7^{ijk} (u_7)^3/a;$$

$$\widetilde{z}_8^{ijk} = z_5^{ijk} (\widetilde{z}_4^{ijk} z_7^{ijk} + z_4^{ijk} \widetilde{z}_7^{ijk}); \qquad \widetilde{z}_9^{ij} = \sum_{k=1, k\neq i,j}^{I} \widetilde{z}_8^{ijk}; \qquad \widetilde{z}_{10}^{ij} = \widetilde{z}_9^{ij} u_6;$$

$$\widetilde{z}_{11}^{ij} = \widetilde{z}_{10}^{ij} u_5 (z_{10}^{ij})^{u_5 - 1}; \qquad \widetilde{z}_{12}^{ij} = -\frac{1}{2} u_5 \widetilde{z}_{11}^{ij} (1 + z_{11}^{ij})^{-\frac{1}{2u_5} - 1}; \qquad \widetilde{z}_{13}^{ij} = z_{13}^{ij}/a;$$

$$\widetilde{z}_{14}^{ij} = -\frac{u_3 \sqrt{2u_4} z_{13}^{ij} z_{14}^{ij}}{a}; \quad \widetilde{z}_{15}^{ij} = -\frac{u_3 \sqrt{2/u_4} z_{13}^{ij} z_{15}^{ij}}{a}; \quad \widetilde{z}_{16}^{ij} = \left.\frac{\partial f_c\left(\left(1+\frac{\rho}{a}\right) r_{ij}\right)}{\partial \rho}\right|_{\rho=0};$$

$$\widetilde{z}_{17}^{ij} = \widetilde{z}_{16}^{ij} z_{14}^{ij} - \widetilde{z}_{16}^{ij} z_{12}^{ij} z_{15}^{ij} + \widetilde{z}_{14}^{ij} z_{16}^{ij} - \widetilde{z}_{12}^{ij} z_{16}^{ij} z_{15}^{ij} - \widetilde{z}_{15}^{ij} z_{16}^{ij} z_{12}^{ij};$$

$$\widetilde{\widetilde{z}}_1^{ijk} = \widetilde{\widetilde{z}}_2^{ijk} = \widetilde{\widetilde{z}}_3^{ijk} = \widetilde{\widetilde{z}}_5^{ijk} = 0; \qquad \widetilde{\widetilde{z}}_4^{ijk} = \left.\frac{\partial^2 f_c\left(\left(1+\frac{\rho}{a}\right) r_{ik}\right)}{\partial \rho^2}\right|_{\rho=0};$$

$$\widetilde{\widetilde{z}}_6^{ijk} = 6 z_6^{ijk}/a^2; \qquad \widetilde{\widetilde{z}}_7^{ijk} = \tfrac{3}{a^2} z_6^{ijk} z_7^{ijk} (u_7)^3 (3 z_6^{ijk} (u_7)^3 + 2);$$

$$\widetilde{\widetilde{z}}_8^{ijk} = z_5^{ijk}(\widetilde{\widetilde{z}}_4^{ijk} z_7^{ijk} + 2\widetilde{z}_4^{ijk}\widetilde{z}_7^{ijk} + z_4^{ijk}\widetilde{\widetilde{z}}_7^{ijk}); \qquad \widetilde{\widetilde{z}}_9^{ij} = \sum_{k=1;k\neq i,j}^{I} \widetilde{\widetilde{z}}_8^{ijk};$$

$$\widetilde{\widetilde{z}}_{10}^{ij} = \widetilde{\widetilde{z}}_9^{ij} u_6; \qquad \widetilde{\widetilde{z}}_{11}^{ij} = u_5(u_5-1)(\widetilde{z}_{10}^{ij})^2(z_{10}^{ij})^{u_5-2} + u_5(\widetilde{\widetilde{z}}_{10}^{ij})(z_{10}^{ij})^{u_5-1};$$

$$\widetilde{\widetilde{z}}_{12}^{ij} = \frac{1+2u_5}{4u_5}(\widetilde{z}_{11}^{ij})^2(1+z_{11}^{ij})^{-\frac{1}{2u_5}-2} - \frac{1}{2u_5}\widetilde{\widetilde{z}}_{11}^{ij}(1+z_{11}^{ij})^{-\frac{1}{2u_5}-1}; \qquad \widetilde{z}_{13}^{ij} = 0;$$

$$\widetilde{\widetilde{z}}_{14}^{ij} = \frac{2(u_3)^2 u_4 (z_{13}^{ij})^2 z_{14}^{ij}}{a^2}; \quad \widetilde{\widetilde{z}}_{15}^{ij} = \frac{2(u_3)^2 (z_{13}^{ij})^2 z_{15}^{ij}}{a^2 u_4}; \quad \widetilde{\widetilde{z}}_{16}^{ij} = \left.\frac{\partial^2 f_c\left(\left(1+\frac{\rho}{a}\right) r_{ij}\right)}{\partial \rho^2}\right|_{\rho=0};$$

$$\widetilde{\widetilde{z}}_{17}^{ij} = \widetilde{\widetilde{z}}_{16}^{ij} z_{14}^{ij} + 2\widetilde{z}_{16}^{ij}\widetilde{z}_{14}^{ij} - \widetilde{\widetilde{z}}_{16}^{ij} z_{12}^{ij} z_{15}^{ij} - 2\widetilde{z}_{16}^{ij}\widetilde{z}_{12}^{ij} z_{15}^{ij} - 2\widetilde{z}_{16}^{ij} z_{12}^{ij}\widetilde{z}_{15}^{ij} +$$
$$+\widetilde{\widetilde{z}}_{14}^{ij} z_{16}^{ij} - \widetilde{\widetilde{z}}_{12}^{ij} z_{16}^{ij} z_{15}^{ij} - 2\widetilde{z}_{15}^{ij} z_{16}^{ij}\widetilde{z}_{12}^{ij} - \widetilde{\widetilde{z}}_{15}^{ij} z_{16}^{ij} z_{12}^{ij}.$$

To compute the second derivative of a function $f_c(r)$ there is a need for smoothing this function. It is proposed to replace the function $f_c(r)$ as follows:

$$f_c(r) = \begin{cases} 0, & r \geq R + R_{cut}, \\ 1, & r \leq R - R_{cut}, \\ C \cdot (f_*)^{\varphi(r)}, & R \leq r < R + R_{cut}, \\ C \cdot \left(2f_* - (f_*)^{\psi(r)}\right), & R - R_{cut} < r \leq R, \end{cases}$$

where $C = \dfrac{1}{2f_*}$, $f_* = \exp(-\dfrac{3}{2})$, $\varphi(r) = \dfrac{R_{cut}^2}{(r-R-R_{cut})^2}$, $\psi(r) = \dfrac{R_{cut}^2}{(r-R+R_{cut})^2}$.
Derivatives of function $f_c(r)$ with respect to ρ are calculated by the formulas:

$$\left.\frac{\partial f_c\left(\left(1+\frac{\rho}{a}\right) r\right)}{\partial \rho}\right|_{\rho=0} = \begin{cases} 0, & r \geq R + R_{cut}, \\ 0, & r \leq R - R_{cut}, \\ C \cdot (f_*)^{\varphi(r)} \ln(f_*) \cdot \widetilde{\varphi}(r), & R \leq r < R + R_{cut}, \\ C \cdot (f_*)^{\psi(r)} \ln(f_*) \cdot \widetilde{\psi}(r), & R - R_{cut} < r \leq R, \end{cases}$$

$$\left.\frac{\partial^2 f_c\left(\left(1+\frac{\rho}{a}\right) r\right)}{\partial \rho^2}\right|_{\rho=0} = \begin{cases} 0, & r \geq R + R_{cut}, \\ 0, & r \leq R - R_{cut}, \\ C \cdot (f_*)^{\varphi(r)} \ln(f_*)\left[\ln(f_*)\widetilde{\varphi}^2(r) + \widetilde{\widetilde{\varphi}}(r)\right], & R \leq r < R + R_{cut}, \\ -C \cdot (f_*)^{\psi(r)} \ln(f_*)\left[\ln(f_*)\widetilde{\psi}^2(r) + \widetilde{\widetilde{\psi}}(r)\right], & R - R_{cut} < r \leq R, \end{cases}$$

where $\widetilde{\varphi}(r) = \dfrac{-2rR_{cut}^2}{a(r-R-R_{cut})^3}$, $\widetilde{\psi}(r) = \dfrac{-2rR_{cut}^2}{a(r-R+R_{cut})^3}$,

$$\widetilde{\widetilde{\varphi}}(r) = \frac{6r^2R_{cut}^2}{a^2(r-R-R_{cut})^4}, \qquad \widetilde{\widetilde{\psi}}(r) = \frac{6r^2R_{cut}^2}{a^2(r-R+R_{cut})^4}.$$

Thus, $B(E)$ is calculated by the formula

$$B(E) = \sum_{i=1}^{I} \sum_{i=1;j\neq i}^{I} \widetilde{\widetilde{z}}_{17}^{ij},$$

where the variables $z_1, z_2, ..., z_{17}$, $\widetilde{z}_1, \widetilde{z}_2, ..., \widetilde{z}_{17}$, $\widetilde{\widetilde{z}}_1, \widetilde{\widetilde{z}}_2, ..., \widetilde{\widetilde{z}}_{17}$ are determined by mentioned above multistep algorithm.

3 Determining the adjoint variables and gradient

We represent the general formulas of Fast Automatic Differentiation below, which will be used to calculate the gradient of function $B(E)$ with respect to parameters of Tersoff Potential specific to modeled substance. Let vectors $z \in R^n$ and $u \in R^m$ satisfy the following system of nonlinear scalar equations (multistep process):

$$z_i = F(i, Z_i, U_i), \qquad\qquad 1 \leq i \leq n \tag{2}$$

where Z_i is the set of vectors z_j, that appear at the right part of equality (2), and U_i is the set of vectors u_j, that appear at the right part of the same equality (2). Usually the vectors $z \in R^n$ and the vectors $u \in R^m$ are called dependent (phase) and independent (control) variables respectively. Let differentiable function $W(z, u)$ define mapping $W : R^n \times R^m \longrightarrow R^1$. Then the composite function $\Omega(u) = W(z(u), u)$ is differentiable, and its gradient with respect to the independent variables u_i is given by the formula

$$\frac{\partial \Omega}{\partial u_i} = W_{u_i}(z, u) + \sum_{q \in \overline{K}_i} F_{u_i}(q, Z_q, U_q) p_q. \tag{3}$$

The multipliers $p_i \in R^n$ are the adjoint variables that are defined by the following system of linear algebraic equations:

$$p_i = W_{z_i}(z, u) + \sum_{q \in \overline{Q}_i} F_{z_i}(q, Z_q, U_q) p_q, \tag{4}$$

where $\overline{Q}_i$ and $\overline{K}_i$ are the index sets:

$$\overline{Q}_i = \{j : 1 \leq j \leq n, \quad z_i \in Z_j\} \qquad\qquad \overline{K}_i = \{j : 1 \leq j \leq n, \quad u_i \in U_j\}.$$

In accordance to (4), for all $i = \overline{1, I}$, $\quad j = \overline{1, I}$, $\quad j \neq i$, $\quad k = \overline{1, I}$, $\quad k \neq i, j$ adjoint variables corresponding to the phase variables $z_1, z_2, ..., z_{17}$, $\widetilde{z}_1, \widetilde{z}_2, ..., \widetilde{z}_{17}$,

$\tilde{\tilde{z}}_1, \tilde{\tilde{z}}_2, ..., \tilde{\tilde{z}}_{17}$ are defined by the equations:

$$p_5^{ijk} = (\tilde{\tilde{z}}_4^{ijk} z_7^{ijk} + 2\tilde{z}_4^{ijk}\tilde{z}_7^{ijk} + z_4^{ijk}\tilde{\tilde{z}}_7^{ijk})\tilde{\tilde{p}}_8^{ijk} + (\tilde{z}_4^{ijk} z_7^{ijk} + z_4^{ijk}\tilde{z}_7^{ijk})\tilde{p}_8^{ijk} + z_4^{ijk} z_7^{ijk} p_8^{ijk};$$

$$p_7^{ijk} = \tilde{\tilde{z}}_4^{ijk} z_5^{ijk}\tilde{\tilde{p}}_8^{ijk} + \tilde{z}_4^{ijk} z_5^{ijk}\tilde{p}_8^{ijk} + z_4^{ijk} z_5^{ijk} p_8^{ijk} +$$
$$+\frac{3}{a^2} z_6^{ijk}(u_7)^3(3z_6^{ijk}(u_7)^3 + 2)\tilde{\tilde{p}}_7^{ijk} + \frac{3}{a} z_6^{ijk}(u_7)^3\tilde{p}_7^{ijk};$$

$$p_8^{ijk} = p_9^{ij}; \qquad p_9^{ij} = u_6 p_{10}^{ij};$$

$$p_{10}^{ij} = \Big[u_5(u_5-1)(u_5-2)(z_{10}^{ij})^{u_5-3}(\tilde{z}_{10}^{ij})^2 + u_5(u_5-1)(z_{10}^{ij})^{u_5-2}\tilde{\tilde{z}}_{10}^{ij}\Big]\tilde{\tilde{p}}_{11}^{ij} +$$
$$+u_5(u_5-1)(z_{10}^{ij})^{u_5-2}\tilde{z}_{10}^{ij}\tilde{p}_{11}^{ij} + u_5(z_{10}^{ij})^{u_5-1}p_{11}^{ij};$$

$$p_{11}^{ij} = -\frac{(1+4u_5)(1+2u_5)}{8(u_5)^3}(1+z_{11}^{ij})^{-\frac{1}{2u_5}-3}(z_{11}^{ij})^2\tilde{\tilde{p}}_{12}^{ij} + \frac{(1+2u_5)}{4(u_5)^2}(1+z_{11}^{ij})^{-\frac{1}{2u_5}-2}\times$$
$$\times\tilde{\tilde{z}}_{11}^{ij}\tilde{\tilde{p}}_{12}^{ij} + \frac{(1+2u_5)}{4(u_5)^2}(1+z_{11}^{ij})^{-\frac{1}{2u_5}-2}\tilde{z}_{11}^{ij}\tilde{p}_{12}^{ij} - \frac{1}{2u_5}(1+z_{11}^{ij})^{-\frac{1}{2u_5}-1}p_{12}^{ij};$$

$$p_{12}^{ij} = (-\tilde{\tilde{z}}_{16}^{ij} z_{15}^{ij} - 2\tilde{z}_{15}^{ij}\tilde{z}_{16}^{ij} - z_{16}^{ijk}\tilde{\tilde{z}}_{15}^{ij})\tilde{\tilde{p}}_{17}^{ij};$$

$$p_{14}^{ij} = \tilde{\tilde{z}}_{16}^{ij}\tilde{\tilde{p}}_{17}^{ij} + \frac{2(u_3)^2 u_4(z_{13}^{ij})^2}{a^2}\tilde{\tilde{p}}_{14}^{ij} - u_3\sqrt{2u_4}\frac{z_{13}^{ij}}{a}\tilde{p}_{14}^{ij};$$

$$p_{15}^{ij} = (-\tilde{\tilde{z}}_{16}^{ij} z_{12}^{ij} - 2\tilde{z}_{12}^{ij}\tilde{z}_{16}^{ij} - z_{16}^{ij}\tilde{\tilde{z}}_{12}^{ij})\tilde{\tilde{p}}_{17}^{ij} + \frac{2(u_3)^2(z_{13}^{ij})^2}{a^2 u_4}\tilde{\tilde{p}}_{15}^{ij} - \sqrt{2/u_4}\frac{u_3 z_{13}^{ij}}{a}\tilde{p}_{15}^{ij};$$

$$p_{16}^{ij} = (\tilde{\tilde{z}}_{14}^{ij} - \tilde{\tilde{z}}_{12}^{ij} z_{15}^{ij} - 2\tilde{z}_{12}^{ij}\tilde{z}_{15}^{ij} - z_{12}^{ij}\tilde{\tilde{z}}_{15}^{ij})\tilde{\tilde{p}}_{17}^{ij}; \qquad p_{17}^{ij} = 0;$$

$$\tilde{p}_7^{ijk} = 2\tilde{z}_4^{ijk} z_5^{ijk}\tilde{\tilde{p}}_8^{ijk} + z_4^{ijk} z_5^{ijk}\tilde{p}_8^{ijk}; \qquad \tilde{p}_8^{ijk} = \tilde{p}_9^{ij};$$

$$\tilde{p}_9^{ij} = u_6\tilde{p}_{10}^{ij}; \qquad \tilde{p}_{10}^{ij} = 2u_5(u_5-1)(z_{10}^{ij})^{u_5-2}\tilde{\tilde{p}}_{11}^{ij} + u_5(z_{10}^{ij})^{u_5-1}\tilde{z}_{10}^{ij}\tilde{p}_{11}^{ij};$$

$$\tilde{p}_{11}^{ij} = \frac{(1+2u_5)}{2(u_5)^2}(1+z_{11}^{ij})^{-1/(2u_5)-2}\tilde{z}_{11}^{ij}\tilde{\tilde{p}}_{12}^{ij} - \frac{(1+z_{11}^{ij})^{-1/(2u_5)-1}}{2u_5}\tilde{p}_{12}^{ij};$$

$$\tilde{p}_{12}^{ij} = -2(\tilde{z}_{16}^{ij} z_{15}^{ij} + \tilde{z}_{15}^{ij} z_{16}^{ij})\tilde{\tilde{p}}_{17}^{ij}; \qquad \tilde{p}_{14}^{ij} = 2\tilde{z}_{16}^{ij}\tilde{\tilde{p}}_{17}^{ij};$$

$$\tilde{p}_{15}^{ij} = -2(\tilde{z}_{16}^{ij} z_{12}^{ij} + \tilde{z}_{12}^{ij} z_{16}^{ij})\tilde{\tilde{p}}_{17}^{ij}; \qquad \tilde{p}_{16}^{ij} = 2(\tilde{z}_{14}^{ij} - \tilde{z}_{12}^{ij} z_{15}^{ij} - \tilde{z}_{15}^{ij} z_{12}^{ij})\tilde{\tilde{p}}_{17}^{ij};$$

$$\tilde{p}_{17}^{ij} = 0; \qquad \tilde{\tilde{p}}_7^{ijk} = z_4^{ijk} z_5^{ijk}\tilde{\tilde{p}}_8^{ijk}; \qquad \tilde{\tilde{p}}_8^{ijk} = \tilde{\tilde{p}}_9^{ij}; \qquad \tilde{\tilde{p}}_9^{ij} = u_6\tilde{\tilde{p}}_{10}^{ij};$$

$$\tilde{\tilde{p}}_{10}^{ij} = u_5(z_{10}^{ij})^{u_5-1}\tilde{\tilde{p}}_{11}^{ij}; \qquad \tilde{\tilde{p}}_{11}^{ij} = -\frac{(1+z_{11}^{ij})^{-1/(2u_5)-1}}{2u_5}\tilde{\tilde{p}}_{12}^{ij};$$

$$\tilde{\tilde{p}}_{12}^{ij} = -z_{15}^{ij} z_{16}^{ij}\tilde{\tilde{p}}_{17}^{ij}; \qquad \tilde{\tilde{p}}_{14}^{ij} = z_{16}^{ij}\tilde{\tilde{p}}_{17}^{ij};$$

$$\tilde{\tilde{p}}_{15}^{ij} = -z_{12}^{ij} z_{16}^{ij}\tilde{\tilde{p}}_{17}^{ij}; \qquad \tilde{\tilde{p}}_{16}^{ij} = (z_{14}^{ij} - z_{12}^{ij} z_{15}^{ij})\tilde{\tilde{p}}_{17}^{ij}; \qquad \tilde{\tilde{p}}_{17}^{ij} = 1;$$

The adjoint variables are calculate in the following order:

$$\widetilde{\widetilde{p}}_{17}^{ij}, \widetilde{\widetilde{p}}_{16}^{ij}, ..., \widetilde{\widetilde{p}}_{7}^{ij}, \widetilde{p}_{17}^{ij}, ..., \widetilde{p}_{7}^{ij}, p_{17}^{ij}, ..., p_{5}^{ij}.$$

Those adjoint variables, whose formulas for calculation aren't provided above, aren't used for calculation of the components of the gradient.

The partial derivatives of function

$$\Omega(\overline{u}) = B(E(\overline{u})) = \sum_{i=1}^{I} \sum_{j=1;j\neq i}^{I} \widetilde{\widetilde{z}}_{17}^{ij}$$

with respect to independent variables u_m, $(m = \overline{1,10})$ (components of gradient), according to equation (3), are determined by the relations:

$$\frac{\partial\Omega}{\partial u_1} = \sum_{i=1}^{I} \sum_{\substack{j=1\\ j\neq i}}^{I} \left(\frac{z_{14}^{ij}}{u_1} p_{14}^{ij} + \frac{z_{15}^{ij}}{u_1} p_{15}^{ij} \right);$$

$$\frac{\partial\Omega}{\partial u_2} = \sum_{i=1}^{I} \sum_{\substack{j=1\\ j\neq i}}^{I} \left(z_{14}^{ij} u_3 \sqrt{2u_4} p_{14}^{ij} + z_{15}^{ij} u_3 \sqrt{2/u_4} p_{15}^{ij} \right);$$

$$\frac{\partial\Omega}{\partial u_3} = \sum_{i=1}^{I} \sum_{\substack{j=1\\ j\neq i}}^{I} \left(z_{14}^{ij} \left(-\sqrt{2u_4}(z_{13}^{ij} - u_2) \right) p_{14}^{ij} + z_{15}^{ij} \left(-\sqrt{2/u_4}(z_{13}^{ij} - u_2) \right) p_{15}^{ij} \right) +$$

$$+ \sum_{i=1}^{I} \sum_{\substack{j=1\\ j\neq i}}^{I} \left(\frac{4u_3u_4(z_{13}^{ij})^2 z_{14}^{ij}}{a^2} \widetilde{\widetilde{p}}_{14}^{ij} - \frac{\sqrt{2u_4} z_{13}^{ij} z_{14}^{ij}}{a} \widetilde{p}_{14}^{ij} \right) +$$

$$+ \sum_{i=1}^{I} \sum_{\substack{j=1\\ j\neq i}}^{I} \left(\frac{4u_3(z_{13}^{ij})^2 z_{15}^{ij}}{a^2 u_4} \widetilde{\widetilde{p}}_{15}^{ij} - \frac{z_{13}^{ij} z_{15}^{ij}}{a} \sqrt{2/u_4} \widetilde{p}_{15}^{ij} \right);$$

$$\frac{\partial\Omega}{\partial u_4} = \sum_{i=1}^{I} \sum_{\substack{j=1\\ j\neq i}}^{I} \left(\left(-\frac{z_{14}^{ij}}{u_4 - 1} - 0.5u_3\sqrt{2/u_4}(z_{13}^{ij} - u_2) z_{14}^{ij} \right) p_{14}^{ij} \right) +$$

$$+ \sum_{i=1}^{I} \sum_{\substack{j=1\\ j\neq i}}^{I} \left(\left(-\frac{z_{15}^{ij}}{u_4(u_4 - 1)} + 0.5(u_3/u_4)\sqrt{2/u_4}(z_{13}^{ij} - u_2) z_{15}^{ij} \right) p_{15}^{ij} \right) +$$

$$+\sum_{i=1}^{I}\sum_{\substack{j=1\\ j\neq i}}^{I}\left(\left(-\frac{u_3}{2a}\sqrt{2/u_4}\,z_{13}^{ij}z_{14}^{ij}\right)\widetilde{p}_{14}^{ij}+\left(\frac{2(u_3)^2(z_{13}^{ij})^2}{a^2}z_{14}^{ij}\right)\widetilde{\widetilde{p}}_{14}^{ij}\right)+$$

$$+\sum_{i=1}^{I}\sum_{\substack{j=1\\ j\neq i}}^{I}\left(\left(\frac{u_3}{2au_4}\sqrt{2/u_4}\,z_{13}^{ij}z_{15}^{ij}\right)\widetilde{p}_{14}^{ij}-\left(\frac{2(u_3)^2(z_{13}^{ij})^2}{a^2(u_4)^2}z_{15}^{ij}\right)\widetilde{\widetilde{p}}_{15}^{ij}\right);$$

$$\frac{\partial\Omega}{\partial u_5}=\sum_{i=1}^{I}\sum_{\substack{j=1\\ j\neq i}}^{I}\left((z_{10}^{ij})^{u_5}\ln(z_{10}^{ij})p_{11}^{ij}+\frac{(1+z_{11}^{ij})^{-1/(2u_5)}}{2(u_5)^2}\ln(1+z_{11}^{ij})p_{12}^{ij}\right)+$$

$$+\sum_{i=1}^{I}\sum_{\substack{j=1\\ j\neq i}}^{I}\left(\left((z_{10}^{ij})^{u_5-1}\widetilde{z}_{10}^{ij}+u_5(z_{10}^{ij})^{u_5-1}\widetilde{z}_{10}^{ij}\ln(z_{10}^{ij})\right)\widetilde{p}_{11}^{ij}\right)+$$

$$+\sum_{i=1}^{I}\sum_{\substack{j=1\\ j\neq i}}^{I}\left(\left((2u_5-1)(z_{10}^{ij})^{u_5-2}+((u_5)^2-u_5)(z_{10}^{ij})^{u_5-2}\ln(z_{10}^{ij})\right)(\widetilde{z}_{10}^{ij})^2\widetilde{\widetilde{p}}_{11}^{ij}\right)+$$

$$+\sum_{i=1}^{I}\sum_{\substack{j=1\\ j\neq i}}^{I}\left(\left((z_{10}^{ij})^{u_5-1}+u_5(z_{10}^{ij})^{u_5-1}\ln(z_{10}^{ij})\right)\widetilde{\widetilde{z}}_{10}^{ij}\widetilde{\widetilde{p}}_{11}^{ij}\right)+$$

$$+\sum_{i=1}^{I}\sum_{\substack{j=1\\ j\neq i}}^{I}\left(\left(\frac{(1+z_{11}^{ij})^{-1/(2u_5)-1}}{2(u_5)^2}-\frac{\ln(1+z_{11}^{ij})}{4(u_5)^3}(1+z_{11}^{ij})^{-1/(2u_5)-1}\right)\widetilde{z}_{11}^{ij}\widetilde{p}_{12}^{ij}\right)+$$

$$+\sum_{i=1}^{I}\sum_{\substack{j=1\\ j\neq i}}^{I}\left(\left(-\frac{(1+u_5)}{2(u_5)^3}(1+z_{11}^{ij})^{-1/(2u_5)-2}\right)(\widetilde{z}_{11}^{ij})^2\widetilde{\widetilde{p}}_{12}^{ij}\right)+$$

$$+\sum_{i=1}^{I}\sum_{\substack{j=1\\ j\neq i}}^{I}\left(\left(\frac{(1+2u_5)\ln(1+z_{11}^{ij})}{8(u_5)^4}(1+z_{11}^{ij})^{-1/(2u_5)-2}\right)(\widetilde{z}_{11}^{ij})^2\widetilde{\widetilde{p}}_{12}^{ij}\right)+$$

$$+\sum_{i=1}^{I}\sum_{\substack{j=1\\ j\neq i}}^{I}\left(\left(\frac{(1+z_{11}^{ij})^{-1/(2u_5)-1}}{2(u_5)^2}-\frac{\ln(1+z_{11}^{ij})}{4(u_5)^3}(1+z_{11}^{ij})^{-1/(2u_5)-1}\right)\widetilde{\widetilde{z}}_{11}^{ij}\widetilde{\widetilde{p}}_{12}^{ij}\right);$$

$$\frac{\partial\Omega}{\partial u_6}=\sum_{i=1}^{I}\sum_{\substack{j=1\\ j\neq i}}^{I}\left(z_9^{ij}p_{10}^{ij}+\tilde{z}_9^{ij}\tilde{p}_{10}^{ij}+\tilde{\tilde{z}}_9^{ij}\tilde{\tilde{p}}_{10}^{ij}\right);$$

$$\frac{\partial\Omega}{\partial u_7}=\sum_{i=1}^{I}\sum_{\substack{j=1\\ j\neq i}}^{I}\sum_{\substack{k=1\\ k\neq i,j}}^{I}\left(3z_6^{ijk}z_7^{ijk}(u_7)^2p_7^{ijk}+\frac{9}{a}z_6^{ijk}z_7^{ijk}(u_7)^2\tilde{p}_7^{ijk}\right)+$$

$$+\sum_{i=1}^{I}\sum_{\substack{j=1\\ j\neq i}}^{I}\sum_{\substack{k=1\\ k\neq i,j}}^{I}\left(\left(\frac{3z_6^{ijk}z_7^{ijk}}{a^2}\left(18z_6^{ijk}(u_7)^5+6(u_7)^2\right)\right)\tilde{\tilde{p}}_7^{ijk}\right);$$

$$\frac{\partial\Omega}{\partial u_8}=\sum_{i=1}^{I}\sum_{\substack{j=1\\ j\neq i}}^{I}\sum_{\substack{k=1\\ k\neq i,j}}^{I}\left(\left(\frac{2u_8}{(u_9)^2}-\frac{2u_8}{(u_9)^2+(u_{10}-z_3^{ijk})^2}\right)p_5^{ijk}\right);$$

$$\frac{\partial\Omega}{\partial u_9}=\sum_{i=1}^{I}\sum_{\substack{j=1\\ j\neq i}}^{I}\sum_{\substack{k=1\\ k\neq i,j}}^{I}\left(\left(\frac{-2(u_8)^2}{(u_9)^3}+\frac{2(u_8)^2u_9}{\left((u_9)^2+(u_{10}-z_3^{ijk})^2\right)^2}\right)p_5^{ijk}\right);$$

$$\frac{\partial\Omega}{\partial u_{10}}=\sum_{i=1}^{I}\sum_{\substack{j=1\\ j\neq i}}^{I}\sum_{\substack{k=1\\ k\neq i,j}}^{I}\left(\left(\frac{2(u_8)^2(u_{10}-z_3^{ijk})}{\left((u_9)^2+(u_{10}-z_3^{ijk})^2\right)^2}\right)p_5^{ijk}\right).$$

The received formulas for calculation of the gradient of function $B(E(u))$ outwardly are represented quite difficult and bulky. Therefore, there is a natural question: whether to use simpler approaches, for example, finite difference method, to calculate the gradient functions $B(E(u))$.

In [5] the comparison of function gradients, calculated by the finite differences and by using Fast Automatic Differentiation formulas (see above), was presented. The results of comparison are the following:

1) when computing the gradient of complicated function using finite differences, one must conduct researches related to the choice of suitable increments of each parameter;

2) for different parameters, the researches must be carried out independently;

3) for the same parameter, the researches must be carried out if its value changed;

4) to calculate the gradient of complicated function using finite differences one must $(m+1)$ times calculate the value of function itself.

In contrary to it, the Fast Automatic Differentiation enables us to calculate gradients of any complicated function with the machine accuracy for arbitrary parameters. The machine time that is needed to calculate the gradient does not exceed three times of calculation of the function itself.

4 Conclusion

In this work an efficient algorithm to calculate gradients of the Bulk modulus and the Shear modulus is presented. The algorithm is based on the modern Fast Automatic Differentiation technique. The formulas to compute the mentioned gradients are derived. These formulas allow us to compute the gradients with the machine accuracy. The computation time that is needed to calculate the gradient does not exceed three times of calculation of the function itself. The comparison of the proposed algorithm and finite differences method to calculate gradients of complicated function is made. The conclusion is made: the calculation of gradient of Bulk modulus and the Shear modulus using finite difference method is linked to enormous difficulties.

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