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Properties of one-sided ideals of topological rings

V.I. Arnautov

Abstract. A continuous ring isomorphism $\nu : (R,\tau) \to (\widehat{R},\widehat{\tau})$ is said to be semi-topological from the left (right) in the class $\Re$ provided (R,τ) is a left ideal (right ideal, ideal) of a topological ring $(\widetilde{R},\widetilde{\tau}) \in \Re$ and $\nu = \widetilde{\nu}\mid_R$ for a topological homomorphism $\widetilde{\nu} : (\widetilde{R},\widetilde{\tau}) \to (\widehat{R},\widehat{\tau})$. The article contains several criteria for a continuous homomorphism to be semi-topological from the left (right).

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A ring (not necessarily an associative one) is said to be a topological ring provided it is equipped with a topology such that all ring operations are continuous in it.

The following isomorphism theorem is often used in general algebra and, in particular, in ring theory: if A is a subring (subgroup) of a ring (group) R and I is an ideal (normal divisor) in R then there exists a ring (group) isomorphism $\nu : A/(A \cap I) \to (A + I)/I$ of quotient rings (quotient groups); in particular, if $A \cap I = 0$ then the ring (group) A is isomorphic to the ring (group) $(A + I)/I$, i.e. the rings (groups) A and $(A + I)/I$ possess the same algebraic properties.

In the case when the category of topological rings (groups) with continuous ring (group) homomorphisms taken for morphisms is considered, the isomorphisms are precisely those mappings which are isomorphisms of groups (rings) and homeomorphisms of topological spaces.

The analogue of the above isomorphism theorem is known to be not valid for the above mentioned categories. The above mentioned mapping ν is known to be no more than a continuous ring (group) isomorphism (see Theorem 1 in [2]).

Hence the morphism $\nu : A/(A \cap I) \to (A + I)/I$ to be an isomorphism of the category of topological rings (groups) the ring (group) A should possess some additional properties, for instance A should be an ideal (normal divisor) or a one-sided ideal of the topological ring (group) (R, τ_R).

The case when A is an ideal of the topological ring (R,τ) was investigated in [1], and the case when A is a normal divisor of the topological group (R,τ) was investigated in [2].

The present paper is a sequel to [1] and [2]. The case when A is a one-sided ideal of the topological ring (R,τ) is investigated in it .

1 Definition. The homomorphism of topological rings $\varphi : (R, \tau) \to (\widehat{R}, \widehat{\tau})$ is as usually called to be a topological isomorphism iff it is both continuous and open mapping.

2 Remark. Let (R, τ) be an arbitrary topological ring I be its arbitrary (closed) ideal. The canonical homomorphism $\xi : (R, \tau) \to (R, \tau)/I$ (i.e. such that $\xi(r) = r + I$) is known to be a topological homomorphism, and if the mapping $\varphi : (R, \tau) \to (\widehat{R}, \widehat{\tau})$ is a topological ring homomorphism and $I = \ker \varphi$ then the topological rings $(\widehat{R}, \widehat{\tau})$ and $(R, \tau)/I$ are topologically isomorphic.

3 Definition. Let $\Re$ be a class of topological rings and (R, τ) and $(\widehat{R}, \widehat{\tau})$ be elements of $\Re$. Similarly to the definition which is given in [1] we say the continuous isomorphism $\varphi : (R, \tau) \to (\widehat{R}, \widehat{\tau})$ is semi-topological from the left (right) in the class $\Re$ provided there exists such a topological ring $(\widetilde{R}, \widetilde{\tau}) \in \Re$ that:

– the topological ring (R, τ) is a left (right) ideal of the topological ring $(\widetilde{R}, \widetilde{\tau})$;

– the isomorphism φ can be extended to a topological homomorphism $\widetilde{\varphi} : (\widetilde{R}, \widetilde{\tau}) \to (\widehat{R}, \widehat{\tau})$.

4 Theorem. *Let $\Re$ be one of the following classes of topological rings:*

1. The class of all (separated) topological rings;

2. The class of all (separated) topological rings admitting a fundamental system of neighbourhoods of zero consisting of subgroups of the additive group of the ring;

3. The class of all (separated) topological rings which are bounded from the right (i.e. for every neighbourhood of zero U there exists such a neighbourhood of zero V that $R \cdot V \subseteq U$);

4. The class of all (separated) topological rings admitting a fundamental system of neighbourhoods of zero consisting of left ideals of the ring.

Then if (R, τ) and $(\widehat{R}, \widehat{\tau}) \in \Re$ and $\varphi : (R, \tau) \to (\widehat{R}, \widehat{\tau})$ is a continuous homomorphism then the following assertions are equivalent:

1. The isomorphism φ is semi-topological from the left in $\Re$;

2. For every element $b \in R$ and an arbitrary neighbourhood of zero U in (R, τ) there exist such neighbourhoods of zero $\widehat{V}$ and V in $(\widehat{R}, \widehat{\tau})$ and (R, τ), respectively, that

$$\varphi^{-1}(\widehat{V}) \cdot b \subseteq U \text{ and } \varphi^{-1}(\widehat{V}) \cdot V \subseteq U.$$

3. There exists a topological ring $(\widetilde{R}, \widetilde{\tau}) \in \Re$ such that the topological ring (R, τ) is a left ideal of the topological ring $(\widetilde{R}, \widetilde{\tau})$ and the isomorphism φ can be extended to a topological homomorphism $\widetilde{\varphi} : (\widetilde{R}, \widetilde{\tau}) \to (\widehat{R}, \widehat{\tau})$ where $(\ker \widetilde{\varphi})^2 = \{0\}$.

Proof. $1 \Rightarrow 2$. Let the isomorphism $\varphi : (R, \tau) \to (\widehat{R}, \widehat{\tau})$ be semi-topological from the left in $\Re$. Hence there exists a topological ring $(\widetilde{R}, \widetilde{\tau}) \in \Re$ such that the topological ring (R, τ) is a left ideal of the topological ring $(\widetilde{R}, \widetilde{\tau})$ and the isomorphism φ can be extended to a topological homomorphism $\widetilde{\varphi} : (\widetilde{R}, \widetilde{\tau}) \to (\widehat{R}, \widehat{\tau})$.

If $b \in R$ and U is an arbitrary neighbourhood of zero in (R, τ) then there exist neighbourhoods of zero $\widetilde{U}$ and $\widetilde{V}$ in $(\widetilde{R}, \widetilde{\tau})$ such that $R \cap \widetilde{U} = U$, $\widetilde{V} \cdot b \subseteq \widetilde{U}$ and $\widetilde{V} \cdot \widetilde{V} \subseteq \widetilde{U}$.

Since (R,τ) is a subring of the topological ring $(\widetilde{R},\widetilde{\tau})$ then $V = R\cap\widetilde{V}$ is a neighbourhood of zero in (R,τ) and since the homomorphism $\widetilde{\varphi}:(\widetilde{R},\widetilde{\tau})\to(\widehat{R},\widehat{\tau})$ is open then $\widehat{V}=\widetilde{\varphi}(\widetilde{V})$ is a neighbourhood of zero in $(\widehat{R},\widehat{\tau})$.

Since φ is an isomorphism and $\widetilde{\varphi}$ is its extension then $R\cap\ker\widetilde{\varphi}=\ker\varphi=\{0\}$. Since R is a left ideal $\widetilde{R}$ and $\ker\widetilde{\varphi}$ is an ideal $\widetilde{R}$ then $(\ker\widetilde{\varphi})\cdot R\subseteq\ker\widetilde{\varphi}\cap R=\{0\}$. Hence

$$\varphi^{-1}(\widehat{V})\cdot b\subseteq\varphi^{-1}\big(\widetilde{\varphi}(\widetilde{V})\big)\cdot b\subseteq\Big(\widetilde{\varphi}^{-1}\big(\widetilde{\varphi}(\widetilde{V})\big)\cdot b\Big)\cap R=\big((\widetilde{V}+\ker\widetilde{\varphi})\cdot b\big)\cap R=$$
$$\big(\widetilde{V}\cdot b+\ker\widetilde{\varphi}\cdot b\big)\cap R\subseteq\big(\widetilde{V}\cdot b+\ker\widetilde{\varphi}\cdot R\big)\cap R=$$
$$\big(\widetilde{V}\cdot b\big)\cap R\subseteq\big(\widetilde{V}\cdot b\big)\cap R\subseteq\widetilde{U}\cap R=U \text{ and}$$
$$\varphi^{-1}(\widehat{V})\cdot V\subseteq\varphi^{-1}\big(\widetilde{\varphi}(\widetilde{V})\big)\cdot V\subseteq\Big(\widetilde{\varphi}^{-1}\big(\widetilde{\varphi}(\widetilde{V})\big)\cdot V\Big)\cap R=\big((\widetilde{V}+\ker\widetilde{\varphi})\cdot V\big)\cap R=$$
$$\big(\widetilde{V}\cdot V+\ker\widetilde{\varphi}\cdot V\big)\cap R\subseteq\big(\widetilde{V}\cdot V+\ker\widetilde{\varphi}\cdot R\big)\cap R=$$
$$\big(\widetilde{V}\cdot V\big)\cap R\subseteq\big(\widetilde{V}\cdot\widetilde{V}\big)\cap R\subseteq\widetilde{U}\cap R=U,$$

that completes the proof of the implication $1\Rightarrow 2$.

$2\Rightarrow 3$. Let (R,τ) and $(\widehat{R},\widehat{\tau})\in\Re$ and $\varphi:(R,\tau)\to(\widehat{R},\widehat{\tau})$ be such a continuous isomorphism that for every element $b\in R$ and every neighbourhood of zero U in (R,τ) there exist neighbourhoods of zero $\widehat{V}$ and V in $(\widehat{R},\widehat{\tau})$ and (R,τ), respectively, that

$$\varphi^{-1}(\widehat{V})\cdot b\subseteq U \text{ and } \varphi^{-1}(\widehat{V})\cdot V\subseteq U.$$

Consider a discrete ring $\widetilde{R}$ such that its additive group is a direct sum of additive groups of rings R and $\widehat{R}$ and the multiplication is defined as follows:
$(r_1,\widehat{r}_1)\cdot(r_2,\widehat{r}_2)=(r_1\cdot r_2,\varphi(r_1)\cdot\widehat{r}_2)$.

One can easily check that $\widetilde{R}$ equipped with these operation is a ring and, if the rings R and $\widehat{R}$ are associative then so is the ring $\widetilde{R}$.

We write $\mathbf{B}$ and $\widehat{\mathbf{B}}$ for the set of all neigbourhoods of zero of the topological ring (R,τ) and $(\widehat{R},\widehat{\tau})$, respectively. For every $V\in\mathbf{B}$ and $\widehat{V}\in\widehat{\mathbf{B}}$ consider the set

$$\widetilde{W}(V,\widehat{V})=\big\{(r-\varphi^{-1}(\widehat{r}),\widehat{r})\mid r\in V,\widehat{r}\in\widehat{V}\big\}.$$

Let us prove that assertions BN1 – BN6 from Theorem 1.2.5 in [3] are valid for the set

$$\widetilde{\mathbf{B}}=\{\widetilde{W}(V,\widehat{V})\mid V\in\mathbf{B},\widehat{V}\in\widehat{\mathbf{B}}\},$$

i.e. it is a fundamental system of neighbourhoods of zero in a certain ring topology (which need not be separated) $\widetilde{\tau}$ on the ring $\widehat{R}$.

Since $0\in V$ and $0\in\widehat{V}$ for every $V\in\mathbf{B}$ and $\widehat{V}\in\widehat{\mathbf{B}}$ then $(0,0)\in\widetilde{W}(V,\widehat{V})$ for every $\widetilde{W}(V,\widehat{V})\in\widetilde{\mathbf{B}}$ and since $(0,0)$ is a zero in the ring $\widetilde{R}$ then the assertion BN1 is valid.

Let $\widetilde{W}(V_1,\widehat{V}_1)$ and $\widetilde{W}(V_2,\widehat{V}_2)\in\widetilde{\mathbf{B}}$. There exist $V_3\in\mathbf{B}$ and $\widehat{V_3}\in\widehat{\mathbf{B}}$ such that $V_3\subseteq V_1\cap V_2$ and $\widehat{V}_3\subseteq\widehat{V}_1\cap\widehat{V}_2$. One can easily see that $\widetilde{W}(V_3,\widehat{V}_3)\subseteq\widetilde{W}(V_1,\widehat{V}_1)\cap\widetilde{W}(V_2,\widehat{V}_2)$, i.e. the assertion BN2 is valid.

Let now $\widetilde{W}(V_1,\widehat{V}_1)\in\widetilde{\mathbf{B}}$. There exist $V_2\in\mathbf{B}$ and $\widehat{V_2}\in\widehat{\mathbf{B}}$ such that $V_2+V_2\subseteq V_1$, $-V_2\subseteq V_1$ and $\widehat{V}_2+\widehat{V}_2\subseteq\widehat{V}_1$, $-\widehat{V}_2\subseteq\widehat{V}_1$. Then $\widetilde{W}(V_2,\widehat{V}_2)+\widetilde{W}(V_2,\widehat{V}_2)=$

$$\begin{gathered}\big\{(r_1-\varphi^{-1}(\widehat{r}_1),\widehat{r}_1)\mid r_1\in V_2,\widehat{r}_1\in\widehat{V}_2\big\}+\big\{(r_2-\varphi^{-1}(\widehat{r}_2),\widehat{r}_2)\mid r_2\in V_2,\widehat{r}_2\in\widehat{V}_2\big\}=\\ \big\{(r_1-\varphi^{-1}(\widehat{r}_1)+r_2-\varphi^{-1}(\widehat{r}_2),\widehat{r}_1+\widehat{r}_2)\mid\\ r_1+r_2\in V_2+V_2\subseteq V_1,\widehat{r}_1+\widehat{r}_2\in\widehat{V}_2+\widehat{V}_2\subseteq\widehat{V}_1\big\}\subseteq\\ \big\{(r_3-\varphi^{-1}(\widehat{r}_3),\widehat{r}_3)\mid r_3\in V_1,\widehat{r}_3\in\widehat{V}_1\big\}=\widetilde{W}(V_2,\widehat{V}_2)\text{ and}\\ -\widetilde{W}(V_2,\widehat{V}_2)=-\big\{(r_1-\varphi^{-1}(\widehat{r}_1),\widehat{r}_1)\mid r_1\in V_2,\widehat{r}_1\in\widehat{V}_2\big\}=\\ \big\{(-r_1-\varphi^{-1}(-\widehat{r}_1),-\widehat{r}_1)\mid -r_1\in -V_2,-\widehat{r}_1\in-\widehat{V}_2\big\}\subseteq\\ \big\{(r-\varphi^{-1}(\widehat{r}),\widehat{r})\mid r\in -V_2\subseteq V_1,\widehat{r}\in-\widehat{V}_2\subseteq V_1\big\}=\widetilde{W}(V_1,\widehat{V}_1),\end{gathered}$$

i.e. the assertions BN3 and BN4 are valid.

Let $\widetilde{W}(V_1,\widehat{V}_1)\in\widetilde{\mathbf{B}}$. There exist $V_2\in\mathbf{B}$ and $\widehat{V_2}\in\widehat{\mathbf{B}}$ such that $V_2-V_2\subseteq V_1$ and $\widehat{V}_2\cdot\widehat{V}_2\subseteq\widehat{V}_1$. Since the assertion 2 is supposed to be valid then there exist $V_3\in\mathbf{B}$ and $\widehat{V_3}\in\widehat{\mathbf{B}}$ such that $V_3\cdot V_3\subseteq V_2$ and $\varphi^{-1}(\widehat{V_3})\cdot V_3\subseteq V_2$. Since the isomorphism $\varphi:(R,\tau)\to(\widehat{R},\widehat{\tau})$ is continuous then without loss of generality we can claim that $V_3\subseteq\widehat{V_3}$.

Than taking into account the definition of the multiplication in the ring $\widetilde{R}$ we get $\widetilde{W}(V_3,\widehat{V}_3)\cdot\widetilde{W}(V_3,\widehat{V}_3)=$

$$\begin{gathered}\big\{\big(r_1-\varphi^{-1}(\widehat{r}_1),\widehat{r}_1\big)\mid r_1\in V_3,\widehat{r}_1\in\widehat{V}_3\big\}\cdot\big\{\big(r_2-\varphi^{-1}(\widehat{r}_2),\widehat{r}_2\big)\mid r_2\in V_3,\widehat{r}_2\in\widehat{V}_3\big\}=\\ \big\{\big((r_1-\varphi^{-1}(\widehat{r}_1))\cdot(r_2-\varphi^{-1}(\widehat{r}_2)),\varphi(r_1-\varphi^{-1}(\widehat{r}_1))\cdot\widehat{r}_2\big)\mid r_1,r_2\in V_3,\widehat{r}_1,\widehat{r}_2\in\widehat{V}_3\big\}=\\ \big\{\big(r_1\cdot(r_2-\varphi^{-1}(\widehat{r}_2))-\varphi^{-1}(\widehat{r}_1)\cdot(r_2-\varphi^{-1}(\widehat{r}_2)),\\ \varphi(r_1)\cdot\widehat{r}_2-\widehat{r}_1\cdot\widehat{r}_2\big)\mid r_1,r_2\in V_3,\widehat{r}_1,\widehat{r}_2\in\widehat{V}_3\big\}=\\ \big\{\big(r_1\cdot r_2-\varphi^{-1}(\widehat{r}_1)\cdot r_2-r_1\cdot\varphi^{-1}(\widehat{r}_2)+\varphi^{-1}(\widehat{r}_1\cdot\widehat{r}_2),\\ \varphi(r_1)\cdot\widehat{r}_2-\widehat{r}_1\cdot\widehat{r}_2\big)\mid r_1,r_2\in V_3,\widehat{r}_1,\widehat{r}_2\in\widehat{V}_3\big\}=\\ \big\{\big(r_1\cdot r_2-\varphi^{-1}(\widehat{r}_1)\cdot r_2-(r_1\cdot\varphi^{-1}(\widehat{r}_2)-\varphi^{-1}(\widehat{r}_1\cdot\widehat{r}_2)),\varphi(r_1)\cdot\widehat{r}_2-\widehat{r}_1\cdot\widehat{r}_2\big)\mid\\ r_1,r_2\in V_3,\widehat{r}_1,\widehat{r}_2\in\widehat{V}_3\big\}.\end{gathered}$$

Taking into account the choice of neighbourhoods V_1, V_2, V_3, $\widehat{V}_1$, $\widehat{V}_2$, $\widehat{V}_3$ and elements r_1, r_2, $\widehat{r}_1$, $\widehat{r}_2$ we obtain

$$\begin{gathered}r_3=r_1\cdot r_2-\varphi^{-1}(\widehat{r}_1)\cdot r_2\in V_3\cdot V_3-\varphi^{-1}(\widehat{V}_3)\cdot V_3\subseteq V_2-V_2\subseteq V_1\text{ and}\\ \widehat{r}_1=\varphi(r_1)\cdot\widehat{r}_2-\widehat{r}_1\cdot\widehat{r}_2\in\varphi(V_3)\cdot\widehat{V}_3-\widehat{V}_3\cdot\widehat{V}_3\subseteq\widehat{V}_3\cdot\widehat{V}_3-\widehat{V}_3\cdot\widehat{V}_3\subseteq\widehat{V}_2-\widehat{V}_2\subseteq\widehat{V}_1,\end{gathered}$$

and hence $\widetilde{W}(V_3,\widehat{V}_3)\cdot\widetilde{W}(V_3,\widehat{V}_3)=$

$$\begin{gathered}\big\{\big(r_1\cdot r_2-\varphi^{-1}(\widehat{r}_1)\cdot r_2-(r_1\cdot\varphi^{-1}(\widehat{r}_2)-\varphi^{-1}(\widehat{r}_1\cdot\widehat{r}_2)),\varphi(r_1)\cdot\widehat{r}_2-\widehat{r}_1\cdot\widehat{r}_2\big)\mid\\ r_1,r_2\in V_3,\widehat{r}_1,\widehat{r}_2\in\widehat{V}_3\big\}=\big\{\big(r_3-\varphi^{-1}(\widehat{r}_3),\widehat{r}_3\big)\mid r_3\in V_1,\widehat{r}_3\in\widehat{V}_3\big\}=\widetilde{W}(V,\widehat{V}),\end{gathered}$$

i.e. the assertion BN5 is valid.

Let $\widetilde{r} = (r, \widehat{r}) \in \widetilde{R}$ and $\widetilde{W}(V_1, \widehat{V}_1) \in \widetilde{\mathbf{B}}$. There exist $V_2 \in \mathbf{B}$ and $\widehat{V_2} \in \widehat{\mathbf{B}}$ such that $r \cdot V_2 \subseteq V_1$ and $\varphi(r) \cdot \widehat{V}_2 \subseteq \widehat{V}_1$. Hence

$$\begin{gathered}
\widetilde{r} \cdot \widetilde{W}(V_2, \widehat{V}_2) = (r, \widehat{r}) \cdot \{\big(a - \varphi^{-1}(\widehat{a}), \widehat{a}\big) \mid a \in V_2,\ \widehat{a} \in \widehat{V}_2\} = \\
\{\big(r \cdot (a - \varphi^{-1}(\widehat{a})), \varphi(r) \cdot \widehat{a}\big) \mid a \in V_2,\ \widehat{a} \in \widehat{V}_2\} = \\
\{\big(r \cdot a - r \cdot \varphi^{-1}(\widehat{a})), \varphi(r) \cdot \widehat{a}\big) \mid a \in V_2,\ \widehat{a} \in \widehat{V}_2\} = \\
\{\big(r \cdot a - \varphi^{-1}(\varphi(r) \cdot \widehat{a})), \varphi(r) \cdot \widehat{a}\big) \mid a \in V_2,\ \widehat{a} \in \widehat{V}_2\} \subseteq \\
\{\big(v - \varphi^{-1}(\widehat{v})), \widehat{v}\big) \mid v \in V_1,\ \widehat{v} \in \widehat{V_1}\} = \widetilde{W}(V_1, \widehat{V}_1),
\end{gathered}$$

since $r \cdot a \in r \cdot V_2 \subseteq V_1$ and $\varphi(r) \cdot \widehat{a} \in \varphi(r) \cdot \widehat{V} \subseteq V_1$. Except that if $\widetilde{r} = (r, \widehat{r}) \in \widetilde{R}$ and $\widetilde{W}(V_1, \widehat{V}_1) \in \widetilde{\mathbf{B}}$ then there exist V_2, $V_3 \in \mathbf{B}$ and $\widehat{V}_2$, $\widehat{V}_3 \in \widehat{\mathbf{B}}$ such that $V_2 - V_2 + V_2 - V_2 \subseteq V_1$ and $\widehat{V}_2 - \widehat{V}_2 \subseteq \widehat{V}_1$, $V_3 \cdot r \subseteq V_2$, $V_3 \cdot \varphi^{-1}(\widehat{r}) \subseteq V_2$, $\widehat{V}_3 \cdot \widehat{r} \subseteq \widehat{V}_2$. Since the inclusions mentioned in the assertion 2 are valid then we may assert without loss of generality that $\varphi^{-1}(\widehat{V}_3) \cdot r \subseteq V_2$ and $\varphi^{-1}(\widehat{V}_3) \cdot \varphi^{-1}(\widehat{r}) \subseteq V_2$ and, since the isomorphism φ is continuous we may claim that $\varphi(V_3) \subseteq \widehat{V}_3$.

Then

$$\begin{gathered}
\widetilde{W}(V_3, \widehat{V}_3) \cdot \widetilde{r} = \{\big(a - \varphi^{-1}(\widehat{a}), \widehat{a}\big) \mid a \in V_3,\ \widehat{a} \in \widehat{V}_3\} \cdot (r, \widehat{r}) = \\
\{\big((a - \varphi^{-1}(\widehat{a})) \cdot r, (\varphi(a - \varphi^{-1}(\widehat{a}))) \cdot \widehat{r}\big) \mid a \in V_3,\ \widehat{a} \in \widehat{V}_3\} = \\
\{\big(a \cdot r - \varphi^{-1}(\widehat{a}) \cdot r, \varphi(a) \cdot \widehat{r} - \varphi(\varphi^{-1}(\widehat{a})) \cdot \widehat{r}\big) \mid a \in V_3,\ \widehat{a} \in \widehat{V}_3\} = \\
\{\big(a \cdot r - \varphi^{-1}(\widehat{a}) \cdot r + \varphi^{-1}(\varphi(a) \cdot \widehat{r}) - \varphi^{-1}(\widehat{a} \cdot \widehat{r}) - \varphi^{-1}(\varphi(a) \cdot \widehat{r}) + \varphi^{-1}(\widehat{a} \cdot \widehat{r}), \\
\varphi(a) \cdot \widehat{r} - \widehat{a} \cdot \widehat{r}\big) \mid a \in V_3,\ \widehat{a} \in \widehat{V}_3\} = \\
\{\big(a \cdot r - \varphi^{-1}(\widehat{a}) \cdot r + a \cdot \varphi^{-1}(\widehat{r}) - \varphi^{-1}(\widehat{a}) \cdot \varphi^{-1}(\widehat{r}) - \varphi^{-1}(\varphi(a) \cdot \widehat{r}) + \varphi^{-1}(\widehat{a} \cdot \widehat{r}), \\
\varphi(a) \cdot \widehat{r} - \widehat{a} \cdot \widehat{r}\big) \mid a \in V_3,\ \widehat{a} \in \widehat{V}_3\} \subseteq \{\big(b - \varphi^{-1}(\widehat{b}), \widehat{b}\big) \mid b \in V_1,\ \widehat{b} \in \widehat{V}_1\} = \widetilde{W}(V_1, \widehat{V}_1),
\end{gathered}$$

since

$$\begin{gathered}
b = a \cdot r - \varphi^{-1}(\widehat{a}) \cdot r + a \cdot \varphi^{-1}(\widehat{r}) - \varphi^{-1}(\widehat{a}) \cdot \varphi^{-1}(\widehat{r}) \in \\
V_3 \cdot r - V_3 \cdot \varphi^{-1}(\widetilde{r}) + \varphi^{-1}(\widehat{V}_3) \cdot r - \varphi^{-1}(\widehat{V}_3) \cdot \varphi^{-1}(\widehat{r}) \subseteq V_2 - V_2 + V_2 - V_2 \subseteq V_1, \\
\widehat{b} = \varphi(a) \cdot \widehat{r} - \widehat{a} \cdot \widehat{r} \in \varphi(V_3) \cdot \widehat{r} - \widehat{V}_3 \cdot \widehat{r} \subseteq \widehat{V}_3 \cdot \widehat{r} - \widehat{V}_3 \cdot \widehat{r} \subseteq \widehat{V}_2 - \widehat{V}_2 \subseteq \widehat{V}_1 \text{ and} \\
-\varphi^{-1}(\widehat{b}) = -\varphi^{-1}(\varphi(a) \cdot \widehat{r}) + \varphi^{-1}(\widehat{a} \cdot \widehat{r}).
\end{gathered}$$

By that the validity of the assertion BN6 has been checked and by Theorem 1.2.5 in [3] the set $\widetilde{\mathbf{B}}$ is a fundamental system of neighbourhoods of zero in a certain ring topology (which need not be separated) $\widetilde{\tau}$ on the ring $\widetilde{R}$.

Now we prove that if the topological rings (R, τ) and $(\widehat{R}, \widehat{\tau})$ are separated then so is $(\widetilde{R}, \widetilde{\tau})$. To do that is sufficient to prove the validity of the assertion BN1' (see Corollary 1.3.7 in [3]) for it.

Let $0 \neq \widetilde{r} = (r, \widehat{r}) \in \widetilde{R}$. If $0 \neq \widehat{r}$ then there exists $\widehat{V} \in \widehat{\mathbf{B}}$ such that $\widehat{r} \notin \widehat{V}$ and hence

$$\widetilde{r} = (r, \widehat{r}) \notin \{(a - \varphi^{-1}(\widehat{a})) \mid a \in R,\ \widehat{a} \in \widehat{V}\} = \{(a, \widehat{a}) \mid a \in R, \widehat{a} \in \widehat{V}\} = \widetilde{W}(R, \widehat{V}).$$

If $0 = \widehat{r}$ then $0 \neq r$ and there exists $V \in \mathbf{B}$ such that $r \notin V$. Hence

$$\widetilde{r} = (r, \widehat{r}) = (r, 0) \notin \{(a - \widehat{a}, \widehat{a}) \mid a \in V,\ \widehat{a} \in \widehat{R}\} = \widetilde{W}(V, \widehat{R}).$$

Hence the assertion BN1' is valid and therefore the topological ring $(\widetilde{R}, \widetilde{\tau})$ is separated.

Let us check that the topological ring (R, τ) is a left ideal of the topological ring $(\widetilde{R}, \widetilde{\tau})$.

Indeed, one can easily prove that the set $A = \{(r, 0) \mid r \in R\}$ is a left ideal of the ring $\widetilde{R}$ and since $A \cap \widetilde{W}(V,\ \widehat{V}) =$

$$\{(r, 0) \mid r \in R\} \cap \{(r - \varphi^{-1}(\widehat{r}), \widehat{r}) \mid r \in V,\ \widehat{r} \in \widehat{V}\} = \{(r, 0) \mid r \in V\}$$

for every $\widehat{V} \in \widehat{\mathbf{B}}$ and $V \in \mathbf{B}$, then topological rings (R, τ) and $(A, \widetilde{\tau}|_A)$ are topologically isomorphic. Then we identify an element $r \in R$ with the element $(r, 0) \in A$ and get that the topological ring (R, τ) is a left ideal of the topological ring $(\widetilde{R}, \widetilde{\tau})$.

We prove now that the isomorphism $\varphi : R \to \widehat{R}$ can be extended to a homomorphism $\widetilde{\varphi} : (\widetilde{R}, \widetilde{\tau}) \to (\widehat{R}, \widehat{\tau})$.

We set $\widetilde{\varphi}(r, \widehat{r}) = \varphi(r)$ for every element $(r, \widehat{r}) \in \widetilde{R}$. It is obvious that the mapping $\widetilde{\varphi} : \widetilde{R} \to \widehat{R}$ is a homomorphism and, taking into account the identification of the element $r \in R$ with the element $(r, 0) \in A$ get that the mapping $\widetilde{\varphi}$ which is defined on $\widetilde{R}$ is an extension of the isomorphism φ.

Except that one can easily verify that $\ker \widetilde{\varphi} = \{(0, \widehat{r}) \mid \widehat{r} \in \widehat{R}\}$, and $(\ker \widetilde{\varphi})^2 = \{0\}$.

Since

$$\widetilde{\varphi}(\widetilde{W}(V, \widehat{V})) \supseteq \widetilde{\varphi}(\{(0 - \varphi^{-1}(\widehat{r}), \widehat{r})) \mid \widehat{r} \in \widehat{V}\}) =$$
$$\{\varphi(\{(-\varphi^{-1}(\widehat{r}) \mid \widehat{r} \in \widehat{V}\} = \{-\widehat{r} \mid \widehat{r} \in \widehat{V}\} = -\widehat{V}$$

and $-\widehat{V} \in \widehat{\mathbf{B}}$ for every $\widehat{V} \in \widehat{\mathbf{B}}$ and $V \in \mathbf{B}$ then by Proposition 1.5.5 in [3] the homomorphism $\widetilde{\varphi} : (\widetilde{R}, \widetilde{\tau}) \to (\widehat{R}, \widehat{\tau})$ is open.

Let now $\widehat{V} \in \widehat{\mathbf{B}}$. There exists a neighbourhood of zero $\widehat{V}_1 \in \widehat{\mathbf{B}}$ such that $\widehat{V}_1 - \widehat{V}_1 \subseteq \widehat{V}$ and since the isomorphism $\varphi : (R, \tau) \to (\widehat{R}, \widehat{\tau})$ is continuous then there exists a neighbourhood of zero $V_1 \in \mathbf{B}$ such that $\varphi(V_1) \subseteq \widehat{V}_1$. Then

$$\widetilde{\varphi}(\widetilde{W}(V_1, \widehat{V}_1)) = \widetilde{\varphi}(\{(r - \varphi^{-1}(\widehat{r}), \widehat{r}) \mid r \in V_1, \widehat{r} \in \widehat{V}_1\} =$$
$$\varphi(\{(r - \varphi^{-1}(\widehat{r}) \mid r \in V_1, \widehat{r} \in \widehat{V}_1\}) = \{\varphi(r) - \widehat{r}) \mid r \in V_1, \widehat{r} \in \widehat{V}_1\} \subseteq \widehat{V}_1 - \widehat{V}_1 \subseteq \widehat{V},$$

and, by Proposition 1.5.5 in [3] the homomorphism $\widetilde{\varphi} : (\widetilde{R}, \widetilde{\tau}) \to (\widehat{R}, \widehat{\tau})$ is continuous.

So the implication $2 \Rightarrow 3$ is proved for the case when $\Re$ is a class of all (separated) topological rings.

Let us prove the implication $2 \Rightarrow 3$ for the rest of the classes which are mentioned in the condition of Theorem.

If $\Re$ is the class of all (separated) topological rings admitting a fundamental system of neighbourhoods of zero consisting of subgroups of the additive group of the ring then to complete the proof of Theorem it is sufficient to check that the above defined sets $\widetilde{W}(V, \widehat{V})$ are subgroups provided so are V and $\widehat{V}$.

Since

$$\widetilde{W}(V, \widehat{V}) - \widetilde{W}(V, \widehat{V}) =$$
$$\{(a - \varphi^{-1}(\widehat{a}), \widehat{a}) \mid a \in V, \widehat{a} \in \widehat{V}\} - \{(b - \varphi^{-1}(\widehat{b}), \widehat{b}) \mid b \in V, \widehat{b} \in \widehat{V}\} =$$
$$\{(a - b - \varphi^{-1}(\widehat{a}) + \varphi^{-1}(\widehat{b}), \widehat{a} - \widehat{b}) \mid a, b \in V, \widehat{a}, \widehat{b} \in \widehat{V}\} =$$
$$\{(a - b - \varphi^{-1}(\widehat{a} - \widehat{b}), \widehat{a} - \widehat{b}) \mid a, b \in V, \widehat{a}, \widehat{b} \in \widehat{V}\} \subseteq$$
$$\{(c - \varphi^{-1}(\widehat{c}), \widehat{c}) \mid c \in V, \widehat{c} \in \widehat{V}\} = \widetilde{W}(V, \widehat{V}),$$

(since $a - b \in V - V = V$ and $\widehat{a} - \widehat{b} \in \widehat{V} - \widehat{V} = \widehat{V}$) then $\widetilde{W}(V, \widehat{V})$ is a subgroup).

Let now $\Re$ be the class the class of all (separated) topological rings which are bounded from the right and (R, τ), $(\widehat{R}, \widehat{\tau}) \in \Re$.

Let us prove that the topological ring $(\widetilde{R}, \widetilde{\tau})$ is also bounded from the right in this case. Indeed, if $\widetilde{W}(V, \widehat{V}) \in \widetilde{\mathbf{B}}$ then there exist $V_1 \in \mathbf{B}$ and $\widehat{V}_1 \in \widehat{\mathbf{B}}$ such that $R \cdot V_1 \subseteq V$ and $\widehat{R} \cdot \widehat{V}_1 \subseteq \widehat{V}$. Then

$$\widetilde{R} \cdot \widetilde{W}(V_1, \widehat{V}_1) = \{(r, \widehat{r}) \mid r \in R, \widehat{r} \in \widehat{R}\} \cdot \{(a - \varphi^{-1}(\widehat{a})) \mid a \in V_1,\ \widehat{a} \in \widehat{V}_1\} =$$
$$\{(r \cdot (a - \varphi^{-1}(\widehat{a})), \varphi(r) \cdot (\widehat{a})) \mid r \in R, a \in V_1, \widehat{a} \in \widehat{V}_1\} =$$
$$\{(r \cdot a - r \cdot \varphi^{-1}(\widehat{a})), \varphi(r) \cdot (\widehat{a})) \mid r \in R, a \in V_1, \widehat{a} \in \widehat{V}_1\} \subseteq$$
$$\{(c - \varphi^{-1}(\widehat{c}), \widehat{c}) \mid c \in V, \widehat{c} \in \widehat{V}\} = \widetilde{W}(V, \widehat{V}),$$

since $r \cdot a \in R \cdot V_1 \subseteq V$, $\varphi(r) \cdot (\widehat{a}) \in \widehat{R} \cdot \widehat{V}_1 \subseteq \widehat{V}$ and $r \cdot \varphi^{-1}(\widehat{a}) = \varphi^{-1}(\varphi(r) \cdot \widehat{a})$.

Let now $\Re$ be the class of all (separated) topological rings, admitting a fundamental system of neighbourhoods of zero consisting of left ideals of the ring. If (R, τ), $(\widehat{R}, \widehat{\tau}) \in \Re$ then these rings are bounded from the right and admit a fundamental system of neighbourhoods of zero consisting of subgroups. Then (see the above two cases) so is $(\widetilde{R}, \widetilde{\tau})$ and, by Proposition 1.6.32 in [3] $(\widetilde{R}, \widetilde{\tau}) \in \Re$.

So the proof of the implication $2 \Rightarrow 3$ is complete for every case mentioned in the condition of Theorem.

To complete the proof of Theorem it is sufficient to prove the implication $3 \Rightarrow 1$. It is obvious since the topological ring $(\widetilde{R}, \widetilde{\tau})$, mentioned in the assertion 3 satisfies the definition 3. □

5 Remark. The following Theorem can be easily obtained from Theorem 4 proceeding to anti-isomorphic rings.

6 Theorem. *Let $\Re$ be one of the following classes of topological rings:*

1. The class of all (separated) topological rings;

2. The class of all (separated) topological rings, admitting a fundamental system of neighbourhoods of zero consisting of subgroups of the additive group of the ring;

3. The class of all (separated) topological rings which are bounded from the left (i.e. for every neighbourhood of zero U there exists a neighbourhood of zero V such that $V \cdot R \subseteq U$);

4. The class of all (separated) topological rings, admitting a fundamental system of neighbourhoods of zero consisting of right ideals of the ring.

Then if (R, τ) and $(\widehat{R}, \widehat{\tau}) \in \Re$ and $\varphi : (R, \tau) \to (\widehat{R}, \widehat{\tau})$ is a continuous homomorphism then the following assertions are equivalent:

1. The isomorphism φ is semi-topological from the right in $\Re$;

2. For every element $b \in R$ and arbitrary neighbourhood of zero U in (R, τ) there exist neighbourhoods of zero $\widehat{V}$ and V in $(\widehat{R}, \widehat{\tau})$ and (R, τ), respectively, such that

$$b \cdot \varphi^{-1}(\widehat{V}) \subseteq U \text{ and } V \cdot \varphi^{-1}(\widehat{V}) \subseteq U.$$

3. There exists a topological ring $(\widetilde{R}, \widetilde{\tau}) \in \Re$, such that the topological ring (R, τ) is a right ideal of the topological ring $(\widetilde{R}, \widetilde{\tau})$, and the isomorphism φ can be extended to a topological homomorphism $\widetilde{\varphi} : (\widetilde{R}, \widetilde{\tau}) \to (\widehat{R}, \widehat{\tau})$ and $(\ker \widetilde{\varphi})^2 = \{0\}$.

7 Remark. The below assertion follows from Theorems 4 and 6 of the present article and from Theorem 1 and Remark 1 from [1].

8 Corollary. *Let $\Re$ be one of the following classes of topological rings:*

1. The class of all (separated) topological rings;

2. The class of all (separated) topological rings, admitting a fundamental system of zero consisting of subgroups of the additive group of the ring;

Then if (R, τ) and $(\widehat{R}, \widehat{\tau}) \in \Re$ and the isomorphism $\varphi : (R, \tau) \to (\widehat{R}, \widehat{\tau})$ is semi-topological both from the right and from the left in the class $\Re$ then it is semi-topological in the class $\Re$.[1]

9 Theorem. *Let $\Re$ be one of the following classes of topological rings:*

1. The class of all (separated) topological rings which are bounded from the left;

2. The class of all (separated) topological rings admitting a fundamental system of neighbourhoods of zero consisting of the right ideals of the ring;

3. The class of all (separated) topological rings which are bounded (i.e. are bounded both from the right and from the left);

4. The class of all (separated) topological rings admitting a fundamental system of neighbourhoods of zero consisting of the ideals of the ring.

Then if (R, τ) and $(\widehat{R}, \widehat{\tau}) \in \Re$ and $\varphi : (R, \tau) \to (\widehat{R}, \widehat{\tau})$ is a continuous isomorphism then the following assertions are equivalent:

1. The isomorphism φ is semi-topological from the left in the class $\Re$;

[1] A continuous isomorphism $\varphi : (R, \tau) \to (\widehat{R}, \widehat{\tau})$ is said to be semi-topological in the class $\Re$ provided there exists a topological ring $(\widetilde{R}, \widetilde{\tau}) \in \Re$ such that the topological ring (R, τ) is an ideal of the topological ring $(\widetilde{R}, \widetilde{\tau})$ and the isomorphism φ can be extended to a topological homomorphism $\widetilde{\varphi} : (\widetilde{R}, \widetilde{\tau}) \to (\widehat{R}, \widehat{\tau})$ (see [1]).

2. For an arbitrary neighbourhood of zero U in (R,τ) there exists a neighbourhood of zero $\widehat{V}$ in $(\widehat{R},\widehat{\tau})$ such that

$$\varphi^{-1}(\widehat{V})\cdot R\subseteq U.$$

3. There exists a topological ring $(\widetilde{R},\widetilde{\tau})\in\Re$ such that the topological ring (R,τ) is a left ideal of the topological ring $(\widetilde{R},\widetilde{\tau})$, the isomorphism φ can be extended to a topological homomorphism $\widetilde{\varphi}:(\widetilde{R},\widetilde{\tau})\to(\widehat{R},\widehat{\tau})$ and $(\ker\widetilde{\varphi})^2=\{0\}$.

Proof. $1\Rightarrow 2$. Let the isomorphism $\varphi:(R,\tau)\to(\widehat{R},\widehat{\tau})$ be semi-topological in the class $\Re$. Then there exists a topological ring $(\widetilde{R},\widetilde{\tau})\in\Re$ such that the topological ring (R,τ) is a left ideal of the topological ring $(\widetilde{R},\widetilde{\tau})$ and the isomorphism φ can be extended to a topological homomorphism $\widetilde{\varphi}:(\widetilde{R},\widetilde{\tau})\to(\widehat{R},\widehat{\tau})$.

Let U be an arbitrary neighbourhood of zero in (R,τ). Since the topological ring (R,τ) is a subring of the topological ring $(\widetilde{R},\widetilde{\tau})$ then there exists a neighbourhood of zero $\widetilde{U}$ in $(\widetilde{R},\widetilde{\tau})$ such that $R\cap\widetilde{U}=U$. Since the rings from the class $\Re$ are bounded from the left then there exists a neighbourhood of zero $\widetilde{V}$ in $(\widetilde{R},\widetilde{\tau})$ such that $\widetilde{V}\cdot\widetilde{R}\subseteq\widetilde{U}$. It follows from the openness of the homomorphism $\widetilde{\varphi}:(\widetilde{R},\widetilde{\tau})\to(\widehat{R},\widehat{\tau})$ that $\widehat{V}=\widetilde{\varphi}(\widetilde{V})$ is a neighbourhood of zero in $(\widehat{R},\widehat{\tau})$.

Since φ is an isomorphism and $\widetilde{\varphi}$ is its extension then $R\cap\ker\widetilde{\varphi}=\ker\varphi=\{0\}$. Since R is a left ideal in $\widetilde{R}$ and $\ker\widetilde{\varphi}$ is an ideal in $\widetilde{R}$ then $(\ker\widetilde{\varphi})\cdot R\subseteq\ker\widetilde{\varphi}\cap R=\{0\}$. Hence

$$\varphi^{-1}(\widehat{V})\cdot R=\varphi^{-1}\big(\widetilde{\varphi}(\widetilde{V})\big)\cdot R\subseteq\Big(\widetilde{\varphi}^{-1}\big(\widetilde{\varphi}(\widetilde{V})\big)\Big)\cdot R=(\widetilde{V}+\ker\widetilde{\varphi})\cdot R=$$
$$\widetilde{V}\cdot R+\ker\widetilde{\varphi}\cdot R=\widetilde{V}\cdot R\subseteq\big(\widetilde{V}\cdot\widetilde{R}\big)\cap R\subseteq\widetilde{U}\cap R=U.$$

So the implication $1\Rightarrow 2$ has been proved.

$2\Rightarrow 3$. The assertion 2 of Theorem 4 obviously follows from the assertion 2 of this theorem. Let $(\widetilde{R},\widetilde{\tau})$ be the topological ring which was constructed in the proof of the implication $2\Rightarrow 3$ in Theorem 4. Let us prove first that the topological ring $(\widetilde{R},\widetilde{\tau})$ is bounded from the left in every case mentioned in the condition of Theorem.

Let $\widetilde{W}(V,\widehat{V})\in\widetilde{\mathbf{B}}$ (see the proof of the implication $2\Rightarrow 3$ of Theorem 4). Then there exists a neighbourhood of zero $V_0\in\mathbf{B}$ such that $V_0-V_0+V_0-V_0\subseteq V$ and $\widehat{V}_0\in\widehat{\mathbf{B}}$ such that and $\widehat{V}_0-\widehat{V}_0\subseteq\widehat{V}$. Since the rings from the class $\Re$ are bounded from the left in every cases mentioned in the condition of Theorem then there exist neighbourhoods of zero $V_1\in\mathbf{B}$ and $\widehat{V}_1\in\widehat{\mathbf{B}}$ such that $V_1\cdot R\subseteq V_0$ and $\widehat{V}_1\cdot\widehat{R}\subseteq\widehat{V}_0$.

By the assertion 2 of present Theorem there exists a neighbourhood of zero $\widehat{V}_2\in\widehat{\mathbf{B}}$ such that $\widehat{V}_2\subseteq\widehat{V}_1$ and $\varphi^{-1}(\widehat{V}_2)\cdot R\subseteq V_0$.

Since the isomorphism $\varphi:(R,\tau)\to(\widehat{R},\widehat{\tau})$ is continuous then there exists a neighbourhood of zero $V_2\in\mathbf{B}$ such that $V_2\subseteq V_1$ and $\varphi(V_2)\subseteq\widehat{V}_2$. Then

$$\widetilde{W}(V_2,\widehat{V}_2)\cdot\widetilde{R}=\{(a-\varphi^{-1}(\widehat{a}),\widehat{a})\mid a\in V_2,\widehat{a}\in\widehat{V}_2\}\cdot\{(r,\widehat{r})\mid r\in R,\widehat{r}\in\widehat{R}\}=$$
$$\big\{\big((a-\varphi^{-1}(\widehat{a}))\cdot r,\varphi(a-\varphi^{-1}(\widehat{a}))\cdot\widehat{r}\big)\mid a\in V_2,\widehat{a}\in\widehat{V}_2,r\in R,\widehat{r}\in\widehat{R}\}=$$
$$\big\{\big((a-\varphi^{-1}(\widehat{a}))\cdot r+\varphi^{-1}(\varphi(a-\varphi^{-1}(\widehat{a}))\cdot\widehat{r})-\varphi^{-1}(\varphi(a-\varphi^{-1}(\widehat{a}))\cdot\widehat{r}),$$
$$\varphi(a-\varphi^{-1}(\widehat{a}))\cdot\widehat{r}\big)\mid a\in V_2,\widehat{a}\in\widehat{V}_2,r\in R,\widehat{r}\in\widehat{R}\}=$$

$$\{\big(a\cdot r-\varphi^{-1}(\widehat{a})\cdot r+a\cdot\varphi^{-1}(\widehat{r})-\varphi^{-1}(\widehat{a})\cdot\varphi^{-1}(\widehat{r})-\varphi^{-1}\big(\varphi(a)\cdot\widehat{r}-\\ \varphi^{-1}(\widehat{a}\cdot\widehat{r})\big),\varphi(a)\cdot\widehat{r}-\widehat{a}\cdot\widehat{r}\big)\mid a\in V_2,\widehat{a}\in\widehat{V}_2,r\in R,\widehat{r}\in\widehat{R}\}\subseteq\\ \{(b-\varphi^{-1}(\widehat{b}),\widehat{b})\mid b\in V,\widehat{b}\in\widehat{V}\}=\widetilde{W}(V,\widehat{V}),$$

since

$$b=a\cdot r-\varphi^{-1}(\widehat{a})\cdot r+a\cdot\varphi^{-1}(\widehat{r})-\varphi^{-1}(\varphi(a))\cdot\varphi^{-1}(\widehat{r})=\\ a\cdot r-\varphi^{-1}(\widehat{a})\cdot r+a\cdot\varphi^{-1}(\widehat{r})-\varphi^{-1}(\widehat{a})\cdot\varphi^{-1}(\widehat{r})\in\\ V_2\cdot R-\varphi^{-1}(\widehat{V}_2)\cdot R+V_2\cdot R-\varphi^{-1}(\widehat{V}_2)\cdot R\subseteq\\ V_1\cdot R-\varphi^{-1}(\widehat{V}_2)\cdot R+V_1\cdot R-\varphi^{-1}(\widehat{V}_2)\cdot R\subseteq V_0-V_0+V_0-V_0\subseteq V\text{ and}\\ \widehat{b}=\varphi(a)\cdot\widehat{r}-\widehat{a}\cdot\widehat{r}\in\varphi(V_2)\cdot\widehat{R}-\widehat{V}_1\cdot\widehat{R}\subseteq\widehat{V}_1\cdot\widehat{R}-\widehat{V}_1\cdot\widehat{R}\subseteq\widehat{V}_0-\widehat{V}_0\subseteq\widehat{V}.$$

The boundedness of the topological ring $(\widetilde{R},\widetilde{\tau})$ follows from that arbitrariness of $\widetilde{W}(V,\widehat{V})$.

To complete the proof of the implication $2\Rightarrow 3$ it is sufficient to check that $(\widetilde{R},\widetilde{\tau})\in\Re$ for every case mentioned in the condition of Theorem provided (R,τ) and $(\widehat{R},\widehat{\tau})\in\Re$.

Indeed, if $\Re$ is the class of all (separated) topological rings which are bounded from the left then it is so.

If $\Re$ is the class of all (separated) topological rings admitting a fundamental system of neighbourhoods of zero consisting of right ideals then topological rings (R,τ) and $(\widehat{R},\widehat{\tau})$ admit a fundamental system of neighbourhoods of zero consisting of subgroups. Hence by Theorem 4 the topological ring $(\widetilde{R},\widetilde{\tau})$ also admits a fundamental system of neighbourhoods of zero consisting of subgroups and it has been proved above that it is bounded from the left. Hence, by Theorem 1.6.32 in [3] the topological ring $(\widetilde{R},\widetilde{\tau})$ admits a fundamental system of neighbourhoods of zero consisting of right ideals, i.e. $(\widetilde{R},\widetilde{\tau})\in\Re$.

Let now $\Re$ be the class of all (separated) topological rings which are bounded. Then the topological rings (R,τ) and $(\widehat{R},\widehat{\tau})$ are bounded from the right and by Theorem 4 the topological ring $(\widetilde{R},\widetilde{\tau})$ is bounded from the right. It has been proved above that it is bounded from the left and hence is bounded, i.e. $(\widetilde{R},\widetilde{\tau})\in\Re$.

If $\Re$ is the class of all (separated) topological rings admitting a fundamental system of neighbourhoods of zero consisting of ideals then topological rings (R,τ) and $(\widehat{R},\widehat{\tau})$ admit a fundamental system of neighbourhoods of zero consisting of left ideals. Then by Theorem 4 the topological ring $(\widetilde{R},\widetilde{\tau})$ also admits a fundamental system of neighbourhoods of zero consisting of left ideals and it has been proved above that it is bounded from the left. Hence (R,τ) and $(\widehat{R},\widehat{\tau})$ are bounded and admit a fundamental system of neighbourhoods of zero consisting of subgroups. Then by Theorem 1.6.32 in [3] it admits a fundamental system of neighbourhoods of zero consisting of ideals, i.e. $(\widetilde{R},\widetilde{\tau})\in\Re$.

So we have proved that $(\widehat{R},\widehat{\tau})\in\Re$ in every case mentioned in the condition of Theorem. This completes the proof of the implication $2\Rightarrow 3$.

To complete the proof of Theorem is sufficient to check the implication $3 \Rightarrow 1$. It is obvious since the topological ring $(\widetilde{R}, \widetilde{\tau})$ mentioned in the assertion 3 of the current Theorem satisfies Definition 3. □

10 Remark. The following Theorem can be easily obtained from Theorem 9 proceeding to anti-isomorphic rings.

11 Theorem. *Let $\Re$ be one of the following classes of topological rings:*

1. The class of all (separated) topological rings which are bounded from the right;

2. The class of all (separated) topological rings admitting a fundamental system of neighbourhoods of zero consisting of left ideals of the ring;

3. The class of all (separated) topological rings which are bounded;

4. The class of all (separated) topological rings admitting a fundamental system of neighbourhoods of zero consisting of ideals of the ring.

Hence if (R, τ) and $(\widehat{R}, \widehat{\tau}) \in \Re$ and $\varphi : (R, \tau) \to (\widehat{R}, \widehat{\tau})$ is a continuous isomorphism then the following assertions are equivalent:

1. The isomorphism φ is semi-topological in the class $\Re$;

2. For every neighbourhood of zero U in (R, τ) there exists a neighbourhood of zero $\widehat{V}$ in $(\widehat{R}, \widehat{\tau})$ such that

$$R \cdot \varphi^{-1}(\widehat{V}) \subseteq U.$$

3. There exists a topological ring $(\widetilde{R}, \widetilde{\tau}) \in \Re$ such that the topological ring (R, τ) is a right ideal of the topological ring $(\widetilde{R}, \widetilde{\tau})$, the isomorphism φ can be extended to a topological homomorphism $\widetilde{\varphi} : (\widetilde{R}, \widetilde{\tau}) \to (\widehat{R}, \widehat{\tau})$ and $(\ker \widetilde{\varphi})^2 = \{0\}$.

12 Remark. The below Theorem is proved similarly to Theorem 3 in [1] and is its two-sided analogue.

13 Theorem. *If $\mathcal{K}$ is the class of all bounded topological rings or the class of all topological rings admitting a fundamental system of neighbourhoods of zero consisting of ideals then the continuous isomorphism $\varphi : (R, \tau) \to (\widehat{R}, \widehat{\tau})$ is semi-topological iff for every neighbourhood of zero U in (R, τ) there exists a neighbourhood of zero $\widehat{V}$ in $(\widehat{R}, \widehat{\tau})$ such that $\varphi^{-1}(\widehat{V}) \cdot R \subseteq U$ and $R \cdot \varphi^{-1}(\widehat{V}) \subseteq U$.*

14 Remark. The below assertion follows from Theorems 9 and 13 and Remark 12 of the present article.

15 Corollary. *Let $\Re$ be one of the following classes of topological rings:*

1. The class of all (separated) bounded topological rings;

2. The class of all (separated) topological rings, admitting a fundamental system of neighbourhoods of zero consisting of ideals of the ring.

Then if (R, τ) and $(\widehat{R}, \widehat{\tau}) \in \Re$ and the isomorphism $\varphi : (R, \tau) \to (\widehat{R}, \widehat{\tau})$ is semi-topological from the right in the class $\Re$ and is semi-topological from the left in the class $\Re$ then it is semi-topological.

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Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
Academiei str. 5, MD-2028 Chisinau
Moldova
e-mail: *arnautov@math.md*

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A test for completeness with respect to implicit reducibility in the chain super-intutionistic logics

I.V. Cucu

Abstract. We examine chain logics $C_2, C_3, \dots$, which are intermediary between classical and intuitionistic logics. They are also the logics of pseudo-Boolean algebras of type $< E_m, \&, \vee, \supset, \neg >$, where E_m is the chain $0 < \tau_1 < \tau_2 < \cdots < \tau_{m-2} < 1$ $(m = 2, 3, \dots)$. The formula F is called to be implicitly expressible in logic L by the system Σ of formulas if the relation

$$L \vdash (F \sim q) \sim ((G_1 \sim H_1) \,\&\, \dots \,\&\, (G_k \sim H_k))$$

is true, where q do not appear in F, and formulas G_i and H_i, for $i = 1, \dots, k$, are explicitly expressible in L via Σ. The formula F is said to be implicitly reducible in logic L to formulas of Σ if there exists a finite sequence of formulas $G_1, G_2, \dots, G_l$ where G_l coincides with F and for $j = 1, \dots, l$ the formula G_j is implicitly expressible in L by $\Sigma \cup \{G_1, \dots, G_{j-1}\}$. The system Σ is called complete relative to implicit reducibility in logic L if any formula is implicitly reducible in L to Σ.
The paper contains the criterion for recognition of completeness with respect to implicit reducibility in the logic C_m, for any $m = 2, 3, \dots$. The criterion is based on 13 closed pre-complete classes of formulas.

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The criterion of functional completeness in classical logic [1, 2] gives an algorithm which permits, for each finite system of Boolean functions given by formulas or tables, to recognise if it is possible to obtain any Boolean function via this system using superpositions or not. Analogous criteria of completeness have been obtained in general k-valued logic, $k > 2$ [2, 3], in propositional intuitionistic logic [4], etc. Each of these criteria is based on a finite number of closed (relative to expressibility in corresponding logic [5]) classes of functions or formulas that are pre-complete (i. e. maximal and non-complete).

In connection with the fact that in general 3-valued logic and even in its fragment – in logic of First Iaśkowski's Matrix [6] there is continuum of closed classes [4, 7]. A.V. Kuznethov [9] introduced the concepts of implicit expressibility, implicit reducibility and parametrical expressibility, which are natural generalizations of usual expressibility. He found a criterion for parametrical expressibility in any general k-valued logic for $k \geq 2$.

The research of the mentioned generalizations of expressibility in nonclassical logics is an actual problem. In the present paper the conditions of implicit reducibility of the set of all formulas in the chain super-intuitionistic logic, which is intermediate between classical and intuitionistic ones, are found. The criterion of completeness with respect to impicit reducibility in these logics is given. This criterion is based on 13-classes of formulas.

Formulas (of propositional logic) are constructed from variables p, q, r (possibly with indexes) by means of logical operations: & (conjunction), $\vee$ (disjunction), $\supset$ (implication), $\neg$ (negation). In the work the formulas are designated with capital letters of the Latin alphabet. Using the mark $\rightleftharpoons$, and reading it as "means" we introduce designations for seven formulas: $1 \rightleftharpoons (p \supset p)$, $0 \rightleftharpoons (p \,\&\, \neg p)$, $\perp\ F \rightleftharpoons (F \vee \neg F)$ (ternondation), $(F \sim G) \rightleftharpoons ((F \supset G) \,\&\, (G \supset F))$ (equivalence), $(F \cdot G) \rightleftharpoons ((F \sim G) \,\&\, \neg\neg G)$, $(F \,\&'\, G) \rightleftharpoons ((F \,\&\, G) \sim \perp (F \sim G)$ and $(F, G, H) \rightleftharpoons ((F \,\&\, G) \vee (F \,\&\, H) \vee (G \,\&\, H))$ (median). In the interpretation of formulas, the symbol $F[\alpha_1, \ldots, \alpha_n]$ designates the result of substitution in the formula F of the values $\alpha_1, \ldots, \alpha_n$ for variables $p_1, \ldots, p_n$, respectively.

Intuitionistical and classical propositional calculuses are based on the mentioned concept of formula. By these calculuses intuitionistical and classical logics are defined. Thus we determine the logic of that calculus as the set of all formulas deducible in the given calculus. The classical logic in this sense coincides, as it is known, with the set of formulas valid on the classical matrix.

In this paper we examine logics that are intermediary between classical logic and intuitionistic one [10, 11]. They are constructed on finite or infinite chains (i.e. linear ordered sets) of true values. It is known that the logic is called a chain [5] if the formula $((p \supset q) \vee (q \supset p))$ is true in it. In the considered m-valued logic $(m = 2, 3, \ldots)$ the variables will take values from the set E_m, where $E_m = \{0, 1, \tau_1, \tau_2, \ldots, \tau_{m-2}\}$ if m is finite and $E_m = \{0, 1, \tau_1, \tau_2, \ldots\}$ if m is infinite. Instead of τ_1 and τ_2 we will write τ and ω, respectively. We remind that the set of all functions as mappings from E_m into E_m is usually called general m-valued logic P_m. Further we consider the linear ordering on the set E_m by the relation $0 < \tau_1 < \tau_2 < \ldots \ldots < \tau_{m-2} < 1$. We define the operations $\&, \vee, \supset$, and $\neg$ on E_m as follows:

$$\begin{array}{l} p \,\&\, q = \min(p, q), \\ p \vee q = \max(p, q), \end{array} \qquad p \supset q = \begin{cases} 1 & \text{if } p \leq q, \\ q & \text{if } p > q, \end{cases} \qquad \neg\, p = p \supset 0.$$

In the considered interpretation of symbols $\&, \vee, \supset$ and $\neg$ each formula expresses some function of general m-valued logic. Let us observe that the function $\lrcorner\, p$ of P_3 defined by the equalities $\lrcorner\, 0 = \lrcorner\, \tau_1 = 1$ and $\lrcorner\, 1 = 0$ is not expressed by any formula. We remind [8, 12] that the pseudo-Boolean algebra is the system $\mathfrak{A} = < M; \&, \vee, \supset, \neg >$ that is a lattice by & and $\vee$, where $\supset$ is relative pseudo-complement and $\neg$ is pseudo-complement. The logic of this algebra is defined as the set of all formulas that are true on $\mathfrak{A}$, i.e. formulas identically equal to the greatest element 1 of this algebra. We will denote the algebra $< E_m; \&, \vee, \supset, \neg >$ $(m = 2, 3, \ldots)$ by Z_m. The logic of this algebra LZ_m is denoted by C_m. It is also possible to define the logic C_1 of one-element algebra which includes the set of all formulas and is contradictory. The smallest chain logic, called Dummett logic [10], coincides with the intersection of all m-valued chain logics with m positive integer number.

Two formulas F and G are called equivalent in logic L (write $L \vdash (F \sim G)$) if the equivalence $F \sim G$ in L is true. Two formulas are equivalent in logic C_m $(m = 1, 2, \ldots)$ if and only if the operators of algebra Z_m, expressed by them, are equal.

Therefore instead of the relation $C_m \vdash (F \sim G)$ we sometimes will use the equality $F = G$ on Z_m. If the formula $F \sim G$ contains only the variables $p_1, p_2, \ldots, p_n$ and the inequality $(F \sim G)\,[p1/\alpha_1, \ldots, p_n/\alpha_n] \neq 1$ is true on Z_m, then we will use the notation $(F \neq G)\,[p1/\alpha_1, \ldots, p_n/\alpha_n]$. The formula F is called explicitly expressible in logic L by the system of formulas of Σ [9] if it is possible to obtain the formula F from variables and formulas of Σ using a finite number of times the weak substitution rule, and the rule of replacement by equivalents in L. The relation of explicit expressibility is transitive. The formula F is called directly expressible via the system of formulas of Σ if it is possible to obtain F from variables and formulas of Σ by using a finite number of times the weak substitution rule. The relation of direct expressibility is transitive.

The formula F is called implicitly expressible in logic L [9] via the system of formulas Σ if there exist the formulas G_i and H_i $(i = 1, \ldots, k)$ explicitly expressible in L by Σ such that the predicate $L \vdash (F \sim q)$, where q is a variable not contained in F, is equivalent to the predicate $L \vdash ((G_1 \sim H_1) \,\&\, \ldots \,\&\, (G_k \sim H_k))$.

Because the relation of implicit expressibility, generally speaking, is not transitive, we are going to introduce a new concept. The formula F is called implicitly reducible in logic L via formulas of Σ if there exists a finite sequence of formulas $G_1, G_2, \ldots, G_l$, where G_l coincides with F and each term of this sequence can be implicitly expressible in L by Σ and terms of the sequence placed before it. We will say that the system Σ' of formulas is implicitly reducible in L to the system Σ if each formula of Σ' is implicitly reducible in L to Σ. It is clear that the relation of implicit reducibility is transitive. The system Σ of formulas is called complete with respect to implicit reducibility in logic L if each formula (in language of this logic) is implicitly reducible in L to Σ. The system Σ of formulas is said to be pre-complete with respect to implicit reducibility in L if Σ is not complete by this reducibility in L, but the system $\Sigma \cup \{F\}$ is complete relative to implicit reducibility in L, for any formula F.

Two functions $f(x_1, x_2, \ldots, x_n)$ and $g(x_1, x_2, \ldots, x_k)$ of P_m are called permutable [13] if the identity $f(g(x_{11}, \ldots, x_{1k}), \ldots, g(x_{n1}, \ldots, x_{nk})) = g(f(x_{11}, \ldots, x_{n1}), \ldots, f(x_{1k}, \ldots, x_{nk}))$ is true. The set of all functions of P_m, permutable with the given function f, is called the centralizer of function f (denoted $\prec f \succ$)[13]. The set of all formulas which in the interpretation on Z_m are permutable with the function f (from P_m) is called the formula centralizer on the algebra Z_m of function f. We say the function $f(x_1, \ldots, x_n)$ of P_m preserves the predicate (relation) $R(x_1, \ldots, x_w)$ if for any possible values of variables $x_{ij} \in E_m$ $(i = 1, \ldots, w;\ j = 1, ..n)$, from the truth of $R(x_{11}, x_{21}, \ldots, x_{w1}), \ldots, R(x_{12}, x_{22}, \ldots, x_{w2}), \ldots, R(x_{1n}, x_{2n}, \ldots, x_{wn})$ follows the truth of $R(f(x_{11}, x_{12}, \ldots, x_{1n}), \ldots, f(x_{21}, x_{22}, \ldots, x_{2n}), \ldots, f(x_{w1}, x_{w2}, \ldots, x_{wn}))$. The centralizer $\prec f(x_1, \ldots, x_n) \succ$ coincides with the set of all functions of P_m which preserve the predicate $f(x_1, \ldots, x_n) = x_{n+1}$, where the variable x_{n+1} differs from $x_1, \ldots, x_n$ [9]. We say that the formula F preserves, on the algebra Z_m, the predicate R if the function of logic C_m, expressed by formula F, preserves R. The predicate could be replaced by the corresponding to it matrix (α_{ij}) $(i = 1, \ldots, w;\ j = 1, \ldots, t)$ of elements of algebra Z_m [14] such that the predicate R is true on all those and only those sets of elements that are columns in this matrix. Let us observe that each formula of the system $\{p \,\&\, q, p \vee q, p \supset q, \neg p\}$ preserves on the algebra Z_m $(m = 3, 4, \ldots)$ the below predicates and matrices, therefore any formula preserves them too:

$$\neg x = \neg y, \quad x \neq \tau_j \quad (j = 1, 2, \ldots, m-2), \tag{1}$$

$$\begin{pmatrix} 0 & \tau & 1 \\ 0 & \tau_j & 1 \end{pmatrix} \quad (j = 1, 2, \ldots, m-2), \tag{2}$$

$$\begin{pmatrix} 0 & \tau & \omega & 1 \\ 0 & \tau_v & \tau_w & 1 \end{pmatrix} \quad (v,\, w = 1, 2, \ldots, m-2;\ v < w), \tag{3}$$

$$\begin{pmatrix} 0 & \tau_j & 1 & 1 \\ 0 & \tau_v & \tau_w & 1 \end{pmatrix} \quad (j,\, v,\, w = 1, 2, \ldots, m-2;\ v < w). \tag{4}$$

We present the next affirmation without any proof.

Affirmation. *If the function f belongs to the class C_m $(m = 2, 3, \ldots)$ then the following identity:*

$$f(\neg\neg x_1, \ldots, \neg\neg x_n) = \neg\neg f(x_1, \ldots, x_n) \tag{5}$$

is true.

Let us observe that the class of all formulas that preserve on Z_m some predicate is closed relative to the explicit expressibility in logic C_m, but it is not necessarily closed relative to the implicit expressibility in this logic [9]. It is easy to see that any class of formulas is closed relative to the implicit reducibility in logic C_m if and only if it is closed relative to the implicit expressibility. We remind that the centralizer of one or another function is closed relative to the implicit expressibility. It is obvious that for each $m = 1, 2, \ldots$, if the class of functions K is closed relative to the implicit expressibility in logic C_m then K is closed relative to the implicit expressibility in any logic C_n where $n \geq m$.

Let us define the functions f_1 and f_2 from P_4 as follows:

$$f_1(0) = 0, \quad f_1(\tau) = 1, \quad f_1(w) = \omega, \quad f_1(1) = 1,$$
$$f_2(0) = 0, \quad f_2(\tau) = \omega, \quad f_2(w) = \omega, \quad f_2(1) = 1.$$

We denote the classes of formulas preserving the predicates $x = 0,\ x = 1,\ \neg x = y,\ x \,\&\, y = z,\ x \vee y = z,\ (x \sim (y \sim z)) = u$ on Z_2, $\lrcorner\lrcorner x = y,\ \perp x = \perp y,\ (x \,\&\, y = z) \,\&\, (\neg x = \neg y), ((x \sim y) \,\&\, \neg\neg y = z) \,\&\, (\neg x = \neg y),\ ((x \,\&\, y) \sim ((x \sim y) \vee \neg(x \sim y)) = z) \,\&\, (\neg x = \neg y)$ on Z_3, $f_1(x) = y,\ f_2(x) = y$, respectively, on Z_4 by symbols $\Omega_0, \Omega_1, \ldots, \Omega_{12}$. Let us observe that the class Ω_5 on algebra Z_2 coincides with known class of linear Boolean functions. Remind that the closure relative to the implicit expressibility in C_2 of classes $\Omega_0, \ldots, \Omega_5$ is based on the fact that they are centralizers of some functions. Analogous closure in C_3 of classes $\Omega_6, \ldots, \Omega_{10}$ is shown in [15]. It follows that these classes are closed relative to the implicit expressibility in any other logic C_m, where $m \geq 3$.

Assertion 1 (A.V. Kuznetsov [9]). *In order that the system Σ of formulas could be complete by the implicit reducibility in logic C_2 it is necessary and sufficient that Σ be not included in any of clases $\Omega_0, \ldots, \Omega_5$.*

According to [15] the next criterion of completeness relative to the implicit reducibility in logic of First Iaśkowski's Matrix is true:

Assertion 2. *In order that the system Σ of formulas could be complete with respect to the implicit reducibility in logic C_3 it is necessary and sufficient that for each $i = 0, \ldots, 10$ should exist a formula of Σ which doesn't belong to the class Ω_i.*

The next criteria of completeness with respect to the reducibility in any chain logic included in C_4 are true:

Theorem 1. *For any $m = 4, 5, \ldots,$ in order that the system Σ of formulas could be complete by the implicit reducibility in logic C_m it is necessary and sufficient that Σ be complete by implicit reducibility in logic C_3 and be not included in the following two formula centralizers on algebra Z_4:*

$$\prec f_1(p) \succ, \quad \prec f_2(p) \succ. \tag{6}$$

Proof. *Necessity.* Let the system Σ be complete with respect to the implicit reducibility in logic C_m $(m \geq 4)$. Then, since the implicit reducibility in logic C_m $(m \geq 2)$ implies the implicit reducibility in C_{m-1}, it results that Σ is complete by the implicit reducibility in C_3. Because formula centralizers are closed relative to the implicit reducibility in logic C_4, then they are closed relative to the implicit reducibility in C_m where $m > 4$. Moreover, they are not complete in C_m, because they don't contain for example the formula $((x \supset y) \,\&\, \neg\neg y)$. So no one of them could contain Σ.

Sufficiency. Let Σ be complete by the implicit reducibility in logic C_3 and be not included in any of two formula centralizers (6). Then Σ is complete by the implicit reducibility in C_2, since there exist, in accordance with Assertion 2, the formulas $F_0, \ldots, F_{10}$ which don't belong to $\Omega_0, .., \Omega_{10}$, and also there exist F_{11}, F_{12}, which don't belong to Ω_{11}, Ω_{12}, respectively. Let us suppose that these formulas don't contain other variables except $p_1, \ldots, p_n$. It is sufficient to prove that every formula of system $\{p \,\&\, q,\ p \vee q,\ p \supset q,\ \neg p\}$ is implicitly reducible to the system Σ of formulas in C_m $(m = 4, 5, \ldots)$. It is known [10] that in any chain logic C_m the relation

$$C_m \vdash (p \vee q) \sim (((p \supset q) \supset q) \,\&\, ((q \supset p) \supset p))$$

is true. The conjunction is implicitly expressible via the implication in any chain logic C_m, because the relation

$$C_m \vdash ((p \,\&\, q) \sim r) \sim (((p \supset (q \supset r)) \sim 1) \& ((r \supset p) \supset 1) \& ((r \supset q) \sim 1))$$

is true. It remains to prove that the formulas $\neg p$ and $p \supset q$ are implicit reducible to the system Σ in any chain logic included in C_4.

This fact results from the next lemmas.

Lemma 1. *If the formula $\neg p$ is implicitly reducible to the system Σ of formulas in logic C_3 then this formula is implicitly reducible to Σ in logic C_m, for any $m = 3, 4, \ldots$*

Lemma 2. *If the formula 0 is implicitly reducible to the system Σ of formulas in logic C_2 then this formula is implicitly reducible to Σ in logic C_m, for any $m = 3, 4, \ldots$*

Lemma 3. *The formula 1 is explicitly expressible through 0 and $\neg p$ in C_m, for any $m = 3, 4, \ldots$*

Lemma 4. *If the formula $\perp p$ is implicitly reducible to the system Σ of formulas in logic C_3 then this formula is implicitly reducible to Σ in logic C_m, for any $m = 3, 4, \ldots$*

Lemma 5. *If the formula $p \,\&\, q$ is implicitly reducible to the system Σ of formulas in logic C_2 then the formula $\neg\neg(p \,\&\, q)$ is explicitly expressible through 0, 1, $\neg p$ and Σ in logic C_m, for any $m = 3, 4, \ldots$*

Lemma 6. *If the formula $\neg p \,\&\, q$ is implicitly reducible to the system Σ of formulas in logic C_3 then the formulas $\neg p \,\&\, q$ and $\neg p \vee q$ are implicitly expressible through* 0, 1, $\neg p$, $\perp p$, $\neg\neg(p \,\&\, q)$ *and Σ in the logic C_m, for any $m = 3, 4, \ldots$*

In order to obtain the implication we further present 5 lemmas without proofs.

Lemma 7. *At least one of 4 following formulas:*

$$p \supset q,\ p \sim q,\ \perp p \vee \perp q,\ (p \,\&\, q) \sim ((p \sim q)\ \vee\ \neg\,(p \sim q)) \tag{7}$$

is explicitly expressible in C_m through formulas of the system

$$\{0,\ 1,\ \neg p,\ \perp p,\ \neg p \,\&\, q,\ \neg p \vee q\} \tag{8}$$

and F_8, for any $m = 3, 4, \ldots$

Lemma 8. *At least one of 3 formulas:*

$$p \supset q,\ \perp p\ \vee \perp q,\ (p \,\&\, q) \sim ((p \sim q)\ \vee\ \neg\,(p \sim q)) \tag{9}$$

is explicitly expressible through formulas of the system (8) and F_8, F_9 in C_m, for any $m = 3, 4, \ldots$

Lemma 9. *At least one of following 4 systems:*

$$\{p \supset q\},\quad \{(p \sim q) \vee q\},\quad \{(p \,\&\, q) \sim ((p \sim q)\ \vee\ \neg\,(p \sim q)), T'\}, \{\perp p \vee \perp q, T'\}, \tag{10}$$

is explicitly expressible through formulas of the system (8) and F_8, F_9 and F_{10} in C_m, for any $m = 3, 4, \ldots$, where

$$T'[\tau, \tau, 1] = T'[\tau, 1, \tau] = \tau,\quad T'[\tau, 1, 1] = 1. \tag{11}$$

Lemma 10. *The implication $(p \supset q)$ is implicitly expressible in C_m, for any $m = 4, 5, \ldots$, through formulas of system (8), formula F_{11} and any of two systems $\{(p \sim q) \vee q\}$ or $\{(p \,\&\, q) \sim ((p \sim q)\ \vee\ \neg\,(p \sim q)), T'\}$, where T' is 3-ary formula, which satisfies (11) conditions.*

Lemma 11. *The formula $p \supset q$ is implicitly expressible in C_m, for any $m = 4, 5, \ldots$, through formulas of (8), formulas F_{11}, F_{12} and the system $\{\perp p \vee \perp q, T'\}$, where T' is the 3-ary formula satisfying conditions (11).*

From the formulated above lemmas it results that conditions of theorem are sufficient, namely the formula $\neg p$ is implicitly reducible to the system Σ of formulas in logic C_m, for any $m = 3, 4, \ldots$. Lemmas 1 – 11 allow us to deduce that the implication $p \supset q$ is implicitlty reducible to Σ in any chain logic C_m included in C_4. So, according to lemmas 1 – 6 the formulas of the system (8) are implicitly reducible to Σ. Lemmas 7 – 9 permit to conclude that at least one of 4 systems of formulas (10) is explicitly expressible in logic C_m through formulas (8) and F_8 – F_{10}. Therefore it remains to observe that one of these systems consists of $p \supset q$, but the implication is implicitly expressible in C_m through any other of 3 systems and formulas F_{11}, F_{12} and (8), in accordance with Lemmas 10 and 11.

From Assertion 1, 2 and Theorem 1 the next criterion of completeness with respect to implicit reducibility in an arbitrary chain logic results.

Theorem 2. *In order that the system of formulas Σ could be complete relative to the implicit reducibility in any chain logic L, including Dummett logic, it is necessary and sufficient that the next conditions be satisfied simultaneously:*

1) if $L \subseteq C_2$ then the system Σ is included neither in Ω_0, nor in Ω_1, nor in Ω_2, nor in Ω_3, nor in Ω_4, nor in Ω_5;

2) if $L \subseteq C_3$ then Σ is also included neither in Ω_6, nor in Ω_7, nor in Ω_8, nor in Ω_9, nor in Ω_{10};

3) if $L \subseteq C_4$ then Σ is also included neither in Ω_{11}, nor in Ω_{12}.

Proof. *Necessity* results from the fact that all these classes are closed relative to the implicit reducibility in C_m and are pairwise incomparable to inclusions.

Sufficiency. Let conditions 1)–3) be satisfied. Then the system Σ is complete relative to implicit reducibility in C_2 according to Assertion 1, and it is complete relative to the implicit reducibility in C_3 by Assertion 2 and it is complete relative to implicit reducibility in any chain logic C_m, included in C_4 according to Theorem 1.

From this criterion the next corollaries follow.

Theorem 3. *For any chain logic L (including Dummett logic) there exists an algorithm that allows to recognize for any finite system Σ of formulas if Σ is complete relative to implicit reducibility in logic L or not.*

From Assertion 1 it results that the classes $\Omega_0, \Omega_1, \ldots, \Omega_5$ and only they are pre-complete relative to implicit reducibility in C_2, and the classes $\Omega_0, \Omega_1, \ldots, \Omega_{10}$ and only they are pre-complete by implicit reducibility in C_3.

Theorem 4. *The next 13 classes: $\Omega_0, \Omega_1, \ldots, \Omega_{12}$ of formulas and only they are pre-complete relative to implicit reducibility in logic C_m, for any $m = 4, 5, \ldots$.*

A system Σ of formulas is called weak complete with respect to implicit reducibility in logic L if the system $\Sigma \cup \{p \supset p,\ p \& \neg p\}$ is complete by implicit reducibility in L.

Theorem 5 (criterion of weak completeness with respect to implicit reducibility in an arbitrary chain logic). *In order that the system Σ of formulas be weak complete relative to implicit reducibility in chain logic L it is necessary and sufficient that the next conditions be satisfied simultaneously:*

1) if $L \subseteq C_2$ then system Σ is included neither in Ω_3, nor in Ω_4, nor in Ω_5;

2) if $L \subseteq C_3$ then system Σ is also included neither in Ω_6, nor in Ω_7, nor in Ω_8, nor in Ω_9, nor in Ω_{10};

3) if $L \subseteq C_4$ then system Σ is also included neither in Ω_{11}, nor in Ω_{12}.

The logics L_1 and L_2 are called equal relative to completeness by implicit reducibility if any system Σ of formulas is complete by implicit reducibility in L_1 if and only if this system is complete by implicit reducibility in L_2.

Theorem 6. *Any chain logic is equal relative to completeness with respect to implicit reducibility to one and only one of the next 4 logics: the absolute contradictory logic, the classical logic, the logic C_3 and C_4 logic.*

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State University of Moldova
A. Mateevici str., 60
Chişinău MD-2009, Moldova
E-mail: *ion_cucu2005@yahoo.com*

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Dynamic Programming Approach for Solving Discrete Optimal Control Problem and its Multicriterion Version *

D. Drucioc, D. Lozovanu, M. Popovici

Abstract. Time discrete systems determined by systems of difference equations are considered. The characterizations of their optimal trajectories with given starting and final states is studied. An algorithm based on dynamic programming technique for determining such trajectories is proposed. In additional multicriterion version for considered control model is formulated and a general algorithm for determining Pareto solution is proposed.

Mathematics subject classification: 90B10, 90C35.
Keywords and phrases: Dynamic networks, discrete optimal control, game control model, Pareto solution.

1 Introduction and Problem formulation

In [1] the following discrete optimal control problem is formulated and studied.

Let L be the dynamical system with the set of the states $X \subseteq \mathbb{R}^n$ where at every moment of time $t = 0, 1, 2, \dots$ the state of L is $x(t) \in X$, $x(t) = (x_1(t), x_2(t), \dots, x_n(t)) \in \mathbb{R}^n$. The dynamics of the system L is described as follows

$$x(t+1) = g_t(x(t), u(t)),\ t = 0, 1, 2 \dots, \tag{1}$$

where

$$x(0) = x_s \tag{2}$$

is the starting point of system L and $u(t) = (u_1(t), u_2(t), \dots, u_m(t)) \in \mathbb{R}^m$ represents the vector of control parameters [2–4]. For vectors of control parameters $u(t)$, $t = 0, 1, 2$, the admissible sets $U_t(x(t))$ are given, i.e.

$$u(t) \in U_t(x(t)), t = 0, 1, 2, \dots\ . \tag{3}$$

We assume that in (1) the vector function

$$g_t(x(t), u(t)) = (g_t^1(x(t), u(t)), g_t^2(x(t), u(t)), \dots, g_t^n(x(t), u(t)))$$

is determined uniquely by $x(t)$ and $u(t)$. So, $x(t+1)$ is determined uniquely by $x(t)$ and $u(t)$ at every moment of time $t = 0, 1, 2, \dots$.

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Let

$$x(0), x(1), \dots, x(t), \dots \tag{4}$$

be a process generated according to (1)–(3).

For each state $x(t)$ we define the numerical determination $F_t(x(t))$ by using the following recursive formula

$$F_{t+1}(x(t+1)) = f_t(x(t), u(t), F_t(x(t))),\ t = 0, 1, 2, \dots$$

and

$$F_0(x(0)) = F_0.$$

In this model $F_t(x(t))$ expresses the cost of system's passage from x_0 to $x(t)$.

Optimization Problem 1. For a given T determine the vectors of control parameters $u(0), u(1), \dots, u(T-1)$, which satisfy the conditions

$$\begin{cases} x(t+1) = g_t(x(t), u(t)), t = 0, 1, 2, \dots, T-1; \\ x(0) = x_0, x(T) = x_f, \\ u(t) \in U_t(x(t)), t = 0, 1, 2 \dots, T-1; \\ F_{t+1}(x(t+1)) = f_t(x(t), u(t), F_t(x(t))), t = 0, 1, 2 \dots, T-1; \\ F_0(x(0)) = F_0 \end{cases} \tag{5}$$

and minimize the object function

$$I_{x_0 x(T)}(u(t)) = F_T(x(T)). \tag{6}$$

Optimization Problem 2. For given T_1 and T_2 determine $T \in [T_1, T_2]$ and a control sequence $u(0), u(1), .., u(T-1)$ which satisfy condition (5) and minimize the object function (6).

Remark 1 . It is obvious that the optimal solution of problem 2 can be obtained by reducing to problem 1 fixing the parameter $T = T_1$, $T = T_1 + 1, \dots, T = T_2$. By choosing the optimal value of solutions of problems of type 1 with $T = T_1, T = T_1 + 1, \dots, T = T_2$ we obtain the solution of problem 2 with $T \in [T_1, T_2]$.

It is easy to observe that a large class of dynamic optimization problems can be represented as a problem mentioned above. As example if

$$f_t(x(t), u(t), F_t(x(t))) = F_t(x(t)) + c_t(x(t), u(t)),$$

where $F_0(x_0) = 0$ and $c_t(x(t), u(t))$ represents the cost of system's passages from state $x(t)$ to state $x(t+1)$, then we obtain the discrete control problems with integral-time which are introduced and treated in [2–7]. Some classes of control problems from [2, 3] may be obtained if

$$F_0(x_0) = 1, \quad f_t(x(t), u(t), F_t(x(t))) = F_t(x(t)) \cdot c_t(x(t), u(t)),\ t = 1, 2, \dots$$

and if

$$F_0(x_0) = 0 \quad f_t(x(t), u(t), F_t(x(t))) = \max\{F_t(x(t)), c_t(x(t), u(t))\}.$$

In this paper we formulate the multicriterion version of the discrete control problem and derive an algorithm for determining Pareto solution. The proposed algorithm represents an extension of single objective problem and its algorithm.

2 Algorithm for determining optimal solution

Let us assume that the starting and final states are fixed, $f_t(x, u, F)$, $t = 0, 1, 2, \dots,$ are non-decreasing function with respect to the third argument, i.e. with respect to F.

$$f_t(x, u, F^{'}) \le f_t(x, u, F^{''}) \text{ if } F^{'} \le F^{''}. \tag{7}$$

Algorithm 1

1. Set $F_0^*(x(0)) = F_0; F_t^*(x(t)) = \infty; x(t) \in X, t = 1, 2, \dots; X_0 = \{x_0\}$.
2. For $t = 1, 2, \dots, T$ determine:

$$X_{t+1} = \{x(t+1) \in X \mid x(t+1) = g_t(x(t), u(t)),$$
$$x(t) \in X_t, u(t) \in U_t(x(t))\}$$

and for every $x(t+1) \in X_{t+1}$ determine

$$F_{t+1}^*(x(t+1)) = \min\{f_t(x(t), u(t), F_t^*(x(t))) \mid x(t+1) = g_t(x(t), u(t)),$$
$$x(t) \in X_t, u(t) \in U_t(x(t))\};$$

3. Find the sequence

$$x_T = x^*(T), x^*(T-1), x^*(T-2), \dots, x^*(1), x^*(0) = x_0,$$
$$u^*(T-1), u^*(T-2), \dots, u^*(1), u^*(0),$$

which satisfy the conditions

$$F_{T-\tau}^*(x^*(T-1)) = f_{T-\tau-1}(x^*(T-\tau-1), u^*(T-\tau-1),$$
$$F_{T-\tau-1}^*(x(T-\tau-1))),\ \tau = 0, 1, 2, \dots, T.$$

Then $u^*(0), u^*(1), u^*(2), \dots, u^*(T-1)$ represent the optimal solution of problem 1.

Theorem 1. *If $f_t(x, u, F)$, $t = 0, 1, 2, \dots, T$, are non-decreasing functions with respect to the third argument F, i.e. the functions $f_t(x, u, F)$, $t = 0, 1, 2 \dots, T$, satisfy condition (7), then the algorithm determines the optimal solution of problem 1. Moreover, an arbitrary leading part $x^*(0), x^*(0), \dots, x^*(k)$ of the optimal trajectory $x^*(0), x^*(0), \dots, x^*(k), \dots, x^*(T)$ is again an optimal one.*

Proof. We prove the theorem by using the induction principle on number of stages T. In the case $T \le 1$ the theorem is evident. We consider that the theorem holds for $T \le k$ and let us prove it for $T = k + 1$.

Assume by contrary that $u^*(0), u^*(1), \dots, u^*(T-2), u^*(T-1)$ is not an optimal solution of problem 1 and $u^{'}(0), u^{'}(1), \dots, u^{'}(T-2), u^{'}(T-1)$ is an optimal solution of problem 1, which differs from $u^*(0), u^*(1), \dots, u^*(T-2), u^*(T-1)$. Then $u^{'}(0), u^{'}(1), \dots, u^{'}(T-2), u^{'}(T-1)$ generate a trajectory $x_0 = x^{'}(0), x^{'}(1), \dots, x^{'}(T) = x_T$ with corresponding numerical evaluations of states

$$F^{'}_{t+1}(x^{'}(t+1)) = f_t(x^{'}(t), u^{'}(t), F^{'}_t(x^{'}(t))), t = 0, 1, 2, \dots, T-1;$$

where

$$F^{'}_0(x^{'}(0)) = F_0 \quad \text{and} \quad F^{'}_T(x^{'}(T)) < F^*_T(x^{'}(T)), \tag{8}$$

because $x^{'}(T) = x^*(T)$. According to the induction principle for problem 1 with $T-1$ stages the algorithm finds the optimal solution. So, for arbitrary $x(T-1) \in X$ we obtain the optimal evaluations $F^*_{T-1}(x(T-1))$ for $x(T-1) \in X$. Therefore

$$F^*_{T-1}(x^{'}(T-1)) \leq F^{'}_{T-1}(x^{'}(T-1)).$$

According to the algorithm

$$f_{T-1}(x^*(T-1), u^*(T-1), F^*_{T-1}(x^*(T-1))) \leq$$
$$\leq f_{T-1}(x^{'}(T-1), u^{'}(T-1), F^*_{T-1}(x^{'}(T-1))). \tag{9}$$

Since $f_t(x, u, F), t = 0, 1, 2\dots$ are non-decreasing functions with respect to F then

$$f_{T-1}(x^{'}(T-1), u^{'}(T-1), F^*_{T-1}(x^{'}(T-1))) \leq$$
$$\leq f_{T-1}(x^{'}(T-1), u^{'}(T-1), F^{'}_{T-1}(x^{'}(T-1))). \tag{10}$$

Using (9) and (10)we obtain

$$F^*_T(x(T)) = f_{T-1}(x^*(T-1), u^*(T-1), F^*_{T-1}(x^*(T-1))) \leq$$
$$\leq f_{T-1}(x^{'}(T-1), u^{'}(T-1), F^*_{T-1}(x^{'}(T-1))) \leq$$
$$\leq f_{T-1}(x^{'}(T-1), u^{'}(T-1), F^{'}_{T-1}(x^{'}(T-1))) = F^{'}_T(x(T)),$$

i.e

$$F^*_T(x(T)) \leq F^{'}_T(x(T)),$$

which contradicts (8). So the algorithm finds the optimal solution of problem 1 with $T = k+1$. □

Theorem 2. *Let X and $U_t(x)$, $x \in X$, $t = 0, 1, 2, \dots, T-1$, be the finite sets, and $M = \max_{x \in X, t=0,1,2,\dots,T-1} |U_t(x)|$. Then the algorithm uses at most $M \cdot |X| \cdot T$ elementary operations (without operations for calculating the values of functions $f_t(x, u, F)$ for given x, u, F).*

Proof. It is sufficient to prove that at step t the algorithm uses not more than $M \cdot |X|$ elementary operations. Indeed for finding the value $F_{t+1}(x(t+1))$ for $x(t+1) \in X$ it is necessary to use $\sum_{x \in X} |U_t(x)|$ operations. Since $\sum_{x \in X} |U_t(x)| \leq |X| \cdot M$ then at step t the algorithm uses not more than $|X| \cdot M$ elementary operations. So in general the algorithm uses not more than $|X| \cdot M \cdot T$ elementary operations. □

3 The discrete optimal control problem on network

Let L be a dynamical system with a finite set of states X, and at every moment of time $t = 0, 1, 2, \dots$ the system L is described by a directed graph $G = (X, E)$, where the vertices $x \in X$ correspond to the states of L and an arbitrary edge $e = (x, y) \in E$ identifies the possibility of the system passage from the state $x = x(t)$ to the state $y = x(t + 1)$. So, the set of edges $E(x) = \{e(x, y)|(x, y) \in E\}$ originated in $x(t)$ corresponds to an admissible set of control parameters $U_t(x(t))$ which determines the next possible state $y = x(t+1)$ of L at the moment of time t. Two states x_0 and x_f are chosen, where $x_0 = x(0)$ is the starting state and $x_f = x(T)$ is the final state of system L. In addition we assume that to each edge $e = (x, y) \in E$ a cost function $c_e(t)$ is associated which depends on time and which expresses the cost of system L to pass from the state $x = x(t)$ to the state $y = x(t + 1)$ at the stage $[t, t + 1]$ (like a *transition*). For given dynamic network we regard the problem of finding a sequence of system transitions $(x(0), x(1)), (x(1), x(2)), \dots, (x(T(x_f - 1)), x(T(x_f)))$ which transfers the system from the starting state $x_0 = x(0)$ to the final state $x_f = x(T(x_f))$ with minimal integral-time cost. Like in Section 1 we will discuss two variants of problem. First when time T is fixed and second when $T \in [T_1, T_2]$. It is easy to observe that for solving these problems we can use algorithm 1. We put $F_0(x(0)) = 0$ and $F_{t+1}(x(t + 1)) = F_t(x(t)) + c_{(x(t),x(t+1))}(t)$. A more general model is obtained if for each edge $e \in E$ a function $f_{e_t}(x(t), F_t(x(t)))$ is associated. Here we put $u(t) = e_t$ and we have the same function like in Section 1, i.e. $f_t(x(t), u(t), F_t(x(t))) = f_{e_t}(x(t), F_t(x(t)))$. For the trajectory $x(0), x(1), \dots, x(t), x(t + 1), \dots$ of system passages we have the following recursive formula $F_{t+1}(x(t + 1)) = f_{e_t}(x(t), F_t(x(t)))$, $t = 0, 1, 2, \dots$, and $F_0(x(0)) = F_0$.

4 Multicriterion Discrete Control Problem: Pareto Optimum

In this section we extend the control model from Section 1 using the concept of cooperative games.

4.1 General Statement of the Problem

We assume that the dynamics of the system L is controlled by p players, who coordinate their actions using the common vector of control parameters $u(t)$. So the dynamics of the system L is described according to (1)–(3).

Let $x(0), x(1), \dots, x(t), \dots$ be a process generated according to (1)–(3) with the given vector of control parameter $u(t)$, $t = 0, 1, 2, \dots$. For each state we define the quantities $F^i_t(x(t))$, $i = 1, 2, \dots, p$, in the following way:

$$F^i_{t+1}(x(t + 1)) = f^i_t(x(t), u(t), F^i_t(x(t))), \tag{10}$$

where

$$F^i_0(x(0)) = F^i_0,\ i = 1, 2, \dots, p, \tag{11}$$

are given representations of the starting state $x(0)$ of the system L; $f_t^i(x(t), u(t), F_t^i(x(t)))$, $t = 0, 1, 2, \dots$, are arbitrary functions. So, $F_t^i(x(t))$ expresses the cost of system's passage from the state $x(0)$ to the state $x(t)$ for player i.

In this model we assume that players choose vectors of control parameters in order to achieve the final state x_f from the starting state x_0 at the moment of time $T(x_f)$, where $T_1 \le T(x_f) \le T_2$.

For the given $u(t)$ the cost of system's passage from x_0 to x_f for player i is calculated on the basis of (1)–(3), (10), (11) and we put

$$I^i_{x_0 x_f}(u(t)) = F^i_{T(x_f)}(x_f),$$

if the trajectory passes through x_f at the time moment $T(x_f)$ such that $T_1 \le T(x_f) \le T_2$; otherwise we put

$$I^i_{x_0 x_f}(u(t)) = \infty.$$

We consider the problem of finding Pareto solution $u^*(t)$, i.e. there is no other vector $u(t)$ for which

$$\Big(I^1_{x_0 x_f}(u(t)), I^2_{x_0 x_f}(u(t)) \dots, I^p_{x_0 x_f}(u(t))\Big) \le$$

$$\le \Big(I^1_{x_0 x_f}(u^*(t)), I^2_{x_0 x_f}(u^*(t)) \dots, I^p_{x_0 x_f}(u^*(t))\Big)$$

and for any $i_0 \in \{1, 2, \dots, p\}$

$$I^{i_0}_{x_0 x_f}(u(t)) < I^{i_0}_{x_0 x_f}(u^*(t)).$$

4.2 Multicriterion Problem on Network and Algorithm for its Solving on T-Partite Networks

We formulate the multicriterion control model on network in general form on the basis of the control model from Section 3.

Let $G = (X, E)$ be a directed graph of transactions for the dynamical system L with the given starting state $x_0 \in X$ and the final state $x_f \in X$. In addition, for the state x_0 starting representations $F_0^1(x_0) = F_0^1$, $F_0^2(x_0) = F_0^2$, $\dots$, $F_0^p(x_0) = F_0^p$ are given, which express the payoff functions of players at the time moment $t = 0$. We define the control u^* on G as a map

$$u : (x, t) \to (y, t + 1) \in X_G(x) \times \{t + 1\} \quad \text{for} \quad x \in X \setminus \{x_f\},\ t = 1, 2, \dots.$$

For an arbitrary control u we define the quantities:

$$I^1_{x_0 x_f}(u),\ I^2_{x_0 x_f}(u),\ \dots, I^p_{x_0 x_f}(u)$$

in the following way.

Let

$$x_0 = x(0), x(1), x(2), \dots, x(T(x_f)) = x_f$$

be a trajectory from x_0 to x_f generated by control u, where $T(x_f)$ is the time moment when the state x_f is reached. Then we put

$$I^i_{x_0 x_f}(u) = F^i_{T(x_f)}(x_f) \quad \text{if} \quad T_1 \le T(x_f) \le T_2,\ i = \overline{1,p},$$

where $F^i_t(x(t))$ are calculated recursively by using the following formula

$$F^i_{t+1}(x(t+1)) = f^i_{(x(t),x(t+1))}(x(t), F^i_t(x(t))),\ t = \overline{0, T(x_f)-1};$$

$$F^i_0(x(0)) = F^i_0,$$

where $f^1_e(\cdot,\cdot),\ f^2_e(\cdot,\cdot),\ \ldots,\ f^p_e(\cdot,\cdot)$ are arbitrary functions. If $T(x_f) \notin [T_1, T_2]$ then we put

$$I^i(u) = \infty,\ i = \overline{1,p}.$$

We regard the problem of finding Pareto solution u^*.

In the following let us show that if the graph G has the structure of $(T+1)$-partite graph and $T_1 = T_2 = T$, then the algorithm from Sect. 2 can be extended for the multicriterion control problem on network.

So, assume that the vertex set X is represented as $X = Z_0 \cup Z_1 \cup \cdots \cup Z_T$, $Z_i \cap Z_j = \emptyset$, $i \ne j$, and the edge set E is divided into T non-empty subsets $E = E_0 \cup E_1 \cup \cdots \cup E_{T-1}$ such that an arbitrary edge $e = (y,z) \in E_t$ begins in $y \in Z_T$ and enters $z \in Z_{t+1}$, $t = \overline{0, T-1}$.

In this case for the nondecreasing function $f^i_e(\cdot,\cdot)$ with respect to the second argument the values $I^i(u) = F^i_t(x_t)$ can be calculated by using the following algorithm.

Algorithm 2

Preliminary step (Step 0): For the starting position $x(0) = x_0$ set $F^i_0(x(0)) = F^i_0$, $i = \overline{1,p}$; for any $x \in X \setminus \{x_0\}$ put $F^i_t(x(t)) = \infty$, $i = \overline{1,p}$, $t = \overline{1,T}$.

General step (Step t, $t \ge 0$): For an arbitrary state $x(t+1) \in X_{t+1}$ find a vertex $x'(t) \in X_t$ such that there is no other vertex $x(t) \in X_t \setminus \{x_f\}$ for which

$$\Big(f^1_{(x(t),x(t+1))}(x(t), F^1_t(x(t))), f^2_{(x(t),x(t+1))}(x(t), F^2_t(x(t))), \ldots$$

$$\ldots, f^p_{(x(t),x(t+1))}(x(t), F^p_t(x(t)))\Big) \le$$

$$\le \Big(f^1_{(x'(t),x(t+1))}(x'(t), F^1_t(x'(t))), f^2_{(x'(t),x(t+1))}(x'(t), F^2_t(x'(t))), \ldots$$

$$\ldots, f^p_{(x'(t),x(t+1))}(x'(t), F^p_t(x'(t)))\Big)$$

and

$$f^{i_0}_{(x(t),x(t+1))}(x(t), F^{i_0}_t(x(t))) < f^{i_0}_{(x'(t),x(t+1))}(x'(t), F^{i_0}_t(x'(t)))$$

for any $i_0 \in \{1, 2, \ldots, p\}$.

Then calculate

$$F^i_{t+1}(x(t+1)) = f^i_{(x'(t),x(t+1))}(x'(t), F^i_t(x'(t))),\ i = \overline{1,p}.$$

If $t < T - 1$ then go to the next step; otherwise STOP. $\square$

If $F^i_t(x(t))$ are known for every vertex $x(t) \in X$ then Pareto optimum u^* can be found starting from the end position x_f by fixing each time $u^*(x(t)) = x(t+1)$ for which

$$F^i_{t+1}(x(t+1)) = f^i_{(x(t),x(t+1))}(x(t), F^i_t(x(t))),\ i = \overline{1,p}.$$

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D. Lozovanu
Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
Academiei str. 5, MD-2028 Chisinau
Moldova
E-mail:*lozovanu@math.md*

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Measure of stability and quasistability to a vector integer programming problem in the l_1 metric*

Vladimir A. Emelichev, Olga V. Karelkina, Kirill G. Kuzmin

Abstract. This paper is devoted to a multicriterion vector integer programming problem with Pareto principle of optimality. Quantitative characteristics of two types of stability under perturbations of the vector criterion parameters with l_1 metric are obtained.

Mathematics subject classification: 90C10, 90C29, 90C31.
Keywords and phrases: Vector integer linear programming problem, Pareto set, efficient solution, stability, quasistability, stability and quasistability radii.

1 Introduction

Multiobjective combinatorial models of decision making are widespread in design, control, economics and many other fields of applied research. Therefore interest of mathematicians in multiobjective problems of discrete optimization keeps very high, as confirmed by the intensive publishing activity (see, for example, bibliography in [1]). One of the areas of investigations in such problems is stability of the problem solution to perturbations of initial data (of the problem parameters). Various settings of stability problem give rise to numerous directions of research. Not touching upon wide spectrum of questions appeared in this area we only refer to the extensive bibliography [2] and to the monographs [3–5].

Present work is concerned with investigations of quantitative characteristics of stability. Such a characteristic, usually called stability radius, is defined as the limit level of perturbations of the problem parameters, which save a given property of a solution set (or of a certain solution). The perturbed parameters are usually coefficients of the scalar or vector criterion. As a rule, the results of investigation of a stability radius are its formal expressions, estimations and algorithms of its calculation. In the case of a single objective, formulae of stability radius are obtained for problems of Boolean programming, problems on systems of subsets and on graphs [6], for some scheduling problems [5,7]. Such formulas are the basis of investigations for algorithmic aspects of the stability analysis of discrete optimization problems (see, for example,[8–10]).

Our research continues the cycle of works, devoted to the stability of the vector (multicriterion) integer programming problems [11–17]. In this paper we analyse the

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discrete analogues of the Hausdorff lower and upper semicontinuity of the Pareto optimal mapping to estimate limit levels of perturbations of the partial criteria coefficients mentioned above of the vector integer programming problem in the case of l_1 metric. Note that analogous results were obtained earlier in [15] for vector integer programming problem in the case of l_∞ metric.

2 Basic definitions

Consider n-criterion problem of the vector integer programming with m variables:

$$Cx = (C_1x, C_2x, \ldots, C_nx)^T \to \min_{x\in X},$$

where $C = [c_{ij}]_{n\times m} \in \mathbf{R}^{n\times m}$, $n, m \in \mathbf{N}$, C_i is i-th row of the matrix C, i.e. $C_i = (c_{i1}, c_{i2}, \ldots, c_{im})$, $i \in N_n = \{1, 2, \ldots, n\}$, X is a finite set of (feasible) solutions in $\mathbf{Z}^m$, $|X| > 1$.

Under a vector integer programming problem we understand the problem of finding the Pareto set, i.e. the set of efficient (Pareto optimal) solutions

$$P^n(C) = \{x \in X :\ \pi(x, C) = \emptyset\},$$

where $\pi(x, C) = \{x' \in X :\ Cx \geq Cx',\ Cx \neq Cx'\}$.

We denote this problem by $Z^n(C)$.

We also define the set of weakly efficient solutions (the Slater set [18])

$$Sl^n(C) = \{x \in X :\ \sigma(x, C) = \emptyset\}$$

and the set of strictly efficient solutions (the Smale set [19])

$$Sm^n(C) = \{x \in X :\ \eta(x, C) = \emptyset\},$$

where

$$\sigma(x, C) = \{x' \in X :\ C_ix > C_ix',\ \ i \in N_n\},$$
$$\eta(x, C) = \{x' \in X \setminus \{x\} :\ Cx \geq Cx'\}.$$

For any matrix $C \in \mathbf{R}^{n\times m}$ the following inclusions are evident

$$Sm^n(C) \subseteq P^n(C) \subseteq Sl^n(C).$$

It is obvious that in the case, where $n = 1$, the considered problem turns into ordinary scalar integer programming problem $Z^1(C)$, $C \in \mathbf{R}^m$, on the bounded set of feasible solutions. The Pareto set coincides with the Slater set ($P^1(C) = Sl^1(C)$) and they turn into the set of optimal solutions.

Adding a perturbing matrix $C' \in \mathbf{R}^{n\times m}$ to the matrix C, we model perturbations of parameters of the problem. Thus, perturbed problem $Z^n(C + C')$ has the form

$$(C + C')x \to \min_{x\in X}.$$

The Pareto set of this problem is $P^n(C+C')$. For any number $k \in \mathbf{N}$ we define two metrics l_1 and l_∞ in space $\mathbf{R}^k$, i.e. under norms of the vector $z = (z_1, z_2, \ldots, z_k) \in \mathbf{R}^k$ we understand correspondingly the numbers

$$||z||_1 = \sum_{i \in N_k} | z_i |, \qquad ||z||_\infty = \max\{| z_i |: \ i \in N_k\}.$$

This allows us formulate the question about quantitative characteristics of the stability. Later in p. 4 and 5 we deduce the corresponding formulas of the limit levels of perturbations. Under the norm of a matrix $C' = [c'_{ij}]_{n \times m}$ we understand the norm of the vector which consists of all elements of the matrix, i.e. the norm of the vector $(c'_{11}, c'_{12}, \ldots, c'_{n,m-1}, c'_{nm})$.

We define the set of perturbing matrices in the space with l_1 metric for an arbitrary number $\varepsilon > 0$:

$$\Omega(\varepsilon) = \{C' \in \mathbf{R}^{n \times m} : \ ||C'||_1 < \varepsilon\}.$$

Definitions 1–4 given below are well known (see, for example,[11, 13, 15], in that case the metric l_∞ is defined in the space of perturbing parameters of a vector integer programming problem).

Definition 1. *The vector integer programming problem $Z^n(C)$, $n \geq 1$, is called stable to perturbations of elements of matrix C if there exists a number $\varepsilon > 0$ such that for any perturbing matrix $C' \in \Omega(\varepsilon)$ the following inclusion holds:*

$$P^n(C + C') \subseteq P^n(C).$$

It is evident that the stability of the problem is equivalent to the Hausdorff upper semicontinuity [3, 4, 20] at the point $C \in \mathbf{R}^{n \times m}$ of the optimal mapping

$$P^n : \ \mathbf{R}^{n \times m} \to 2^X, \tag{1}$$

i.e. the point-to-set (set-valued) mapping that assigns the Pareto set $P^n(C)$ to each collection of the problem parameters from metric space $\mathbf{R}^{n \times m}$.

Let us consider a quantitative evaluation of stability .

Definition 2. *Under stability radius of the vector integer programming problem $Z^n(C)$, $n \geq 1$, we understand the number*

$$\rho_1^n(C) = \sup\{\varepsilon > 0 : \ \forall C' \in \Omega(\varepsilon) \ \ (P^n(C + C') \subseteq P^n(C))\}$$

if the problem $Z^n(C)$ is stable, and $\rho_1^n(C) = 0$ otherwise.

In other words, the stability radius of the problem $Z^n(C)$ is the limit level of perturbations of elements of matrix C in the space $\mathbf{R}^{n \times m}$ with metric l_1, which does not lead to appearance of new efficient solutions.

It is clear that the problem $Z^n(C)$ is always stable and its stability radius is equal to infinity if the equation $P^n(C) = X$ holds. The problem $Z^n(C)$, for which the set $\bar{P}^n(C) = X \setminus P^n(C)$ is non-empty, is called non-trivial.

Now consider the case where the stability of problem $Z^n(C)$ is defined as the discrete analogue of the Hausdorff lower semicontinuity at the point C of optimal mapping (1). For the vector integer programming problem, the lower semicontinuity means that there exists a neighborhood of the point C in space $\mathbf{R}^{n\times m}$ where the Pareto set can only expand.

Definition 3. *The vector integer programming problem $Z^n(C)$, $n \geq 1$, is called quasistable to perturbations of the elements of matrix C), if there exists a number $\varepsilon > 0$ such that for any perturbing matrix $C' \in \Omega(\varepsilon)$ the following inclusion holds*

$$P^n(C) \subseteq P^n(C + C').$$

Definition 4. *Under the quasistability radius of the vector integer programming problem $Z^n(C)$, $n \geq 1$, we understand the number*

$$\rho_2^n(C) = \sup\{\varepsilon > 0 : \ \forall C' \in \Omega(\varepsilon) \ \ (P^n(C) \subseteq P^n(C + C'))\},$$

if the problem $Z^n(C)$ is quasistable, and $\rho_2^n(C) = 0$ otherwise.

In that way, the quasistability radius determines the limit level of perturbations preserving all efficient solutions of the initial problem.

3 Auxiliary statements

For any solution $x \in \bar{P}^n(C)$ we define the set

$$P_x(C) = P^n(C) \cap \sigma(x, C).$$

The following properties are obvious.

Property 1. *If $P^n(C) = Sl^n(C)$, then $P_x(C) \neq \emptyset$ for any solution $x \in \bar{P}^n(C)$.*

By definition, put $[z]^+ = \max\{0, z\}$, where $z \in \mathbf{R}$.

Property 2. *If the inequality*

$$(C_i + C_i')(x - x') \leq 0 \tag{2}$$

holds for any index $i \in N_n$, then

$$[C_i(x - x')]^+ \leq ||C_i'||_1 ||x - x'||_\infty. \tag{3}$$

Clearly, inequality (3) holds for $C_i(x - x') \leq 0$. If $C_i(x - x') > 0$, then it follows from (2) and linearity of function $C_i(x - x')$ that

$$[C_i(x - x')]^+ = C_i(x - x') = (C_i + C_i')(x - x') - C_i'(x - x') \leq$$

$$\leq -C_i'(x - x') \leq ||C_i'||_1 ||x - x'||_\infty.$$

Property 3. *If $x \in \bar{P}^n(C)$ and*

$$P^n(C) \cap \sigma(x, C + C') = \emptyset, \tag{4}$$

then there exists a solution $x^* \in \bar{P}^n(C)$ such that $x^* \in Sl^n(C + C')$.

Since $\sigma(x, C + C') = \emptyset$, we can put $x^* = x$. If $\sigma(x, C + C') \neq \emptyset$, then taking into account external stability of the Slater set (see, for example,[18]) there exists a solution $x^* \in \sigma(x, C + C')$ such that $x^* \in Sl^n(C + C')$. It follows from (4) that $x^* \in \bar{P}^n(C)$.

Denote

$$\frac{C_p(x - x')}{C_q(x - x')}$$

by $\gamma(x, x', p, q)$ for any $p, q \in N_n$, $x \in \bar{P}^n(C)$, $x' \in P_x(C)$. It is clear that the values $\gamma(x, x', p, q)$ and $||C_p||_1$ are positive for any parameters x, x', p, q under the assumption $P^n(C) = Sl^n(C)$.

Lemma 1. *Let $P^n(C) = Sl^n(C)$, $x \in \bar{P}^n(C)$, $p, q \in N_n$ and number ψ be positive and such that*

$$||C_q||_1 \max\{\gamma(x, x', p, q) : \ x' \in P_x(C)\} \leq \psi. \tag{5}$$

Then for any number $\varepsilon > \psi$ there exist $C' \in \Omega(\varepsilon)$ and $x^* \in \bar{P}^n(C)$ such that

$$x^* \in Sl^n(C + C'). \tag{6}$$

Proof. It follows directly from Lemma that the inequalities

$$\varepsilon > \psi \geq ||C_q||_1 \zeta(x),$$

where $\zeta(x) = \max\{\gamma(x, x', p, q) : \ x' \in P_x(C)\}$ hold. According to Corollary 1, the set $P_x(C)$ is not empty. It is obvious that there exists number $\delta > 0$ such that

$$\varepsilon > (1 + \delta)||C_q||_1 \zeta(x) > \psi. \tag{7}$$

We define the perturbing matrix $C' = [c'_{ij}]_{n \times m}$ by

$$c'_{ij} = \begin{cases} -\alpha_j, & \text{if} \quad i = p, \ j \in N_m, \\ 0, & \text{if} \quad i \in N_n \setminus \{p\}, \ j \in N_m, \end{cases}$$

where $\alpha_j = (1 + \delta)\ c_{qj}\ \zeta(x)$. Hence, taking into account (7), we have

$$C' \in \Omega(\varepsilon),$$

$$C'_p = -(1 + \delta) C_q \zeta(x), \tag{8}$$

$$C'_i = (0, 0, \dots, 0) \in \mathbf{R}^m, \qquad i \in N_n \backslash \{p\}. \tag{9}$$

Let us show that equality (4) holds, i.e. there are no solutions from $P^n(C)$ belonging to $\sigma(x, C + C')$. Let $x^0 \in P^n(C)$.

Case 1: $x^0 \in P_x(C)$. Combining equality (8) and the definition of $\zeta(x)$, we obtain

$$(C_p + C'_p)(x - x^0) = C_p(x - x^0) - (1 + \delta)\zeta(x)C_q(x - x^0) \leq$$
$$\leq C_p(x - x^0) - (1 + \delta)\gamma(x, x^0, p, q)C_q(x - x^0) = -\delta C_p(x - x^0) < 0.$$

Hence, $x^0 \notin \sigma(x, C + C')$.

Case 2: $x^0 \in P^n(C) \setminus P_x(C)$. If there exists an index $s \in N_n \setminus \{p\}$ such that $C_s(x - x^0) < 0$, then it follows from (9) that the inequality $(C_s + C'_s)(x - x^0) < 0$ holds.

If for any index $i \in N_n \setminus \{p\}$ the inequality $C_i(x - x^0) \geq 0$ holds, then it follows from $x \in \bar{P}^n(C)$ and $x^0 \in P^n(C) \setminus P_x(C))$ that the inequality $C_p(x - x^0) < 0$ is true. Hence, we have from equality (8):

$$(C_p + C'_p)(x - x^0) \leq C_p(x - x^0) < 0.$$

Consequently we obtain $x^0 \notin \sigma(x, C + C')$ in this case.

Thus, equality (4) holds. Hence it follows from Corollary 3 that there exists a solution $x^* \in \bar{P}^n(C)$ such that inclusion (6) holds.

Lemma 1 is proved.

From Lemma 4.3 [4] (see also Theorem 3.2 [15]) we obtain

Lemma 2. *For any solution $x \in Sl^n(C) \setminus P^n(C)$ and for any number $\varepsilon > 0$ there exists a matrix $C^* \in \Omega(\varepsilon)$ such that $x \in P^n(C + C^*)$.*

Lemma 3. *Let for number ξ and for solutions x and x' the inequalities*

$$0 < \xi ||x - x'||_\infty \leq \sum_{i \in N_n} [C_i(x - x')]^+ \tag{10}$$

hold. Then for any perturbing matrix $C' \in \Omega(\xi)$ we have $x \notin \pi(x', C + C')$.

Proof. Suppose, to the contrary, that there exists a perturbing pair $C' \in \Omega(\xi)$ such that $x \in \pi(x', C + C')$. Then for any index $i \in N_n$ inequality (2) is valid. Hence, it follows from Corollary 2 that inequality (3) holds. Since $C' \in \Omega(\xi)$, we have

$$\sum_{i \in N_n} [C_i(x - x')]^+ \leq \sum_{i \in N_n} ||C'_i||_1 ||x - x'||_\infty =$$
$$= ||C'||_1 ||x - x'||_\infty < \xi ||x - x'||_\infty,$$

which gives a contradiction with condition (10).

Lemma 3 is proved.

Lemma 4. *Let $x, x' \in X$, $x \neq x'$. Let vector $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$ consist of positive elements such that*

$$\eta_i ||x - x'||_\infty > [C_i(x - x')]^+, \qquad i \in N_n. \tag{11}$$

Then for any number $\varepsilon > \|\eta\|_1$ there exists a perturbing matrix $C' \in \Omega(\varepsilon)$ such that $x \in \pi(x', C + C')$.

Proof. It is enough to build a perturbing matrix $C' \in \Omega(\varepsilon)$ such that

$$(C_i + C'_i)(x - x') < 0, \qquad i \in N_n. \tag{12}$$

Let $q = \arg\max\{|x_j - x'_j| : \ j \in N_n\}$. Define the elements of perturbing matrix $C' = [c'_{ij}]_{n\times m}$ by the formula

$$c'_{ij} = \begin{cases} \beta_i \ \mathrm{sign}(x'_q - x_q), & \text{if} \quad i \in N_m, \ j = q, \\ 0, & \text{if} \quad i \in N_n, \ j \neq q. \end{cases}$$

It is clear that $C' \in \Omega(\varepsilon)$. It follows from the above formula that

$$C'_i(x-x') = \sum_{j\in N_m} c'_{ij}(x_j - x'_j) = c'_{iq}(x_q - x'_q) = -\beta_i|x_q - x'_q| = -\beta_i||x-x'||_\infty, \quad i \in N_n$$

holds. Hence, combining the linearity of the function $C_i(x - x')$ and ratio (11), we prove inequalities (12):

$$(C_i + C'_i)(x - x') = C_i(x - x') + C'_i(x - x') =$$
$$= C_i(x - x') - \beta_i||x - x'||_\infty \leq [C_i(x - x')]^+ - \beta_i||x - x'||_\infty < 0, \qquad i \in N_n.$$

Lemma 4 is proved.

4 Stability radius

It is well known [21] (see also [3,4,13,15]) that necessary and sufficient condition for non-stability of the problem $Z^n(C)$ is that the Pareto set $P^n(C)$ does not coincide with the Slater set $Sl^n(C)$. In this case the stability radius of the problem $Z^n(C)$ is equal to zero.

It remains to consider the case where $P^n(C) = Sl^n(C)$.

Theorem 1. *Let*

$$P^n(C) = Sl^n(C),$$
$$\varphi = \min_{x\in \bar{P}^n(C)} \ \max_{x'\in P_x(C)} \ \min_{i\in N_n} \frac{C_i(x - x')}{||x - x'||_\infty},$$
$$\psi = \min_{x\in \bar{P}^n(C)} \ \min_{(i,k)\in N_n\times N_n} \ \max_{x'\in P_x(C)} \frac{C_i(x - x')}{C_k(x - x')}||C_k||_1.$$

Then the stability radius $\rho_1^n(C)$ of any non-trivial vector integer programming problem $Z^n(C)$, $n \geq 1$, has the following bounds: $0 < \varphi \leq \rho_1^n(C) \leq \psi$.

Proof. It follows from Corollary 1 that for any solution $x \in \bar{P}^n(C)$ the set $P_x(C)$ is nonempty. Hence, $\varphi > 0$.

We now prove that $\rho_1(C) \geq \varphi$. Let $C' \in \Omega(\varphi)$. Then it follows directly from the definition of φ that for any $x \in \bar{P}^n(C)$ there exists a solution $x^0 \in P_x(C)$ such that:

$$\|C'_i\|_1 \leq \|C'\|_1 < \varphi \leq \frac{C_i(x - x^0)}{\|x - x^0\|_\infty}, \quad i \in N_n$$

holds. Therefore, we have

$$(C_i + C_i')(x - x^0) \geq C_i(x - x^0) - \|C_i'\|_1 \|x - x^0\|_\infty > 0, \quad i \in N_n,$$

i.e. $x^0 \in \pi(x, C + C')$. Hence, $x \in \bar{P}^n(C + C')$. Thus, for any matrix $C' \in \Omega(\varphi)$ the inclusion $P^n(C + C') \subseteq P^n(C)$ holds. Consequently, $\rho_1^n(C) \geq \varphi$.

In particular, let us prove that $\rho_1^n(C) \leq \psi$. Suppose that $\varepsilon > \psi$. It follows from the definition of ψ that there exist indices $p, q \in N_n$ and solution $x \in \bar{P}^n(C)$ such that the inequality (5) is fulfilled. It follows from lemma 1 that for any number ε_1, where $\varepsilon > \varepsilon_1 > \psi > 0$, there exist $C' \in \Omega(\varepsilon_1)$ and $x^* \in \bar{P}^n(C)$ such that $x^* \in Sl^n(C + C')$.

There are only two cases.

Case 1: $x^* \in P^n(C + C')$. Since $x^* \in \bar{P}^n(C))$, it follows that $P^n(C + C') \nsubseteq P^n(C)$, $C' \in \Omega(\varepsilon)$.

Case 2: $x^* \in Sl^n(C + C') \setminus P^n(C + C')$. It follows from Lemma 2 that for $\varepsilon_2 := \varepsilon - \varepsilon_1 > 0$ there exists a matrix $C'' \in \Omega(\varepsilon_2)$ such that $x^* \in P^n(C + C' + C'')$. In other words, for any number $\varepsilon = \varepsilon_1 + \varepsilon_2 > \psi$ there exists matrix $C^0 = C' + C''$ such that $P^n(C + C^0) \nsubseteq P^n(C)$, $C^0 \in \Omega(\varepsilon)$.

Combining the results of considered above cases, we see that the inequality $\rho_1^n(C) < \varepsilon$ holds for any $\varepsilon > \psi$. Consequently, $\rho_1^n(C) \leq \psi$.

Theorem 1 is proved.

As corollaries of Theorem 1 and of the mentioned above criterion of stability of the non-trivial problem $Z^n(C)$, we obtain the following results.

Corollary 1. *For the stability radius of any non-trivial vector integer programming problem $Z^n(C)$, $n \geq 1$, we have*

$$\min_{x \in \bar{P}^n(C)} \max_{x' \in P^n(C) \cap \pi(x, C)} \min_{i \in N_n} \frac{C_i(x - x')}{||x - x'||_\infty} \leq \rho_1^n(C) \leq \min_{i \in N_n} \|C_i\|_1 \leq \|C\|_1. \tag{13}$$

If the lower bound is equal to zero, then $\rho^n(C) = 0$.

Proof. Suppose that $i = k$ in the expression ψ (see Theorem 1). Then

$$\rho_1^n(C) \leq \|C_i\|_1, \qquad i \in N_n.$$

Hence, the upper bound is valid in (13).

Now we show that

$$\rho_1^n(C) \geq \varphi', \tag{14}$$

where φ' is the left-hand side of (13).

At first we consider the case $P^n(C) \neq Sl^n(C)$. Then $\rho_1^n(C) = 0$. Let us show that $\varphi' = 0$. It is obvious that there exists solution $x^0 \in \bar{P}^n(C) \cap Sl^n(C)$. Therefore for any solution $x' \in P^n(C) \cap \pi(x, C)$ there exists an index $s \in N_n$ for which $C_s(x^0 - x') = 0$. Hence, $\varphi' = 0$.

Now we will consider the case $P^n(C) = Sl^n(C)$ and prove that $\varphi' = \varphi$ (see Theorem 1). By definition, put

$$\tau(x, x') = \min\Big\{ \frac{C_i(x - x')}{||x - x'||_\infty} : \ i \in N_n \Big\}.$$

According to the evident inclusion

$$P_x(C) \subseteq P^n(C) \cap \pi(x, C),$$

we define the set

$$Q(x) = (P^n(C) \cap \pi(x, C)) \setminus P_x(C).$$

From the above notations it is clear that

$$\tau(x, x') \begin{cases} = 0, & \text{if} \quad x' \in Q(x), \\ > 0 & \text{if} \quad x' \in P_x(C). \end{cases}$$

Since the set $P_x(C)$ is nonempty (in view of Corollary 1), it is clear that $\varphi' = \varphi$.

We will prove that $\rho_1^n(C) = 0$ for $\varphi' = 0$. In this case it follows directly from the definition of φ' that there exists a solution $x^0 \in \bar{P}^n(C)$ such that for any solution $x' \in P^n(C) \cap \pi(x^0, C)$ there exists an index $s \in N_n$ for which $C_s(x^0 - x') = 0$. In other words, there is no solution $x' \in P^n(C) \cap \pi(x^0, C)$ belonging to $\sigma(x^0, C)$. Hence, $P^n(C) \cap \sigma(x^0, C) = \emptyset$. Thus, it follows from corollary 3 that there exists a solution $x^* \in \bar{P}^n(C)$ such that $x^* \in Sl^n(C)$, i.e. $P^n(C) \neq Sl^n(C)$. According to the stability criterion of the problem $Z^n(C)$, we have that the problem $Z^n(C)$ is non-stable. Consequently, $\rho_1^n(C) = 0$.

Corollary 1 is proved.

From Corollary 1, we have the following statement.

Corollary 2. *Let the vector integer programming problem $Z^n(C)$, $n \geq 1$, be non trivial. Let matrix $C \in \mathbf{R}^{n \times m}$ contain at least one null row. Then the problem $Z^n(C)$ is nonstable.*

Corollary 3. *If the vector integer programming problem $Z^n(C)$, $n \geq 1$, has a unique efficient solution x^0, then*

$$\rho_1^n(C) = \min_{x \in \bar{P}^n(C)} \ \min_{i \in N_n} \frac{C_i(x - x^0)}{||x - x^0||_\infty}. \tag{15}$$

Proof. We denote the right-hand side of formula (15) by θ. It follows from Corollary 1 that the inequality $\rho_1^n(C) \geq \theta$ holds. Hence, to prove corollary 3 it is enough to show that $\rho_1^n(C) \leq \theta$.

Let $\varepsilon > \theta$. It follows from the definition of θ that there exist a solution $x^* \in \bar{P}^n(C)$ and an index $s \in N_n$ such that

$$C_s(x^* - x^0) = \theta ||x^* - x^0||_\infty. \tag{16}$$

Let $q = \arg\max\{|x^*_j - x^0_j| : \ j \in N_n\}$ and the elements of perturbing matrix $C' = [c'_{ij}]_{n\times m}$ be defined by the formula:

$$c'_{ij} = \begin{cases} \gamma \ \mathrm{sign}(x^0_q - x^*_q), & \text{if} \quad i = s, \ j = q, \\ 0, & \text{if} \quad i \in N_n \setminus \{s\}, \ j \in N_m \setminus \{q\}, \end{cases}$$

where $\theta < \gamma < \varepsilon$.

It is obvious that $C' \in \Omega(\varepsilon)$. From the construction of matrix C' we have

$$C'_s(x^* - x^0) = \sum_{j\in N_m} c'_{sj}(x^* - x^0) = c'_{sq}(x^* - x^0) = -\gamma|x^*_q - x^0_q| = -\gamma||x^* - x^0||_\infty.$$

From the above qualities and equality (16), we have

$$(C_s + C'_s)(x^* - x^0) = C_s(x^* - x^0) - \gamma||x^* - x^0||_\infty = (\theta - \gamma)||x^* - x^0||_\infty < 0,$$

i.e. $x^0 \notin \pi(x^*, C+C')$. If $\pi(x^*, C+C') = \emptyset$, then $x^* \in P^n(C+C')$. If $\pi(x^*, C+C') \neq \emptyset$, then due to external stability of the Pareto set $P^n(C + C')$ (see, for example,[18]) there exists a solution $\hat{x} \in \pi(x^*, C + C')$ such that $\hat{x} \in P^n(C + C')$.

Thus in the case, $P^n(C) = \{x^0\}$, for any number $\varepsilon > \theta$ there exist matrix $C' \in \Omega(\varepsilon)$ and solution $x' \neq x$ such that $x' \in P^n(C+C')$, i.e. $P^n(C+C') \not\subseteq P^n(C)$. Thus, for any number $\varepsilon > \theta$ the inequality $\rho^n_1(C) < \varepsilon$ holds. Hence, $\rho^n_1(C) \leq \theta$.

Corollary 3 is proved.

It follows from Corollary 3 that the lower bound φ in Theorem 1 is attainable for $|P^n(C)| = 1$.

Since $P^1(C) = Sl^1(C)$, as a corollary of Theorem 1, we have

Corollary 4. *Singlecriterion (scalar) integer programming problem $Z^1(C)$ ($C \in \mathbf{R}^m$) is always stable.*

5 Quasistability radius

Theorem 2. *The quasistability radius $\rho^n_2(C)$ of the vector integer programming problem $Z^n(C)$, $n \geq 1$, is expressed by the formula*

$$\rho^n_2(C) = \min_{x'\in P^n(C)} \ \min_{x\in X\setminus\{x'\}} \sum_{i\in N_n} \frac{[C_i(x - x')]^+}{||x - x'||_\infty}. \tag{17}$$

Proof. It is evident that the right-hand side of formula (17) is nonnegative for any matrix C. We denote it by ξ.

First let us prove the inequality $\rho^n_2(C) \geq \xi$. If $\xi = 0$, then the inequality is evident.

Let $\xi > 0$ and $C' \in \Omega(\xi)$. It follows from the definition of the value ξ that for any vectors $x' \in P^n(C)$ and $x \in X \setminus \{x'\}$ the inequality

$$\xi||x - x'||_\infty \leq \sum_{i\in N_n} [C_i(x - x')]^+$$

holds. Hence, it follows from Lemma 3 that $x \notin \pi(x', C + C')$, i.e. in view of $x' \notin \pi(x', C + C')$ the set $\pi(x', C + C')$ is non-empty. Therefore $x' \in P^n(C)$ belongs to the set $P^n(C+C')$ for any perturbing matrix $C' \in \Omega(\xi)$, i.e. $P^n(C) \subseteq P^n(C+C')$. Consequently, $\rho_2^n(C) \geq \xi$.

Now we show that $\rho_2^n(C) \leq \xi$. Let $\varepsilon > \xi$. It follows directly from the definition of the number ξ that there exist solutions $x' \in P^n(C)$ and $x \neq x'$ such that

$$\xi||x - x'||_\infty = \sum_{i \in N_n} [C_i(x - x')]^+.$$

Hence, it is obvious that there exist positive numbers $\eta_i, i \in N_n$, such that

$$\eta_i||x - x'||_\infty > [C_i(x - x')]^+, \quad i \in N_n, \qquad \varepsilon > \sum_{i \in N_n} \eta_i > \xi.$$

Thus, it follows from Lemma 4 that there exists a matrix $C' \in \Omega(\varepsilon)$ for which $x \in \pi(x', C + C')$, i.e. $x' \notin P^n(C + C')$. This means that the inequality $\rho_2^n(C) < \varepsilon$ holds for any number $\varepsilon > \xi$. Consequently, $\rho_2^n(C) \leq \xi$.

Theorem 2 is proved.

Any problem on a system of subsets of a finite set is equivalent to a boolean programming problem. Thus formula (17) easily moves to the well-known [17, 22] formula of the quasistability radius of the vector integer programming problem with linear criteria.

Corollary 5. *A necessary and sufficient condition for the quasistability of the vector integer programming problem $Z^n(C)$, $n \geq 1$, is the equality $P^n(C) = Sm^n(C)$ [3].*

Corollary 6. *Singlecriterion (scalar) vector integer programming problem $Z^1(C)$ ($C \in \mathbf{R}^m$) is quasistable if and only if it has a unique optimal solution.*

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Belarussian State University
ave. F. Skoriny, 4, Minsk 220050
Belarus
E-mail: *emelichev@bsu.by*
okarel@mail.ru; *kuzminkg@mail.ru*

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On determining the minimum cost flows in dynamic networks *

Maria Fonoberova

Abstract. The dynamic minimum cost flow problem that generalizes the static one is studied. We assume that the supply and demand function and capacities of edges depend on time. One very important case of the minimum cost flow problem with nonlinear cost functions, defined on edges, that do not depend on flow but depend on time is studied.

Mathematics subject classification: 90B10, 90C35, 90C27.
Keywords and phrases: Dynamic networks, dynamic flows, minimum cost flows.

1 Introduction and Problem Formulation

In this paper we study the dynamic version of the minimum cost flow problem on networks, which generalizes the well-known static minimum cost flow problem. We use dynamic network flow models instead of the static ones due to the fact that dynamic flows are much more closer to reality than static flows that can not properly consider the evolution of the system under study over time. For our problem the time is an essential component: the flows of some commodity take time to pass from one location to another and the structure of network changes over time.

The minimum cost flow problem on networks is of special interest not only from the practical point of view but it also has a great theoretical importance in the investigation and solving of various optimization problems on graphs. It can be used for the research and solving of the distribution problem, the synthesis problem of communication networks or the allocation problem. The field of applications of the considered problem considerably enlarges in the case when the cost functions are nonlinear.

In this paper we consider the flow problem on dynamic networks with nonlinear cost functions, defined on edges. Moreover, we assume that the supply and demand function and capacities of edges also depend on time. We study one very important case of the minimum cost flow problem with cost functions that do not depend on flow but depend on time and derive different approaches for its solving.

Let a dynamic network $N = (V, E, q, c, \tau, \varphi)$, which consists of directed graph $G = (V, E)$ with set of vertices $V = V_+ \cup V_- \cup V_0$, where V_+, V_- and V_0 are sets of sources, sinks and intermediate nodes, respectively, and set of arcs E, be given. Without loosing generality, we assume that no edges enter sources or exit sinks. We

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consider the discrete time model, in which all times are integral and bounded by horizon T, which defines the set $\mathbb{T} = \{0, 1, \dots, T\}$ of time moments we consider. The functions in network N are defined as follows: demand and supply function q: $V \times \mathbb{T} \to R$, capacity function c: $E \times \mathbb{T} \to R_+$, transit time function τ: $E \to R_+$, and cost function φ: $E \times \mathbb{T} \times R_+ \to R_+$. The demand and supply function $q_v(t)$ satisfies the following conditions:

a) there exists $v \in V$ with $q_v(0) < 0$;

b) if $q_v(t) < 0$ for a node $v \in V$ then $q_v(t) = 0$, $t = 1, 2, \dots, T$;

c) $\sum\limits_{t \in \mathbb{T}} \sum\limits_{v \in V} q_v(t) = 0$.

Nodes $v \in V$ with $\sum_{t \in \mathbb{T}} q_v(t) < 0$ are called sources, nodes $v \in V$ with $\sum_{t \in \mathbb{T}} q_v(t) > 0$ are called sinks and nodes $v \in V$ with $\sum_{t \in \mathbb{T}} q_v(t) = 0$ are called intermediate.

A dynamic flow on N is a function x: $E \times \mathbb{T} \to R_+$ that satisfies the following conditions:

$$\sum_{\substack{e \in E^+(v) \\ t - \tau_e \geq 0}} x_e(t - \tau_e) - \sum_{e \in E^-(v)} x_e(t) = q_v(t), \ \forall\, t \in \mathbb{T}, \ \forall\, v \in V; \tag{1}$$

$$x_e(t) = 0, \ \forall\, e \in E, \ t = \overline{T - \tau_e + 1, T}; \tag{2}$$

where $E^+(v) = \{(u, v) \,|\, (u, v) \in E\}$, $\quad E^-(v) = \{(v, u) \,|\, (v, u) \in E\}$.

Here the function x defines the value $x_e(t)$ of flow entering edge e at time t. It is easy to observe that the flow does not enter edge e at time t if it has to leave the edge after time T; this is ensured by condition (2). Conditions (1) represent flow conservation constraints.

Feasible dynamic flow also has to verify the following capacity constraints:

$$0 \leq x_e(t) \leq c_e(t), \quad \forall\, t \in \mathbb{T}, \ \forall\, e \in E. \tag{3}$$

Hereinafter we will show that the problem with capacity constraints can be reduced to the one without restrictions on edge capacities.

The considered problem consists in minimizing the integral cost F of transporting all the flow on N:

$$F = \sum_{e \in E} \sum_{t \in \mathbb{T}} \varphi_e(x_e(t), t) \to \min. \tag{4}$$

In the case when $\tau_e = 0$, $\forall\, e \in E$ and $T = 0$ the formulated problem becomes the classical problem on a static network.

2 The Time-Expanded Network Method

To solve the formulated dynamic problem we reduce it to a static one on an auxiliary time-expanded network N^T. The essence of such a network is that it contains copies of the vertices of the dynamic network for each moment of time, and the transit times and flows are implicit in the edges linking those copies. The time-expanded network $N^T = (V^T, E^T, c^T, q^T, \varphi^T)$ is defined as follows:

1. $V^T\colon\ = \{v(t)\,|\,v \in V,\ t \in \mathbb{T}\}$;
2. $E^T\colon\ = \{(v(t), w(t+\tau_e))\,|\,e = (v,w) \in E,\ 0 \le t \le T - \tau_e\}$;
3. $c^T_{e(t)}\colon\ = c_e(t)$ for $e(t) \in E^T$;
4. $\varphi^T_{e(t)}(x^T_{e(t)})\colon\ = \varphi_e(x_e(t), t)$ for $e(t) \in E^T$;
5. $q^T_{v(t)}\colon\ = q_v(t)$ for $v(t) \in V^T$.

If we define a flow correspondence to be $x^T_{e(t)}\colon\ = x_e(t)$, the minimum-cost flow problem on dynamic networks can be solved by using the solution of the static minimum cost flow problem on the time-expanded network. It is shown in [2] that for each minimum-cost flow in the dynamic network there is a corresponding minimum-cost flow in the static network and vice versa. In such a way, to solve the considered problem, we have to build the time-expanded network N^T for the given dynamic network N, to solve the classical minimum-cost flow problem on the static network N^T and to reconstruct the solution of the static problem on N^T to the dynamic problem on N.

Remark 1. In the case of the acyclic network the constructed time-expanded network can be reduced to the network of the smaller size, using the following algorithm, based on the process of elimination of irrelevant nodes from the time-expanded network [5]:

Algorithm

1. To build the time-expanded network N^{T^*} for the given dynamic network N.

2. To perform a breadth-first parse of the nodes for each source from the time-expanded network. The result of this step is the set $V_-(V_-^{T^*})$ of the nodes that can be reached from at least a source in V^{T^*}.

3. To perform a breadth-first parse of the nodes beginning with the sink for each sink and to parse the edges in the direction opposite to their normal orientation. The result of this step is the set $V_+(V_+^{T^*})$ of nodes from which at least a sink in V^{T^*} can be reached.

4. The reduced network will consist of a subset of nodes V^{T^*} and edges from E^{T^*} determined in the following way
$$V'^{T^*} = V^{T^*} \cap V_-(V_-^{T^*}) \cap V_+(V_+^{T^*}), \qquad E'^{T^*} = E^{T^*} \cap (V'^{T^*} \times V'^{T^*}).$$

5. ${q'}^{T^*}_{v(t)}\colon\ = q_v(t)$ for $v(t) \in V'^{T^*}$;

6. ${c'}^{T^*}_{e(t)}\colon\ = c_e(t)$ for $e(t) \in E'^{T^*}$;

7. ${\varphi'}^{T^*}_{e(t)}(x^T_{e(t)})\colon\ = \varphi_e(x_e(t), t)$ for $e(t) \in E^T$. $\square$

In the next section we derive the procedure of the reduction of the considered problem (1)–(4) to the one without condition (3).

3 Reduction of the Considered Problem to the One without Restrictions on Edge Capacities

The procedure of the reduction of the linear minimum-cost flow problem with restrictions on edge capacities to the one without restrictions on edge capacities was proposed in [1]. In this paper we derive this procedure for the minimum-cost flow problem with nondecreasing and nonnegative cost functions.

It is more optimal to reduce the considered dynamic problem to a static one and after that to reduce it to the problem without restrictions on edge capacities. Therefore let us consider problem (1)-(4) on the static network and let us show that this problem can be reduced to a problem on a new network H. For facility we will use the same notations as in the dynamic network but will discard all time information. The new network H will consist of set of vertices W, $|W| = n + m$, and set of arcs F, $|F| = 2m$. The graph (W, F) is a bipartite graph with two parts E and V, i.e. $W = E \bigcup V$ and there are only arcs leaving from vertices of set E and entering vertices of set V. By $[u, v]$ we denote a vertex which corresponds to arc (u, v) in G. If there is an arc (u, v) in G, then there are arcs $([u, v], u)$ and $([u, v], v)$ in F and $\varphi_{([u,v],u)}(x_{([u,v],u)}) = 0$ and $\varphi_{([u,v],v)}(x_{([u,v],v)}) = \varphi_{(u,v)}(x_{(u,v)})$. We associate value $c_{(u,v)}$ with vertices $[u, v]$ and value $\sum_{(u,v)\in E} c_{(u,v)} - q_v$ with vertices $v \in V$. In a new problem we have to find flows $x_{([u,v],w)}$ that solve the following problem:

$$\sum_{(u,v)\in E} \varphi_{([u,v],v)}(x_{([u,v],v)}) \to \min \tag{5}$$

$$x_{([u,v],u)} + x_{([u,v],v)} = c_{(u,v)} \tag{6}$$

$$\sum_{v\in V} [x_{([u,v],u)} + x_{([v,u],u)}] = \sum_{v\in V} c_{(u,v)} - q_u \tag{7}$$

$$x_{([u,v],w)} \geq 0 \tag{8}$$

Now we show that these problems are equivalent. Let us consider that flow $x_{(u,v)}$ is a feasible one for the initial problem. Set

$$x_{([u,v],v)} = x_{(u,v)} \tag{9}$$

$$x_{([u,v],u)} = c_{(u,v)} - x_{(u,v)} \tag{10}$$

In such a way flows in the new problem are nonnegative, so condition (8) is true. Besides, as

$$x_{([u,v],v)} + x_{([u,v],u)} = c_{(u,v)},$$

$$\sum_{v\in V} [x_{([u,v],u)} + x_{([v,u],u)}] = \sum_{v\in V} c_{(u,v)} - \sum_{v\in V} x_{(u,v)} + \sum_{v\in V} x_{(v,u)},$$

then conditions (6) and (7) are true.

Vice versa let us consider that the new problem is feasible. If we define the flow by formula (9), then it is obvious that condition (2) is true. Further in view of (6)

$$\sum_{v \in V} x_{(u,v)} - \sum_{v \in V} x_{(v,u)} = \sum_{v \in V} [x_{([u,v],v)} - x_{([v,u],u)}] =$$

$$= \sum_{v \in V} [c_{(u,v)} - x_{([u,v],u)}] - \sum_{v \in V} x_{([v,u],u)}.$$

Using (7) we reduce the first part of this equality to q_u and hence condition (1) is true.

It is evident that costs of feasible flows in these two problems are equal, so in such a way we reduced the problem with restrictions on edge capacities to the problem without restrictions on edge capacities.

We would like to note that the same argumentation can be held to solve the considered network problem in the case when there are two-side restrictions on edge capacity:

$$r_e \leq x_e \leq c_e, \quad \forall\, e \in E,$$

where r_e and c_e are lower and upper boundaries of the capacity of the edge e at time t correspondingly. This case can easily be reduced to the one with only one-side restrictions [1]. We introduce one additional artificial source b_1 and one additional artificial sink b_2. For every arc $e = (u, v)$, where $r_e \neq 0$ we introduce arcs (b_1, v) and (u, b_2) with r and 0 as the upper and lower boundaries of the capacity of the edges. We reduce c to $c - r$, but r to 0. We also introduce the arc (b_2, b_1) with $c_{(b_2,b_1)} = \infty$ and $r_{(b_2,b_1)} = 0$.

4 The Minimum Cost Flow Problem with Cost Functions that Do Not Depend on Flow

Further we will study the minimum cost flow problem without restrictions on edge capacities and with cost functions that do not depend on flow, i.e. when the cost functions are constant on the constructed time-expanded network. Obviously that in the case of constant cost functions the structure of optimal solution does not depend on flow distribution on network. When there is only one source and one sink the considered problem becomes a problem of finding the shortest path from a source to a sink. For solving this problem there is a plenty of algorithms [1, 3].

The formulated minimum cost flow problem with constant cost functions is related to network synthesis problems and Steiner trees. The network synthesis problem is formulated as follows. Let the graph $G = (V, E)$, $|V| = n$, with source $\tilde{v} \in V$ be given. Moreover with every arc $e \in E$ a length φ_e is associated. The problem consists in finding the graph $G^* = (V, E^*)$, $E^* \subset E$, in which there is a path from the vertex $\tilde{v}$ to every other vertex $u \in V \setminus \{\tilde{v}\}$ and the total length of its arcs is minimal.

It is easy to show that the optimal graph G^* is a tree rooted at the source. One of the methods for solving this problem is to generate all trees with the root $\tilde{v}$ that allow flows, to calculate the cost of the flow and to select the tree with the minimal cost. Evidently this method can be applied only in the case when the number of vertices $u \in V \setminus \{\tilde{v}\}$, for which $q_u > 0$ is not too big. Nevertheless this approach can be used for some practical problems.

The more general network synthesis problem is formulated as follows. Let the directed graph $G = (V, E)$, $|V| = n$ be given. With every arc $e \in E$ of this graph a length φ_e is associated. Moreover a subset of vertices $\widetilde{V}$, $|\widetilde{V}| = p$ $(p < n)$, is given, for which for every $u \in V \setminus \widetilde{V}$ there is a path $P(v, u)$ from the vertex $v \in \widetilde{V}$ to u. The problem consists in finding the graph $G^* = (V, E^*)$, $E^* \subset E$, which satisfies this condition and the total length of which is minimal. Evidently the optimal graph G^* is a tree with the base $\widetilde{V} \subset V$. An algorithm for finding the minimal tree with the given base is proposed in [6].

The particular case of the network synthesis problem is the Steiner problem, which is formulated as follows. Let the directed graph $G = (V, E)$ with the root vertex $\tilde{v}$ and the subset of vertices $U \subset V$, where a nonnegative length φ_e is associated with each arc $e \in E$, be given. It is necessary to find a tree $T^* = (V^*, E^*)$, $V^* \subset V$, that contains subset of vertices U, i. e. $U \subset V^*$, and for which the sum of lengths of its edges is minimal. In our problem subset U represents a set of stocks on the network. Though the problem of constructing Steiner tree is NP-complete, many heuristic algorithms have been designated to approximate the result within polynomial time [4, 7].

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Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
Academiei str. 5, MD-2028 Chisinau
Moldova
E-mail: *fonoberov@math.md*

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Functional bases of centro-affine invariants for the three-dimensional quadratic differential systems

Natalia Gherstega

Abstract. Functional bases of centro-affine invariants are constructed for the three-dimensional differential systems with polynomial right-hand sides of order less than three.

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Keywords and phrases: Differential system, Lie algebra of operators, functional basis of centro-affine invariants.

1 On the number of elements in a functional basis of invariants

It is known [1–3] that in the study of a polynomial differential system with the aid of Lie algebras and the orbit's theory an important role belongs to invariants and comitants of the systems [4–5]. Functional basis of invariants (comitants) should be especially mentioned. This can be explained by the fact that knowledge of Lie algebra of operators allows us to determine beforehand the exact number of elements in a minimal basis. In this article using the general theorem of algebraic invariants theory [4] functional bases of centro-affine invariants are studied for different three-dimensional differential systems with right-hand sides of order less than three.

Consider the three-dimensional differential system

$$\dot{x}^j = a^j + a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta \quad (j, \alpha, \beta = \overline{1,3}), \tag{1}$$

where the coefficient tensor $a^j_{\alpha\beta}$ is symmetrical in lower indices, in which the complete convolution holds, and the group of centro-affine transformations $GL(3,\mathbb{R})$: $\overline{x}^j = q^j_r x^r$ $(\Delta = \det(q^j_r) \neq 0;\ \ j, r = \overline{1,3})$.

The Lie algebra of operators [1] for linear representation of the group $GL(3,\mathbb{R})$ in the space of coefficients of system (1) is given by the following operators:

$$d_i = D_i^{(0)} + D_i^{(1)} + D_i^{(2)} \quad (i = \overline{1,9}), \tag{2}$$

where

$$D_1^{(0)} = a^1 \frac{\partial}{\partial a^1}, \quad D_2^{(0)} = a^2 \frac{\partial}{\partial a^2}, \quad D_3^{(0)} = a^3 \frac{\partial}{\partial a^3},$$

$$D_4^{(0)} = a^2 \frac{\partial}{\partial a^1}, \quad D_5^{(0)} = a^3 \frac{\partial}{\partial a^1}, \quad D_6^{(0)} = a^1 \frac{\partial}{\partial a^2},$$

$$D_7^{(0)} = a^3\frac{\partial}{\partial a^2}, \quad D_8^{(0)} = a^1\frac{\partial}{\partial a^3}, \quad D_9^{(0)} = a^2\frac{\partial}{\partial a^3}; \tag{3}$$

$$\begin{gathered}
D_1^{(1)} = a_2^1\frac{\partial}{\partial a_2^1} + a_3^1\frac{\partial}{\partial a_3^1} - a_1^2\frac{\partial}{\partial a_1^2} - a_1^3\frac{\partial}{\partial a_1^3},\\
D_2^{(1)} = -a_2^1\frac{\partial}{\partial a_2^1} + a_1^2\frac{\partial}{\partial a_1^2} + a_3^2\frac{\partial}{\partial a_3^2} - a_2^3\frac{\partial}{\partial a_2^3},\\
D_3^{(1)} = -a_3^1\frac{\partial}{\partial a_3^1} - a_3^2\frac{\partial}{\partial a_3^2} + a_1^3\frac{\partial}{\partial a_1^3} + a_2^3\frac{\partial}{\partial a_2^3},\\
D_4^{(1)} = a_1^2\frac{\partial}{\partial a_1^1} + (a_2^2 - a_1^1)\frac{\partial}{\partial a_2^1} + a_3^2\frac{\partial}{\partial a_3^1} - a_1^2\frac{\partial}{\partial a_2^2} - a_1^3\frac{\partial}{\partial a_2^3},\\
D_5^{(1)} = a_1^3\frac{\partial}{\partial a_1^1} + a_2^3\frac{\partial}{\partial a_2^1} + (a_3^3 - a_1^1)\frac{\partial}{\partial a_3^1} - a_1^2\frac{\partial}{\partial a_3^2} - a_1^3\frac{\partial}{\partial a_3^3},\\
D_6^{(1)} = -a_2^1\frac{\partial}{\partial a_1^1} + (a_1^1 - a_2^2)\frac{\partial}{\partial a_1^2} + a_2^1\frac{\partial}{\partial a_2^2} + a_3^1\frac{\partial}{\partial a_3^2} - a_2^3\frac{\partial}{\partial a_1^3},\\
D_7^{(1)} = -a_2^1\frac{\partial}{\partial a_3^1} + a_1^3\frac{\partial}{\partial a_1^2} + a_2^3\frac{\partial}{\partial a_2^2} + (a_3^3 - a_2^2)\frac{\partial}{\partial a_3^2} - a_2^3\frac{\partial}{\partial a_3^3},\\
D_8^{(1)} = -a_3^1\frac{\partial}{\partial a_1^1} - a_3^2\frac{\partial}{\partial a_1^2} + (a_1^1 - a_3^3)\frac{\partial}{\partial a_1^3} + a_2^1\frac{\partial}{\partial a_2^3} + a_3^1\frac{\partial}{\partial a_3^3},\\
D_9^{(1)} = -a_3^1\frac{\partial}{\partial a_2^1} - a_3^2\frac{\partial}{\partial a_2^2} + a_1^2\frac{\partial}{\partial a_1^3} + (a_2^2 - a_3^3)\frac{\partial}{\partial a_2^3} + a_3^2\frac{\partial}{\partial a_3^3};
\end{gathered} \tag{4}$$

$$\begin{gathered}
D_1^{(2)} = -a_{11}^1\frac{\partial}{\partial a_{11}^1} + a_{22}^1\frac{\partial}{\partial a_{22}^1} + a_{23}^1\frac{\partial}{\partial a_{23}^1} + a_{33}^1\frac{\partial}{\partial a_{33}^1} - 2a_{11}^2\frac{\partial}{\partial a_{11}^2} -\\
-a_{12}^2\frac{\partial}{\partial a_{12}^2} - a_{13}^2\frac{\partial}{\partial a_{13}^2} - 2a_{11}^3\frac{\partial}{\partial a_{11}^3} - a_{12}^3\frac{\partial}{\partial a_{12}^3} - a_{13}^3\frac{\partial}{\partial a_{13}^3},\\
D_2^{(2)} = -a_{12}^1\frac{\partial}{\partial a_{12}^1} - 2a_{22}^1\frac{\partial}{\partial a_{22}^1} - a_{23}^1\frac{\partial}{\partial a_{23}^1} + a_{11}^2\frac{\partial}{\partial a_{11}^2} + a_{13}^2\frac{\partial}{\partial a_{13}^2} -\\
-a_{22}^2\frac{\partial}{\partial a_{22}^2} + a_{33}^2\frac{\partial}{\partial a_{33}^2} - a_{12}^3\frac{\partial}{\partial a_{12}^3} - 2a_{22}^3\frac{\partial}{\partial a_{22}^3} - a_{23}^3\frac{\partial}{\partial a_{23}^3},\\
D_3^{(2)} = -a_{13}^1\frac{\partial}{\partial a_{13}^1} - a_{23}^1\frac{\partial}{\partial a_{23}^1} - 2a_{33}^1\frac{\partial}{\partial a_{33}^1} - a_{13}^2\frac{\partial}{\partial a_{13}^2} - a_{23}^2\frac{\partial}{\partial a_{23}^2} -\\
-2a_{33}^2\frac{\partial}{\partial a_{33}^2} + a_{11}^3\frac{\partial}{\partial a_{11}^3} + a_{12}^3\frac{\partial}{\partial a_{12}^3} + a_{22}^3\frac{\partial}{\partial a_{22}^3} - a_{33}^3\frac{\partial}{\partial a_{33}^3},
\end{gathered}$$

$$D_4^{(2)} = a_{11}^2\frac{\partial}{\partial a_{11}^1} + (a_{12}^2 - a_{11}^1)\frac{\partial}{\partial a_{12}^1} + a_{13}^2\frac{\partial}{\partial a_{13}^1} + (a_{22}^2 - 2a_{12}^1)\frac{\partial}{\partial a_{22}^1} + (a_{23}^2 - a_{13}^1)\frac{\partial}{\partial a_{23}^1} +$$

$$+ a_{33}^2\frac{\partial}{\partial a_{33}^1} - a_{11}^2\frac{\partial}{\partial a_{12}^2} - 2a_{12}^2\frac{\partial}{\partial a_{22}^2} - a_{13}^2\frac{\partial}{\partial a_{23}^2} - a_{11}^3\frac{\partial}{\partial a_{12}^3} - 2a_{12}^3\frac{\partial}{\partial a_{22}^3} - a_{13}^3\frac{\partial}{\partial a_{23}^3},$$

$$D_5^{(2)} = a_{11}^3\frac{\partial}{\partial a_{11}^1} + a_{12}^3\frac{\partial}{\partial a_{12}^1} + (a_{13}^3 - a_{11}^1)\frac{\partial}{\partial a_{13}^1} + a_{22}^3\frac{\partial}{\partial a_{22}^1} + (a_{23}^3 - a_{12}^1)\frac{\partial}{\partial a_{23}^1} +$$

$$+ (a_{33}^3 - 2a_{13}^1)\frac{\partial}{\partial a_{33}^1} - a_{11}^2\frac{\partial}{\partial a_{13}^2} - a_{12}^2\frac{\partial}{\partial a_{23}^2} - 2a_{13}^2\frac{\partial}{\partial a_{33}^2} - a_{11}^3\frac{\partial}{\partial a_{13}^3} - a_{12}^3\frac{\partial}{\partial a_{23}^3} - 2a_{13}^3\frac{\partial}{\partial a_{33}^3},$$

$$D_6^{(2)} = -2a_{12}^1\frac{\partial}{\partial a_{11}^1} - a_{22}^1\frac{\partial}{\partial a_{12}^1} - a_{23}^1\frac{\partial}{\partial a_{13}^1} + (a_{11}^1 - 2a_{12}^2)\frac{\partial}{\partial a_{11}^2} + (a_{12}^1 - a_{22}^2)\frac{\partial}{\partial a_{12}^2} +$$

$$+ (a_{13}^1 - a_{23}^2)\frac{\partial}{\partial a_{13}^2} + a_{22}^1\frac{\partial}{\partial a_{22}^2} + a_{23}^1\frac{\partial}{\partial a_{23}^2} + a_{33}^1\frac{\partial}{\partial a_{33}^2} - 2a_{12}^3\frac{\partial}{\partial a_{11}^3} - a_{22}^3\frac{\partial}{\partial a_{12}^3} - a_{23}^3\frac{\partial}{\partial a_{13}^3},$$

$$D_7^{(2)} = -a_{12}^1\frac{\partial}{\partial a_{13}^1} - a_{22}^1\frac{\partial}{\partial a_{23}^1} - 2a_{23}^1\frac{\partial}{\partial a_{33}^1} + a_{11}^3\frac{\partial}{\partial a_{11}^2} + a_{12}^3\frac{\partial}{\partial a_{12}^2} + (a_{13}^3 - a_{12}^2)\frac{\partial}{\partial a_{13}^2} +$$

$$+ a_{22}^3\frac{\partial}{\partial a_{22}^2} + (a_{23}^3 - a_{22}^2)\frac{\partial}{\partial a_{23}^2} + (a_{33}^3 - 2a_{23}^2)\frac{\partial}{\partial a_{33}^2} - a_{12}^3\frac{\partial}{\partial a_{13}^3} - a_{22}^3\frac{\partial}{\partial a_{23}^3} - 2a_{23}^3\frac{\partial}{\partial a_{33}^3},$$

$$D_8^{(2)} = -2a_{13}^1\frac{\partial}{\partial a_{11}^1} - a_{23}^1\frac{\partial}{\partial a_{12}^1} - a_{33}^1\frac{\partial}{\partial a_{13}^1} - 2a_{13}^2\frac{\partial}{\partial a_{11}^2} - a_{23}^2\frac{\partial}{\partial a_{12}^2} - a_{33}^2\frac{\partial}{\partial a_{13}^2} +$$

$$+ (a_{11}^1 - 2a_{13}^3)\frac{\partial}{\partial a_{11}^3} + (a_{12}^1 - a_{23}^3)\frac{\partial}{\partial a_{12}^3} + (a_{13}^1 - a_{33}^3)\frac{\partial}{\partial a_{13}^3} + a_{22}^1\frac{\partial}{\partial a_{22}^3} + a_{23}^1\frac{\partial}{\partial a_{23}^3} + a_{33}^1\frac{\partial}{\partial a_{33}^3},$$

$$D_9^{(2)} = -a_{13}^1\frac{\partial}{\partial a_{12}^1} - 2a_{23}^1\frac{\partial}{\partial a_{22}^1} - a_{33}^1\frac{\partial}{\partial a_{23}^1} - a_{13}^2\frac{\partial}{\partial a_{12}^2} - 2a_{23}^2\frac{\partial}{\partial a_{22}^2} - a_{33}^2\frac{\partial}{\partial a_{23}^2} +$$

$$+ a_{11}^2\frac{\partial}{\partial a_{11}^3} + (a_{12}^2 - a_{13}^3)\frac{\partial}{\partial a_{12}^3} + a_{13}^2\frac{\partial}{\partial a_{13}^3} + (a_{22}^2 - 2a_{23}^3)\frac{\partial}{\partial a_{22}^3} + (a_{23}^2 - a_{33}^3)\frac{\partial}{\partial a_{23}^3} + a_{33}^2\frac{\partial}{\partial a_{33}^3}. \tag{5}$$

According to [2] is proved the following

Theorem 1. *The polynomial $\theta(a)$ in the coefficients of the system (1) is a centro-affine invariant [3] of the system (1) with weight g iff the equalities*

$$d_i(\theta) = -g\theta \quad (i = \overline{1,3}), \quad d_j(\theta) = 0 \quad (j = \overline{4,9}), \tag{6}$$

hold, where d_i $(i = \overline{1,3})$ and d_j $(j = \overline{4,9})$ are the operators (2)–(5).

Definition 1. *The set of polynomial invariants $\{\theta_s(a),\ s \in B\}$ of the system (1) with respect to the $GL(3,\mathbb{R})$-group is called a functional basis of invariants of the system (1) with respect to this group if any invariant $\theta(a)$ of the system (1) with respect to the $GL(3,\mathbb{R})$-group can be written as a univocal function of the invariants $\theta_s(a)$. (Here B is some set of finite or transfinite natural numbers.)*

Definition 2. *A functional basis of invariants of the system (1) with respect to the $GL(3,\mathbb{R})$- group is called minimal if any invariant could not be removed out, overwise it is not a functional basis anymore.*

With the aid of Theorem 1 we obtaine

Lemma 1. *The number of elements μ in a functional basis of centro-affine invariants for the system (1) is equal to 22 (i.e. $\mu = 22$).*

Proof. We observe, according to equalities (6), that any invariant $\theta(a)$ satisfies a non-homogeneous linear system of partial differential equations of the first order. In the theory of equations (see for example [6]) it is known that the number of functionally independent solutions (invariants) of the system (6) is equal to

$$\mu = N - rank M_1 + 1, \tag{7}$$

where N is the number of coefficients in the system (1), and M_1 is the matrix, constructed on coordinate vectors of the operators (2)–(5). As for coefficients of the system (1) we have $N = 30$, and the general rank of the matrix M_1 is equal to 9, according to equality (7) we obtain that the number of functionally independent solutions (invariants) of the system (6) is equal to 22. Lemma 1 is proved.

Remark, with the aid of respective combinations of the operators (3)-(5), the truth of equalities of the type (7) can be showed for any subsystem of the system (1). Taking into consideration this fact and equality (7) we obtain that for μ holds

Lemma 2. *The number of elements μ in the basis of centro-affine invariants for the three-dimensional differential system is given in the Table 1.*

Table 1

μ	*Differential system*	*Number of the system*
0	$\dot{x}^j = a^j \quad (j = \overline{1,3})$	(8)
3	$\dot{x}^j = a^j_\alpha x^\alpha \quad (j,\alpha = \overline{1,3})$	(9)
10	$\dot{x}^j = a^j_{\alpha\beta} x^\alpha x^\beta \quad (j,\alpha,\beta = \overline{1,3})$	(10)
4	$\dot{x}^j = a^j + a^j_\alpha x^\alpha \quad (j,\alpha = \overline{1,3})$	(11)
13	$\dot{x}^j = a^j + a^j_{\alpha\beta} x^\alpha x^\beta \quad (j,\alpha,\beta = \overline{1,3})$	(12)
19	$\dot{x}^j = a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta \quad (j,\alpha,\beta = \overline{1,3})$	(13)

Remark 1. For the system (8) we have $\mu = 0$, as writing the equation (6) with the operators $D_i^{(0)}(\theta) = -g\theta \ \ (i = \overline{1,3}),\ \ D_j(\theta) = 0 \ \ (j = \overline{4,9})$ from (3) we obtain that the constant is the unique solution of this system. Such invariants will not be considered further.

To construct invariants of the system (1) and (9)–(13) we will use their notation with the aid of the convolution and alternation [4]. Further the unit three-vector ε^{pqr} with coordinates $\varepsilon^{123} = -\varepsilon^{132} = \varepsilon^{312} = -\varepsilon^{321} = \varepsilon^{231} = -\varepsilon^{213} = 1$ and $\varepsilon^{pqr} = 0 \ \ (p, q, r = \overline{1,3})$ will be used in other cases.

2 Centro-affine invariants of functional bases for the systems (9)–(13) and (1)

Theorem 2. *The expressions*

$$\theta_1 = a^\alpha_\alpha, \quad \theta_2 = a^\alpha_\beta a^\beta_\alpha, \quad \theta_3 = a^\alpha_\gamma a^\beta_\alpha a^\gamma_\beta, \tag{14}$$

form a functional basis of centro-affine invariants of the system (9).

Proof. We observe that the invariants (14) satisfy $D_i^{(1)}(\theta_j) = 0 \ \ (i = \overline{1,9};\ \ j = \overline{1,3})$, where $D_i^{(1)}$ is from (4). One can verify that the Jacobi matrix for the polynomials from (14) has the general rank 3. Hence the indicated invariants are functionally independent and according to Table 1 form a functional basis of centro-affine invariants of the system (9). Theorem 2 is proved.

From (5) with the aid of information from Table 1 we obtain

Theorem 3. *The expressions*

$$\begin{gathered}
i_1 = a^\alpha_{p\gamma} a^\beta_{q\alpha} a^\gamma_{r\beta} \varepsilon^{pqr}, \quad i_2 = a^\alpha_{ps} a^\beta_{qt} a^\gamma_{ru} a^\delta_{\alpha\delta} a^\mu_{\beta\mu} a^\nu_{\gamma\nu} \varepsilon^{pqr} \varepsilon^{stu}, \\
i_3 = a^\alpha_{\beta p} a^\beta_{\delta q} a^\gamma_{rs} a^\delta_{\nu t} a^\mu_{\gamma u} a^\nu_{\alpha\mu} \varepsilon^{pqr} \varepsilon^{stu}, \quad i_4 = a^\alpha_{\delta p} a^\beta_{\nu q} a^\gamma_{\beta r} a^\delta_{\mu s} a^\mu_{\gamma t} a^\nu_{\alpha u} \varepsilon^{pqr} \varepsilon^{stu}, \\
i_5 = a^\alpha_{ps} a^\beta_{qt} a^\gamma_{\alpha r} a^\delta_{\nu u} a^\mu_{\beta\mu} a^\nu_{\gamma\delta} \varepsilon^{pqr} \varepsilon^{stu}, \quad i_6 = a^\alpha_{ps} a^\beta_{qt} a^\gamma_{\delta r} a^\delta_{\alpha u} a^\mu_{\beta\nu} a^\nu_{\gamma\mu} \varepsilon^{pqr} \varepsilon^{stu}, \\
i_7 = a^\alpha_{ps} a^\beta_{qt} a^\gamma_{\beta r} a^\delta_{\mu u} a^\mu_{\gamma\nu} a^\nu_{\alpha\delta} \varepsilon^{pqr} \varepsilon^{stu}, \quad i_8 = a^\alpha_{ps} a^\beta_{qt} a^\gamma_{\alpha r} a^\delta_{\delta u} a^\mu_{\beta\gamma} a^\nu_{\mu\nu} \varepsilon^{pqr} \varepsilon^{stu}, \\
i_9 = a^\alpha_{ps} a^\beta_{qt} a^\gamma_{\beta r} a^\delta_{\gamma u} a^\mu_{\alpha\mu} a^\nu_{\delta\nu} \varepsilon^{pqr} \varepsilon^{stu}, \quad i_{10} = a^\alpha_{ps} a^\beta_{qt} a^\gamma_{\nu r} a^\delta_{\beta u} a^\mu_{\delta\mu} a^\nu_{\gamma\alpha} \varepsilon^{pqr} \varepsilon^{stu}
\end{gathered} \tag{15}$$

form a functional basis of centro-affine invariants of the system (10).

Jacobi matrix for the invariants $i_1 - i_{10}$ is calculated when

$$\begin{gathered}
a^1_{11} = a^1_{12} = a^1_{13} = a^1_{22} = 1, \quad a^1_{23} = 2, \quad a^1_{33} = 3, \quad a^2_{11} = -1, \\
a^2_{12} = 6, \quad a^2_{13} = -1, \quad a^2_{22} = 0, \quad a^2_{23} = 5, \quad a^2_{33} = 0, \\
a^3_{11} = a^3_{12} = a^3_{13} = 1, \quad a^3_{22} = 7, \quad a^3_{23} = 4, \quad a^3_{33} = 0,
\end{gathered}$$

its rank is equal to 10, that shows the functional independence of $i_1 - i_{10}$.

Using the operators $D_i^{(0)} + D_i^{(1)}$ $(i = \overline{1,9})$ from (3)–(4) and $\mu = 4$ from Table 1, is proved

Theorem 4. *The expressions (14) and*

$$\theta_4 = a^\alpha_\mu a^\beta_\alpha a^\gamma_\nu a^\delta a^\mu a^\nu \varepsilon_{\beta\gamma\delta} \tag{16}$$

form a functional bases of centro-affine invariants of the system (11).

Using the operators $D_i^{(0)} + D_i^{(2)}$ $(i = \overline{1,9})$ from (3), (5) and $\mu = 13$ from Table 1, is proved

Theorem 5. *The expressions (15) with any three invariants from the following four*

$$i_{11} = a^\alpha_{\alpha\beta}a^\beta, \quad i_{12} = a^\alpha_{\alpha\beta}a^\beta_{\gamma\delta}a^\gamma a^\delta, \quad i_{13} = a^\alpha_{\beta\gamma}a^\beta_{\alpha\delta}a^\gamma a^\delta, \quad i_{14} = a^\alpha_{\beta\nu}a^\beta_{\alpha\gamma}a^\gamma_{\delta\mu}a^\delta a^\mu a^\nu \tag{17}$$

form a functional basis of centro-affine invariants of the system (12).

Jacobi matrix for elements of a basis of centro-affine invariants of the system (12) from (15) and (17) is calculated when

$$a^1 = 3, \quad a^2 = 5, \quad a^3 = 7, \quad a^1_{11} = a^1_{12} = a^1_{13} = a^1_{22} = 1, \quad a^1_{23} = 2,$$

$$a^1_{33} = 3, \quad a^2_{11} = -1, \quad a^2_{12} = 6, \quad a^2_{13} = -1, \quad a^2_{22} = 0, \quad a^2_{23} = 5,$$

$$a^2_{33} = 0, \quad a^3_{11} = a^3_{12} = a^3_{13} = 1, \quad a^3_{22} = 7, \quad a^3_{23} = 4, \quad a^3_{33} = 0,$$

its rank is equal to 13, that shows the functional independence of the indicated invariants.

Remark 2. The invariants (17) for the system (12) are obtained from tensorial expressions of the comitants K_1, K_6, K_7 and K_{17}, respectively, from the monograph [4, p. 141–142] after the substitution x^- for a^-.

Using the operators $D_i^{(1)} + D_i^{(2)}$ $(i = \overline{1,9})$ from (4)–(5) and $\mu = 19$ from Table 1 is proved

Theorem 6. *The expressions (14), (15) and*

$$i_{15} = a^\alpha_p a^\beta_{\alpha q} a^\gamma_{\beta r} a^\delta_{\gamma\delta}\varepsilon^{pqr}, \quad i_{16} = a^\alpha_p a^\beta_{\delta q} a^\gamma_{\gamma r} a^\delta_{\alpha\beta}\varepsilon^{pqr}, \quad i_{17} = a^\alpha_p a^\beta_{\delta q} a^\gamma_{\alpha r} a^\delta_{\beta\gamma}\varepsilon^{pqr},$$

$$i_{18} = a^\alpha_\gamma a^\beta_{\alpha p} a^\gamma_{\beta q} a^\delta_{\delta r}\varepsilon^{pqr}, \quad i_{19} = a^\alpha_p a^\beta_\alpha a^\gamma_{\beta q} a^\delta_{\mu r} a^\mu_{\gamma\delta}\varepsilon^{pqr}, \quad i_{20} = a^\alpha_p a^\beta_\gamma a^\gamma_{\mu q} a^\delta_{\beta r} a^\mu_{\alpha\delta}\varepsilon^{pqr}, \tag{18}$$

form a functional basis of the centro-affine invariants of the system (13).

Jacobi matrix for elements of a basis of centro-affine invariants of the system (13) from (14), (15) and (18) is calculated when

$$a^1_1 = -7, \quad a^1_2 = 5, \quad a^1_3 = -9, \quad a^2_1 = 4, \quad a^2_2 = 5, \quad a^2_3 = 7,$$

$$a_1^3 = -3, \quad a_2^3 = -2, \quad a_3^3 = 5, \quad a_{11}^1 = a_{12}^1 = a_{13}^1 = a_{22}^1 = 1,$$

$$a_{23}^1 = 2, \quad a_{33}^1 = 3, \quad a_{11}^2 = -1, \quad a_{12}^2 = 6, \quad a_{13}^2 = -1, \quad a_{22}^2 = 0,$$

$$a_{23}^2 = 5, \quad a_{33}^2 = 0, \quad a_{11}^3 = a_{12}^3 = a_{13}^3 = 1, \quad a_{22}^3 = 7, \quad a_{23}^3 = 4, \quad a_{33}^3 = 0,$$

it's rank is equal to 19, that shows the functional independence of indicated invariants.

Remark 3. The invariants

$$i_{21} = a_\beta^\alpha a_{\alpha\gamma}^\beta a^\gamma, \quad i_{22} = a_\gamma^\alpha a_{\alpha\beta}^\beta a^\gamma, \quad i_{23} = a_\gamma^\alpha a_\delta^\beta a_{\alpha\beta}^\gamma a^\delta,$$

$$i_{24} = a_\beta^\alpha a_{\alpha\gamma}^\beta a_{\delta\mu}^\gamma a^\delta a^\mu, \quad i_{25} = a_\gamma^\alpha a_{\alpha\beta}^\beta a_{\delta\mu}^\gamma a^\delta a^\mu \tag{19}$$

for the system (1) are obtained from tensorial expressions of the comitants K_3, K_4, K_8, K_{12} and K_{13}, respectively, from the monograph [4, p. 141–142] after the substitution x^- for a^-.

Using the operators $D_i^{(0)} + D_i^{(1)} + D_i^{(2)}$ $(i = \overline{1,9})$ from (3)–(5), Remark 3 and the statement of Lemma 1 about the number of centro-affine invariants in functional basis (22 elements) for the system (1), is proved

Theorem 7. *The expressions (14)–(16) and (18), with i_{11} from (17) and i_{21} from (19) form a functional basis of centro-affine invariants of the system (1).*

Jacobi matrix for elements of a basis of centro-affine invariants of the system (1) from (14)–(16) and (18) with i_{11} from (17), i_{21} from (19) is calculated when

$$a^1 = 3, \quad a^2 = 5, \quad a^3 = 7, \quad a_1^1 = -7, \quad a_2^1 = 5, \quad a_3^1 = -9,$$

$$a_1^2 = 4, \quad a_2^2 = 5, \quad a_3^2 = 7, \quad a_1^3 = -3, \quad a_2^3 = -2, \quad a_3^3 = 5,$$

$$a_{11}^1 = a_{12}^1 = a_{13}^1 = a_{22}^1 = 1, \quad a_{23}^1 = 2, \quad a_{33}^1 = 3,$$

$$a_{11}^2 = -1, \quad a_{12}^2 = 6, \quad a_{13}^2 = -1, \quad a_{22}^2 = 0, \quad a_{23}^2 = 5,$$

$$a_{33}^2 = 0, \quad a_{11}^3 = a_{12}^3 = a_{13}^3 = 1, \quad a_{22}^3 = 7, \quad a_{23}^3 = 4, \quad a_{33}^3 = 0,$$

its rank is equal to 22, that shows the functional independence of indicated invariants.

One can verify

Remark 4. With the aid of the invariants (14)-(19) it is possible to construct other functional bases of centro-affine invariants of the system (1) consisting of 22 elements.

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Department of Mathematics
State University of Tiraspol
str. Gh. Iablocichin 5
MD-2069 Chisinau
Moldova
E-mail: *gherstega@mail.ru*

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Limits of solutions to the semilinear wave equation with small parameter

A. Perjan

Abstract. We study the existence of the limits of solution to singularly perturbed initial boundary value problem of hyperbolic - parabolic type with boundary Dirichlet condition for the semilinear wave equation. We prove the convergence of solutions and also the convergence of gradients of solutions to perturbed problem to the corresponding solutions to the unperturbed problem as the small parameter tends to zero. We show that the derivatives of solution relative to time-variable possess the boundary layer function of the exponential type in the neighborhood of $t = 0$.

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1 Introduction

Let $\Omega \in \mathbb{R}^3$ be an open and bounded set with the smooth boundary $\partial\Omega$. Consider the following initial boundary value problem for the wave equation, which in what follows will be called (P_ε):

$$\begin{cases} \varepsilon u_{tt}(x,t) + u_t(x,t) - \Delta u(x,t) + u^3(x,t) = f(x,t), \quad x \in \Omega, t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \overline{\Omega}, \\ u(x,t)\Big|_{x\in\partial\Omega} = 0, \quad t \geq 0, \end{cases}$$

where ε is a small positive parameter.

We will study the behaviour of the solutions to the problem (P_ε) as $\varepsilon \to 0$. It is natural to expect that the solutions to the problem (P_ε) tend to the corresponding solutions to the unperturbed problem (P_0):

$$\begin{cases} v_t(x,t) - \Delta v(x,t) + v^3(x,t) = f(x,t), \quad x \in \Omega, t > 0, \\ v(x,0) = u_0(x), \quad x \in \overline{\Omega}, \\ v(x,t)\Big|_{x\in\partial\Omega} = 0, \quad t \geq 0, \end{cases}$$

as $\varepsilon \to 0$. The main results are contained in Theorem 5. Under some conditions on u_0, u_1 and f we will prove that

$$u \to v \quad \text{in} \quad C([0,T]; L^2(\Omega)), \quad \text{as} \quad \varepsilon \to 0, \tag{1}$$

$$u \to v \quad \text{in} \quad L^\infty(0,T;H_0^1(\Omega)), \quad \text{as} \quad \varepsilon \to 0, \tag{2}$$

$$u' - v' - \alpha e^{-t/\varepsilon} \to 0 \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)), \quad \text{as} \quad \varepsilon \to 0, \tag{3}$$

where $\alpha = f(0) - u_1 + \Delta u_0 - u_0^3$. The relationship (3) shows that the derivative u' has the singular behaviour relative to the small values of the parameter ε in neighborhood of the set $\{(x,t) | x \in \Omega, t = 0\}$. It means that the set $\{(x,t) | x \in \Omega, t = 0\}$ is *the boundary layer* for u' and the function α is *the boundary layer function* for u'. The proofs of the relations (1), (2) and (3) are based on two key points. The first one is the relationship between the solutions to the problem (P_0) and (P_ε) in the linear case (see Lemma 3 and Theorem 3). The second key point represents *apriori* estimates of solutions to the problem (P_ε), which are uniform relative to small parameter ε (see Lemma 2).

The singularly perturbed nonlinear problems of hyperbolic-parabolic type were studied by many authors. Without pretending to the complete list of the works in this area, we mention here only the works [1] - [6] in which the larger references can be found.

In that follows we need to use some notations. Let X be a Banach space. For $k \in \mathbb{N}$, $p \in [1, \infty)$ and $(a,b) \subset (-\infty, +\infty)$ we denote by $W^{k,p}(a,b;X)$ the usual Sobolev spaces of the vectorial distributions $W^{k,p}(a,b;X) = \{f \in D'(a,b,X); f^{(l)} \in L^p(a,b;X), l = 0,1,\dots,k\}$ equipped with the norm

$$||f||_{W^{k,p}(a,b;X)} = \Big(\sum_{l=0}^{k} ||f^{(l)}||^p_{L^p(a,b;X)} \Big)^{1/p}.$$

For each $k \in \mathbb{N}, W^{k,\infty}(a,b;X)$ is the Banach space equipped with the norm

$$||f||_{W^{k,\infty}(a,b;X)} = \max_{0 \le l \le k} ||f^{(l)}||_{L^\infty(a,b;X)}.$$

In the following for $k \in \mathbb{N}$ we denote by $H^k(\Omega)$ $(L^2(\Omega) = H^0(\Omega))$ the usual real Hilbert spaces equipped with the following scalar products and norms:

$$\Big(u, v\Big)_{H^k(\Omega)} = \int_\Omega \sum_{|\alpha| \le k} \partial^\alpha u(x) \partial^\alpha v(x) dx, \quad [u,v] = (u,v)_{H_0^1(\Omega)},$$

$$(u,v) = \int_\Omega u(x) v(x) dx, \quad |u| = ||u||_{L^2(\Omega)}, \quad ||u|| = ||u||_{H_0^1(\Omega)}.$$

By $H^{-k}(\Omega)$ we denote the dual space to $H^k(\Omega)$, i.e. $H^{-k}(\Omega) = (H_0^k(\Omega))'$. We will write $\langle \cdot, \cdot \rangle$ to denote the pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

2 Solvability of the problems $(\mathbf{P}_\varepsilon)$ and $(\mathbf{P_0})$

First of all we shall remind the definitions of solutions to the problems (P_ε) and (P_0) and also the existence theorems for solutions to these problems.

Definition 1. *We say a function* $u \in L^2(0,T;H_0^1(\Omega))$ *with* $u' \in L^2(0,T;L^2(\Omega))$, $u'' \in L^2(0,T:H^{-1}(\Omega))$ *is a solution to the problem* (P_ε) *provided*

$$\varepsilon\langle u''(t),\eta\rangle + (u'(t),\eta) + [u(t),\eta] + (u^3(t),\eta) = (f(t),\eta), \quad \forall \eta \in H_0^1(\Omega), \tag{4}$$

a.e. $t \in [0,T]$ *and*

$$u(0) = u_0, \quad u'(0) = u_1. \tag{5}$$

Definition 2. *We say a function* $v \in L^2(0,T;H_0^1(\Omega))$ *with* $v' \in L^2(0,T;H^{-1}(\Omega))$ *is a solution to the problem* (P_0) *provided*

$$\langle v'(t),\eta\rangle + [v(t),\eta] + (v^3(t),\eta) = (f(t),\eta), \quad \forall \eta \in H_0^1(\Omega), \tag{6}$$

a.e. $t \in [0,T]$ *and*

$$v(0) = u_0. \tag{7}$$

Remark 1. *In view of the conditions* $u \in L^2(0,T;H_0^1(\Omega))$, $u' \in L^2(0,T;L^2(\Omega))$, *and* $u'' \in L^2(0,T;H^{-1}(\Omega))$ *we have* $u \in C([0,T];L^2(\Omega))$ *and* $u' \in C([0,T];H^{-1}(\Omega))$. *Consequently, we will understand the equalities (5) in the following sense:* $|u(t)-u_0| \to 0, ||u'(t)-u_1||_{H^{-1}(\Omega)} \to 0$ *as* $t \to 0$. *Similarly, in view of the conditions* $v \in L^2(0,T;H_0^1(\Omega))$ *with* $v' \in L^2(0,T:H^{-1}(\Omega))$, *we have* $v \in C([0,T];L^2(\Omega))$, *consequently, we will understand the equality (7) in the following sense:* $|v(t)-u_0| \to 0$ *as* $t \to 0$.

Theorem 1 [7]. *Let* $T > 0$. *If* $f \in W^{1,1}(0,T;L^2(\Omega))$, $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_1 \in H_0^1(\Omega)$, *then there exists a unique solution to the problem* (P_ε) *such that* $u \in W^{1,\infty}(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;H^2(\Omega))$, $u'' \in L^\infty(0,T;L^2(\Omega))$, $u''' \in L^\infty(0,T;H^{-1}(\Omega))$.

Theorem 2 [8]. *Let* $T > 0$. *If* $f \in W^{1,1}(0,T;L^2(\Omega))$, $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, *then there exists a unique solution* $v \in W^{1,\infty}(0,T;L^2(\Omega))$ *to the problem* (P_0) *and the estimates*

$$|v(t)| + \Big(\int_0^t ||v(\tau)||^2 d\tau\Big)^{1/2} + \Big(\int_0^t |v^2(\tau)|^2 d\tau\Big)^{1/2} \leq$$

$$\leq |u_0| + \int_0^t |f(\tau)| d\tau, \quad \forall t \in [0,T], \tag{8}$$

$$||v'||_{L^\infty(0,t;L^2(\Omega)} + \Big(\int_0^t ||v'(\tau)||^2 d\tau\Big)^{1/2} + \Big(\int_0^t (v'^2(\tau),v^2(\tau)) d\tau\Big)^{1/2} \leq$$

$$\leq |\Delta u_0 + f(0) - u_0^3| + \int_0^t |f'(\tau)| d\tau, \quad \forall t \in [0,T], \tag{9}$$

are true.

Remark 2. *If* $f \in W^{1,1}(0,T;L^2(\Omega)), u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_1 \in H_0^1(\Omega)$, *then according to the conclusion of Theorem 1 in fact* $u \in C^1([0,T];L^2(\Omega)) \cap$

$C([0,T];H_0^1(\Omega))$. Consequently, the term $\varepsilon\langle u''(t),\eta\rangle$ in (4) can be expressed in the form $\varepsilon(u''(t),\eta)$ and we will understand the equalities (5) in the following sense: $||u(t)-u_0||\to 0, |u'(t)-u_1|\to 0$ as $t\to 0$. Similarly, in view of the conclusion of Theorem 2, $v\in C([0,T];L^2(\Omega)), v'\in L^\infty(0,T;L^2(\Omega))$, the term $\langle v'(t),\eta\rangle$ in (6) can be expressed in the form $(v'(t),\eta)$.

3 Apriori estimates for solutions to the problem $(\mathbf{P}_\varepsilon)$

In this section we shall prove an *apriori* estimates for the solutions to the problem (P_ε) which are uniform relative to the small values of parameter ε. Before proving the estimates for the solutions to problem (P_ε) we recall the following well-known lemma.

Lemma 1 (see for example [9]). *Let $\psi\in L^1(a,b)(-\infty<a<b<\infty)$ with $\psi\ge 0$ a. e. on (a,b) and let c be a fixed real constant. If $h\in C([a,b])$ verifies*

$$\frac{1}{2}h^2(t)\le\frac{1}{2}c^2+\int_a^t\psi(s)h(s)ds,\quad\forall t\in[a,b],$$

then

$$|h(t)|\le|c|+\int_a^t\psi(s)ds,\quad\forall t\in[a,b],$$

also holds.

Denote by $u(t)=u(t,\cdot)$,

$$E_0(u,t)=\varepsilon|u'(t)|^2+|u(t)|^2+||u(t)||^2+2(1-\varepsilon)\int_0^t|u'(\tau)|^2d\tau+$$

$$+2\varepsilon(u(t),u'(t))+2\int_0^t||u(\tau)||^2d\tau+2\int_0^t|u^2(\tau)|^2d\tau+\frac{1}{2}|u^2(t)|^2$$

and

$$E_1(u,t)=\varepsilon^2|u'(t)|^2+\frac{1}{2}|u(t)|^2+\varepsilon||u(t))||^2+\varepsilon\Big(u'(t),u(t)\Big)+$$

$$+\varepsilon\int_0^t|u'(\tau)|^2d\tau+\int_0^t||u(\tau)||^2d\tau.$$

Lemma 2. *Let $f\in W^{1,1}(0,\infty;L^2(\Omega))$, $u_0\in H_0^1(\Omega)\cap H^2(\Omega)$, $u_1\in H_0^1(\Omega)$. Then there exists the positive constant $C=C(\Omega)$ such that for any solution u to the problem (P_ε) the following estimates*

$$E_0^{1/2}(u,t)\le CM_0,\quad t\in[0,\infty),\quad 0<\varepsilon<1,\tag{10}$$

$$E_1^{1/2}(u',t)\le CM_1,\quad a.e.\quad t\in[0,\infty),\quad 0<\varepsilon\le 1/2,\tag{11}$$

hold, where

$$M_0=M_0(||u_0||,|u_1|,||f||_{W^{1,1}(0,\infty;L^2(\Omega))}),\quad M_0(0,0,0)=0,$$

$$M_1 = M_1(||u_0||_{H^2(\Omega)}, ||u_1||, ||f||_{W^{1,1}(0,\infty;L^2(\Omega))}), \quad M_1(0,0,0) = 0. \tag{12}$$

If in addition $f \in W^{2,1}(0,\infty; L^2(\Omega))$, $u_1, \alpha \in H_0^1(\Omega) \cap H^2(\Omega)$, *then there exists* $\varepsilon_0 = \varepsilon_0(\Omega, M_0) \in (0,1)$ *such that the function*

$$z(t) = u'(t) + \alpha e^{-t/\varepsilon} \tag{13}$$

satisfies the estimate

$$||z||_{L^\infty(0,\infty;H_0^1(\Omega))} + ||z'||_{L^\infty(0,\infty;L^2(\Omega))} + ||z||_{W^{1,2}(0,\infty;H_0^1(\Omega))} \le CM_2, \tag{14}$$

for $0 < \varepsilon \le \varepsilon_0$, *where*

$$\alpha = f(0) - u_1 + \Delta u_0 - u_0^3, \tag{15}$$

$$M_2 = M_2(||f||_{W^{2,1}(0,\infty;L^2(\Omega))}, ||u_1||_{H^2(\Omega)}, ||\alpha||_{H^2(\Omega)}), \quad M_2(0,0,0) = 0. \tag{16}$$

Proof. In what follows let us agree to denote all constants depending only on Ω by the same constant C. The direct computations show that for every solution to the problem (P_ε) the following equality

$$\frac{d}{dt}E_0(u,t) = 2\Big(f(t), u(t) + u'(t)\Big), \quad a.e. \quad t \in [0,\infty) \tag{17}$$

is fulfilled. For $\varepsilon \in (0,1)$ we have that $E_0(u,t) \ge 0$ and $|u(t)| \le (E_0(t,u))^{1/2}$. Then integrating the equality (17) on $(0,t)$ we get

$$E_0(u,t) = E_0(u,0) + 2\Big(f(t) - f(0), u(t)\Big) + 2\Big(f(0), u(t) - u(0)\Big) +$$

$$+2\int_0^t \Big(f(\tau) - f'(\tau), u(\tau)\Big)d\tau \le E_0(u,0) + \frac{1}{2}|u(t)|^2 + 8\Big(\int_0^t |f'(\tau)|d\tau\Big)^2 +$$

$$+|u_0|^2 + 9|f(0)|^2 + 2\int_0^t \Big(|f(\tau)| + |f'(\tau)|\Big)E_0^{1/2}(u,\tau)d\tau, \quad t \in [0,\infty).$$

From the last inequality we have that

$$E_0(u,t) \le 2(c_0+5)^2 M_0^2 +$$

$$+4\int_0^t \Big(|f(\tau)| + |f'(\tau)|\Big)E_0^{1/2}(u,\tau)d\tau\Big), \quad t \in [0,\infty), \tag{18}$$

where c_0 is the constant from the inequality $|u|^2 \le c_0||u||^2, u \in H_0^1(\Omega)$. Since $E_0(u,t) \in C([0,\infty))$ due to Lemma 1, from (18) the estimate (10) follows.

To prove the estimate (11) let us denote by $u_h(t) = h^{-1}(u(t+h) - u(t))$, $h > 0$. For any solution of the problem (P_ε) the equality

$$\frac{d}{dt}E_1(u_h,t) = \Big(F_h(t), 2\varepsilon u_h'(t) + u_h(t)\Big), \quad a.e. \quad t \in [0,\infty),$$

is true, where

$$F_h(t) = f_h(t) - u_h(t)\Big(u^2(t+h) + u(t+h)u(t) + u^2(t)\Big).$$

Integrating the last equality on $(0,t)$, we obtain

$$E_1(u_h,t) = E_1(u_h,0) + \int_0^t \Big(F_h(\tau), 2\varepsilon u_h'(\tau) + u_h(\tau)\Big)d\tau, \quad t \in [0,\infty).$$

As $|u_h(\tau) + 2\varepsilon u_h'(\tau)| \le 2E_1^{1/2}(u_h,\tau)$, then from the last equality we get

$$E_1(u_h,t) \le E_1(u_h,0) + 2\int_0^t |F_h(\tau)|E_1^{1/2}(u_h,\tau)d\tau, \quad t \in [0,\infty).$$

Using Lemma 1, from the last inequality we obtain the estimate

$$E_1^{1/2}(u_h,t) \le E_1^{1/2}(u_h,0) + \int_0^t |F_h(\tau)|d\tau, \quad t \in [0,\infty). \tag{19}$$

Since for $1 \le p < \infty, k \in \mathbb{N}$ and $u \in W^{1,p}(0,T;H^k(\Omega))$ the inequality

$$\int_0^t ||u_h(\tau)||^p_{H^k(\Omega)}d\tau \le \int_0^t ||u'(\tau)||^p_{H^k(\Omega)}d\tau, \quad t \in [0,\infty), \tag{20}$$

is true, then we obtain

$$\begin{aligned}\int_0^t |F_h(\tau)|d\tau \le \int_0^t |f'(\tau)|d\tau + 2\Big(\int_0^t |u'(\tau)|^2 d\tau\Big)^{1/2}\Big[\Big(\int_0^t |u^2(\tau+h)|^2 d\tau\Big)^{1/2} +\\ +\Big(\int_0^t |u^2(\tau)|^2 d\tau\Big)^{1/2}\Big] \le M_0 + (1-\varepsilon)^{-1}E_0(u,t) \le\\ \le CM_0\Big(1+M_0\Big), \quad t \in [0,\infty), \quad 0 < \varepsilon \le 1/2.\end{aligned} \tag{21}$$

As $u'(0) = u_1, \varepsilon u''(0) = f(0) - u_1 + \Delta u_0 - u_0^3$, and $|u_0^3| \le 4\sqrt{3}||u_0||^3$, then

$$E_1^{1/2}(u',0) \le C\Big(M_0 + M_0^3 + ||u_0||_{H^2(\Omega)} + ||u_1||\Big). \tag{22}$$

Using the estimates (21), (22) and passing to the limit in the inequality (19) as $h \to 0$ we obtain the estimate (11).

Now let us prove the estimate (14). Under the conditions on f, u_0 and u_1 we have that $z \in W^{1,\infty}(0,\infty;H_0^1(\Omega)) \cap L^\infty(0,\infty;H^2(\Omega))$, $z'' \in L^\infty(0,\infty;L^2(\Omega))$ and z is the solution to the problem

$$\begin{cases} \varepsilon\Big(z''(t),\eta\Big) + \Big(z'(t),\eta\Big) + [z(t),\eta] + 3\Big(u^2(t)z(t),\eta\Big) =\\ = \Big(f_1(t,\varepsilon),\eta\Big), \quad \forall\eta \in H_0^1(\Omega), \quad a.e. \quad t \in (0,\infty),\\ z(0) = z_0, \quad z'(0) = 0,\end{cases} \tag{23}$$

where

$$f_1(t,\varepsilon) = f'(t) + \Big(3u^2(t)\alpha - \Delta\alpha\Big)e^{-t/\varepsilon}, \quad z_0 = f(0) - u_0^3 + \Delta u_0.$$

Denote by

$$E_2(z,t) = \varepsilon^2|z'(t)|^2 + \frac{1}{2}|z(t)|^2 + \varepsilon||z(t))||^2 + \varepsilon\Big(z'(t), z(t)\Big) +$$

$$+\varepsilon\int_0^t |z'(\tau)|^2 d\tau + \int_0^t ||z(\tau)||^2 d\tau + 3\int_0^t \Big(u^2(\tau)z(\tau), z(\tau)\Big)d\tau.$$

For the solution z to the problem (23) we have

$$\frac{d}{dt}E_2(z,t) = \Big(f_1(t,\varepsilon), z(t) + 2\varepsilon z'(t)\Big) - 6\varepsilon\Big(z'(t), u^2(t)z(t)\Big), \quad a.e. \quad t\in(0,\infty).$$

Integrating the last equality on $(0,t)$ we obtain

$$E_2(z,t) = E_2(z,0) + \int_0^t \Big(f_1(\tau,\varepsilon), z(\tau) + 2\varepsilon z'(\tau)\Big)d\tau -$$

$$-6\varepsilon\int_0^t \Big(z'(\tau), u^2(\tau)z(\tau)\Big)d\tau, \quad t\in[0,\infty). \tag{24}$$

Using Holder's inequality, the estimate (10) and the inequality

$$||z||_{L^6(\Omega)} \le \gamma||z||, \quad \forall z\in H_0^1(\Omega), \quad \gamma = (48)^{1/6}, \tag{25}$$

we get the estimate

$$\Big|\Big(z'(\tau), u^2(\tau)z(\tau)\Big)\Big| \le |z'(\tau)|||z(\tau)||_{L^6(\Omega)}||u(\tau)||^3_{L^6(\Omega)} \le CM_0^3|z'(\tau)|||z(\tau)||,$$

from which it follows that

$$6\varepsilon\Big|\int_0^t \Big(z'(\tau), u^2(\tau)z(\tau)\Big)d\tau\Big| \le \frac{\varepsilon}{2}\int_0^t |z'(\tau)|^2 d\tau +$$

$$+CM_0^6\varepsilon\int_0^t ||z(\tau||^2 d\tau \le \frac{1}{2}E_2(z,t), \quad t\in[0,\infty), \quad 0<\varepsilon\le\varepsilon_0, \tag{26}$$

where $\varepsilon_0 = min\{1/2, (2C)^{-1}M_0^{-6}\}$. As $|z(\tau) + 2\varepsilon z'(\tau)| \le 2E_2^{1/2}(z,\tau)$, then due to Lemma 1 from (24) and (26) follows the estimate

$$E_2^{1/2}(z,t) \le 2E_2^{1/2}(z,0) + 2\int_0^t |f_1(\tau,\varepsilon)|d\tau, \quad t\in[0,\infty), \quad 0<\varepsilon\le\varepsilon_0. \tag{27}$$

The inequality $|u^2(\tau)\alpha| \le \gamma^4||\alpha||||u(\tau)||^2$ permits to get the estimate

$$\int_0^t |f_1(\tau,\varepsilon)|d\tau \le M_0 + CM_0^2||\alpha|| + ||\alpha||_{H^2(\Omega)},\ t\in[0,\infty),\ 0<\varepsilon<1. \tag{28}$$

As
$$E_2^{1/2}(z,0) \le C||f(0) - u_0^3 + \Delta u_0|| \le CM_1, \quad 0 < \varepsilon < 1,$$
then from (27) and (28) follows the estimate
$$E_2^{1/2}(z,t) \le CM_2, \quad t \in [0,\infty), \quad 0 < \varepsilon \le \varepsilon_0. \tag{29}$$
Further, if z is a solution to the problem (23), then the function $z_h(t) = h^{-1}(z(t+h) - z(t)), h > 0$ is the solution to the problem
$$\begin{cases} \varepsilon\Big(z_h''(t),\eta\Big) + \Big(z_h'(t),\eta\Big) + [z_h(t),\eta] + 3\Big(u^2(t)z_h(t),\eta\Big) = \\ = \Big(G_h(t,\varepsilon),\eta\Big), \quad \forall \eta \in H_0^1(\Omega), \quad a.e. \quad t \in (0,\infty), \\ z_h(0) = z_{0h}, \quad z_h'(0) = z_{1h}, \end{cases} \tag{30}$$
where
$$G_h(t,\varepsilon) = f_{1h}(t,\varepsilon) - 3u_h(t)z(t+h)\big(u(t+h)+u(t)\big),$$
$$z_{0h} = h^{-1}(z(h) - z_0), \quad z_{1h} = h^{-1}z'(h).$$
In exactly the same way as the inequality (27) was obtained we get the inequality
$$E_2^{1/2}(z_h,t) \le 2E_2^{1/2}(z_h,0) + 2\int_0^t |G_h(\tau,\varepsilon)|d\tau,\ t \in [0,\infty),\ 0 < \varepsilon \le \varepsilon_0. \tag{31}$$
As $u'(t) = z(t) - \alpha e^{-t/\varepsilon}$, then using Holder's inequality, the inequalities (25), (20) and the estimates (10), (29) we obtain
$$\int_0^t |u_h(\tau)z(\tau+h)\big(u(\tau+h)+u(\tau)\big)|d\tau \le$$
$$\le \int_0^t ||u_h(\tau)||_{L^6(\Omega)}||z(\tau+h)||_{L^6(\Omega)}\Big(||u(\tau+h)||_{L^6(\Omega)} + ||u(\tau)||_{L^6(\Omega)}\Big)d\tau \le$$
$$\le CM_0\int_0^t ||u_h(\tau)||||z(\tau+h)||d\tau \le$$
$$\le CM_0E_1^{1/2}(u',t)E_2^{1/2}(z,t+h) \le CM_2, \quad t \in [0,\infty), \quad 0 < \varepsilon \le \varepsilon_0. \tag{32}$$
Using Holder's inequality, the inequalitis (20), (25) and the estimates (10), (11) we will estimate f_{1h} as follows
$$\int_0^t |f_{1h}(\tau)|d\tau \le \int_0^t |f_1'(\tau,\varepsilon)d\tau \le \int_0^t |f''(\tau)|d\tau +$$
$$+\frac{1}{\varepsilon}\int_0^t e^{-\tau/\varepsilon}(|\Delta\alpha| + 3|\alpha u^2(\tau)|)d\tau + 6\int_0^t e^{-\tau/\varepsilon}|\alpha u(\tau)u'(\tau)|d\tau \le$$
$$\le M_2 + C||\alpha||_{L^6(\Omega)}\int_0^t e^{-\tau/\varepsilon}||u(\tau)||_{L^6(\Omega)}\Big(\frac{1}{\varepsilon}||u(\tau)||_{L^6(\Omega)} +$$

$$+||u'(\tau)||_{L^6(\Omega)}\Big)d\tau \le CM_2, \quad t\in[0,\infty), \quad 0<\varepsilon\le 1/2. \tag{33}$$

The estimates (32) and (33) imply the following estimate for G_h

$$\int_0^t |G_h(\tau)|d\tau \le CM_2, \quad t\in[0,\infty), \quad 0<\varepsilon\le\varepsilon_0. \tag{34}$$

As

$$E_2^{1/2}(z',0) = |f'(0)+\Delta u_1 - 3u_0^2u_1| \le CM_2, \tag{35}$$

then, using the estimates (34), (35) and passing to the limit in the inequality (31) as $h\to 0$, we obtain the estimate

$$E_2^{1/2}(z',t) \le CM_2, \quad t\in(0,\infty), \quad 0<\varepsilon\le\varepsilon_0. \tag{36}$$

From (29) and (36) follows the estimate

$$||z||_{W^{1,\infty}(0,\infty;L^2(\Omega))} + ||z||_{W^{1,2}(0,\infty;H_0^1(\Omega))} \le CM_2, \quad 0<\varepsilon\le\varepsilon_0. \tag{37}$$

Finally, let us estimate $||z||_{L^\infty(0,\infty;H_0^1(\Omega))}$. To this end we denote by

$$E_3(z,t) = \varepsilon|z'(t)|^2 + |z(t)|^2 + ||z(t)||^2 + 2(1-\varepsilon)\int_0^t |z'(\tau)|^2 d\tau +$$

$$+2\varepsilon(z(t),z'(t)) + 2\int_0^t ||z(\tau)||^2 d\tau + 6\int_0^t \Big(u^2(\tau),z^2(\tau)\Big)d\tau + 3\Big(u^2(t),z^2(t)\Big).$$

If z is a solution to the problem (23), then

$$\frac{d}{dt}E_3(z,t) = 2\Big(f_1(t,\varepsilon), z(t)+z'(t)\Big) + 6\Big(u(t)u'(t), z^2(t)\Big), \quad a.e. \quad t\in(0,\infty).$$

Integrating the last equality on $(0,t)$, similarly as the inequality (18) was obtained, we get

$$E_3(z,t) \le C\Big(E_3(z,0) + ||f_1'||^2_{L^1(0,\infty;L^2(\Omega))} + \int_0^t \Big|\Big(u(\tau)u'(\tau), z^2(\tau)\Big)\Big|d\tau +$$

$$+\int_0^t \Big(|f_1(\tau,\varepsilon)| + |f_1'(\tau,\varepsilon)|\Big)E_3^{1/2}(z,\tau)d\tau\Big), \quad t\in[0,\infty), \quad 0<\varepsilon<1. \tag{38}$$

In the obvious way we obtain the estimate

$$E_3(z,0) + ||f_1'||^2_{L^1(0,\infty;L^2(\Omega))} \le CM_2. \tag{39}$$

Using Holder's inequality, the inequality (25) and estimates (10), (11), (29), we get the estimate

$$\int_0^t \Big|\Big(u(\tau)u'(\tau), z^2(\tau)\Big)\Big|d\tau \le \int_0^t |u'(\tau)|||z(\tau)||^2_{L^6(\Omega)}||u(\tau)||_{L^6(\Omega)}d\tau \le$$

$$\le \gamma^3 \int_0^t |u'(\tau)|||z(\tau)||^2||u(\tau)||d\tau \le$$

$$\le CM_1M_0E_2(z,t) \le CM_2, \quad t \in [0,\infty), \quad 0 < \varepsilon \le \varepsilon_0. \tag{40}$$

Due to Lemma 1 from (38), (39) and (40) follows the estimate

$$||z||_{L^\infty(0,t;H_0^1(\Omega))} \le E_3^{1/2}(z,t) \le C\Big(M_2 + \int_0^t (|f_1(\tau,\varepsilon)| + |f_1'(\tau,\varepsilon)|)d\tau\Big) \le$$

$$\le CM_2, \quad t \in [0,\infty), \quad 0 < \varepsilon \le \varepsilon_0. \tag{41}$$

The estimates (37), (41) imply (14). Lemma 2 is proved.

Corollary 1. *If $f \in W^{2,1}(0,\infty;L^2(\Omega)), u_1, f(0) - u_0^3 + \Delta u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, then, in fact, the function $u(t)$ satisfies the estimates*

$$||u||_{W^{1,p}(0,\infty;H_0^1(\Omega))} \le CM_2, \quad 0 < \varepsilon \le \varepsilon_0, \quad p \in [2,\infty], \tag{42}$$

$$\varepsilon||u''||_{L^\infty(0,\infty;L^2(\Omega))} + ||\Delta u||_{L^\infty(0,\infty;L^2(\Omega))} \le CM_2, \quad 0 < \varepsilon \le \varepsilon_0. \tag{43}$$

4 Relationship between solutions to the problems ($\mathbf{P}_\varepsilon$) and ($\mathbf{P_0}$) in the linear case

In this section we shall give the relationship between solutions to the problem (P_ε) and (P_0) in the linear case, i. e. in the case when the term u^3 in the problems (P_ε) and (P_0) is missing. This relation was inspired by the work [10]. At first we shall give some properties of the kernel $K(t,\tau,\varepsilon)$ of transformation which realizes this connection.

For $\varepsilon > 0$ denote

$$K(t,\tau,\varepsilon) = \frac{1}{2\sqrt{\pi}\varepsilon}\Big(K_1(t,\tau,\varepsilon) + 3K_2(t,\tau,\varepsilon) - 2K_3(t,\tau,\varepsilon)\Big),$$

where

$$K_1(t,\tau,\varepsilon) = \exp\Big\{\frac{3t-2\tau}{4\varepsilon}\Big\}\lambda\Big(\frac{2t-\tau}{2\sqrt{\varepsilon t}}\Big),$$

$$K_2(t,\tau,\varepsilon) = \exp\Big\{\frac{3t+6\tau}{4\varepsilon}\Big\}\lambda\Big(\frac{2t+\tau}{2\sqrt{\varepsilon t}}\Big),$$

$$K_3(t,\tau,\varepsilon) = \exp\Big\{\frac{\tau}{\varepsilon}\Big\}\lambda\Big(\frac{t+\tau}{2\sqrt{\varepsilon t}}\Big), \quad \lambda(s) = \int_s^\infty e^{-\eta^2}d\eta.$$

Lemma 3 [11]. *The function $K(t,\tau,\varepsilon)$ possesses the following properties*:

(i) *For any fixed $\varepsilon > 0$ $K \in C(\{t \ge 0\} \times \{\tau \ge 0\}) \cap C^\infty(R_+ \times R_+)$;*

(ii) $K_t(t,\tau,\varepsilon) = \varepsilon K_{\tau\tau}(t,\tau,\varepsilon) - K_\tau(t,\tau,\varepsilon), \quad t > 0, \tau > 0;$

(iii) $\varepsilon K_\tau(t,0,\varepsilon) - K(t,0,\varepsilon) = 0, \quad t \geq 0;$

(iv) $K(0,\tau,\varepsilon) = \dfrac{1}{2\varepsilon}\exp\Big\{-\dfrac{\tau}{2\varepsilon}\Big\}, \quad \tau \geq 0;$

(v) *For each fixed* $t > 0$, $s, q \in \mathbb{N}$ *there exist constants* $C_1(s,q,t,\varepsilon) > 0$ *and* $C_2(s,q,t) > 0$ *such that*

$$|\partial_t^s \partial_\tau^q K(t,\tau,\varepsilon)| \leq C_1(s,q,t,\varepsilon)\exp\{-C_2(s,q,t)\tau/\varepsilon\}, \quad \tau > 0;$$

(vi) $K(t,\tau,\varepsilon) > 0, \quad t \geq 0, \quad \tau \geq 0;$

(vii) *Let* ε *be fixed,* $0 < \varepsilon \ll 1$ *and* H *be a Hilbert space. For any* $\varphi : [0,\infty) \to H$ *continuous on* $[0,\infty)$ *such that* $|\varphi(t)| \leq M\exp\{Ct\}$, $t \geq 0$, *the relationship*

$$\lim_{t\to 0}\int_0^\infty K(t,\tau,\varepsilon)\varphi(\tau)d\tau = \int_0^\infty e^{-\tau}\varphi(2\varepsilon\tau)d\tau,$$

is valid in H;

(viii) $\int_0^\infty K(t,\tau,\varepsilon)d\tau = 1, \quad t \geq 0;$

(ix)

$$\int_0^\infty K(t,\tau,\varepsilon)|t-\tau|^q d\tau \leq C\varepsilon^{q/2}\big(1+t^{q/2}\big), \quad q \in [0,1].$$

(x) *Let* $f \in W^{1,\infty}(0,\infty;H)$. *Then there exists positive constant* C *such that*

$$\Big|\Big|f(t) - \int_0^\infty K(t,\tau,\varepsilon)f(\tau)d\tau\Big|\Big|_H \leq C\sqrt{\varepsilon}(1+\sqrt{t})\|f'\|_{L^\infty(0,\infty;H)}, \quad t \geq 0.$$

(xi) *There exists* $C > 0$ *such that*

$$\int_0^t\int_0^\infty K(\tau,\theta,\varepsilon)\exp\Big\{-\frac{\theta}{\varepsilon}\Big\}d\theta d\tau \leq C\varepsilon, \quad t \geq 0, \quad \varepsilon > 0.$$

Theorem 3. *Suppose that* $f \in L^\infty(0,\infty;L^2(\Omega))$ and $u \in W^{2,\infty}(0,\infty;L^2(\Omega)) \cap L^\infty(0,\infty;H_0^1(\Omega))$ *is the solution to the problem*:

$$\varepsilon(u''(t),\eta) + (u'(t),\eta) + [u(t),\eta] = (f(t),\eta), \quad \forall\eta \in H_0^1(\Omega), \tag{44}$$

a.e. $t \in [0,\infty)$,

$$u(0) = u_0, \quad u'(0) = u_1. \tag{45}$$

Then the function v_0 *which is defined by*

$$v_0(t) = \int_0^\infty K(t,\tau,\varepsilon)u(\tau)d\tau \tag{46}$$

is the solution to the problem

$$(v_0'(t), \eta) + [v_0(t), \eta] = (f_0(t, \varepsilon), \eta), \quad \forall \eta \in H_0^1(\Omega), \quad \forall t > 0, \tag{47}$$

$$v_0 = \varphi_\varepsilon, \tag{48}$$

where

$$f_0(t, \varepsilon) = F_0(t, \varepsilon) + \int_0^\infty K(t, \tau, \varepsilon) f(\tau) d\tau,$$

$$F_0(t, \varepsilon) = \frac{1}{\sqrt{\pi}} \Big[2 \exp\Big\{ \frac{3t}{4\varepsilon} \Big\} \lambda\Big(\sqrt{\frac{t}{\varepsilon}} \Big) - \lambda\Big(\frac{1}{2} \sqrt{\frac{t}{\varepsilon}} \Big) \Big] u_1, \quad \varphi_\varepsilon = \int_0^\infty e^{-\tau} u(2\varepsilon\tau) d\tau.$$

Moreover, $v_0 \in W^{2,\infty}(0, \infty; L^2(\Omega)) \cap L^\infty(0, \infty; H_0^1(\Omega))$.

Proof. As u is the solution to the problem (44), (45) and $u, u', u'' \in L^\infty(0, \infty; L^2(\Omega))$, then $v_0 \in W^{2,\infty}(0, \infty; L^2(\Omega)) \cap L^\infty(0, \infty; H_0^1(\Omega))$, $u \in C^1([0, \infty); L^2(\Omega))$ and $|u(t) - u_0| \to 0, |u'(t) - u_1| \to 0$ as $t \to 0$. Therefore, integrating by parts and using the properties (**i**) - (**iii**) and (**v**) of Lemma 3, we get

$$(v_0'(t), \eta) = \Big(\int_0^\infty K_t(t, \tau, \varepsilon) u(\tau) d\tau, \eta \Big) =$$

$$\Big(\int_0^\infty \Big(\varepsilon K_{\tau\tau}(t, \tau, \varepsilon) - K_\tau(t, \tau, \varepsilon) \Big) u(\tau) d\tau, \eta \Big) =$$

$$= \Big(\int_0^\infty K(t, \tau, \varepsilon) \Big(\varepsilon u''(\tau) + u'(\tau) \Big) d\tau + \varepsilon K(t, 0, \varepsilon) u_1, \eta \Big) =$$

$$= \Big(f_0(t, \varepsilon) u_1, \eta \Big) - [v(t), \eta], \quad \forall \eta \in H_0^1(\Omega), \quad \forall t > 0.$$

Thus $v_0(t)$, which is defined by (46) satisfies the equation (47). From property (**vii**) of Lemma 3 the validity of the initial condition (48) follows. Thus Theorem 3 is proved.

5 Limits of solutions to the problem $(\mathbf{P}_\varepsilon)$ as $\varepsilon \to 0$

In this section we shall study the behavior of solutions to the problem (P_ε) as $\varepsilon \to 0$.

Theorem 4. *Suppose that* $f \in W^{2,1}(0, \infty; L^2(\Omega)), u_0, u_1, f(0) - u_0^3 + \Delta u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, *then there exist constants* $C = C(\Omega)$ *and* $\varepsilon_0 = \varepsilon_0(\Omega, M_0)$ *such that the following estimates*

$$||u - v||_{C([0,t]; L^2(\Omega))} \leq C M_2 \big(t^{3/2} + 1 \big) \sqrt{\varepsilon}, \quad t \geq 0, \quad 0 < \varepsilon \leq \varepsilon_0, \tag{49}$$

$$||u - v||_{L^\infty(0,t; H_0^1(\Omega))} \leq C M_2 \big(1 + t^{3/2} \big) \varepsilon^{1/4}, \quad t \geq 0, \quad 0 < \varepsilon \leq \varepsilon_0, \tag{50}$$

are fulfilled. If in addition $f \in W^{2,\infty}(0, \infty; L^2(\Omega))$, *then*

$$||u' - v' - \alpha h e^{-t/\varepsilon}||_{L^\infty(0,t; L^2(\Omega))} \leq C M_3 \varepsilon^{1/4} (1 + t^{5/2}), \quad t \geq 0, \, 0 < \varepsilon \leq \varepsilon_0, \tag{51}$$

is fulfilled, where u is a solution to the problem (P_ε), v is the solution to the problem (P_0) and $M_3 = M_2 + ||f''||_{L^\infty(0,\infty;L^2(\Omega))}$.

Proof. *Proof of estimate (49).* If u is the solution to the problem (P_ε), then according to Theorem 3 the function

$$w(t) = \int_0^\infty K(t,\tau,\varepsilon)u(\tau)d\tau$$

is the solution to the problem

$$\begin{cases} \Big(w'(t),\eta\Big) + [w(t),\eta] = \Big(F(t,\varepsilon),\eta\Big), \quad \forall \eta \in H_0^1(\Omega), \quad t>0, \\ w(0) = \varphi_\varepsilon, \end{cases} \tag{52}$$

where

$$F(t,\varepsilon) = F_0(t,\varepsilon) + \int_0^\infty K(t,\tau,\varepsilon)f(\tau)d\tau - \int_0^\infty K(t,\tau,\varepsilon)u^3(\tau)d\tau.$$

Using the estimates (11), (41) and properties (**viii**) and (**x**) from Lemma 3 we obtain the following estimates

$$|u(t)-w(t)| \le C\sqrt{\varepsilon}(1+\sqrt{t})||u'||_{L^\infty(0,\infty;L^2(\Omega))} \le$$

$$\le CM_1\sqrt{\varepsilon}(1+\sqrt{t}), \quad t \ge 0, \quad 0<\varepsilon \le 1/2, \tag{53}$$

and

$$||u(t)-w(t)|| \le CM_2\varepsilon^{1/2}(1+\sqrt{t}), \quad t\ge 0, \quad 0<\varepsilon\le\varepsilon_0. \tag{54}$$

Therefore, as $||w(t)|| \le M_0$, then

$$|u^3(t)-w^3(t)| \le C||u(t)-w(t)||_{L^6(\Omega)}\Big(||u(t)||^2_{L^6(\Omega)} + ||w(t)||^2_{L^6(\Omega)}\Big) \le$$

$$\le C\gamma^3||u(t)-w(t)||||u(t)||^2 \le CM_2\sqrt{\varepsilon}(1+\sqrt{t}), \quad t\ge 0,\ 0<\varepsilon\le\varepsilon_0. \tag{55}$$

Denote by $y(t) = v(t) - w(t)$, where v is the solution to the problem (P_0) and w is the solution to the problem (52). Then the function y is the solution to the following problem:

$$\begin{cases} \Big(y'(t),\eta\Big) + [y(t),\eta] + \Big((v^2(t)+v(t)w(t)+w^2(t))y(t),\eta\Big) = \\ = \Big(F_1(t,\varepsilon),\eta\Big), \quad \forall\eta\in H_0^1(\Omega), \quad t>0, \\ y(0) = u_0 - \varphi_\varepsilon, \end{cases} \tag{56}$$

where

$$F_1(t,\varepsilon) = f(t) - F_0(t,\varepsilon) - \int_0^\infty K(t,\tau,\varepsilon)f(\tau)d\tau + \int_0^\infty K(t,\tau,\varepsilon)u^3(\tau)d\tau - w^3(t),$$

Due to the estimate (8) (in the linear case) for the function y we get the estimate

$$|y(t)| \leq |y(0)| + \int_0^t |F_1(\tau,\varepsilon)|d\tau, \quad t \geq 0. \tag{57}$$

From the estimate (11) it follows that

$$|y(0)| \leq \int_0^\infty e^{-\tau}|u_0 - u(2\varepsilon\tau)|d\tau \leq$$

$$\leq \int_0^\infty e^{-\tau}\int_0^{2\varepsilon\tau} |u'(s)|ds d\tau \leq C\varepsilon ||u'||_{L^\infty(0,\infty;L^2(\Omega))} \leq C\varepsilon M_1. \tag{58}$$

As $q(s) = e^{s^2}\lambda(s) \leq C$, for $s \in [0,\infty]$ then

$$\int_0^t |F_0(\tau,\varepsilon)|d\tau \leq C|u_1|\int_0^t \Big[\exp\Big\{\frac{3\tau}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{\tau}{\varepsilon}}\Big) + \lambda\Big(\frac{1}{2}\sqrt{\frac{\tau}{\varepsilon}}\Big)\Big]d\tau =$$

$$= C|u_1|\int_0^t \exp\Big\{-\frac{\tau}{4\varepsilon}\Big\}\Big[q\Big(\sqrt{\frac{\tau}{\varepsilon}}\Big) + q\Big(\frac{1}{2}\sqrt{\frac{\tau}{\varepsilon}}\Big)\Big]d\tau \leq C\varepsilon|u_1|, \quad t \geq 0. \tag{59}$$

Using the properties (**viii**) and (**x**) from Lemma 3, we have

$$\int_0^t \Big|f(\tau) - \int_0^\infty K(\tau,s,\varepsilon)f(s)ds\Big|d\tau \leq$$

$$\leq C\sqrt{\varepsilon}||f'||_{L^\infty(0,\infty:L^2(\Omega))}(1+t^{3/2}) \leq C\sqrt{\varepsilon}M_2(1+t^{3/2}), \quad t \geq 0. \tag{60}$$

Let us evaluate the difference

$$I(t) = w^3(t) - \int_0^\infty K(t,\tau,\varepsilon)u^3(\tau)d\tau.$$

Due to inequality (25) and the estimates (10), (41), we get

$$|(u^3(s))'| = 3|u'(s)u^2(s)| \leq 3||u'(s)||_{L^6(\Omega)}||u(s)||^2_{L^6(\Omega)} \leq$$

$$\leq 3\gamma^3||u'(s)||||u(s)||^2 \leq CM_2, \quad s \in [0,\infty), \quad 0 < \varepsilon \leq \varepsilon_0,$$

and, consequently,

$$\Big|u^3(t) - \int_0^\infty K(t,\tau,\varepsilon)u^3(\tau)d\tau\Big| \leq CM_2\sqrt{\varepsilon}(1+\sqrt{t}), \quad t \geq 0, \quad 0 < \varepsilon \leq \varepsilon_0.$$

Hence, from the last estimate and (55), we get

$$|I(t)| \leq |w^3(t) - u^3(t)| + \Big|u^3(t) - \int_0^\infty K(t,\tau,\varepsilon)u^3(\tau)d\tau\Big| \leq$$

$$\leq CM_2\sqrt{\varepsilon}\big(1+\sqrt{t}\big), \quad t \geq 0, \quad 0 < \varepsilon \leq \varepsilon_0.$$

Therefore

$$\int_0^t |I(\tau)|d\tau \le CM_2\sqrt{\varepsilon}\big(1+t^{3/2}\big), \quad t\ge 0, \quad 0<\varepsilon\le\varepsilon_0. \tag{61}$$

Gathering the estimates (59), (60) and (61), we have

$$\int_0^t |F_1(\tau,\varepsilon)|d\tau \le CM_2\sqrt{\varepsilon}(1+t^{3/2}), \quad t\ge 0, \quad 0<\varepsilon\le\varepsilon_0.$$

Using the last estimate and (58), from (57) follows the estimate

$$|y(t)| \le CM_2\sqrt{\varepsilon}(1+t^{3/2}), \quad t\ge 0, \quad 0<\varepsilon\le\varepsilon_0. \tag{62}$$

Finally, the estimates (53) and (62) involve (49).

Proof of estimate (50). To prove the estimate (50) we have to evaluate $||y||_{L^\infty(0,\infty;H_0^1(\Omega))}$. To this end we observe that due to (9) for y' is true the estimate

$$|y'(t)| \le |Y_0| + \int_0^t |F_1'(\tau,\varepsilon) - a'(\tau)y(\tau)|d\tau, \quad t\in[0,\infty), \tag{63}$$

where $a(t) = v^2(t) + v(t)w(t) + w^2(t)$, $Y_0 = \Delta y(0) + F_1(0,\varepsilon) - a(0)y(0)$. Using the estimate (43), we get

$$|\Delta y(0)| \le CM_2. \tag{64}$$

Due to the inequalities (11) and (25) we obtain that

$$|F_1(0,\varepsilon)| \le C\Big(||f||_{W^{1,1}(0,\infty;L^2(\Omega))} + |u_1| + ||u_0||^3\Big) \le CM_2 \tag{65}$$

and

$$|a(0)y(0)| = |u_0^3 - \varphi_\varepsilon^3| \le CM_2. \tag{66}$$

The estimates (64), (65) and (66) imply

$$|Y_0| \le CM_2. \tag{67}$$

As

$$\varepsilon K_\tau(t,\tau,\varepsilon) - K(t,\tau,\varepsilon) = -\frac{3}{4\varepsilon\sqrt{\pi}}\Big(K_1(t,\tau,\varepsilon) - K_2(t,\tau,\varepsilon)\Big)$$

and

$$\int_0^\infty \Big(K_1(t,\tau,\varepsilon) + K_2(t,\tau,\varepsilon)\Big)d\tau \le C\varepsilon\Big(\lambda\big(-\frac{1}{2}\sqrt{t/\varepsilon}\big) + e^{3t/4\varepsilon}\lambda\big(\sqrt{t/\varepsilon}\big)\Big) \le C\varepsilon,$$

then for $f\in W^{1,\infty}(0,\infty;L^2(\Omega))$, due to the properties (**ii**), (**iii**) and (**v**) from Lemma 3, we obtain the estimate

$$\Big|\int_0^\infty K_t(t,\tau,\varepsilon)f(\tau)d\tau\Big| = \Big|\int_0^\infty \big(\varepsilon K_\tau(t,\tau,\varepsilon) - K(t,\tau,\varepsilon)\big)f'(\tau)d\tau\Big| \le$$

$$\leq C\varepsilon^{-1}\int_0^\infty \big(K(t,\tau,\varepsilon)+K_2(t,\tau,\varepsilon)\big)|f'(\tau)|d\tau \leq$$

$$\leq C||f||_{W^{1,\infty}(0,\infty;L^2(\Omega))} \leq CM_2. \tag{68}$$

Similarly, using the inequalities (25), (42) and the properties (**ii**), (**iii**), (**v**) and (**viii**) from Lemma 3, we obtain the estimates

$$\Big|\int_0^\infty K_t(t,\tau,\varepsilon)u^3(\tau)d\tau\Big| \leq C||u||_{W^{1,\infty}(0,\infty;H_0^1(\Omega))} \leq CM_2, \quad 0<\varepsilon\leq\varepsilon_0, \tag{69}$$

$$|(w^3)'(t)| \leq C||w'(t)||||w(t)||^2 \leq$$

$$\leq CM_0\Big|\Big|\int_0^\infty K_t(t,\tau,\varepsilon)u(\tau)d\tau\Big|\Big| \leq CM_2 \quad 0<\varepsilon\leq\varepsilon_0, \quad t\in[0,\infty). \tag{70}$$

By direct computation we can show that

$$\int_0^\infty |F_0'(\tau,\varepsilon)|d\tau \leq C|u_1| \leq CM_2, \quad t\in[0,\infty). \tag{71}$$

The estimates (68), (69), (70) and (71) involve the estimate

$$\int_0^t |F_1'(\tau,\varepsilon)|d\tau \leq CM_2, \quad 0<\varepsilon\leq\varepsilon_0, \quad t\in[0,\infty). \tag{72}$$

Thanks to the inequality (25) and Holder's inequality we have

$$|a'(t)y(t)| \leq C\Big(||v(t)||||v'(t)|| + ||v'(t)||||w(t)|| +$$

$$+||v(t)||||w'(t)|| + ||w(t)||||w'(t)||\Big)||y(t)||, \quad t\in[0,\infty).$$

Therefore, the estimates (8), (9), (62) and (33) imply

$$\int_0^t |a'(\tau)y(\tau)|d\tau \leq CM_2(1+t^{3/2}), \quad 0<\varepsilon\leq\varepsilon_0, \quad t\in[0,\infty). \tag{73}$$

Using the estimates (67), (72) and (73), from (63) we obtain

$$|y'(t)| \leq CM_2(1+t^{3/2}), \quad 0<\varepsilon\leq\varepsilon_0, \quad t\in[0,\infty). \tag{74}$$

As

$$(y'(t),y(t)) + ||y(t)||^2 + (a(t),y(t)) = (F_1(t,\varepsilon),y(t))$$

$$|F_1(t,\varepsilon)| \leq CM_2, \quad 0<\varepsilon\leq\varepsilon_0, \quad t\in[0,\infty),$$

then due to (74) we get

$$||y(t)||^2 \leq |y(t)|\big(|F_1(t)|+|y'(t)|\big) \leq CM_2\sqrt{\varepsilon}(1+t^{3/2})^2, \quad 0<\varepsilon\leq\varepsilon_0, \quad t\in[0,\infty).$$

From the last estimates and (54) the estimate (50) follows.

Proof of estimate (51). Denote by $z(t) = u'(t) + \alpha e^{-t/\varepsilon}$, where α is defined in (13). Then $z(t)$ is a solution to the problem (23). According to Theorem 3 the function $w_1(t)$, which is defined as

$$w_1(t) = \int_0^\infty K(t,\tau,\varepsilon) z(\tau) d\tau$$

is a solution to the problem

$$\begin{cases} (w_1'(t),\eta) + [w_1(t),\eta] = (\mathcal{F}(t,\varepsilon),\eta), \quad \forall \eta \in H_0^1(\Omega), \quad t > 0, \\ w_1(0) = w_{1\varepsilon}, \end{cases}$$

where

$$\mathcal{F}(t,\varepsilon) = \int_0^\infty K(t,\tau,\varepsilon) f_1(\tau,\varepsilon) d\tau - 3\int_0^\infty K(t,\tau,\varepsilon) u^2(\tau) z(\tau) d\tau.$$

$$w_{1\varepsilon} = \int_0^\infty e^{-\tau} z(2\varepsilon\tau) d\tau.$$

Using the estimates (11), (14) and properties (**viii**), (**ix**) and (**x**) from Lemma 3, similarly as the estimates (53) and (54) were obtained, we obtain the following estimates

$$\begin{aligned} |z(t) - w_1(t)| &\le C\sqrt{\varepsilon}(1+\sqrt{t})||z'||_{L^\infty(0,\infty;L^2(\Omega))} \le \\ &\le CM_2\sqrt{\varepsilon}(1+\sqrt{t}), \quad t \ge 0, \quad 0 < \varepsilon \le \varepsilon_0, \end{aligned} \tag{75}$$

and

$$\begin{aligned} ||z(t) - w_1(t)|| &\le \int_0^\infty K(t,\tau,\varepsilon) \Big| \int_\tau^t ||z'(s)|| ds \Big| d\tau \le \\ &\le ||z'(t)||_{L^2(0,\infty;H_0^1(\Omega))} \int_0^\infty K(t,\tau,\varepsilon) |t-\tau|^{1/2} d\tau \le \\ &\le CM_2 \varepsilon^{1/4}(1+t^{1/4}), \quad t \ge 0, \quad 0 < \varepsilon \le \varepsilon_0. \end{aligned} \tag{76}$$

Under the conditions on f, u_0 and u_1, if v is a solution to the problem (P_0), then the function $v_1(t) = v'(t)$ is a solution to the problem

$$\begin{cases} \Big(v_1'(t),\eta\Big) + [v_1(t),\eta] + 3\Big(v^2(t) v_1(t),\eta\Big) = \\ = \Big(f'(t),\eta\Big), \quad \forall \eta \in H_0^1(\Omega), \quad a.e. \quad t \in (0,\infty), \\ v_1(0) = z_0. \end{cases}$$

Denote by $R(t) = w_1(t) - v_1(t)$. Then the function R is a solution to the problem

$$\begin{cases} \Big(R'(t),\eta\Big) + [R(t),\eta] + 3\Big(v^2(t) R(t),\eta\Big) = \\ = \Big(\mathcal{F}_1(t,\varepsilon),\eta\Big), \quad \forall \eta \in H_0^1(\Omega), \quad a.e. \quad t \in (0,\infty), \\ R(0) = w_{1\varepsilon} - z_0, \end{cases}$$

where

$$\mathcal{F}_1(t,\varepsilon)=\mathcal{F}(t,\varepsilon)+3v^2(t)w_1(t)-f'(t).$$

For the function R the estimate

$$|R(t)|\leq |R(0)|+\int_0^t|\mathcal{F}_1(\tau,\varepsilon)|d\tau,\quad t\geq 0, \tag{77}$$

holds. Using the estimate (14) we get

$$|R(0)|\leq\int_0^\infty e^{-\tau}|z_0-z(2\varepsilon\tau)|d\tau\leq C\varepsilon||z'||_{L^\infty(0,\infty;L^2(\Omega))}\leq C\varepsilon M_2. \tag{78}$$

To estimate the second term of the right-hand side of (77) we will present $\mathcal{F}_1(t,\varepsilon)$ in the following form:

$$\mathcal{F}_1(t,\varepsilon)=I_1(t,\varepsilon)+I_2(t,\varepsilon)+3I_3(t,\varepsilon),$$

where

$$I_1(t,\varepsilon)=\int_0^\infty K(t,\tau,\varepsilon)f'(\tau)d\tau-f'(t),$$

$$I_2(t,\varepsilon)=\int_0^\infty K(t,\tau,\varepsilon)\Big(3u^2(\tau)\alpha-\Delta\alpha\Big)e^{-\tau/\varepsilon}d\tau,$$

$$I_3(t,\varepsilon)=v^2(t)w_1(t)-\int_0^\infty K(t,\tau,\varepsilon)u^2(\tau)z(\tau)d\tau.$$

Using the properties (**viii**) and (**x**) from Lemma 3, we obtain the estimate

$$|I_1(t,\varepsilon)|\leq C||f''||_{L^\infty(0,\infty;L^2(\Omega))}\sqrt{\varepsilon}(1+\sqrt{t}),\quad t\geq 0,\quad \varepsilon>0$$

and, consequently,

$$\int_0^t|I_1(\tau,\varepsilon)|d\tau\leq C||f''||_{L^\infty(0,\infty;L^2(\Omega))}\sqrt{\varepsilon}(1+t^{3/2}),\quad t\geq 0,\quad \varepsilon>0. \tag{79}$$

In view of the inequality (25) and the estimate (10) we have

$$|u^2(t)\alpha|\leq ||u(t)||^2_{L^6(\Omega)}||\alpha||_{L^6(\Omega)}\leq\gamma^3||u(t)||^2||\alpha||\leq CM_2,\quad t\geq 0.$$

Therefore the property (**xi**) from Lemma 3 permits to estimate $I_2(t,\varepsilon)$

$$\int_0^t|I_2(\tau,\varepsilon)|d\tau\leq CM_2\int_0^t\int_0^\infty K(\tau,s,\varepsilon)e^{-s/\varepsilon}dsd\tau\leq$$

$$\leq CM_2\varepsilon,\quad t\geq 0,\quad 0<\varepsilon\leq\varepsilon_0. \tag{80}$$

Further, using the inequalities (25), (50) and (76), we get

$$|v^2(t)w_1(t)-u^2(t)z(t)|\leq C\Big(||w_1(t)||||u(t)-v(t)||(||u(t)||+||v(t)||)+$$

$$+||w_1(t)-z(t)||||u(t)||^2\Big) \le CM_2\Big(||u(t)-v(t)||+||w_1(t)-z(t)||\Big) \le$$
$$\le CM_2\varepsilon^{1/4}(1+t^{3/2}), \quad t\ge 0, \quad 0<\varepsilon\le\varepsilon_0. \tag{81}$$

Using the estimates (10), (14), (42) and property (**ix**) from Lemma 3, we obtain the following estimate

$$\int_0^\infty K(t,\tau,\varepsilon)\big|u^2(\tau)z(\tau)-u^2(t)z(t)\big|d\tau \le$$
$$\le \int_0^\infty K(t,\tau,\varepsilon)\Big|\int_\tau^t |2u(s)u'(s)z(s)+u^2(s)z'(s)|ds\Big|d\tau \le$$
$$\le CM_2\int_0^\infty K(t,\tau,\varepsilon)\Big|\int_\tau^t (1+|z'(s)|)ds\Big|d\tau \le$$
$$\le CM_2\varepsilon^{1/4}(1+t^{1/4}), \quad t\ge 0, \quad 0<\varepsilon\le\varepsilon_0. \tag{82}$$

The estimates (81), (82) imply

$$\int_0^t |I_3(\tau)|d\tau \le CM_2\varepsilon^{1/4}(1+t^{5/2}), \quad t\ge 0, \quad 0<\varepsilon\le\varepsilon_0.$$

From the last estimate and (79), (80) it follows that

$$\int_0^t |\mathcal{F}_1(\tau,\varepsilon)|d\tau \le C\mathcal{M}_2\varepsilon^{1/4}(1+t^{5/2}), \quad t\ge 0, \quad 0<\varepsilon\le\varepsilon_0. \tag{83}$$

From (77), due to (78) and (83) it follows that

$$|R(t)| \le C\mathcal{M}_2\varepsilon^{1/4}(1+t^{5/2}), \quad t\ge 0, \quad 0<\varepsilon\le\varepsilon_0.$$

The last estimate and (75) imply the estimate (51). Theorem 4 is proved.

Theorem 5. *Let $T>0$. Suppose that $f\in W^{2,1}(0,T;L^2(\Omega)), u_0, u_1, f(0)-u_0^3+\Delta u_0 \in H_0^1(\Omega)\cap H^2(\Omega)$, then there exist constants $C=C(\Omega,T)$ and $\varepsilon_0=\varepsilon_0(\Omega,M_0)$ such that the following estimates*

$$||u-v||_{C([0,T];L^2(\Omega))} \le C\mathcal{M}_2\sqrt{\varepsilon}, \quad 0<\varepsilon\le\varepsilon_0, \tag{84}$$

$$||u-v||_{L^\infty(0,T;H_0^1(\Omega))} \le C\mathcal{M}_2\varepsilon^{1/4}, \quad 0<\varepsilon\le\varepsilon_0, \tag{85}$$

are fulfilled. If in addition $f\in W^{2,\infty}(0,T;L^2(\Omega))$, then

$$||u'-v'-\alpha h e^{-t/\varepsilon}||_{L^\infty(0,T;L^2(\Omega))} \le C\mathcal{M}_3\varepsilon^{1/4}, \quad 0<\varepsilon\le\varepsilon_0, \tag{86}$$

is fulfilled, where u is the solution to the problem (P_ε), v is the solution to the problem (P_0). The constants $\mathcal{M}_i, i=1,2,3$, depend on the same values as M_i, the difference being that the norms $||f||_{W^{k,l}(0,\infty;L^2(\Omega))}$ in M_i are replaced with the norms $||f||_{W^{k,l}(0,T;L^2(\Omega))}$.

Proof. For any function $f \in W^{2,l}(0,T;L^2(\Omega)), l \in [1,\infty]$, there exits the extension $\tilde{f} \in W^{2,l}(0,\infty;L^2(\Omega))$ such that

$$||\tilde{f}||_{W^{2,l}(0,\infty;L^2(\Omega))} \leq C(T)||\tilde{f}||_{W^{2,l}(0,T;L^2(\Omega))} \tag{87}$$

(see, for instance, [12]). If $\tilde{u}$ is the solution to the problem (4), (5) with the same initial conditions u_0, u_1 and the right-hand side $\tilde{f}$, then according to Theorem 1 we have that $u(t) = \tilde{u}(t)$ for $t \in [0,T]$. Similarly, if $\tilde{v}$ is a solution to the problem (6), (7) with the same initial condition u_0 and the right-hand side $\tilde{f}$, then according to Theorem 2 we have that $v(t) = \tilde{v}(t)$ for $t \in [0,T]$.

Consequently, using the estimates (49), (50), (51) and (87) we obtain the estimates (84), (85) and (86). Theorem 5 is proved.

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Moldova State University
Faculty of Mathimatic and Computer Science
str. Mateevici 60, Chişinău MD−2009
Moldova
E-mail: perjan@usm.md

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Unsteady flow of an Oldroyd-B fluid induced by a constantly accelerating plate

Corina Fetecău, Constantin Fetecău

Abstract. We study the start-up flow of an Oldroyd-B fluid between two infinite parallel plates, one of them at rest and the other one being subject, after time zero, to a constant acceleration A. The solutions that are obtained satisfy both the associate partial differential equations and all imposed initial and boundary conditions. They reduce to those for a Maxwell, Second grade or Navier-Stokes fluid as a limiting case.

Mathematics subject classification: 76A05.
Keywords and phrases: Oldroyd-B fluid, velocity field, tangential tension, constantly accelerating plate.

1 Introduction

In a recent paper [1], we established exact solutions for the motion of a second grade fluid and of a Maxwell one between two infinite parallel plates, one of them being subject to a constant acceleration A. It is the goal of this work to extend these results to a larger class of non-Newtonian fluids, namely Oldroyd-B fluids. The constitutive equations of an incompressible Oldroyd-B fluid, as they were presented by Rajagopal [2], are

$$\mathbf{T} = -\,p\mathbf{I} + \mathbf{S}, \quad \mathbf{S} + \lambda(\dot{\mathbf{S}} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^{\mathrm{T}}) = \mu[\mathbf{A} + \lambda_r(\dot{\mathbf{A}} - \mathbf{L}\mathbf{A} - \mathbf{A}\mathbf{L}^{\mathrm{T}})], \tag{1.1}$$

where $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{S}$ the extra-stress tensor, $\mathbf{L}$ the velocity gradient, $\mathbf{A} = \mathbf{L} + \mathbf{L}^{\mathrm{T}}$ is the first Rivlin-Ericksen tensor, $-p\mathbf{I}$ denotes the indeterminate spherical stress, μ is the dynamic viscosity, λ and $\lambda_r (< \lambda)$ are relaxation and retardation times and the superposed dot indicates the material time derivative.

This model includes as special cases the Maxwell model and linearly viscous fluid model. Consequently, their solutions will appear as special cases of our solutions. Furthermore, the solutions for a second grade fluid can be also obtained. Recently, the Oldroyd-B fluids have received a lot of attention from both the theoreticians and the experimentalists in rheology. They can describe many of the non-Newtonian characteristics exhibited by polymeric materials such as stress-relaxation, normal stress differences in simple shear flows and non-linear creep.

2 Governing equations

In the following we shall consider unidirectional motions of the form [1, 3, 4]

$$\mathbf{v} = \mathbf{v}(y,\, t) = v(y,\, t)\, i, \tag{2.1}$$

where i is the unit vector along the $x-$coordinate direction of the system of Cartesian coordinates x, y and z. For this velocity field, the constraint of incompressibility is automatically satisfied. We shall also suppose that the extra-stress tensor $\mathbf{S}$ depends on y and t only, i.e., $\mathbf{S} = \mathbf{S}(\mathrm{y}, \mathrm{t})$. If the fluid has been at rest till the moment $t = 0$, then the initial condition

$$\mathbf{S}(y,\, 0) = \mathbf{0}, \tag{2.2}$$

together with $(1.1)_2$ lead to $S_{xz} = S_{yz} = S_{yy} = S_{zz} = 0$ [5] and

$$(1 + \lambda\partial_t)\tau = \mu(1 + \lambda_r\partial_t)\partial_y v, \quad (1 + \lambda\partial_t)\sigma - 2\lambda\tau\partial_y v = -2\mu\lambda_r(\partial_y v)^2, \tag{2.3}$$

where $\tau = S_{xy}$ and $\sigma = S_{xx}$.

The balance of linear momentum, in the absence of body forces and of a pressure gradient in the x-direction, reduce to

$$\partial_y\tau = \rho\partial_t v, \quad \partial_y p = \partial_z p = 0\,. \tag{2.4}$$

Eliminating τ between Eqs. $(2.3)_1$ and $(2.4)_1$, we attain to the linear partial differential equation

$$\lambda\partial_t^2 v(y,t) + \partial_t v(y,t) = \nu(1 + \lambda_r\partial_t)\partial_y^2 v(y,t), \tag{2.5}$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid and ρ its constant density.

3 Couette flow induced by a constantly accelerating plate

Consider an incompressible Oldroyd-B fluid at rest between two infinite parallel plates at a distance h apart. Suppose that the lower plate is subject, after time zero, to a constant acceleration A in a direction parallel to the upper one, which is stationary. The governing equation is (2.5) while the initial and boundary conditions are [5]

$$v(y,0) = \partial_t v(y,0) = 0; \quad y \in [0,\, h)\,, \tag{3.1}$$

respectively,

$$v(0,t) = At, \quad v(h,t) = 0; \quad t > 0\,. \tag{3.2}$$

Multiplying both sides of Eq. (2.5) by $\sin(\lambda_n y)$, integrating between the limits $y = 0$ and $y = h$ and having (3.1) and (3.2) in mind, we find that [6]

$$\lambda\ddot{v}_{sn}(t) + (1 + \alpha\lambda_n^2)\dot{v}_{sn}(t) + \nu\lambda_n^2 v_{sn}(t) = \lambda_n A(\nu t + \alpha); \quad t > 0\,, \tag{3.3}$$

where $\alpha = \nu\lambda_r$, $\lambda_n = n\pi/h$ and the finite Fourier sine transforms $v_{sn}(t)$, $(n = 1, 2, 3, \dots)$, of $v(y, t)$ have to satisfy the conditions

$$v_{sn}(0) = \partial_t v_{sn}(0) = 0; \quad n = 1, 2, 3, \dots \tag{3.4}$$

The solutions of the ordinary differential equations (3.3) with the initial conditions (3.4) are

$$v_{sn}(t) = \frac{A}{\lambda_n} \begin{cases} t - \dfrac{1}{\nu\lambda_n^2}\left[1 - \dfrac{r_{2n}r_{3n}\exp(r_{1n}t) - r_{1n}r_{4n}\exp(r_{2n}t)}{r_{2n} - r_{1n}}\,\lambda\right] \\ \qquad \text{if } \lambda_n \in (0, \infty)\backslash\{a, b\} \\ t - \dfrac{1}{\nu\lambda_n^2}\left\{1 - \left[\dfrac{1 - \alpha^2\lambda_n^4}{4\lambda}\,t + 1\right]\exp\left(-\dfrac{1 + \alpha\lambda_n^2}{2\lambda}\,t\right)\right\} \\ \qquad \text{if } \lambda_n \in \{a, b\}, \end{cases} \tag{3.5}$$

where

$$r_{1n},\, r_{2n} = \frac{-(1 + \alpha\lambda_n^2) \pm \sqrt{(1 + \alpha\lambda_n^2)^2 - 4\nu\lambda\lambda_n^2}}{2\lambda},$$

$$r_{3n},\, r_{4n} = \frac{1 - \alpha\lambda_n^2 \pm \sqrt{(1 + \alpha\lambda_n^2)^2 - 4\nu\lambda\lambda_n^2}}{2\lambda},$$

$$a = \frac{1}{\sqrt{\nu}(\sqrt{\lambda} + \sqrt{\lambda - \lambda_r})}$$

and

$$b = \frac{1}{\sqrt{\nu}(\sqrt{\lambda} - \sqrt{\lambda - \lambda_r})}.$$

Now, using the Fourier's sine formula [6], we find that

$$v(y, t) = \left(1 - \frac{y}{h}\right) At - \frac{2A}{\nu h}\sum_{n=1}^{\infty}\frac{\sin(\lambda_n y)}{\lambda_n^3} +$$

$$+ \frac{2A}{\nu h}\exp\left(-\frac{t}{2\lambda}\right)\sum_{n=1}^{\infty}\Phi_n(t)\frac{\sin(\lambda_n y)}{\lambda_n^3}\,, \tag{3.6}$$

where

$$\Phi_n(t) = \exp\left(-\frac{\alpha\lambda_n^2}{2\lambda}\,t\right)\begin{cases} ch\left(\dfrac{\beta_n t}{2\lambda}\right) + \dfrac{1+\nu(\lambda_r - 2\lambda)\lambda_n^2}{\beta_n} sh\left(\dfrac{\beta_n t}{2\lambda}\right) & \text{if } \lambda_n \in (0,\, a)\cup(b,\, \infty) \\ \dfrac{1-\alpha^2\lambda_n^4}{4\lambda}\,t + 1 & \text{if } \lambda_n \in \{a,\, b\} \\ \cos\left(\dfrac{\gamma_n t}{2\lambda}\right) + \dfrac{1+\nu(\lambda_r - 2\lambda)\lambda_n^2}{\gamma_n}\sin\left(\dfrac{\gamma_n t}{2\lambda}\right) & \text{if } \lambda_n \in (a,\, b), \end{cases}$$

where

$$\beta_n = \sqrt{(1+\alpha\lambda_n^2)^2 - 4\nu\lambda\lambda_n^2}$$

and

$$\gamma_n = \sqrt{4\nu\lambda\lambda_n^2 - (1+\alpha\lambda_n^2)^2}.$$

From $(2.3)_1$ and (2.2) it easily results that

$$\tau(y,t) = \frac{\mu}{\lambda}\exp\left(-\frac{t}{\lambda}\right)\int_0^t \exp\left(\frac{\tau}{\lambda}\right)(1+\lambda_r\partial_\tau)\partial_y v(y,\tau)d\tau. \tag{3.7}$$

By substituting (3.6) into (3.7) we find that

$$\tau(y,\, t) = -\frac{\mu A}{h}\left\{t + (\lambda_r - \lambda)\left[1 - \exp\left(-\frac{t}{\lambda}\right)\right]\right\} -$$

$$-\frac{2\rho A}{h}\sum_{n=1}^{\infty}\frac{\cos(\lambda_n y)}{\lambda_n^2} + \frac{2\rho A}{h}\exp\left(-\frac{t}{\lambda}\right)\sum_{n=1}^{\infty}\Psi_n(t)\frac{\cos(\lambda_n y)}{\lambda_n^2}\,, \tag{3.8}$$

where

$$\Psi_n(t) = \exp\left(-\frac{\alpha\lambda_n^2}{2\lambda}\,t\right)\begin{cases} ch\left(\dfrac{\beta_n t}{2\lambda}\right) + \dfrac{1-\alpha\lambda_n^2}{\beta_n} sh\left(\dfrac{\beta_n t}{2\lambda}\right) & if \;\; \lambda_n \in (0,\, a)\cup(b,\, \infty) \\ \dfrac{1+\alpha\lambda_n^2}{2\lambda}\left[1 - \lambda_r\dfrac{1+\alpha\lambda_n^2}{2\lambda}\right] t + 1 & \text{if } \lambda_n \in \{a,\, b\} \\ \cos\left(\frac{\gamma_n t}{2\lambda}\right) + \dfrac{1-\alpha\lambda_n^2}{\gamma_n}\sin\left(\dfrac{\gamma_n t}{2\lambda}\right) & \text{if } \lambda_n \in (a,\, b). \end{cases}$$

As soon as the velocity field $v(y,\, t)$ and the tangential tension $\tau(y,\, t)$ have been determined, we can find the normal tension $\sigma(y,\, t)$ using $(2.3)_2$ and (2.2). The hydrostatic pressure as it results from (2.4), is an arbitrary function of t.

4 Limiting cases

1. Taking the limit of Eqs. (3.6) and (3.8) as $\lambda_r \to 0$, we attain to the similar solutions for a Maxwell fluid (see [1], Eq. (4.8), for the velocity field)

$$v(y,\,t) = \left(1-\frac{y}{h}\right)\,At - \frac{2A}{\nu h}\sum_{n=1}^{\infty}\frac{\sin(\lambda_n y)}{\lambda_n^3} +$$

$$+\frac{2A}{\nu h}\exp\left(-\frac{t}{2\lambda}\right)\sum_{n=1}^{\infty}V_n(t)\frac{\sin(\lambda_n y)}{\lambda_n^3} \tag{4.1}$$

and

$$\tau(y,\,t) = -\frac{\mu A}{h}\left\{t+\lambda\left[\exp\left(-\frac{t}{\lambda}\right)-1\right]\right\} - \frac{2\rho A}{h}\sum_{n=1}^{\infty}\frac{\cos(\lambda_n y)}{\lambda_n^2} +$$

$$+\frac{2\rho A}{h}\exp\left(-\frac{t}{2\lambda}\right)\sum_{n=1}^{\infty}T_n(t)\frac{\cos(\lambda_n y)}{\lambda_n^2}, \tag{4.2}$$

where

$$V_n(t) = \begin{cases} ch\left(\dfrac{\alpha_n t}{2\lambda}\right) + \dfrac{1-2\nu\lambda\lambda_n^2}{\alpha_n} sh\left(\dfrac{\alpha_n t}{2\lambda}\right) & \text{if } \lambda_n \in \left(0,\ \dfrac{1}{2\sqrt{\nu\lambda}}\right) \\[2ex] \dfrac{1}{4\lambda}\,t+1 & \text{if } \lambda_n = \dfrac{1}{2\sqrt{\nu\lambda}} \\[2ex] \cos\left(\dfrac{\delta_n t}{2\lambda}\right) + \dfrac{1-2\nu\lambda\lambda_n^2}{\delta_n}\sin\left(\dfrac{\delta_n t}{2\lambda}\right) & \text{if } \lambda_n \in \left(\dfrac{1}{2\sqrt{\nu\lambda}}\,,\ \infty\right) \end{cases},$$

$$T_n(t) = \begin{cases} ch\left(\dfrac{\alpha_n t}{2\lambda}\right) + \dfrac{1}{\alpha_n} sh\left(\dfrac{\alpha_n t}{2\lambda}\right) & \text{if } \lambda_n \in \left(0,\ \dfrac{1}{2\sqrt{\nu\lambda}}\right) \\[2ex] \dfrac{1}{2\lambda}\,t+1 & \text{if } \lambda_n = \dfrac{1}{2\sqrt{\nu\lambda}} \\[2ex] \cos\left(\dfrac{\delta_n t}{2\lambda}\right) + \dfrac{1}{\delta_n}\sin\left(\dfrac{\delta_n t}{2\lambda}\right) & \text{if } \lambda_n \in \left(\dfrac{1}{2\sqrt{\nu\lambda}},\ \infty\right) \end{cases},$$

$\alpha_n = \sqrt{1-4\nu\lambda\lambda_n^2}$ and $\delta_n = \sqrt{4\nu\lambda\lambda_n^2-1}$.

2. In the special case when both λ_r and $\lambda \to 0$ into Eqs. (3.6) and (3.8), or $\lambda \to 0$ in Eqs. (4.1) and (4.2), we get the simple solutions (see [1], Eq. (3.10) for the velocity field)

$$v(y,\,t) = \left(1 - \frac{y}{h}\right)\,At - \frac{2A}{h}\sum_{n=1}^{\infty}\left[1 - e^{-\nu\lambda_n^2 t}\right]\frac{\sin(\lambda_n y)}{\lambda_n^3}\,, \tag{4.3}$$

respectively,

$$\tau(y,\,t) = -\frac{\mu A}{h}t - \frac{2\rho A}{h}\sum_{n=1}^{\infty}\left[1 - e^{-\nu\lambda_n^2 t}\right]\frac{\cos(\lambda_n y)}{\lambda_n}\,, \tag{4.4}$$

for a Navier-Stokes fluid.

3. Finally, by formally letting $\lambda_r \to 0$ into (3.6) and (3.8) (but using only the first lines of $\Phi_n(\cdot)$ and $\Psi_n(\cdot)$), we also attain to the solutions (see [1], Eq. (3.9) for the velocity field)

$$v(y,\,t) = \left(1 - \frac{y}{h}\right)\,At - \frac{2A}{\nu h}\sum_{n=1}^{\infty}\left[1 - \exp\left(-\frac{\nu\lambda_n^2}{1+\alpha\lambda_n^2}\,t\right)\right]\frac{\sin(\lambda_n y)}{\lambda_n^3} \tag{4.5}$$

and

$$\tau(y,\,t) = -\frac{\mu A}{h}(t+\lambda_r) - \frac{2\rho A}{h}\sum_{n=1}^{\infty}\left[1 - \frac{1}{1+\alpha\lambda_n^2}\exp\left(-\frac{\nu\lambda_n^2}{1+\alpha\lambda_n^2}\,t\right)\right]\frac{\cos(\lambda_n y)}{\lambda_n^2}, \tag{4.6}$$

corresponding to a second grade fluid.

5 Conclusions

In the present paper we have established the velocity filed and the associated tangential tension, corresponding to an unsteady lineal flow of an incompressible Oldroyd-B fluid between two infinite parallel plates. One of plates is held fixed and other one is subject, after time zero, to a constant acceleration A. Direct computations show that $v(y,t)$ and $\tau(y,t)$, given by (3.6) and (3.8), satisfy both the associate partial differential equations (2.5) and $(2.3)_1$ and all imposed initial and boundary conditions, the differentiation term by term in sums being clearly permissible. These solutions reduce to those for Maxwell, Navier-Stokes and Second grade fluids as limiting cases.

The solutions (3.6) and (3.8) as well as those for a Maxwell fluid (4.1) and (4.2), contain sine and cosine terms. This indicates that in contrast with the Navier-Stokes and second grade fluids, whose solutions (4.3)-(4.6) do not contain such terms, oscillations are set up in the fluid. The amplitudes of these oscillations decay exponentially in time, the damping being proportional to $\exp(-t/2\lambda)$.

Finally, it is important to underline that the Oldroyd-B model does not contain as a special case the second grade fluid model. However, for some special classes of motions, as that considered here, the governing equations also include the equations of motion for the second grade fluid. Consequently, in all these cases, the similar solutions corresponding to second grade fluids can be also obtained as limiting cases of those for Oldroyd-B fluids.

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C. Fetecău
Department of Mathematics
"Gh. Asachi" Technical Universiti
Iasi 700506, Romania
E-mail: *fetecau@math.tuiasi.ro*

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Corina Fetecău
Department of Theoretical Mechanics
"Gh. Asachi" Technical Universiti
Iasi 700506, Romania
E-mail: *cfetecau@yahoo.de*

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About commutative Moufang loops of finite special rank

A. Babiy, N. Sandu

Abstract. Commutative Moufang loops of finite special rank are characterized with the help of various associative subloops and with the help of various abelian subgroups of their multiplication group.

Mathematics subject classification: 20N05.
Keywords and phrases: Commutative Moufang loop, multiplication group of a loop, ZA-loop, special rank.

By analogy with group theory [1], the (*special*) *rank* of loop Q is the least positive number rQ with the following property: any finitely generated subloop of loop Q can be generated by rQ elements; if there are not such numbers, then we sat $rQ = \infty$. As to the structure and subject is concerned this paper is analogous to paper [2], where the commutative Moufang loops (abbreviated CMLs) with maximum condition for subloops are characterized. All notions and results of the CML theory we need are in detail described in the paper [2] (see also [3]). The present work characterizes a CML of finite rank with the help of various associative subloops and with the help of various abelian subgroups of its multiplication group.

Takes place

Lemma 1. *The following statements are equivalent for an arbitrary CML Q:*

1) CML Q satisfies the minimum condition for subloops;

2) CML Q is a direct product of a finite number of quasicyclic groups, belonging to the center of CML Q, and a finite CML;

3) CML Q satisfies the minimum condition for invariant subloops;

4) CML Q satisfies the minimum condition for non-invariant associative subloops;

5) if the CML Q contains a centrally nilpotent of class n subloop, then it satisfies the minimum condition for centrally nilpotent of class n subloops;

6) if the CML Q contains a centrally solvable of class s subloop, then it satisfies the minimum condition for centrally solvable of class s subloops;

7) at least one maximal associative subloop of the CML Q satisfies the minimum condition for subloops.

The equivalence of conditions 1), 2), 3) is proved in [4], the equivalence of conditions 1), 4), 5), 6) is proved in [5] and the equivalence of conditions 1), 7) is proved in [6].

Lemma 2. *If H is an invariant subloop of an arbitrary loop Q, then*

$$rQ \leq rH + r(Q/H).$$

Proof. Let t, s be ranks of loops H and Q/H respectively. Let us take in Q a finite set M and consider the subloop $xH|x \in M >$ in the quotient loop Q/H. As it is finitely generated, then it is possible to select in loop $< M >$ such a subset S of order s that

$$< xH|x \in M >=< xH|x \in S > .$$

Let us fix some notation for any $x \in M$

$$x = x_s x_h, x_s \in < S >, x_h \in < H > .$$

As $< x_h|x \in M >$ is a finitely generated subloop of loop H, then it is generated by a certain subset T of order $\leq t$. Obviously, $< M >=< T \cup S >$, i.e. $rQ \leq t + s$. This completes the proof of Lemma 2.

Further we will need a well known statement.

Lemma 3. *A primary abelian group has a finite rank r if and only if it decomposes into a direct product of r cyclic and quasicyclic groups and, consequently, satisfies the minimum condition for subgroups.*

We will also need the marvellous theorem of M.I. Kargapolov [7].

Lemma 4. *Let A be an invariant abelian subgroup of group G. If all abelian subgroups of group G have finite ranks, then the abelian subgroups of group G/A also have finite ranks.*

Lemma 5 *A periodic CML has a finite rank if and only if it satisfies the maximum condition for subloops.*

Proof. An arbitrary periodic CML Q decomposes into a direct product of its maximal p-subloops Q_p. In addition, for $p \neq 3$, Q_p are abelian groups [8]. Therefore, if Q has a finite rank rQ, then the number of subloops Q_p is finite and ranks rQ_p do not exceed the rank rQ. Conversely, if the CML Q satisfies the minimum condition for subloops, then the number of subloops Q_p is finite and each subloop Q_p satisfies the minimum condition for subloops. Hence it is enough to consider the case when the CML Q is a 3-loop.

Let us suppose that the CML Q has a finite rank. Then all its associative subloops have finite ranks and, by Lemma 3, satisfy the minimum condition for subloops. Then by 5) of Lemma 1 the CML Q satisfies the minimum condition for subloops as well.

Conversely, if the CML Q satisfies the minimum condition for subloops, then by 2) of Lemma 1 it decomposes into a direct product of a finite number of quasicyclic groups and a finite CML. Then it follows from Lemma 3 that the CML Q has a finite rank. This completes the proof of Lemma 5.

Lemma 6 *Let Q be a non-periodic CML with the center $Z(Q)$ of finite rank. Then $Z(Q)$ decomposes into the direct product $Z(Q) = T \times L$, where T is a non-periodic abelian group, L is an abelian group without torsion, and the quotient loop Q/L is a periodic CML.*

Proof. It is shown in [3] that for any element $x \in Q$, $x^3 \in Z(Q)$. Hence the center $Z(Q)$ is a non-periodic abelian group. The periodic part T of group $Z(Q)$ decomposes into a direct product of its primary components and as T has a finite rank, their number is finite. Therefore T has a finite exponent. Then by Corollary 10.1.13 from [9] T stands as a direct factor the $Z(Q) = T \times L$, where L is an abelian group without torsion. CML $Q/Z(Q)$ has the exponent three. Then it follows from the relations $Q/Z(Q) \cong (Q/L)/(Z(Q)/L) = (Q/L)/((T \times L)/L) \cong (Q/L)/T$ that the CML Q/L is periodic as the extension of the periodic CML T with the help of the periodic CML $Q/Z(Q)$. This completes the proof of Lemma 6.

We remind that the *multiplication group* $\mathfrak{M}(Q)$ of an arbitrary CML Q is the group, generated by all permutations $L(x)$, where $L(x)y = xy$.

Lemma 7. *A CML Q has a finite rank rQ if and only if the rank $r\mathfrak{M}$ of its multiplication group $\mathfrak{M}$ is finite.*

Proof. Let a CML Q have a finite rank and let $\mathfrak{N}$ be a subgroup of the group $\mathfrak{M}$, generated by a finite set of elements $\varphi_1, \dots, \varphi_t$. By the definition of group $\mathfrak{M}$, any element φ_i $(i = 1, \dots, t)$ is a product of a finite number of permutations $L(a_{il})$. Let H be a subloop of CML Q, generated by all elements a_{ij}. The subloop H can be generated by no more that rQ elements $b_1, \dots, b_m$. Let us now suppose that element a_{ij}, written via the generators $b_1, \dots, b_m$, has the form $a_{ij} = uv$, where u, v contain in the notation less generators $b_1, \dots, b_m$ than a_{ij}. Let us show that the subgroup $\mathfrak{N}$ is generated by the permutations $L(b_1), \dots, L(b_m)$. Indeed, the CLM is characterized by the identity $x^2 \cdot yz = xy \cdot yz$ [3]. Then $L(x^2)L(y)z = L(xy)L(x)z$, $L(x^2)L(y) = L(xy)L(x)$, $L(xy) = L(x^2)L(y)L^{-1}(x)$, $L(xy) = L(x)L(x)L(y)L^{-1}(x)$. Hence $L(a_{ij}) = L(uv) = L(u)L(u)L(v)L^{-1}(u)$. Continuing this process with the permutations $L(u), L(v)$, we will obtain after a finite number of steps that the permutation $L(a_{ij})$ is expressed through the permutations $L(b_1), \dots, L(b_m)$. Consequently, $r\mathfrak{M} \leq rQ$.

Let now H be a finitely generated subloop of CML Q, let $\{h_1, \dots, h_k\}$ be a certain minimal system of its generators and let the rank $r\mathfrak{M}$ be finite. Let us now consider the subgroup $\mathfrak{N}$ of group $\mathfrak{M}$ with a system of generating elements $\{L(h_1), \dots, L(h_k)\}$. This system is minimal because if any of the generators is expressed through the others, for example $L(h_1) = L^{-1}(h_2)L(h_3)$, then $L(h_1)1 = L^{-1}(h_2)L(h_3)1$, $h_1 = h_2^{-1}h_3$, thus contradicting the minimality of system $\{h_1, \dots, h_k\}$. As the group $\mathfrak{M}$ is locally nilpotent [4], then the commutator-group $\mathfrak{M}'$ of group $\mathfrak{M}$ is contained into the Frattini subgroup. Then the quotient group $\mathfrak{M}/\mathfrak{M}'$ decomposes into a direct product of cyclic groups and their number coincides with the number of the generators of a minimal system [9]. Therefore any minimal system of generators has the same number of elements. Consequently, the system $\{h_1, \dots, h_k\}$ contains no more than $r\mathfrak{M}$ elements, hence $rQ \leq r\mathfrak{M}$. This completes the proof of Lemma 7.

Lemma 8. *The multiplication group $\mathfrak{M}$ of a periodic CML Q has a finite rank if and only if it satisfies the minimum condition for subloops.*

Proof. The multiplication group $\mathfrak{M}$ of a periodic CML Q is locally nilpotent and decomposes into a direct product of its maximal p-subgroups $\mathfrak{M}_p$ [4]. If the group $\mathfrak{M}$ has a finite rank or satisfies the minimum condition for subgroups, then it is obvious that the number of subgroups $\mathfrak{M}_p$ is finite. Let the group $\mathfrak{M}$ have a finite rank. Then each subgroup $\mathfrak{M}_p$ has a finite rank. It is shown in [10] that a locally nilpotent p-group has a finite rank if and only if it satisfies the minimum condition for subgroups. Therefore subgroups $\mathfrak{M}_p$, and then the group $\mathfrak{M}$ too, satisfy the minimum condition for subgroups. Conversely, if the group $\mathfrak{M}$ satisfies the minimum condition for subgroups, then each subgroup $\mathfrak{M}_p$ satisfies this condition. Hence they have finite ranks, then the group $\mathfrak{M}$ has a finite rank as well.

Lemma 9. *Let $\mathfrak{A}$ be the maximal abelian subgroup of the multiplication group $\mathfrak{M}$ of a CML. Then any non-unitary invariant subgroup $\mathfrak{N}$ of the group $\mathfrak{M}$ has a non-unitary intersection with $\mathfrak{A}$.*

Proof. Let us suppose the contrary, that $\mathfrak{A} \cap \mathfrak{N} = \{e\}$. As $\mathfrak{A}$ is the maximal abelian subgroup, then there exist such elements $\alpha \in \mathfrak{A}, \nu \in \mathfrak{N}$, that $[\alpha, \nu] \neq e$. The subgroup $\mathfrak{N}$ is invariant in $\mathfrak{M}$, therefore $[\alpha, \nu] \in \mathfrak{N}$. The commutator-group $\mathfrak{M}'$ is a 3-group [3], then there exists such an element $\beta \in \mathfrak{N}$ that $\beta^3 = e$. We denote $\mathfrak{C} =< \mathfrak{A}, \beta >$. The group $\mathfrak{M}$ is locally nilpotent [4] and as the subgroup $\mathfrak{A}$ is maximal in $\mathfrak{C}$, then it is invariant in $\mathfrak{C}$ [9]. The subgroup $\mathfrak{N}$ is invariant in $\mathfrak{M}$, then the subgroup $\mathfrak{C} \cap \mathfrak{N}$ is invariant in $\mathfrak{C}$ as well. By the supposition $\mathfrak{A} \cap \mathfrak{N} = \{e\}$, there are two different the unitary element invariant in $\mathfrak{C}$ subgroups $\mathfrak{C} \cap \mathfrak{N}$ and $\mathfrak{A}$ intersect on the unitary element. Therefore $\mathfrak{C}$ is a direct product of the groups $\mathfrak{C} \cap \mathfrak{N}$ and $\mathfrak{A}$. Then any element of $\mathfrak{A}$ commutes with elements of $\mathfrak{C} \cap \mathfrak{N}$. But it contradicts the maximality of the abelian group $\mathfrak{A}$. This completes the proof of Lemma 9.

Proposition 1. *The following conditions are equivalent for an arbitrary non-associative CML Q with the multiplication group $\mathfrak{M}$:*

1) the group $\mathfrak{M}$ satisfies the minimum condition for subgroups;

2) the group $\mathfrak{M}$ satisfies the minimum condition for non-invariant subgroups;

3) at least one maximal abelian subgroup of the group $\mathfrak{M}$ satisfies the minimum condition for subgroups.

Proof. The quotient group $\mathfrak{M}/Z(\mathfrak{M})$ is a 3-group [3]. Hence, if α is a certain element of infinite order from $\mathfrak{M}$, then for a certain n, $\alpha^n \in Z(\mathfrak{M})$. If the group $\mathfrak{M}$ satisfies the condition 2) or 3), then there is an abelian subgroup $\mathfrak{A}$ in $\mathfrak{M}$ which contains the center $Z(\mathfrak{M})$ and satisfies the minimum condition for subgroups. Therefore the group $\mathfrak{M}$ is periodic. Then $\mathfrak{M}$ decomposes into a direct product of its maximal p-subgroups $\mathfrak{M}_p$, in addition for $p \neq 3$, $\mathfrak{M}_p \subseteq Z(\mathfrak{M})$. It follows from the definition of group $\mathfrak{A}$ that $\mathfrak{M}_p \subseteq \mathfrak{A}$ for $p \neq 3$. Therefore it is enough to consider the case when $\mathfrak{M}$ is a 3-group.

Let now the condition 2) hold in the group $\mathfrak{M}$ and let us suppose that the condition 1) does not hold in $\mathfrak{M}$. Then it follows from Theorem 4.11 from [11] that the group $\mathfrak{M}$ contains an invariant subgroup $\mathfrak{A}$, within which all its cyclic subgroups are invariant in $\mathfrak{M}$, such that the quotient group $\mathfrak{M}/\mathfrak{A}$ is a cyclic group. Let us suppose that the CML Q is generated by the elements $b, a_1, a_2, \ldots$. Taking into account the construction of group $\mathfrak{M}$ we will suppose that the permutations $L(a_i)$, $i = 1, 2, \ldots$ (perhaps $L(b)$ too) belong to subgroup $\mathfrak{A}$. Hence they generate in $\mathfrak{M}$ invariant subgroups. Then for an arbitrary fixed element $x \in Q$ and a certain natural number n, $L^{-1}(x)L(a_i)L(x) = L^n(a_i)$. Further, $L(a_i)L(x) = L(x)L^n(a_i)$, $L(a_i)L(x)y = L(x)L^n(a_i)y$, $a_i \cdot xy = a \cdot a_i^n y$. If $y = 1$, then $a_i x = x a_i^n$, $x a_i = x a_i^n, a_i^{n-1} = 1$. The last equality holds true only for $n = 1$. Then $a_i \cdot xy = x \cdot a_i y$, i.e. $a_i \in Z(Q)$. It follows easily from here that the CML Q is associative. Contradiction. Consequently, the CML Q satisfies the condition 1).

Let us now suppose that the condition 3) holds true in group $\mathfrak{M}$. The group $\mathfrak{M}$ is locally nilpotent [4], then by Theorem 1.8 from [11] it has a central system. Now, taking into account Lemma 9, the further proof of implication 3) $\to$ 1) can be completed exactly repeating the proof of Theorem 1.19 from [11]. Further, as the implications 1) $\to$ 2), 1) $\to$ 3) are obvious, the proposition is proved.

Proposition 2. *The following conditions are equivalent for an arbitrary non-associative commutative Moufang ZA-loop Q:*

1) CML Q has a finite rank;

2) if the CLM Q contains an invariant (resp. non-invariant) centrally nilpotent of class n subloop, then all invariant (or non-invariant) associative subloops of invariant (resp. non-invariant) centrally nilpotent of class n subloops of the CML Q have finite ranks;

3) if the CLM Q contains an invariant (resp. non-invariant) centrally solvable of class s subloop, then all invariant (or non-invariant) associative subloops of invariant (resp. non-invariant) centrally solvable of class s subloops of the CML Q have finite ranks;

4) if the CLM Q contains an invariant (resp. non-invariant) centrally nilpotent of class n subloop, then all invariant (or non-invariant) associative subloops of at least one maximal invariant (resp. non-invariant) centrally nilpotent of class n subloops of the CML Q have finite ranks;

5) if the CLM Q contains an invariant (resp. non-invariant) centrally solvable of class s subloop, then all invariant (or non-invariant) associative subloops of at least one maximal invariant (resp. non-invariant) centrally solvable of class s subloops of the CML Q have finite ranks;

6) the center $Z(Q)$ of CML Q has a finite rank;

7) the group $\mathfrak{M}$ has a finite rank;

8) if the group $\mathfrak{M}$ contains an invariant (resp. non-invariant) nilpotent of class n subgroup, then all invariant (or non-invariant) abelian subgroups of invariant (resp. non-invariant) nilpotent of class n subgroups of group $\mathfrak{M}$ have finite ranks;

9) if the group $\mathfrak{M}$ contains an invariant (resp. non-invariant) solvable of class s subgroup, then all invariant (or non-invariant) abelian subgroups of invariant (resp.

non-invariant) solvable of class s subgroup of group $\mathfrak{M}$ have finite ranks;

10) if the group $\mathfrak{M}$ contains an invariant (resp. non-invariant) nilpotent of class n subgroup, then all invariant (or non-invariant) abelian subgroups of at least one maximal invariant (resp. non-invariant) nilpotent of class n subgroups of group $\mathfrak{M}$ have finite ranks;

11) if the group $\mathfrak{M}$ contains an invariant (resp. non-invariant) solvable of class s subgroup, then all invariant (or non-invariant) abelian subgroups of at least one maximal invariant (resp. non-invariant) solvable of class s subgroup of group $\mathfrak{M}$ have finite ranks;

12) the center $Z(\mathfrak{M})$ of group $\mathfrak{M}$ has a finite rank.

Proof. It is easy to notice that if the CML Q satisfies one of the conditions 2) – 5), then there are invariant (or non-invariant) associative subloops of finite rank in Q which contain the center $Z(Q)$. Therefore the implications 1) → 2) → 6), 1) → 3) → 6), 1) → 4) → 6), 1) → 5) → 6) hold true. By analogy the implications 7) → 8) → 12), 7) → 9) → 12), 7) → 10) → 12), 7) → 11) → 12) also hold true. Further, it is shown in [2] that the relation $Z(Q) \cong Z(\mathfrak{M})$ holds true for an arbitrary CML Q. Hence the implications 6) ↔ 12) hold as well.

Let us now prove the justice of implication 6) → 1). Let us suppose that CML Q is periodic. The center $Z(Q)$ has a finite rank, then by Lemma 5 it satisfies the minimum condition for subloops. It is shown in [6] that the ZA-loop Q also satisfies this condition, and the justice of condition follows from Lemma 1.

However, if CML Q is non-periodic, then it follows from Lemma 6 that $Z(Q) = T \times L$, where T is a periodic abelian group, L is an abelian group without torsion, and that Q/L is periodic. It is shown in [2] that under the homomorphism $Q \to Q/L$ the center $Z(Q)$ is mapped in the center $Z(Q/L)$. But $Z(Q)/L = (T \times L)/L \cong T$. Hence the center $Z(Q/L)$ of the periodic CML Q/L has a finite rank. Then by the previous case CML Q/L has a finite rank too. As subloop L has a finite rank, by Lemma 2 the CML Q has a finite rank too, i.e. the condition 1) holds true in Q.

Finally, the implication 1) → 7) follows from Lemma 7.

Theorem 1. *The following conditions are equivalent for an arbitrary non-associative CML Q with the multiplication group $\mathfrak{M}$:*

1) CML Q has a finite rank;

2) if the CLM Q contains a centrally nilpotent of class n subloop, then all invariant (or non-invariant) associative subloops of centrally nilpotent of class n subloops of CML Q have a finite rank;

3) if the CLM Q contains a centrally solvable of class s subloop, then all non-invariant associative subloops of centrally solvable of class s subloops of CML Q have a finite rank;

4) non-invariant associative subloops of invariant subloops of CML Q has a finite rank;

5) at least one maximal associative subloop of CML Q has a finite rank;

6) the group $\mathfrak{M}$ has a finite rank;

7) if the group $\mathfrak{M}$ contains a nilpotent of class n subgroup, then all non-invariant abelian subgroups of nilpotent of class n subgroups of group $\mathfrak{M}$ have a finite rank;

8) if the group $\mathfrak{M}$ contains a solvable of class s subgroup, then all non-invariant abelian subgroups of solvable of class s subgroups of group $\mathfrak{M}$ have a finite rank;

9) non-invariant abelian subgroups of invariant subgroups of group $\mathfrak{M}$ has a finite rank;

10) at least one maximal abelian subgroup of the group $\mathfrak{M}$ has a finite rank.

Proof. Implications 1) $\rightarrow$ 2), 1) $\rightarrow$ 3), 1) $\rightarrow$ 4), 1) $\rightarrow$ 5) are obvious. Let us prove the justice of the converse implications. Let us firstly consider the case when the CML Q is periodic. Any centrally nilpotent CML is a ZA-loop. Hence it follows from 2) of Proposition 2 that if CML Q satisies the condition 2), then all its centrally nilpotent of class n subloops have finite ranks. By Lemma 5 they satisfy the minimum condition for subloops. Then by Lemma 1 the CML Q satisfies this condition too, and it follows from Lemma 5 that the condition 1) holds in CML Q.

If the condition 3) (resp. 4)) holds in CML Q, then it follows from Lemma 5 and 4) of Lemma 1 that centrally solvable of class s (resp. invariant) subloops of CML Q satisfy the minimum condition for subloops in CML Q. Then by Lemma 1 the CML Q satisfies this condition as well, and it follows from Lemma 5 that the condition 1) holds in CML Q. Implication 5) $\rightarrow$ 1) is proved by analogy.

Let now CML Q be non-periodic. It is easy to notice that the statements hold true. If the CML Q satisfies one of the conditions 2) – 5), then in CML Q there are associative subloops of finite rank, containing $Z(Q)$. Therefore the center $Z(Q)$ has a finite rank. By analogy, if CML Q satisfies one of the conditions 7) – 10), then the center $Z(\mathfrak{M})$ of group $\mathfrak{M}$ has a finite rank.

Let us now suppose that one of the conditions 2) – 5) holds true in CML Q. Let L be an abelian group without torsion, considered in Lemma 6. It is proved in [2] that if a subloop H of CML Q is centrally nilpotent of class n or centrally solvable of class s, or maximally associative, or invariant, either non-invariant, then the image of subloop H under the homomorphism $Q \rightarrow Q/L$ will be of the same type. The abelian group L has a finite rank. If K is an associative subloop of finite rank of CML Q, containing L, then it follows from Lemma 3 that the abelian group K/L has also a finite rank. It follows from here that if the CML Q satisfies one of the conditions 2) – 5), then CML Q/L will satisfy this condition as well. By Lemma 6 the CML Q/L is periodic. Then by the previous case the CML K/L has a finite rank. Further, as the subloop L has a finite rank, then it follows from Lemma 2 that the CML Q has a finite rank as well. Consequently, the implications 2) $\rightarrow$ 1), 3) $\rightarrow$ 1), 4) $\rightarrow$ 1), 5) $\rightarrow$ 1) hold true.

Implications 1) $\leftrightarrow$ 6) are Lemma 7. Implications 6) $\rightarrow$ 7), 6) $\rightarrow$ 8), 6) $\rightarrow$ 9), 6) $\rightarrow$ 10) are obvious. Let us prove the converse implications. It follows from Proposition 1 that conditions 7), 8), 9) hold true not only for non-invariant abelian subgroups, but for all abelian subgroups as well. It is shown in [12] that if in a locally nilpotent group all abelian subgroups have finite ranks, then the group itself has a finite rank. The multiplication group $\mathfrak{M}$ is locally nilpotent [4]. Therefore if the condition 7) (resp. 8) or 9)) holds true in CML Q, then all nilpotent of class n subgroups (resp. all solvable of class s subgroups, or all invariant subgroups, either at least one maximal abelian subgroup) of group $\mathfrak{M}$ have finite ranks. Let us suppose that the CML Q is periodic. Then by Lemma 8 all nilpotent of class n subgroups (resp. all solvable of class s subgroups, or all invariant subgroups, either at least one maximal abelian subgroup) of group $\mathfrak{M}$ satisfy the minimum condition for subgroups. It is shown in [13, 14] that a locally nilpotent group, containing a nilpotent of class n subgroup (or solvable of class s subgroup) and satisfying the minimum condition for nilpotent of class n subgroups (or solvable of class s subgroups), satisfies the minimum condition for subgroups. If a locally nilpotent group satisfies the minimum condition for invariant subgroups, then it satisfies the minimum condition for subgroups too [15]. Therefore, if we take into account Proposition 1, then the multiplication group $\mathfrak{M}$ satisfies the minimum condition for subgroups and by Lemma 8, it has a finite rank. Consequently, if the CML Q is periodic, then the implications 7) $\rightarrow$ 6), 8) $\rightarrow$ 6), 9) $\rightarrow$ 6), 10) $\rightarrow$ 6) hold true.

It is obvious that if the CML Q satisfies one of the conditions 7) – 10), in the group $\mathfrak{M}$ there are abelian groups of finite rank, containing the center $Z(\mathfrak{M})$. Hence $Z(\mathfrak{M})$ has a finite rank. It is shown in [2] that $Z(\mathfrak{M}) \cong Z(Q)$. Then it follows from Lemma 6 that the group $Z(\mathfrak{M})$ decomposes into a direct product of periodic abelian group $\mathfrak{T}$ and abelian group without torsion $\mathfrak{L}$. The CML Q is non-periodic. Then the group $\mathfrak{M}$ is also non-periodic [4]. It is shown in [3] that the quotient group $\mathfrak{M}/Z(\mathfrak{M})$ is a 3-group. Hence, if α is an element of infinite order from $\mathfrak{M}$, then $\alpha^n \in \mathfrak{L}$. Therefore the group $\mathfrak{M}/\mathfrak{L}$ is periodic.

If $\mathfrak{N}$ is a nilpotent of class n subgroup or solvable of class s subgroup, or invariant subgroup, or non-invariant subgroup, either maximal abelian subgroup, then, as it was shown in [2], the image of $\mathfrak{N}$ under the homomorphism $\mathfrak{M} \rightarrow \mathfrak{M}/\mathfrak{L}$ will be the same. Further, if $\mathfrak{K}$ is an abelian subgroup of finite rank, containing $\mathfrak{L}$, then by Lemma 3 the quotient group $\mathfrak{K}/\mathfrak{L}$ has a finite rank. Hence, if the group $\mathfrak{M}$ satisfies one of the conditions 7) – 10), then the group $\mathfrak{M}/\mathfrak{L}$ satisfies the same condition. We have proved above that the group $\mathfrak{M}/\mathfrak{L}$ is periodic. Then, as shown while considering the case when CML Q is periodic, the group $\mathfrak{M}/\mathfrak{L}$ has a finite rank. The subgroup $\mathfrak{L}$ has a finite rank, then it follows from Lemma 2 that the group $\mathfrak{M}$ has a finite rank too. Consequently, the implications 7) $\rightarrow$ 6), 8) $\rightarrow$ 6), 9) $\rightarrow$ 6), 10) $\rightarrow$ 6) hold for CML Q. This completes the proof of Theorem 1.

We notice that in [16] is shown the equivalence of conditions 6), 7) of theorem for arbitrary periodic locally nilpotent group. The multiplication group $\mathfrak{M}$ of an arbitrary CML Q is locally nilpotent and if the CML Q is non-periodic, then the group $\mathfrak{M}$ is also non-periodic. However, unlike the multiplication group $\mathfrak{M}$, in [16] there is an example of non-periodic locally nilpotent group for which the conditions 6), 7) of the theorem are not equivalent.

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Tiraspol State University
str. Iablochkin, 5
Chişinău MD-2069
Moldova
E-mail: *sandumn@yahoo.com*

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An application of Briot-Bouquet differential subordinations

Gh. Oros, Georgia Irina Oros

Abstract. Let $f \in \mathcal{A}$. We consider the following integral operator:

$$F(z) = \frac{2}{z}\int_0^z f(t)dt. \tag{1}$$

By using this integral operator we obtain a Briot-Bouquet differential subordination.

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1 Introduction and preliminaries

We let $\mathcal{H}[U]$ denote the class of holomorphic functions in the unit disc

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we let

$$\mathcal{H}[a,n] = \{f \in \mathcal{H}[U], \ f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots, \ z \in U\}$$

and

$$\mathcal{A}_n = \{f \in \mathcal{H}[U], \ f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots, \ z \in U\}$$

with $\mathcal{A}_1 = \mathcal{A}$.

A function $f \in \mathcal{H}[a,1]$ is convex in U if it is univalent and $f(U)$ is convex.

It is well known that f is convex if and only if $f'(0) \neq 0$ and

$$\mathrm{Re}\ \frac{zf''(z)}{f'(z)} + 1 > 0, \ z \in U.$$

If f and g are analytic functions in U, then we say that f is subordinate to g, written $f \prec g$, or $f(z) \prec g(z)$, if there is a function w analytic in U with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g[w(z)]$ for $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

In order to prove the new results we shall use the following lemma.

Lemma A [1, Theorem 3.2b, p. 83]. *Let h be a convex function in U, with $h(0) = a$, and let n be a positive integer. Suppose that the Briot-Bouquet differential equation*

$$q(z) + \frac{nzq'(z)}{q(z)+1} = h(z)$$

has a univalent solution q that satisfies $q(z) \prec h(z)$.

If $p \in \mathcal{H}[a,n]$ satisfies

$$p(z) + \frac{zp'(z)}{p(z)+1} \prec h(z) \tag{2}$$

then $p(z) \prec q(z)$, and q is the best (a,n) dominant of (2).

2 Main results

Lemma B. *The function $h(z) = 1 + Rz + \dfrac{Rz}{2+Rz}$ is convex.*

Proof. We show that the function $h(z) = 1+Rz+\dfrac{Rz}{2+Rz}$ is convex for all $R \in (0,1]$.

We study the function

$$h(z) = 1 + z + \frac{z}{2+z}, \quad h'(z) = 1 + \frac{2}{(2+z)^2}, \quad h''(z) = -\frac{4}{(2+z)^3}.$$

We calculate

$$\mathrm{Re}\ \left[1 + \frac{zh''(z)}{h'(z)}\right] = \mathrm{Re}\ \left[1 - 4\frac{z}{(2+z)(z^2+4z+6)}\right].$$

We take $z = e^{i\theta}$, $\theta \in [0, 2\pi]$ and we obtain

$$\mathrm{Re}\ \left[1 + \frac{zh''(z)}{h'(z)}\right] = \mathrm{Re}\ \left[1 - 4\frac{e^{i\theta}}{(2+e^{i\theta})(e^{2i\theta}+4e^{i\theta}+6)}\right] =$$

$$= \mathrm{Re}\ \left[1 - 4\frac{\cos\theta + i\sin\theta}{(2+\cos\theta+i\sin\theta)(\cos 2\theta + i\sin 2\theta + 4\cos\theta + 4i\sin\theta + 6)}\right] =$$

$$= 1 - 4\frac{(2\cos\theta+1)(2\cos^2\theta + 4\cos\theta + 5) + 4\sin^2\theta(\cos\theta+2)}{(4\cos\theta+5)[(2\cos^2\theta+4\cos\theta+5)^2 + 4\sin^2\theta(\cos\theta+2)^2]}.$$

We let $\cos\theta = t$, $t \in [-1,1]$. Then

$$\mathrm{Re}\ \left[1 + \frac{zh''(z)}{h'(z)}\right] = \frac{96t^3+336t^2+372t+153}{96t^3+344t^2+444t+205} > 0 \text{ for all } t \in [-1,1]$$

which shows that h is a convex function for $R = 1$, hence it is convex for any $0 \le R \le 1$.

Remark 1. The equation $32t^3 + 112t^2 + 126t + 51 = 0$ has the root $t = -1,905$.

Theorem. *Let $0 < R \le 1$, $q(z) = 1 + Rz$, with* $\mathrm{Re}\ q(z) > 0$ *and*

$$h(z) = 1 + Rz + \frac{Rz}{2+Rz} \tag{3}$$

be convex in U.

If $f \in \mathcal{A}$ *and*

$$\frac{zf'(z)}{f(z)} \prec h(z) \tag{4}$$

then

$$\frac{zF'(z)}{F(z)} \prec 1 + Rz$$

where F *is given by (1).*

Proof. From (1) we have

$$zF(z) = 2\int_0^z f(t)dt, \quad z \in U.$$

By using the derivate of this equality, with respect to z, after a short calculation, we obtain $zF'(z) + F(z) = 2f(z)$. This equality is equivalent to

$$F(z)\left[1 + \frac{zF'(z)}{F(z)}\right] = 2f(z). \tag{5}$$

If we let

$$p(z) = \frac{zF'(z)}{F(z)} \tag{6}$$

then (5) becomes

$$F(z)[1 + p(z)] = 2f(z). \tag{7}$$

By using the derivate of (7) with respect to z, after a short calculation, we obtain

$$\frac{zF'(z)}{F(z)} + \frac{zp'(z)}{1 + p(z)} = \frac{zf'(z)}{f(z)}$$

which, using (6), is equivalent to

$$p(z) + \frac{zp'(z)}{1 + p(z)} = \frac{zf'(z)}{f(z)}.$$

Using (4), we have

$$p(z) + \frac{zp'(z)}{1 + p(z)} \prec h(z).$$

According to Lemma B the function h given by (3) is convex and by applying Lemma A we deduce that $p(z) \prec q(z)$, which shows that F satisfies

$$\frac{zF'(z)}{F(z)} \prec 1 + Rz$$

and $q(z) = 1 + Rz$, is the best dominant.

From our theorem we deduce the following result:

Corollary. *Let n be a positive integer, $0 < R \le 1$, $q(z) = 1 + Rz$, with* $\mathrm{Re}\, q(z) > 0$, *and*

$$h(z) = 1 + Rz + \frac{Rz}{2 + Rz},$$

be convex in U.

If $f \in \mathcal{H}[0, n]$ and

$$\frac{zf'(z)}{f(z)} \prec h(z)$$

then

$$\left| \frac{zF'(z)}{F(z)} - 1 \right| < R,$$

where F is given by (1).

Remark. For $R = 1$, $n = 1$, the Corollary was obtained in [2].

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Department of Mathematics
University of Oradea
Str. Armatei Române, 5
410087 Oradea, Romania

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Junior spatial groups of $(22\underline{1})$-symmetry

A.A Shenesheutskaia

Abstract. The connection between junior groups of three independent kinds of antisymmetry transformations and junior groups of $(22\underline{1})$-symmetry, derived from space Fedorov groups was established. This connection allowed us to find all these groups.

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I. The problem of generalization of 230 space Fedorov groups with 32 crystallographic P-symmetries has a great theoretic and applied significance when receiving junior groups of these P-symmetries we assign indices to points of the space. By means of adequate physical explanation of these indices the junior groups of these P-symmetries coincide with the space groups of magnetic symmetry of crystals. On other hand, by these groups it is possible to model all various symmetry groups of the six-dimensional Euclidean space which keep invariant a three-dimensional plane in this space, i.e. groups of the category G_{63} [1].

The most point of the problem of derivation of junior groups from 230 space Fedorov groups for 32 crystallographic P-symmetries was solved by Kishinev geometricians [1, 2]. To finish this problem it remains to receive only junior space groups of $(22\underline{1})$-symmetry. This article is devoted to the resolving of this extensive problem.

II. The symbol $22\underline{1}$ is the symbol of thesenior space point group of symmetry and antisymmetry generated by rotations around three pairwise orthogonal two-fold rotational axes and by antiidentical transformation $\underline{1}$. One of the 32 crystallographic P-symmetries, modeling the symmetry group of a rectangular parallelepiped, i.e. the symmetry group mmm, generated by reflections in three pairwise orthogonal planes, is denoted by this symbol ($\underline{1}$ is interpreted as the reflection in a point, i.e. as an inversion) [3].

The groups mmm and $E_3 = \{\underline{1}\} \times \{1'\} \times \{^*1\}$ are isomorphic, where the group E_3 is the direct product of three groups of order 2, generated by antiidentical transformations of the kind 1, kind 2 and kind 3, respectively. The existence of such isomorphism between groups $22\underline{1}$ and E_3 makes it possible to reduce the problem of searching for junior space groups of $(22\underline{1})$-symmetry to the problem of searching for junior space groups of three-fold antisymmetry. Thus, to resolve the problem we need only junior groups of three independent kinds (i.e. groups of the type M^3), isomorphic to Fedorov groups.

To find these groups it is enough to know the number q of independent generators of Fedorov group by the change of which for respective transformations of antisymmetry of one kind we receive only junior groups. Then the initial group generates groups of the type M^l only when $l \leq q$. Such groups are derived from classic ones by means of the Shubnikov–Zamorzaev method: l or more symmetry transformations in the system of generators of Fedorov group are replaced with respective transformations of antisymmetry of different kinds, among which l kinds are independent [4].

To apply this method it is convenient to use the catalogue of Fedorov groups in Zamorzaev symbolism, which reflects the full system of generators of these groups. However if first we derive from each Fedorov group $(2^q-1)\times\ldots\times(2^q-2^{l-1})$ groups of type M^l and then we compare them in order to find identical and to eliminate extra ones, it is not quite relevant. That's why it is rational to divide the main problem into more simple problems. To receive groups of the type M^l it is convenient to proceed as follows:

a) to derive all possible point groups $M^m (m \leq l)$ from 32 generating groups G_{30} which are subgroups of 230 groups G_3;

b) to make the same procedure with 14 translation subgroups of groups G_3;

c) to finish the derivation of space groups of the type M^l, using the results of a) and b) [4].

However, to different junior space groups of the type M^3 obtained from one family identical groups of $(22\underline{1})$-symmetry may correspond, as the group $E_3 = (e,\ \underline{1},\ 1',\ {}^*1,\ \underline{1}',\ {}^*\underline{1},\ {}^*1',\ {}^*\underline{1}')$ contains 7 different kinds of antisymmetry transformations, and in the group $mmm = m_1m_2m_3 = (e, m_1, m_2, m_3, m_1m_2 = 2_{12}, m_1m_3 = 2_{13}, m_2m_3 = 2_{23}, m_1m_2m_3 = i_{123})$ only three transformations are essentially different, for example, $m_1, 2_{12}, i_{123}$, as the transformations m_1, m_2, m_3 and $2_{12}, 2_{13}, 2_{23}$ play the same geometrical role, respectively.

Consequently, for example, to the group $\{a, b, c\}(2' \cdot {}^*m : 2)$ and to five groups, obtained from this group by all permutations of signs –, /, * (which mean transformations of antisymmetry of kind 1, kind 2 and kind 3, respectively),

$\{\underline{a}, b, c\}({}^*2 \cdot m' : 2)$; $\{a', b, c\}(\underline{2} \cdot {}^*m : 2)$;

$\{a', b, c\}({}^*2 \cdot \underline{m} : 2)$; $\{{}^*a, b, c\}(\underline{2} \cdot m' : 2)$; $\{{}^*a, b, c\}(2' \cdot \underline{m} : 2)$,

i.e. to six different junior groups of three-fold antisymmetry of the family $18s$ correspond the following six identical groups of $(22\underline{1})$-symmetry:

$\{a^1, b, c\}(2^2 \cdot {}^3m : 2)$; $\{a^1, b, c\}(2^3 \cdot {}^2m : 2)$; $\{a^2, b, c\}(2^1 \cdot {}^3m : 2)$;

$\{a^2, b, c\}(2^3 \cdot {}^1m : 2)$; $\{a^3, b, c\}(2^1 \cdot {}^2m : 2)$; $\{a^3, b, c\}(2^2 \cdot {}^1m : 2)$.

Thus, only one group of $(22\underline{1})$-symmetry corresponds to six different groups of the type M^3.

Consequently, to obtain all different junior groups of $(22\underline{1})$-symmetry one needs to obtain all different junior groups of three-fold antisymmetry, to unite them in nests and to examine only representatives of these nests.

At the same time it is convenient to use the distribution of 230 space Fedorov groups in 34 different equivalence classes [5]:

1. $1s$; **2.** $2s$; **3.** $3s, 2a$; **4.** $4s, 26s, 1h, 33h, 3a, 7a, 42a$; **5.** $5s$;

6. $6s, 16s, 22s, 35s, 47s, 48s, 53s, 54s, 55s, 56s, 57s, 71s, 4h, 7h, 9h, 10h, 15h, 25h, 29h, 30h, 31h, 32h, 34h, 5a, 10a, 11a, 25a, 27a, 33a, 36a, 37a, 38a, 41a, 43a, 44a, 45a, 50a, 52a, 84a, 85a, 103a$; **7.** $7s$; **8.** $8s, 10s, 32s, 62a$; **9.** $9s$; **10.** $11s, 24h, 6a$; **11.** $12s$; **12.** $13s, 17h$; **13.** $14s, 15s, 24s, 58s, 6h, 11h, 20h, 23h, 35h, 36h, 15a, 16a, 23a, 54a, 55a, 60a, 61a$; **14.** $17s, 22h, 20a$; **15.** $18s$; **16.** $19s, 36s, 14a$; **17.** $20s$; **18.** $21s$; **19.** $23s, 40s, 41s, 42s, 49s, 63s, 65s, 66s, 69s, 73s, 27h, 42h, 43h, 44h, 45h, 46h, 47h, 53h, 54h, 12a, 32a, 34a, 35a, 39a, 40a, 48a, 49a, 51a, 53a, 76a, 77a, 79a, 80a, 81a, 82a, 83a, 86a, 96a, 99a, 100a, 101a, 102a$; **20.** $25s, 29s, 31s, 34s, 50s, 72s, 12h, 13h, 14h, 26h, 28h, 37h, 48h$; **21.** $27s$; **22.** $28s$; **23.** $37s$; **24.** $38s$; **25.** $61s$; **26.** $3h$; **27.** $5h$; **28.** $8h$; **29.** $19h$; **30.** $21h$; **31.** $1a$; **32.** $8a$; **33.** $21a$; **34.** $29a$.

Groups of one class have isomorphic so called antisymmetric characteristics and as a result generate the same quantity of groups of the type M^3. That's why it is enough to study only representatives of these classes: $1s, 2s, 3s, 4s, 5s, 6s, 7s, 8s, 9s, 11s, 12s, 13s, 14s, 17s, 18s, 19s, 20s, 21s, 23s, 25s, 27s, 28s, 37s, 38s, 61s, 3h, 5h, 8h, 19h, 21h, 1a, 8a, 21a, 29a$.

As in the groups $23s, 27s, 38s, 61s, 8a$ the number q of generators which we may replace by transformations of antisymmetry at the same time is smaller than 3, then we exclude these groups. Consequently, we have to examine 29 Fedorov groups, but not 34: $1s, 2s, 3s, 4s, 5s, 6s, 7s, 8s, 9s, 11s, 12s, 13s, 14s, 17s, 18s, 19s, 20s, 21s, 25s, 28s, 37s, 3h, 5h, 8h, 19h, 21h, 1a, 21a, 29a$.

So, only 141 Fedorov groups generate junior groups of three-fold antisymmetry.

III. By means of the above method all possible junior groups of three-fold antisymmetry were derived from each of the enumerated group. These groups were unibed in nests, which contain six, three, two or one group.

The obtained results were reduced in the following table:

Representatives of the equivalence classes in Fedorov symbolism	Quantity of the groups in the given equivalence class (including the group representative)	Quantity of the groups of type M^3, derived from the given group representative	Quantity of the groups of type M^3 derived from the groups from given equivalence class (including the group representative)	Quantity of the nests (junior groups of $(22\underline{1})$-symmetry) derived from the given group representative	Quantity of the nests (junior groups of $(22\underline{1})$-symmetry), derived from the groups from given equivalence class (including the group representative)
1	2	3	4	5	6
$1s$	1	$1\times 1 = 1$	1	1	1
$2s$	1	$2\times 3 + 2\times 1 = 8$	8	2+2=4	4
$3s$	2	$19\times 6 + 17\times 3 + 3\times 1 = 168$	336	19+17+3=39	78
$4s$	7	$4\times 6 + 6\times 3 = 42$	294	4+6=10	70
$5s$	1	$34\times 6 + 20\times 3 + 2\times 1 = 266$	266	34+20+2=56	56
$6s$	41	$12\times 6 + 4\times 3 = 84$	3444	12+4=16	656
$7s$	1	$151\times 6 + 79\times 3 + 5\times 1 = 1148$	1148	151+79+5=235	235

1	2	3	4	5	6
$8s$	4	$108 \times 6 + 36 \times 3 = 756$	3024	108+36=144	576
$9s$	1	$95 \times 6 + 61 \times 3 +$ $+3 \times 1 = 756$	756	95+61+3=159	159
$11s$	3	$3 \times 6 + 3 \times 3 +$ $+1 \times 1 = 28$	84	3+3+1=7	21
$12s$	1	$28 \times 6 + 14 \times 3 = 210$	210	28+14=42	42
$13s$	2	$602 \times 6 + 112 \times 3 = 3948$	7896	602+112=714	1428
$14s$	17	$212 \times 6 + 24 \times 3 = 1344$	22848	212+24=236	4012
$17s$	3	$204 \times 6 + 12 \times 3 = 1260$	3780	204+12=216	648
$18s$	1	$1051 \times 6 + 238 \times 3 +$ $+1 \times 2 + 6 \times 1 = 7028$	7028	1051+238+ +1+6=1296	1296
$19s$	3	$1228 \times 6 + 148 \times 3 = 7812$	23436	1228+148=1376	4128
$20s$	1	$73 \times 6 + 21 \times 3 +$ $+3 \times 1 = 504$	504	73+21+3=97	97
$21s$	1	$702 \times 6 + 69 \times 3 +$ $+1 \times 2 + 3 \times 1 = 4424$	4424	702+69+1+3=775	775
$25s$	29	$28 \times 6 = 168$	4872	28	812
$28s$	8	$104 \times 6 + 30 \times 3 = 714$	5712	104+30=134	1072
$37s$	4	$420 \times 6 = 2520$	10080	420	1680
$3h$	1	$52 \times 6 + 36 \times 3 = 420$	420	52+36=88	88
$5h$	2	$49 \times 6 + 21 \times 3 = 357$	714	49+21=70	140
$8h$	1	$1 \times 6 + 5 \times 3 = 21$	21	1+5=6	6
$19h$	1	$8 \times 6 + 16 \times 3 +$ $+2 \times 1 = 98$	98	8+16+2=26	26
$21h$	1	$1140 \times 6 + 128 \times 3 = 7224$	7224	1140+128=1268	1268
$1a$	1	$2 \times 3 + 1 \times 1 = 7$	7	2+1=3	3
$21a$	1	$68 \times 6 + 12 \times 3 +$ $+2 \times 2 = 448$	448	68+12+2=82	82
$29a$	1	$9 \times 6 + 1 \times 2 = 56$	56	9+1=10	10
Σ	141	41820	109139	7556	19469

Thus, the full number of junior groups of three independent kinds, derived from 141 space Fedorov groups G_3, is equal to 109139, and the full number of junior groups of the $(22\underline{1})$-symmetry, generated by these 141 space Fedorov groups G_3, is equal to 19469.

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State University of Moldova
A. Mateevici str. 60
Chisinau MD-2009, Moldova

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