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Early History of the Theory of Rings in Novosibirsk

L. A. Bokut

Abstract. It is a note on early history of ring theory in Novosibirsk. We mostly cover the first 10-15 years of the existence of the A. I.Malcev department of algebra and mathematical logic and A. I.Shirshov (1921–1981) laboratory of ring theory at the Sobolev Institute of Mathematics. By all means, this note is far from being complete, see also a survey by L. A. Bokut, I. P. Shestakov [16]. This article is written in a cooperation with E. N. Kuzmin (1938–2011) who was the active participant of events discussed below.

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1 Introduction

These notes are written thanks to an initiative of Dr. Larissa Sbitneva. At the opening ceremony of the 5th International conference on Nonassociative Algebra and its Applications, Oaxtepec, Morelos, Mexico, 2003, she asked me to say a few words on the history of ring theory in Novosibirsk. Some other participants of the conference supported this idea. I will restrict myself mostly to the first 10–15 years of the existence of the department of algebra and mathematical logic at the Sobolev Institute of Mathematics, Novosibirsk. By all means, these notes are far from being complete, see also a survey L. A. Bokut, I. P. Shestakov [16]. This article is written in cooperation with E. N. Kuzmin who was an active participant of events discussed below.

2 A. I. Malcev (1909–1967) and A. I. Shirshov (1921–1981) are the founders of ring theory in Novosibirsk

Let me recall that in 1957 prominent Russian mathematicians and mechanicians S. A. Khristianovich (1908–2000), M. A. Lavrentev (1900–1980), and S. L. Sobolev (1908–1986) came up with the idea of organizing a Siberian branch of the Soviet Academy of Sciences. Their idea was supported by the Russian leader at that time, N. S. Khruschev. As a result, the Russian government decided to create some 20 academic research institutes together with Novosibirsk State University and build a special town, now known as Akademgorodok, near Novosibirsk.

Thus, (now Sobolev) Institute of Mathematics was founded in 1957 by S. L. Sobolev, who had then been its director until 1983. He invited A. I. Malcev from Ivanovo (near Moscow) Pedagogical Institute to organize a department of algebra and mathematical logic.

A. I. Malcev was a graduate student (1934–1937) of a great Russian mathematician A. N. Kolmogorov (1903–1987), who recognized his very first result, the locality (compactness) theorem in mathematical logic, as the beginning of a new branch of mathematics [53]. This prediction had been fully established. Later Malcev was recognized as "a man who showed a road from logic to algebra" (A. Robinson). By the way, Malcev graduated from the Moscow State University, the "mehmat", in 1931 and began to work at Ivanovo in the same year. It should be mentioned that students of the MSU had to spend 4 years for undergraduate studies, had no diploma works and had no scientific advisors at that time. Malcev studied himself mathematical logic and philosophy at the MSU. He had proved the locality theorem of mathematical logic in 1934 and had sent a manuscript with the proof to Kolmogorov. As the result, Kolmogorov invited him immediately for graduate study in . . . algebra (it was a surprise for Malcev) at the MSU. Malcev had defended Candidate of Science Thesis at the MSU in 1937 (on the theory of abelian groups) and Doctor of Science Thesis at the Steklov Mathematical Institute, Kazan, December, 1941 (on the theory of representations of infinite dimensional algebras and infinite groups), with N. G. Chebotarev (1894–1947) (Kazan) and V. A. Tartakovskii (1901–1973) (Leningrad) as official experts. By the way, S. L. Sobolev was the director of the Steklov Mathematical Institute during the war in 1941–42 (the Institute had to move from Moscow to Kazan; since 1943, Sobolev was the first deputy-director of the Laboratory N2 of the Academy of Sciences of the USSR, now the Kurchatov Institute for Nuclear Research).

In nonassociative algebra, Malcev is known as an author of the Levy-Malcev theorem for Lie algebras, as the originator of the theory of Malcev algebras and binary-Lie algebras. He made profound contributions to the theory of Lie groups. Speaking about associative algebras, he was an author of the Malcev-Wedderburn theorem on finite dimensional associative algebras, a founder with O. Ore of the theory of imbedding of rings into skew fields (and semigroups into groups), an author of the Malcev-Neumann division ring construction, a founder of the representation theory of infinite algebras (and infinite groups) by matrices over fields. His collected papers have been published in two volumes [71, 72].

Also S. L. Sobolev invited A. I. Shirshov, a pupil of A. G. Kurosh (1908–1971), from Moscow State University to be the first deputy-director of the new institute. No doubt, the invitation was supported by Malcev who knew Shirshov's results very well. Malcev was an official expert on Shirshov's Doctor of Science Thesis, MSU, 1958, and admired it very much; as it happened, we with E. N. Kuzmin were at the defence meeting and remember that Malcev called Shirshov's Thesis "brilliant" (the other expert was V. M. Glushkov (1923–1982) (Kiev), a prominent specialist in algebra and cybernetics; by the way, his colleagues were trying to check some of Shirshov's calculations by computer). It worth to be mentioned, that A. I. Shirshov was the first deputy-dean of the faculty of mechanics and mathematics (the "mehmat") of the MSU at that time (the dean was A. N. Kolmogorov).

Novosibirsk was the home region for Shirshov, he had been born at Kolyvan and grown up at Aleisk, small towns near (by the Siberian scale) Novosibirsk [110]. What is more, he studied for one year (1939–1940) at Tomsk State University, that is also near Novosibirsk, and he had begun his high school teacher career at Aleisk. By the way, Shirshov was a high school teacher for 7 years during 1940–1950, with three years interruption, 1942–1945, for the Second World War. Shirshov graduated from Voroshilovograd (Lugansk) Pedagogical Institute in Ukraine by the distance education in 1949. He had started his graduate study at the MSU in 1950, had defended his Candidate of Science Thesis in 1953, and his Doctor of Science Thesis in 1958.

A. I. Shirshov is known for his contributions to the theories of free Lie algebras (Shirshov-Witt theorem on subalgebras, Lyndon-Shirshov words, the Composition-Diamond lemma, Gröbner-Shirshov bases), of PI-algebras (the Shirshov height theorem), of Jordan and alternative algebras (solution of the Kurosh problem, the Shirshov theorem on special Jordan algebras). His collected papers had been published in the book [107].

Shirshov had five students at the MSU: L. A. Bokut, G. V. Dorofeev, E. N. Kuzmin, V. N. Latyshev, and K. A. Zhevlakov (we graduated from the MSU in 1958–1961). Three of us (Kuzmin, Zhevlakov, and me) left Moscow for Novosibirsk with Shirshov, two others remained in Moscow. We had a number of students at Institute of Mathematics, Novosibirsk State University, Moscow State University and Moscow Pedagogical Institute: I. P. Shestakov, A. M. Slinko, A. A. Nikitin, I. M. Miheev, R. E. Roomeldi (1949–1999), A. S. Markovichev (students of Zhevlakov, and after his death, students of Shirshov); V. T. Filippov (1948–2001), F. S. Kerdman, Sh. M. Kasymov, O. Saudi (Syria) (students of Kuzmin, the first one, Filippov, joint with Shirshov); S. V. Pchelintsev (student of Dorofeev); V. E. Barbaumov, S. A. Pikhtilkov, Mekei Abish (Mongolia), I. L. Guseva, T. Gateva (Bulgaria), V. V. Borisenko, N. A. Iyudu, V. V. Schigolev (students of Latyshev); I. V. L'vov (1947–2003), G. P. Kukin (1948–2004), Yu. N. Maltsev, A. V. Yagzhev (1950–2001), V. K. Kharchenko, A. Z. Ananin, E. M. Zjabko (he had been excluded from the NSU after two years of education, see below), V. N. Gerasimov, Ts. Dashdorzh (Mongolia), R. Gonchigdorzh (Mongolia), A. N. Grishkov, A. A. Urman, V. V. Talapov, B. V. Tarasov, G. V. Kryazhovskikh, A. I. Valitskas, O. K. Bobkov, V. V. Vdovin, A. V. Chehonadskikh, A.Ṡ. Stern, A. Ya. Vais, N. G. Nesterenko, A. V. Sidorov, E. P. Petrov, A. T. Kolotov, A. R. Kemer, E. I. Zelmanov (my students, last three joint with Shirshov). Next generation of Shirshov's school include V. N. Zhelyabin (student of Shestakov and Slinko); Yu. A. Medvedev, A. V. Iltyakov, O. N. Smirnov, U. U. Umirbaev, I. M. Isaev, S. R. Sverchkov, V. G. Skosyrskii (1956–1995), S. V. Polikarpov, N. A. Pisarenko, S. Yu. Vasilovskii (students of Shestakov); A. P. Pozhidaev (student of Filippov); A. Ya. Belov (undergraduate student of Pchelintsev, Belov participated A. V. Mikhalev and V. N. Latyshev's seminar on ring theory at the MSU for many years); A. N. Koryukin (student of Kharchenko), and many others. My students, V. B. Kulchinovskii (joint with S. N. Vasilev, Irkutsk), E. N. Poroshenko, P. S. Kolesnikov (joint with I. V. L'vov and E. I. Zelmanov), E. S. Chibrikov, I. A. Firdman, I. A. Dolguntseva (joint with P. S. Kolesnikov) have got Candidate of Science Degrees at the Sobolev IM and the NSU. P. S. Kolesnikov has got P. Deligne grant (2006–2008) for his study of associative conformal algebras.

There were a lot of activities in algebra and logic at Novosibirsk and the USSR in 1960th. Some well known algebraists and logicians had visited Malcev and his group at Novosibirsk in the 1960s: P. S. Novikov (Moscow), A. Tarsky (Berkeley), B. Neumann (Canberra), P. G. Kontorovich (Sverdlovsk), L. A. Kaluzhnin (Kiev), D. A. Suprunenko (Minsk), V. M. Glushkov (Kiev), B. I. Plotkin (Riga), V. A. Andrunakievich (Kishinev), L. A. Skornyakov (Moscow), S. I. Adyan (Moscow), A. I. Kostrikin (Moscow), V. P. Platonov (Minsk), V. D. Belousov (Kishinev), A. L. Shmelkin (Moscow), L. N. Shevrin (Sverdlovsk), Yu. M. Ryabuhin (Kishinev), V. I. Arnautov (Kishinev). There was 5-th All-Union Algebra Colloquium at Novosibirsk in 1963 headed by A. I. Malcev. All leading specialists in Algebra and Logic of the USSR came to it, including A. G. Kurosh (Moscow). The preceding All-Union Algebra Colloquiums were: Moscow, 1958, 1959, A. G. Kurosh (Chair); Sverdlovsk, 1960, P. G. Kontorovich (Chair); Kiev, Ukraine, 1962, L. A. Kaluzhnin (Chair). Further All-Union Colloquiums were: Minsk, Belorussia, 1964, D. A. Suprunenko (Chair); Kishinev, Moldavia, 1965, V. A. Andrunakievich (Chair); Riga, Latvia, 1967, B. I. Plotkin (Chair); Gomel, Belorussia, 1968, V. A. Chunikhin (Chair); Novosibirsk, 1969, A. I. Shirshov (Chair). The last All-Union Mathematical Congress held in Leningrad at 1961 with algebra section headed by D. K. Faddeev. N. Jacobson (Yale) visited this Congress. A. I. Malcev was the head of Algebra Section of the Moscow International Mathematical Congress (1966). S. Amitsur (Jerusalem) and P. M. Cohn (London) came to this Congress. There was an All-Union Topological Conference at Novosibirsk in 1967 headed by A. I. Malcev. All leading specialists in topology from the USSR came to it, including P. S. Aleksandrov (Moscow). Also K. Kuratovsky and A. Mostovsky from Poland and M. Katetov from Czech-Slovakia had participated in the Topological Conference.

All of this stimulated the Novosibirsk Ring Theory group in a great respect.

Last but not least, N. Jacobson's profound books on Ring Theory [41]–[45] influenced all members of Shirshov's school very much.

3 Alternative algebras. K.A.Zhevlakov (1939–1972)

K. A. Zhevlakov came to Novosibirsk after graduating from the MSU in 1961. In his master degree work, Zhevlakov [129] proved a result all of us liked very much. He proved an analogue for alternative algebras of the Nagata–Higman (–Dubnov–Ivanov [24]) theorem: the solvability of any alternative algebra with an identity $x^n = 0$ of characteristic $p > n$ (or $p = 0$). After moving to Novosibirsk at 1961, he was trying to solve the analogous problem for Jordan algebras. Time was not ripe for this problem; it was solved for characteristic 0 by Efim Zelmanov 30 years later ([121], 1991); for characteristic $> 2n$, it was solved by V. Skosyrskii and E. Zelmanov

([102], 1983) only in the case of special Jordan algebras. We with E. N. Kuzmin remember that Zhevlakov had spent about two years trying to solve this problem (actually, Kuzmin and Zhevlakov had shared a room at an apartment at that time). Sometimes he thought that he had found a positive solution, other times he believed that he constructed a counter-example to the problem. But each time, he was able to find a mistake in his reasonings. At last, A. I. Malcev and A. I. Shirshov convinced him to abandon this problem. I remember how Malcev was once telling to Zhevlakov that the structure theory of rings, for example, alternative, is a good and respectable issue. It should be mentioned the first among Shirshov's students adored combinatorial problems of ring theory more then structural problems. Probably it was due to the influence of Shirshov's beautiful combinatorial papers. Malcev was trying to change this one-sided point of view. I should say also that N. Jacobson's book "Structure of rings" was very important for all members of Novosibirsk ring theory group. As the result, one can see a harmonious combination of both theories in papers by K. A. Zhevlakov and E. N. Kuzmin on the structure theory of alternative and Malcev algebras, later on in papers by I. P. Shestakov, V. T. Filippov, A. N. Grishkov, S. V. Pchelintsev on the same classes of algebras and on binary-Lie and $(-1, 1)$-algebras, and at last in works by E. I. Zelmanov on the structure theory of Jordan and Lie algebras with brilliant applications to group theory.

K. A. Zhevlakov made fast progress in the structure theory of alternative algebras, including the structure of alternative Artinian algebras [130], the existence of Jacobson radical in the class of alternative algebras [131], and so on (see [132]). He defended his Candidate of Science Thesis in 1965 and Doctor of Science Thesis in 1967, soon after Malcev's death. His work had been supported by S. P. Novikov, a 1970 Fields Laureate, and he had got a prestigious Lenin Komsomol Prize in 1970. K. A. Zhevlakov attracted to ring theory a group of undergraduate students including I. V. L'vov, Yu. N. Maltsev, G. P. Kukin, A. M. Slinko, A. A. Nikitin, I. P. Shestakov. The first three became soon my students and participated in my seminar "Associative rings and Lie algebras", and the other three participated in Zhevlakov's seminar on nonassociative rings. It should be mentioned that at the time we are speaking about (1960s) we had a hierarchy of seminars. At the top was "Algebra and Logic" seminar directed by A. I. Malcev before his death, then "Ring theory" seminar directed by A. I. Shirshov, and two student seminars in ring theory. The same was in the group theory (M. I. Kargapolov (1928–1976), Yu. I. Merzlyakov (1940–1995), V. N. Remeslennikov, A. I. Kokorin (1929–1987), V. M. Koputov, V. D. Mazurov), in model theory and mathematical logic (A. D. Taimanov (1917–1990), Yu. L. Ershov, A. V. Gladkii, D. M. Smirnov (1918–2005), M. I. Taizlin, D. A. Zakharov (1925–1996), L. L. Maksimova, I. A. Lavrov). I have to mention also Boris Abramovich Trakhtenbrot (born 1921) who was a student of the prominent mathematician P. S. Novikov (1900–1976), he headed a seminar in logic and computer science and was a chair of the automata theory department at the IM.

K. A. Zhevlakov has left a strong scientific trace in Novosibirsk school of ring theory. A well known book [132] (English translation [133]) had been based on lectures by Shirshov at the MSU and Zhevlakov at the NSU.

I. P. Shestakov made a great progress in the theory of alternative algebras, especially for free alternative algebras [99, 100] (the latter publication is a summary of his Doctor of Science Thesis, 1978). He had proved that the basis rank of the variety of alternative algebras is infinite (it was a solution of Shirshov's problem, see Ch. 7 below for some details) [101]. In a joint paper ([102], 1990), I. P. Shestakov and E. I. Zelmanov had described prime alternative super algebras over a field of characteristic not 2, 3, and had applied this result to a proof of nilpotency of the Jacobson radical of any free alternative algebra over a field of characteristic 0. The latter result was a solution of a Zhevlakov problem. A description of prime alternative algebras had been done earlier by M. Slater, a student of I. Herstein ([98], 1972).

Yu. A. Medvedev, a student of Shestakov, had proved that a periodic loop is locally finite if it is embeddable into an alternative PI-algebra [80].

Many results for alternative algebras had been done also by G. V. Dorofeev (see Ch. 7), S. V. Pchelintsev, V. T. Filippov, A. V. Iltyakov, S. R. Sverchkov, Yu. A. Medvedev, and others.

Recently I. P. Shestakov and U. U. Umirbaev [103]–[105] has solved one of the fundamental problems for polynomial automorphisms. In 1942 H. W. E. Jung had proved that any automorphism of an algebra $k[x, y]$ of polynomials over a field of characteristic 0 is tame (a product of elementary automorphisms). In 1972 M. Nagata had conjectured that the following polynomial automorphism over complex numbers

$$(x, y, z) \to (x - 2(xz + y^2)y - (xz + y^2)^2 z, y + (xz + y^2)z, z)$$

is not tame. At last in 2003 Shestakov and Umirbaev have proved that the Nagata's conjecture is true!

4 Jordan algebras

Some radicals in the class of (special) Jordan algebras had been studied by A. M. Slinko [114, 115]. He had proved that the Baer (lower) radical is locally nilpotent in special Jordan algebras, and the Levitzki (local nilpotent) radical is ideal-hereditary in the class of Jordan algebras.

The class of special Jordan algebras is not a variety, P. M. Cohn [18], but it is a quasi-variety. S. R. Sverchkov [116] had proved that this quasi-variety can not be defined by a finite number of quasi-identities. It is an analogue of a well known Malcev's result (1940) for the class of semigroups embeddable into groups.

V. N. Zhelyabin [127, 128] had proved theorems on splitting of the Jacobson radical for Jordan and alternative algebras over a Hensel ring that are analogous to the ones obtained for associative algebras by G. Azumaya (1951).

"Russian revolution in Jordan algebras" (these are K. McCrimmon's words) had been made by Efim Isaakovich Zelmanov at the end of 1970s-beginning of 1980s. His firsts of these results had been done before Shirshov's death. He had settled a long standing gap in the theory of Jordan algebras with minimal condition proving that the Jacobson radical is nilpotent in such algebras ([119], 1978). Zelmanov had proved local nilpotency of Jordan nil algebras of bounded index ([120], 1979). Previously it was proved by Shirshov (1957) for special Jordan algebras. Then he had described Jordan division algebras giving a positive response to a longstanding problem of Jacobson ([121], 1979). Also he had described prime Jordan algebras without nonzero nil ideals ([121], 1979). Some of these results of Zelmanov's I had announced in my talk at a Conference on Division Rings, Oberwolfach, 1978. P. M. Cohn and G. Bergman were among the participants. P. M. Cohn was very astonished by Zelmanov's results. I had given a manuscript of my talk to G. Bergman and he had sent it to N. Jacobson. I believe it was the first information about Zelmanov's results to the West mathematicians. Later Jacobson [42] had lectured Zelmanov's first results on structure theory of Jordan algebras with a great enthusiasm. He had also lectured on Skosyrskii's theorem [112] that the Levitzki radical of a special Jordan algebra J is the intersection of J with the Levitzki radical of an envelope.

A. I. Shirshov was very proud of Zelmanov's results. It was long before Zelmanov had obtained a solution of the Restricted Burnside Problem and long before he had got a Fields Medal. But Shirshov had understand a phenomenon of Zelmanov very well. He had told me once: "People will remember us for we save Zelmanov for science". By the way, I must say that Shirshov was very unhappy that Zelmanov had failed (!) to defend his Candidate of Science Thesis "Jordan Division Algebras" at a Science Counsel at the Institute of Mathematics on 25 of October, 1980. On this very day Shirshov's mother had died (they were living together for many years) and this very day was the last day that Shirshov had visited his dear Institute of Mathematics, when he was the first deputy-director since 1958 to 1973. Later on Zelmanov was successful in this business due to the help of S. L. Sobolev in May 1981, after Shirshov's death on 28 of February, 1981. Shirshov's Ring Theory Department had been divided into two laboratories: my laboratory "Associative and Lie rings" (with Ananin, Gerasimov, Kharchenko, Lvov, Zelmanov) and Shestakov's laboratory "Nonassociative rings" (with Filippov, Gainov, Kuzmin, Medvedev, Skosyrskii, together with two specialists in group theory, N. S. Romanovskii and S. A. Syskin). I would like to say my thanks to the first deputy-director of the IM at that time, a prominent specialist in Riemanian Geometry Viktor Andreevich Toponogov (1930–2004) for his help to establish my laboratory. In three years, Zelmanov had finished his "Jordan revolution" and had written his Doctor of Science Thesis "Jordan Systems and Graded Simple Lie algebras". He had successfully defend this Thesis at a Science Counsel headed by D. K. Faddeev, deputy-head was Z. I. Borevich, at Leningrad State University in 1985 (with some supports from A. I. Kostrikin, V. N. Latyshev, V. P. Platonov). The last Chapter 5 of his Dr.Sc. Thesis was "Burnside Type Problems: Algebraic Algebras" (algebraic Jordan algebras and algebraic Lie algebras). It was a beginning of his thoughts on Lie nil (Engel) algebras and finally on the Restricted Burnside Problem for finite groups, that was successfully finished in another 4 years [125, 126].

A lot of results for Jordan algebras had been proved by Yu. A. Medvedev [82–

84] at the end of 1980th. The results in the paper [82] continue the researches of I. P. Shestakov [Mat. Sb., Nov. Ser. 122 (164), No. 1 (9), 31–40 (1983)] concerning polynomial identities in finitely generated Jordan and alternative algebras. Let J be a finitely generated Jordan PI-algebra over a commutative ring R with $\frac{1}{2}$. Then:

1) The universal multiplicative enveloping algebra of J is a PI-algebra as well.

2) If the ring R is Noetherian then the nil radical of the algebra J is nilpotent.

3) The algebra of the multiplications of J is an associative PI-algebra.

In the paper [83], Medvedev proved that an absolute zero divisor in a finitely generated Jordan algebra generates a nilpotent ideal.

Medvedev's work [84] was based on the results and methods of his earlier study of Jordan A-algebras [Algebra Logika 26, No. 6, 731–755 (1987)]. In particular, he proved: The free Jordan algebra from more than two generators is not prime and has a nonzero center.

5 Malcev algebras and binary-Lie algebras

In 1955, A. I. Malcev [70] invented two classes of nonassociative algebras: Moufang–Lie algebras and binary-Lie algebras. A. A. Sagle [97] changed name "Moufang-Lie algebras" to "Malcev algebras". A great contribution to the theory of Malcev algebras had been made by Evgenii Nikiforovich Kuzmin (born in 1938). In the middle of the 1960s-beginning of 1970s, he proved some fundamental results on structure theory of Malcev algebras and on connections of Malcev algebras and local analytic Moufang loops [59–61], see also [63]. His results included a description of central simple finite dimensional (f.d.) Malcev algebras over a field of characteristic > 3. He had also proved the existence of local analytic Moufang loop with any given tangent f.d. Malcev algebra over the real field. Some of these results had been presented in a joint talk with A. I. Malcev at the All-Union Topological Conference in Novosibirsk a few days before Malcev's death. Kuzmin had defended his Doctor of Science Thesis on the subject in 1972. F. S. Kerdman, a student of Kuznim, had studied global analytic Moufang loops and their connections with Malcev algebras [48]. Later on Kuzmin's student Valerii Terentevich Filippov (1952–2001) was very successful in his study of Malcev algebras and alternative algebras. He had described central simple infinite dimensional Malcev algebras in [27]: all of them are Lie algebras. Also he invented a new class of algebras, the n-Lie algebras [28], which are now called Filippov algebras. Later Sh. M. Kasymov, a student of Kuzmin from Uzbekistan, had proved that Cartan subalgebras of any f.d. n-Lie algebra are conjugated in a case of algebraically closed field of characteristic 0 [46].

A. N. Grishkov [38] and E. N. Kuzmin [62] had independently proved an analogue of Levi's theorem for Malcev algebras.

Malcev algebras became a popular subject in Novosibirsk. Some important results on the subject have been made by I. P. Shestakov, A. N. Grishkov, S. V. Pchelintsev.

A lot of papers for binary-Lie algebras have been published by E. N. Kuzmin, A. N. Grishkov, V. T. Filippov, I. P. Shestakov. Kuzmin [58] had proved an analogue of Engel theorem for binary-Lie algebras. Grishkov [39] had established that any simple finite dimensional binary-Lie algebra over an algebraically closed field of characteristic 0 is Malcev algebra.

6 Other classes of non-associative algebras

The class of mono-composition algebras was invented by Alexei Timofeevich Gainov (born 1929). He was an Ivanovo student of A. I. Malcev. His first result [31] was a characterization of binary-Lie algebras by two identities. Gainov moved to Novosibirsk in 1960. He introduced mono-composition algebras as a generalization of the composition algebras [32].

Raul Roomeldi (1949–1999) had graduated from Tartu University (Estonia). He was a graduate student of Zhevlakov at the NSU and after his death a student of Shirshov. He had proved an analogue of the Nagata–Higman (–Dubnov–Ivanov) theorem for $(-1,1)$-algebras [94].

I. M. Miheev, a student of Zhevlakov, had proved an analogue of the Wedderburn principal theorem for $(-1,1)$-algebras [85]. He had resolved a long-standing question of A. A. Albert that there exists a simple, right alternative (infinite dimensional) algebra that is not alternative [86]. Later V. G. Skosyrskii [113] had proved that any simple, right alternative algebra either alternative or nil.

S. V. Pchelincev, a student of Dorofeev, had proved that the associator ideal of a free finitely generated $(-1,1)$-algebra is nilpotent [89].

A. A. Nikitin had proved an analogue of Wedderburn's principal theorem for (γ,δ)-algebras over a field of characteristic > 5 [87].

A. S. Markovichev had proved that radicals in (γ,δ)-algebras are hereditary [79].

7 Varieties of non-associative algebras

Georgii Vladimirovich Dorofeev (1938–2008) was as it was mentioned above a student of Shirshov at the MSU. His first result was an example of a solvable alternative algebra that is not nilpotent [21]. He constructed an identity that is valid on any 3-generated alternative algebras of characteristic 0 but not valid in the class of all alternative algebras [22]. This identity leads naturally to the question of whether the basis rank of the class of alternative algebras is finite or infinite. The question was known in Novosibirsk as a problem of Shirshov.

Later on I. P. Shestakov [100] proved that the basis rank is infinite. It means that the series

$$\mathrm{Alt}_1 \subseteq \mathrm{Alt}_2 \subseteq \cdots \subseteq \mathrm{Alt}_n \subseteq \mathrm{Alt}_{n+1} \subseteq \dots$$

does not stabilize at a finite step, where $\mathrm{Alt_n}$ is the variety of alternative algebras generated by he free alternative algebra of the rank n. V. T. Filippov [28, 29] had proved that the above series (over an associative-commutative ring) is strictly increasing at any step but possibly $n = 3$. Actually, both Shestakov's and Filippov's papers contain analogous results for the Malcev algebras.

At the end of 1970th, Dorofeev found identities that characterize the join of some important varieties of non-associative algebras [23].

Valerii Anatolievich Parfenov (1944–2016) was a student of Shirshov in Novosibirsk. He proved [90] that varieties of Lie algebras over a field of characteristic zero consist of a free semigroup under the Malcev-Neumann multiplication. It is a Lie algebra analogue of the Neumann-Shmelkin theorem for groups. The same kind of results have been obtained by my student Alexandr Aronovich Urman (born 1944) [117] for commutative varieties of (anticommutative) non-associative algebras.

8 Varieties of associative algebras

Viktor Nikolaevich Latyshev (born 1934) was a student of Shirshov at the MSU. He is a specialist in PI-algebras. Since his university years, he had been working on the Specht problem, whether every associative algebra over a field of characteristic zero is finitely based in the sense of identities. It should be noted that the Specht problem had been one of the most appreciated problems in Shirshov's school. A. I. Malcev also knew this problem and certainly recognized it as a central problem of the theory of varieties of associative algebras. Latyshev had been working on the problem for many years, doing more and more cases of varieties that are finitely based [65,66]. Among other aspects, his works kept the Specht problem alive not only in the USSR, but also in Bulgaria, (see M. B. Gavrilov (1940–1998) [36], G. K. Genov [33], A. P. Popov [92, 93], V. S. Drensky [25, 26]). There were close relations of Novosibirsk and Moscow algebraists with algebraists of this country. The Specht problem was solved positively by Alexandr Robertovich Kemer in 1986 (see [47]). Recently A. Ya. Belov, A. V. Grishin and V. V. Shchigolev published important results on the analogue of the Specht problem for associative algebras in finite characteristic. In general, the last problem has negative solution even for finitely generated algebras over finite fields, but for finitely generated algebras over an infinite field of finite characteristic the solution is still positive (see [6]).

Igor Vladimorovich Lvov (1947–2003) was my student. His main results belong to the theory of PI-algebras. He proved that any finite associative ring is finitely based, the Lvov-Kruse theorem. Lvov proved it in 1969, and published in 1973 [67]. The last result is also valid for finite alternative rings (I. V. Lvov [69]), finite Lie rings (Yu. A. Bahturin, A. Yu. Olshanskii, students of A. L. Shmelkin [4]), finite Jordan rings (Yu. A. Medvedev [81]), but not valid in general for finite (nonassociative) rings (S. V. Polin, a student of A. G. Kurosh [91]). All these positive results are analogues of the Oates-Powell theorem for finite groups [88] (Sheila Oates-Macdonald and M. B. Powel were students of G. Higman). Later I. V. Lvov, A. Z. Ananin, Yu. N. Maltsev, and V. T. Markov (a student of A. V. Mikhalev from the MSU) proved the following result in the middle of 1970th: Let M be a variety of associative algebras over an infinite field k. Then the following properties are equivalent: (1)

All finitely generated (f.g.) algebras from M are representable by matrices over commutative algebras; (2) All f.g. algebras from M are weakly Noetherian (i.e., any two-sided ideal is finitely generated); (3) All f.g. algebras from M are residually finite; (4) All f.g. algebras from M are Hopfian; (5) M has an identity $xy^nx = \sum_{i+j>0} \alpha_{ij} y^i x y^{n-i-j} x y^j$ $(\alpha_{ij} \in k)$ (see a survey by Bokut, Kharchenko, Lvov [13] translated in [54]). A variety M with these properties is sometimes called a Hilbert-Malcev variety. Just before his death, Lvov published [69] a detailed proof of A. Smoktunovich's result on the existence of simple nil associative algebra.

Yu. N. Maltsev in his Candidate of Science Thesis, 1973, had proved the following interesting results. All identities of an algebra of all upper triangular $n \times n$ matrices over a field of characteristic zero are the consequences of only one, $[x_1, x_2][x_3, x_4] \dots [x_{2n-1}, x_{2n}] = 0$ [73]. If R is an algebra that is nil over a right (algebra) ideal A satisfying an identity of degree d, then R satisfies a standard identity of degree d provided R has no nonzero nil ideals [74]. Actually Zelmanov [118] had later proved that if an algebra has no nonzero nil ideals and is nil over a PI-subalgebra then it is a PI-algebra. A ring R is said to be an H-extension of its subring A if, for every $x \in R$, there is a natural $n > 1$ such that $x^n - x \in A$. If A is commutative, then the ring R satisfies the identity $[[x_1, x_2], [x_3, x_4]] = 0$; if R is an algebra and A a right ideal of R satisfying an identity, then R satisfies an identity [75]. Also Maltsev had described varieties of associative algebras with the commutative product of subvarieties [76], just non-commutative varieties of rings (*Sib. Mat. J.*, 17, 803–810 (1976)) and found a basis of identities of the second order matrices over a finite field (*Algebra Logika*, 17, 18–20 (1978)). Later (1986) he had defended Doctor of Science Thesis at the LSU, Yu. N. Maltsev, *Critical rings and varieties of associative rings* (see [77, 78]).

9 Lie algebras, associative algebras, and groups

In 1958, published in [7], I found a basis of a free Lie algebra that is compatible with the derived series (see also [95]). It gives a basis of a free solvable Lie algebra. In the same paper, a basis of any free polynilpotent Lie algebra had been found. These results are based on a Shirshov's result from his Candidate of Science Thesis [106], published in [107], on series of bases of free Lie algebras (see also [96]). Some applications of my basis had been found by V. N. Latyshev [65], A. L. Shmelkin [111], and Yu. M. Gorchakov [37]. In 1959, published in [8], I generalized a result by J. Dixmier [20] on nilpotent Lie algebras. Those were my master degree results. In my Candidate of Science Thesis, 1963, I proved that any Lie algebra can be imbedded into an algebraically closed Lie algebra (in the sense that any equation over the algebra has a solution in this algebra) [9]. It was initiated by P. M. Cohn's result [19] that any Lie algebra is embeddable into a division Lie algebra. The proof used Shirshov's method [108] on what is now called the Gröbner-Shirshov bases for ideals of free Lie algebras. In [10], I had actually found Gröbner-Shirshov bases for P. S. Novikov's groups, and based on it, I had fully analyzed the conjugacy problem for these groups. As a result, I proved that for any Turing degree of insolvability there exists a Novikov's group with this degree of the conjugacy problem.

In [11], I had found an example of a semigroup S such that the multiplicative semigroup of a semigroup algebra of S (namely, $GF(2)\langle S\rangle$) can be imbedded into a group but the algebra can not be imbedded into any division ring. Up to now, this is the only known example of a semigroup with the property. The proof is based on a (relative) Gröbner-Shirshov basis of the universal group of the multiplicative semigroup of the algebra $GF(2)\langle\langle S\rangle\rangle$, of infinite power series over S with coefficients in $G(2)$. In particular, it gave a solution of a Malcev's problem (see [71], p. 6).

Last two results consist of my Doctor of Science Thesis, 1969.

In [12], I had proved that some recursively presented Lie algebras can be imbedded into finitely presented Lie algebras. It gave the existence of a finitely presented Lie algebra with the unsolvable word (equality) problem (solution of a problem of Shirshov [106]). A proof is based on Gröobner-Shirshov bases for Lie algebras.

Explicit examples of finitely presented Lie algebras with the unsolvable word problem had been found by my student Georgii Petrovich Kukin (1948–2004) [55]. In [56], he proved that the Cartesian subalgebra of the free product of Lie algebras is free. Also he had found a description of any subalgebra of the free (amalgamated) product of Lie algebras by means of generators and defining relations [57]. Recently E. S. Chibrikov [17] has solved Kukin's problem of an explicit construction of a left normed basis of a free Lie algebra.

Our joint book with G. P. Kukin [15] contains some of results mentioned above in this chapter, see also my survey [14].

My student since 1970 Vladislav Kirillovich Kharchenko in his Master of Science Diploma, 1974, had proved that if the ring of invariants R^G of an associative ring R with a finite group G of automorphisms is a PI-ring, then R is also a PI-ring, provided R has no additive $|G|$-torsion [49]. He had described [50] the structure of prime rings satisfying a generalized identity with automorphisms. This generalized a theorem of W. S. Martindale (1969) and was in the same spirit as a theorem of S. A. Amitsur on rings with involution (1969). In the same paper he had answered in the affirmative a question studied by G. M. Bergman and I. M. Isaacs (1973): Let G be a finite group of automorphisms of a ring R without nilpotent elements; then $R^G \neq (0)$. There were main results of his Candidate of Science Thesis, 1976. Kharchenko had published a survey "Groups and Lie algebras acting on noncommutative rings" [51], 1980, and had got his Doctor of Science Degree on the subject in 1984 at the Leningrad State University. Later Kharchenko published his results on non-commutative Galois theory in his well known book [52].

Victor Nikolaevich Gerasimov was also my student since 1970 (in fact, Gerasimov and Kharchenko were classmates). His 1974 Master of Science Diploma [34] contains a deep study of one-relator associative algebras. From his results it follows that the Hilbert series of any one-relator homogeneous associative algebra is rational [5]. His Candidate of Science Thesis, V. N. Gerasimov, *Free associative algebras and inverting homomorphisms of rings*, had been translated by the AMS, together with ones by N. G. Nesterenko, *Representations of algebras by triangular matrices*, and

A. I. Valitskas, *Embedding rings in (Jacobson) radical rings and rational identities of (Jacobson) radical algebras* [35].

Last but not least, Aleksandr Zigfridovich Ananin was my student since 1971. His first paper with my other student Evgenii Mikhailovich Zjabko [1], 1974 had contained a solution of a well known C. Faith problem. Let me give a review by W. G. Leavitt, see (MR0360721 (50 13168)) of this paper that shows the real significance of it (remember that the authors were 2nd year undergraduate students): "For a ring R, consider the property: $(*)$ For an arbitrary pair $x, y \in R$ there exist positive integers $m(x,y)$, $n(x,y)$ such that $x^{m(x,y)}$ commutes with $y^{n(x,y)}$. The authors show in a very ingenious way that if R has property $(*)$ and no nil ideals then R is commutative. Even more, it is shown that if R is arbitrary with $(*)$ then the set I of all nilpotent elements of R is an ideal of R, with R/I commutative. This paper is the last in a long sequence of commutativity theorems by various authors, the previous best result being that of A. I. Lihtman [Mat. Sb. (N. S.) 83 (125) (1970), 513–523; MR 42 6023] who proved the same two theorems for the special case of $(*)$ in which m and n are functions, respectively, of x and y alone." Later this theorem had been reproved by I. Herstein [40]. Zjabko was a very promising mathematician. It was a big tragedy for him and for us, that Zjabko had been excluded (1973) from the NSU for "dirtiness in his dormitory room" despite our efforts with Shirshov to save him (he was an excellent student but "worse luck", he was a Jew (!?)). Later Ananin had proved important results on (triangular) matrix representable varieties of associative algebras [2, 3].

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L. A. Bokut
Sobolev Institute of Mathematics
Russian Academy of Sciences Siberian Branch
Novosibirsk 630090, Russia
E-mail: *bokut@math.nsc.ru*

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Pretorsions in modules and associated closure operators

A. I. Kashu

Abstract. This article contains the results on the pretorsions of the module category R-Mod and on the closure operators defined by them. The pretorsions of R-Mod can be described in diverse forms: by classes of modules, filters of left ideals of R, closure operators, dense submodules, etc. In the set $\mathbb{PT}$ of pretorsions of R-Mod the main operations are studied, as well as their expressions in terms of classes of modules, filters, operators, etc. The approximations of pretorsions by jansian pretorsions and by torsions are mentioned.

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1 Introduction. Preliminary notions and facts

In this work the pretorsions of a module category R-Mod and the associated closure operators are studied. The main operations in the set $\mathbb{PT}$ of pretorsions of R-Mod are investigated. The multilateral descriptions of pretorsions of R-Mod are accentuated. Pretorsions of R-Mod can be considered as subfunctors of the identity functor of R-Mod (r); as pretorsion classes of R-Mod $(\mathcal{T}_r)$; as filters of left ideals of R $(\mathcal{E}_r)$; as closure operators of the lattice $\mathbb{L}({}_R R)$ of left ideals of R (t_r); as closure operators of the category R-Mod (C^r); as functions defined by dense submodules $(\boldsymbol{\mathfrak{F}}_s^r)$.

The main operations in $\mathbb{PR}$ are investigated and the representations of them by corresponding constructions $(\mathcal{T}_r, \mathcal{E}_r, C^r$, etc.) are indicated. For the given pretorsion $r \in \mathbb{PT}$ the least jansian pretorsion or torsion containing r is shown.

Let R be a ring with unit $1 \neq 0$ and R-Mod be the category of unitary left R-modules. A *preradical* r of R-Mod is a subfunctor of identity functor of R-Mod, i.e. $r(M) \subseteq M$ for every $M \in R$-Mod and $f\big(r(M)\big) \subseteq r(M')$ for every R-morphism $f \colon M \to M'$ of R-Mod. A preradical r is *hereditary* (or *pretorsion*) if $r(N) = r(M) \cap N$ for every $N \in \mathbb{L}(M)$, where $\mathbb{L}(M)$ is the lattice of submodules of M [1–4].

We denote by $\mathbb{PR}$ the class of all preradicals of R-Mod, and by $\mathbb{PT}$ the class (set) of all pretorsions of R-Mod. Every preradical $r \in \mathbb{PR}$ defines two classes of modules:

$\mathcal{T}_r = \{M \in R\text{-Mod} \mid r(M) = M\}$ – the class of *r-torsion* modules;

$\mathcal{F}_r = \{M \in R\text{-Mod} \mid r(M) = 0\}$ – the class of *r-torsionfree* modules.

The class $\mathcal{K} \subseteq R$-Mod is called *pretorsion class* if it is closed under homomorphic images and direct sums. If $\mathcal{K} \subseteq R$-Mod is closed under submodules, it is called *hereditary* class. It is well known the following description of pretorsions by classes of modules [1–4].

Proposition 1.1. *There exists a monotone bijection between the pretorsions of R-Mod and* ***hereditary pretorsion classes*** *of R-Mod. It is defined by the rules:* $r \rightsquigarrow \mathcal{T}_r,\ \mathcal{T} \rightsquigarrow r^{\mathcal{T}}$, *where* $r^{\mathcal{T}}(M) = \sum\limits_{\alpha \in \mathfrak{A}} \{N_\alpha \in \mathbb{L}(M) \mid N_\alpha \in \mathcal{T}\}$.

An important peculiarity of pretorsions consists in the fact that they can be characterized by the special sets of left ideals of R ([1–4]). A set of left ideals $\mathcal{E} \subseteq \mathbb{L}({}_R R)$ is called a *preradical filter* (left linear topology, topologizing filter) if the following conditions are satisfied:

(a_1) If $I \in \mathcal{E}$ and $a \in R$, then $(I : a) = \{x \in R \mid xa \in I\} \in \mathcal{E}$;

(a_2) If $I \in \mathcal{E}$ and $I \subseteq J$, $J \in \mathbb{L}({}_R R)$, then $J \in \mathcal{E}$;

(a_3) If $I, J \in \mathcal{E}$, then $I \cap J \in \mathcal{E}$.

Proposition 1.2. *There exists a monotone bijection between the pretorsions of R-Mod and the* ***preradical filters*** *of R. It is defined by the mappings:*

$r \rightsquigarrow \mathcal{E}_r = \{I \in \mathbb{L}({}_R R) \mid r(R/I) = R/I\}$;

$\mathcal{E} \rightsquigarrow r_{\mathcal{E}}, \quad r_{\mathcal{E}}(M) = \{m \in M \mid (0 : m) \in \mathcal{E}\}$.

Remark. From the Proposition 1.2 follows that $\mathbb{PT}$ is a set, in contrast to $\mathbb{PR}$ which in general case is a class.

Therefore investigating the pretorsions we can use the diverse form of their expressions: $r, \mathcal{T}_r, \mathcal{E}_r$. The other three forms of presentation of pretorsions will be indicated in the following account.

2 Operations in the set of pretorsions $\mathbb{PT}$

In the set $\mathbb{PT}$ of pretorsions of R-Mod can be defined the following operations:

– *the meet* $\bigwedge\limits_{\alpha \in \mathfrak{A}} r_\alpha$, where $\big(\bigwedge\limits_{\alpha \in \mathfrak{A}} r_\alpha\big)(M) = \bigcap\limits_{\alpha \in \mathfrak{A}} r_\alpha(M)$, $\{r_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq \mathbb{PT}$;

– *the join* $\bigvee\limits_{\alpha \in \mathfrak{A}} r_\alpha$, where $\bigvee\limits_{\alpha \in \mathfrak{A}} r_\alpha = \bigwedge \{s \in \mathbb{PT} \mid s \geq r_\alpha \ \ \forall\, \alpha \in \mathfrak{A}\}$;

– *the product* $r \cdot s$, where $(r \cdot s)(M) = r\big(s(M)\big)$;

– *the coproduct* $r \# s$, where $[(r \# s)(M)]/s(M) = r\big(M/s(M)\big)$.

Remarks. 1. The product $r \cdot s$ of two pretorsions coincides with their meet $r \wedge s$, since using the heredity of r we have:

$$(r \cdot s)(M) = r\,(s(M)) = r(M) \cap s(M) = (r \wedge s)(M).$$

So in continuation we consider the set $\mathbb{PT}\,(\wedge, \vee, \#)$ equipped by three operations, where $\mathbb{PT}\,(\wedge,\vee)$ is a complete lattice.

2. In [1] the operation $(r : s)$ is defined in $\mathbb{PR}$ by the rule $[(r : s)(M)]/r(M) = r\big(M/s(M)\big)$, so $(r : s) = s\,\#\,r$. Our notation is more convenient and more coordinated with the other notations.

A series of properties of the defined operations are indicated in [1, 4], etc.

Now we will show how can be expressed the operations of $\mathbb{PT}\,(\wedge,\vee,\#)$ by the classes of modules $\mathcal{T}_r$, corresponding to the pretorsions $r \in \mathbb{PT}$. For that we remind that P. Gabriel [5] defined the product $\mathbb{C}\cdot\mathbb{D}$ of two closed (fermeé) classes of modules as follows:

$$\mathbb{C} \cdot \mathbb{D} = \{M \in R\text{-Mod} \mid M/\,\mathbb{D}M \in \mathbb{C}\},$$

where $\mathbb{D}M = \sum\limits_{\alpha\in\mathfrak{A}} \{N_\alpha \in \mathbb{L}(M) \mid N_\alpha \in \mathbb{D}\}$. We will preserve this rule, changing only the notation for hereditary pretorsion classes:

$$\mathcal{T}_r \,\#\, \mathcal{T}_s = \{M \in R\text{-Mod} \mid M/\,s(M) \in \mathcal{T}_r\}.$$

In parallels with the operations in $\mathbb{PT}$, we define the following operations on the classes of modules of the form $\mathcal{T}_r$, where $r \in \mathbb{PT}$:

- *the meet*: $\bigwedge\limits_{\alpha\in\mathfrak{A}} \mathcal{T}_{r_\alpha} = \bigcap\limits_{\alpha\in\mathfrak{A}} \mathcal{T}_{r_\alpha}$;
- *the join*: $\bigvee\limits_{\alpha\in\mathfrak{A}} \mathcal{T}_{r_\alpha} = \bigcap \{\mathcal{T}_s \mid \mathcal{T}_s \supseteq \mathcal{T}_{r_\alpha} \ \ \forall\, \alpha \in \mathfrak{A}\}$;
- *the coproduct*: $\mathcal{T}_r \,\#\, \mathcal{T}_s = \{M \in R\text{-Mod} \mid M/\,s(M) \in \mathcal{T}_r\}$.

Now we indicate the concordance between the operations of $\mathbb{PT}$ and the operations with the hereditary pretorsion classes of R-Mod.

Proposition 2.1. $\mathcal{T}_{\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha} = \bigwedge\limits_{\alpha\in\mathfrak{A}} \mathcal{T}_{r_\alpha}$ *for every family* $\{r_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq \mathbb{PT}$.

Proof. By the definitions we have:

$$M \in \mathcal{T}_{\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha} \Leftrightarrow \big(\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha\big)(M) = M \Leftrightarrow \bigcap_{\alpha\in\mathfrak{A}} r_\alpha(M) = M \Leftrightarrow r_\alpha(M) = M \ \ \forall\,\alpha \in \mathfrak{A} \Leftrightarrow$$

$$\Leftrightarrow M \in \mathcal{T}_{r_\alpha} \ \ \forall\,\alpha \in \mathfrak{A} \Leftrightarrow M \in \bigwedge_{\alpha\in\mathfrak{A}} \mathcal{T}_{r_\alpha}. \qquad \square$$

Similarly from the definitions follows the

Proposition 2.2. $\mathcal{T}_{\bigvee_{\alpha\in\mathfrak{A}} r_\alpha} = \bigvee\limits_{\alpha\in\mathfrak{A}} \mathcal{T}_{r_\alpha}$. $\square$

Proposition 2.3. $\mathcal{T}_{r\#s} = \mathcal{T}_r \# \mathcal{T}_s$ *for every pretorsions* $r, s \in \mathbb{PT}$.

Proof. By the definition of coproduct we obtain:

$$M \in \mathcal{T}_{r\#s} \Leftrightarrow (r \# s)(M) = M \Leftrightarrow [(r \# s)(M)]/s(M) = M/s(M) \Leftrightarrow$$

$$\Leftrightarrow r\big(M/s(M)\big) = M/s(M) \Leftrightarrow M/s(M) \in \mathcal{T}_r \Leftrightarrow M \in \mathcal{T}_r \# \mathcal{T}_s. \qquad \square$$

In continuation we will consider the expression of operations of $\mathbb{PT}$ by the corresponding *preradical filters* $\mathcal{E}_r$ of pretorsions $r \in \mathbb{PT}$. Denote $\boldsymbol{\mathfrak{PF}}$ the set of all preradical filters of R and define in this set the following operations:

- *the meet*: $\bigwedge\limits_{\alpha\in\mathfrak{A}} \mathcal{E}_{r_\alpha} = \bigcap\limits_{\alpha\in\mathfrak{A}} \mathcal{E}_{r_\alpha}$;
- *the join*: $\bigvee\limits_{\alpha\in\mathfrak{A}} \mathcal{E}_{r_\alpha} = \bigcap \{\mathcal{E} \in \boldsymbol{\mathfrak{PF}} \mid \mathcal{E} \supseteq \mathcal{E}_{r_\alpha} \ \ \forall\, \alpha \in \mathfrak{A}\}$;
- *the coproduct*: $\mathcal{E}_r \# \mathcal{E}_s = \{I \in \mathbb{L}(_RR) \mid \exists\ H \in \mathcal{E}_r,\ I \subseteq H$ such that $(I : a) \in \mathcal{E}_s \ \ \forall\, \alpha \in H\}$.

Remark. The latter operation is defined in [4] by changing the order of terms. Our notation is harmonized with the previous ones.

Now we show the relations between these operations and the operations of $\mathbb{PT}$.

Proposition 2.4. $\mathcal{E}_{\bigwedge\limits_{\alpha\in\mathfrak{A}} r_\alpha} = \bigwedge\limits_{\alpha\in\mathfrak{A}} \mathcal{E}_{r_\alpha}$ *for every family* $\{r_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq \mathbb{PT}$.

Proof follows from the Proposition 2.1. $\square$

Proposition 2.5. $\mathcal{E}_{\bigvee\limits_{\alpha\in\mathfrak{A}} r_\alpha} = \bigvee\limits_{\alpha\in\mathfrak{A}} \mathcal{E}_{r_\alpha}$.

Proof follows from the Proposition 2.2. $\square$

Proposition 2.6. $\mathcal{E}_{r\#s} = \mathcal{E}_r \# \mathcal{E}_s$ *for every* $r, s \in \mathbb{PT}$.

Proof. ($\subseteq$) Let $I \in \mathcal{E}_{r\#s}$. Then from the Proposition 2.3 follows:

$$R/I \in \mathcal{T}_{r\#s} = \mathcal{T}_r \# \mathcal{T}_s = \{M \in R\text{-Mod} \mid M/s(M) \in \mathcal{T}_r\}.$$

Therefore $(R/I)\,/\,s(R/I) \in \mathcal{T}_r$.

Now we consider the left ideal $H \subseteq R$ defined by the rule $(H/I) = s(R/I)$. Then $(R/I)\,/\,(H/I) \in \mathcal{T}_r$, so $R/H \in \mathcal{T}_r$, i.e. $H \in \mathcal{E}_r$. Moreover, from the definition of H we have $H/I \in \mathcal{T}_s$.

So we have a left ideal $H \in \mathcal{E}_r$, $I \subseteq H$ with the condition $H/I \in \mathcal{T}_s$ (i.e. $(I : a) \in \mathcal{E}_s$ for every $a \in H$). By the definition this means that $I \in \mathcal{E}_r \# \mathcal{E}_s$.

($\supseteq$) Let $I \in \mathcal{E}_r \# \mathcal{E}_s$, i.e. there exists a left ideal $H \subseteq R$ such that $I \subseteq H$ and $H/I \in \mathcal{T}_s$. Consider the left ideal $H' \subseteq R$ defined by the rule $H'/I = s(R/I)$. From the condition $H/I \in \mathcal{T}_s$ follows that $H/I \subseteq s(R/I) = H'/I$, so $H \subseteq H'$. Since $H \in \mathcal{E}$, now we have $H' \in \mathcal{E}_r$, i.e. $R/H' \in \mathcal{T}_r$.

From the other hand, by Proposition 2.3 and definitions we have:

$$\mathcal{E}_{r\# s} = \big\{ I \in \mathbb{L}(_R R) \mid R/I \in \mathfrak{T}_{r\# s} = \mathfrak{T}_r \# \mathfrak{T}_s =$$
$$= \{ M \in R\text{-Mod} \mid M/ s(M) \in \mathfrak{T}_r \} \big\} = \{ I \in \mathbb{L}(_R R) \mid (R/I)/ s(R/I) \in \mathfrak{T}_r \} =$$
$$= \{ I \in \mathbb{L}(_R R) \mid (R/I)/ (H'/ I) \in \mathfrak{T}_r \} = \{ I \in \mathbb{L}(_R R) \mid R/H' \in \mathfrak{T}_r \}.$$

Now from the relation $R/H' \in \mathfrak{T}_r$ obtained above follows that $I \in \mathcal{E}_{r\# s}$. □

3 Pretorsions and closure operators in $\mathbb{L}(_R R)$

In this section we will indicate a new form of expression for pretorsions of R-Mod by some closure operators of the lattice $\mathbb{L}(_R R)$ of left ideals of R. With this intention we consider a mapping $t\colon \mathbb{L}(_R R) \to \mathbb{L}(_R R)$ and the following conditions on t:

1°) $t(I) \supseteq I$ (*extension*);

2°) $t\big(t(I)\big) = t(I)$ (*idempotency*);

3°) $I \subseteq J \Rightarrow t(I) \subseteq t(J)$ (*monotony*);

4°) $t(I : a) = \big(t(I) : a\big)$ $\forall\, a \in R$ (*modularity*);

5°) $t(I \cap J) = t(I) \cap t(J)$ (*linearity*).

It is well known that the conditions 1°)−3°) define the ordinary notion of *closure operator* of the lattice $\mathbb{L}(_R R)$.

Definition 3.1. If the mapping t satisfies the conditions 1°)−4°), then it is called the *modular closure operator* of $\mathbb{L}(_R R)$ [3, 6]. If t satisfies the conditions 1°), 3°), 4°), 5°), then it will be called the *modular preclosure operator* of $\mathbb{L}(_R R)$.

There exists a monotone bijection between the *torsions* of R-Mod and the modular closure operators of $\mathbb{L}(_R R)$ [3, 6]. This bijection is obtained as follows:

$$r \rightsquigarrow t_r, \quad t_r(I) = \{ a \in R \mid (I : a) \in \mathcal{E}_r \};$$
$$t \rightsquigarrow r_t, \quad r_t(M) = \{ m \in M \mid t(0 : m) = R \}.$$

Now we will show the generalization of this result for the case of pretorsions [7].

Proposition 3.1. *Let $r \in \mathbb{PT}$ and $\mathcal{E}_r$ be the associated preradical filter. Define the operator t_r of $\mathbb{L}(_R R)$ by the rule:*

$$t_r(I) = \{ a \in R \mid (I : a) \in \mathcal{E}_r \}.$$

Then t_r is a modular preclosure operator of $\mathbb{L}(_R R)$.

Proof. Verify the conditions 1°), 3°), 4°), 5°) for t_r.

1°) If $a \in I$, then $(I : a) = R$, $R \in \mathcal{E}_r$, so $a \in t_r(I)$.

3°) If $I \subseteq J$ and $a \in t_r(I)$, then $(I : a) \in \mathcal{E}_r$. From the relation $(I : a) \subseteq (J : a)$ by (a_2) it follows that $(J : a) \in \mathcal{E}_r$, so $a \in t_r(J)$.

4°) By the definitions we have:

$$t_r(I:a) = \{x \in R \mid \big((I:a):x\big) = (I:xa) \in \mathcal{E}_r\};$$
$$\big(t_r(I):a\big) = \{x \in R \mid xa \in t_r(I)\} = \{x \in R \mid (I:xa) \in \mathcal{E}_r\},$$

so 4°) is true.

5°) The expressions of 5°) have the form:

$$t_r(I \cap J) = \{a \in R \mid \big((I \cap J):a\big) \in \mathcal{E}_r\} = \{a \in R \mid (I:a) \cap (J:a) \in \mathcal{E}_r\};$$
$$t_r(I) \cap t_r(J) = \{a \in R \mid (I:a) \in \mathcal{E}_r\} \cap \{a \in R \mid (J:a) \in \mathcal{E}_r\} =$$
$$= \{a \in R \mid (I:a) \cap (J:a) \in \mathcal{E}_r\},$$

therefore 5°) is true. □

Proposition 3.2. *Let t be a modular preclosure operator of $\mathbb{L}({}_RR)$. Define the function r_t by the rule:*

$$r_t(M) = \{m \in M \mid t(0:m) = R\}$$

for every $M \in R$-Mod. Then r_t is a pretorsion of R-Mod.

Proof. It is obvious that the set $r_t(M)$ forms a submodule of M. Moreover, for every R-morphism $f\colon\ M \to M'$ we have $f\big(r_t(M)\big) = \{f(m) \mid t(0:m) = R\}$. Since $\big(0:f(m)\big) \supseteq (0:m)$, we obtain $t\big(0:f(m)\big) \supseteq t(0:m) = R$, so $t\big(0:f(m)\big) = R$, i.e. $f(m) \in r_t(M')$. Therefore $f\big(r_t(M)\big) \subseteq r_t(M')$ and r_t is a preradical of R-Mod.

Finally, for every $N \in \mathbb{L}(M)$ we have:

$$r_t(M) \cap N = \{n \in N \mid n \in r_t(M)\} = \{n \in N \mid t(0:n) = R\} = r_t(N),$$

so r_t is hereditary, i.e. $r_t \in \mathbb{PT}$. □

Theorem 3.3. *The mappings $r \rightsquigarrow r_t$ and $t \rightsquigarrow r_t$ define a monotone bijection between the pretorsions of R-Mod and the modular preclosure operators of $\mathbb{L}({}_RR)$.*

Proof. Taking into account the Propositions 3.1 and 3.2, it is sufficient to prove that the indicated mappings define a bijection, i.e. $r = r_{t_r}$ and $t = t_{r_t}$.

Verify the first relation:

$$r_{t_r}(M) = \{m \in M \mid t_r(0:m) = R\} = \{m \in M \mid \{a \in R \mid (0:am) \in \mathcal{E}_r\} = R\} =$$
$$= \{m \in M \mid (0:am) \in \mathcal{E}_r \ \forall\, a \in R\} = \{m \in M \mid \big((0:m):a\big) \in \mathcal{E}_r \ \forall\, a \in R\} =$$
$$= \{m \in M \mid (0:m) \in \mathcal{E}_r\} = r(M),$$

so $r = r_{t_r}$.

On the other hand, for every modular preclosure operator t of $\mathbb{L}({}_RR)$ we have:

$$t_{r_t}(I) = \{a \in R \mid (I:a) \in \mathcal{E}_{r_t}\},$$

where $\mathcal{E}_{r_t} = \{I \in \mathbb{L}({}_RR) \mid t(I) = R\}$. Now using the modularity 4°) we obtain:

$$t_{r_t}(I) = \{a \in R \mid t(I:a) = R\} = \{a \in R \mid \big(t(I):a\big) = R\} =$$
$$= \{a \in R \mid a \in t(I)\} = t(I),$$

therefore $t = t_{r_t}$. □

We remark the fact that the preradical filter of a pretorsion r_t has the form $\mathcal{E}_{r_t} = \{I \in \mathbb{L}({}_RR) \mid t(I) = R\}$, i.e. it coincides with the set of t-dense left ideals of R.

In continuation we show haw can be obtained from the Theorem 3.3 the similar result for the *torsions*, which was formulated above. We remind that by definition a torsion is a hereditary radical. As the pretorsions, they can be described by the filters of left ideals of R. Supplementing the conditions $(a_1)-(a_3)$ which define the preradical filters (see Section 1), we now consider the following conditions on the set of left ideals $\mathcal{E} \subseteq \mathbb{L}({}_RR)$:

(a_4) If $I_\alpha \in \mathcal{E},\ \alpha \in \mathfrak{A}$, then $\bigcap\limits_{\alpha\in\mathfrak{A}} I_\alpha \in \mathcal{E}$;

(a_5) If $I \subseteq J,\ J \in \mathcal{E}$ and $(I : j) \in \mathcal{E}$ for every $j \in J$, then $I \in \mathcal{E}$.

If $r \in \mathbb{PT}$ and $\mathcal{E}_r$ satisfies the condition (a_4), then r is called *jansian pretorsion*. Such pretorsions will be considered in Section 7.

The set of left ideals $\mathcal{E} \subseteq \mathbb{L}({}_RR)$ is called a *radical filter* (Gabriel filter, left Gabriel topology) if it satisfies the conditions $(a_1), (a_2)$ and (a_5). The description of torsions of R-Mod by the radical filters of $\mathbb{L}({}_RR)$ consists in the following [1–5].

Proposition 3.4. *The mappings*

$$r \rightsquigarrow \mathcal{E}_r, \quad \mathcal{E}_r = \{I \in \mathbb{L}({}_RR) \mid r(R/I) = R/I\};$$
$$\mathcal{E} \rightsquigarrow r_{\mathcal{E}}, \quad r_{\mathcal{E}}(M) = \{m \in M \mid (0 : m) \in \mathcal{E}\}$$

define a monotone bijection between the torsions of R-Mod and radical filters of $\mathbb{L}({}_RR)$. □

Now we will indicate the transition from the pretorsions to the torsions of R-Mod in terms of the modular preclosure operators of $\mathbb{L}({}_RR)$.

Proposition 3.5. *Let $r \in \mathbb{PT}$ and t_r be the associated modular preclosure operator of $\mathbb{L}({}_RR)$. Then the following conditions are equivalent:*

1) *r is a torsion;*

2) *t_r satisfies the condition 2°), i.e. it is idempotent.*

Proof. 1) ⇒ 2) If r is a torsion with radical filter $\mathcal{E}_r$, then by the definitions we have:

$$t_r(I) = \{a \in R \mid (I : a) \in \mathcal{E}_r\};$$
$$t_r\big(t_r(I)\big) = \{b \in R \mid \big(t_r(I) : b\big) \in \mathcal{E}_r\}.$$

Let $b \in t_r\big(t_r(I)\big)$. From $I \subseteq t_r(I)$ follows $(I : b) \subseteq \big(t_r(I) : b\big) \in \mathcal{E}_r$. Moreover, for every $d \in \big(t_r(I) : b\big)$ we have $\big((I : b) : d\big) \in \mathcal{E}_r$. Indeed, from $d \in \big(t_r(I) : b\big)$ follows $db \in t_r(I)$, i.e. $(0 : db) \in \mathcal{E}_r$. Therefore $\big((I : b) : d\big) = (I : db) \in \mathcal{E}_r$, so $\big((I : b) : d\big) \in \mathcal{E}_r$.

Now we can use the condition (a_5) in the situation $(I:b) \subseteq \big(t_r(I):b\big) \in \mathcal{E}_r$, from which follows that $(I:b) \in \mathcal{E}_r$, which means that $b \in t_r(I)$. So we have $t_r\big(t_r(I)\big) \subseteq t_r(I)$, which implies the condition 2°).

2) $\Rightarrow$ 1) Suppose that the operator t_r is idempotent. By the definitions we have:

$$t_r(I) = \{a \in R \mid (I:a) \in \mathcal{E}_r\}, \quad t_r\big(t_r(I)\big) = \{b \in R \mid \big(t_r(I):b\big) \in \mathcal{E}_r\}.$$

Therefore the idempotence of t_r means that from the $\big(t_r(I):b\big) \in \mathcal{E}_r$ follows $(I:b) \in \mathcal{E}_r$.

It is sufficient to prove that the filter $\mathcal{E}_r$ satisfies the condition (a_5). Suppose that $I \subseteq J$, $J \in \mathcal{E}_r$ and $(I:j) \in \mathcal{E}_r$ for every $j \in J$. From the last condition we have $J \subseteq t_r(I)$ and from the $J \in \mathcal{E}_r$ we obtain $t_r(I) \in \mathcal{E}_r$, therefore $\big(t_r(I):b\big) \in \mathcal{E}_r$ for every $b \in R$. By the idempotence of t_r now follows $(I:b) \in \mathcal{E}_r$ for every $b \in R$, therefore $I \in \mathcal{E}_r$. So the condition (a_5) is satisfied for $\mathcal{E}_r$, i.e. r is a torsion. □

Applying Theorem 3.3 and Proposition 3.5, we obtain the mentioned above result on torsions ([3, 6]).

Corollary 3.6. *The mappings $r \rightsquigarrow t_r$ and $t \rightsquigarrow r_t$ define a monotone bijection between the torsions of R-Mod and modular closure operators of $\mathbb{L}({}_RR)$.* □

4 Pretorsions and closure operators of *R*-Mod

An important aspect of pretorsions of R-Mod, closely related by the previous, consists in the description of pretorsions with the help of some *closure operators of the category R-Mod*. We remind firstly the necessary definitions and facts ([8–10]).

A *closure operator* of R-Mod is defined as a function C, which associates to every pair $N \subseteq M$, where $N \in \mathbb{L}(M)$ and $M \in R$-Mod, a submodule of M denoted by $C_M(N)$, such that the following conditions are satisfied:

(c_1) $N \subseteq C_M(N)$ (*extension*);

(c_2) $N_1 \subseteq N_2 \Rightarrow C_M(N_1) \subseteq C_M(N_2)$ (*monotony*);

(c_3) $f\big(C_M(N)\big) \subseteq C_{M'}\big(f(N)\big)$ for every R-morphism $f\colon M \to M'$ and $N \subseteq M$ (*continuity*).

We denote by $\mathbb{CO}$ the class of all closure operators of R-Mod. Define in this class the following operations:

– *the meet* $\bigwedge\limits_{\alpha\in\mathfrak{A}} C_\alpha$, where $\big(\bigwedge\limits_{\alpha\in\mathfrak{A}} C_\alpha\big)_M(N) = \bigcap\limits_{\alpha\in\mathfrak{A}} \big[\big(C_\alpha\big)_M(N)\big]$;

– *the join* $\bigvee\limits_{\alpha\in\mathfrak{A}} C_\alpha$, where $\big(\bigvee\limits_{\alpha\in\mathfrak{A}} C_\alpha\big)_M(N) = \sum\limits_{\alpha\in\mathfrak{A}} \big[\big(C_\alpha\big)_M(N)\big]$;

– *the product* $C \cdot D$, where $\big(C \cdot D\big)_M(N) = C_M\big(D_M(N)\big)$;

– *the coproduct* $C \# D$, where $\big(C \# D\big)_M(N) = C_{D_M(N)}(N)$.

We remind also the main types of closure operators of R-Mod. An operator $C \in \mathbb{CO}$ is called:

– *weakly hereditary*, if $C_M(N) = C_{C_M(N)}(N)$;

– *idempotent*, if $C_M(N) = C_M\big(C_M(N)\big)$;

– *hereditary*, if $C_N(L) = C_M(L) \cap N$, where $L \subseteq N \subseteq M$;

– *cohereditary*, if $\big(C_M(N) + K\big)/K = C_{M/K}\big((N+K)/K\big)$, where $K, N \in \mathbb{L}(M)$;

– *maximal*, if $C_M(N)/N = C_{M/N}(\bar{0})$ $\big($or: $C_M(N)/K = C_{M/K}(N/K)$, where $K \subseteq N \subseteq M\big)$;

– *minimal*, if $C_M(N) = C_M(0) + N$ $\big($or: $C_M(N) = C_M(L) + N$, where $L \subseteq N \subseteq M\big)$.

There exists a close relation between the class of preradicals $\mathbb{PR}$ and the class of closure operators $\mathbb{CO}$ of R-Mod, which is expressed by the following mappings:

1) $\Phi\colon\ \mathbb{CO} \to \mathbb{PR}$, where $\Phi(C) = r_C$, $r_C(M) = C_M(0)$;

2) $\Psi_1\colon\ \mathbb{PR} \to \mathbb{CO}$, where $\Psi_1(r) = C^r$, $[(C^r)_M(N)]/N = r(M/N)$;

3) $\Psi_2\colon\ \mathbb{PR} \to \mathbb{CO}$, where $\Psi_2(r) = C_r$, $(C_r)_M(N) = N + r(M)$.

The class of maximal closure operators $\mathbb{M}ax\,(\mathbb{CO})$ coincides with the operators of the form C^r, $r \in \mathbb{PR}$, and the pair (Φ, Ψ_1) establishes the bijection $\mathbb{M}ax\,(\mathbb{CO}) \cong \mathbb{PR}$. Dually, the class of minimal closure operators $\mathbb{M}in\,(\mathbb{CO})$ coincides with the class of closure operators of the form C_r, $r \in \mathbb{PR}$, and the pair (Φ, Ψ_2) defines a bijection $\mathbb{M}in\,(\mathbb{CO}) \cong \mathbb{PR}$.

In continuation we remind the effect of the defined above mappings to the class $\mathbb{PT}$ of *pretorsions* of R-Mod. The following statements are proved in [9] (Part IV, Propositions 2.7, 3.5).

Proposition 4.1. 1) *The pair of mappings* (Φ, Ψ_1) *defines a monotone bijection between the pretorsions of R-Mod and the maximal and hereditary closure operators of R-Mod.*

2) *The pair* (Φ, Ψ_2) *determines a monotone bijection between the pretorsions of R-Mod and the minimal and hereditary closure operators of R-Mod.* □

Denoting by $\mathbb{M}ax\,(\mathbb{HCO})$ the class of maximal and hereditary closure operators of $\mathbb{CO}$, we have the bijection $\mathbb{PT} \cong \mathbb{M}ax\,(\mathbb{HCO})$.

Let $r \in \mathbb{PT}$ and $\mathcal{E}_r$ be the associated preradical filter. Then the maximal and hereditary closure operator C^r of R-Mod is defined by the rule $[C^r_M(N)]/N = r(M/N)$ and can be expressed by the filter $\mathcal{E}_r$ as follows.

Lemma 4.2. *$C^r_M(N) = \{m \in M \mid (N : m) \in \mathcal{E}_r\}$, where $(N : m) = \{a \in R \mid am \in N\}$.*

Proof. It is obvious that the set $\{m \in M \mid (N : m) \in \mathcal{E}_r\}$ is a submodule of M, containing N. Since

$$r(M/N) = \{m + N \in M/N \mid \big(0 : (m+N)\big) = (N : m) \in \mathcal{E}_r\},$$

by the definition of $C^r_M(N)$ follows the statement. □

For the subsequent investigations we need the following conditions on the closure operator $C \in \mathbb{CO}$:

(c_4) $\big(C_M(N) : m\big) = C_R(N : m)$ for every $N \in \mathbb{L}(M)$ and $m \in M$ (*modularity*);

(c_5) $C_M(N \cap L) = C_M(N) \cap C_M(L)$ for every $N, L \in \mathbb{L}(M)$ (*linearity*).

Proposition 4.3. *Let $r \in \mathbb{PT}$ and C^r be the respective maximal and hereditary closure operator of R-Mod. Then C^r satisfies the conditions (c_4) and (c_5), i.e. it is modular and linear.*

Proof. (c_4) From the definitions and Lemma 4.2 we have:

$$\big(C^r_M(N) : m\big) = \{a \in R \mid am \in C^r_M(N)\} = \{a \in R \mid (N : am) =$$
$$= \big((N : m) : a\big) \in \mathcal{E}_r\},$$
$$C^r_R(N : m) = \{a \in R \mid \big((N : m) : a\big) = (N : am) \in \mathcal{E}_r\},$$

so (c_4) is true.

(c_5) The expressions of (c_5) have the form:

$$C^r_M(N \cap L) = \{m \in M \mid \big((N \cap L) : m\big) = (N : m) \cap (L : m) \in \mathcal{E}_r\},$$
$$C^r_M(N) \cap C^r_M(L) = \{m \in M \mid (N : m) \in \mathcal{E}_r\} \cap \{m \in M \mid (L : m) \in \mathcal{E}_r\} =$$
$$= \{m \in M \mid (N : m) \cap (L : m) \in \mathcal{E}_r\}.$$ □

Now we mention the relation of these results with the facts of Section 3. Let $r \in \mathbb{PT}$ with the corresponding closure operator C^r. If we consider the action of C^r on the lattice $\mathbb{L}({}_RR)$ (i.e. we fix $M = {}_RR$), then we obtain a closure operator C^r_R of $\mathbb{L}({}_RR)$.

Corollary 4.4. *If $r \in \mathbb{PT}$, then the operator t_r of $\mathbb{L}({}_RR)$ defined by the rule $t_r(I) = \{a \in R \mid (I : a) \in \mathcal{E}_r\}$ coincides with the operator C^r_R, therefore C^r_R is a modular preclosure operator of $\mathbb{L}({}_RR)$.*

Proof. From the Lemma 4.2 we have $C^r_R(I) = \{a \in R \mid (I : a) \in \mathcal{E}_r\}$, therefore $C^r_R = t_r$. From Proposition 3.1 it now follows that C^r_R is a modular preclosure operator of $\mathbb{L}({}_RR)$. □

Now we show the similar results on the *torsions* of R-Mod. For that we use the following

Lemma 4.5. *Let $r \in \mathbb{PT}$ and C^r be the associated maximal closure operator. Then the following conditions are equivalent:*

1) *r is a torsion;*
2) *C^r is an idempotent closure operator.*

Proof. 1) $\Rightarrow$ 2) If r is a torsion, then $\mathcal{E}_r$ is a radical filter, so it satisfies the condition (a_5). Let $m \in C^r_M\big(C^r_M(N)\big)$. Then $\big(C^r_M(N) : m\big) \in \mathcal{E}_r$ and it is obvious that $(N : m) \subseteq \big(C^r_M(N) : m\big)$. Moreover, for every $a \in \big(C^r_M(N) : m\big)$ we have $am \in C^r_M(N)$, so $(N : am) = \big((N : m) : a\big) \in \mathcal{E}_r$. Now we can apply the condition (a_5) in the situation $(N : m) \subseteq \big(C^r_M(N) : m\big) \in \mathcal{E}_r$, concluding that $(N : m) \in \mathcal{E}_r$, i.e. $m \in C^r_M(N)$. This proves the relation $C^r_M\big(C^r_M(N)\big) \subseteq \big(C^r_M(N)$, which is sufficient for the idempotence of C^r.

2) $\Rightarrow$ 1) If C^r is idempotent, then the operator $C^r_R = t_r$ of $\mathbb{L}(_RR)$ satisfies the condition 2°), i.e. it is idempotent. From the Proposition 3.5 this is equivalent to the fact that r is a torsion. □

From the Proposition 4.1 and Lemma 4.5 follows the

Corollary 4.6. *The pair of mappings (Φ, Ψ_1) define a monotone bijection between the torsions of R-Mod and maximal, hereditary and idempotent closure operators of R-Mod.* □

It is interesting that the closure operators of the form C^r, where $r \in \mathbb{PT}$ (i.e. maximal and hereditary) can be characterized by the conditions (c_4) and (c_5) indicated above. By Proposition 4.3 every closure operator of such type satisfies the conditions (c_4) and (c_5). Now we show that the inverse statement is also true.

Proposition 4.7. *Let $C \in \mathbb{CO}$ and C satisfies the conditions (c_4) and (c_5), i.e. it is modular and linear. Then the set of C-dense left ideals $\mathcal{E}_C = \{I \in \mathbb{L}(_RR) \mid C_R(I) = R\}$ is a preradical filter, the pretorsion defined by $\mathcal{E}_C$ coincides with $r_C = \Phi(C)$ and $C = C^{r_C}$.*

Proof. Verify the conditions $(a_1) - (a_3)$ for $\mathcal{E}_C$.

(a_1) If $I \in \mathcal{E}_C$ and $a \in R$, then $C_R(I) = R$ and from (c_4) we have
$$C_R(I : a) = \big(C_R(I) : a\big) = (R : a) = R,$$
therefore $(I : a) \in \mathcal{E}_C$.

(a_2) If $I \in \mathcal{E}_C$ and $I \subseteq J$, then $C_R(I) = R$ and from (c_2) we have $C_R(I) \subseteq C_R(J)$, so $C_R(J) = R$, i.e. $J \in \mathcal{E}_C$.

(a_3) If $I, J \in \mathcal{E}_C$, then $C_R(I) = C_R(J) = R$, so from (c_5) we obtain $C_R(I \cap J) = C_R(I) \cap C_R(J) = R$, i.e. $I \cap J \in \mathcal{E}_C$.

This proves that $\mathcal{E}_C$ is a preradical filter, therefore it defines a pretorsion $r_{\mathcal{E}_C}$. It coincides with $r_C = \Phi(C)$, since from the definitions and (c_4) we have:

$r_{\mathcal{E}_C}(M) = \{m \in M \mid (0:m) \in \mathcal{E}_C\} = \{m \in M \mid C_R(0:m) = R\} =$
$= \{m \in M \mid \big(C_M(0):m\big) = R\} = \{m \in M \mid m \in C_M(0)\} = C_M(0) = r_C(M).$

The similar arguments show that $C^{r_C} = C$. Indeed, for every $N \subseteq M$ using (c_4) we obtain:

$(C^{r_C})_M(N) = \{m \in M \mid (N:m) \in \mathcal{E}_C\} = \{m \in M \mid C_R(N:m) = R\} =$
$= \{m \in M \mid \big(C_M(N):m\big) = R\} = \{m \in M \mid m \in C_M(N)\} = C_M(N).$ □

From Propositions 4.3 and 4.7 follows the

Corollary 4.8. *The pair of mappings* (Φ, Ψ_1) *defines a monotone bijection between the pretorsions of* R*-Mod and the modular and linear closure operators of* $\mathbb{CO}$. □

5 Relations between the operations of $\mathbb{PT}$ and $\mathbb{CO}$

By Proposition 4.1 the pair of mappings (Φ, Ψ_1) defines a monotone bijection $\mathbb{PT} \cong \mathbb{M}ax\,(\mathbb{HCO})$. Now we specify the form of operations in $\mathbb{M}ax\,(\mathbb{HCO})$:

- *the meet*: $\big(\bigwedge\limits_{\alpha \in \mathfrak{A}} C_\alpha\big)_M(N) = \bigcap\limits_{\alpha \in \mathfrak{A}} \big[\big(C_\alpha\big)_M(N)\big];$
- *the join*: $\bigvee\limits_{\alpha \in \mathfrak{A}} C_\alpha = \bigwedge \{D \in \mathbb{M}ax\,(\mathbb{HCO}) \mid D \supseteq C_\alpha \;\; \forall\, \alpha \in \mathfrak{A}\};$
- *the product*: $(C \cdot D)_M(N) = C_M\big(D_M(N)\big).$

In the case of pretorsions the relation $r \cdot s = r \wedge s$ was mentioned (Section 2). Similarly, in the case of *hereditary* closure operators the coproduct coincides with the meet.

Lemma 5.1. *If* $C, D \in \mathbb{CO}$ *and* C *is hereditary, then* $C \,\#\, D = C \wedge D$.

Proof. For every $N \subseteq M$ from the heredity of C used in the situation $N \subseteq D_M(N) \subseteq M$ we obtain:

$$(C \,\#\, D)_M(N) = C_{D_M(N)}(N) = C_M(N) \cap D_M(N) = (C \wedge D)_M(N). \qquad \square$$

For this reason in the case of hereditary closure operators we consider only three operations: meet, join and product, so we have the bijection: $\mathbb{PT}(\wedge, \vee, \#) \cong \mathbb{M}ax\,(\mathbb{HCO})(\wedge, \vee, \cdot)$. The following statements show the concordance of operations in this bijection.

Proposition 5.2. $C^{\bigwedge_{\alpha \in \mathfrak{A}} r_\alpha} = \bigwedge\limits_{\alpha \in \mathfrak{A}} C^{r_\alpha}$ *for every family* $\{r_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq \mathbb{PT}$.

Proof. Since $\mathcal{E}_{\bigwedge_{\alpha \in \mathfrak{A}} r_\alpha} = \bigwedge\limits_{\alpha \in \mathfrak{A}} \mathcal{E}_{r_\alpha}$ (Proposition 2.4) we have:

$$\Big(C^{\bigwedge_{\alpha \in \mathfrak{A}} r_\alpha}\Big)_M(N) = \{m \in M \mid (N:m) \in \mathcal{E}_{\bigwedge_{\alpha \in \mathfrak{A}} r_\alpha}\};$$

$$\big(\bigwedge_{\alpha \in \mathfrak{A}} C^{r_\alpha}\big)_M(N) = \bigcap_{\alpha \in \mathfrak{A}} \big[C_M^{r_\alpha}(N)\big] = \bigcap_{\alpha \in \mathfrak{A}} \big[\{m \in M \mid (N:m) \in \mathcal{E}_{r_\alpha}\}\big] =$$
$$= \{m \in M \mid (N:m) \in \bigwedge_{\alpha \in \mathfrak{A}} \mathcal{E}_{r_\alpha} = \mathcal{E}_{\bigwedge_{\alpha \in \mathfrak{A}} r_\alpha}\}. \qquad \square$$

Proposition 5.3. $C^{\bigvee_{\alpha\in\mathfrak{A}} r_\alpha} = \bigvee_{\alpha\in\mathfrak{A}} C^{r_\alpha}$ *for every family* $\{r_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq \mathbb{PT}$.

Proof follows from the Proposition 2.5. □

Proposition 5.4. $C^{r \# s} = C^r \cdot C^s$ *for any pretorsions* $r, s \in \mathbb{PT}$.

Proof. We verify the relation $C_M^{r\# s}(N) = C_M^r\big(C_M^s(N)\big)$, where $N \subseteq M$.

($\subseteq$) Let $m \in C_M^{r\# s}(N)$. Then from the Proposition 2.6 and from the definitions we have:

$(N : m) \in \mathcal{E}_{r\# s} = \mathcal{E}_r \# \mathcal{E}_s =$

$= \{I \in \mathbb{L}({}_R R) \mid \exists\ H \in \mathcal{E}_r,\ I \in H \ \text{ such that } \ (I : a) \in \mathcal{E}_s \ \ \forall\ a \in H\}.$

So there exists $H \in \mathcal{E}_r$ such that $(N : m) \subseteq H$ and $\big((N : m) : a\big) = (N : am) \in \mathcal{E}_s$ for every $a \in H$. Therefore for every element $am + N \in (Hm + N)/N$ we have $\big(0 : (am + N)\big) = (N : am) \in \mathcal{E}_s$, which means that $(Hm + N)/N \in \mathcal{T}_s$. But then $(Hm + N)/N \subseteq s(M/N) = C_M^s(N)/N$, so $Hm \subseteq C_M^s(N)$ and $H \subseteq \big(C_M^s(N) : m\big)$. Since $H \in \mathcal{E}_r$, now we have $\big(C_M^s(N) : m\big) \in \mathcal{E}_r$, which means that $m \in C_M^r\big(C_M^s(N)\big)$.

($\supseteq$) Let $m \in C_M^r\big(C_M^s(N)\big)$. Then $\big(C_M^s(N) : m\big) \in \mathcal{E}_r$ and denoting $H = \big(C_M^s(N) : m\big)$ we have $H \in \mathcal{E}_r$ and $Hm \subseteq C_M^s(N)$. From the relation $N \subseteq C_M^s(N)$ follows $(N : m) \subseteq \big(C_M^s(N) : m\big) = H$. Moreover, for every $a \in H$ we have $am \in C_M^s(N)$, i.e. $(N : am) = \big((N : m) : a\big) \in \mathcal{E}_s$. By the definition this means that $(N : m) \in \mathcal{E}_r \# \mathcal{E}_s = \mathcal{E}_{r\# s}$, therefore $m \in C_M^{r\# s}(N)$. □

From the previous statements we conclude that the mapping Ψ_1 preserves the meets and joins, but it converts the coproduct into the product.

6 Characterization of pretorsions by dense submodules

Let $C \in \mathbb{CO}$. For every $M \in R$-Mod we denote:

$\boldsymbol{\mathfrak{F}}_1^C(M) = \{N \in \mathbb{L}(M) \mid C_M(N) = M\}$ – the set of C-*dense* submodules of M;

$\boldsymbol{\mathfrak{F}}_2^C(M) = \{N \in \mathbb{L}(M) \mid C_M(N) = N\}$ – the set of C-*closed* submodules of M.

Thus the operator $C \in \mathbb{CO}$ defines two functions $\boldsymbol{\mathfrak{F}}_1^C$ and $\boldsymbol{\mathfrak{F}}_2^C$, which distinguish in every module M the set of C-dense submodules $\boldsymbol{\mathfrak{F}}_1^C(M)$ and the set of C-closed submodules $\boldsymbol{\mathfrak{F}}_2^C(M)$. In some cases by the help of these functions the operator C can be reestablished. More exactly, C can be restored by $\boldsymbol{\mathfrak{F}}_1^C$ if and only if it is weakly hereditary. Dually, C can be reestablished by $\boldsymbol{\mathfrak{F}}_2^C$ if and only if it is idempotent ([9], Part I).

Now we remind some results on the function $\mathfrak{F}_1^C$ defined by C-dense submodules. For every $C \in \mathbb{CO}$ the function $\mathfrak{F}_1^C$ satisfies the following conditions:

1) If $N \in \mathfrak{F}_1^C(M_\alpha),\ M_\alpha \subseteq M,\ \alpha \in \mathfrak{A}$, then $N \in \mathfrak{F}_1^C\big(\sum_{\alpha\in\mathfrak{A}} M_\alpha\big)$;

2) If $N \subseteq P \subseteq M$ and $N \in \mathfrak{F}_1^C(P)$, then $N + K \in \mathfrak{F}_1^C(P + K)$ for every $K \subseteq M$;

3) If $f\colon M \to M'$ is an R-morphism and $N \in \mathfrak{F}_1^C(M)$, then $f(N) \in \mathfrak{F}_1^C\big(f(M)\big)$.

An abstract function $\mathfrak{F}$ which separates in every module M a set of submodules $\mathfrak{F}(M)$ is called a *function of type* $\mathfrak{F}_1$, if it satisfies the conditions 1) – 3). Then $\mathfrak{F}$ defines a closure operator $C^{\mathfrak{F}}$ by the rule:

$$(C^{\mathfrak{F}})_M(N) = \sum_{\alpha\in\mathfrak{A}} \{M_\alpha \subseteq M \mid N \in \mathfrak{F}(M_\alpha)\}.$$

The description of the weakly hereditary closure operators by the functions of type $\mathfrak{F}_1$ consists in the following ([9], Part I, Theorem 2.6).

Proposition 6.1. *The mappings $C \rightsquigarrow \mathfrak{F}_1^C$ and $\mathfrak{F} \rightsquigarrow C^{\mathfrak{F}}$ define a monotone bijection between the **weakly hereditary** closure operators of $\mathbb{CO}$ and the functions of type $\mathfrak{F}_1$ of R-Mod.*

By the restriction of this bijection we obtain the similar result for the *hereditary* closure operators of $\mathbb{CO}$. For that the following condition on the abstract function $\mathfrak{F}$ is considered:

(Her) If $N \subseteq P \subseteq M$ and $N \in \mathfrak{F}(M)$, then $N \in \mathfrak{F}(P)$.

Proposition 6.2. *The mappings $C \rightsquigarrow \mathfrak{F}_1^C$ and $\mathfrak{F} \rightsquigarrow C^{\mathfrak{F}}$ define a monotone bijection between the **hereditary** closure operators of $\mathbb{CO}$ and the abstract functions of type $\mathfrak{F}_1$ of R-Mod, which satisfy the condition* (Her) ([9]*, Part* II*, Corollary* 2.3)*.*

In a similar way from the Proposition 6.1 the description of *weakly hereditary and maximal* closure operators can be obtained. With this aim the following condition on a function $\mathfrak{F}$ is considered:

(Max) If $K \subseteq N \subseteq M$ and $N/K \in \mathfrak{F}(M/K)$, then $N \in \mathfrak{F}(M)$.

Proposition 6.3. *The mappings $C \rightsquigarrow \mathfrak{F}_1^C$ and $\mathfrak{F} \rightsquigarrow C^{\mathfrak{F}}$ define a monotone bijection between the **weakly hereditary and maximal** closure operators of $\mathbb{CO}$ and the abstract functions of type $\mathfrak{F}_1$, which satisfy the condition* (Max) ([9]*, Part* II*, Corollary* 3.3)*.*

From Propositions 6.2 and 6.3 we have

Corollary 6.4. *The mappings $C \rightsquigarrow \mathfrak{F}_1^C$ and $\mathfrak{F} \rightsquigarrow C^{\mathfrak{F}}$ establish a monotone bijection between the **hereditary and maximal** closure operators of $\mathbb{CO}$ and the abstract functions of type $\mathfrak{F}_1$, which satisfy the conditions* (Her) *and* (Max)*.*

Now we can use the fact that the pretorsions of R-Mod are described by the maximal and hereditary closure operators of R-Mod, since by Proposition 4.1 we have the bijection: $\mathbb{PT} \cong \mathbb{M}ax(\mathbb{HCO})$. In one's turn the operators of $\mathbb{M}ax(\mathbb{HCO})$ by Corollary 6.4 can be characterized by the abstract functions of type $\mathfrak{F}_1$ with the conditions (Max) and (Her). Therefore the followihg is true.

Proposition 6.5. *There exists a monotone bijection between the pretorsions of R-Mod and the abstract functions of type $\mathfrak{F}_1$, which satisfy the conditions* (Max) *and* (Her)*.*

This bijection has the form:

$r \rightsquigarrow \mathfrak{F}_1^r$, where $\mathfrak{F}_1^r(M) = \{N \in \mathbb{L}(M) \mid (N:m) \in \mathcal{E}_r \ \ \forall\, m \in M\}$;

$\mathfrak{F} \rightsquigarrow r_{\mathfrak{F}}$, where $r_{\mathfrak{F}}(M) = \sum \{M_\alpha \in \mathbb{L}(M) \mid 0 \in \mathfrak{F}(M_\alpha)\}$.

We mention also the fact that for every pretorsion $r \in \mathbb{PT}$ we have $\mathfrak{F}_1^r({}_RR) = \mathcal{E}_r$.

From the exposed above results follows that every pretorsion $r \in \mathbb{PT}$ can be described not only by the class $\mathcal{T}_r$ and the filter $\mathcal{E}_r$, but also by the operator t_r of $\mathbb{L}({}_RR)$, by the operator C^r of R-Mod and by the function $\mathfrak{F}_1^r$, which selects the dense submodules.

7 On some approximations of pretorsions

Concluding this work, we mention some simple methods of approximations of pretorsions by *jansian pretorsions* and by *torsions* of R-Mod. By approximations we means the constructions of the least jansian pretorsion or of the least torsion, which contains the given pretorsion.

Let $r \in \mathbb{PT}$. We denote $L_r = \cap\ \{I_\alpha \in \mathbb{L}({}_RR) \mid I_\alpha \in \mathcal{E}_r\}$. Then L_r is an ideal of R and it is called the *kernel* of r. The following conditions for $r \in \mathbb{PT}$ are equivalent ([1, 3, 4]):

1) r is jansian (see condition (a_4), Section 3);

2) $L \in \mathcal{E}_r$;

3) the class $\mathcal{T}_r$ is closed under products: if $M_\alpha \in \mathcal{T}_r$ $(\alpha \in \mathfrak{A})$, then $\prod\limits_{\alpha \in \mathfrak{A}} M_\alpha \in \mathcal{T}_r$.

If r is a jansian pretorsion, then $\mathcal{E}_r = \{I \in \mathbb{L}({}_RR) \mid I \supseteq L_r\}$.

There exists an *antimonotone bijection* between the jansian pretorsions of R-Mod and two sided ideals of R. It is defined by the rules:

$$r \rightsquigarrow L_r, \quad I \rightsquigarrow \mathcal{E}_I = \{I_\alpha \in \mathbb{L}({}_RR) \mid I_\alpha \supseteq I\}.$$

It is obvious that if the pretorsion $r \in \mathbb{PT}$ is jansian, then the associated maximal and hereditary closure operator C^r acts as follows: $C^r_M(N) = \{m \in M \mid (N:m) \supseteq L_r\}$.

It is easy to show how can be expressed by C^r the condition that the pretorsion $r \in \mathbb{PT}$ is jansian. For that we consider the following condition to an arbitrary $C \in \mathbb{CO}$:

(c_6) $C_M\big(\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha\big) = \bigcap\limits_{\alpha\in\mathfrak{A}} C_M(N_\alpha)$ for every family $\{N_\alpha \mid \alpha\in\mathfrak{A}\}\subseteq \mathbb{L}(M)$ (*complete linearity*).

Proposition 7.1. *For every $r\in\mathbb{PT}$ the following conditions are equivalent:*

1) *r is a jansian pretorsion;*
2) *the closure operator C^r satisfies the condition (c_6).*

Proof. 1) $\Rightarrow$ 2) If r is jansian, then:

$$m\in C^r_M\big(\bigcap_{\alpha\in\mathfrak{A}} N_\alpha\big) \Leftrightarrow \big((\bigcap_{\alpha\in\mathfrak{A}} N_\alpha):m\big)\supseteq L_r \Leftrightarrow \bigcap_{\alpha\in\mathfrak{A}}(N_\alpha:m)\supseteq L_r \Leftrightarrow$$
$$\Leftrightarrow m\in \bigcap_{\alpha\in\mathfrak{A}} C^r_M(N_\alpha), \text{ so is true } (c_6).$$

2) $\Rightarrow$ 1) If C^r is complete linear, then $C^r_R\big(\bigcap\limits_{I_\alpha\in\mathcal{E}_r} I_\alpha\big) = \bigcap\limits_{I_\alpha\in\mathcal{E}_r}[C^r_R(I_\alpha)] = R$, so $\bigcap\limits_{I_\alpha\in\mathcal{E}_r} I_\alpha = L_r\in\mathcal{E}_r$, i.e. r is jansian. □

Let $r\in\mathbb{PT}$ and L_r be the kernel of the pretorsion r. Then the ideal L_r defines a jansian pretorsion $\hat{r}$, determined by the preradical filter $\mathcal{E}_{\hat{r}} = \{I\in\mathbb{L}(_RR)\mid I\supseteq L_r\}$, i.e. $\hat{r}(M)=\{m\in M\mid (0:m)\supseteq L_r\}$ for every $M\in R$-Mod.

Proposition 7.2. *$\hat{r}$ is the least jansian pretorsion containing the pretorsion $r\in\mathbb{PT}$.*

Proof. Since $\mathcal{E}_r\subseteq\mathcal{E}_{\hat{r}}$, we have $r\le\hat{r}$ and $\hat{r}$ is a jansian pretorsion with the kernel L_r. If $s\in\mathbb{PT}$ is jansian and $r\le s$, than $\mathcal{E}_r\le\mathcal{E}_s$, so $L_r\supseteq L_s$, therefore $\hat{r}\le s$. This means that $\hat{r}$ is the least jansian pretorsion containing r. □

Taking into account this property, $\hat{r}$ is called the *jansian hull* of the pretorsion $r\in\mathbb{PT}$ [4]. For an ideal I of R we denote by r_I the jansian pretorsion defined by I, so that $r_I(M)=\{m\in M\mid (0:m)\supseteq I\}$.

Proposition 7.3. $\bigwedge\limits_{\alpha\in\mathfrak{A}}\hat{r}_\alpha = r_{\sum_{\alpha\in\mathfrak{A}} L_{r_\alpha}}$ *for every family $\{r_\alpha\mid\alpha\in\mathfrak{A}\}\subseteq\mathbb{PT}$.*

Proof. We compare the respective preradical filters:

$$\mathcal{E}_{\bigwedge_{\alpha\in\mathfrak{A}}\hat{r}_\alpha} = \bigcap_{\alpha\in\mathfrak{A}}\mathcal{E}_{\hat{r}_\alpha} = \{I\in\mathbb{L}(_RR)\mid I\in\mathcal{E}_{\hat{r}_\alpha}\ \ \forall\,\alpha\in\mathfrak{A}\} =$$
$$= \{I\in\mathbb{L}(_RR)\mid I\supseteq L_{r_\alpha}\ \ \forall\,\alpha\in\mathfrak{A}\} = \{I\in\mathbb{L}(_RR)\mid I\supseteq\sum_{\alpha\in\mathfrak{A}} L_{r_\alpha}\} = \mathcal{E}_{r_{\sum_{\alpha\in\mathfrak{A}} L_{r_\alpha}}}.$$ □

In continuation we show the other type of approximation of a pretorsion $r\in\mathbb{PT}$, namely by the help of *torsions*. Every pretorsion $r\in\mathbb{PT}$ is accompanied by two classes of modules:

$$\mathcal{T}_r=\{M\in R\text{-Mod}\mid r(M)=M\},\quad \mathcal{F}_r=\{M\in R\text{-Mod}\mid r(M)=0\}.$$

It is well known that the class $\mathcal{T}_r$ uniquely reestablishes the pretorsion r, while the class $\mathcal{F}_r$ not always determines r.

To clarify the situation it is convenient to use the following operators of "orthogonality", which act to the abstract classes of modules $\mathcal{K} \subseteq R$-Mod ([1–3]):

$$\mathcal{K}^{\uparrow} = \{X \in R\text{-Mod} \mid Hom_R(X,Y) = 0 \ \ \forall\, Y \in \mathcal{K}\},$$

$$\mathcal{K}^{\downarrow} = \{Y \in R\text{-Mod} \mid Hom_R(X,Y) = 0 \ \ \forall\, X \in \mathcal{K}\}.$$

For every $\mathcal{K} \subseteq R$-Mod the class $\mathcal{K}^{\uparrow}$ is a *torsion class* (i.e. it is closed under homomorphic image, direct sums and extensions), and $\mathcal{K}^{\downarrow}$ is a *torsionfree class* (i.e. it is closed under submodules, direct products and extensions). Moreover, $\mathcal{K}^{\downarrow\uparrow}$ is the least torsion class containing $\mathcal{K}$, and $\mathcal{K}^{\uparrow\downarrow}$ is the least torsionfree class containing $\mathcal{K}$. If r is an idempotent radical, then $\mathcal{T}_r = \mathcal{F}_r^{\uparrow}$ and $\mathcal{F}_r = \mathcal{T}_r^{\downarrow}$. In this case $\mathcal{T}_r$ is hereditary if and only if $\mathcal{F}_r$ is stable and this means that r is a torsion.

Lemma 7.4. *If r is a pretorsion, then the class $\mathcal{F}_r = \mathcal{T}_r^{\downarrow}$ is closed under submodules, direct products, extensions and injective envelopes, i.e. $\mathcal{F}_r$ is a torsionfree stable class.*

Proof. The first three properties of the class $\mathcal{F}_r = \mathcal{T}_r^{\downarrow}$ are obvious, since every class of the form $\mathcal{K}^{\downarrow}$ is torsionfree. We verify the stability of $\mathcal{F}_r$: $M \in \mathcal{F}_r$ implies $E(M) \in \mathcal{F}_r$, where $E(M)$ is the injective envelope of M.

Let $M \in \mathcal{F}_r$, i.e. $r(M) = \{m \in M \mid (0:m) \in \mathcal{E}_r\} = 0$. Suppose that $r\big(E(M)\big) \neq 0$. Then there exists an element $0 \neq x \in E(M)$ such that $(0:x) \in \mathcal{E}_r$. Since $Rx \neq 0$, we have $Rx \cap M \neq 0$, so there exists an element $0 \neq m = ax \in M$, where $a \in R$, for which $(0:m) = (0:ax) = \big((0:x):a\big) \in \mathcal{E}_r$, therefore $0 \neq m \in r(M)$, contradiction. This shows that $r\big(E(M)\big) = 0$, i.e. $E(M) \in \mathcal{F}_r$ and the class $\mathcal{F}_r$ is stable. □

Now we remind the relation between the torsions r of R-Mod and the associated classes $\mathcal{T}_r$ and $\mathcal{F}_r$ ([1–3, 6]).

Lemma 7.5. 1) *The mappings $r \rightsquigarrow \mathcal{T}_r$, and $\mathcal{T} \rightsquigarrow r^{\mathcal{T}}$, where $r^{\mathcal{T}}(M) = \sum\limits_{\alpha \in \mathfrak{A}} \{N_\alpha \in \mathbb{L}(M) \mid N_\alpha \in \mathcal{T}\}$, define a monotone bijection between the torsions of R-Mod and the hereditary torsion classes of R-Mod.*

2) *The mappings $r \rightsquigarrow \mathcal{F}_r$, and $\mathcal{F} \rightsquigarrow r_{\mathcal{F}}$, where $r_{\mathcal{F}}(M) = \bigcap\limits_{\alpha \in \mathfrak{A}} \{N_\alpha \in \mathbb{L}(M) \mid M/N_\alpha \in \mathcal{F}\}$, establish an antimonotone bijection between the torsions of R-Mod and the stable torsionfree classes of R-Mod.*

Let $r \in \mathbb{PT}$. By the Lemma 7.4 the class $\mathcal{F}_r$ is a stable torsionfree class, so by the Lemma 7.5 $\mathcal{F}_r$ defines a *torsion* $\widetilde{r}$ such that $\mathcal{T}_{\widetilde{r}} = \mathcal{F}_r^{\uparrow} = \mathcal{T}_r^{\downarrow\uparrow}$ and $\mathcal{F}_{\widetilde{r}} = \mathcal{F}_r$, i.e.

$$\widetilde{r}(M) = \bigcap_{\alpha \in \mathfrak{A}} \{N_\alpha \in \mathbb{L}(M) \mid M/N_\alpha \in \mathcal{F}_r\}.$$

Proposition 7.6. *Let $r \in \mathbb{PT}$. Then the torsion $\widetilde{r}$, defined by the class $\mathcal{F}_r$, is the least torsion containing r.*

Proof. By the definitions the class of modules $\mathcal{T}_{\widetilde{r}} = \mathcal{F}_r^{\uparrow} = \mathcal{T}_r^{\downarrow\uparrow}$ is the least hereditary torsion class, which contains $\mathcal{T}_r$. Therefore $\widetilde{r}$ is the least torsion containing r. □

The torsion $\widetilde{r}$ constructed above is called the *torsion hull* of the pretorsion $r \in \mathbb{PT}$. Then $\mathcal{E}_{\widetilde{r}}$ is the least *radical filter* of R, containing the preradical filter $\mathcal{E}_r$. It is obvious that class of modules $\mathcal{T}_{\widetilde{r}}$ can be directly described by the class $\mathcal{T}_r$, as well as the radical filter $\mathcal{E}_{\widetilde{r}}$ can be expressed by $\mathcal{E}_r$. For example: $\mathcal{E}_{\widetilde{r}} = \{I \in \mathbb{L}(_R R) \mid \forall\ J \supset I,\quad J \neq R,\quad \exists\, a \notin J$ such that $(J : a) \in \mathcal{E}_r\}$ ([2], Chapter VI, Proposition 5.4).

In particular, for the pretorsion $\mathbb{Z}$ defined by the preradical filter of *essential* left ideals $\mathcal{E}_{\mathbb{Z}} = \{I \in \mathbb{L}(_R R) \mid I \subseteq'\ _R R\}$, the corresponding torsion hull is $\mathbb{Z}_2$ with the radical filter (*Goldie topology*):

$\mathcal{E}_{\mathbb{Z}_2} = \{I \in \mathbb{L}(_R R) \mid \exists\ J \in \mathcal{E}_{\mathbb{Z}}$ such that $I \subset J$ and $(I : b) \in \mathcal{E}_{\mathbb{Z}}\ \ \forall\ b \neq J\}$ ([2], Chapter VI, Proposition 6.3).

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A. I. Kashu
Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
5 Academiei str. Chişinău, MD−2028
Moldova

E-mail: *alexei.kashu@math.md*

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Properties of accessible subrings of pseudonormed rings when taking quotient rings

S. A. Aleschenko, V. I. Arnautov

Abstract. Let (R,ξ) and $(\bar{R},\bar{\xi})$ be pseudonormed rings, $\varphi : R \to \bar{R}$ be a ring isomorphism. We prove that $\varphi : (R,\xi) \to (\bar{R},\bar{\xi})$ is a superposition of a finite number of semi-isometric isomorphisms if and only if it is a narrowing on an accessible subring of some isometric homomorphism.

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We will say that a pseudonormed ring is a ring R which may be non-associative and has a pseudonorm (see [1], Definition 2.3.1).

The following isomorphism theorem is widely applied in the general algebra and, in particular, in the ring theory:

Theorem 1. *If A is a subring of a ring R and I is an ideal of the ring R then the quotient rings $A/(A \bigcap I)$ and $(A+I)/I$ are isomorphic rings. In particular, if $A \bigcap I = 0$, then the ring A is isomorphic to the ring $(A+I)/I$, i.e. the rings A and $(A+I)/I$ possess identical algebraic properties.*

Since it is necessary to take into account properties of pseudonorms when studying the pseudonormed rings then one needs to consider isomorphisms which keep pseudonorms. Such isomorphisms are called isometric isomorphisms.

The isomorphism theorem does not always take place for pseudonormed rings. The following theorem was proved in the work [2]:

Theorem 2. *Let (R,ξ) and $(\bar{R},\bar{\xi})$ be pseudonormed rings, $\varphi : R \to \bar{R}$ be a ring isomorphism. The inequality $\bar{\xi}(\varphi(r)) \le \xi(r)$ is satisfied for all $r \in R$ if and only if:*

– there exists a pseudonormed ring $(\widehat{R},\widehat{\xi})$ such that (R,ξ) is a subring of the pseudonormed ring $(\widehat{R},\widehat{\xi})$;

– the isomorphism φ can be extended up to an isometric homomorphism $\widehat{\varphi} : \left(\widehat{R},\widehat{\xi}\right) \to \left(\bar{R},\bar{\xi}\right)$ of the pseudonormed rings, i. e. $\bar{\xi}\left(\widehat{\varphi}\left(\widehat{r}\right)\right) = \inf\left\{\widehat{\xi}\left(\widehat{r}+a\right) | a \in \ker\widehat{\varphi}\right\}$ for all $\widehat{r} \in \widehat{R}$.

As it's shown in Theorem 2 it is impossible to tell anything more than the validity of the inequality $\bar{\xi}(\varphi(r)) \le \xi(r)$ in the case when A is a subring of a pseudonormed ring (R,ξ).

The case when A is an ideal of a pseudonormed ring (R, ξ) was studied in the work [2], the case when A is a one-sided ideal of a pseudonormed ring (R, ξ) was studied in the work [3].

The following definition was introduced in [2]:

Definition 1. *Let (R, ξ) and $(\bar{R}, \bar{\xi})$ be pseudonormed rings and $\varphi : R \to \bar{R}$ be a ring isomorphism. The isomorphism $\varphi : (R, \xi) \to (\bar{R}, \bar{\xi})$ is called a semi-isometric isomorphism if there exists a pseudonormed ring $(\widehat{R}, \widehat{\xi})$ such that the following conditions are valid:*

1) the ring R is an ideal in the ring $\widehat{R}$;

2) $\widehat{\xi}(r) = \xi(r)$ for any $r \in R$;

3) the isomorphism φ can be extended up to an isometric homomorphism $\widehat{\varphi} : (\widehat{R}, \widehat{\xi}) \to (\bar{R}, \bar{\xi})$ of the pseudonormed rings.

The following theorem was proved in [2]:

Theorem 3. *Let (R, ξ) and $(\bar{R}, \bar{\xi})$ be pseudonormed rings and $\varphi : R \to \bar{R}$ be a ring isomorphism. Then the isomorphism $\varphi : (R, \xi) \to (\bar{R}, \bar{\xi})$ is a semi-isometric isomorphism of the pseudonormed rings iff the inequalities $\xi(a \cdot b) \leq \bar{\xi}(\varphi(a)) \cdot \xi(b)$, $\xi(b \cdot a) \leq \bar{\xi}(\varphi(a)) \cdot \xi(b)$ and $\bar{\xi}(\varphi(a)) \leq \xi(a)$ are true for any $a, b \in R$.*

This paper is a continuation of [2] and [3] and it's devoted to the study of the case when A is an accessible subring of a pseudonormed ring (R, ξ) (see Definition 2). It's shown that a ring isomorphism is a superposition of semi-isometric isomorphisms iff it is a narrowing on the accessible subring A of some isometric homomorphism.

Definition 2. *As usual, a subring A of a rings R is called an accessible subring of the stage no more than n of the ring R if there exists a chain $A = R_0 \subseteq R_1 \subseteq R_2 \subseteq \ldots \subseteq R_n = R$ of subrings of the ring R such that R_i is an ideal in R_{i+1} for $i = 0, 1, \ldots n-1$. Further we shall designate it as $A = R_0 \lhd R_1 \lhd R_2 \lhd \ldots \lhd R_n = R$.*

Proposition 1. *Let: 1) $(\widehat{R}, \widehat{\xi})$ be a pseudonormed ring; 2) R be an ideal in $\widehat{R}$; 3) $\widehat{I}$ be a closed ideal in $(\widehat{R}, \widehat{\xi})$ and $I = \widehat{I} \bigcap R$; 4) $\widetilde{I} = [I]_{(\widehat{R}, \widehat{\xi})}$ and $\widetilde{R} = R + \widetilde{I}$; 5) $\bar{\varepsilon} : R/I \to (R + \widehat{I})/I$ be the natural embedding; 6) $\widehat{\omega} : \widehat{R} \to \widehat{R}/I$ and $\widetilde{\omega} : \widehat{R}/I \to \widehat{R}/\widetilde{I}$ be canonical homomorphisms. Then $\widetilde{\omega}|_{R/I} : (\bar{R}, \bar{\xi}) = (R, \widehat{\xi}|_R)/I \to (\widetilde{R}, \widehat{\xi}|_{\widetilde{R}})/\widetilde{I} = (\bar{\bar{R}}, \bar{\bar{\xi}})$ is an isometric isomorphism.*

Proof. Let's consider the following diagram 1.

$$\begin{array}{ccccc}
R \subseteq & & \widetilde{R} = R + \widetilde{I} & \subseteq & \widehat{R} \\
\widehat{\omega}|_R \downarrow & & \widehat{\omega}|_{\widetilde{R}} \downarrow & & \widehat{\omega} \downarrow \\
R/I & \xrightarrow{\bar{\varepsilon}} & \widetilde{R}/I & \subseteq & \widehat{R}/I \\
\Big\| & & \widetilde{\omega}|_{\widetilde{R}/I} \downarrow & & \widetilde{\omega} \downarrow \\
R/I & \xrightarrow{\widetilde{\omega}|_{R/I}} & \widetilde{R}/\widetilde{I} & \subseteq & \widehat{R}/\widetilde{I}
\end{array}$$

As $I \subseteq \widetilde{I}$ then $inf\{\widehat{\xi}(r+i)|i \in I\} \geq inf\{\widehat{\xi}(r+i)|i \in \widetilde{I}\}$ for any $r \in R$. Therefore $\bar{\xi}(\bar{r}) \geq \bar{\bar{\xi}}(\widetilde{\omega}(\bar{r}))$ for any $\bar{r} \in \bar{R}$.

We show that the reverse inequality is true.

Let $\bar{r}$ be any element in the ring $\bar{R} = R/I$ and ε be any positive number. If $r \in R$ is an element such that $\bar{r} = r + I$ then there exists an element $\widetilde{i}_0 \in \widetilde{I}$ such that $\bar{\bar{\xi}}(\widetilde{\omega}(\bar{r})) + \frac{\varepsilon}{2} \geq \widehat{\xi}(r + \widetilde{i}_0)$. Since $\widetilde{i}_0 \in \widetilde{I} = [I]_{(\widehat{R},\widehat{\xi})}$ then there exists an element $i_0 \in I$ such that $\widehat{\xi}(i_0 - \widetilde{i}_0) < \frac{\varepsilon}{2}$. Hence we have the inequality

$$\bar{\xi}(\bar{r}) = inf\{\widehat{\xi}(r+i)|i \in I\} \leq \widehat{\xi}(r+i_0) = \widehat{\xi}(r + \widetilde{i}_0 - \widetilde{i}_0 + i_0) \leq$$
$$\widehat{\xi}(r+\widetilde{i}_0) + \widehat{\xi}(i_0 - \widetilde{i}_0) < \bar{\bar{\xi}}(\widetilde{\omega}(\bar{r})) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \bar{\bar{\xi}}(\widetilde{\omega}(\bar{r})) + \varepsilon.$$

Passing to the limit in these inequalities when $\varepsilon \to 0$, we obtain $\bar{\xi}(\bar{r}) \leq \bar{\bar{\xi}}(\widetilde{\omega}(\bar{r}))$.

Thus it follows from the inequalities $\bar{\xi}(\bar{r}) \geq \bar{\bar{\xi}}(\widetilde{\omega}(\bar{r}))$ and $\bar{\xi}(\bar{r}) \leq \bar{\bar{\xi}}(\widetilde{\omega}(\bar{r}))$ we have the equality $\bar{\xi}(\bar{r}) = \bar{\bar{\xi}}(\widetilde{\omega}(\bar{r}))$, i.e. $\widetilde{\omega}|_{R/I} : (\bar{R}, \bar{\xi}) = (R, \widehat{\xi}|_R)/I \to (\widetilde{R}, \widehat{\xi}|_{\widetilde{R}})/\widetilde{I} = (\bar{\bar{R}}, \bar{\bar{\xi}})$ is an isometric isomorphism.

The proposition is proved.

Theorem 4. *Let (R, ξ) and $(\bar{R}, \bar{\xi})$ be pseudonormed rings and $\varphi : R \to \bar{R}$ be a ring isomorphism. Then the following statements are equivalent:*

1. There exists a pseudonormed ring $(\widehat{R}, \widehat{\xi})$ such that (R, ξ) is an accessible subring of the stage no more than n of the pseudonormed ring $(\widehat{R}, \widehat{\xi})$ and the isomorphism φ can be extended up to an isometric homomorphism $\widehat{\varphi} : (\widehat{R}, \widehat{\xi}) \to (\bar{R}, \bar{\xi})$.

2. φ is a superposition of n semi-isometric isomorphisms, i.e. there exist pseudonormed rings $(R, \xi) = (R_0, \xi_0), (R_1, \xi_1), \ldots, (R_n, \xi_n) = (\bar{R}, \bar{\xi})$ and semi-isometric isomorphisms $\varphi_i : (R_i, \xi_i) \to (R_{i+1}, \xi_{i+1})$ for $i = 0, 1, \ldots, n-1$ such that $\varphi = \varphi_n \circ \varphi_{n-1} \circ \ldots \circ \varphi_0$.

Proof 1 ⇒ 2. Let $R = \widehat{R}_0 \lhd \widehat{R}_1 \lhd \widehat{R}_2 \lhd \ldots \lhd \widehat{R}_n = \widehat{R}$ be a chain of subrings such that $\widehat{R}_i$ is an ideal in $\widehat{R}_{i+1}$ for $i = 0, 1, \ldots n-1$ and the isomorphism $\varphi : R \to \bar{R}$ can be extended up to an isometric homomorphism $\widehat{\varphi} : (\widehat{R}, \widehat{\xi}) \to (\bar{R}, \bar{\xi})$.

If $\widehat{I} = \ker \widehat{\varphi}$ and $\widetilde{\omega} : R_{k+1} \to R_{k+1}/\widehat{I}$ is the canonical homomorphism (i.e. $\widetilde{\omega}(r) = r + \widehat{I}$) then there exists an isometric isomorphism $\eta : (\widehat{R_n}, \widehat{\xi_n})/\widehat{I} \to (\bar{R}, \bar{\xi})$ such that $\widehat{\varphi} = \eta \circ \widetilde{\omega}$.

Let's consider the following diagram 2 (mappings entering into the diagram are defined below).

$$\begin{array}{ccccccccccc}
R = \widehat{R}_0 & \lhd & \ldots & \lhd & \widehat{R}_k & \lhd & \widehat{R}_{k+1} & = & \widehat{R}_{k+1} & = & \widehat{R} \\
\| & & & & \omega|_{\widehat{R}_k}\downarrow & & \omega\downarrow & & & & \\
R & \xrightarrow{\varphi_0} & \ldots & \xrightarrow{\varphi_{k-1}} & \widehat{R}_k/I & \lhd & \widehat{R}_{k+1}/I & & \downarrow\widetilde{\omega} & & \downarrow\widehat{\varphi} \\
\varphi\downarrow & & & & \varphi_k\downarrow & & \widehat{\omega}\downarrow & & & & \\
\bar{R} & = & & & \widehat{R}_{k+1}/\widehat{I} & = & \widehat{R}_{k+1}/\widehat{I} & = & \widehat{R}_{k+1}/\widehat{I} & \xrightarrow{\eta} & \bar{R}
\end{array}$$

The further proof will be done by induction on the number n.

If $n = 1$ then (R, ξ) is an accessible subring of the stage 1 (i.e. it is an ideal) of the pseudonormed ring $(\widehat{R}, \widehat{\xi})$ and the isomorphism φ can be extended up to an isometric homomorphism $\widehat{\varphi} : (\widehat{R}, \widehat{\xi}) \to (\bar{R}, \bar{\xi})$, and hence $\varphi : (R, \xi) \to (\bar{R}, \bar{\xi})$ is a semi-isometric isomorphism.

Let's assume that the theorem is true for $n = k$, and let $n = k + 1$. Since $\widehat{R}_k$ and $\widehat{I}$ are ideals in $\widehat{R}_{k+1}$ then $I = \widehat{R}_k \bigcap \widehat{I}$ is an ideal in $\widehat{R}_{k+1}$ too.

In the beginning let's consider the case when $I = \widehat{R}_k \bigcap \widehat{I}$ is a closed ideal in $(\widehat{R}_{k+1}, \widehat{\xi})$. If $\omega : \widehat{R}_{k+1} \to \widehat{R}_{k+1}/I$ is the canonical homomorphism, then $\omega|_{\widehat{R}_k} : (\widehat{R}_k, \widehat{\xi}|_{\widehat{R}_k}) \to (\widehat{R}_k, \widehat{\xi}|_{\widehat{R}_k})/I$ is an isometric homomorphism. As $\widehat{R}_0 \bigcap \ker \omega|_{\widehat{R}_k} = \widehat{R}_0 \bigcap I = \widehat{R}_0 \bigcap \widehat{I} = \widehat{R}_0 \bigcap \ker \widehat{\varphi} = \ker \varphi = \{0\}$ and $\widehat{R}_k = \widehat{R}_k \bigcap \widehat{R} = \widehat{R}_k \bigcap (R + \widehat{I}) = R + (\widehat{R}_k \bigcap \widehat{I}) = R + I$ then $\omega|_{\widehat{R}_0} : \widehat{R}_0 \to \widehat{R}_k/I$ is an isomorphism and by the assumption $\omega|_{\widehat{R}_0}$ is a superposition of k semi-isometric isomorphisms, i.e. there are pseudonormed rings $(R, \xi) = (R_0, \xi_0), (R_1, \xi_1), \ldots, (R_k, \xi_k) = (\widehat{R}_k, \widehat{\xi}|_{R_k})/I$ and isometric isomorphisms $\varphi_i : (R_i, \xi_i) \to (R_{i+1}, \xi_{i+1})$ for $i = 0, 1, \ldots, k-1$ such that $\omega|_{\widehat{R}_0} = \varphi_{k-1} \circ \varphi_{k-2} \circ \ldots \circ \varphi_0$.

As $I = \widehat{I} \bigcap R_k = (\ker \widehat{\varphi}) \bigcap R_k = \ker(\widehat{\varphi}|_{R_k})$ and $\overline{R} = \varphi(R) = \widehat{\varphi}(R)$ then $\widehat{R}_k + \widehat{I} = \widehat{R}_0 + \widehat{I} = \widehat{R}_{k+1}$, and so $\varphi_k = \widehat{\omega}|_{\widehat{R}_k/I} : \widehat{R}_k/I \to \widehat{R}_{k+1}/\widehat{I}$ is an isomorphism.

Since $\widehat{R}_k/I$ is an ideal in $\widehat{R}_{k+1}/I$ then $\varphi_k : (\widehat{R}_k, \widehat{\xi}|_{\widehat{R}_k})/I \to (\widehat{R}_{k+1}, \widehat{\xi})/\widehat{I}$ is a semi-isometric isomorphism. Hence $\eta \circ \varphi_k : (\widehat{R}_k, \widehat{\xi}|_{\widehat{R}_k})/I \to (\overline{R}_{k+1}, \overline{\xi})$ is a semi-isometric isomorphism, and $(\eta \circ \varphi_k) \circ \varphi_{k-1} \circ \varphi_{k-2} \circ \ldots \circ \varphi_0 = \eta \circ \varphi_k \circ \omega|_{\widehat{R}_0} = \eta \circ \widetilde{\omega}|_{R_0} = \widehat{\varphi}|_{R_0} = \varphi$, i.e. the isomorphism φ is a superposition of $k+1$ semi-isometric isomorphisms in the case when I is a closed ideal in $(\widehat{R}_{k+1}, \widehat{\xi})$.

Let's consider now the case when $I = \widehat{R}_k \bigcap \widehat{I}$ is non-closed ideal in $(\widehat{R}_{k+1}, \widehat{\xi})$. Let's designate $\widetilde{I} = [I]_{(\widehat{R}_{k+1}, \widehat{\xi})}$ and consider the diagram 3 which is obtained by adding one line to the diagram 2 (definitions of unknown by now rings and mappings see below).

$$
\begin{array}{ccccccccccc}
R = \widehat{R}_0 & \lhd & \ldots & \lhd & \widehat{R}_k & \lhd & \widehat{R}_{k+1} & = & \widehat{R}_{k+1} & = & \widehat{R} \\
\| & & & & \omega|_{\widehat{R}_k}\downarrow & & \omega\downarrow & & & & \\
R & \xrightarrow{\varphi_0} & \ldots & \xrightarrow{\varphi_{k-1}} & \widehat{R}_k/I & \lhd & \widehat{R}_{k+1}/I & & & & \\
 & & & & \bar{\eta}\downarrow & & \omega'\downarrow & & & & \\
\varphi\downarrow & & & & (\widehat{R}_k + \widetilde{I})/\widetilde{I} & \lhd & \widehat{R}_{k+1}/\widetilde{I} & & \downarrow\widetilde{\omega} & & \downarrow\widehat{\varphi} \\
 & & & & \varphi'_k\downarrow & & \bar{\omega}\downarrow & & & & \\
\bar{R} & & = & & \widehat{R}_{k+1}/\widehat{I} & = & \widehat{R}_{k+1}/\widehat{I} & = & \widehat{R}_{k+1}/\widehat{I} & \xrightarrow{\eta} & \bar{R}
\end{array}
$$

As $\widehat{R}_k$ is an ideal in $\widehat{R}_{k+1}$ then $I = \widehat{R}_k \bigcap \widehat{I}$ is an ideal in $\widehat{R}$, and hence $\widetilde{I}$ is a closed ideal in $(\widehat{R}, \widehat{\xi}) = (\widehat{R}_{k+1}, \widehat{\xi})$. Then $(\widehat{R}_{k+1}, \widehat{\xi})/\widetilde{I}$ and $(\widehat{R}_k + \widetilde{I}, \widehat{\xi}|_{\widehat{R}_k + \widetilde{I}})/\widetilde{I}$ are pseudonormed rings. If $\omega : \widehat{R} \to \widehat{R}/I$, $\omega' : \widehat{R}/I \to \widehat{R}/\widetilde{I}$ and $\bar{\omega} : \widehat{R}/\widetilde{I} \to \widehat{R}/\widehat{I}$ are the canonical homomorphisms then $\widetilde{\omega} = \overline{\omega} \circ \omega' \circ \omega$. As $(\widehat{R}_k + \widetilde{I})/\widetilde{I}$ is an ideal in $\widehat{R}_{k+1}/\widetilde{I}$ then $\varphi'_k = \bar{\omega}|_{(\widehat{R}_k+\widetilde{I})/\widetilde{I}} : (\widehat{R}_k + \widetilde{I}, \widehat{\xi}|_{\widehat{R}_k+\widetilde{I}})/\widetilde{I} \to (\widehat{R}_{k+1}, \widehat{\xi})/\widehat{I}$ is a semi-isometric isomorphism.

According to Proposition 1 $\bar{\eta} = \omega'|_{(\widehat{R}_k/I)} : (\widehat{R}_k, \widehat{\xi}|_{\widehat{R}_k})/I \to (\widehat{R}_k + \widetilde{I}, \widehat{\xi}|_{\widehat{R}_k+\widetilde{I}})/\widetilde{I}$ is an isometric isomorphism and hence $\overline{\eta} \circ \omega|_{\widehat{R}_k} : (\widehat{R}_k, \widehat{\xi}|_{\widehat{R}_k}) \to (\widehat{R}_k + \widetilde{I}, \widehat{\xi}|_{\widehat{R}_k+\widetilde{I}})/\widetilde{I}$ is an isometric homomorphism.

By the induction hypothesis, there exist pseudonormed rings $(R, \xi) = (R_0, \xi_0)$, $(R_1, \xi_1), \ldots, (R_k, \xi_k) = (\widehat{R}_k, \widehat{\xi}|_{\widehat{R}_k})/I$ and semi-isometric isomorphisms $\varphi_i : (R_i, \xi_i) \to (R_{i+1}, \xi_{i+1})$ for $i = 0, 1, 2, \ldots, k-1$ such that $\overline{\eta} \circ \omega|_{\widehat{R}_0} = \varphi_{k-1} \circ \varphi_{k-2} \circ \ldots \circ \varphi_0$.

Since $\eta, \overline{\eta}$ are isometric isomorphisms and φ'_k is a semi-isometric isomorphism then $\varphi''_k = \eta \circ \varphi'_k \circ \overline{\eta} : (\widehat{R}_k, \widehat{\xi}|_{\widehat{R}_k})/I \to (\overline{R}, \bar{\xi})$ is a semi-isometric isomorphism, at that $\varphi = \widehat{\varphi}|_R = \eta \circ \widetilde{\omega}|_R = \eta \circ \overline{\omega} \circ \omega' \circ \omega|_R = \eta \circ \varphi'_k \circ \overline{\eta} \circ \omega|_R = \varphi''_k \circ \overline{\eta} \circ \omega|_R = \varphi''_k \circ \varphi_{k-1} \circ \varphi_{k-2} \circ \ldots \circ \varphi_0$, i.e. the isomorphism φ is a superposition of $k+1$ semi-isometric isomorphisms in the case when I is a non-closed ideal in $(\widehat{R}_{k+1}, \widehat{\xi})$.

Thus we have proved that 2 follows from 1 for any natural number n.

Proof 2 ⇒ 1. Let's assume there are pseudonormed rings
$$(R, \xi) = (R_0, \xi_0), (R_1, \xi_1), (R_2, \xi_2) \ldots, (R_n, \xi_n) = (\bar{R}, \bar{\xi})$$
and semi-isometric isomorphisms $\varphi_i : (R_i, \xi_i) \to (R_{i+1}, \xi_{i+1})$ for $i = 0, 1, \ldots, n-1$ such that φ is the superposition of these semi-isometric isomorphisms, i.e. $\varphi = \varphi_{n-1} \circ \varphi_{n-2} \circ \ldots \circ \varphi_0$.

For any $0 \le i \le j \le n$ we consider the isomorphism $f_{i,j}$ such that $f_{i,j} = \varphi_{j-1} \circ \ldots \circ \varphi_i : R_i \to R_j$ for $i < j$ and $f_{i,i} : R_i \to R_i$ is the identical mapping.

The further proof will be done in some stages.

I. The construction of the ring $\widehat{R}$ and checking of some its algebraic properties.

Let's define on the set $\widehat{R} = \{(r_0, r_1, \ldots, r_n) \mid r_i \in R_i, i = 0, 1, \ldots, n\}$ the operations of addition and multiplication as follows:
$$(a_0, a_1, \ldots, a_n) + (b_0, b_1, \ldots, b_n) = (a_0 + b_0, a_1 + b_1, \ldots, a_n + b_n)$$
and
$$(a_0, a_1, \ldots, a_n) \cdot (b_0, b_1, \ldots, b_n) = (r_0, r_1, \ldots, r_n),$$
where $r_i = a_i \cdot b_i$ for $i \in \{0, n\}$ and $r_i = a_i \cdot b_i + (f_{0,i}(a_0) - a_i) \cdot \varphi_i^{-1}(b_{i+1}) + \varphi_i^{-1}(a_{i+1}) \cdot (f_{0,i}(b_0) - b_i)$ for $1 \le i \le n-1$.

As the mappings $\varphi_i : R_i \to R_{i+1}$ and $f_{0,i} : R_0 \to R_i$ are isomorphisms then it's easily checked that:

I.1. $\widehat{R}$ is a non-associative ring with respect to these operations (even if the initial rings are associative).

I.2. For any $0 \le k < n$ the set $\widehat{R}_k = \{(r_0, \ldots, r_n) \in \widehat{R} \mid r_i = 0 \text{ if } i > k\}$ is an ideal in the ring $\widehat{R}_{k+1} = \{(r_0, \ldots, r_n) \in \widehat{R} \mid r_i = 0 \text{ if } i > k+1\}$.

I.3. $\widehat{R}_0 = \{(r_0, \ldots, r_n) \in \widehat{R} \mid r_i = 0 \text{ if } i \ge 1\}$ is an accessible subring of the stage no more than n in the ring $\widehat{R}_n = \widehat{R}$;

I.4. The mapping $\psi : \widehat{R}_0 \to R_0 = R$ which transfers the element $(a, 0, \ldots, 0) \in \widehat{R}_0$ into the element $a \in R_0$ is isomorphic.

I.5. From the definition of the operations of addition and multiplication in $\widehat{R}$ it follows that $\widehat{I} = \{(0, r_1, \ldots r_n) \mid r_i \in R_i, i = 1, \ldots, n\}$ is an ideal in the ring $\widehat{R}$ and $\widehat{R}_0 \bigcap \widehat{I} = \{0\}$ and $\widehat{R}_0 + \widehat{I} = \widehat{R}$.

I.6. If $\widehat{\varphi} : \widehat{R} \to \bar{R}$ is a mapping such that $\widehat{\varphi}(r_0, r_1, \ldots, r_n) = \varphi(r_0)$ for any $(r_0, r_1, \ldots, r_n) \in \widehat{R}$ then $\widehat{\varphi} : \widehat{R} \to \bar{R}$ is a ring homomorphism, and besides $\ker \widehat{\varphi} = \widehat{I}$ and $\widehat{\varphi}|_R = \varphi$.

Identifying any elements $(a, 0, \ldots, 0) \in \widehat{R}_0$ with the elements $a \in R_0$, we shall identify the ring $\widehat{R}_0$ with the ring R_0. Therefore we can consider that $R = R_0$ is an accessible subring of the stage no more than n of the ring $\widehat{R}_n = \widehat{R}$.

II. The definition of a pseudonorm $\widehat{\xi}$ on the ring $\widehat{R}$ and checking of some properties of the pseudonormed ring $(\widehat{R}, \widehat{\xi})$.

Let's define $\widehat{\xi}((r_0, r_1, \ldots, r_n)) = \sum\limits_{i=0}^{n-1} \xi_i(r_i - \varphi_i^{-1}(r_{i+1})) + \xi_n(r_n)$.

II.1. Let's check that $\widehat{\xi}$ is a pseudonorm on the ring $\widehat{R}$.

It's easy follows from the definition of the function $\widehat{\xi}$ that $\widehat{\xi}((-r_0, -r_1, \ldots, -r_n)) = \widehat{\xi}((r_0, r_1, \ldots, r_n)) \geq 0$ for any $(r_0, r_1, \ldots, r_n) \in \widehat{R}$ and $\widehat{\xi}((r_0, r_1, \ldots, r_n)) = 0$ if and only if $(r_0, r_1, \ldots, r_n) = (0, 0, \ldots, 0)$.

Let $a = (a_0, a_1, \ldots, a_n) \in \widehat{R}$ and $b = (b_0, b_1, \ldots, b_n) \in \widehat{R}$. Then

$$\widehat{\xi}(a + b) = \sum_{i=0}^{n-1} \xi_i(a_i + b_i - \varphi_i^{-1}(a_{i+1} + b_{i+1})) + \xi_n(a_n + b_n) \leq$$

$$\sum_{i=0}^{n-1} (\xi_i(a_i - \varphi_i^{-1}(a_{i+1})) + \xi_i(b_i - \varphi_i^{-1}(b_{i+1}))) + \xi_n(a_n) + \xi_n(b_n) = \widehat{\xi}(a) + \widehat{\xi}(b).$$

If $r = (r_0, r_1, \ldots, r_n) = a \cdot b = (a_0, a_1, \ldots, a_n) \cdot (b_0, b_1, \ldots, b_n)$ then $r_0 = a_0 \cdot b_0$, $r_n = a_n \cdot b_n$, $r_i = a_i \cdot b_i + (f_{0,i}(a_0) - a_i) \cdot \varphi_i^{-1}(b_{i+1}) + \varphi_i^{-1}(a_{i+1}) \cdot (f_{0,i}(b_0) - b_i)$ for $i \in \{1, 2, \ldots, n-1\}$ and

$$\widehat{\xi}(a \cdot b) = \widehat{\xi}((r_0, r_1, \ldots, r_n)) = \xi_n(r_n) + \sum_{i=0}^{n-1} \xi_i(r_i - \varphi_i^{-1}(r_{i+1})).$$

Let's consider each term of this sum. It's obvious that $\xi_n(r_n) \leq \xi_n(a_n) \cdot \xi_n(b_n)$.

Let $h_i = a_i - \varphi_i^{-1}(a_{i+1})$ and $h'_i = b_i - \varphi_i^{-1}(b_{i+1})$ for $i \in \{0, 1, \ldots, n-1\}$; $h_n = a_n$ and $h'_n = b_n$. Taking in consideration the definitions of mapping $f_{i,j}$ by induction on the number $j - i$ it's easy proved that

$$f_{i,j}(a_i) - a_j = f_{i,j}(a_i) - \varphi_{j-1}(\varphi_{j-1}^{-1}(a_j)) =$$
$$f_{i,j}(a_i) - f_{i,j}(\varphi_i^{-1}(a_{i+1})) + f_{i,j}(\varphi_i^{-1}(a_{i+1})) - f_{j-1,j}(\varphi_{j-1}^{-1}(a_j)) =$$
$$f_{i,j}(a_i - \varphi_i^{-1}(a_{i+1})) + f_{i,j}(\varphi_i^{-1}(a_{i+1})) - f_{j-1,j}(\varphi_{j-1}^{-1}(a_j)) = f_{i,j}(h_i) +$$
$$f_{i,j}(\varphi_i^{-1}(a_{i+1})) - f_{j-1,j}(\varphi_{j-1}^{-1}(a_j)) = \ldots = f_{i,j}(h_i) + f_{i+1,j}(h_{i+1}) + \ldots + f_{j-1,j}(h_{j-1})$$

for any $0 \leq i < j \leq n$. Then for $i \in \{1, 2, \ldots, n-2\}$ we have

$$\xi_i(r_i - \varphi_i^{-1}(r_{i+1})) = \xi_i(a_i \cdot b_i + (f_{0,i}(a_0) - a_i) \cdot \varphi_i^{-1}(b_{i+1}) + \varphi_i^{-1}(a_{i+1}) \cdot (f_{0,i}(b_0) - b_i) -$$
$$\varphi_i^{-1}(a_{i+1} \cdot b_{i+1} + (f_{0,i+1}(a_0) - a_{i+1}) \cdot \varphi_{i+1}^{-1}(b_{i+2}) + \varphi_{i+1}^{-1}(a_{i+2}) \cdot (f_{0,i+1}(b_0) - b_{i+1}))) =$$
$$\xi_i(a_i \cdot b_i + \sum_{k=0}^{i-1} f_{k,i}(h_k) \cdot \varphi_i^{-1}(b_{i+1}) + \varphi_i^{-1}(a_{i+1}) \cdot \sum_{k=0}^{i-1} f_{k,i}(h'_k) - \varphi_i^{-1}(a_{i+1}) \cdot \varphi_i^{-1}(b_{i+1}) -$$
$$\varphi_i^{-1}(\sum_{k=0}^{i} f_{k,i+1}(h_k) \cdot \varphi_{i+1}^{-1}(b_{i+2}) + \varphi_{i+1}^{-1}(a_{i+2}) \cdot \sum_{k=0}^{i} f_{k,i+1}(h'_k))) = \xi_i(a_i \cdot b_i + \sum_{k=0}^{i-1} f_{k,i}(h_k) \cdot$$

$\varphi_i^{-1}(b_{i+1}-\varphi_{i+1}^{-1}(b_{i+2}))+\varphi_i^{-1}(a_{i+1}-\varphi_{i+1}^{-1}(a_{i+2}))\cdot\sum\limits_{k=0}^{i-1} f_{k,i}(h'_k)-(a_i-h_i)\cdot(b_i-h'_i)-h_i\cdot\varphi_i^{-1}(b_{i+1}-h'_{i+1})-\varphi_i^{-1}(a_{i+1}-h_{i+1})\cdot h'_i)=\xi_i(\sum\limits_{k=0}^{i-1} f_{k,i}(h_k)\cdot\varphi_i^{-1}(h'_{i+1})+\varphi_i^{-1}(h_{i+1})\cdot\sum\limits_{k=0}^{i-1} f_{k,i}(h'_k)+h_i\cdot(b_i-\varphi_i^{-1}(b_{i+1}))+(a_i-\varphi_i^{-1}(a_{i+1}))\cdot h'_i-h_i\cdot h'_i+h_i\cdot\varphi_i^{-1}(h'_{i+1})+\varphi_i^{-1}(h_{i+1})\cdot h'_i)=\xi_i(\sum\limits_{k=0}^{i-1} f_{k,i}(h_k)\cdot\varphi_i^{-1}(h'_{i+1})+\varphi_i^{-1}(h_{i+1})\cdot\sum\limits_{k=0}^{i-1} f_{k,i}(h'_k)+h_i\cdot h'_i+h_i\cdot\varphi_i^{-1}(h'_{i+1})+\varphi_i^{-1}(h_{i+1})\cdot h'_i)$.

If $i=n-1$ then
$\xi_{n-1}(r_{n-1}-\varphi_{n-1}^{-1}(r_n))=\xi_{n-1}(a_{n-1}\cdot b_{n-1}+(f_{0,n-1}(a_0)-a_{n-1})\cdot\varphi_{n-1}^{-1}(b_n)+\varphi_{n-1}^{-1}(a_n)\cdot(f_{0,n-1}(b_0)-b_{n-1})-\varphi_{n-1}^{-1}(a_n\cdot b_n))=\xi_{n-1}(a_{n-1}\cdot b_{n-1}+\sum\limits_{k=0}^{n-2} f_{k,n-1}(h_k)\cdot\varphi_{n-1}^{-1}(h'_n)+\varphi_{n-1}^{-1}(h_n)\cdot\sum\limits_{k=0}^{n-2} f_{k,n-1}(h'_k)-(a_{n-1}-h_{n-1})\cdot(b_{n-1}-h'_{n-1}))=\xi_{n-1}(\sum\limits_{k=0}^{n-2} f_{k,n-1}(h_k)\cdot\varphi_{n-1}^{-1}(h'_n)+\varphi_{n-1}^{-1}(h_n)\cdot\sum\limits_{k=0}^{n-2} f_{k,n-1}(h'_k)+h_{n-1}\cdot(h'_{n-1}+\varphi_{n-1}^{-1}(h'_n))+(h_{n-1}+\varphi_{n-1}^{-1}(h_n))\cdot h'_{n-1}-h_{n-1}\cdot h'_{n-1})=\xi_{n-1}(\sum\limits_{k=0}^{n-2} f_{k,n-1}(h_k)\cdot\varphi_{n-1}^{-1}(h'_n)+\varphi_{n-1}^{-1}(h_n)\cdot\sum\limits_{k=0}^{n-2} f_{k,n-1}(h'_k)+h_{n-1}\cdot h'_{n-1}+h_{n-1}\cdot\varphi_{n-1}^{-1}(h'_n)+\varphi_{n-1}^{-1}(h_n)\cdot h'_{n-1})$.

Since the isomorphism $\varphi_i:(R_i,\xi_i)\to(R_{i+1},\xi_{i+1})$ is a semi-isometric then according to Theorem 3 the following inequalities are true:

$$\frac{\xi_i(a_i\cdot b_i)}{\xi_i(b_i)}\le\xi_{i+1}(\varphi_i(a_i))\le\xi_i(a_i)\text{ and }\frac{\xi_i(a_i\cdot b_i)}{\xi_i(a_i)}\le\xi_{i+1}(\varphi_i(b_i))\le\xi_i(b_i).$$

It's follows from the definition of the isomorphisms $f_{k,i}$:

$$\xi_i(f_{k,i}(h_k))\le\xi_k(h_k)\text{ and }\xi_i(f_{k,i}(h'_k))\le\xi_k(h'_k)$$

for any $0\le k\le i\le n$. Then for $i\in\{1,2,\dots,n-1\}$ we have
$\xi_i(\sum\limits_{k=0}^{i-1} f_{k,i}(h_k)\cdot\varphi_i^{-1}(h'_{i+1})+\varphi_i^{-1}(h_{i+1})\cdot\sum\limits_{k=0}^{i-1} f_{k,i}(h'_k)+h_i\cdot h'_i+h_i\cdot\varphi_i^{-1}(h'_{i+1})+\varphi_i^{-1}(h_{i+1})\cdot h'_i)\le\sum\limits_{k=0}^{i-1}\xi_i(f_{k,i}(h_k))\cdot\xi_{i+1}(h'_{i+1})+\sum\limits_{k=0}^{i-1}\xi_{i+1}(h_{i+1})\cdot\xi_i(f_{k,i}(h'_k))+\xi_i(h_i)\cdot\xi_i(h'_i)+\xi_i(h_i)\cdot\xi_{i+1}(h'_{i+1})+\xi_{i+1}(h_{i+1})\cdot\xi_i(h'_i)\le\sum\limits_{k=0}^{i-1}\xi_k(h_k)\cdot\xi_{i+1}(h'_{i+1})+\sum\limits_{k=0}^{i-1}\xi_{i+1}(h_{i+1})\cdot\xi_k(h'_k)+\xi_i(h_i)\cdot\xi_i(h'_i)+\xi_i(h_i)\cdot\xi_{i+1}(h'_{i+1})+\xi_{i+1}(h_{i+1})\cdot\xi_i(h'_i)$.

If $i=0$ then
$\xi_0(r_0-\varphi_1^{-1}(r_1))=\xi_0(a_0\cdot b_0-\varphi_0^{-1}(a_1\cdot b_1+(\varphi_0(a_0)-a_1)\cdot\varphi_1^{-1}(b_2)+\varphi_1^{-1}(a_2)\cdot(\varphi_0(b_0)-b_1)))=\xi_0(a_0\cdot b_0-\varphi_0^{-1}(a_1)\cdot\varphi_0^{-1}(b_1)-(a_0-\varphi_0^{-1}(a_1))\cdot\varphi_0^{-1}(\varphi_1^{-1}(b_2))-\varphi_0^{-1}(\varphi_1^{-1}(a_2))\cdot(b_0-\varphi_0^{-1}(b_1)))=\xi_0(a_0\cdot b_0-(a_0-h_0)\cdot(b_0-h'_0)-h_0\cdot\varphi_0^{-1}(b_1-h'_1)-\varphi_0^{-1}(a_1-h_1)\cdot h'_0)=\xi_0(h_0\cdot h'_0+h_0\cdot\varphi_0^{-1}(h'_1)+\varphi_0^{-1}(h_1)\cdot h'_0)\le\xi_0(h_0)\cdot\xi_0(h'_0)+\xi_0(h_0)\cdot\xi_1(h'_1)+\xi_1(h_1)\cdot\xi_0(h'_0)$.

It follows from the proven inequalities that

$$\widehat{\xi}(a\cdot b) \leq \xi_0(h_0)\cdot\xi_0(h'_0)+\xi_0(h_0)\cdot\xi_1(h'_1)+\xi_1(h_1)\cdot\xi_0(h'_0)+\sum_{i=1}^{n-1}\Big(\sum_{k=0}^{i-1}\xi_k(h_k)\cdot\xi_{i+1}(h'_{i+1})+$$

$$\sum_{k=0}^{i-1}\xi_{i+1}(h_{i+1})\cdot\xi_k(h'_k)+\xi_i(h_i)\cdot\xi_i(h'_i)+\xi_i(h_i)\cdot\xi_{i+1}(h'_{i+1})+\xi_{i+1}(h_{i+1})\cdot\xi_i(h'_i)\Big)+$$

$$\xi_n(a_n)\cdot\xi_n(b_n)=\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\xi_i(h_i)\cdot\xi_j(h'_j)+\xi_n(a_n)\cdot\sum_{j=0}^{n-1}\xi_j(h'_j)+\sum_{i=0}^{n-1}\xi_i(h_i)\cdot\xi_n(b_n)$$

$$+\xi_n(a_n)\cdot\xi_n(b_n)=\left(\sum_{i=0}^{n-1}\xi_i(h_i)+\xi_n(a_n)\right)\cdot\left(\sum_{j=0}^{n-1}\xi_j(h'_j)+\xi_n(b_n)\right)=\widehat{\xi}(a)\cdot\widehat{\xi}(b).$$

Thus we have shown the inequality $\widehat{\xi}(a\cdot b) \leq \widehat{\xi}(a)\cdot\widehat{\xi}(b)$ for any $a, b \in \widehat{R}$. Therefore $(\widehat{R}, \widehat{\xi})$ ia a pseudonormed ring.

II.2. Since $\widehat{\xi}(r, 0, \dots, 0) = \xi_0(r-0)+\xi_1(0)+\ldots+\xi_n(0) = \xi(r)$ for any $r \in R$ and any element $r \in R$ is identifying with the element $(r, 0, \dots, 0) \in \widehat{R}_0$ then $\widehat{\xi}|_R = \xi$.

II.3. Let's show that $\widehat{\varphi} : (\widehat{R}, \widehat{\xi}) \to (\bar{R}, \bar{\xi})$ is an isometric homomorphism, i.e. $\bar{\xi}\left(\widehat{\varphi}\left(\widehat{r}\right)\right) = \inf\left\{\widehat{\xi}\left(\widehat{r}+\widehat{a}\right)\Big|\widehat{a}\in\ker\widehat{\varphi}\right\}$ for all $\widehat{r} \in \widehat{R}$. Let $\widehat{r} = (r_0, r_1, \dots, r_n) \in \widehat{R}$ and $\widehat{b} = (0, f_{0,1}(r_0) - r_1, \dots, f_{0,n}(r_0) - r_n)$. Then $\widehat{b} \in \widehat{I}$ and so

$$\inf\left\{\widehat{\xi}(\widehat{r}+\widehat{a})\Big|\widehat{a}\in\ker\widehat{\varphi}\right\} \leq \widehat{\xi}(\widehat{r}+\widehat{b}) = \widehat{\xi}((r_0, r_1, \dots, r_n)+$$

$$(0, f_{0,1}(r_0)-r_1, \dots, f_{0,n}(r_0)-r_n)) = \widehat{\xi}((r_0, f_{0,1}(r_0), \dots, f_{0,n}(r_0))) =$$

$$\xi_0(r_0-\varphi_0^{-1}(f_{0,1}(r_0)))+\xi_1(f_{0,1}(r_0)-$$

$$\varphi_1^{-1}(f_{0,2}(r_0)))+\ldots+\xi_{n-1}(f_{0,n-1}(r_0)-\varphi_{n-1}^{-1}(f_{0,n}(r_0)))+\xi_n(f_{0,n}(r_0)) =$$

$$\xi_0(0)+\xi_1(0)+\ldots+\xi_{n-1}(0)+\xi_n(\varphi(r_0)) = \bar{\xi}(\varphi(r_0)) = \bar{\xi}(\widehat{\varphi}(\widehat{r})).$$

On the other hand, since $f_{0,n} = \varphi$ and $\xi_i(d_i) \geq \xi_n(f_{i,n}(d_n))$ for every $d_i \in R_i$ and any $i \in \{0, 1, \dots, n\}$ then for every element $\widehat{a} = (o, a_1, \dots, a_n) \in \widehat{I}$ we have

$$\widehat{\xi}(\widehat{r}+\widehat{a}) = \widehat{\xi}((r_0, r_1+a_1, \dots, r_n+a_n) = \xi_0(r_0-\varphi_0^{-1}(r_1+a_1))+$$

$$\sum_{i=1}^{n-1}\xi_i(r_i+a_i-\varphi_i^{-1}(r_{i+1}+a_{i+1}))+\xi_n(r_n+a_n)\geq\xi_n(f_{0,n}(r_0)-f_{0,n}(\varphi_0^{-1}(r_1+a_1)))+$$

$$\sum_{i=1}^{n-1}\xi_n(f_{i,n}(r_i+a_i)-f_{i,n}(\varphi_i^{-1}(r_{i+1}+a_{i+1})))+\xi_n(r_n+a_n)=\xi_n(f_{0,n}(r_0)-f_{1,n}(r_1+a_1))+$$

$$\sum_{i=1}^{n-1}\xi_n(f_{i,n}(r_i+a_i)-f_{i+1,n}(r_{i+1}+a_{i+1}))+\xi_n(r_n+a_n)\geq$$

$$\xi_n\left(f_{0,n}(r_0)-f_{1,n}(r_1+a_1)+\sum_{i=1}^{n-1}(f_{i,n}(r_i+a_i)-f_{i+1,n}(r_{i+1}+a_{i+1}))+r_n+a_n\right)=$$

$$\xi_n\left(f_{0,n}(r_0)\right)=\xi_n(\varphi(r_0))=\bar{\xi}(\widehat{\varphi}(\widehat{r})).$$

Since $\widehat{a}\in\widehat{I}$ is any element then $\inf\left\{\widehat{\xi}\,(\widehat{r}+\widehat{a})\,\middle|\,\widehat{a}\in\ker\widehat{\varphi}\right\}\geq\bar{\xi}(\widehat{\varphi}(\widehat{r}))$ and so $\inf\left\{\widehat{\xi}\,(\widehat{r}+\widehat{a})\,\middle|\,\widehat{a}\in\ker\widehat{\varphi}\right\}=\bar{\xi}(\widehat{\varphi}(\widehat{r}))$. Therefore $\widehat{\varphi}:(\widehat{R},\widehat{\xi})\to(\bar{R},\bar{\xi})$ is an isometric homomorphism.

The theorem is completely proved.

Designation 1. *Let R be a ring. Put $R^1=R$ and for any natural number n define R^n as the subgroup generated by the set $\{a\cdot b|a\in R^s,b\in R^t,0<s,t<n,s+t=n\}$. It's easy to note that R^n is an ideal in the ring R.*

Definition 3. *A ring R is called a nilpotent ring if $R^n=0$ for some natural number n. The minimal one from these natural numbers is called the index of nilpotence.*

Theorem 5. *Let (R,ξ) and $(\bar{R},\bar{\xi})$ be associative pseudonormed rings, $\varphi:R\to\bar{R}$ be a ring isomorphism and $R^n=0$. Then the following statements are equivalent:*

1. $\bar{\xi}(\varphi(r))\leq\xi(r)$ for any $r\in R$.

2. φ is a superposition of n semi-isometric isomorphisms, i.e. there exist pseudonormed rings $(R,\xi)=(R_0,\xi_0),(R_1,\xi_1),\ldots,(R_n,\xi_n)=(\bar{R},\bar{\xi})$ and semi-isometric isomorphisms $\varphi_i:(R_i,\xi_i)\to(R_{i+1},\xi_{i+1})$ for $i=0,1,\ldots,n-1$ such that $\varphi=\varphi_n\circ\varphi_{n-1}\circ\ldots\circ\varphi_0$.

3. There exists a non-associative pseudonormed ring $(\widehat{R},\widehat{\xi})$ such that (R,ξ) is an accessible subring of the stage no more than n of the pseudonormed ring $(\widehat{R},\widehat{\xi})$ and the isomorphism φ can be extended up to an isometric homomorphism $\widehat{\varphi}:(\widehat{R},\widehat{\xi})\to(\bar{R},\bar{\xi})$.

Proof 1 ⇒ 2.

Let $R_k=R$ for $k=0,1,\cdots,n-1$ and $R_n=\bar{R}$; let $\varphi_{n-1}=\varphi:R\to\bar{R}$ and $\varphi_k=\varepsilon:R\to R$ be the identical mapping for $k=0,1,\cdots,n-2$; let $\xi_0(r)=\xi(r)$, $\xi_n(\bar{r})=\bar{\xi}(\bar{r})$, $\xi_{n-1}(r)=\bar{\xi}(\varphi(r))$ and

$$\xi_k(r) = sup\left\{\bar{\xi}(\varphi(r)), \frac{\xi_{k-1}(r \cdot a)}{\xi_{k-1}(a)}, \frac{\xi_{k-1}(a \cdot r)}{\xi_{k-1}(a)}\Big| a \in R\backslash\{0\}\right\}$$

for $k = 1, 2, \cdots, n-2$.

Let's prove by induction on the number k that each function ξ_k is a pseudonorm on the ring R_k.

It's obvious that $\xi_k(-r) = \xi_k(r) \geq 0$ for any $r \in R_k$ and $\xi_k(r) = 0$ if and only if $r = 0$. Let's show the validity of inequalities $\xi_k(r_1 + r_2) \leq \xi_k(r_1) + \xi_k(r_2)$ and $\xi_k(r_1 \cdot r_2) \leq \xi_k(r_1) \cdot \xi_k(r_2)$ for any $r_1, r_2 \in R_k$.

Indeed, for any $a \in R\backslash\{0\}$ we have

$$\frac{\xi_{k-1}((r_1 + r_2) \cdot a)}{\xi_{k-1}(a)} \leq \frac{\xi_{k-1}(r_1 \cdot a)}{\xi_{k-1}(a)} + \frac{\xi_{k-1}(r_2 \cdot a)}{\xi_{k-1}(a)} \leq$$

$$sup\left\{\frac{\xi_{k-1}(r_1 \cdot b)}{\xi_{k-1}(b)}\Big| b \in R\backslash\{0\}\right\} + sup\left\{\frac{\xi_{k-1}(r_2 \cdot b)}{\xi_{k-1}(b)}\Big| b \in R\backslash\{0\}\right\} \leq \xi_k(r_1) + \xi_k(r_2),$$

$$\frac{\xi_{k-1}(a \cdot (r_1 + r_2))}{\xi_{k-1}(a)} \leq \frac{\xi_{k-1}(a \cdot r_1)}{\xi_{k-1}(a)} + \frac{\xi_{k-1}(a \cdot r_2)}{\xi_{k-1}(a)} \leq$$

$$sup\left\{\frac{\xi_{k-1}(b \cdot r_1)}{\xi_{k-1}(b)}\Big| b \in R\backslash\{0\}\right\} + sup\left\{\frac{\xi_{k-1}(b \cdot r_2)}{\xi_{k-1}(b)}\Big| b \in R\backslash\{0\}\right\} \leq \xi_k(r_1) + \xi_k(r_2)$$

and

$$\bar{\xi}(\varphi(r_1 + r_2)) = \bar{\xi}(\varphi(r_1) + \varphi(r_2)) \leq \bar{\xi}(\varphi(r_1)) + \bar{\xi}(\varphi(r_2)) \leq \xi_k(r_1) + \xi_k(r_2).$$

Therefore

$$\xi_k(r_1+r_2) = sup\left\{\bar{\xi}(\varphi(r_1 + r_2)), \frac{\xi_{k-1}((r_1 + r_2) \cdot a)}{\xi_{k-1}(a)}, \frac{\xi_{k-1}(a \cdot (r_1 + r_2))}{\xi_{k-1}(a)}\Big| a \in R\backslash\{0\}\right\} \leq$$

$\xi_k(r_1) + \xi_k(r_2)$.

For any $a \in R\backslash\{0\}$ we have

$$\frac{\xi_{k-1}((r_1 \cdot r_2) \cdot a)}{\xi_{k-1}(a)} = \frac{\xi_{k-1}(r_1 \cdot (r_2 \cdot a))}{\xi_{k-1}(r_2 \cdot a))} \cdot \frac{\xi_{k-1}(r_2 \cdot a)}{\xi_{k-1}(a)} \leq$$

$$sup\left\{\frac{\xi_{k-1}(r_1 \cdot b)}{\xi_{k-1}(b)}\Big| b \in R\backslash\{0\}\right\} \cdot sup\left\{\frac{\xi_{k-1}(r_2 \cdot c)}{\xi_{k-1}(c)}\Big| c \in R\backslash\{0\}\right\} \leq \xi_k(r_1) \cdot \xi_k(r_2),$$

$$\frac{\xi_{k-1}(a \cdot (r_1 \cdot r_2))}{\xi_{k-1}(a)} \leq \frac{\xi_{k-1}(a \cdot r_1)}{\xi_{k-1}(a)} \cdot \frac{\xi_{k-1}((a \cdot r_1) \cdot r_2)}{\xi_{k-1}(a \cdot r_1)} \leq$$

$$sup\left\{\frac{\xi_{k-1}(b \cdot r_1)}{\xi_{k-1}(b)}\Big| b \in R\backslash\{0\}\right\} \cdot sup\left\{\frac{\xi_{k-1}(c \cdot r_2)}{\xi_{k-1}(c)}\Big| c \in R\backslash\{0\}\right\} \leq \xi_k(r_1) \cdot \xi_k(r_2)$$

and

$$\bar{\xi}(\varphi(r_1 \cdot r_2)) = \bar{\xi}(\varphi(r_1) \cdot \varphi(r_2)) \leq \bar{\xi}(\varphi(r_1)) \cdot \bar{\xi}(\varphi(r_2)) \leq \xi_k(r_1) \cdot \xi_k(r_2).$$

Therefore

$$\xi_k(r_1 \cdot r_2) = sup\left\{\bar{\xi}(\varphi(r_1 \cdot r_2)), \frac{\xi_{k-1}((r_1 \cdot r_2) \cdot a)}{\xi_{k-1}(a)}, \frac{\xi_{k-1}(a \cdot (r_1 \cdot r_2))}{\xi_{k-1}(a)} \Big| a \in R\backslash\{0\}\right\} \leq$$

$\xi_k(r_1) \cdot \xi_k(r_2)$.

Thus the function ξ_k is a pseudonorm on the ring R_k.

Let's prove that $\varphi_k : (R_k, \xi_k) \to (R_{k+1}, \xi_{k+1})$ is a semi-isometric isomorphism for $k = 0, 1, \cdots, n-2$.

Let's check the validity of inequality $\xi_{k+1}(\varphi_k(r)) \leq \xi_k(r)$.

Since

$$\bar{\xi}(\varphi(r)) \leq \xi_k(r), \frac{\xi_k(r \cdot a)}{\xi_k(a)} \leq \xi_k(r) \text{ and } \frac{\xi_k(a \cdot r)}{\xi_k(a)} \leq \xi_k(r)$$

for any $a \in R\backslash\{0\}$ then

$$sup\left\{\bar{\xi}(\varphi(r)), \frac{\xi_{k-1}(r \cdot a)}{\xi_{k-1}(a)}, \frac{\xi_{k-1}(a \cdot r)}{\xi_{k-1}(a)} \Big| a \in R\backslash\{0\}\right\} \leq \xi_k(r)$$

and

$$\xi_{k+1}(\varphi_k(r)) = \xi_{k+1}(\varepsilon(r)) = \xi_{k+1}(r) \leq \xi_k(r)$$

for any $r \in R_k$.

Let's show that the inequalities $\xi_k(r \cdot q) \leq \xi_{k+1}(\varphi_k(r)) \cdot \xi_k(q)$ and $\xi_k(q \cdot r) \leq \xi_{k+1}(\varphi_k(r)) \cdot \xi_k(q)$ are true.

Indeed, for any $q \neq 0$ we have

$$\frac{\xi_k(r \cdot q)}{\xi_k(q)} \leq sup\left\{\frac{\xi_k(r \cdot a)}{\xi_k(a)} \Big| a \in R\backslash\{0\}\right\} \leq$$

$$sup\left\{\bar{\xi}(\varphi(r)), \frac{\xi_k(r \cdot a)}{\xi_k(a)}, \frac{\xi_k(a \cdot r)}{\xi_k(a)} \Big| a \in R\backslash\{0\}\right\} = \xi_{k+1}(r)$$

and

$$\frac{\xi_k(q \cdot r)}{\xi_k(q)} \leq sup\left\{\frac{\xi_k(a \cdot r)}{\xi_k(a)} \Big| a \in R\backslash\{0\}\right\} \leq$$

$$sup\left\{\bar{\xi}(\varphi(r)), \frac{\xi_k(r \cdot a)}{\xi_k(a)}, \frac{\xi_k(a \cdot r)}{\xi_k(a)} \Big| a \in R\backslash\{0\}\right\} = \xi_{k+1}(r).$$

Thus

$$\xi_k(r \cdot q) \leq \xi_{k+1}(r) \cdot \xi_k(q) = \xi_{k+1}(\varepsilon(r)) \cdot \xi_k(q) = \xi_{k+1}(\varphi_k(r)) \cdot \xi_k(q)$$

and

$$\xi_k(q \cdot r) \leq \xi_{k+1}(r) \cdot \xi_k(q) = \xi_{k+1}(\varepsilon(r)) \cdot \xi_k(q) = \xi_{k+1}(\varphi_k(r)) \cdot \xi_k(q).$$

All conditions of Theorem 3 are satisfied. Therefore $\varphi_k : (R_k, \xi_k) \to (R_{k+1}, \xi_{k+1})$ is a semi-isometric isomorphism for $k = 0, 1, \cdots, n-2$.

Let's consider $\varphi_{n-1} : (R_{n-1}, \xi_{n-1}) \to (R_n, \xi_n)$. Since $\xi_{n-1}(r) = \bar{\xi}(\varphi(r))$ for any $r \in R$ that the isomorphism $\varphi_{n-1} = \varphi : (R_{n-1}, \xi_{n-1}) = (R, \xi_{n-1}) \to (R_n, \xi_n) = (\bar{R}, \bar{\xi})$ is isometric.

Therefore there exist pseudonormed rings $(R, \xi) = (R_0, \xi_0), (R_1, \xi_1), \ldots, (R_n, \xi_n) = (\bar{R}, \bar{\xi})$ and semi-isometric isomorphisms $\varphi_i : (R_i, \xi_i) \to (R_{i+1}, \xi_{i+1})$ for $i = 0, 1, \ldots, n-1$ such that $\varphi = \varphi_n \circ \varphi_{n-1} \circ \ldots \circ \varphi_0$.

The implication **1** $\Rightarrow$ **2** is proved.

The implication **2** $\Rightarrow$ **3** follows from Theorem 4. The implication **3** $\Rightarrow$ **1** follows from Theorem 2.

The theorem is completely proved.

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S. A. Aleschenko
Transnistrian T. G. Shevchenko State University
str. 25 Octombrie, 128, MD-3300 Tiraspol
Moldova

E-mail: *alesch.svet@gmail.com*

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V. I. Arnautov
Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
str. Academiei, 5, MD-2028 Chişinău
Moldova

E-mail: *arnautov@math.md*

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Radicals and generalizations of derivations

E. P. Cojuhari, B. J. Gardner

Abstract. By results of Slin'ko and of Anderson, the locally nilpotent and nil radicals of algebras over a field of characteristic 0 are preserved by derivations. This note deals with radical preservation by various generalizations of derivations.

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1 Introduction

It was shown by Slin'ko [17] that if d is a derivation on an associative algebra A over a field of characteristic 0, then $d(\mathcal{L}(A)) \subseteq \mathcal{L}(A)$ and $d(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$, where $\mathcal{L}$ and $\mathcal{N}$ are, respectively, the locally nilpotent and nil radical classes. This generalized a similar result proved earlier by Anderson [3] for a restricted class of algebras. The behaviour of the Jacobson radical is quite different; e.g. if K is a field, the Jacobson radical of the ring $K[[X]]$ of formal power series is the principal ideal generated by X, and this is not invariant under formal differentiation.

A contrasting result for algebras over a field of prime characteristic was obtained by Krempa [13]: a hereditary radical class $\mathcal{R}$ is preserved by all derivations of all algebras if and only if $\mathcal{R}$ consists of (hereditarily) idempotent algebras.

In this note we shall examine several generalizations of derivations and their effects on certain radicals, mostly $\mathcal{L}$ and $\mathcal{N}$, and also their effects on idempotent ideals. Idempotent ideals are invariant under ordinary derivations, there are plenty of radical classes consisting of idempotent rings (including the class of *all* idempotent rings) and even the prime radical of a ring can be idempotent, so idempotent ideals are pertinent to our investigation.

Confining attention to algebras over fields (as in [3, 13] and [17]) avoids some complications, notably with ideal structure, but leaves some interesting questions unexamined. We shall prove a number of results about (additively) torsion-free rings A by using, or first proving, the results in the special case of an algebra over a field of characteristic 0 and extending them to the general case by means of the divisible hull $D(A)$ of A. It is possible to extend some results without using $D(A)$, though not all, but we use a uniform approach.

All our rings and algebras are associative, but similar questions could be pursued for non-associative structures of various kinds. Indeed Krempa's investigations in [13] were more broadly based, and among other things he established a strong connection between derivations and the ADS condition for Lie algebras.

Now for the types of mappings whose effects we shall study.

A *derivation* on a ring is an additive endomorphism d such that $d(ab) = d(a)b + ad(b)$ for all a, b.

A *higher derivation* is a sequence $(d_0, d_1, \dots, d_n, \dots)$ of additive endomorphisms such that for each n we have $d_n(ab) = \sum_{i+j=n} d_i(a)d_j(b)$ for all a, b (so that in particular, d_0 is a ring endomorphism).

For ring endomorphisms α, β, an (α, β)-*derivation* is an additive endomorphism d such that $d(ab) = d(a)\beta(b) + \alpha(a)d(b)$ for all a, b. (Thus for a higher derivation, as $d_1(ab) = d_1(a)d_0(b) + d_0(a)d_1(b)$ for all a, b, d_1 is a (d_0, d_0)-derivation).

Finally, a D-*structure* for a ring A with identity 1 and a monoid G with identity e is a family of mappings $\sigma_{x,y} : A \to A$, where $x, y \in G$, satisfying

Condition (A)

(0) For each $x \in G$ and $a \in R$, we have $\sigma_{x,y}(a) = 0$ for almost all $y \in G$.

(i) Each $\sigma_{x,y}$ is an additive endomorphism.

(ii) $\sigma_{x,y}(ab) = \sum_{z \in G} \sigma_{x,z}(a)\sigma_{z,y}(b)$.

(iii) $\sigma_{xy,z} = \sum_{uv=z} \sigma_{x,u} \circ \sigma_{y,v}$.

(iv_1) $\sigma_{x,y}(1) = 0$ if $x \neq y$; (iv_2) $\sigma_{x,x}(1) = 1$;

(iv_3) $\sigma_{e,x}(a) = 0$ if $x \neq e$; (iv_4) $\sigma_{e,e}(a) = a$.

For unexplained terms and ideas, see [9] for rings and radicals, [8] for abelian groups.

2 Known results

The first result is well known and elementary.

Proposition 1. *If I is an idempotent ideal of a ring R and d is a derivation on R then $d(I) \subseteq I$.*

The following two results were proved for algebras over fields of characteristic 0, but they can be extended to all rings that are additively torsion-free, as we shall see in the next section.

Theorem 1. *(Anderson [3]) Let A be an algebra over a field K of characteristic 0 with DCC on ideals. For every hereditary radical class $\mathcal{R}$ we have $d(\mathcal{R}(A)) \subseteq \mathcal{R}(A)$ for all K-linear derivations d on A.*

Theorem 2. *(Slin'ko [17]) Let $\mathcal{L}(A)$, $\mathcal{N}(A)$ denote, respectively, the locally nilpotent and nil radicals of an algebra A over a field K of characteristic 0. Then $d(\mathcal{L}(A)) \subseteq \mathcal{L}(A)$ and $d(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$ for all K-linear derivations d on A.*

The situation with algebras over a field of positive characteristic is rather different.

Theorem 3. *(Krempa [13]) Let $\mathcal{V}$ be a variety of algebras over a field of prime characteristic p which is closed under tensoring by commutative-associative algebras. Let $\mathcal{R}$ be a hereditary radical class in $\mathcal{V}$. Then $d(\mathcal{R}(A)) \subseteq \mathcal{R}(A)$ for all derivations d of all algebras $A \in \mathcal{V}$ if and only if $\mathcal{R}$ consists of idempotent algebras.*

The varieties of associative and commutative-associate algebras satisfy the conditions of $\mathcal{V}$ in this theorem.

3 Some results involving additive structure

For an (additively written) abelian group G, a positive integer n and a prime p, let

$$nG = \{nx : x \in G\}; \quad G[n] = \{x \in G : nx = 0\}; \quad G_p = \bigcup_{n \in \mathbb{Z}^+} G[p^n].$$

All of the indicated subsets are subgroups, and if G is the additive group of a ring they are all ideals. Moreover, if G is a torsion group then $G = \bigoplus_p G_p$ (where the sum is taken over all primes p) and if G is the additive group of a torsion ring this is also a ring direct sum. In general $\bigoplus_p G_p$ is the *torsion subgroup* of G, which we shall call $T(G)$. When G is the additive group of a ring, $T(G)$ is an ideal, which we shall call the *torsion ideal.* In what follows, when referring to additive aspects of rings, we shall not distinguish notationally between a ring and its additive group. Thus, for instance, if A is a ring then $A[n] = \{a \in A : na = 0\} \triangleleft A$.

Proposition 2. *Let A be a ring, $I = nA$, $A[n]$, A_p or $T(A)$. If d is a derivation on A, then $d(I) \subseteq I$ and we get a derivation $\overline{d}$ on A/I by defining $\overline{d}(a + I) = d(a) + I$ for all $a \in A$.*

Proof. Since d is an additive endomorphism we have $d(I) \subseteq I$ so $\overline{d}$ is well-defined. The rest is straightforward. □

Proposition 3. *If A is a torsion ring and d is a derivation on A, then for each prime p, the restriction of d defines a derivation d_p of A_p. Conversely, if e_p is a derivation on A_p for each p, then we get a derivation e on A by defining $e(\sum_p a_p) = \sum_p e_p(a_p)$, where a_p is the component of a in A_p for each p.*

Proof. The first part follows from Proposition 2. For the second part, if $a = \sum a_p, b = \sum b_p \in A$, then

$$e(ab) = e\left(\sum a_p b_p\right) = \sum e_p(a_p b_p) = \sum \left(e_p(a_p)b_p + a_p e_p(b_p)\right)$$

$$= \sum e_p(a_p) \sum b_p + \sum a_p \sum e_p(b_p) = e(a)b + ae(b),$$

and clearly $e(a + b) = e(a) + e(b)$. □

Corollary 1. *Let A be a torsion ring, $\mathcal{R}$ a radical class of rings. Then $\mathcal{R}(A)$ is preserved by all derivations on A if and only if for every p, $\mathcal{R}(A_p)$ is preserved by all derivations on A_p.*

Proof. First note that $\mathcal{R}(A) = \bigoplus_p \mathcal{R}(A_p)$. If $\mathcal{R}(A)$ is preserved by derivations and δ is a derivation on A_p, then δ extends to a derivation d on A, so $d(\mathcal{R}(A)) \subseteq \mathcal{R}(A)$. Also $d(A_p) \subseteq A_p$. Hence

$$\delta(\mathcal{R}(A_p)) = \delta(A_p \cap \mathcal{R}(A)) = d(A_p \cap \mathcal{R}(A)) \subseteq A_p \cap \mathcal{R}(A) = \mathcal{R}(A_p).$$

If the action of $\mathcal{R}$ is preserved by derivations in all the A_p and e is any derivation on A, then

$$e(\mathcal{R}(A)) = e(\bigoplus_p \mathcal{R}(A_p)) = \bigoplus_p e_p(\mathcal{R}(A_p)) \subseteq \bigoplus_p \mathcal{R}(A_p) = \mathcal{R}(A).$$

□

Thus the radical-preservation problem for torsion rings reduces to that for p-rings. A p-ring R satisfying the stronger condition $pR = 0$ is an algebra over the field $\mathbb{Z}_p$ and all its ring ideals are $\mathbb{Z}_p$-algebra ideals. It is not known whether the preservation property for $\mathbb{Z}_p$-algebras (for some or all radicals) has much influence on that for p-rings generally. We shall prove one theorem related to this question.

Proposition 4. *For every p-ring A we have $pA \subseteq \mathcal{L}(A) \subseteq \mathcal{N}(A)$, whence $\mathcal{L}(A/pA) = \mathcal{L}(A)/pA$ and $\mathcal{N}(A/pA) = \mathcal{N}(A)/pA$*

Proof. We only have to show that pA is locally nilpotent. For this it suffices to prove that if S is a finite sub*set* of pA then there is a positive integer m such that all products of elements of S with m or more factors are zero. (This is straightforward but tedious to prove by brute force; it is also contained in Theorem 4.1.5, p.186 of [9].) If $a, b \in A$, then $(pa)b = \underbrace{(a + a + \cdots + a)}_{p \text{ terms}} b = \underbrace{ab + ab + \cdots + ab}_{p \text{ terms}} = p(ab)$ and similarly $a(pb) = p(ab)$. Hence $pa \cdot pb = p(a \cdot pb) = p(p(ab)) = p^2ab$ and so on. If $a_1, a_2, \ldots, a_n \in A$, then for $y_1, y_2, \ldots, y_m \in \{a_1, a_2, \ldots, a_n\}$ we have $py_1 \cdot py_2 \cdot \ldots \cdot py_m = p^m y_1 y_2 \ldots y_m = 0$ if $p^m \geq \max\{o(a_1), o(a_2), \ldots, o(a_n)\}$, where $o(a_i)$ is the (additive) order of a_i for each i. □

In fact the same proof shows that if $\mathcal{R}$ is any radical class with $\mathcal{L} \subseteq \mathcal{R}$, then $\mathcal{R}(A/pA) = \mathcal{R}(A)/pA$. This gives us

Theorem 4. *Let d be a derivation on a p-ring A, $\overline{d}$ the induced derivation on A/pA. Let $\mathcal{R}$ be a radical class containing $\mathcal{L}$. If $d(\mathcal{R}(A) \subseteq \mathcal{R}(A)$, then $\overline{d}(\mathcal{R}(A/pA)) \subseteq \mathcal{R}(A/pa)$.*

Now let A be a torsion-free ring. Its *divisible hull* $D(A)$ is a minimal divisible group containing A. For each $a \in A$ and each non-zero integer n there is an element $\alpha \in D(A)$ such that $n\alpha = a$, and as $D(A)$ is torsion-free, α is unique. It is therefore natural to give α the name $\frac{a}{n}$. Then $\frac{a}{n} = \frac{b}{m}$ if and only if $ma = nb$. In $D(A)$ we similarly define elements $\frac{x}{k}$ for $x \in D(A)$ and non-zero $k \in \mathbb{Z}$. We get a ring on $D(A)$

by defining $\frac{a}{n}\frac{c}{k} = \frac{ac}{nk}$ and this ring has a subring $\left\{\frac{a}{1} : a \in A\right\}$ which we identify with A. We make $D(A)$ into an algebra over the field $\mathbb{Q}$ by defining $\frac{m}{n}x = \frac{mx}{n}$ for $m, n, k \in \mathbb{Z}$, $x \in D(A)$. In particular, $\frac{m}{n}\frac{a}{k} = \frac{ma}{nk}$ for $a \in A$. For all this cf. Theorem 119.1, p.284 of [8], Vol. II.

Proposition 5. *Let A be a torsion-free ring. Then $\mathcal{L}(D(A)) = D(\mathcal{L}(A))$ and $\mathcal{N}(D(A)) = D(\mathcal{N}(A))$.*

Proof. We shall prove the result for $\mathcal{L}$. The proof for $\mathcal{N}$ is similar but simpler.

Let $I = \mathcal{L}(A)$. For $n \in \mathbb{Z}^+$ let $I_n = \{a \in A : na \in I\}$. Then $I_n \triangleleft A$. If $a_1, a_2, \dots, a_k \in I_n$ then $na_1, na_2, \dots, na_k$ are in the locally nilpotent ideal I, so there is a positive integer ℓ such that every ℓ-fold product of na_is is zero. Such a product has the form $n^\ell c_1 c_2 \dots c_\ell$, so since A is torsion-free, $c_1 c_2 \dots c_\ell = 0$. But the c_j are arbitrary elements of $\{a_1, a_2, \dots, a_k\}$, so by Theorem 4.1.5 of [9] referred to above, I_n is locally nilpotent, whence $I_n \subseteq I$ and thus $I_n = I$. This being so for every n, I, as an additive subgroup, is *pure* in A. If $a \in A$, $c \in I$, m, n are non-zero integers and $\frac{a}{n} = \frac{c}{m}$, then $ma = nc \in I$, so $a \in I$. Thus without ambiguity we can identify $D(I)$ with the obvious subring of $D(A)$. It is easily seen that $D(I) \triangleleft D(A)$.

If $\frac{c_1}{k_1}, \frac{c_2}{k_2}, \dots, \frac{c_t}{k_t} \in D(I)$ $(c_j \in I, k_j \in \mathbb{Z})$, then long enough products of c_js are zero. But such products are multiples, by non-zero integers, of products of $\frac{c_j}{k_j}$s of the same length. It follows that $D(I)$ is locally nilpotent and thus $D(I) \subseteq \mathcal{L}(D(A))$.

Let $J/D(I)$ be a locally nilpotent ideal of $D(A)/D(I)$. Then J is a locally nilpotent ideal of $D(A)$, so $J \cap A$ is a locally nilpotent ideal of A and hence $J \cap A \subseteq I$. If $\frac{g}{s} \in J$, where $g \in A$, $s \in \mathbb{Z}$, then $g = s\frac{g}{s} \in J \cap A \subseteq I$, so $\frac{g}{s} \in D(I)$ and so $J/D(I) = 0$. Thus $\mathcal{L}(D(A))/D(I) = 0$. It follows that $\mathcal{L}(D(A)) \subseteq D(I)$, so the two ideals are equal, i.e. $\mathcal{L}(D(A)) = D(\mathcal{L}(A))$. □

It follows that $\mathcal{L}(A) = A \cap \mathcal{L}(D(A))$ and $\mathcal{N}(A) = A \cap \mathcal{N}(D(A))$.

Note that the corresponding result for the Jacobson radical is false. For instance, if $A = \left\{\frac{2n}{2m+1} : n, m \in \mathbb{Z}\right\}$, then $\mathbb{Q}$ is a divisible hull for A, A is its own Jacobson radical and $\mathbb{Q}$ has zero radical.

Lemma 1. *If G is a torsion-free abelian group, each of its endomorphisms has a unique extension to an endomorphism of $D(G)$ and this is a $\mathbb{Q}$-linear transformation of $D(A)$ as a $\mathbb{Q}$-vector space.*

Proof. For an endomorphism f of G define $\hat{f} : D(G) \to D(G)$ by setting $\hat{f}\left(\frac{a}{n}\right) = \frac{f(a)}{n}$ for all $a \in G$, $n \in \mathbb{Z}\backslash\{0\}$. Then $\hat{f}$ is well-defined, as if $\frac{a}{n} = \frac{b}{m}$, then $mf(a) = f(ma) = f(nb) = nf(b)$, i.e. $\frac{f(a)}{n} = \frac{f(b)}{m}$. Then for all $a,\ c \in G$, $n,\ k \in \mathbb{Z}\backslash\{0\}$

we have $\hat{f}\left(\frac{a}{n}+\frac{c}{k}\right) = \hat{f}\left(\frac{ka+nc}{nk}\right) = \frac{f(ka+nc)}{nk} = \frac{kf(a)+nf(c)}{nk} = \frac{kf(a)}{nk} + \frac{nf(c)}{nk} = \frac{f(a)}{n} + \frac{f(c)}{k} = \hat{f}\left(\frac{a}{n}\right) + \hat{f}\left(\frac{c}{k}\right)$. Also $\hat{f}\left(\frac{m}{n}\frac{a}{k}\right) = \hat{f}\left(\frac{ma}{nk}\right) = \frac{f(ma)}{nk} = \frac{mf(a)}{nk} = \frac{m}{n}\frac{f(a)}{k} = \frac{m}{n}\hat{f}\left(\frac{a}{k}\right)$ for $a \in A, m, n, k \in \mathbb{Z}$. If $\tilde{f}$ is any extension of f, then $G \subseteq Ker(\hat{f} - \tilde{f})$, so $Im(\hat{f} - \tilde{f})$ is a torsion group and hence zero. □

Corollary 2. *Let A be a torsion-free ring.*
(i) Every derivation d on A has a unique extension to $D(A)$ and this is $\mathbb{Q}$-linear.
(ii) Every higher derivation on A has a unique extension to $D(A)$ and all its maps are $\mathbb{Q}$-linear.
(ii) If α and β are endomorphisms of A, then every (α, β)-derivation on A has a unique extension to an $(\hat{\alpha}, \hat{\beta})$-derivation on $D(A)$ and this is $\mathbb{Q}$-linear.

Proof. All the maps involved in (i), (ii) and (iii) are additive endomorphisms of A, and so have unique extensions to additive endomorphisms of $D(A)$. We just need to show that these endomorphisms have all other properties required of them.

(ii) Let $(d_0, d_1, \ldots, d_n \ldots)$ be a higher derivation on A. For each n let $\hat{d_n}$ be the extension of d_n to $D(A)$ as in the lemma. For each $a, b \in A$ and non-zero $k, \ell \in \mathbb{Z}$, we have $\hat{d_n}\left(\frac{a}{k}\frac{b}{\ell}\right) = \hat{d_n}\left(\frac{ab}{k\ell}\right) = \frac{d_n(ab)}{k\ell} = \frac{\sum_{i+j=n} d_i(a)d_j(b)}{k\ell} = \sum_{i+j=n} \frac{d_i(a)}{k}\frac{d_j(b)}{\ell} = \sum_{i+j=n} \hat{d_i}\left(\frac{a}{k}\right)\hat{d_j}\left(\frac{b}{\ell}\right)$.

Similar arguments show that extensions of ring endomorphisms and extensions of derivations are derivations.

(iii) Let d be an (α, β)-derivation on A. Then for all $a, b \in A$ and non-zero $k, \ell \in \mathbb{Z}$, we have

$$\hat{d}\left(\frac{a}{k}\right)\hat{\beta}\left(\frac{b}{\ell}\right) + \hat{\alpha}\left(\frac{a}{k}\right)\hat{d}\left(\frac{b}{\ell}\right) = \frac{d(a)}{k}\frac{\beta(b)}{\ell} + \frac{\alpha(a)}{k}\frac{d(b)}{\ell} = \frac{d(a)\beta(b) + \alpha(a)d(b)}{k\ell}$$
$$= \frac{d(ab)}{k\ell} = \hat{d}\left(\frac{a}{k}\frac{b}{\ell}\right).$$

□

Note that not every derivation on $D(A)$ is an extension of one on A: consider inner derivations, for example.

Now if A is a torsion-free ring, d a derivation on A, then by Corollary 2 d extends to a $\mathbb{Q}$-linear derivation $\hat{d}$ on $D(A)$, so

$$d(\mathcal{L}(A)) = d(A \cap \mathcal{L}(D(A))) = \hat{d}(A \cap \mathcal{L}(D(A))) \subseteq \hat{d}(\mathcal{L}(D(A))) \subseteq \mathcal{L}(D(A))$$

and $d(\mathcal{L}(A)) \subseteq A$, so

$$d(\mathcal{L}(A)) \subseteq A \cap \mathcal{L}(D(A)) = \mathcal{L}(A).$$

We can argue similarly for $\mathcal{N}(A)$. Thus we have

Theorem 5. *If d is a derivation on a torsion-free ring A then $d(\mathcal{L}(A)) \subseteq \mathcal{L}(A)$ and $d(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$.*

4 Preservation by higher derivations

Proposition 6. *Let $(d_0, d_1, \dots, d_n, \dots)$ be a higher derivation on a ring A, I an idempotent ideal of A with $d_0(I) \subseteq I$. Then $d_n(I) \subseteq I$ for all n.*

Proof. If $d_n(I) \subseteq I$ then for all $a, b \in I$ we have

$$d_{n+1}(ab) = d_0(a)d_{n+1}(b) + d_1(a)d_n(b) + d_2(a)d_{n-1}(b) + \cdots + d_{n-1}(a)d_2(b) + d_n(a)d_1(b) + d_{n+1}(a)d_0(b) \in I$$

if $d_0(I), d_1(I), \dots, d_n(I) \subset I$. □

Theorem 6. *Let A be a torsion-free ring, $(d_0, d_1, \dots, d_n, \dots)$ a higher derivation on A. If d_0 is an automorphism, then $d_n(\mathcal{L}(A)) \subseteq \mathcal{L}(A)$ and $d_n(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$ for all n.*

Proof. We first treat the case where A is an algebra over a field of characteristic 0. Note that $\mathcal{L}(A)$ and $\mathcal{N}(A)$ (where A is treated as a ring) are algebra ideals (as happens with all radicals) and so coincide with these radicals of A treated as an algebra (see [7]).

It has been proved by many authors e.g. Heerema [11], Miller [15], Abu-Saymeh [1],[2], Mirzavaziri [16], Hazewinkel [10]) that in the circumstances of the theorem, if $d_0 = id$ then each d_n $(n \geq 1)$ is a linear combination of compositions of derivations, whence the result follows from Theorem 2. In general we have

$$\begin{aligned} d_0^{-1} \circ d_n(ab) =& d_0^{-1}(d_0(a)d_n(b) + d_1(a)d_{n-1}(b) + \cdots + d_{n-1}(a)d_1(b) + \\ d_n(a)d_0(b)) =& d_0^{-1} \circ d_0(a)d_0^{-1} \circ d_n(b) + d_0^{-1} \circ d_1(a)d_0^{-1} \circ d_{n-1}(b) + \cdots + \\ & d_0^{-1} \circ d_{n-1}(a)d_0^{-1} \circ d_1(b) + d_0^{-1} \circ d_n(a)d_0^{-1} \circ d_0(b) \end{aligned}$$

for all $n \geq 1$, so $(d_0^{-1} \circ d_0, d_0^{-1} \circ d_1, \dots, d_0^{-1} \circ d_n, \dots)$ is a higher derivation with the identity as its zeroth term, whence $d_0^{-1} \circ d_n(\mathcal{L}(A)) \subseteq \mathcal{L}(A)$ for all n. But $\mathcal{L}(A)$ is invariant under automorphisms, so

$$d_n(\mathcal{L}(A)) = d_0 \circ d_0^{-1} \circ d_n(\mathcal{L}(A)) \subseteq d_0(\mathcal{L}(A)) = \mathcal{L}(A).$$

The argument for $\mathcal{N}(A)$ is the same.

Now turning to a general torsion-free ring A, by Corollary 2 (ii) we can extend our higher derivation uniquely to a higher derivation $(\hat{d_0}, \hat{d_1}, \dots, \hat{d_n}, \dots)$ of $D(A)$, which is an algebra over the field $\mathbb{Q}$ of rational numbers. It is easy to see that if d_0 is an automorphism of A, then $\hat{d_0}$ is an automorphism of $D(A)$. Hence by Proposition 5 and the first part of the proof we have

$$\hat{d_n}(\mathcal{L}(D(A)) = \hat{d_n}(D(\mathcal{L}(A))) \subseteq D(\mathcal{L}(A)) \quad \text{for every } n.$$

Thus if $a \in \mathcal{L}(A)$, then

$$d_n(a) = \hat{d_n}\left(\frac{a}{1}\right) \in D(\mathcal{L}(A)) \cap A = \mathcal{L}(A)$$

for each n.

Again, the argument for $\mathcal{N}$ is the same. □

A natural question is whether for a higher derivation $(d_0, d_1, \ldots, d_n, \ldots)$, in particular on a torsion-free ring, if d_0 preserves one of our radicals the latter must be preserved by every d_n. We have an example of similar behaviour in a ring with prime characteristic p; the radical involved is not $\mathcal{L}$ or $\mathcal{N}$, but it is a hereditary supernilpotent radical.

Example 1. (Cf. Krempa [12]) Let $\mathcal{U}$ be the upper radical class defined by the field K_p with p elements. We get a higher derivation $(d_0, d_1, \ldots, d_n, \ldots)$ on $K_p[X]$ by defining $d_i(a_0 + a_1X + \cdots + a_kX^k) = a_iX^i$ for all i. Now $\mathcal{U}$ is special, so if $\alpha \in \mathcal{U}(K_p[X])$ then α is taken to 0 by each homomorphism from $K_p[X]$ to K_p. In particular $d_0(\alpha) = 0$ (as the function which assigns the zeroth coefficient is a homomorphism). Thus $d_0(\mathcal{U}(K_p[X])) \subseteq \mathcal{U}(K_p[X])$. But $X - X^p \in \mathcal{U}(K_p[X])$ and $d_1(X - X^p) = X$. If X were in $\mathcal{U}(K_p[X])$ then the principal ideal (X) would be in $\mathcal{U}$. But K_p is a homomorphic image of (X) *via* $X \mapsto 1$. Thus $X \notin \mathcal{U}(K_p[X])$ so $d_1(\mathcal{U}(K_p[X])) \nsubseteq \mathcal{U}(K_p[X])$.

For *commutative* rings we have a preservation result which does not depend on additive properties.

Theorem 7. *Let A be a commutative ring, $(d_0, d_1, \ldots, d_n, \ldots)$ a higher derivation on A. Then $d_n(\mathcal{L}(A)) \subseteq \mathcal{L}(A)$ and $d_n(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$ for all n.*

Proof. Since A is commutative, $\mathcal{L}(A) = \mathcal{N}(A) =$ the set of nilpotent elements of A. The correspondence $a \mapsto \sum\limits_{n=0}^{\infty} d_n(a)X^n$ defines a homomorphism $f : A \to A[[X]]$ (the formal power series ring). If a is nilpotent then so is $f(a)$ and then, by commutativity, so are its coefficients. (This is presumably well known. Here is an outline of a proof. If $(\sum\limits_{n=0}^{\infty} a_nX^n)^m = 0$, then $a_0^m = 0$. By commutativity, $\sum\limits_{n=1}^{\infty} a_nX^n = \sum\limits_{n=0}^{\infty} a_nX^n - a$ is also nilpotent, whence a_1 is nilpotent, and so on.) Thus each $d_n(a)$ is nilpotent and therefore in $\mathcal{L}(A)$. □

Presumably this result does not hold in the absence of any restriction on A, though we do not have an example to show this. The following example shows that higher derivations do not necessarily take nilpotent elements to nilpotent elements.

Example 2. *We use an example of [4]. Let R be a ring with identity, $A = M_2(R)[X]$. We get a higher derivation on $A[X]$ by defining $d_n(c_0 + c_1X + \ldots) = c_nX^n$ for all n. Then $(e_{12} + (e_{11} - e_{22})X - e_{21}X^2)^2 = 0$, but $d_1(e_{12} + (e_{11} - e_{22})X - e_{21}X^2) = e_{11} - e_{22}$, which is a unit.*

Not much seems to be known about representing the terms of a general higher derivation by combinations of some kind of derivations. Loy [14] remarks that if $(d_0, d_1, \ldots, d_n, \ldots)$ is a higher derivation, d_0 is idempotent and $d_0 \circ d_n = d_n \circ d_0$ for all n, then the d_n are expressible as linear combinations of compositions of $(d_0, d_0)-$derivations δ with $d_0 \circ \delta = \delta \circ d_0$.

Note that there are related results expressing the maps of certain D-structures in terms of endomorphisms and derivations of various kinds in Section 6 of [5] and Section 3 of [6].

5 Preservation by $(\alpha, \beta)-$derivations

It might be expected that ideals preserved by α and β and by derivations might be preserved by $(\alpha, \beta)-$ derivations. The situation is more complicated, however. The case of idempotent ideal is easy.

Proposition 7. *If I is an idempotent ideal of a ring A, d an $(\alpha, \beta)-$derivation on A, where $\alpha(I) \subseteq I$ and $\beta(I) \subseteq I$, then $d(I) \subseteq I$.*

Proof. For $a, b \in I$ we have $d(ab) = d(a)\beta(b) + \alpha(a)d(b) \in I$ as $\beta(b), \alpha(a) \in I$. □

Theorem 8. *If α is an automorphism of a torsion-free ring A then $d(\mathcal{L}(A)) \subseteq \mathcal{L}(A)$ and $d(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$ for all $(\alpha, \alpha)-$derivations d of A.*

Proof. The proof uses Corollary 2 and is like part of that of Theorem 6: $\alpha^{-1} \circ d$ is an ordinary derivation, so $\alpha^{-1} \circ d(\mathcal{L}(A)) \subseteq \mathcal{L}(A)$. Hence $d(\mathcal{L}(A)) \subseteq \alpha(\mathcal{L}(A)) \subseteq \mathcal{L}(A)$. The same argument gives the result for the nil radical. □

We do not know if there is an analogous theorem for (α, β)-derivations when α and β are *distinct* automorphisms. We do however have counterexamples when α and β are non-automorphisms, distinct or not.

Example 3. *Let K be a field (any characteristic),*

$$A = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} : a, b \in K \right\}$$

and define $f, \delta : A \to A$ by setting $f\left(\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, $\delta\left(\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\right) = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}$ for all $a, b \in K$. Then f is an endomorphism and δ is an (f, f)-derivation. We have $\mathcal{L}(A) = \mathcal{N}(A) = \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}$ and the radicals are preserved by f but not by δ.

Example 4. *For a field K we consider the ring $\begin{bmatrix} K & K \\ 0 & K \end{bmatrix}$ of upper triangular 2×2 matrices. Let $\alpha\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, $\beta\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$*

and $d\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} 0 & b \\ 0 & b \end{bmatrix}$ *for all* $a, b, c \in K$. *Clearly* α *and* β *are endomorphisms. For all* a, b, c, d, e *and* $f \in K$ *we have* $d\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right)\beta\left(\begin{bmatrix} d & e \\ 0 & f \end{bmatrix}\right) + \alpha\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right)d\left(\begin{bmatrix} d & e \\ 0 & f \end{bmatrix}\right) = \begin{bmatrix} 0 & b \\ 0 & b \end{bmatrix}\begin{bmatrix} f & 0 \\ 0 & f \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} 0 & e \\ 0 & e \end{bmatrix} = \begin{bmatrix} 0 & bf \\ 0 & bf \end{bmatrix} + \begin{bmatrix} 0 & ae \\ 0 & ae \end{bmatrix} = \begin{bmatrix} 0 & bf+ae \\ 0 & bf+ae \end{bmatrix} = d\left(\begin{bmatrix} ad & ae+bf \\ 0 & cf \end{bmatrix}\right) = d\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} d & e \\ 0 & f \end{bmatrix}\right)$, *so* d *is an* $(\alpha, \beta)-$*derivation. Now* $\mathcal{L}\left(\begin{bmatrix} K & K \\ 0 & K \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} K & K \\ 0 & K \end{bmatrix}\right) = \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}$ *and* $\alpha\left(\begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}\right) = \beta\left(\begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}\right) = 0$ *so both radicals are preserved by* α *and* β. *However, if* $b \neq 0$ *then* $d\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & b \\ 0 & b \end{bmatrix} \notin \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}$, *so the radicals are not preserved by* d.

6 Preservation by D-structures

Preservation by all mappings of an arbitrary D-structure is a very demanding condition. We shall see that even for algebras over a field of characteristic 0, the locally nilpotent and nil radicals need not be preserved. We begin the section however with a positive result.

Theorem 9. *Let* σ *be a D-structure defined by a ring* A *and a free monoid* $G = \{e, x, x^2, \ldots, x^n, \ldots\}$ *and write* σ_{nm} *for* σ_{x^n,x^m}. *Suppose further that* $\sigma_{nm} = 0$ *for* $n < m$. *If* I *is an idempotent ideal of* A *and* $\sigma_{11}(I) \subseteq I$ *then* $\sigma_{ij}(I) \subseteq I$ *for all* i, j.

Proof. The conditions imposed imply that σ_{11} is an endomorphism and $\sigma_{nn} = \sigma_{11}^n$ for all n (see [5], Proposition 3.1 and (6.9)). Clearly we need only consider σ_{ij} for $i \geq j$, and prove that $\sigma_{ij}(ab) \in I$ for all $a, b \in I$. It is given that $\sigma_{11}(I) \subseteq I$. Now for all $a, b \in I$ we have $\sigma_{10}(ab) = \sigma_{11}(a)\sigma_{10}(b) + \sigma_{10}(a)\sigma_{00}(b) \in I$, since $\sigma_{11}(I) \subseteq I$. Thus $\sigma_{1j}(I) \subseteq I$ for all $j \leq 1$. Now we proceed by induction.

Suppose $\sigma_{ij}(I) \subseteq I$ for all $j \leq i$ when $i < n$. Then $\sigma_{nn}(I) \subseteq I$ as $\sigma_{nn} = \sigma_{11}^n$. If $j < n$ then

$$\sigma_{nj}(ab) = \sum_{n \geq k \geq j} \sigma_{nk}(a)\sigma_{kj}(b) = \sigma_{nn}(a)\sigma_{nj}(b) + \sigma_{nj}(a)\sigma_{jj}(b) + \sum_{n > k > j} \sigma_{nk}(a)\sigma_{kj}(b).$$

But $\sigma_{nn}(a)$ and $\sigma_{jj}(b) \in I$ and for $k < n$ we have $\sigma_{kj}(b) \in I$ by the inductive hypothesis. Hence $\sigma_{nj}(I) \subseteq I$ for all $j \leq n$. We have proved that for every i and for all $j \leq i$, we have $\sigma_{ij}(I) \subseteq I$, and this is what we need. □

It is not known how the mappings of a D-structure treat idempotent ideals in general.

Even in the presence of DCC for ideals, the mappings of a D-structure need not preserve the locally nilpotent or the nil radical of an algebra over a field of characteristic 0.

Example 5. *The ring* $\begin{bmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix}$ *is a* $\mathbb{Q}-$*algebra and has DCC on ideals. Also* $\mathcal{L}\left(\begin{bmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix}\right) = \begin{bmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{bmatrix}$. *For the cyclic group* $G = \{e, x\}$ *of order* 2 *we get a D-structure as follows:* $\sigma_{x,x}\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, $\sigma_{x,e}\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} 0 & c-a \\ 0 & b \end{bmatrix}$ *for all* $a, b, c \in \mathbb{Q}$, $\sigma_{e,e} = id, \sigma_{x,e} = 0$. *Then* $\sigma_{x,x}$ *preserves the radicals, but* $\sigma_{x,e}$ *does not.*

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E. P. Cojuhari
Institute of Mathematics and Computer Science
5 Academiei St, MD 2028 Chişinău
Moldova

Department of Mathematics
Technical University of Moldova
168 Ştefan cel Mare Blvd, MD 2004 Chişinău
Moldova
E-mail: *elena.cojuhari@mate.utm.md*

B. J. Gardner
Discipline of Mathematics
University of Tasmania PB 37
Hobart TAS 7001, Australia
E-mail: *barry.gardner@utas.edu.au*

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Some examples of topological modules

Mihail Ursul, Adela Tripe

Abstract. In the paper examples of modules which do not admit topologies of different types are constructed.

Mathematics subject classification: 16W80.
Keywords and phrases: Discrete topology; Bohr topology; antidiscrete topology; topological module; topological ring; elementary p-group .

1 Introduction

In the monograph [2] (Chapter 5) the problem of topologization of rings and modules is discussed. The aim of this paper is to construct examples of modules which do not admit some types of topologies.

2 Notation and conventions

An elementary p-group A, where p is a prime number is an abelian group with identity $px = 0$. By [6], Theorem 17.2 (Prüfer, Baer) A is a direct sum of cyclic groups of order p. Rings are assumed to be associative with identity and modules unitary. Topological rings are assumed to be Hausdorff, but topological modules are not assumed to be Hausdorff.

Let R be a ring and M an (R, R)-bimodule. The product $R \times M$ is endowed with the multiplication

$$(r, m)(r', m') = (rr', rm' + mr').$$

If R is a topological ring and M a topological (R, R)-bimodule, then $R \times M$ endowed with the product topology becomes a topological ring. It is called the trivial extension of R by M and is denoted by $R \ltimes M$ (see [8]).

3 Preliminaries

The problem of topologization of a module is stated as follows: Let ${}_RM$ be a left R-module and $\mathcal{T}$ be a ring topology on R. Does there exist a group topology $\mathcal{U}$ such that ${}_{(R,\mathcal{T})}(M, \mathcal{U})$ is a topological module? This problem has a satisfactory solution in the case when $\mathcal{T}$ is the discrete topology.

It can be considered another problem: Let ${}_RM$ be a left R-module. Let $\mathcal{U}$ be a group topology on M. Does there exist a ring topology $\mathcal{T}$ such that ${}_{(R,\mathcal{T})}(M,\mathcal{U})$ is a topological module?

Recall that a left R-module M, where R is a topological ring and M a topological group is called a topological module if the mapping

$$R \times M \to M, (r, m) \mapsto rm$$

is continuous.

As a corollary we obtain that if $(R, \mathcal{T}_d)$ is a ring with discrete topology $\mathcal{T}_d$ and $\mathcal{U}$ is a group topoloogy on M, then ${}_{(R,\mathcal{T}_d)}(M,\mathcal{U})$ is a topological module if and only if the mapping $M \to M, m \mapsto rm$ is continuous for every $r \in R$.

It follows from these statements Theorem 5.1.2, [2]: Every infinite module ${}_RM$ admits a nondiscrete Hausdorff R-module topology if R is viewed as a topological ring with the discrete topology.

A short proof: Consider on M the maximal totally bounded group topology. It is well-known that every endomorphism of M is continuous [4], [5].

4 Examples

Example 1. A topological ring and an overring such that the topology of ring cannot be extended to the overring.

Let R be a second countable connected Boolean topological ring with identity. (The existence of such topological rings has been proved in [3]). Let M be a maximal ideal of R. Then M is a dense subspace of R. Indeed, otherwise M will be open and R/M will be a discrete connected space of cardinality 2, a contradiction.

Consider the simple R-module $N = R/M$ and the trivial extension $R \ltimes N$. Then $(R, 0)$ is a subring of index 2 of $R \ltimes N$ and we can identify it with R. We claim that the topology of $(R, 0)$ cannot be extended to a Hausdorff topology of $R \ltimes N$. Indeed, otherwise $(0, N) = (R, 0)(0, 1 + M)$ will be a nonzero connected discrete topological group, a contradiction.

Remark 1. An example of a ring having a subring whose topology cannot be extended has been constructed in [7].

Lemma 1 (folklore)**.** *If A is a dense subgroup of a connected abelian group G, then A is generated by each of its neighborhoods V of zero.*

Example 2. A countable topological ring R and a countable R-module ${}_RM$ such that every module topology is the antidiscrete topology.

Let S be a connected second countable Boolean topological ring with identity and R a dense countable subring containing identity. By Lemma 1 the additive group of R is generated by each of its neighborhoods of zero. Let M be a maximal ideal of R and $N = R/M$. Then the unique module topology on N will be antidiscrete.

Indeed, if $\mathcal{T}$ is a module topology on N and L the intersection of all neighborhoods of zero, then L is a submodule. If $L = 0$, then $(N, \mathcal{T})$ is Hausdorff, hence N is a nonzero discrete group generated by each its neighborhood of zero, a contradiction. Therefore, $L = N$, hence $\mathcal{T}$ is the antidiscrete topology.

Now let $L = \oplus_{i\in\mathbb{N}} N_i$, where $N_i = N (i \in \mathbb{N})$. We claim that every module topology on L is the antidiscrete topology.

Indeed, assume that $\mathcal{T}$ is a module topology and let P be the intersection of all neighborhoods of zero of $(L, \mathcal{T})$. Then $P \supseteq N_i$ for every $i \in \mathbb{N}$. Since P is a submodule, $P = L$, hence $\mathcal{T}$ is the antidiscrete topology.

Another example of this kind has been constructed in [1].

Next example is related to the example 3.4 from [1].

Example 3. Let p be a prime number, A a countable elementary p-group, $\mathcal{T}_d$ be the discrete topology on $\mathrm{End}\, A$, and $\mathcal{T}_{Bohr}$ the Bohr topology on A , i.e., the finest totally bounded group topology on A (see [5]). We notice that $(A, \mathcal{T}_{Bohr})$ has a fundamental system of neighborhoods of zero consisting of all subgroups of finite index.
Then:

(i) ${}_{\mathrm{End}\, A}A$ is a simple module.

(ii) Every $(\mathrm{End}\, A, \mathcal{T}_d)$-module topology on ${}_{\mathrm{End}\, A}A$ is Hausdorff or discrete.

(iii) Every endomorphism α of $(A, \mathcal{T}_{Bohr})$ is contintuous.

(iv) ${}_{(\mathrm{End}\, A, \mathcal{T}_d)}(A, \mathcal{T}_{Bohr})$ is a topological module.

(v) $\mathcal{T}_{Bohr} \leq \mathcal{T}$ for each Hausdorff $(\mathrm{End}\, A, \mathcal{T}_d)$-module topology $\mathcal{T}$ on A.

(vi) Every nondiscrete Hausdorff topological module ${}_{(\mathrm{End}\, A, \mathcal{T}_d)}(A, \mathcal{T})$ has no nontrivial convergent sequence.

(vii) Every compact subspace of ${}_{(\mathrm{End}\, A, \mathcal{T}_d)}(A, \mathcal{T}_{Bohr})$ is finite.

(viii) The topology $\mathcal{T}_{Bohr}$ is maximal in the set of all nondiscrete Hausdorff $(\mathrm{End}\, A, \mathcal{T}_d)$-module topologies on A.

Proofs:

(i) Indeed, let $0 \neq a \in A$ and $b \in A$. There exists $\alpha \in \mathrm{End}\, A$ such that $\alpha a = b$. Therefore ${}_{\mathrm{End}\, A}A$ is a simple module.

(ii) Follows from (i).

(iii) This property was proved in [4], p. 39 for arbitrary abelian groups. We recall here the proof: If H is a subgroup of finite index of A, then $\alpha^{-1}(H)$ is a subgroup of finite index of A.

(iv) Follows from (iii).

(v) Indeed, let H be a subgroup of A of finite index. Let $H \oplus H' = A$. Put $\alpha \in \operatorname{End} A$, $\alpha \restriction_H = 0, \alpha \restriction_{H'} = 1_{H'}$. Then α is a continuous endomorphism of $(A, \mathcal{T})$. It follows that $H = \ker \alpha$ is closed in $(A, \mathcal{T})$. Since H has a finite index, it is open in $(A, \mathcal{T})$. We have proved that $\mathcal{T}_{Bohr} \leq \mathcal{T}$.

(vi) Assume the contrary. Let $\{a_n\}_{n \in \mathbb{N}}$ be a convergent sequence and let $\lim\limits_{n \to \infty} a_n = a$. Then $\lim\limits_{n \to \infty} (a_n - a) = 0$. Therefore we can assume without loss of generality that $a = 0$.

Since $\{a_n\}_{n \in \mathbb{N}}$ is a nontrivial sequence there exists $k_1 \in \mathbb{N}$ such that $a_{k_1} \neq 0$. The group A has a structure of a vector $\mathbb{F}_p$-space. Assume that the vectors $a_{k_1}, \cdots, a_{k_{n-1}}$, where $k_1 < \cdots < k_{n-1}$, are linearly independent. Since the subgroup B generated by the elements $a_{k_1}, \cdots, a_{k_{n-1}}$ is finite, there exists $k_n \in \mathbb{N}$ such that $k_{n-1} < k_n$ and $a_{k_n} \notin B$. Since $\lim\limits_{n \to \infty} a_{k_n} = 0$, we can assume without loss of generality that $\{a_n\}_{n \in \mathbb{N}}$ is a linearly independent system. Let $0 \neq b \in A$ and let $\alpha \in \operatorname{End} A, \alpha a_n = b$ for every $n \in \mathbb{N}$. Since α is a continuous endomorphism of $A, 0 = \lim\limits_{n \to \infty} \alpha a_n = b$, a contradiction.

(vii) Assume on the contrary that ${}_{(\operatorname{End} A, \mathcal{T}_d)}(A, \mathcal{T}_{Bohr})$ contains an infinite compact subset K. Since K is countable, it contains nontrivial convergent sequence. A contradiction with (vi).

(viii) Assume the contrary: let $\mathcal{T}$ be a nondiscrete Hausdorff $(\operatorname{End} A, \mathcal{T}_d)$-module topology and $\mathcal{T} \geq \mathcal{T}_{Bohr}, \mathcal{T} \neq \mathcal{T}_{Bohr}$. Let H be a subgroup of A such that $H \in \mathcal{T}, H \notin \mathcal{T}_{Bohr}$. Let $H \oplus H' = A$. Then H and H' are infinite and countable. Let α be an isomorphism of H on H' and $\beta \in \operatorname{End} A, \beta(h \oplus h') = \alpha(h)(h, h' \in H)$. There exists a neighborhood U of zero of ${}_{(\operatorname{End} A, \mathcal{T}_d)}(A, \mathcal{T})$ such that $U \subseteq H$ and $\beta(U) = \alpha(U) \subseteq H$. Thus $\alpha(U) = 0$, a contradiction.

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Mihail Ursul
Department of Mathematics and Computer Science
University of Technology, Lae, Papua New Guinea
E-mail: *mihail.ursul@gmail.com*

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Adela Tripe
Department of Mathematics and Computer Science
University of Oradea, Oradea, Romania
E-mail: *adela.tripe@gmail.com*

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Unrefinable chains when taking the infimum in the lattice of ring topologies for a nilpotent ring

V. I. Arnautov, G. N. Ermakova

Abstract. A nilpotent ring $\widehat{R}$ and two ring topologies $\widehat{\tau}''$ and $\widehat{\tau}*$ on $\widehat{R}$ are constructed such that $\widehat{\tau}*$ is a coatom (i.e. between the discrete topology τ_d and $\widehat{\tau}*$ there no exists ring topologies) and such that between $\inf\{\widehat{\tau}'', \widehat{\tau}_d\}$ and $\inf\{\widehat{\tau}'', \widehat{\tau}*\}$ there exists an infinite chain of ring topologies in the lattice of all ring topologies of the ring $\widehat{R}$.

Mathematics subject classification: 22A05, 06B30, 22A30.
Keywords and phrases: Nilpotent ring, ring topology, lattice of ring topologies, unrefinable chains, coatoms, infimum of ring topologies.

1 Introduction

As is known, in any modular lattice, the lengths of any finite unrefinable chains with the same ends are equal and the lengths of finite unrefinable chains do not become greater if we take the infimum or the supremum in these lattices.

The lattice of all ring topologies for a nilpotent ring need not be modular [1]. However, as is shown in [2], in the lattice of all ring topologies on a nilpotent ring, the lengths of any finite unrefinable chains which have the same ends are equal.

Given the above, it was natural to expect that the lengths of any finite unrefinable chains do not become greater if for a nilpotent ring we take the infimum or the supremum in the lattice of all ring topologies. However, as shown in this article, it is not the case if we take the infimum.

An example of a nilpotent ring R and such ring topologies $\widehat{\tau}''$ and $\widehat{\tau}*$ that $\widehat{\tau}*$ is a coatom in the lattice of all ring topologies of the ring R (i.e. between the discrete topology τ_d and $\widehat{\tau}*$ there exist no ring topologies) is constructed, and an infinite chain of ring topologies, which are less than $\widehat{\tau}'' = \inf\{\tau'', \tau_d\}$ and more than $\inf\{\widehat{\tau}'', \widehat{\tau}*\}$, exists.

To present the further results we need the following known result (see [3], page 39 and page 51):

Theorem 1. *Let $\mathcal{B}$ be a collection of subsets of a ring R such that the following conditions are satisfied:*

1) $\{0\} = \bigcap\limits_{V \in \mathcal{B}} V$ *;*

2) for any $V_1, V_2 \in \mathcal{B}$ *there exists* $V_3 \in \mathcal{B}$ *such that* $V_3 \subseteq V_1 \bigcap V_2$*;*

3) for any $V_1 \in \mathcal{B}$ *there exists* $V_2 \in \mathcal{B}$ *such that* $V_2 + V_2 \subseteq V_1$*;*

4) for any $V_1 \in \mathcal{B}$ there exists $V_2 \in \mathcal{B}$ such that $-V_2 \subseteq V_1$;

5) for any $V_1 \in \mathcal{B}$ there exists $V_2 \in \mathcal{B}$ such that $V_2 \cdot V_2 \subseteq V_1$;

6) for any $V_1 \in \mathcal{B}$ and any element $r \in R$ there exists $V_2 \in \mathcal{B}$ such that $r \cdot V_2 \subseteq V_1$ and $V_2 \cdot r \subseteq V_1$.

Then there exists a unique ring topology τ on the ring R for which (R, τ) is a Hausdorff space and the collection $\mathcal{B}$ is a basis of neighborhoods of zero [1].

2 Basic results

To state basic results we need the following notations:

Notations 2.

2.1. $\mathbb{N}$ is the set of all natural numbers, $\mathbb{Z}$ is the set of all integers and $\mathbb{R}(+, \cdot)$ is the field of real numbers;

2.2. R is the set of all matrices of the dimension 3×3 over the field $\mathbb{R}$ of real numbers of the form $\begin{pmatrix} 0 & a_{1,2} & a_{1,3} \\ 0 & 0 & a_{2,3} \\ 0 & 0 & 0 \end{pmatrix}$

$$R' = \left\{ \begin{pmatrix} 0 & a_{1,2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Big| a_{1,2} \in \mathbb{R} \right\};$$

$$R'' = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{2,3} \\ 0 & 0 & 0 \end{pmatrix} \Big| a_{2,3} \in \mathbb{R} \right\};$$

$R(A) = \left\{ \begin{pmatrix} 0 & 0 & a_{1,3} \\ 0 & 0 & a_{2,3} \\ 0 & 0 & 0 \end{pmatrix} \Big| a_{1,3} \in A,\ a_{2,3} \in \mathbb{R} \right\}$ for any subgroup $A(+)$ of the group $\mathbb{R}(+)$ of the field $\mathbb{R}(+, \cdot)$;

2.3. $R_i = R$, $R'_i = R'$ and $R''_i = R''$ for every natural number i;

2.4. $R_i(A) = R(A)$ for every natural number i and any subgroup $A(+)$ of the group $\mathbb{R}(+)$ of the field $\mathbb{R}(+, \cdot)$;

2.5. $\widehat{R} = \sum\limits_{i=1}^{\infty} R_i$, $\widehat{R}' = \sum\limits_{i=1}^{\infty} R'_i$ and $\widehat{R}'' = \sum\limits_{i=1}^{\infty} R''_i$; $\widehat{R}(A) = \sum\limits_{i=1}^{\infty} R_i(A)$;

2.6. $\widehat{V}_n = \{\widehat{g} \in \widehat{R} \mid pr_i(\widehat{g}) = 0 \text{ if } i \leq n\}$ for any $n \in \mathbb{N}$;

2.7. $\widehat{R}_k(A) = \{\widehat{g} \in \widehat{R} | pr_k(\widehat{g}) \in R_k(A) \text{ and } pr_j(\widehat{g}) = \{0\} \text{ if } j \neq k\}$, where $k \in \mathbb{N}$ and $A(+)$ is a subgroup of the group $\mathbb{R}(+)$.

Remark 3. It is easy to see that R with the usual operation of matrix is a ring and $R^3 = 0$ and $(R')^2 = (R'')^2 = (R(A))^2 = 0$.

[1] As usual, the set V is called a neighborhood of an element a in the topological space (X, τ) if $a \in U \subseteq V$ for some $U \in \tau$.

In addition, since

$$\begin{pmatrix} 0 & a_{1,2} & a_{1,3} \\ 0 & 0 & a_{2,3} \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & b_{1,2} & b_{1,3} \\ 0 & 0 & b_{2,3} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a_{1,2} \cdot b_{2,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then it is obvious that $R^3 = 0$ and $(R')^2 = (R'')^2 = (R(A))^2 = 0$.

Proposition 4. *For the ring $\widehat{R}(+,\cdot)$ the following statements are true:*

1. *The collection $\mathcal{B}' = \{\widehat{V}_i \bigcap \widehat{R}' | i \in \mathbb{N}\}$ satisfies the conditions of Theorem 1, and hence, it is a basis of neighborhoods of zero for a ring topology $\widehat{\tau}'$ on the ring $\widehat{R}(+,\cdot)$;*

2. *The collection $\mathcal{B}'' = \{\widehat{V}_i \bigcap \widehat{R}'' | i \in \mathbb{N}\}$ satisfies the conditions of Theorem 1, and hence, is a basis of neighborhoods of zero for a ring topology $\widehat{\tau}''$ on the ring $\widehat{R}(+,\cdot)$;*

3. *If A is a subgroup of the group $\mathbb{R}(+)$ of the field $\mathbb{R}(+,\cdot)$, then the collection $\mathcal{B}(A) = \{\widehat{R}(A) \bigcap \widehat{V}_n | n \in \mathbb{N}\}$ satisfies all the conditions of Theorem 1, and hence, it is a basis of neighborhoods of zero for a ring topology $\widehat{\tau}(A)$ on the ring $\widehat{R}(+,\cdot)$.*

Proof. In addition, taking into consideration the definitions of sets $\widehat{V}_n$, $\widehat{R}'$, $\widehat{R}''$, and $\widehat{R}(A)$ we obtain that any set from the collection $\mathcal{B}' \bigcup \mathcal{B}'' \bigcup \mathcal{B}(A, \mathcal{F})$ is a subring of the ring $\widehat{R}(+,\cdot)$, and hence, any collection $\mathcal{B}'$, $\mathcal{B}''$, and $\mathcal{B}(A, \mathcal{F})$ satisfies conditions 1, 2, 3, 4 and 5 of Theorem 1.

To complete the proof of the theorem it remains to verify that for any of the mentioned collections the condition 6 of Theorem 1 are also satisfied.

Let now $\widehat{g} \in \widehat{R}$, then there exists a natural number n such that $pr_i(\widehat{g}) = 0$ for $i > m$.

If $\widehat{V}_k \bigcap \widehat{R}' \in \mathcal{B}'$ and $m = max\{k, n\}$, then $\widehat{g} \cdot \widehat{a} = 0$ and $\widehat{a} \cdot \widehat{g} = 0$ for any $\widehat{a} \in \widehat{V}_m \bigcap \widehat{R}'$, and hence, $\widehat{g} \cdot (\widehat{V}_m \bigcap \widehat{R}') \subseteq \widehat{V}_k \bigcap \widehat{R}'$ and $(\widehat{V}_m \bigcap \widehat{R}') \cdot \widehat{g} \subseteq \widehat{V}_k \bigcap \widehat{R}'$, i.e. the condition 6 of Theorem 1 holds for the collection $\mathcal{B}'$.

Analogously, if $\widehat{V}_k \bigcap \widehat{R}'' \in \mathcal{B}''$ and $m = max\{k, n\}$, then $\widehat{g} \cdot \widehat{a} = 0 \in \widehat{V}_k \bigcap \widehat{R}''$ for any $\widehat{a} \in \widehat{V}_m \bigcap \widehat{R}''$, and $\widehat{a} \cdot \widehat{g} = 0 \in \widehat{V}_k \bigcap \widehat{R}''$ for any $\widehat{a} \in \widehat{V}_m \bigcap \widehat{R}''$. Then $\widehat{g} \cdot (\widehat{V}_m \bigcap \widehat{R}'') \subseteq \widehat{V}_k \bigcap \widehat{R}''$ and $\widehat{g} \cdot (\widehat{V}_m \bigcap \widehat{R}'') \subseteq \widehat{V}_k \bigcap \widehat{R}''$, i.e. the condition 6 of Theorem 1 holds for the collection $\mathcal{B}''$.

If $\widehat{V}(A) \bigcap \widehat{V}_k \in \mathcal{B}(A)$ and $m = max\{n, k\}$, then $\widehat{V}(A) \bigcap V_m \subseteq \widehat{V}(A) \bigcap \widehat{V}_k$ and $\widehat{a} \cdot \widehat{g} = 0$ for any $\widehat{a} \in \widehat{V}(A) \bigcap \widehat{V}_m$, and $\widehat{V}(A, F) \bigcap \widehat{V}_k \in \mathcal{B}(A)$ and $m = max\{n, k\}$. Then $\widehat{V}(A) \bigcap \widehat{V}_m \subseteq \widehat{V}(A) \bigcap \widehat{V}_k$ and $\widehat{a} \cdot \widehat{g} = 0$ for any $\widehat{a} \in \widehat{V}(A) \bigcap \widehat{V}_m$.

Hence, $\widehat{g} \cdot (\widehat{V}(A) \bigcap \widehat{V}_m) = \{0\} \subseteq \widehat{V}(A) \bigcap \widehat{V}_k$ and $(\widehat{V}(A) \bigcap \widehat{V}_m) \cdot \widehat{g} = \{0\} \subseteq \widehat{V}(A) \bigcap \widehat{V}_k$, i.e. the condition 6 of Theorem 1 holds for the collection $\mathcal{B}(A)$.

By this, the proposition is completely proved. □

Proposition 5. *Let $\widehat{\tau}'$ and $\widehat{\tau}''$ be ring topologies on the ring $\widehat{R}$, defined in Proposition 5, and $n \in \mathbb{N}$. If τ is a non-discrete ring topology on the ring $\widehat{R}$ such that*

$\tau \geq \widehat{\tau}'$, then for any neighborhood W of zero in the topological ring $(\widehat{R}, \inf\{\tau, \widehat{\tau}''\})$ there exists a natural number $k \geq n$ such that $\widehat{R}_k(\mathbb{R}) \subseteq W$. (see 2.7)

Proof. Let W be a neighborhood of zero in the topological ring $(\widehat{R}, \inf\{\tau, \widehat{\tau}''\})$, and let W_1 be a neighborhood of zero in the topological ring $(\widehat{R}, \inf\{\tau, \widehat{\tau}''\})$ such that $W_1 \cdot W_1 + W_1 \subseteq W$. Then W_1 is a neighborhood of zero in each of the topological ring $(\widehat{R}, \tau)$ and $(\widehat{R}, \widehat{\tau}'')$, and hence, there exists a natural number $n_0 \in \mathbb{N}$ such that $n_0 \geq n$ and $\widehat{V}_{n_0} \bigcap \widehat{R}'' \subseteq W_1$. Since $\tau \geq \widehat{\tau}'$, then $\widehat{R}' \bigcap \widehat{V}_{n_0}$ is a neighborhood of zero in the topological ring $(\widehat{R}, \tau)$. Hence $\widehat{R}' \bigcap \widehat{V}_{n_0} \bigcap W_1$ is a neighborhood of zero in the topological ring $(\widehat{R}, \tau)$.

Since τ is a non-discrete topology, then $\widehat{R}' \bigcap \widehat{V}_{n_0} \bigcap W_1 \neq \{0\}$.

If $0 \neq \widehat{g}_0 \in \widehat{R}' \bigcap \widehat{V}_{n_0} \bigcap W_1 \neq \{0\}$, then there exists a natural number $k \geq n_0 \geq n$ such that $pr_k(\widehat{g}_0) \neq 0$.

Since $\widehat{g}_0 \in \widehat{R}'$, then $pr_k(\widehat{g}_0) = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and $a \neq 0$. Now if $\widehat{g}_1 \in \widehat{R}_k(\mathbb{R})$ then $pr_k(\widehat{g}_1) = \begin{pmatrix} 0 & 0 & r \\ 0 & 0 & a_{2,3} \\ 0 & 0 & 0 \end{pmatrix}$ and $pr_i(\widehat{g}_1) = 0$ for $i \neq k$.

If $\widehat{g}_2 \in \widehat{R}''$ and $\widehat{g}_3 \in \widehat{R}''$ are such elements that $pr_k(\widehat{g}_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a^{-1} \cdot r \\ 0 & 0 & 0 \end{pmatrix}$, $pr_k(\widehat{g}_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{2,3} \\ 0 & 0 & 0 \end{pmatrix}$, and $pr_i(\widehat{g}_2) = pr_i(\widehat{g}_3) = 0$ for $i \neq k$, then $\widehat{g}_2 \in \widehat{R}''_k \bigcap \widehat{V}_{n_0} \subseteq W_1$. Then $\widehat{g}_0 \cdot \widehat{g}_2 + \widehat{g}_3 \in W_1 \cdot W_1 + W_1 \subseteq W$. As

$$pr_k(\widehat{g}_1) = \begin{pmatrix} 0 & 0 & r \\ 0 & 0 & a_{2,3} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a^{-1} \cdot r \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & r \\ 0 & 0 & a_{2,3} \\ 0 & 0 & 0 \end{pmatrix} =$$

$pr_k(\widehat{g}_0) \cdot pr_k(\widehat{g}_2) + pr_k(\widehat{g}_3)$ and $pr_i(\widehat{g}_1) = 0 = pr_i(\widehat{g}_0) \cdot pr_i(\widehat{g}_2) + pr_i(\widehat{g}_2)$ for $i \neq k$ then $\widehat{g}_1 = \widehat{g}_0 \cdot \widehat{g}_2 + \widehat{g}_3 \in W$. From the arbitrariness of the element $\widehat{g}_1$ it follows then that $\widehat{R}_k(\mathbb{R}) \subseteq W$.

By this, the proposition is completely proved. □

Theorem 6. *Let $\widehat{\tau}'$ and $\widehat{\tau}''$ be ring topologies on the ring $\widehat{R}$, defined in Proposition 5. Then the following statements are true:*

1. *If τ is a ring topology on the ring $\widehat{R}$ such that $\tau \geq \widehat{\tau}'$, then*

$$\sup\{\widehat{\tau}(A), \inf\{\widehat{\tau}'', \tau\}\} > \sup\{\widehat{\tau}(B), \inf\{\widehat{\tau}'', \tau\}\}.$$

for any subgroups $A \subset B$ of the group $\mathbb{R}(+)$.

2. *If $\widehat{\tau}_d$ is the discrete topology on the ring $\widehat{R}$, and $\widehat{\tau}*$ is a coatom in the lattice of all ring topologies on the ring $\widehat{R}$ such that $\widehat{\tau}* \geq \widehat{\tau}'$, then between the topologies* $\inf\{\widehat{\tau}_d, \widehat{\tau}''\}$ *and* $\inf\{\widehat{\tau}*, \widehat{\tau}''\}$, *there exists a chain of ring topologies on the ring $\widehat{R}$ which is infinitely decreasing and infinitely increasing.*

Proof. Proof of Statement 7.1. Since $A \subset B$, then (see the notation at the beginning of this article) $\widehat{V}_n(A) \subseteq \widehat{V}_n(B)$ for any a natural number n. Then (see Proposition 5) $\widehat{\tau}(A) \geq \widehat{\tau}(B)$, and hence,

$$\sup\{\widehat{\tau}(A), \inf\{\widehat{\tau}'', \tau\}\} \geq \sup\{\widehat{\tau}(B), \inf\{\widehat{\tau}'', \tau\}\}.$$

We will show that

$$\sup\{\widehat{\tau}(A), \inf\{\widehat{\tau}'', \tau\}\} > \sup\{\widehat{\tau}(B), \inf\{\widehat{\tau}'', \tau\}\}.$$

Assume the contrary, i.e. that

$$\sup\{\widehat{\tau}(A), \inf\{\widehat{\tau}'', \tau\}\} = \sup\{\widehat{\tau}(B), \inf\{\widehat{\tau}'', \tau\}\}.$$

Then $\widehat{R}(A)$ is a neighborhood of zero in the topological ring $(\widehat{R}, \widehat{\tau}(A))$, and hence, $\widehat{R}(A)$ is a neighborhood of zero in the topological ring $(\widehat{R}, \sup\{\widehat{\tau}(B), \inf\{\widehat{\tau}'', \tau\}\})$. Then there exists a neighborhood W of zero in the topological ring $(\widehat{R}, \inf\{\widehat{\tau}'', \tau\})$ and a natural number $n \in \mathbb{N}$ such that $W \bigcap (\widehat{V}(B) \bigcap \widehat{V}_n) \subseteq \widehat{R}(A)$.

By Proposition 5, there exists a natural number $k \geq n$ such that $\widehat{R}_k(\mathbb{R}) \subseteq W$, and hence, $\widehat{R}_k(B) \subseteq \widehat{R}_k(\mathbb{R}) \subseteq W$. As $k \geq n$ then $\widehat{R}_k(B) \subseteq \widehat{V}_n$.

Since $k > m$, then (see 3.7)

$$R_k(B) = pr_k(\widehat{R}_k(B)) \subseteq pr_k(\widehat{R}(A)) = R_k(A),$$

but this contradicts $B \not\subseteq A$.

By this, Statement 1 is proved.

Proof of Statement 2. There exists a chain $\{A_i \mid i \in \mathbb{Z}\}$ of subgroups A_i of the group $\mathbb{R}(+)$ such that $A_i \subseteq A_{i+1}$ for any $i \in \mathbb{Z}$, i.e. this chain of subgroups is infinitely decreasing and infinitely increasing.

For any subgroup A_i let us consider the ring topology $\widehat{\tau}(A_i)$ on the ring $\widehat{R}$. Since $\widehat{\tau}* \geq \widehat{\tau}'$, then by statement 1, of this theorem

$$\sup\{\widehat{\tau}(A_i), \inf\{\widehat{\tau}'', \widehat{\tau}*\}\} > \sup\{\widehat{\tau}(A_{i+1}), \inf\{\widehat{\tau}'', \widehat{\tau}*\}\},$$

and hence, the chain of ring topologies $\sup\{\widehat{\tau}(A_i), \inf\{\widehat{\tau}'', \widehat{\tau}*\}\}$ is infinitely decreasing and infinitely increasing.

To complete the proof of the theorem it remains to verify that

$$\inf\{\widehat{\tau}*, \widehat{\tau}''\} \leq \sup\{\widehat{\tau}(A_i), \inf\{\widehat{\tau}*, \widehat{\tau}''\}\} \leq \inf\{\widehat{\tau}_d, \widehat{\tau}''\}$$

for any subgroup $A_i(+)$ of the group $\mathbb{R}(+)$, where $i \in \mathbb{Z}$.

In fact, from the definition of the sets $R(A)$ and R'' (see 3.2) it follows that $R(\{0\}) = R''$, and hence, $\widehat{\tau}(\{0\}) = \widehat{\tau}'' = \inf\{\widehat{\tau}_d, \widehat{\tau}''\}$. Then

$$\inf\{\widehat{\tau*}, \widehat{\tau''}\} \leq \sup\{\widehat{\tau}(\mathbb{R}), \inf\{\widehat{\tau}*, \widehat{\tau}''\}\} \leq \sup\{\widehat{\tau}(A_i), \inf\{\widehat{\tau}*, \widehat{\tau}''\}\} \leq$$

$$\sup\{\widehat{\tau}(\{0\}), \inf\{\widehat{\tau}_d, \widehat{\tau}''\}\} = \inf\{\widehat{\tau}'', \widehat{\tau}_d, \widehat{\tau}''\} = \inf\{\widehat{\tau}_d, \widehat{\tau}''\}$$

By this, the theorem is proved. □

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V. I. Arnautov
Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
5 Academiei str., MD-2028, Chisinau
Moldova
E-mail: *arnautov@math.md*

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G. N. Ermacova
Transnistrian State University
25 October str., 128, Tiraspol, 278000
Moldova
E-mail: *galla0808@yandex.ru*

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On the inverse operations in the class of preradicals of a module category, II

Ion Jardan

Abstract. In the present work a new operation, called left coquotient with respect to meet, in the class of preradicals $\mathbb{PR}$ of the category R-Mod of left R-modules is defined and investigated. It is dual to the studied earlier left quotient with respect to join [2]. Main properties of this operation and relations with lattice operations in $\mathbb{PR}$ are shown. Connections with some constructions in the large complete lattice $\mathbb{PR}$ are studied and some particular cases are mentioned.

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1 Introduction and preliminary facts

This work is devoted to the theory of radicals of modules ([1], [4]-[7]) and contains the investigation of a new operation in the class of preradicals of a module category.

Let R be a ring with unity and R-Mod be the category of unitary left R-modules. We remind that a *preradical* r of R-Mod is a subfunctor of identity functor of R-Mod, i.e. r associates to every module $M \in R$-Mod a submodule $r(M) \subseteq M$ such that $f(r(M)) \subseteq r(M')$ for every R-morphism $f: M \to M'$.

We denote by $\mathbb{PR}$ the class of all preradicals of the category R-Mod. In this class four operation are defined [4]:

1) the *meet* $\underset{\alpha \in \mathfrak{A}}{\wedge} r_\alpha$ of a family of preradicals $\{r_\alpha\}_{\alpha \in \mathfrak{A}}$:
$$\left(\underset{\alpha \in \mathfrak{A}}{\wedge} r_\alpha\right)(M) \overset{def}{=} \bigcap_{\alpha \in \mathfrak{A}} r_\alpha(M),\ M \in R\text{-Mod};$$

2) the *join* $\underset{\alpha \in \mathfrak{A}}{\vee} r_\alpha$ of a family of preradicals $\{r_\alpha\}_{\alpha \in \mathfrak{A}}$:
$$\left(\underset{\alpha \in \mathfrak{A}}{\vee} r_\alpha\right)(M) \overset{def}{=} \sum_{\alpha \in \mathfrak{A}} r_\alpha(M),\ M \in R\text{-Mod};$$

3) the *product* $r \cdot s$ of preradicals $r, s \in \mathbb{PR}$:
$$(r \cdot s)(M) \overset{def}{=} r(s(M)),\ M \in R\text{-Mod};$$

4) the *coproduct* $r \# s$ of preradicals $r, s \in \mathbb{PR}$:
$$[(r \# s)(M)]/s(M) \overset{def}{=} r(M/s(M)),\ M \in R\text{-Mod}.$$

In the class $\mathbb{PR}$ the partial order relation " $\leq$ " is defined by the rule:

$$r_1 \leq r_2 \overset{def}{\Leftrightarrow} r_1(M) \subseteq r_2(M) \text{ for every } M \in R\text{-Mod}.$$

The class $\mathbb{PR}$ is a large complete lattice with respect to the operations of meet and join.

We remark that in the book [4] the coproduct is denoted by $(r:s)$ and is defined by the rule $[(r:s)(M)]/r(M) = s(M/r(M))$, so $(r \# s) = (s:r)$.

The following properties of distributivity hold [4]:

(1) $(\wedge r_\alpha) \cdot s = \wedge (r_\alpha \cdot s)$; (2) $(\vee r_\alpha) \cdot s = \vee (r_\alpha \cdot s)$;

(3) $(\wedge r_\alpha) \# s = \wedge (r_\alpha \# s)$; (4) $(\vee r_\alpha) \# s = \vee (r_\alpha \# s)$

for every family $\{r_\alpha\}_{\alpha \in \mathfrak{A}} \subseteq \mathbb{PR}$ and $s \in \mathbb{PR}$.

Using these relations some new inverse operations can be defined in the class $\mathbb{PR}$. One of them, the left quotient of product with respect to join, was defined and investigated in [2]. In this work we will study another inverse operation, namely the left coquotient of coproduct with respect to meet. In the case of pretorsions it was investigated by J. S. Golan by other methods in [1] (see [3]). Similar questions are discussed in [8], [9] and [10].

Now we remind the principal types of preradicals. A preradical $r \in \mathbb{PR}$ is called:

- *idempotent preradical*, if $r(r(M)) = r(M)$ for every $M \in R$-Mod (or if $r \cdot r = r$);
- *radical*, if $r(M/r(M)) = 0$ for every $M \in R$-Mod (or if $r \# r = r$);
- *idempotent radical*, if both previous conditions are fulfilled;
- *pretorsion (hereditary preradical)*, if $r(N) = N \bigcap r(M)$ for every $N \subseteq M$, $M \in R$-Mod;
- *cohereditary*, if $r(M/N) = (r(M) + N)/N$, for every $N \subseteq M \in R$-Mod;
- *torsion*, if r is a hereditary radical;
- *coprime*, if $r \neq 0$ and for any $t_1, t_2 \in \mathbb{PR}$, $t_1 \# t_2 \geq r$ implies $t_1 \geq r$ or $t_2 \geq r$ [9];
- $\vee$-*coprime*, if for any $t_1, t_2 \in \mathbb{PR}$, $t_1 \vee t_2 \geq r$ implies $t_1 \geq r$ or $t_2 \geq r$ [9];
- *coirreducible*, if for any $t_1, t_2 \in \mathbb{PR}$, $t_1 \vee t_2 = r$ implies $t_1 = r$ or $t_2 = r$ [9].

The operations of meet and join are commutative and associative, while the operations of product and coproduct are associative. By means of these operations four preradicals are obtained which are arranged in the following order:

$$r \cdot s \leq r \wedge s \leq r \vee s \leq r \# s$$

for every $r, s \in \mathbb{PR}$.

During this work we will use the following facts and notions from general theory of preradicals (see [4]$-$[7]).

Lemma 1.1. *(Monotony of the product) For any $s_1, s_2 \in \mathbb{PR}$, $s_1 \leq s_2$ implies that $r \cdot s_1 \leq r \cdot s_2$ and $s_1 \cdot r \leq s_2 \cdot r$ for every $r \in \mathbb{PR}$.* □

Lemma 1.2. *(Monotony of the coproduct) For any* $s_1, s_2 \in \mathbb{PR}$, $s_1 \leq s_2$ *implies that* $r \# s_1 \leq r \# s_2$ *and* $s_1 \# r \leq s_2 \# r$ *for every* $r \in \mathbb{PR}$. □

Lemma 1.3. *If the preradical* r *is cohereditary, then* $r \# s = r \vee s$ *for every* $s \in \mathbb{PR}$. □

Lemma 1.4. *For every* $r, s, t \in \mathbb{PR}$ *we have:*

1) $(r \cdot s) \# t \geq (r \# t) \cdot (s \# t)$;

2) $(r \# s) \cdot t \leq (r \cdot t) \# (s \cdot t)$. □

Definition 1.1. The *totalizer* of preradical r is the preradical

$$t\,(r) = \wedge \{ r_\alpha \in \mathbb{PR} \,|\, r_\alpha \# r = 1 \} .$$

Definition 1.2. The *pseudocomplement* of r in $\mathbb{PR}$ is a preradical $r^\perp \in \mathbb{PR}$ with the properties:

1) $r \wedge r^\perp = 0$;

2) If $s \in \mathbb{PR}$ is such that $s > r^\perp$, then $r \wedge s \neq 0$.

Lemma 1.5. *Each* $r \in \mathbb{PR}$ *has a unique pseudocomplement* $r^\perp$ *such that if* $s \in \mathbb{PR}$ *and* $r \wedge s = 0$, *then* $s \leq r^\perp$. □

Definition 1.3. The *supplement* of r in $\mathbb{PR}$ is a preradical $r^* \in \mathbb{PR}$ with the properties:

1) $r \vee r^* = 1$;

2) If $s \in \mathbb{PR}$ is such that $s < r^*$, then $r \vee s \neq 1$.

Lemma 1.6. *Let* $r \in \mathbb{PR}$ *and* r *possesses the supplement* r^*. *If* $s \in \mathbb{PR}$ *and* $r \vee s = 1$, *then* $s \geq r^*$. □

2 Left coquotient with respect to meet

Now we introduce and investigate the inverse operation of coproduct with respect to meet in the class of preradicals $\mathbb{PR}$ of category R-Mod.

Definition 2.1. Let $r, s \in \mathbb{PR}$. The *left coquotient with respect to meet of* r *by* s is defined as the least preradical among $r_\alpha \in \mathbb{PR}$ with the property $r_\alpha \# s \geq r$. We denote this preradical by $r \,{}^{\wedge}\!/_{\#}\, s$.

We will call r the *numerator* and s the *denominator* of the coquotient $r \,{}^{\wedge}\!/_{\#}\, s$.

Now we mention the existence of the left coquotient for every pair of preradicals.

Lemma 2.1. *For every* $r, s \in \mathbb{PR}$ *there exists the left coquotient* $r \,{}^{\wedge}\!/_{\#}\, s$ *with respect to meet, and it can be presented in the form* $r \,{}^{\wedge}\!/_{\#}\, s = \wedge \{ r_\alpha \in \mathbb{PR} \,|\, r_\alpha \# s \geq r \}$.

Proof. Since $1 \# s \geq r$ for every $s \in \mathbb{PR}$, the family of preradicals $\{r_\alpha \mid r_\alpha \# s \geq r\}$ is not empty. By the distributivity of coproduct with respect to meet of preradicals we have $\left(\bigwedge\limits_{r_\alpha \# s \geq r} r_\alpha\right) \# s = \bigwedge\limits_{r_\alpha \# s \geq r} (r_\alpha \# s)$. Since $r_\alpha \# s \geq r$ for every preradical r_α it follows that $\bigwedge\limits_{r_\alpha \# s \geq r} (r_\alpha \# s) \geq r$, i.e. $\left(\bigwedge\limits_{r_\alpha \# s \geq r} r_\alpha\right) \# s \geq r$. Therefore the preradical $\bigwedge\limits_{r_\alpha \# s \geq r} r_\alpha$ is one of r_α and it is the least among r_α with the property $r_\alpha \# s \geq r$. So $r \,{}^{\wedge}\!\!/_{\#}\, s = \wedge\{r_\alpha \in \mathbb{PR} \mid r_\alpha \# s \geq r\}$. $\square$

Moreover, from the proof of Lemma 2.1 it follows that $(r \,{}^{\wedge}\!\!/_{\#}\, s) \# s \geq r$. We will often use this relation futher.

Lemma 2.2. *For every* $r, s \in \mathbb{PR}$ *we have* $r \,{}^{\wedge}\!\!/_{\#}\, s \leq r$.

Proof. By Lemma 2.1 $r \,{}^{\wedge}\!\!/_{\#}\, s = \wedge\{r_\alpha \in \mathbb{PR} \mid r_\alpha \# s \geq r\}$. Since $r \# s \geq r$ it follows that r is one of preradicals r_α. Therefore $r \geq \wedge\{r_\alpha \in \mathbb{PR} \mid r_\alpha \# s \geq r\}$, i.e. $r \geq r \,{}^{\wedge}\!\!/_{\#}\, s$. $\square$

Now we indicate the behaviour of the left coquotient with respect to the order relation $(\leq)$ of $\mathbb{PR}$.

Proposition 2.3. *(Monotony in the numerator) If* $r_1, r_2 \in \mathbb{PR}$ *and* $r_1 \leq r_2$, *then* $r_1 \,{}^{\wedge}\!\!/_{\#}\, s \leq r_2 \,{}^{\wedge}\!\!/_{\#}\, s$ *for every* $s \in \mathbb{PR}$.

Proof. From Lemma 2.1 we have $r_1 \,{}^{\wedge}\!\!/_{\#}\, s = \wedge\{r_\alpha \in \mathbb{PR} \mid r_\alpha \# s \geq r_1\}$ and $r_2 \,{}^{\wedge}\!\!/_{\#}\, s = \wedge\left\{r'_\beta \in \mathbb{PR} \,\middle|\, r'_\beta \# s \geq r_2\right\}$. The relations $r_1 \leq r_2$ and $r'_\beta \# s \geq r_2$ imply $r'_\beta \# s \geq r_1$, so each r'_β is one of preradicals r_α. This proves that $\wedge\{r_\alpha \in \mathbb{PR} \mid r_\alpha \# s \geq r_1\} \leq \wedge\left\{r'_\beta \in \mathbb{PR} \,\middle|\, r'_\beta \# s \geq r_2\right\}$, so $r_1 \,{}^{\wedge}\!\!/_{\#}\, s \leq r_2 \,{}^{\wedge}\!\!/_{\#}\, s$. $\square$

Proposition 2.4. *(Antimonotony in the denominator) If* $s_1, s_2 \in \mathbb{PR}$ *and* $s_1 \leq s_2$, *then* $r \,{}^{\wedge}\!\!/_{\#}\, s_1 \geq r \,{}^{\wedge}\!\!/_{\#}\, s_2$ *for every* $s \in \mathbb{PR}$.

Proof. From Lemma 2.1 we have $r \,{}^{\wedge}\!\!/_{\#}\, s_1 = \wedge\{r_\alpha \in \mathbb{PR} \mid r_\alpha \# s_1 \geq r\}$ and $r \,{}^{\wedge}\!\!/_{\#}\, s_2 = \wedge\left\{r'_\beta \in \mathbb{PR} \,\middle|\, r'_\beta \# s_2 \geq r\right\}$. Let $s_1 \leq s_2$. Then from the monotony of coproduct we have $r_\alpha \# s_1 \leq r_\alpha \# s_2$. Since $r_\alpha \# s_1 \geq r$, we obtain $r_\alpha \# s_2 \geq r$. So each preradical r_α is one of preradicals r'_β, therefore

$$\wedge\{r_\alpha \in \mathbb{PR} \mid r_\alpha \# s_1 \geq r\} \geq \wedge\left\{r'_\beta \in \mathbb{PR} \,\middle|\, r'_\beta \# s_2 \geq r\right\},$$

i.e. $r \,{}^{\wedge}\!\!/_{\#}\, s_1 \geq r \,{}^{\wedge}\!\!/_{\#}\, s_2$. $\square$

The following fact is very useful for the further investigations.

Proposition 2.5. *For every* $r, s, t \in \mathbb{PR}$ *we have:*

$$r \leq t \# s \Leftrightarrow r \,{}^{\wedge}\!\!/_{\#}\, s \leq t.$$

Proof. ($\Rightarrow$) By Lemma 2.1 $r \,{}^{\wedge}\!\!/_{\#}\, s = \wedge\{r_\alpha \in \mathbb{PR} \mid r_\alpha \# s \geq r\}$. If $t \# s \geq r$, then t is one of preradicals r_α, therefore $t \geq \wedge\{r_\alpha \in \mathbb{PR} \mid r_\alpha \# s \geq r\} = r \,{}^{\wedge}\!\!/_{\#}\, s$.

($\Leftarrow$) Let $t \geq r \,{}^{\wedge}\!\!/_{\#}\, s$. From the monotony of coproduct $t \# s \geq (r \,{}^{\wedge}\!\!/_{\#}\, s) \# s$ and by definition of left coquotient we have $(r \,{}^{\wedge}\!\!/_{\#}\, s) \# s \geq r$, therefore $t \# s \geq r$. □

In continuation we show some properties of the studied operation.

Proposition 2.6. *For every preradicals* $r, s \in \mathbb{PR}$ *we have:*

$$(r \# s) \,{}^{\wedge}\!\!/_{\#}\, s \leq r.$$

Proof. From Lemma 2.1 we have $(r \# s) \,{}^{\wedge}\!\!/_{\#}\, s = \wedge\{t_\alpha \in \mathbb{PR} \mid t_\alpha \# s \geq r \# s\}$. Since $r \# s \geq r \# s$, the preradical r is one of preradicals t_α, therefore we obtain $r \geq \wedge\{t_\alpha \in \mathbb{PR} \mid t_\alpha \# s \leq r \# s\}$, i.e. $r \geq (r \# s) \,{}^{\wedge}\!\!/_{\#}\, s$. □

Proposition 2.7. *For every* $r, s, t \in \mathbb{PR}$ *the following relations are true:*

1) $(r \,{}^{\wedge}\!\!/_{\#}\, s) \,{}^{\wedge}\!\!/_{\#}\, t \;=\; r \,{}^{\wedge}\!\!/_{\#}\, (t \# s)$;

2) $(r \# s) \,{}^{\wedge}\!\!/_{\#}\, t \;\leq\; r \# (s \,{}^{\wedge}\!\!/_{\#}\, t)$.

Proof. 1) From Lemma 2.1 we have $r \,{}^{\wedge}\!\!/_{\#}\, (t \# s) \;=\; \wedge\{r_\alpha \in \mathbb{PR} \mid r_\alpha \# (t \# s) \geq r\}$ and $(r \,{}^{\wedge}\!\!/_{\#}\, s) \,{}^{\wedge}\!\!/_{\#}\, t = \wedge\{t_\beta \in \mathbb{PR} \mid t_\beta \# t \geq r \,{}^{\wedge}\!\!/_{\#}\, s\}$.

($\leq$) Let $r_\alpha \# (t \# s) \geq r$. Then $(r_\alpha \# t) \# s \geq r$ and from Proposition 2.5 we obtain $r_\alpha \# t \geq r \,{}^{\wedge}\!\!/_{\#}\, s$. So any preradical r_α is one of preradicals t_β, therefore we obtain $\wedge\{r_\alpha \in \mathbb{PR} \mid r_\alpha \# (t \# s) \geq r\} \;\geq\; \wedge\{t_\beta \in \mathbb{PR} \mid t_\beta \# t \geq r \,{}^{\wedge}\!\!/_{\#}\, s\}$, i.e. $r \,{}^{\wedge}\!\!/_{\#}\, (t \# s) \geq (r \,{}^{\wedge}\!\!/_{\#}\, s) \,{}^{\wedge}\!\!/_{\#}\, t$.

($\geq$) Let $t_\beta \# t \;\geq\; r \,{}^{\wedge}\!\!/_{\#}\, s$. Using the monotony of coproduct we obtain $(t_\beta \# t) \# s \geq (r \,{}^{\wedge}\!\!/_{\#}\, s) \# s$, but from the definition of left coquotient $(r \,{}^{\wedge}\!\!/_{\#}\, s) \# s \geq r$, so $t_\beta \# (t \# s) = (t_\beta \# t) \# s \geq r$. This shows that each preradical t_β is one of preradicals r_α, therefore $\wedge\{t_\beta \in \mathbb{PR} \mid t_\beta \# t \geq r \,{}^{\wedge}\!\!/_{\#}\, s\} \geq \wedge\{r_\alpha \in \mathbb{PR} \mid r_\alpha \# (t \# s) \geq r\}$, i.e $(r \,{}^{\wedge}\!\!/_{\#}\, s) \,{}^{\wedge}\!\!/_{\#}\, t \;\geq\; r \,{}^{\wedge}\!\!/_{\#}\, (t \# s)$.

2) By definition of left coquotient $s \;\leq\; (s \,{}^{\wedge}\!\!/_{\#}\, t) \# t$. Using the monotony of coproduct we have $r \# s \leq r \# [(s \,{}^{\wedge}\!\!/_{\#}\, t) \# t] = [r \# (s \,{}^{\wedge}\!\!/_{\#}\, t)] \# t$, and from Proposition 2.5 we obtain $(r \# s) \,{}^{\wedge}\!\!/_{\#}\, t \;\leq\; r \# (s \,{}^{\wedge}\!\!/_{\#}\, t)$. □

Proposition 2.8. *For every* $r, s, t \in \mathbb{PR}$ *the following relations hold:*

1) $(r \,{}^{\wedge}\!\!/_{\#}\, t) \,{}^{\wedge}\!\!/_{\#}\, (s \,{}^{\wedge}\!\!/_{\#}\, t) \leq r \,{}^{\wedge}\!\!/_{\#}\, s$;

2) $(r \# t) \,{}^{\wedge}\!\!/_{\#}\, (s \# t) \leq r \,{}^{\wedge}\!\!/_{\#}\, s$.

Proof. 1) From Proposition 2.5 the relation of this statement is equivalent to the relation $r \,{}^{\wedge}\!\!/_{\#}\, t \leq (r \,{}^{\wedge}\!\!/_{\#}\, s) \# (s \,{}^{\wedge}\!\!/_{\#}\, t)$.

By definition of left coquotient $r \leq (r \,{}^{\wedge}\!\!/_{\#}\, s) \# s$ and $s \leq (s \,{}^{\wedge}\!\!/_{\#}\, t) \# t$, therefore from the monotony and the associativity of coproduct we obtain $r \leq (r \,{}^{\wedge}\!\!/_{\#}\, s) \# s \leq (r \,{}^{\wedge}\!\!/_{\#}\, s) \# [(s \,{}^{\wedge}\!\!/_{\#}\, t) \# t] \;=\; [(r \,{}^{\wedge}\!\!/_{\#}\, s) \# (s \,{}^{\wedge}\!\!/_{\#}\, t)] \# t$. Applying Proposition 2.5 we have $r \,{}^{\wedge}\!\!/_{\#}\, t \leq (r \,{}^{\wedge}\!\!/_{\#}\, s) \# (s \,{}^{\wedge}\!\!/_{\#}\, t)$.

2) From Proposition 2.5 the relation of this statement is equivalent to the relation $r \# t \leq (r \,{}^{\wedge}\!/_{\#}\, s) \# (s \# t)$.

By definition of left coquotient $r \leq (r \,{}^{\wedge}\!/_{\#}\, s) \# s$. Using the monotony of coproduct we obtain $r \# t \leq [(r \,{}^{\wedge}\!/_{\#}\, s) \# s\,] \# t = (r \,{}^{\wedge}\!/_{\#}\, s) \# (s \# t)$. □

Now we will discuss the question of relations beetween the left coquotient with respect to meet and the lattice operations of $\mathbb{PR}$.

Proposition 2.9. *(The left distributivity of left coquotient* $r \,{}^{\wedge}\!/_{\#}\, s$ *relative to join) Let* $s \in \mathbb{PR}$. *Then for every family of preradicals* $\{ r_\alpha |\, \alpha \in \mathfrak{A}\}$ *the following relation holds:*

$$\Big(\bigvee_{\alpha \in \mathfrak{A}} r_\alpha \Big) \,{}^{\wedge}\!/_{\#}\, s = \bigvee_{\alpha \in \mathfrak{A}} (r_\alpha \,{}^{\wedge}\!/_{\#}\, s).$$

Proof. ($\leq$) By definition of left coquotient we have $r_\alpha \leq (r_\alpha \,{}^{\wedge}\!/_{\#}\, s) \# s$ for every $\alpha \in \mathfrak{A}$. Then $\bigvee_{\alpha \in \mathfrak{A}} r_\alpha \leq \bigvee_{\alpha \in \mathfrak{A}} [(r_\alpha \,{}^{\wedge}\!/_{\#}\, s) \# s\,]$. From the distributivity of coproduct of preradicals relative to join it follows that $\bigvee_{\alpha \in \mathfrak{A}} r_\alpha \leq \Big[\bigvee_{\alpha \in \mathfrak{A}} (r_\alpha \,{}^{\wedge}\!/_{\#}\, s) \Big] \# s$. Using Proposition 2.5 we obtain $\Big(\bigvee_{\alpha \in \mathfrak{A}} r_\alpha \Big) \,{}^{\wedge}\!/_{\#}\, s \leq \bigvee_{\alpha \in \mathfrak{A}} (r_\alpha \,{}^{\wedge}\!/_{\#}\, s)$.

($\geq$) From Lemma 2.1 we have $\Big(\bigvee_{\alpha \in \mathfrak{A}} r_\alpha \Big) \,{}^{\wedge}\!/_{\#}\, s = \wedge \Big\{ t_\beta \in \mathbb{PR} \,|\, t_\beta \# s \geq \bigvee_{\alpha \in \mathfrak{A}} r_\alpha \Big\}$ and $\bigvee_{\alpha \in \mathfrak{A}} (r_\alpha \,{}^{\wedge}\!/_{\#}\, s) = \bigvee_{\alpha \in \mathfrak{A}} \Big(\bigwedge_{r'_\gamma \# s \geq r_\alpha} r'_\gamma \Big)$.

Let $t_\beta \# s \geq \bigvee_{\alpha \in \mathfrak{A}} r_\alpha$. Since $\bigvee_{\alpha \in \mathfrak{A}} r_\alpha \geq r_\alpha$ for every $\alpha \in \mathfrak{A}$ we have $t_\beta \# s \geq r_\alpha$, so each preradical t_β is one of preradicals r'_γ. This implies the relation $\wedge \Big\{ t_\beta \in \mathbb{PR} \,|\, t_\beta \# s \geq \bigvee_{\alpha \in \mathfrak{A}} r_\alpha \Big\} \geq \wedge \{ r'_\gamma \in \mathbb{PR} \,|\, r'_\gamma \# s \geq r_\alpha \}$ for every $\alpha \in \mathfrak{A}$, therefore $\wedge \Big\{ t_\beta \in \mathbb{PR} \,|\, t_\beta \# s \geq \bigvee_{\alpha \in \mathfrak{A}} r_\alpha \Big\} \geq \bigvee_{\alpha \in \mathfrak{A}} (\wedge \{ r'_\gamma \in \mathbb{PR} \,|\, r'_\gamma \# s \geq r_\alpha \})$, which means that $\Big(\bigvee_{\alpha \in \mathfrak{A}} r_\alpha \Big) \,{}^{\wedge}\!/_{\#}\, s \geq \bigvee_{\alpha \in \mathfrak{A}} (r_\alpha \,{}^{\wedge}\!/_{\#}\, s)$. □

Proposition 2.10. *In the class* $\mathbb{PR}$ *the following relations are true:*

1) $\Big(\bigwedge_{\alpha \in \mathfrak{A}} r_\alpha \Big) \,{}^{\wedge}\!/_{\#}\, s \leq \bigwedge_{\alpha \in \mathfrak{A}} (r_\alpha \,{}^{\wedge}\!/_{\#}\, s)$;

2) $r \,{}^{\wedge}\!/_{\#} \Big(\bigwedge_{\alpha \in \mathfrak{A}} s_\alpha \Big) \geq \bigvee_{\alpha \in \mathfrak{A}} (r \,{}^{\wedge}\!/_{\#}\, s_\alpha)$;

3) $r \,{}^{\wedge}\!/_{\#} \Big(\bigvee_{\alpha \in \mathfrak{A}} s_\alpha \Big) \leq \bigwedge_{\alpha \in \mathfrak{A}} (r \,{}^{\wedge}\!/_{\#}\, s_\alpha)$.

Proof. 1) By the definition of left coquotient we have $r_\alpha \leq (r_\alpha \,{}^{\wedge}\!/_{\#}\, s) \# s$ for every $\alpha \in \mathfrak{A}$, therefore $\bigwedge_{\alpha \in \mathfrak{A}} r_\alpha \leq \bigwedge_{\alpha \in \mathfrak{A}} [(r_\alpha \,{}^{\wedge}\!/_{\#}\, s) \# s]$. From the distributivity of coproduct of preradicals relative to meet it follows that $\bigwedge\limits_{\alpha\in\mathfrak{A}} r_\alpha \leq \left[\bigwedge\limits_{\alpha\in\mathfrak{A}} (r_\alpha \,{}^{\wedge}\!\!/_{\#}\, s)\right] \# s$ and using Proposition 2.5 we obtain $\left(\bigwedge\limits_{\alpha\in\mathfrak{A}} r_\alpha\right) {}^{\wedge}\!\!/_{\#}\, s \leq \bigwedge\limits_{\alpha\in\mathfrak{A}} (r_\alpha \,{}^{\wedge}\!\!/_{\#}\, s)$.

2) For every $\alpha \in \mathfrak{A}$ we have $\bigwedge\limits_{\alpha\in\mathfrak{A}} s_\alpha \leq s_\alpha$. From the antimonotony in the denominator of left coquotient it follows that $r \,{}^{\wedge}\!\!/_{\#} \left(\bigwedge\limits_{\alpha\in\mathfrak{A}} s_\alpha\right) \geq r \,{}^{\wedge}\!\!/_{\#}\, s_\alpha$ for all $\alpha \in \mathfrak{A}$, therefore $r \,{}^{\wedge}\!\!/_{\#} \left(\bigwedge\limits_{\alpha\in\mathfrak{A}} s_\alpha\right) \geq \bigvee\limits_{\alpha\in\mathfrak{A}} (r \,{}^{\wedge}\!\!/_{\#}\, s_\alpha)$.

3) For every $\alpha \in \mathfrak{A}$ we have $\bigvee\limits_{\alpha\in\mathfrak{A}} s_\alpha \geq s_\alpha$. From the antimonotony in the denominator of left coquotient it follows that $r \,{}^{\wedge}\!\!/_{\#} \left(\bigvee\limits_{\alpha\in\mathfrak{A}} s_\alpha\right) \leq r \,{}^{\wedge}\!\!/_{\#}\, s_\alpha$ for all $\alpha \in \mathfrak{A}$, therefore $r \,{}^{\wedge}\!\!/_{\#} \left(\bigvee\limits_{\alpha\in\mathfrak{A}} s_\alpha\right) \leq \bigwedge\limits_{\alpha\in\mathfrak{A}} (r \,{}^{\wedge}\!\!/_{\#}\, s_\alpha)$. □

3 The left coquotient $r \,{}^{\wedge}\!\!/_{\#}\, s$ in particular cases

In this section we study some particular cases of left coquotient with respect to meet, its relations with special constructions in large complete lattice $\mathbb{PR}$ and the connection with some types of preradicals (coprime, $\vee$-coprime, coirreducible), as well as the arrangement (relative position) of preradicals obtained by the studied operation.

Proposition 3.1. *For every preradicals $r, s \in \mathbb{PR}$ the following conditions are equivalent:*

1) $r \leq s$;

2) $r \,{}^{\wedge}\!\!/_{\#}\, s = 0$.

Proof. 1) $\Rightarrow$ 2) Let $r \leq s$. So $r \leq 0 \# s$ and from Proposition 2.5 we obtain $r \,{}^{\wedge}\!\!/_{\#}\, s \leq 0$, therefore $r \,{}^{\wedge}\!\!/_{\#}\, s = 0$.

2) $\Rightarrow$ 1) Let $r \,{}^{\wedge}\!\!/_{\#}\, s = 0$. By definition of left coquotient we have $(r \,{}^{\wedge}\!\!/_{\#}\, s) \# s \geq r$, so $0 \# s \geq r$, i.e $s \geq r$. □

Proposition 3.2. *Let $r, s \in \mathbb{PR}$. Then:*

1) $1 \,{}^{\wedge}\!\!/_{\#}\, s = t\,(s)$ *(see Def. 1.1);*

2) $r \,{}^{\wedge}\!\!/_{\#}\, 0 = r$.

Proof. From the definition of left coquotient we have:

1) $1 \,{}^{\wedge}\!\!/_{\#}\, s = \wedge \{r_\alpha \in \mathbb{PR} \mid r_\alpha \# s \geq 1\} = \wedge \{r_\alpha \in \mathbb{PR} \mid r_\alpha \# s = 1\} = t\,(s)$;

2) $r \,{}^{\wedge}\!\!/_{\#}\, 0 = \wedge \{r_\alpha \in \mathbb{PR} \mid r_\alpha \# 0 \geq r\} = \wedge \{r_\alpha \in \mathbb{PR} \mid r_\alpha \geq r\} = r$. □

From Propositions 3.1 and 3.2 such particular cases follow:

(1) $0 \hat{/}_{\#} 0 = 0$; (2) $r \hat{/}_{\#} r = 0$ for every $r \in \mathbb{PR}$;

(3) $s \hat{/}_{\#} 1 = 0$ for every $s \in \mathbb{PR}$; (4) $1 \hat{/}_{\#} 1 = t\,(1) = 0$.

As in Proposition 3.1 $(r \hat{/}_{\#} r) \# r = 0 \# r = r$ for every $r \in \mathbb{PR}$.

Moreover, the distributivity of coproduct of preradicals relative to meet implies $t\,(s) \# s = \left(\underset{r_\alpha \# s = 1}{\wedge} r_\alpha \right) \# s = \underset{r_\alpha \# s = 1}{\wedge} (r_\alpha \# s) = 1$ for every $s \in \mathbb{PR}$.

Now we will indicate the relations between the totalizer $t\,(r)$ of preradical r and such constructions in the large complete lattice $\mathbb{PR}$ as pseudocomplement and supplement (see Def. 1.2, Def. 1.3).

Proposition 3.3. *For every preradical $s \in \mathbb{PR}$ we have $t\,(s) \geq s^{\perp}$.*

Proof. By definition $t\,(s) = \wedge \{ r_\alpha \mid r_\alpha \# s = 1 \}$. The pseudocomplement $s^{\perp}$ of preradical s by definition has the property $s \wedge s^{\perp} = 0$. Since $s \cdot s^{\perp} \leq s \wedge s^{\perp} = 0$, we obtain $s \cdot s^{\perp} = 0$. We have that $t\,(s) \# s = 1$, so $s^{\perp} = 1 \cdot s^{\perp} = (t\,(s) \# s) \cdot s^{\perp}$. From Lemma 1.4 $(t\,(s) \# s) \cdot s^{\perp} \leq \big(t\,(s) \cdot s^{\perp}\big) \# \big(s \cdot s^{\perp}\big) = \big(t\,(s) \cdot s^{\perp}\big) \# 0 = t\,(s) \cdot s^{\perp}$. Therefore $s^{\perp} \leq t\,(s) \cdot s^{\perp}$, but $t\,(s) \cdot s^{\perp} \leq t\,(s)$, so $s^{\perp} \leq t\,(s)$. □

Moreover, we have $s^{\perp} \leq t\,(s) \cdot s^{\perp}$, but $s^{\perp} \geq t\,(s) \cdot s^{\perp}$, so $s^{\perp} = t\,(s) \cdot s^{\perp}$.

Proposition 3.4. *Let $s \in \mathbb{PR}$ and s have the supplement s^*. Then $t\,(s) \leq s^*$.*

Proof. By definition $t\,(s) = \wedge \{ r_\alpha \mid r_\alpha \# s = 1 \}$. The supplement s^* of s from the definition has the property $s \vee s^* = 1$. Since $s^* \# s \geq s^* \vee s = s \vee s^* = 1$, we obtain $s^* \# s = 1$. So s^* is one of preradicals r_α, therefore $s^* \geq \wedge \{ r_\alpha \mid r_\alpha \# s = 1 \}$, i.e. $s^* \geq t\,(s)$. □

Moreover, from Proposition 2.3 $r \hat{/}_{\#} s \leq 1 \hat{/}_{\#} s = t\,(s)$, therefore $r \hat{/}_{\#} s \leq s^*$.

The next two statements show when the cancellation properties for left coquotient hold (see Proposition 2.6).

Proposition 3.5. *Let $r, s \in \mathbb{PR}$. The following conditions are equivalent:*

1) $r = (r \# s) \hat{/}_{\#} s$;

2) $r = t \hat{/}_{\#} s$ *for some preradical* $t \in \mathbb{PR}$.

Proof. 1) $\Rightarrow$ 2) If $r = (r \# s) \hat{/}_{\#} s$, then $r = t \hat{/}_{\#} s$ with $t = r \# s$.

2) $\Rightarrow$ 1) Let $r = t \hat{/}_{\#} s$ for some preradical t. By definition of left coquotient $(t \hat{/}_{\#} s) \# s \geq t$. From Proposition 2.3 we obtain $[(t \hat{/}_{\#} s) \# s] \hat{/}_{\#} s \geq t \hat{/}_{\#} s$. But from Proposition 2.6 $[(t \hat{/}_{\#} s) \# s] \hat{/}_{\#} s \leq t \hat{/}_{\#} s$, therefore we have $[(t \hat{/}_{\#} s) \# s] \hat{/}_{\#} s = t \hat{/}_{\#} s$. Since $t \hat{/}_{\#} s = r$, we obtain $(r \# s) \hat{/}_{\#} s = r$. □

Proposition 3.6. *Let $r, s \in \mathbb{PR}$. The following conditions are equivalent:*

1) $r = (r \hat{/}_{\#} s) \# s$;

2) $r = t \# s$ *for some preradical* $t \in \mathbb{PR}$.

Proof. 1) $\Rightarrow$ 2) If $r = (r \,{}^{\wedge}\!/_{\#}\, s) \# s$, then $r = t \# s$ with $t = r \,{}^{\wedge}\!/_{\#}\, s$.

2) $\Rightarrow$ 1) Let $r = t \# s$ for some preradical t. By Proposition 2.6 $(t \# s) \,{}^{\wedge}\!/_{\#}\, s \leq t$. From the monotony of coproduct it follows that $[(t \# s) \,{}^{\wedge}\!/_{\#}\, s] \# s \leq t \# s$. But from the definition of left coquotient $[(t \# s) \,{}^{\wedge}\!/_{\#}\, s] \# s \geq t \# s$, therefore $[(t \# s) \,{}^{\wedge}\!/_{\#}\, s] \# s = t \# s$. Since $t \# s = r$, we have $(r \,{}^{\wedge}\!/_{\#}\, s) \# s = r$. □

Now we will study the behaviour of the left coquotient $r \,{}^{\wedge}\!/_{\#}\, s$ in the cases of such types of preradicals as coprime, $\vee$-coprime and coirreducible.

Proposition 3.7. *The preradical r is coprime if and only if for every preradical s we have $r \,{}^{\wedge}\!/_{\#}\, s = 0$ or $r \,{}^{\wedge}\!/_{\#}\, s = r$.*

Proof. ($\Rightarrow$) Let $r \neq 0$. By definition $(r \,{}^{\wedge}\!/_{\#}\, s) \# s \geq r$ and if r is coprime, then we have $r \,{}^{\wedge}\!/_{\#}\, s \geq r$ or $s \geq r$. If $r \,{}^{\wedge}\!/_{\#}\, s \geq r$, then since by Lemma 2.2 $r \,{}^{\wedge}\!/_{\#}\, s \leq r$, it follows that $r \,{}^{\wedge}\!/_{\#}\, s = r$. If $s \geq r$, then from Proposition 3.1 we have $r \,{}^{\wedge}\!/_{\#}\, s = 0$.

($\Leftarrow$) Let $t_1 \# t_2 \geq r$ for some preradicals $t_1, t_2 \in \mathbb{PR}$. From Proposition 2.5 we obtain $t_1 \geq r \,{}^{\wedge}\!/_{\#}\, t_2$. For the preradical t_2 from the condition of this proposition we have $r \,{}^{\wedge}\!/_{\#}\, t_2 = 0$ or $r \,{}^{\wedge}\!/_{\#}\, t_2 = r$. If $r \,{}^{\wedge}\!/_{\#}\, t_2 = 0$, then from Proposition 3.1 it follows that $t_2 \geq r$. If $r \,{}^{\wedge}\!/_{\#}\, t_2 = r$, then $t_1 \geq r \,{}^{\wedge}\!/_{\#}\, t_2 = r$. So for every $t_1, t_2 \in \mathbb{PR}$ with $t_1 \# t_2 \geq r$ we have $t_1 \geq r$ or $t_2 \geq r$, which means that the preradical r is coprime. □

Proposition 3.8. *If the preradical r is $\vee$-coprime, then the coquotient $r \,{}^{\wedge}\!/_{\#}\, s$ is $\vee$-coprime for every $s \in \mathbb{PR}$.*

Proof. Suppose that $t_1 \vee t_2 \geq r \,{}^{\wedge}\!/_{\#}\, s$, for some $t_1, t_2 \in \mathbb{PR}$. Then from Proposition 2.5 we obtain $(t_1 \vee t_2) \# s \geq r$. From the distributivity of coproduct of preradicals relative to join we have $(t_1 \# s) \vee (t_2 \# s) \geq r$. If r is $\vee$-coprime, then $t_1 \# s \geq r$ or $t_2 \# s \geq r$. From Proposition 2.5 we obtain that $t_1 \geq r \,{}^{\wedge}\!/_{\#}\, s$ or $t_2 \geq r \,{}^{\wedge}\!/_{\#}\, s$. So for every preradicals $t_1, t_2 \in \mathbb{PR}$ with $t_1 \vee t_2 \geq r \,{}^{\wedge}\!/_{\#}\, s$ we have $t_1 \geq r \,{}^{\wedge}\!/_{\#}\, s$ or $t_2 \geq r \,{}^{\wedge}\!/_{\#}\, s$, which means that the preradical $r \,{}^{\wedge}\!/_{\#}\, s$ is $\vee$-coprime. □

Proposition 3.9. *Let $r, s \in \mathbb{PR}$ and $r = t \# s$ for some preradical $t \in \mathbb{PR}$. If the preradical r is coirreducible, then the preradical $r \,{}^{\wedge}\!/_{\#}\, s$ is coirreducible.*

Proof. Let $t_1 \vee t_2 = r \,{}^{\wedge}\!/_{\#}\, s$ for some preradicals $t_1, t_2 \in \mathbb{PR}$. If $r = t \# s$ for some preradical t, then by Proposition 3.6 $r = (r \,{}^{\wedge}\!/_{\#}\, s) \# s$, so $r = (t_1 \vee t_2) \# s$. From the distributivity of coproduct of preradicals relative to join $r = (t_1 \# s) \vee (t_2 \# s)$. If r is coirreducible, then $t_1 \# s = r$ or $t_2 \# s = r$.

If $t_1 \# s = r$, then from Proposition 2.5 we have $t_1 \geq r \,{}^{\wedge}\!/_{\#}\, s$. But $t_1 \leq r \,{}^{\wedge}\!/_{\#}\, s$, because $t_1 \vee t_2 = r \,{}^{\wedge}\!/_{\#}\, s$, therefore $t_1 = r \,{}^{\wedge}\!/_{\#}\, s$.

If $t_2 \# s = r$, then similarly we obtain $t_2 = r \,{}^{\wedge}\!/_{\#}\, s$.

So for every preradicals $t_1, t_2 \in \mathbb{PR}$ with $t_1 \vee t_2 = r \,{}^{\wedge}\!/_{\#}\, s$ we have $t_1 = r \,{}^{\wedge}\!/_{\#}\, s$ or $t_2 = r \,{}^{\wedge}\!/_{\#}\, s$, which means that the preradical $r \,{}^{\wedge}\!/_{\#}\, s$ is coirreducible. □

The operation of left coquotient with respect to meet implies some order relations between the associated preradicals. To see that we firstly prove

Proposition 3.10. *For every* $r, s, t \in \mathbb{PR}$ *the following relations are true:*

1) $r \,{}^{\wedge}\!/_{\#}\, s = (r \vee s) \,{}^{\wedge}\!/_{\#}\, s;$

2) $(r \,{}^{\wedge}\!/_{\#}\, s) \# s \geq r \vee s.$

Proof. 1) From Proposition 2.9 we have that $(r \vee s) \,{}^{\wedge}\!/_{\#}\, s = (r \,{}^{\wedge}\!/_{\#}\, s) \vee (s \,{}^{\wedge}\!/_{\#}\, s)$, but $s \,{}^{\wedge}\!/_{\#}\, s = 0$, so $(r \vee s) \,{}^{\wedge}\!/_{\#}\, s = (r \,{}^{\wedge}\!/_{\#}\, s) \vee 0 = r \,{}^{\wedge}\!/_{\#}\, s$.

Moreover, since $r \# s \geq r \vee s$ from Proposition 2.3 we obtain

$$(r \# s) \,{}^{\wedge}\!/_{\#}\, s \geq (r \vee s) \,{}^{\wedge}\!/_{\#}\, s = r \,{}^{\wedge}\!/_{\#}\, s.$$

2) By 1) we have $r \,{}^{\wedge}\!/_{\#}\, s = (r \vee s) \,{}^{\wedge}\!/_{\#}\, s$ and so $(r \,{}^{\wedge}\!/_{\#}\, s) \# s = ((r \vee s) \,{}^{\wedge}\!/_{\#}\, s) \# s$. From the definition of left coquotient we have $((r \vee s) \,{}^{\wedge}\!/_{\#}\, s) \# s \geq r \vee s$, therefore $(r \,{}^{\wedge}\!/_{\#}\, s) \# s \geq r \vee s$. □

Corollary 3.11. 1) *For every preradicals* $r, s \in \mathbb{PR}$ *the following relations hold:*

$$r \,{}^{\wedge}\!/_{\#}\, s \leq (r \# s) \,{}^{\wedge}\!/_{\#}\, s \leq r \leq r \vee s \leq (r \,{}^{\wedge}\!/_{\#} s) \# s \leq r \# s;$$

2) *If* r *is cohereditary, then*

$$r \,{}^{\wedge}\!/_{\#}\, s = (r \# s) \,{}^{\wedge}\!/_{\#}\, s \leq r \leq r \vee s = (r \,{}^{\wedge}\!/_{\#} s) \# s = r \# s$$

for every $s \in \mathbb{PR}$. □

We remark that the operations of left quotient with respect to join and left coquotient with respect to meet are complete in the sense of existence for any two preradicals.

In conclusion, we can say that in this work is introduced and studied a new (complete) operation (*left coquotient with respect to meet*) in the class of preradicals $\mathbb{PR}$ of R-Mod, which is dual the previous operation (*left quotient with respect to join*) and possesses similar properties. The indicated facts dualise the results of paper [2]. In the particular case of pretorsions as corrolaries we obtain a series of results of J. S. Golan [1], as is indicated in [3].

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Ion Jardan
Technical University of Moldova
Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
E-mail: *jordanion79@gmail.com; ion.jardan@mate.utm.md*

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On LCA groups whose ring of continuous endomorphisms satisfies DCC on closed ideals

Valeriu Popa

Abstract. We determine the structure of LCA (locally compact abelian) groups X with the property that the ring $E(X)$ of continuous endomorphisms of X, taken with the compact-open topology, satisfies DCC (descending chain condition) on different types of closed ideals.

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Introduction

A well known theorem of L. Fuchs [7, Theorem 111.3] asserts that the endomorphism ring of an (abstract) abelian group X is right (respectively, left) artinian if and only if X is the direct sum of a finite group and finitely many copies of the additive group of rational numbers. F. Szász observed [15] that the same conclusion about the structure of X remains true under weaker hypothesis that the endomorphism ring of X satisfies DCC on principal right (respectively, left) ideals.

The purpose of the present paper is to extend these results to the more general setting obtained by considering LCA groups and their rings of continuous endomorphisms. To be precise, let $\mathcal{L}$ be the class of all LCA groups. For $X \in \mathcal{L}$, let $E(X)$ denote the ring of continuous endomorphisms of X, endowed with the compact-open topology. We shall determine here the explicite structure of groups $X \in \mathcal{L}$ with the property that the ring $E(X)$ satisfies DCC on closed right (respectively, left) ideals, and we shall show that the corresponding class of groups coincides with the class of those groups $X \in \mathcal{L}$ whose ring $E(X)$ satisfies DCC on topologically principal right (respectively, left) ideals. We shall also determine the groups $X \in \mathcal{L}$ for which $E(X)$ is right (respectively, left) artinian.

1 Notation

Throughout the following, $\mathbb{N}$ is the set of natural numbers (including zero), $\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$, and $\mathbb{P}$ is the set of prime numbers.

The groups in $\mathcal{L}$ which we shall mention frequently are the reals $\mathbb{R}$, the p-adic numbers $\mathbb{Q}_p$, the p-adic integers $\mathbb{Z}_p$ (all with their usual topologies), the rationals

$\mathbb{Q}$, the quasi-cyclic groups $\mathbb{Z}(p^\infty)$ and the cyclic groups $\mathbb{Z}(p^n)$ of order p^n (all with the discrete topology), where $p \in \mathbb{P}$ and $n \in \mathbb{N}$.

For $X \in \mathcal{L}$, we let 1_X, $c(X)$, $d(X)$, $k(X)$, $m(X)$, $t(X)$, and X^* denote respectively the identity map on X, the connected component of zero in X, the maximal divisible subgroup of X, the subgroup of compact elements of X, the smallest closed subgroup K of X such that the quotient group X/K is torsion-free, the torsion subgroup of X, and the character group of X.

We denote by $E(X)$ the ring of continuous endomorphisms of X and by $H(X, Y)$, where Y is another group in $\mathcal{L}$, the group of continuous homomorphisms from X to Y, both endowed with the compact-open topology.

For $n \in \mathbb{N}$ and $p \in \mathbb{P}$, we let $nX = \{nx \mid x \in X\}$, $X[n] = \{x \in X \mid nx = 0\}$, $X_p = \{x \in X \mid \lim_{k\to\infty} p^k x = 0\}$, and $S(X) = \{q \in \mathbb{P} \mid (k(X)/c(X))_q \neq 0\}$.

For $a \in X$ and $S \subset X$, $\langle a \rangle$ is the subgroup of X generated by a, $\overline{S}$ is the closure of S in X, and $A(X^*, S) = \{\gamma \in X^* \mid \gamma(x) = 0 \text{ for all } x \in S\}$.

Also, we write $X = A \oplus B$ (respectively, $X = A \dotplus B$) in case X is a topological (respectively, an algebraic) direct sum of its subgroups A and B.

If $(X_i)_{i\in I}$ is a family of groups in $\mathcal{L}$, we write $\prod_{i\in I} X_i$ for the topological direct product of the groups X_i and $\prod_{i\in I}(X_i; U_i)$ for the topological local direct product of the groups X_i relative to the compact open subgroups $U_i \subset X_i$. We recall that $\prod_{i\in I}(X_i; U_i)$ consists of all $(x_i)_{i\in I} \in \prod_{i\in I} X_i$ with $x_i \in U_i$ for all but finitely many i, topologized by declaring all neighbourhoods of zero in the topological group $\prod_{i\in I} U_i$ to be a fundamental system of neighbourhoods of zero in $\prod_{i\in I}(X_i; U_i)$.

If F is a field, $\mathbb{M}_n(F)$ stands for the ring of all $n \times n$ matrices with entries in F. The symbol $\cong$ denotes topological group (ring) isomorphism.

2 Topological Morita context rings

In our study of groups $X \in \mathcal{L}$ with the property that $E(X)$ satisfies DCC on different types of closed ideals, we will frequently make use of topological Morita context rings. Here we recall this construction and derive several facts about its closed ideals.

Let $\mathcal{M} = \big(R, S, {}_RP_S, {}_SQ_R, [\cdot,\cdot]_R, [\cdot,\cdot]_S\big)$ be a topological Morita context, that is R and S are topological rings with identity, ${}_RP_S$ is a unital topological (R,S)-bimodule, ${}_SQ_R$ is a unital topological (S,R)-bimodule, $[\cdot,\cdot]_R : {}_RP_S \times {}_SQ_R \to {}_RR_R$ is a continuous (R,R)-bilinear S-balanced mapping, and $[\cdot,\cdot]_S : {}_SQ_R \times {}_RP_S \to {}_SS_S$ is a continuous (S,S)-bilinear R-balanced mapping such that

$$[p,q]_R p' = p[q,p']_S \quad \text{and} \quad [q,p]_S q' = q[p,q']_R$$

for all $r \in R, s \in S, p, p' \in P$ and $q, q' \in Q$. By analogy with the case of abstract Morita contexts, we can associate to $\mathcal{M}$ a topological ring, called the topological Morita context ring of $\mathcal{M}$. Specifically, we endow the set

$$M = \left\{ \begin{pmatrix} r & p \\ q & s \end{pmatrix} \mid r \in R, p \in P, q \in Q, s \in S \right\}$$

with the product topology of $R\times P\times Q\times S$, and define addition and multiplication on M by setting:

$$\begin{pmatrix} r_1 & p_1 \\ q_1 & s_1 \end{pmatrix} + \begin{pmatrix} r_2 & p_2 \\ q_2 & s_2 \end{pmatrix} = \begin{pmatrix} r_1+r_2 & p_1+p_2 \\ q_1+q_2 & s_1+s_2 \end{pmatrix}$$

and

$$\begin{pmatrix} r_1 & p_1 \\ q_1 & s_1 \end{pmatrix} \begin{pmatrix} r_2 & p_2 \\ q_2 & s_2 \end{pmatrix} = \begin{pmatrix} r_1r_2+[p_1,q_2]_R & r_1p_2+p_1s_2 \\ q_1r_2+s_1q_2 & [q_1,p_2]_S+s_1s_2 \end{pmatrix}$$

for all $r_1, r_2 \in R$, $p_1, p_2 \in P$, $q_1, q_2 \in Q$, and $s_1, s_2 \in S$. As is well known, the algebraic properties of operations of R, S, P and Q, and of mappings $[\cdot,\cdot]_R$ and $[\cdot,\cdot]_S$ ensure that, with respect to the above addition and multiplication, M is a ring with identity. It turns out that, in the considered topological situation, these operations on M are also compatible with the topology of M. To see this, it suffices in view of [3, Ch. I, §4, Proposition 1] to observe that composing the mentioned operations on M with the canonical projections on the components of M we get continuous mappings, because of the continuity of operations on R, S, P and Q, and of mappings $[\cdot,\cdot]_R$ and $[\cdot,\cdot]_S$. Thus M becomes a topological ring with identity, which we will denote by $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$. We will use frequently the special cases $\begin{pmatrix} R & 0 \\ Q & S \end{pmatrix}$ and $\begin{pmatrix} R & P \\ 0 & S \end{pmatrix}$ corresponding respectively to $P=\{0\}$ or $Q=\{0\}$.

As we will be working with closed ideals of $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$, it is desirable to relate them to closed subobjects of the components R, P, Q, and S. For this purpose, we need to introduce four mappings of $\mathcal{M}$. Recall that if A and B are topological rings and if $h : A \to B$ is a continuous ring homomorphism, then any topological right (respectively, left) B-module X can be viewed as a topological right (respectively, left) A-module via the scalar multiplication given by $xa = xh(a)$ (respectively, $ax = h(a)x$) for all $a \in A$ and $x \in X$. For example, if $h_R : R\times S \to R$ and $h_S : R\times S \to S$ are the canonical projections, then R, S, P, Q and hence their products can be considered as topological right (respectively, left) modules over the topological direct product ring $R\times S$. We will use the following continuous mappings:

$$\varphi_{R,Q,P} : {}_{R\times S}((R\times Q)_R \times {}_RP)_S \to {}_{R\times S}(P\times S)_S,\ ((r,q),p) \to (rp, [q,p]_S),$$
$$\varphi_{P,S,Q} : {}_{R\times S}((P\times S)_S \times {}_SQ)_R \to {}_{R\times S}(R\times Q)_R,\ ((p,s),q) \to ([p,q]_R, sq),$$
$$\varphi_{P,Q,S} : {}_R(P_S \times {}_S(Q\times S))_{R\times S} \to {}_R(R\times P)_{R\times S},\ (p,(q,s)) \to ([p,q]_R, ps),$$
$$\varphi_{Q,R,P} : {}_S(Q_R \times {}_R(R\times P))_{R\times S} \to {}_S(Q\times S)_{R\times S},\ (q,(r,p)) \to (qr, [q,p]_S).$$

It is easy to see that $\varphi_{R,Q,P}$ is R-balanced and $(R\times S, S)$-bilinear, $\varphi_{P,S,Q}$ is S-balanced and $(R\times S, R)$-bilinear, $\varphi_{P,Q,S}$ is S-balanced and $(R, R\times S)$-bilinear, and $\varphi_{Q,R,P}$ is R-balanced and $(S, R\times S)$-bilinear.

We have:

Lemma 1. *Let* $\big(R, S, {}_RP_S, {}_SQ_R, [\cdot,\cdot]_R, [\cdot,\cdot]_S\big)$ *be a topological Morita context.*

(i) The closed right ideals of $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$ *are of the form*

$$\begin{pmatrix} A & B \end{pmatrix} = \left\{ \begin{pmatrix} r & p \\ q & s \end{pmatrix} \mid (r,q) \in A, (p,s) \in B \right\},$$

where A *is a closed submodule of* $(R \times Q)_R$ *and* B *is a closed submodule of* $(P \times S)_S$ *such that* $\varphi_{P,S,Q}(B \times Q) \subset A$ *and* $\varphi_{R,Q,P}(A \times P) \subset B$.

(ii) The closed left ideals of $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$ *are of the form*

$$\begin{pmatrix} C \\ D \end{pmatrix} = \left\{ \begin{pmatrix} r & p \\ q & s \end{pmatrix} \mid (r,p) \in C, (q,s) \in D \right\},$$

where C *is a closed submodule of* ${}_R(R \times P)$ *and* D *is a closed submodule of* ${}_S(Q \times S)$ *such that* $\varphi_{P,Q,S}(P \times D) \subset C$ *and* $\varphi_{Q,R,P}(Q \times C) \subset D$.

(iii) The closed ideals of $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$ *are of the form*

$$\begin{pmatrix} I & U \\ V & J \end{pmatrix} = \left\{ \begin{pmatrix} r & p \\ q & s \end{pmatrix} \mid r \in I, p \in U, q \in V, s \in J \right\},$$

where I *is a closed ideal of* R, J *is a closed ideal of* S, U *is a closed subbimodule of* ${}_RP_S$, V *is a closed subbimodule of* ${}_SQ_R$, *and the following conditions hold:* $[U,Q]_R \subset I$, $[P,V]_R \subset I$, $[Q,U]_S \subset J$, $[V,P]_S \subset J$, $IP \subset U$, $PJ \subset U$, $QI \subset V$, $JQ \subset V$.

Proof. (i) Let A and B be as stated in (i). Clearly, the additive group of $\begin{pmatrix} A & B \end{pmatrix}$ is a closed subgroup of the additive group of $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$. Given any $\begin{pmatrix} r_0 & p_0 \\ q_0 & s_0 \end{pmatrix} \in \begin{pmatrix} A & B \end{pmatrix}$ and $\begin{pmatrix} r & p \\ q & s \end{pmatrix} \in \begin{pmatrix} R & P \\ Q & S \end{pmatrix}$, we also have

$$(r_0 r, q_0 r) \in A, \quad ([p_0, q]_R, s_0 q) = \varphi_{P,S,Q}((p_0, s_0), q) \in A,$$

$$(p_0 s, s_0 s) \in B \quad \text{and} \quad (r_0 p, [q_0, p]_S) = \varphi_{R,Q,P}((r_0, q_0), p) \in B,$$

so

$$\begin{pmatrix} r_0 & p_0 \\ q_0 & s_0 \end{pmatrix} \begin{pmatrix} r & p \\ q & s \end{pmatrix} = \begin{pmatrix} r_0 r + [p_0, q]_R & r_0 p + p_0 s \\ q_0 r + s_0 q & [q_0, p]_S + s_0 s \end{pmatrix} \in \begin{pmatrix} A & B \end{pmatrix},$$

and hence $\begin{pmatrix} A & B \end{pmatrix}$ is a closed right ideal of $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$.

To show the converse, we first make the following observations. Since, clearly, $r \mapsto \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}$ is a continuous ring homomorphism from R into $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$, $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$ can be regarded as a topological right R-module. Then $\begin{pmatrix} R & 0 \\ Q & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & P \\ 0 & S \end{pmatrix}$ become topological submodules of $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}_R$, and $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}_R$ can be written in the form

$$\begin{pmatrix} R & P \\ Q & S \end{pmatrix}_R = \begin{pmatrix} R & 0 \\ Q & 0 \end{pmatrix}_R \oplus \begin{pmatrix} 0 & P \\ 0 & S \end{pmatrix}_R.$$

In particular, the mapping

$$\pi_{R\times Q} : \begin{pmatrix} R & P \\ Q & S \end{pmatrix}_R \to (R \times Q)_R, \ \begin{pmatrix} r & p \\ q & s \end{pmatrix} \mapsto (r, q),$$

is a continuous morphism of R-modules whose restriction to $\begin{pmatrix} R & 0 \\ Q & 0 \end{pmatrix}_R$ is an isomorphism of topological R-modules. Similarly, by using the ring homomorfism $s \mapsto \begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix}$ from S into $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$, $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$ can be given the structure of topological right S-module. Then $\begin{pmatrix} R & 0 \\ Q & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & P \\ 0 & S \end{pmatrix}$ become topological submodules of $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}_S$, and $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}_S$ can be written in the form

$$\begin{pmatrix} R & P \\ Q & S \end{pmatrix}_S = \begin{pmatrix} R & 0 \\ Q & 0 \end{pmatrix}_S \oplus \begin{pmatrix} 0 & P \\ 0 & S \end{pmatrix}_S.$$

In particular, the mapping

$$\pi_{P\times S} : \begin{pmatrix} R & P \\ Q & S \end{pmatrix}_S \to (P \times S)_S, \ \begin{pmatrix} r & p \\ q & s \end{pmatrix} \mapsto (p, s),$$

is a continuous morphism of S-modules whose restriction to $\begin{pmatrix} 0 & P \\ 0 & S \end{pmatrix}_S$ is an isomorphism of topological S-modules.

Now, let Y be an arbitrary closed right ideal of $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$. It is clear that $Y_R \subset \begin{pmatrix} R & P \\ Q & S \end{pmatrix}_R$ and $Y_S \subset \begin{pmatrix} R & P \\ Q & S \end{pmatrix}_S$. Given any $\begin{pmatrix} r & p \\ q & s \end{pmatrix} \in Y$, we have

$$\begin{pmatrix} r & 0 \\ q & 0 \end{pmatrix} = \begin{pmatrix} r & p \\ q & s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in Y$$

and

$$\begin{pmatrix} 0 & p \\ 0 & s \end{pmatrix} = \begin{pmatrix} r & p \\ q & s \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in Y.$$

It follows that

$$Y_R = (Y \cap \begin{pmatrix} R & 0 \\ Q & 0 \end{pmatrix})_R \oplus (Y \cap \begin{pmatrix} 0 & P \\ 0 & S \end{pmatrix})_R.$$

and

$$Y_S = (Y \cap \begin{pmatrix} R & 0 \\ Q & 0 \end{pmatrix})_S \oplus (Y \cap \begin{pmatrix} 0 & P \\ 0 & S \end{pmatrix})_S.$$

In particular, $A = \pi_{R\times Q}(Y) = \pi_{R\times Q}(Y \cap \begin{pmatrix} R & 0 \\ Q & 0 \end{pmatrix})$ is a closed submodule of $(R\times Q)_R$ and $B = \pi_{P\times S}(Y) = \pi_{P\times S}(Y \cap \begin{pmatrix} 0 & P \\ 0 & S \end{pmatrix})$ is a closed submodule of $(P \times S)_S$.

It only remains for us to show that $\varphi_{P,S,Q}(B\times Q) \subset A$ and $\varphi_{R,Q,P}(A\times P) \subset B$. Pick arbitrary $(p, s) \in B$ and $q' \in Q$. Then $\begin{pmatrix} 0 & p \\ 0 & s \end{pmatrix} \in Y$, so

$$\begin{pmatrix} [p, q']_R & 0 \\ sq' & 0 \end{pmatrix} = \begin{pmatrix} 0 & p \\ 0 & s \end{pmatrix} \begin{pmatrix} 0 & 0 \\ q' & 0 \end{pmatrix} \in Y,$$

and hence $([p, q']_R, sq') \in A$. Since $(p, s) \in B$ and $q' \in Q$ were arbitrary, we conclude that $\varphi_{P,S,Q}(B\times Q) \subset A$. Next pick arbitrary $(r, q) \in A$ and $p' \in P$. Then $\begin{pmatrix} r & 0 \\ q & 0 \end{pmatrix} \in Y$, so

$$\begin{pmatrix} 0 & rp' \\ 0 & [q, p']_S \end{pmatrix} = \begin{pmatrix} r & p \\ q & s \end{pmatrix} \begin{pmatrix} 0 & p' \\ 0 & 0 \end{pmatrix} \in Y,$$

and hence $(rp', [q, p']_S) \in B$. It follows that $\varphi_{R,Q,P}(A \times P) \subset B$.

(ii) The proof of (ii) is similar to that of (i).

(iii) The fact that $\begin{pmatrix} I & U \\ V & J \end{pmatrix}$ is a closed ideal of $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$ is clear. For the converse, pick an arbitrary closed ideal Y of $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$. Given any $\begin{pmatrix} r & p \\ q & s \end{pmatrix} \in Y$, we have

$$\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r & p \\ q & s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in Y$$

$$\begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r & p \\ q & s \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in Y$$

$$\begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r & p \\ q & s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in Y$$

and

$$\begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r & p \\ q & s \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in Y.$$

Set $I' = Y \cap \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}$, $U' = Y \cap \begin{pmatrix} 0 & P \\ 0 & 0 \end{pmatrix}$, $V' = Y \cap \begin{pmatrix} 0 & 0 \\ Q & 0 \end{pmatrix}$ and $J' = Y \cap \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}$. It follows that the additive group of Y is a topological direct sum of the additive groups of I', U', V' and J', proving the closeness of $I = \pi_R(I')$, $U = \pi_P(U')$, $V = \pi_Q(V')$, and $J = \pi_S(J')$, where π_R, π_P, π_Q, and π_S are the canonical projections of $\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$ onto R, P, Q, and S respectively. It is also clear that I is an ideal of R, J is an ideal of S, U is a subbimodule of P, and V is a subbimodule of Q. Finally, the inclusions in (iii) follow from the inclusions in (i) and (ii). □

Specializing to $\begin{pmatrix} R & P \\ 0 & S \end{pmatrix}$, we obtain the following corollary.

Corollary 1. *Let R and S be topological rings with identity, and let P be a unital topological (R, S)-bimodule.*

(i) The closed right ideals of $\begin{pmatrix} R & P \\ 0 & S \end{pmatrix}$ are of the form

$$\left\{ \begin{pmatrix} r & p \\ 0 & s \end{pmatrix} \mid r \in I, (p, s) \in B \right\},$$

where I is a closed right ideal of R and B is a closed submodule of $(P \times S)_S$ such that $IP \times \{0\} \subset B$.

(ii) The closed left ideals of $\begin{pmatrix} R & P \\ 0 & S \end{pmatrix}$ are of the form

$$\left\{ \begin{pmatrix} r & p \\ 0 & s \end{pmatrix} \mid s \in J, (r, p) \in C \right\},$$

where J is a closed left ideal of S and C is a closed submodule of ${}_R(R \times P)$ such that $\{0\} \times PJ \subset C$.

(iii) The closed ideals of $\begin{pmatrix} R & P \\ 0 & S \end{pmatrix}$ are of the form

$$\left\{ \begin{pmatrix} r & p \\ 0 & s \end{pmatrix} \mid r \in I, s \in J, p \in U \right\},$$

where I is a closed ideal of R, J is a closed ideal of S, and U is a closed subbimodule of ${}_R P_S$ such that $IP + PJ \subset U$.

Next we consider chain conditions in $\begin{pmatrix} R & P \\ 0 & S \end{pmatrix}$. In accordance with [10, (1.22)], we have:

Lemma 2. *Let R and S be topological rings with identity, and let P be a unital topological (R, S)-bimodule. The ring $\begin{pmatrix} R & P \\ 0 & S \end{pmatrix}$ satisfies DCC on closed right (respectively, left) ideals if and only if so does R (respectively, S), and the right S-module $(P \times S)_S$ (respectively, left R-module ${}_R(R \times P)$) satisfies DCC on closed submodules.*

The same statement is true if we replace throughout DCC by ACC.

Proof. Assume $\begin{pmatrix} R & P \\ 0 & S \end{pmatrix}$ satisfies DCC on closed right ideals, and let $(I_n)_n \subset R_R$ and $(B_n)_n \subset (P \times S)_S$ be descending chains of closed submodules. Passing to the chain $\big((I_n \times \{0\} \quad B_n)\big)_n$ of $\begin{pmatrix} R & P \\ 0 & S \end{pmatrix}$, we see that $(I_n)_n$ and $(B_n)_n$ must stabilise.

For the converse, let $(Y_n)_n$ be a descending chain of closed right ideals of $\begin{pmatrix} R & P \\ 0 & S \end{pmatrix}$. For each n, we can write $Y_n = \big(I_n \times \{0\} \quad B_n\big)$, where $I_n \subset R_R$ and $B_n \subset (P \times S)_S$ are closed submodules such that $I_n \supset I_{n+1}$ and $B_n \supset B_{n+1}$. As $(I_n)_n$ and $(B_n)_n$ are stationary, $(Y_n)_n$ must be stationary as well. □

We close this section by pointing out the specific topological Morita context rings, which we will be working with. Let $X \in \mathcal{L}$. To any two closed subgroups A and B of X such that $X = A \oplus B$, we associate the topological Morita context

$$\mathcal{M}(A,B) = \big(E(A), E(B), {}_{E(A)}H(B,A)_{E(B)}, {}_{E(B)}H(A,B)_{E(A)}, [\cdot,\cdot]_{E(A)}, [\cdot,\cdot]_{E(B)}\big),$$

where $[f,g]_{E(A)} = f \circ g$ and $[g,f]_{E(B)} = g \circ f$ for all $f \in H(B,A)$ and $g \in H(A,B)$. We write $\begin{pmatrix} E(A) & H(B,A) \\ H(A,B) & E(B) \end{pmatrix}$ for the topological Morita context ring of $\mathcal{M}(A,B)$.

Lemma 3. *Let X be a group in $\mathcal{L}$ which can be written in the form $X = A \oplus B$ for some closed subgroups A and B of X. Then*

$$E(X) \cong \begin{pmatrix} E(A) & H(B,A) \\ H(A,B) & E(B) \end{pmatrix}.$$

If A is topologically fully invariant in X, then

$$E(X) \cong \begin{pmatrix} E(A) & H(B,A) \\ 0 & E(B) \end{pmatrix}.$$

If A and B are both topologically fully invariant in X, then

$$E(X) \cong E(A) \times E(B).$$

Proof. Let $\eta_A : A \to X$, $\eta_B : B \to X$ and $\pi_A : X \to A$, $\pi_B : X \to B$ denote respectively the canonical injections and the canonical projections corresponding to the above decomposition of X. Define

$$\xi : E(X) \to \begin{pmatrix} E(A) & H(B,A) \\ H(A,B) & E(B) \end{pmatrix}$$

by setting

$$\xi(u) = \begin{pmatrix} \pi_A \circ u \circ \eta_A & \pi_A \circ u \circ \eta_B \\ \pi_B \circ u \circ \eta_A & \pi_B \circ u \circ \eta_B \end{pmatrix}$$

for all $u \in E(X)$. It is easy to see that ξ establishes a topological ring isomorphism between $E(X)$ and $\begin{pmatrix} E(A) & H(B,A) \\ H(A,B) & E(B) \end{pmatrix}$.

If A is topologically fully invariant, then $\pi_B \circ u \circ \eta_A = 0$ for all $u \in E(X)$, so $\mathrm{im}(\xi) = \begin{pmatrix} E(A) & H(B,A) \\ 0 & E(B) \end{pmatrix}$. If B is topologically fully invariant as well, then $\mathrm{im}(\xi) = \begin{pmatrix} E(A) & 0 \\ 0 & E(B) \end{pmatrix}$. □

3 Reduction to topological p-primary groups

In this section, we establish some necessary conditions in order for the ring $E(X)$ of a group $X \in \mathcal{L}$ satisfy *DCC* on topologically principal ideals, i.e. on ideals of the form $\overline{(f)}$ with $f \in E(X)$.

We begin by recalling that for any group $X \in \mathcal{L}$, $E(X)$ and $E(X^*)$ are topologically anti-isomorphic [11, (2.1)]. Recall also that the group X is called residual if $\overline{d(X)} \subset k(X)$ and $c(X) \subset m(X)$, and that X is called topologically torsion in case $\lim_{n\in\mathbb{N}}(n!)x = 0$ for all $x \in X$.

Theorem 1. *Let X be a residual group in $\mathcal{L}$ such that the collection*

$$\mathcal{E} = \{\overline{nE(X)} \mid n \in \mathbb{N}_0\}$$

has a minimal element with respect to set inclusion. Then X is a topological torsion group, and there exists a finite subset S of $S(X)$ such that the following conditions hold:

(i) For each $p \in S(X) \setminus S$, X_p is densely divisible and torsionfree;

(ii) For each $p \in S$, there exists an $n(p) \in \mathbb{N}$ such that $m(X_p) = X_p[p^{n(p)}]$ and $\overline{d(X_p)} = \overline{p^{n(p)}X_p}$.

Proof. Let $\overline{n_0E(X)}$, where $n_0 \in \mathbb{N}_0$, be a minimal element of $\mathcal{E}$. Then

$$\overline{n_0E(X)} = \overline{pn_0E(X)} \tag{1}$$

for all $p \in \mathbb{P}$. Our first objective is to show that $\overline{n_0X}$ and $\overline{n_0X^*}$ are densely divisible. Fix any $q \in \mathbb{P}$. We show first that

$$\overline{n_0X} = \overline{q\overline{n_0X}} \qquad \text{and} \qquad \overline{n_0X^*} = \overline{q\overline{n_0X^*}}.$$

To this end, pick any $x \in X$ and define $\delta_x : E(X) \to X$ by setting $\delta_x(u) = u(x)$ for all $u \in E(X)$. In view of the equality (1), we can find a net $(u_i^{(q)})_{i\in I_q}$ of elements in $E(X)$ such that $n_01_X = \lim_{i\in I_q} qn_0u_i^{(q)}$. Since δ_x is a continuous [5, Ch. X, §3, Theorem 3, Corollary 1] group homomorphism, it follows that

$$n_0x = \delta_x(n_01_X) = \lim_{i\in I_q} \delta_x(qn_0u_i^{(q)}) = \lim_{i\in I_q} qn_0u_i^{(q)}(x),$$

and so $n_0x \in \overline{qn_0X}$. As x was arbitrarily chosen in X, this gives $n_0X \subset \overline{qn_0X}$, so $\overline{n_0X} \subset \overline{qn_0X}$. It follows that $\overline{n_0X} = \overline{qn_0X}$ because the reverse inclusion is obvious.

On the other hand, the multiplication by q being continuous, we have $q\overline{n_0 X} \subset \overline{qn_0 X}$ [3, Ch. I, §2, Theorem 1], whence $\overline{q\overline{n_0 X}} \subset \overline{qn_0 X}$. As the opposite inclusion is obvious, it follows that $\overline{q\overline{n_0 X}} = \overline{qn_0 X} = \overline{n_0 X}$. Further, since $E(X)$ and $E(X^*)$ are topologically anti-isomorphic, the equality (1) also gives $\overline{n_0 E(X^*)} = \overline{pn_0 E(X^*)}$ for all $p \in \mathbb{P}$. Applying the preceding argument to X^*, we conclude that $\overline{n_0 X^*} = \overline{q\overline{n_0 X^*}}$.

Now we show that $\overline{n_0 X}$ and $\overline{n_0 X^*}$ are densely divisible. By [8, (24.22) and (22.17)], we have

$$(\overline{n_0 X})^*[q] = A((\overline{n_0 X})^*, \overline{q\overline{n_0 X}}) = A((\overline{n_0 X})^*, \overline{n_0 X}) = \{0\}.$$

Analogously, $(\overline{n_0 X^*})^*[q] = \{0\}$. Since $q \in \mathbb{P}$ was arbitrary, it follows that $(\overline{n_0 X})^*$ and $(\overline{n_0 X^*})^*$ are torsion-free, so $\overline{n_0 X}$ and $\overline{n_0 X^*}$ are densely divisible by [13, (5.2)]. In particular, $\overline{d(X)} \supset \overline{n_0 X}$ and $\overline{d(X^*)} \supset \overline{n_0 X^*}$, whence $\overline{d(X)} = \overline{n_0 X}$ and $\overline{d(X^*)} = \overline{n_0 X^*}$ because the opposite inclusions are obvious. By taking annihilators, we also obtain

$$m(X) = A(X, \overline{d(X)}) = A(X, \overline{n_0 X}) = X[n_0]$$

and $m(X^*) = X^*[n_0]$. Finally, since X and X^* are residual groups, we must have

$$c(X) \subset m(X) = X[n_0] \quad \text{and} \quad c(X^*) \subset m(X^*) = X^*[n_0],$$

so $c(X) = \{0\} = c(X^*)$ because $X[n_0]$ and $X^*[n_0]$ are totally disconnected [8, (24.21)]. This implies that X is a topological torsion group [1, (3.5)], and hence $X \cong \prod_{p \in S(X)}(X_p; U_p)$, where, for each $p \in S(X)$, U_p is a compact open subgroup of X_p [1, (3.13)]. Let

$$n_0 = p_1^{n_1} \cdots p_t^{n_t} \quad \text{and} \quad S = \{p_1, \ldots, p_t\},$$

where $p_1, \ldots, p_t$ are the distinct prime divisors of n_0 and $t, n_1, \ldots, n_t \in \mathbb{N}_0$. We can write

$$X = X_{p_1} \oplus \cdots \oplus X_{p_t} \oplus G \quad \text{and} \quad X^* = X^*_{p_1} \oplus \cdots \oplus X^*_{p_t} \oplus H,$$

where $G = \overline{\sum_{p \nmid n_0} X_p} \cong \prod_{p \nmid n_0}(X_p; U_p)$ and $H = \overline{\sum_{p \nmid n_0} X^*_p} \cong \prod_{p \nmid n_0}(X^*_p; A(X^*_p, U_p))$. It is clear that G and $H \cong G^*$ are torsion-free, so (i) holds [13, (5.2)]. For each $i = 1, \ldots, t$, we also have $m(X_{p_i}) = X_{p_i}[p_i^{n_i}]$ and $m(X^*_{p_i}) = X^*_{p_i}[p_i^{n_i}]$, so (ii) holds as well. □

In order to deal with general groups $X \in \mathcal{L}$, we need the following lemma which is inspired by [7, p. 236, (b)] and [9, Lemma 64.1].

Lemma 4. *Let X be a group in $\mathcal{L}$ for which there exist two sequences $(A_n)_{n \in \mathbb{N}}$ and $(B_n)_{n \in \mathbb{N}}$ of non-zero closed subgroups such that*

$$X = A_0 \oplus \cdots \oplus A_n \oplus B_n \quad \text{and} \quad B_n = A_{n+1} \oplus B_{n+1}$$

for all $n \in \mathbb{N}$. Then $E(X)$ fails to satisfy DCC on topologically principal right (respectively, left) ideals.

Proof. For $n \in \mathbb{N}$, let $\varepsilon_n \in E(X)$ denote the canonical projection of X onto B_n. As in the proof of [7, p. 236, (b)] or [9, Lemma 64.1], one can see that $\big(\varepsilon_n E(X)\big)_{n\in\mathbb{N}}$ and $\big(E(X)\varepsilon_n\big)_{n\in\mathbb{N}}$ are strictly descending chains of right, respectively, left ideals. It remains to observe that, for every $n \in \mathbb{N}$, $\varepsilon_n E(X)$ and $E(X)\varepsilon_n$ are closed in $E(X)$ because ε_n is idempotent. □

For general groups in $\mathcal{L}$, we have:

Theorem 2. *Let X be a group in $\mathcal{L}$ such that $E(X)$ satisfies DCC on topologically principal ideals. Then $X = U \oplus V \oplus W \oplus Y$, where $U \cong \mathbb{R}^d$ for some $d \in \mathbb{N}$, $V \cong \mathbb{Q}^{(\mu)}$ and $W \cong (\mathbb{Q}^*)^\nu$ for some cardinal numbers μ and ν, and Y is a topological torsion group in $\mathcal{L}$ satisfying the following conditions:*

(i) $S(Y) = S(X)$ is finite;

(ii) for each $p \in S(Y)$, there exists $\overline{n(p) \in \mathbb{N}}$ such that $m(Y_p) = Y[p^{n(p)}]$ and $\overline{d(Y_p)} = \overline{p^{n(p)}Y_p}$.

Proof. By [1, (9.3)], we can write $X = U \oplus V \oplus W \oplus Y$, where $U \cong \mathbb{R}^d$ for some $d \in \mathbb{N}$, $V \cong \mathbb{Q}^{(\mu)}$ and $W \cong (\mathbb{Q}^*)^\nu$ for some cardinal numbers μ and ν, and Y is residual. In particular, $k(X) = W \oplus k(Y)$ and $c(X) \cap k(X) = W \oplus \big(c(Y) \cap k(Y)\big)$, so $k(X)/\big(c(X) \cap k(X)\big) \cong k(Y)/\big(c(Y) \cap k(Y)\big)$, and hence $S(Y) = S(X)$. Our first aim is to show that the collection $\mathcal{E} = \{\overline{nE(Y)} \mid n \in \mathbb{N}_0\}$ has a minimal element with respect to inclusion. Let $Z = U \oplus V \oplus W$, so

$$E(X) \cong \begin{pmatrix} E(Z) & H(Y,Z) \\ H(Z,Y) & E(Y) \end{pmatrix},$$

as it follows from Lemma 3. For $n \in \mathbb{N}_0$, let $\mathcal{I}_n$ be the closed ideal of $\begin{pmatrix} E(Z) & H(Y,Z) \\ H(Z,Y) & E(Y) \end{pmatrix}$ generated by $\begin{pmatrix} 0 & 0 \\ 0 & n1_Y \end{pmatrix}$. We assert that

$$\mathcal{I}_n = \begin{pmatrix} \overline{\big(H(Y,Z)H(Z,Y)\big)} & H(Y,Z) \\ H(Z,Y) & \overline{nE(Y)} \end{pmatrix},$$

where $\overline{\big(H(Y,Z)H(Z,Y)\big)} \subset E(Z)$. To see that

$$\mathcal{I}_n \subset \begin{pmatrix} \overline{\big(H(Y,Z)H(Z,Y)\big)} & H(Y,Z) \\ H(Z,Y) & \overline{nE(Y)} \end{pmatrix},$$

it suffices to show that

$$\begin{pmatrix} \overline{\big(H(Y,Z)H(Z,Y)\big)} & H(Y,Z) \\ H(Z,Y) & \overline{nE(Y)} \end{pmatrix}$$

is a closed ideal of $\begin{pmatrix} E(Z) & H(Y,Z) \\ H(Z,Y) & E(Y) \end{pmatrix}$. We will show the later by applying Lemma 1(iii). Clearly, we have

$$\overline{\big(H(Y,Z)H(Z,Y)\big)}H(Y,Z) \subset H(Y,Z),$$

$$H(Y,Z)\overline{nE(Y)} \subset H(Y,Z),$$

$$H(Z,Y)\overline{\big(H(Y,Z)H(Z,Y)\big)} \subset H(Z,Y),$$

$$\overline{nE(Y)}H(Z,Y) \subset H(Z,Y),$$

and

$$\big[H(Y,Z), H(Z,Y)\big]_{E(Z)} \subset \overline{\big(H(Y,Z)H(Z,Y)\big)}.$$

Further, since $\frac{1}{n}1_Z$ is a continuous endomorphism of Z, every $f \in H(Y,Z)$ and $g \in H(Z,Y)$ can be written in the form $f = n(\frac{1}{n}f)$ and $g = n(\frac{1}{n}g)$. Consequently, we also have

$$\big[H(Z,Y), H(Y,Z)\big]_{E(Y)} \subset \overline{nE(Y)}.$$

It follows that Lemma 1(iii) is applicable, so

$$\begin{pmatrix} \overline{\big(H(Y,Z)H(Z,Y)\big)} & H(Y,Z) \\ H(Z,Y) & \overline{nE(Y)} \end{pmatrix}$$

is a closed ideal of $\begin{pmatrix} E(Z) & H(Y,Z) \\ H(Z,Y) & E(Y) \end{pmatrix}$, and hence

$$\mathcal{I}_n \subset \begin{pmatrix} \overline{\big(H(Y,Z)H(Z,Y)\big)} & H(Y,Z) \\ H(Z,Y) & \overline{nE(Y)} \end{pmatrix}.$$

On the other hand, given any $f \in H(Y,Z)$ and $g \in H(Z,Y)$, we have

$$\begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{n}f \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & n1_Y \end{pmatrix} \in \mathcal{I}_n,$$

$$\begin{pmatrix} 0 & 0 \\ g & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & n1_Y \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \frac{1}{n}g & 0 \end{pmatrix} \in \mathcal{I}_n,$$

and

$$\begin{pmatrix} fg & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ g & 0 \end{pmatrix} \in \mathcal{I}_n,$$

so

$$\mathcal{I}_n \supset \begin{pmatrix} \overline{\big(H(Y,Z)H(Z,Y)\big)} & H(Y,Z) \\ H(Z,Y) & \overline{nE(Y)} \end{pmatrix},$$

and hence

$$\mathcal{I}_n = \begin{pmatrix} \overline{\big(H(Y,Z)H(Z,Y)\big)} & H(Y,Z) \\ H(Z,Y) & \overline{nE(Y)} \end{pmatrix}.$$

Now, since $\begin{pmatrix} E(Z) & H(Y,Z) \\ H(Z,Y) & E(Y) \end{pmatrix}$ satisfies DCC on topologically principal ideals, we conclude that the collection $\{\mathcal{I}_n \mid n \in \mathbb{N}_0\}$ has a minimal element, which implies that the collection

$$\mathcal{E} = \{\overline{nE(Y)} \mid n \in \mathbb{N}_0\}$$

has a minimal element as well. It follows that Theorem 1 is applicable to Y. In particular, Y is a topological torsion group, so

$$Y \cong \prod_{p \in S(Y)} (Y_p; O_p),$$

where, for each $p \in S(Y)$, O_p is a compact open subgroup of Y_p [1, (3.13)]. It remains to observe that if $S(Y)$ were infinite, say $S(Y) = \{p_0, p_1, \ldots\}$, then we could construct, by setting $A_n = Y_{p_n}$ and $B_n = \overline{\sum_{i>n} Y_{p_i}}$, two sequences $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ of closed subgroups of Y as in Lemma 4, a contradiction. □

4 The necessary condition in case of topological p-primary groups

As we saw in the preceding section, the problem of determining the groups $X \in \mathcal{L}$ for which the ring $E(X)$ satisfies DCC on topologically principal right (respectively, left) ideals reduces to the case of topological p-primary groups. In the present section, we deal with this last type of groups.

We begin by extending and sharpening a result of L. Robertson, which asserts that $\mathbb{Q}_p$ is splitting in the class of torsion-free groups in $\mathcal{L}$ (see [1, Proposition 6.23]).

Theorem 3. *Let $X \in \mathcal{L}$ and let D be a closed subgroup of X such that $D \cong \mathbb{Q}_p$ for some $p \in \mathbb{P}$. The following conditions are equivalent:*

(i) D splits topologically from X.

(ii) $D \not\subset \big(c(X) \cap k(X)\big) + m(X)$.

Proof. Assume (i). Then we can write $X = D \oplus G$ for some closed subgroup G of X. Since $X/G \cong D$ is torsion-free, we have $m(X) \subset G$. Also, since X/G is totally disconnected, we have $c(X) \subset G$. Consequently, $c(X) + m(X) \subset G$ and hence (ii) holds.

Assume (ii). By [1, (9.3)], we can write $X = U \oplus V \oplus W \oplus Y$, where $U \cong \mathbb{R}^d$ for some $d \in \mathbb{N}$, $V \cong \mathbb{Q}^{(\mu)}$ and $W \cong (\mathbb{Q}^*)^\nu$ for some cardinal numbers μ and ν, and Y is residual. Since $D = k(D)$ and $k(X) = W \oplus Y$, we have $D \subset W \oplus Y$. Consequently, it suffices to show that D splits topologically from $W \oplus Y$.

Now, since Y is residual, we have $c(Y) \subset m(Y) = m(X)$, which implies

$$(c(X) \cap k(X)) + m(X) = W \oplus m(Y).$$

Our assumption then gives $D \not\subset W \oplus m(Y)$, and hence $W \oplus Y \setminus W \oplus m(Y)$ must contain elements of D. Denote by $\varphi : W \oplus Y \to (W \oplus Y)/(W \oplus m(Y))$ the canonical projection, and let f be the restriction of φ to D. By [8, (5.27)], we have $D/\ker(f) \cong f(D)$. Since $(W \oplus Y)/(W \oplus m(Y)) \cong Y/m(Y)$ is torsion-free and since every quotient of $\mathbb{Q}_p$ by a proper closed subgroup is torsion, we conclude that

$$D \cap (W \oplus m(Y)) = \ker f = \{0\}.$$

In particular, $f(D) \cong \mathbb{Q}_p$, and hence $f(D)$ splits topologically from $(W \oplus Y)/(W \oplus m(Y))$ [1, (6.23)]. Write $(W \oplus Y)/(W \oplus m(Y)) = f(D) \oplus G$ for some closed subgroup G of $(W \oplus Y)/(W \oplus m(Y))$, and set $G_0 = \varphi^{-1}(G)$. We assert that $W \oplus Y = D \oplus G_0$. Indeed, it is clear that G_0 is a closed subgroup of $W \oplus Y$. If $a \in D \cap G_0$, then $\varphi(a) \in \varphi(D) \cap \varphi(G_0) = f(D) \cap G = \{0\}$, so $a \in D \cap \big(W \oplus m(Y)\big) = \{0\}$. Further, given any $z \in W \oplus Y$, we have $\varphi(z) = \varphi(a) + \varphi(b)$ for some $a \in D$ and $b \in G_0$. Consequently, $z - a - b = t$ for some $t \in W \oplus m(Y)$, and hence $z = a + b + t$. Since $b + t \in G_0$, we conclude that $W \oplus Y = D \dotplus G_0$. Since $\mathbb{Q}_p$ is σ-compact, it then follows from [1, (6.5)] that $W \oplus Y = D \oplus G_0$. □

Corollary 2. *Let X be a group in $\mathcal{L}$ such that $t(X)$ is reduced and closed in X. If D is a closed subgroup of X satisfying $D \cong \mathbb{Q}_p$, then D splits topologically from X.*

Proof. As in the proof of Lemma 3, write $X = U \oplus V \oplus W \oplus Y$, where $U \cong \mathbb{R}^d$ for some $d \in \mathbb{N}$, $V \cong \mathbb{Q}^{(\mu)}$ and $W \cong (\mathbb{Q}^*)^\nu$ for some cardinal numbers μ and ν, and Y is residual. Since $t(X)$ is closed in X, we have $m(X) = t(X) = t(Y)$, so $(c(X) \cap k(X)) + m(X) = W \oplus t(Y)$. It is also clear that $D \subset k(X) = W \oplus Y$. In order to apply Theorem 3, we have to show that $D \not\subset W \oplus t(Y)$. Assume this is not so, and let $\varepsilon \in E(X)$ denote the canonical projection of X onto Y. It follows that $\varepsilon(D)$ is a subgroup of $t(Y)$. Since $\varepsilon(D)$ is divisible and $t(Y)$ is reduced, we get $\varepsilon(D) = \{0\}$, so $D \subset W$, which is a contradiction because W is compact and D is not. □

We continue with the following

Lemma 5. *Let $p \in \mathbb{P}$, and let X be a non-reduced topological p-primary group in $\mathcal{L}$ such that $t(X) = X[p^{n_0}]$ for some $n_0 \in \mathbb{N}$. For any non-zero $a \in d(X)$, let D_a be the smallest divisible subgroup of X containing a. Then $\overline{D_a} \cong \mathbb{Q}_p$ and $X = \overline{D_a} \oplus G$ for some closed subgroup G of X.*

Proof. Since $t(X) = X[p^{n_0}]$, $d(X)$ cannot contain copies of $\mathbb{Z}(p^\infty)$, so D_a is algebraically isomorphic to $\mathbb{Q}$. It follows from [2, Theorem 1] that $\overline{D_a}$ is divisible. Since X is a topological p-primary group, there exists a topological group isomorphism f from $\mathbb{Z}_p$ onto $\overline{\langle a \rangle}$. Let $\eta : \overline{\langle a \rangle} \to \overline{D_a}$ denote the canonical injection, and set $h = \eta \circ f$. Since $\mathbb{Z}_p$ is open in $\mathbb{Q}_p$, h extends to a continuous group homomorphism

$h_0 : \mathbb{Q}_p \to \overline{D_a}$ [8, (A.7)]. Now, since $\mathbb{Q}_p$ is the minimal divisible extension of $\mathbb{Z}_p$, $\mathbb{Z}_p$ is essential in $\mathbb{Q}_p$ [6, Lemma 24.3], and hence $\ker(h_0) = \{0\}$ [6, Lemma 24.2]. We deduce that h_0 is a topological isomorphism from $\mathbb{Q}_p$ onto a closed subgroup of $\overline{D_a}$ [1, (4.21)]. Now, since $h_0(\mathbb{Q}_p)$ is divisible and $a \in h_0(\mathbb{Q}_p)$, we must have $h_0(\mathbb{Q}_p) = \overline{D_a}$, so $\overline{D_a} \cong \mathbb{Q}_p$. It remains to apply Corollary 2. □

Now we can concretize the structure of topological p-primary groups in $\mathcal{L}$ with the property in question.

Theorem 4. *Let $p \in \mathbb{P}$, and let X be a topological p-primary group in $\mathcal{L}$ such that $E(X)$ satisfies DCC on topologically principal right (respectively, left) ideals. Then*

$$X \cong \prod_{i=0}^{k(p)} \mathbb{Z}(p^{r_i(p)}) \times \mathbb{Q}_p^{l(p)}$$

for some $k(p), r_0(p), \ldots, r_{k(p)}(p), l(p) \in \mathbb{N}$.

Proof. By Theorem 1, there exists an $n(p) \in \mathbb{N}$ such that $m(X) = X[p^{n(p)}]$ and $\overline{d(X)} = \overline{p^{n(p)}X}$. We will distinguish two cases: $d(X) = \{0\}$ and $d(X) \neq \{0\}$.

First assume $d(X) = \{0\}$, so $X = X[p^{n(p)}]$. To decompose X, pick an element of maximal order $x_0 \in X$, and set $A_0 = \langle x_0 \rangle$. Clearly, $A_0 \cong \mathbb{Z}(p^{r_0(p)})$ for some $r_0(p) \in \mathbb{N}$. By [12, Lemma 2], we can write $X = A_0 \oplus B_0$ for some closed subgroup B_0 of X. If $B_0 \neq \{0\}$, choose an element of maximal order $x_1 \in B_0$ and write $X = A_0 \oplus A_1 \oplus B_1$, where $A_1 \cong \mathbb{Z}(p^{r_1(p)})$ for some $r_1(p) \in \mathbb{N}$ and B_1 is a closed subgroup of B_0. As Lemma 4 shows, if we continue in this way, we must arrive at a step $k(p)$ with $B_{k(p)} = \{0\}$.

Next assume $d(X) \neq \{0\}$. Picking any non-zero $y_0 \in d(X)$, let D_0 be the closure of the smallest divisible subgroup of X containing y_0. By Lemma 5, $D_0 \cong \mathbb{Q}_p$ and $X = D_0 \oplus G_0$ for some closed subgroup G_0 of X. If $d(G_0) \neq 0$, pick any non-zero $y_1 \in d(G_0)$ and let D_1 be the closure of the smallest divisible subgroup of D_0 containing y_1. As above, we have $D_1 \cong \mathbb{Q}_p$ and $X = D_0 \oplus D_1 \oplus G_1$ for some closed subgroup G_1 of G_0. By Lemma 4 again, this procedure must stop after a finite number, say $l(p)$, of steps, and so

$$X = D_0 \oplus \cdots \oplus D_{l(p)-1} \oplus G_{l(p)},$$

where $G_{l(p)}$ is reduced. This shows that

$$d(X) = D_0 \oplus \cdots \oplus D_{l(p)-1} = \overline{d(X)} \quad \text{and} \quad X[p^{n(p)}] \subset G_{l(p)}.$$

Therefore

$$\begin{aligned} p^{n(p)} G_{l(p)} &\subset \overline{p^{n(p)}X} \cap G_{l(p)} = \overline{d(X)} \cap G_{l(p)} \\ &= (D_0 \oplus \cdots \oplus D_{l(p)-1}) \cap G_{l(p)} = \{0\}, \end{aligned}$$

so $G_{l(p)} = X[p^{n(p)}]$, and hence

$$X = D_0 \oplus \cdots \oplus D_{l(p)-1} \oplus X[p^{n(p)}].$$

Since $D_0 \oplus \cdots \oplus D_{l(p)-1}$ and $X[p^{n(p)}]$ are fully invariant in X, we deduce from Lemma 3 that

$$E(X) \cong E(D_0 \oplus \cdots \oplus D_{l(p)-1}) \times E(X[p^{n(p)}]),$$

and hence $E(X[p^{n(p)}])$ satisfies DCC on topologically principal ideals. It follows that the first case applies to $X[p^{n(p)}]$, completing the proof. □

5 Characterizations

In this last section, we establish our results. We begin with two lemmas, which are needed in the proof of the main result. For the former, recall that every divisible torsion-free abelian group D can be considered as a vector space over the field of rational numbers, $\mathbb{Q}$, and this $\mathbb{Q}$-vector space structure is the only one existing on D. Moreover, every group homomorphism between such groups is in fact a homomorphism of $\mathbb{Q}$-vector spaces.

We have:

Lemma 6. *Let $d, n, l_1, \ldots, l_n \in \mathbb{N}$ and $p_1, \ldots, p_n \in \mathbb{P}$. The $\mathbb{Q}$-vector spaces $\mathbb{R}^d \times \prod_{i=1}^n \mathbb{Q}_{p_i}^{l_i}$ and $(\mathbb{Q}^*)^d$ satisfy both ACC and DCC on closed $\mathbb{Q}$-subspaces.*

Proof. It is clear that in either of $\mathbb{Q}$-vector spaces $\mathbb{R}^d$ and $\mathbb{Q}_p^l$, where $d, l \in \mathbb{N}$ and $p \in \mathbb{P}$, the closed $\mathbb{Q}$-subspaces are in fact $\mathbb{R}$-subspaces and respectively $\mathbb{Q}_p$-subspaces. As $\dim_{\mathbb{R}}(\mathbb{R}^d) = d$ and $\dim_{\mathbb{Q}_p}(\mathbb{Q}_p^l) = l$, we conclude that $\mathbb{R}^d$ and $\mathbb{Q}_p^l$ satisfy ACC and DCC on closed $\mathbb{Q}$-subspaces. Now, write the $\mathbb{Q}$-vector space $G = \mathbb{R}^d \times \prod_{i=1}^n \mathbb{Q}_{p_i}^{l_i}$ in the form

$$G = G_0 \oplus G_1 \oplus \cdots \oplus G_n,$$

where $G_0 \cong \mathbb{R}^d$, $G_1 \cong \mathbb{Q}_{p_1}^{l_1}, \ldots, G_n \cong \mathbb{Q}_{p_n}^{l_n}$. Given a closed $\mathbb{Q}$-subspace H of G, it is clear that $c(H) \subset c(G) = G_0$. It is also clear that, for any $x \in G_0 \cap H$, the $\mathbb{Q}$-subspace $\mathbb{Q}x \subset G_0 \cap H$, so $\mathbb{R}x = \overline{\mathbb{Q}x} \subset G_0 \cap H$, and hence $G_0 \cap H$ is connected [3, Ch. 1, §11, Proposition 2]. It follows that $c(H) = G_0 \cap H$. Further, since H is torsion-free, we can write $H = H_0 \oplus K$ (a topological direct sum of topological groups), where $H_0 = c(H)$ [1, (6.13)]. Moreover, since $H_0 \subset G_0$, we have $K \subset G_1 \oplus \cdots \oplus G_n$, so $K = H_1 \oplus \cdots \oplus H_n$, where $H_i \subset G_i$ for all $i = 1, \ldots, n$ [1, (3.13)]. Thus we obtained a decomposition of H as a topological direct sum $H = H_0 \oplus H_1 \oplus \cdots \oplus H_n$ of $\mathbb{Q}$-vector spaces. Since the $\mathbb{Q}$-vector spaces $G_0, G_1, \ldots, G_n$ satisfy ACC and DCC on closed $\mathbb{Q}$-subspaces, we conclude that so does G.

Now let us consider the case of $(\mathbb{Q}^*)^d$. It suffices to observe that a closed subgroup C of $(\mathbb{Q}^*)^d$ is a $\mathbb{Q}$-subspace if and only if its annihilator $A(\mathbb{Q}^d, C)$ is a $\mathbb{Q}$-subspace of $\mathbb{Q}^d$. Indeed, if C is a $\mathbb{Q}$-subspace of $(\mathbb{Q}^*)^d$ and $x \in A(\mathbb{Q}^d, C)$, then $\gamma(\frac{p}{q}x) = \frac{p}{q}\gamma(x) = 0$ for all $\gamma \in C$ and $\frac{p}{q} \in \mathbb{Q}$. Consequently, $\frac{p}{q}x \in A(\mathbb{Q}^d, C)$ for all $\frac{p}{q} \in \mathbb{Q}$, so $A(\mathbb{Q}^d, C)$

is a $\mathbb{Q}$-subspace of $\mathbb{Q}^d$. In a similar way, if $A(\mathbb{Q}^d, C)$ is a $\mathbb{Q}$-subspace of $\mathbb{Q}^d$, then C is a closed $\mathbb{Q}$-subspace of $(\mathbb{Q}^*)^d$. Since $\mathbb{Q}^d$ is of finite dimension, the proof is complete. □

Lemma 7. *Let R be a topological ring, M a topological (right or left) R-module, and C a closed submodule of M.*

(i) If M satisfies DCC on closed submodules, then so do C and M/C.

(ii) If C is either compact or open in M and if C and M/C satisfy DCC on closed submodules, then so does M.

Proof. The proof follows the same pattern as in the abstract case (see, for examle,[9, Proposition 27.1]). The requirement in (ii) that C is either compact or open in M assures that the image through the canonical projection of any closed submodule of M is closed in M/C. □

We are now prepared to prove our main result.

Theorem 5. *For a group $X \in \mathcal{L}$, the following statements are equivalent:*

(i) $E(X)$ satisfies both ACC and DCC on closed right ideals.

(ii) $E(X)$ satisfies DCC on closed right ideals.

(iii) $E(X)$ satisfies DCC on topologically principal right ideals, i.e. ideals of the form $\overline{fE(X)}$ with $f \in E(X)$.

(iv) $X \cong \mathbb{R}^d \times \mathbb{Q}^n \times (\mathbb{Q}^*)^m \times \prod_{p\in S_1} \mathbb{Q}_p^{l(p)} \times \prod_{p\in S_2} \prod_{i=0}^{k(p)} \mathbb{Z}(p^{r_i(p)})$, where S_1, S_2 are finite subsets of $\mathbb{P}$, and d, n, m, the $k(p)$'s, the $r_i(p)$'s and the $l(p)$'s are natural numbers.

Proof. Clearly, (i) implies (ii) and (ii) implies (iii). The fact that (iii) implies (iv) follows from Theorem 2 and Theorem 4.

Now assume (iv). We can write $X = D \oplus T$, where

$$D \cong \mathbb{R}^d \times \mathbb{Q}^n \times (\mathbb{Q}^*)^m \times \prod_{p\in S_1} \mathbb{Q}_p^{l(p)} \quad \text{and} \quad T \cong \prod_{p\in S_2} \prod_{i=0}^{k(p)} \mathbb{Z}(p^{r_i(p)}).$$

It is clear that $D = d(X)$ and $T = t(X)$, so D and T are topologically fully invariant subgroups of X. It follows from Lemma 3 that $E(X) \cong E(D) \times E(T)$. Since $E(T)$ is finite and since every right ideal $\mathcal{J}$ of $E(D) \times E(T)$ is of the form $\mathcal{J} = \mathcal{J}_d \times \mathcal{J}_t$, where $\mathcal{J}_d$ is a right ideal of $E(D)$ and $\mathcal{J}_t$ is a right ideal of $E(T)$, it suffices to show that $E(D)$ satisfies ACC and DCC on closed right ideals. In order to do this, write $D = M \oplus W$, where

$$M \cong \mathbb{Q}^n \quad \text{and} \quad W \cong \mathbb{R}^d \times (\mathbb{Q}^*)^m \times \prod_{p\in S_1} \mathbb{Q}_p^{l(p)}.$$

We have $W = c(D) + k(D)$, so W is topologically fully invariant in D, and hence

$$E(D) \cong \begin{pmatrix} E(W) & H(M,W) \\ 0 & E(M) \end{pmatrix}$$

by Lemma 3 again. It follows from Lemma 2 that we will achieve our goal if we show that $E(W)_{E(W)}$ and $\big(H(M,W)\times E(M)\big)_{E(M)}$ satisfy ACC and DCC on closed submodules.

First we consider the case of $\big(H(M,W) \times E(M)\big)_{E(M)}$. Since $E(M) \cong \mathbb{M}_n(\mathbb{Q})$, we deduce that $E(M)$ is discrete and satisfies ACC and DCC on right ideals. As then $H(M,W) \times \{0\}$ is open in $H(M,W) \times E(M)$, it suffices by Lemma 7 to show that $H(M,W)$ satisfies ACC and DCC on closed $E(M)$-submodules. To this end, we write

$$W = V \oplus K \oplus L, \tag{2}$$

where $V \cong \mathbb{R}^d$, $K \cong (\mathbb{Q}^*)^m$, and $L = \oplus_{p\in S_1} L_p$ with $L_p \cong \mathbb{Q}_p^{l(p)}$ for all $p \in S_1$. We know from [8, (23.34)(d)] that

$$H(M,W) \cong H(M,V) \times H(M,K) \times \prod_{p\in S_1} H(M,L_p) \tag{3}$$

as topological groups, and hence as topological $E(M)$-modules because the corresponding canonical isomorphism in (3) is easily seen to be an isomorphism of $E(M)$-modules . Now, since M is discrete and K is compact, it follows by the Ascoli theorem that $H(M,K)$ is compact. Therefore to see that $H(M,W)$ satisfies ACC and DCC on closed $E(M)$-submodules, it suffices by Lemma 7 to show that so do $H(M,K)$ and $H(M,V) \times \prod_{p\in S_1} H(M,L_p)$. For this purpose, we will consider $H(M,K)$ and $H(M,V) \times \prod_{p\in S_1} H(M,L_p)$ as vector spaces over $\mathbb{Q}$, by using the inclusion $\lambda \mapsto \lambda I_n$ of $\mathbb{Q}$ into $\mathbb{M}_n(\mathbb{Q}) \cong E(M)$. It is then clear that the closed $E(M)$-submodules of $H(M,K)$ and those of $H(M,V)\times\prod_{p\in S_1} H(M,L_p)$ are closed $\mathbb{Q}$-subspaces, so it will suffice to show that $H(M,K)$ and $H(M,V)\times\prod_{p\in S_1} H(M,L_p)$ satisfy both ACC and DCC on closed $\mathbb{Q}$-subspaces. Now, since $H(\mathbb{Q},\mathbb{Q}^*) \cong \mathbb{Q}^*$, $H(\mathbb{Q},\mathbb{R}) \cong \mathbb{R}$, and $H(\mathbb{Q},\mathbb{Q}_p) \cong \mathbb{Q}_p$ for all $p \in \mathbb{P}$, we deduce from [8, (23.34)(c, d)] that

$$H(M,K) \cong (\mathbb{Q}^*)^{nm} \ \text{ and } \ H(M,V) \times \prod_{p\in S_1} H(M,L_p) \cong \mathbb{R}^{nd} \times \prod_{p\in S_1} \mathbb{Q}_p^{nl(p)}$$

as topological groups, and hence as topological vector spaces over $\mathbb{Q}$. It follows from Lemma 6 that both $H(M,K)$ and $H(M,V) \times \prod_{p\in S_1} H(M,L_p)$ satisfy ACC and DCC on closed $\mathbb{Q}$-subspaces. This proves that $H(M,W)\times E(M)$ satisfies ACC and DCC on closed $E(M)$-submodules.

Further, we consider the case of $E(W)$. Since $K \oplus L = k(W)$ is topologically fully invariant in W, we deduce from (2) and Lemma 3 that

$$E(W) \cong \begin{pmatrix} E(K\oplus L) & H(V, K\oplus L) \\ 0 & E(V) \end{pmatrix}.$$

By Lemma 2, we have to show that the modules $E(K\oplus L)_{E(K\oplus L)}$ and $\big(H(V,K\oplus L)\times E(V)\big)_{E(V)}$ satisfy ACC and DCC on closed submodules.

First we consider the case of $\big(H(V,K\oplus L)\times E(V)\big)_{E(V)}$. By use of the inclusion $\lambda\mapsto\lambda I_d\in\mathbb{M}_d(\mathbb{R})\cong E(V)$, the group $H(V,K\oplus L)\times E(V)$ can be given a topological vector space structure over the field of reals, $\mathbb{R}$. It is clear that every $E(V)$-submodules of $H(V,K\oplus L)\times E(V)$ becomes an $\mathbb{R}$-subspace. So to achieve our goal, it suffices to show that $H(V,K\oplus L)\times E(V)$ is of finite dimension. This is clear for $E(V)$. On the other hand, $H(V,K\oplus L)=H(V,K)$ because $V=c(V)$ and $c(L)=\{0\}$. Since, by [8, (23.34)(c,d)], $H(V,K)\cong\mathbb{R}^{md}$ as topological groups and hence as topological $\mathbb{R}$-spaces, $H(V,K)$ has finite dimension as well.

Next consider the case of $E(K\oplus L)=E(K\oplus\oplus_{p\in S_1}L_p)$. We will proceed by induction on $n=card(S_1)$. If $S_1=\varnothing$, then $E(K\oplus L)=E(K)$. Since $E(K)$ and $E(K^*)$ are topologically anti-isomrphic, and since $E(K^*)\cong\mathbb{M}_m(\mathbb{Q})^{opp}$, the fact that $E(K)$ satisfies ACC and DCC on closed right ideals is clear. Assume $S_1=\{p\}$, so $L=L_p$. Since $K=c(K\oplus L_p)$ is topologically fully invariant in $K\oplus L_p$, it follows that

$$E(K\oplus L)=E(K\oplus L_p)\cong\begin{pmatrix}E(K) & H(L_p,K)\\ 0 & E(L_p)\end{pmatrix}.$$

To see that $E(K\oplus L_p)_{E(K\oplus L_p)}$ satisfies ACC and DCC on closed submodules, it suffices to show that so do $E(K)_{E(K)}$ and $\big(H(L_p,K)\times E(L_p)\big)_{E(L_p)}$. The case of $E(K)$ is clear. Further, by use of the inclusion $\lambda\mapsto\lambda I_{l(p)}$ of the field $\mathbb{Q}_p$ of p-adic numbers into $\mathbb{M}_{l(p)}(\mathbb{Q}_p)\cong E(L_p)$, the group $H(L_p,K)\times E(L_p)$ can be given a vector space structure over $\mathbb{Q}_p$. Since every $E(L_p)$-submodule of $\big(H(L_p,K)\times E(L_p)\big)_{E(L_p)}$ is a $\mathbb{Q}_p$-vector space, it suffices to show that $\big(H(L_p,K)\times E(L_p)\big)_{\mathbb{Q}_p}$ has finite dimension. This is clear for $E(L_p)_{\mathbb{Q}_p}$ because $E(L_p)\cong\mathbb{M}_{l(p)}(\mathbb{Q}_p)$. Also, since $H(L_p,K)\cong H(K^*,L_p^*)\cong H(\mathbb{Q},\mathbb{Q}_p)^{ml(p)}\cong\mathbb{Q}_p^{ml(p)}$, we have $\dim_{\mathbb{Q}_p}H(L_p,K)=ml(p)$, proving the case $n=1$. Assume $n\geq 2$ and that for every proper subset S' of S_1, the ring $E(K\oplus\oplus_{p\in S'}L_p)$ satisfies ACC and DCC on closed right ideals. Pick any $p\in S_1$. We have

$$E(K\oplus L)\cong\begin{pmatrix}E(K\oplus\oplus_{q\in S_1\setminus\{p\}}L_q) & H(L_p,K\oplus\oplus_{q\in S_1\setminus\{p\}}L_q)\\ 0 & E(L_p)\end{pmatrix}.$$

By the induction hypothesis, the ring $E(K\oplus\oplus_{q\in S_1\setminus\{p\}}L_q)$ satisfies ACC and DCC on closed right ideals. Observing that

$$H(L_p,K\oplus\oplus_{q\in S_1\setminus\{p\}}L_q)=H(L_p,K),$$

we conclude from the preceding case that $H(L_p,K\oplus\oplus_{q\in S_1\setminus\{p\}}L_q)_{E(L_p)}$ satisfies ACC and DCC on closed submodules, Consequently, Lemma 2 is applicable, and the proof is complete. □

Corollary 3. *For a group $X\in\mathcal{L}$, the following statements are equivalent:*

(i) $E(X)$ satisfies both ACC and DCC on closed left ideals.

(ii) $E(X)$ *satisfies DCC on closed left ideals.*

(iii) $E(X)$ *satisfies DCC on topologically principal left ideals, i.e. ideals of the form* $\overline{E(X)f}$ *with* $f \in E(X)$.

(iv) $X \cong \mathbb{R}^d \times \mathbb{Q}^n \times (\mathbb{Q}^*)^m \times \prod_{p \in S_1} \mathbb{Q}_p^{l(p)} \times \prod_{p \in S_2} \prod_{i=0}^{k(p)} \mathbb{Z}(p^{r_i(p)})$, *where* S_1, S_2 *are finite subsets of* $\mathbb{P}$, *and* d, n, m, *the* $k(p)$*'s, the* $r_i(p)$*'s and the* $l(p)$*'s are natural numbers.*

In particular, $E(X)$ *satisfies DCC on closed left ideals if and only if it satisfies DCC on closed right ideals.*

Proof. The assertion follows from the fact that $E(X)$ and $E(X^*)$ are topologically anti-isomorphic. □

Specializing to the case of discrete groups, we see that the result of L. Fuchs and F. Szász, mentioned in Introduction, can be supplemented as follows.

Corollary 4. *For a discrete group* $X \in \mathcal{L}$, *the following statements are equivalent:*

(i) $E(X)$ *is right* (*respectively, left*) *artinian.*

(ii) $E(X)$ *satisfies DCC on principal right* (*respectively, left*) *ideals.*

(iii) $E(X)$ *satisfies DCC on closed right* (*respectively, left*) *ideals.*

(iv) $E(X)$ *satisfies DCC on topologically principal right* (*respectively, left*) *ideals.*

(v) $X \cong \mathbb{Q}^n \times \prod_{p \in S} \mathbb{Z}(p^{k(p)})$, *where* $n \in \mathbb{N}$, S *is a finite subset of* $\mathbb{P}$ *and* $k(p) \in \mathbb{N}$ *for all* $p \in S$.

Proof. Since (i) ⇒ (ii) ⇒ (iv) ⇒ (v) and (i) ⇒ (iii) ⇒ (iv) ⇒ (v), it remains to apply [7, Theorem 111.3]. □

In the following, we drop the assumption that the ideals are closed. First, we consider the problem of determining the groups $X \in \mathcal{L}$ for which the ring $E(X)$ is right (respectively, left) artinian. We need the following

Lemma 8. *Let* Y *be one of the groups* $\mathbb{R}^d$, $(\mathbb{Q}^*)^m$, *or* $\mathbb{Q}_p^{l(p)}$, *where* $d, m, l(p) \in \mathbb{N}_0$ *and* $p \in \mathbb{P}$. *For any* $n \in \mathbb{N}_0$, *the module* $H(\mathbb{Q}^n, Y)_{E(\mathbb{Q}^n)}$ *fails to be artinian.*

Proof. Let C be a $\mathbb{Q}$-basis of Y and $\{\gamma_k \mid k \in \mathbb{N}\}$ a countable subset of C. For $i \in \mathbb{N}$, let

$$H_i = \{h \in H(\mathbb{Q}^n, Y) \mid \mathrm{im}(h) \subset \langle \gamma_k \mid k \geq i \rangle_{\mathbb{Q}}\},$$

where $\langle \gamma_k \mid k \geq i \rangle_{\mathbb{Q}}$ is the $\mathbb{Q}$-subspace of Y generated by the γ_k with $k \geq i$. Then $(H_i)_{i \in \mathbb{N}}$ is a strictly decreasing sequence of $E(\mathbb{Q}^n)$-submodules of $H(\mathbb{Q}^n, Y)_{E(\mathbb{Q}^n)}$. □

We have:

Corollary 5. *For a group $X \in \mathcal{L}$, the following statements are equivalent:*

(i) *$E(X)$ is right artinian.*

(ii) *X is topologically isomorphic with one of the groups*
$\mathbb{R}^d \times (\mathbb{Q}^*)^n \times \prod_{p\in S_1} \mathbb{Q}_p^{l(p)} \times \prod_{p\in S_2} \prod_{i=0}^{k(p)} \mathbb{Z}(p^{r_i(p)})$,
or $\mathbb{Q}^n \times \prod_{p\in S_2} \prod_{i=0}^{k(p)} \mathbb{Z}(p^{r_i(p)})$, *where S_1, S_2 are finite subsets of $\mathbb{P}$ and $d, n, k(p), l(p), r_i(p) \in \mathbb{N}$ for all $i \in \{0, \dots, k(p)\}$ and $p \in S_1 \cup S_2$.*

Proof. Assume (i). Then, clearly, $E(X)$ satisfies DCC on closed right ideals, so

$$X \cong \mathbb{R}^d \times \mathbb{Q}^n \times (\mathbb{Q}^*)^m \times \prod_{p\in S_1} \mathbb{Q}_p^{l(p)} \times \prod_{p\in S_2} \prod_{i=0}^{k(p)} \mathbb{Z}(p^{r_i(p)})$$

for some finite subsets S_1, S_2 of $\mathbb{P}$ and natural numbers $d, n, m, k(p), l(p)$, and $r_i(p)$ with $i \in \{0, \dots, k(p)\}$ and $p \in S_1 \cup S_2$. Writing $X = D \oplus T$, where $D \cong \mathbb{R}^d \times \mathbb{Q}^n \times (\mathbb{Q}^*)^m \times \prod_{p\in S_1} \mathbb{Q}_p^{l(p)}$ and $T \cong \prod_{p\in S_2} \prod_{i=0}^{k(p)} \mathbb{Z}(p^{r_i(p)})$. we have $E(X) \cong E(D) \times E(T)$. It follows that $E(D)$ is right artinian. Now, write $D = M \oplus W$, where $M \cong \mathbb{Q}^n$ and $W \cong \mathbb{R}^d \times (\mathbb{Q}^*)^m \times \prod_{p\in S_1} \mathbb{Q}_p^{l(p)}$. Hence $E(D) \cong \begin{pmatrix} E(W) & H(M,W) \\ 0 & E(M) \end{pmatrix}$, where $H(M,W)_{E(\mathbb{Q}^n)}$ is topologically isomorphic with $H(\mathbb{Q}^n, \mathbb{R}^d)_{E(\mathbb{Q}^n)} \times H(\mathbb{Q}^n, (\mathbb{Q}^*)^m)_{E(\mathbb{Q}^n)} \times \prod_{p\in S_1} H(\mathbb{Q}^n, \mathbb{Q}_p^{l(p)})_{E(\mathbb{Q}^n)}$, as easily follows from [8, (23,34)(d)]. If M and W were both non-zero, it would follow from Lemma 8 and [10, (1,2)] that $E(D)$ is not right artinian. This contradiction proves (ii).

To see the converse, we have, by Corollary 4, to consider only the case of $X = \mathbb{R}^d \times (\mathbb{Q}^*)^n \times \prod_{p\in S_1} \mathbb{Q}_p^{l(p)} \times \prod_{p\in S_2} \prod_{i=0}^{k(p)} \mathbb{Z}(p^{r_i(p)})$. Then writing $X = C \oplus T$, where $C \cong \mathbb{R}^d \times (\mathbb{Q}^*)^m \times \prod_{p\in S_1} \mathbb{Q}_p^{l(p)}$ and $T \cong \prod_{p\in S_2} \prod_{i=0}^{k(p)} \mathbb{Z}(p^{r_i(p)})$. we have $E(X) \cong E(C) \times E(T)$. Consequently, it suffices to show that $E(C)$ is right artinian. Write $C = V \oplus K \oplus L$, where $V \cong \mathbb{R}^d$, $K \cong (\mathbb{Q}^*)^n$ and $L = \oplus_{p\in S_1} L_p$ with $L_p \cong \mathbb{Q}_p^{l(p)}$ for all $p \in S_1$. Then $E(C) \cong \begin{pmatrix} E(K) & H(V \oplus L, K) \\ 0 & E(V \oplus L) \end{pmatrix}$. Since $E(K) \cong \mathbb{M}_d(\mathbb{Q})^{opp}$ and $E(V \oplus L) \cong \mathbb{M}_d(\mathbb{R}) \times \prod_{p\in S_1} \mathbb{M}_{l(p)}(\mathbb{Q}_p)$, it suffices by [10, (1.2)] to show that $H(V \oplus L, K)_{E(V\oplus L)}$ is artinian. It is clear from [8, (23,34)(c)] that

$$H(V \oplus L, K)_{E(V\oplus L)} \cong H(V, K)_{E(V\oplus L)} \times \prod_{p\in S_1} H(L_p, K)_{E(V\oplus L)},$$

where the scalar multiplication of the modules $H(V, K)_{E(V\oplus L)}$ and respectively $H(L_p, K)_{E(V\oplus L)}$ with $p \in S_1$ is given by using the projection of $E(V \oplus L) \cong E(V) \times \prod_{q\in S_1} E(L_q)$ onto $E(V)$ respectively $E(L_p)$. Thus it suffices to show that $H(V, K)_{E(V)}$ and respectively $H(L_p, K)_{E(L_p)}$ with $p \in S_1$ are artinian. Now, since the field $\mathbb{R}$ embeds in $E(V)$ and the field $\mathbb{Q}_p$ embeds in $E(L_p)$, $H(V, K)$ can be considered as a vector space over $\mathbb{R}$ and $H(L_p, K)_{E(L_p)}$ as a vector space over $\mathbb{Q}_p$. The conclusion follows because these spaces are finite dimensional. □

Corollary 6. *For a group $X \in \mathcal{L}$, the following statements are equivalent:*

(i) *$E(X)$ is left artinian.*

(ii) *X is topologically isomorphic with one of the groups*

$\mathbb{R}^d \times \mathbb{Q}^n \times \prod_{p\in S_1} \mathbb{Q}_p^{l(p)} \times \prod_{p\in S_2} \prod_{i=0}^{k(p)} \mathbb{Z}(p^{r_i(p)})$,

or $(\mathbb{Q}^*)^n \times \prod_{p\in S_2} \prod_{i=0}^{k(p)} \mathbb{Z}(p^{r_i(p)})$, *where S_1, S_2 are finite subsets of $\mathbb{P}$ and $d, n, k(p), l(p), r_i(p) \in \mathbb{N}$ for all $i \in \{0, \dots, k(p)\}$ and $p \in S_1 \cup S_2$.*

Proof. Since $E(X)$ and $E(X^*)$ are topologically anti-isomorphic, the assertion follows from Corollary 5 and duality. □

We close the paper by determining the groups $X \in \mathcal{L}$ with the property that $E(X)$ satisfies DCC on principal right (respectively, left) ideals. It turns out that this last condition on $E(X)$ is equivalent to those of Theorem 5. First we establish the following

Lemma 9. *Let $X = \mathbb{Q}^n$ and $Y = \mathbb{R}^d \times (\mathbb{Q}^*)^m \times \prod_{p\in S_1} \mathbb{Q}_p^{l(p)}$, where S is a subset of $\mathbb{P}$ and $d, n, m,$ and $l(p)$ for $p \in S$ are natural numbers. If $u, v \in H(X, Y)$ satisfy $v = u \circ w$ for some $w \in E(X)$ and $\dim_{\mathbb{Q}} \operatorname{im}(v) = \dim_{\mathbb{Q}} \operatorname{im}(u)$, then $v = u \circ w'$ for some invertible $w' \in E(X)$.*

Proof. It is clear that the morphisms in $H(X, Y)$ are $\mathbb{Q}$-linear mappings. Since $\dim \operatorname{im}(v) = \dim \operatorname{im}(u)$, it follows by rank-nullity connection [14, Theorem 2.12] that $\ker(u)$ and $\ker(v)$ have the same dimension, say k. Let $e_1, \dots, e_n$ and $e'_1, \dots, e'_n$ be bases in X such that $e_1, \dots, e_k$ is a basis in $ker(u)$ and $e'_1, \dots, e'_k$ is a basis in $ker(v)$. Clearly, $v(e'_i) = u(w(e'_i))$ for all $i = 1, \dots, n$. We define $w' \in E(X)$ by setting

$$w'(e'_i) = \begin{cases} e_i, & \text{if} \quad i = 1, \dots, k; \\ w(e_i), & \text{if} \quad i = k+1, \dots, n. \end{cases}$$

Then w' is invertible and $(u \circ w')(e'_i) = v(e'_i)$ for all $i = 1, \dots, n$, so $v = u \circ w'$. □

We have:

Corollary 7. *For a group $X \in \mathcal{L}$, the following statements are equivalent:*

(i) *$E(X)$ satisfies DCC on principal right (respectively, left) ideals.*

(ii) *$X \cong \mathbb{R}^d \times \mathbb{Q}^n \times (\mathbb{Q}^*)^m \times \prod_{p\in S_1} \mathbb{Q}_p^{l(p)} \times \prod_{p\in S_2} \prod_{i=0}^{k(p)} \mathbb{Z}(p^{r_i(p)})$, where S_1, S_2 are disjoint finite subsets of $\mathbb{P}$, and $d, n, m,$ the $k(p)$'s, the $r_i(p)$'s and the $l(p)$'s are natural numbers.*

Proof. The fact that (i) implies (ii) follows from Theorem 5. Assume (ii) and write $X = D \oplus T$, where $D \cong \mathbb{R}^d \times \mathbb{Q}^n \times (\mathbb{Q}^*)^m \times \prod_{p \in S_1} \mathbb{Q}_p^{l(p)}$ and $T \cong \prod_{p \in S_2} \prod_{i=0}^{k(p)} \mathbb{Z}(p^{r_i(p)})$. Since $E(X) \cong E(D) \times E(T)$, it suffices to show that $E(D)$ satisfies DCC on principal right (respectively, left) ideals. We will first consider the case of principal right ideals. Write $D = M \oplus W$, where

$$M \cong \mathbb{Q}^n \quad \text{and} \quad W \cong \mathbb{R}^d \times (\mathbb{Q}^*)^m \times \prod_{p \in S_1} \mathbb{Q}_p^{l(p)}.$$

Since W is topologically fully invariant in D, it follows that

$$E(D) \cong \begin{pmatrix} E(W) & H(M,W) \\ 0 & E(M) \end{pmatrix}.$$

Let

$$\begin{pmatrix} f_1 & g_1 \\ 0 & h_1 \end{pmatrix} \begin{pmatrix} E(W) & H(M,W) \\ 0 & E(M) \end{pmatrix} \supset \ldots \supset \begin{pmatrix} f_i & g_i \\ 0 & h_i \end{pmatrix} \begin{pmatrix} E(W) & H(M,W) \\ 0 & E(M) \end{pmatrix} \supset \ldots$$

be a descending chain of principal right ideals. For any $i \in \mathbb{N}_0$, we have

$$\begin{pmatrix} f_i & g_i \\ 0 & h_i \end{pmatrix} \begin{pmatrix} E(W) & H(M,W) \\ 0 & E(M) \end{pmatrix} = \begin{pmatrix} f_iE(W) & f_iH(M,W) + g_iE(M) \\ 0 & h_iE(M) \end{pmatrix},$$

so $\big(f_iE(W)\big)_i$, $\big(f_iH(M,W) + g_iE(M)\big)_i$, and respectively $\big(h_iE(M)\big)_i$ are descending chains of submodules in $E(W)_{E(W)}$, $H(M,W)_{E(M)}$, and respectively $E(M)_{E(M)}$. Moreover, the chain $\big(f_iH(M,W)\big)_i$ of submodules of $H(M,W)_{E(M)}$ decreases as well, because so does the chain $\big(f_iE(W)\big)_i$. Now, since $E(W)$ and $E(M)$ are artinian rings by Corollary 5, the chains $\big(f_iE(W)\big)_i$ and $\big(h_iE(M)\big)_i$ are stationary. It remains to show that the chain $\big(f_iH(M,W) + g_iE(M)\big)_i$ stabilises as well. Fix any $i_0 \in \mathbb{N}_0$ such that $f_iE(W) = f_{i_0}E(W)$ for all $i \geq i_0$. Using this representation, we get easily $f_iH(M,W) = f_{i_0}H(M,W)$ for all $i \geq i_0$. Observe also that, without loss of generality, we may consider $g_iE(M) \supset g_{i+1}E(M)$ for all $i \geq i_0$. Indeed, given any such i, we can write $g_{i+1} = f_i \circ u_i + g_i \circ v_i$ for some $u_i, v_i \in E(M)$. It follows easily that, for $g'_{i+1} = g_i \circ v_i$, we have

$$f_{i+1}H(M,W) + g_{i+1}E(M) = f_{i+1}H(M,W) + g'_{i+1}E(M).$$

Thus, replacing g_{i+1} with g'_{i+1}, we get our claim by induction. Now, we clearly have $\operatorname{im}(g_i) \supset \operatorname{im}(g_{i+1})$, so

$$\dim \operatorname{im}(g_{i_0}) \geq \dim \operatorname{im}(g_{i_0+1}) \geq \ldots,$$

and hence there is $j_0 \geq i_0$ such that $\dim \operatorname{im}(g_i) = \dim \operatorname{im}(g_{j_0})$ for all $i \geq j_0$. It follows from Lemma 9 that for every $i \geq j_0$ there is an invertible $w_i \in E(M)$ such that $g_i = g_{j_0} \circ w_i$, whence $g_{j_0} = g_i \circ w_i^{-1}$. Consequently, the chain $\big(f_iH(M,W) + g_iE(M)\big)_i$ stabilises.

Next we consider the case of left principal ideals. Because of the form of D, it is clear that the preceding argument can be applied to $E(D^*)$ to conclude that $E(D^*)$ satisfies DCC on principal right ideals. As $E(D)$ and $E(D^*)$ are topologically anti-isomorphic, it follows that $E(D)$ must satisfy DCC on principal left ideals. □

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Valeriu Popa
Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
Academiei str. 5, MD-2028, Chişinău
Moldova
E-mail: *vpopa@math.md*

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The Lyapunov quantities and the center conditions for a class of bidimensional polynomial systems of differential equations with nonlinearities of the fourth degree

Iurie Calin, Stanislav Ciubotaru

Abstract. For the autonomous bidimensional polynomial systems of differential equations with nonlinearities of the fourth degree the $GL(2,\mathbb{R})$-invariant recurrence equations for determination of the Lyapunov quantities were established. Moreover, the general form of Lyapunov quantities for the mentioned systems is obtained. For a class of such systems the necessary and sufficient $GL(2,\mathbb{R})$-invariant conditions for the existence of center are given.

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Let us consider the system of differential equations with nonlinearities of the fourth degree

$$\frac{dx}{dt} = \mathbf{P}_1(x,y) + \mathbf{P}_4(x,y) = \mathbf{P}(x,y), \quad \frac{dy}{dt} = \mathbf{Q}_1(x,y) + \mathbf{Q}_4(x,y) = \mathbf{Q}(x,y), \tag{1}$$

where $\mathbf{P}_i(x,y)$, $\mathbf{Q}_i(x,y)$ are homogeneous polynomials of degree i in x and y with real coefficients.

The goal of this paper is to determine the invariant recurrence formulas for construction of the Lyapunov quantities for the system of differential equations with nonlinearities of the fourth degree and to establish the invariant center conditions for a class of these systems. The center-focus problem is one of the most important problem in the Qualitative Theory of Differential Equations. This problem is completely solved only for the bidimensional quadratic systems and for the systems with nonlinearities of the third degree [1–3]. Also, this problem was solved for some classes of cubic differential systems [4–7]. In [8] the center problem for a linear center perturbed by homogeneous polynomials, more exactly for the systems of the form

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -x + \mathbf{Q}_4(x,y)$$

was solved. In [9], the authors give some sufficient conditions for the integrability in polar coordinates of a bidimensional polynomial systems with linear part of center type and non-linear part given by homogeneous polynomials of the fourth degree.

Also they establish a conjecture that if it turns to be true then the integrable cases they found are the only possible ones. In [10] the author gives some center conditions for a class of bidimensional polynomial systems of the fourth degree.

1 Definitions and notations

The system (1) can be written in the following coefficient form:

$$\begin{aligned}\frac{dx}{dt} &= \mathrm{c}x + \mathrm{d}y + \mathrm{g}x^4 + 4\mathrm{h}x^3y + 6\mathrm{k}x^2y^2 + 4\mathrm{l}xy^3 + \mathrm{m}y^4,\\ \frac{dy}{dt} &= \mathrm{e}x + \mathrm{f}y + \mathrm{n}x^4 + 4\mathrm{p}x^3y + 6\mathrm{q}x^2y^2 + 4\mathrm{r}xy^3 + \mathrm{s}y^4.\end{aligned} \tag{2}$$

We denote by A the 14-dimensional coefficient space of the system (1), by $\mathbf{a} \in A$ the vector of coefficients $\mathbf{a} = (\mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{f}, \mathrm{g}, \mathrm{h}, \mathrm{k}, \mathrm{l}, \mathrm{m}, \mathrm{n}, \mathrm{p}, \mathrm{q}, \mathrm{r}, \mathrm{s})$, by $\boldsymbol{q} \in \mathcal{Q} \subseteq Aff(2, \mathbb{R})$ a nondegenerate linear transformation of the phase plane of system (1), by $\mathbf{q}$ the transformation matrix and by $r_{\boldsymbol{q}}(\mathbf{a})$ the linear representation of the coefficients of transformed system in the space A.

Definition 1 (see [11, 12])**.** *A polynomial* $\mathcal{K}(\mathbf{a}, \mathbf{x})$ *in coefficients of system* (1) *and coordinates of the vector* $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ *is called a comitant of system* (1) *with respect to the group* $\mathcal{Q}$ *if there exists a function* $\lambda : \mathcal{Q} \to \mathbb{R}$ *such that*

$$\mathcal{K}(r_{\boldsymbol{q}}(\mathbf{a}), \mathbf{q}\mathbf{x}) \equiv \lambda(\boldsymbol{q})\mathcal{K}(\mathbf{a}, \mathbf{x})$$

for every $\boldsymbol{q} \in \mathcal{Q}$, $\mathbf{a} \in A$ *and* $\mathbf{x} \in \mathbb{R}^2$.

If $\mathcal{Q}$ is the group $GL(2, \mathbb{R})$ of nondegenerate linear transformations

$$\mathbf{u} = \mathbf{q}\mathbf{x}, \quad \Delta_{\mathbf{q}} = \det \mathbf{q} \neq 0 \tag{3}$$

of the phase plane of system (1), where $\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}$ is a vector of new phase variables and $\mathbf{q} = \begin{pmatrix} q_1^1 & q_2^1 \\ q_1^2 & q_2^2 \end{pmatrix}$ is the transformation matrix, then the comitant is called $GL(2, \mathbb{R})$-comitant or center-affine comitant. In what follows only $GL(2, \mathbb{R})$-comitants are considered. If a comitant does not depend on coordinates of the vector $\mathbf{x}$, then it is called invariant.

The function $\lambda(\boldsymbol{q})$ is called a multiplicator. It is known [11] that the function $\lambda(\boldsymbol{q})$ has the form $\lambda(\boldsymbol{q}) = \Delta_{\mathbf{q}}^{-\chi}$, where χ is an integer, which is called the weight of the comitant $\mathcal{K}(\mathbf{a}, \mathbf{x})$. If $\chi = 0$, then the comitant is called absolute, otherwise it is called relative.

We say that a comitant $\mathcal{K}(\mathbf{a}, \mathbf{x})$ has the character $(\rho;\ \chi;\ \delta)$ if it has the weight χ, the degree δ with respect to the coefficients of the system (1) and the degree ρ with respect to the coordinates of the vector $\mathbf{x}$.

Definition 2 (see [13]). *Let φ and ψ be homogeneous polynomials in coordinates of the vector* $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ *of the degrees ρ_1 and ρ_2, respectively. The polynomial*

$$(\varphi, \psi)^{(j)} = \frac{(\rho_1 - j)!(\rho_2 - j)!}{\rho_1!\rho_2!} \sum_{i=0}^{j} (-1)^i \binom{j}{i} \frac{\partial^j \varphi}{\partial x^{j-i} \partial y^i} \frac{\partial^j \psi}{\partial x^i \partial y^{j-i}}$$

is called the transvectant of index j of polynomials φ and ψ.

Using this formula we have the following remarks.

Remark 1 (see [14]). *If polynomials φ and ψ are $GL(2,\mathbb{R})$-comitants of system* (1) *with the characters $(\rho_\varphi;\ \chi_\varphi;\ \delta_\varphi)$ and $(\rho_\psi;\ \chi_\psi;\ \delta_\psi)$, respectively, then the transvectant of index $j \le \min\{\rho_\varphi, \rho_\psi\}$ is a $GL(2,\mathbb{R})$-comitant of system* (1) *with the character $(\rho_\varphi + \rho_\psi - 2j;\ \chi_\varphi + \chi_\psi + j;\ \delta_\varphi + d_\psi)$. If $j > \min\{\rho_\varphi, \rho_\psi\}$, then $(\varphi, \psi)^{(j)} = 0$.*

Remark 2. *If homogeneous polynomials f, g, φ and ψ have the degrees m, n, μ and 0 (m, n, $\mu \in \mathbb{N}^*$), respectively, with respect to x and y and $l, q \in \mathbb{N}$, $\alpha \in \mathbb{R}$, then*

a) $(\alpha f, g)^{(k)} = (f, \alpha g)^{(k)} = \alpha (f, g)^{(k)}$, ***b)*** $(f^q, f)^{(2l+1)} = 0$,

c) $(f + g, \varphi)^{(k)} = (f, \varphi)^{(k)} + (g, \varphi)^{(k)}$, ***d)*** $(\psi, f)^{(k)} = 0$,

e) $(f \cdot g, \varphi)^{(1)} = \dfrac{m}{m+n}(f, \varphi)^{(1)} g + \dfrac{n}{m+n}(g, \varphi)^{(1)} f.$

Remark 3. *If homogeneous polynomials f and φ have the degrees $m \in N^*$ and 2, respectively, with respect to x and y, then*

$$((f, \varphi)^{(1)}, \varphi)^{(1)} = \frac{m-1}{m}(f, \varphi)^{(2)} \varphi - \frac{1}{2} f (\varphi, \varphi)^{(2)}.$$

The $GL(2,\mathbb{R})$-comitants of the first degree with respect to the coefficients of the system (1) have the form

$$R_i = \mathbf{P}_i(x, y) y - \mathbf{Q}_i(x, y) x,\ \ S_i = \frac{1}{i}\left(\frac{\partial \mathbf{P}_i(x, y)}{\partial x} + \frac{\partial \mathbf{Q}_i(x, y)}{\partial y} \right),\ \ i = 1, 4. \qquad (4)$$

By using the comitants R_i and S_i, $i = 1, 4$, the system (1) can be written [15] in the form

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{2}\frac{\partial R_1}{\partial y} + \frac{1}{2} S_1 x + \frac{1}{5}\frac{\partial R_4}{\partial y} + \frac{4}{5} S_4 x, \\ \frac{dy}{dt} &= -\frac{1}{2}\frac{\partial R_1}{\partial x} + \frac{1}{2} S_1 y - \frac{1}{5}\frac{\partial R_4}{\partial x} + \frac{4}{5} S_4 y. \end{aligned} \qquad (5)$$

For every homogeneous $GL(2,\mathbb{R})$-comitant $\mathcal{K}(x, y)$ with degree $s \in \mathbb{N}^*$ of the system (1) from (5) we obtain the total derivative of $\mathcal{K}(x, y)$ with respect to t [16]:

$$\frac{d\mathcal{K}}{dt} = \frac{\partial \mathcal{K}}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial \mathcal{K}}{\partial y} \cdot \frac{dy}{dt} = \frac{\partial \mathcal{K}}{\partial x}\left(\frac{1}{2}\frac{\partial R_1}{\partial y} + \frac{1}{2} S_1 x + \frac{1}{5}\frac{\partial R_4}{\partial y} + \frac{4}{5} S_4 x \right) +$$

$$+\frac{\partial \mathcal{K}}{\partial y}\left(-\frac{1}{2}\frac{\partial R_1}{\partial x}+\frac{1}{2}S_1 y-\frac{1}{5}\frac{\partial R_4}{\partial x}+\frac{4}{5}S_4 y\right)= \tag{6}$$
$$= s(\mathcal{K},R_1)^{(1)}+\frac{s}{2}\mathcal{K}S_1+s(\mathcal{K},R_4)^{(1)}+\frac{4s}{5}\mathcal{K}S_4,$$

where $(\mathcal{K},R_i)^{(1)}$ is a Jacobian (the transvectant of the first index) of $GL(2,\mathbb{R})$-comitants $\mathcal{K}$ and R_i. The representation (6) shows that the derivative with respect to t of every homogeneous $GL(2,\mathbb{R})$-comitant with the degree $s\geq 1$ of the system (1) is a $GL(2,\mathbb{R})$-comitant too.

By using the comitants R_i and S_i $(i=1,4)$, and the notion of the transvectant the following $GL(2,\mathbb{R})$-comitants and invariants of the system (1) were constructed (in the list below, the bracket "$[\![$" is used in order to avoid placing the otherwise necessary parenthesis "("):

$$I_1=S_1,\quad I_2=(R_1,R_1)^{(2)},\quad I_3=[\![S_4,R_1)^{(2)},R_1)^{(1)},(S_4,R_1)^{(2)})^{(1)},$$
$$I_4=[\![R_4,R_1)^{(2)},R_1)^{(2)},R_1)^{(1)},((R_4,R_1)^{(2)},R_1)^{(2)})^{(1)},$$
$$K_1=(S_4,R_1)^{(1)},\quad K_2=((S_4,R_1)^{(2)},R_1)^{(1)},\quad K_3=(R_4,S_4)^{(3)},$$
$$K_4=(K_3^2,S_4)^{(3)},\quad K_5=((K_3,S_4)^{(2)},R_1)^{(2)}$$
$$J_1=((R_4,R_4)^{(4)},R_1)^{(2)},\quad J_2=((R_4,S_4)^{(3)},R_1)^{(2)},\quad J_3=((S_4,S_4)^{(2)},R_1)^{(2)},$$
$$J_4=[\![R_4,R_4)^{(2)},R_1)^{(2)},R_1)^{(2)},R_1)^{(2)},\quad J_5=[\![R_4,S_4)^{(2)},R_1)^{(2)},R_1)^{(2)},$$
$$J_6=(K_4,K_5)^{(1)}.$$

2 Lyapunov quantities for bidimensional polynomial systems of differential equations with nonlinearities of the fourth degree with $S_1=0,\ I_2\neq 0$

We will consider the system (1) with the conditions $S_1=0, I_2>0$. These conditions mean that the eigenvalues of the Jacobian matrix at the singular point $(0,0)$ are pure imaginary, i.e., the system has the center or a weak focus at $(0,0)$. In these conditions the system (1) can be reduced, via a linear transformation and time rescaling, to the system

$$\frac{dx}{dt}=y+\mathbf{P}_4(x,y),\qquad \frac{dy}{dt}=-x+\mathbf{Q}_4(x,y), \tag{7}$$

which can be written in the form

$$\frac{dx}{dt}=\frac{1}{2}\frac{\partial R_1}{\partial y}+\frac{1}{5}\frac{\partial R_4}{\partial y}+\frac{4}{5}S_4 x,\qquad \frac{dy}{dt}=-\frac{1}{2}\frac{\partial R_1}{\partial x}-\frac{1}{5}\frac{\partial R_4}{\partial x}+\frac{4}{5}S_4 y, \tag{8}$$

where $R_1=x^2+y^2$.

Let us consider the formal power series of the form

$$F(x,y)=x^2+y^2+\sum_{j=3}^{\infty}F_j(x,y)$$

where for each j, $F_j(x,y)$ is a homogeneous polynomial of degree j, so that the derivative of $F(x,y)$ along the solutions of the system (7) (or (8)) satisfies

$$\frac{dF(x,y)}{dt} = \sum_{k=2}^{\infty} G_{2k}(x^2+y^2)^k,$$

where G_{2k} are the polynomials in the coefficients of the system (7), called *Lyapunov quantities* [17].

For establishing the center conditions for the system (7) we will determine Lyapunov quantities. The polynomials $F_j(x,y)$ and the constants G_{2k} can be determined from the identity:

$$\frac{\partial\left(x^2+y^2+\sum\limits_{j=3}^{\infty}F_j(x,y)\right)}{\partial x}(y+\mathbf{P}_4(x,y))+$$
$$+\frac{\partial\left(x^2+y^2+\sum\limits_{j=3}^{\infty}F_j(x,y)\right)}{\partial y}(-x+\mathbf{Q}_4(x,y)) \equiv \sum_{k=2}^{\infty} G_{2k}(x^2+y^2)^k. \qquad (9)$$

Because for the system (7) $R_1 = x^2+y^2$ and by using (8), the identity (9) can be written in the form:

$$\frac{\partial\left(R_1+\sum\limits_{j=3}^{\infty}F_j(x,y)\right)}{\partial x}\left(\frac{1}{2}\frac{\partial R_1}{\partial y}+\frac{1}{5}\frac{\partial R_4}{\partial y}+\frac{4}{5}S_4x\right)+$$
$$+\frac{\partial\left(R_1+\sum\limits_{j=3}^{\infty}F_j(x,y)\right)}{\partial y}\left(-\frac{1}{2}\frac{\partial R_1}{\partial x}-\frac{1}{5}\frac{\partial R_4}{\partial x}+\frac{4}{5}S_4y\right) \equiv \sum_{k=2}^{\infty} G_{2k}R_1^k. \qquad (10)$$

Next, we analyze the identity (10) which is more general than the identity (9), taking $S_1 = 0,\ I_2 = (R_1,R_1)^{(2)} \neq 0$. By using the notion of the transvectant and Euler formula, the left side of the identity (10) can be written into the form:

$$\frac{1}{5}\left(\frac{\partial R_1}{\partial x}\cdot\frac{\partial R_4}{\partial y}-\frac{\partial R_1}{\partial y}\cdot\frac{\partial R_4}{\partial x}\right)+\frac{4}{5}S_4\left(\frac{\partial R_1}{\partial x}\cdot x-\frac{\partial R_1}{\partial y}\cdot y\right)+$$
$$+\frac{1}{2}\sum_{j=3}^{\infty}\left(\frac{\partial F_j(x,y)}{\partial x}\cdot\frac{\partial R_1}{\partial y}-\frac{\partial F_j(x,y)}{\partial y}\cdot\frac{\partial R_1}{\partial x}\right)+$$
$$+\frac{1}{5}\sum_{j=3}^{\infty}\left(\frac{\partial F_j(x,y)}{\partial x}\cdot\frac{\partial R_4}{\partial y}-\frac{\partial F_j(x,y)}{\partial y}\cdot\frac{\partial R_4}{\partial x}\right)+$$
$$+\frac{4}{5}S_4\sum_{j=3}^{\infty}\left(\frac{\partial F_j(x,y)}{\partial x}\cdot x+\frac{\partial F_j(x,y)}{\partial y}\cdot y\right)=$$

$$= 2(R_1, R_4)^{(1)} + 2 \cdot \frac{4}{5} R_1 S_4 + \sum_{j=3}^{\infty} j \cdot (F_j, R_1)^{(1)} + \sum_{j=3}^{\infty} j \cdot (F_j, R_4)^{(1)} + \frac{4}{5} \sum_{j=3}^{\infty} j \cdot F_j S_4,$$

and the identity (10) is reduced to the form:

$$\sum_{j=3}^{\infty} j \cdot (F_j, R_1)^{(1)} + \sum_{j=2}^{\infty} j \cdot W(F_j) \equiv \sum_{k=2}^{\infty} G_{2k} R_1^k, \tag{11}$$

where $F_2 = R_1$, $W(F_j) = (F_j, R_4)^{(1)} + \frac{4}{5} F_j S_4$.

Equaling in (11) polynomials with the same degree with respect to the coordinates of the vector (x, y), the identity (11) can be reduced to the system of differential equations in partial derivatives:

$$\begin{aligned}
&3(F_3, R_1)^{(1)} = 0,\\
&4(F_4, R_1)^{(1)} = G_4 R_1^2,\\
&5(F_5, R_1)^{(1)} + 2W(F_2) = 0,\\
&6(F_6, R_1)^{(1)} + 3W(F_3) = G_6 R_1^3,\\
&7(F_7, R_1)^{(1)} + 4W(F_4) = 0,\\
&8(F_8, R_1)^{(1)} + 5W(F_5) = G_8 R_1^4,\\
&9(F_9, R_1)^{(1)} + 6W(F_6) = 0,\\
&10(F_{10}, R_1)^{(1)} + 7W(F_7) = G_{10} R_1^5,\\
&11(F_{11}, R_1)^{(1)} + 8W(F_8) = 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&j(F_j, R_1)^{(1)} + (j-3)W(F_{j-3}) = \begin{cases} 0, & \text{for } j = 2l+1,\ l \in \mathbb{N}^*,\\ G_j R_1^{\frac{j}{2}}, & \text{for } j = 2l+2,\ l \in \mathbb{N}^*, \end{cases}\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned} \tag{12}$$

Equations of the form $j(F_j, R_1)^{(1)} = 0$, in the case when j is an odd number, have the solution $F_j \equiv 0$ in the class of homogeneous polynomials with real coefficients. In the case when j is an even number, the equations $j(F_j, R_1)^{(1)} = G_j R_1^{\frac{j}{2}}$ admit the solution of the form $F_j = C R_1^{\frac{j}{2}}$ and then $G_j = 0$, where C is an arbitrary real constant. Assuming $C = 0$, we can consider in this case that $F_j \equiv 0$. From the first equation of the system (12), it follows that $F_3 \equiv 0$. This implies $W(F_3) \equiv 0$ and so, $F_6 \equiv 0$ and $G_6 = 0$. In turn, $F_6 \equiv 0$ implies $W(F_6) \equiv 0$, and then $F_9 \equiv 0$ and so on. From the second equation of the system (12), it follows that $F_4 \equiv 0$ and $G_4 = 0$. From $F_4 \equiv 0$, it turns out that $W(F_4) \equiv 0$ and then $F_7 \equiv 0$. In turn, $F_7 \equiv 0$ implies $W(F_7) \equiv 0$ and then $F_{10} \equiv 0$ and $G_{10} = 0$, and so on. Basing on those mentioned, the system (12) is reduced to the following system:

$$5(F_5, R_1)^{(1)} + 2W(F_2) = 0,$$

$$
\begin{aligned}
&8(F_8, R_1)^{(1)} + 5W(F_5) = G_8 R_1^4,\\
&11(F_{11}, R_1)^{(1)} + 8W(F_8) = 0,\\
&14(F_{14}, R_1)^{(1)} + 11W(F_{11}) = G_{14} R_1^7,\\
&17(F_{17}, R_1)^{(1)} + 14W(F_{14}) = 0,\\
&20(F_{20}, R_1)^{(1)} + 17W(F_{17}) = G_{20} R_1^{10},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&(3m+2)(F_{3m+2}, R_1)^{(1)} + (3m-1)W(F_{3m-1}) =\\
&\qquad = \begin{cases} 0, & \text{for } m = 2l-1,\ l \in \mathbb{N}^*,\\ G_{3m+2} R_1^{\frac{3m+2}{2}}, & \text{for } m = 2l,\ l \in \mathbb{N}^*, \end{cases} \qquad (13)\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
$$

From the system (13) it follows that only the homogeneous polynomials $F_{3m-1}(\mathbf{a}, \mathbf{x})$, $m \in \mathbb{N}^*$ and the Lyapunov quantities $G_{6l+2}(\mathbf{a})$, $l \in \mathbb{N}^*$ participate in solving the center-focus problem for the system (1). By solving consecutively the equations of the system (13) the polynomials F_5, F_8, F_{11}, F_{14}, F_{17}, F_{20}, $\dots$, and respectively the Lyapunov quantities G_8, G_{14}, G_{20}, $\dots$, are determined.

$$
F_5 = \sum_{j=0}^{2} \frac{2 \cdot 5! \cdot 2^{j+1} \cdot R_1^j \cdot [\![W(F_2), \overbrace{R_1)^{(2)}, \dots, R_1)^{(2)}}^{j}, R_1)^{(1)}}{(4-2j)! \cdot \prod_{i=0}^{j} \left((5-2i)^2 \cdot (R_1, R_1)^{(2)}\right)},
$$

$$
F_8 = \sum_{j=0}^{3} \frac{5 \cdot 8! \cdot 2^{j+1} \cdot R_1^j \cdot [\![W(F_5), \overbrace{R_1)^{(2)}, \dots, R_1)^{(2)}}^{j}, R_1)^{(1)}}{(7-2j)! \cdot \prod_{i=0}^{j} \left((8-2i)^2 \cdot (R_1, R_1)^{(2)}\right)},
$$

$$
F_{11} = \sum_{j=0}^{5} \frac{8 \cdot 11! \cdot 2^{j+1} \cdot R_1^j \cdot [\![W(F_8), \overbrace{R_1)^{(2)}, \dots, R_1)^{(2)}}^{j}, R_1)^{(1)}}{(10-2j)! \cdot \prod_{i=0}^{j} \left((11-2i)^2 \cdot (R_1, R_1)^{(2)}\right)},
$$

$$
F_{14} = \sum_{j=0}^{6} \frac{11 \cdot 14! \cdot 2^{j+1} \cdot R_1^j \cdot [\![W(F_{11}), \overbrace{R_1)^{(2)}, \dots, R_1)^{(2)}}^{j}, R_1)^{(1)}}{(13-2j)! \cdot \prod_{i=0}^{j} \left((14-2i)^2 \cdot (R_1, R_1)^{(2)}\right)},
$$

$$
F_{17} = \sum_{j=0}^{8} \frac{14 \cdot 17! \cdot 2^{j+1} \cdot R_1^j \cdot [\![W(F_{14}), \overbrace{R_1)^{(2)}, \dots, R_1)^{(2)}}^{j}, R_1)^{(1)}}{(16-2j)! \cdot \prod_{i=0}^{j} \left((17-2i)^2 \cdot (R_1, R_1)^{(2)}\right)},
$$

$$F_{20} = \sum_{j=0}^{9} \frac{17 \cdot 20! \cdot 2^{j+1} \cdot R_1^j \cdot [\![W(F_{17}), \overbrace{R_1)^{(2)}, \dots, R_1)^{(2)}}^{j}, R_1)^{(1)}}{(19-2j)! \cdot \prod\limits_{i=0}^{j} \Big((20-2i)^2 \cdot (R_1, R_1)^{(2)}\Big)},$$

. . . .

$$F_{3m+2} =$$

$$= \sum_{j=0}^{[\frac{3m+1}{2}]} \frac{(3m-1) \cdot (3m+2)! \cdot 2^{j+1} \cdot R_1^j \cdot [\![W(F_{3m-1}), \overbrace{R_1)^{(2)}, \dots, R_1)^{(2)}}^{j}, R_1)^{(1)}}{(3m-2j+1)! \cdot \prod\limits_{i=0}^{j} \Big((3m-2i+2)^2 \cdot (R_1, R_1)^{(2)}\Big)}, \quad (14)$$

. . . .

where $m \in \mathbb{N}^*$, $W(F_i) = (F_i, R_4)^{(1)} + \frac{4}{5} F_i S_4$.

$$G_8 = \frac{5 \cdot 8! \cdot 2^4 \cdot [\![W(F_5), \overbrace{R_1)^{(2)}, \dots, R_1)^{(2)}}^{4}}{\prod\limits_{i=0}^{3} \Big((8-2i)^2 \cdot (R_1, R_1)^{(2)}\Big)},$$

$$G_{14} = \frac{11 \cdot 14! \cdot 2^7 \cdot [\![W(F_{11}), \overbrace{R_1)^{(2)}, \dots, R_1)^{(2)}}^{7}}{\prod\limits_{i=0}^{6} \Big((14-2i)^2 \cdot (R_1, R_1)^{(2)}\Big)},$$

$$G_{20} = \frac{17 \cdot 20! \cdot 2^{10} \cdot [\![W(F_{17}), \overbrace{R_1)^{(2)}, \dots, R_1)^{(2)}}^{10}}{\prod\limits_{i=0}^{9} \Big((20-2i)^2 \cdot (R_1, R_1)^{(2)}\Big)},$$

. . . .

$$G_{6l+2} =$$

$$= \frac{(6l-1) \cdot (6l+2)! \cdot 2^{3l+1} \cdot [\![W(F_{6l-1}), \overbrace{R_1)^{(2)}, \dots, R_1)^{(2)}}^{3l+1}}{\prod\limits_{i=0}^{3l} \Big((6l-2i+2)^2 \cdot (R_1, R_1)^{(2)}\Big)}, \quad (15)$$

. . . .

where $l \in \mathbb{N}^*$, $W(F_i) = (F_i, R_4)^{(1)} + \frac{4}{5} F_i S_4$.

Next we show that the polynomials F_{3m+2} (14) and Lyapunov quantities G_{6l+2} (15) satisfy the equations of system (13). Replacing in the right side of (13) the expresion for F_{3m+2} (14) and by using Remarks 1, 2 and 3 we obtain:

$$(3m+2)(3m-1)(3m+2)!\times$$
$$\times\sum_{j=0}^{[\frac{3m+1}{2}]}\frac{2^{j+1}\cdot\Big(R_1^j\cdot[\![W(F_{3m-1}),\overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j},R_1)^{(1)},R_1\Big)^{(1)}}{(3m-2j+1)!\cdot\prod_{i=0}^{j}\Big((3m-2i+2)^2\cdot(R_1,R_1)^{(2)}\Big)}+$$
$$+(3m-1)W(F_{3m-1})=$$

applying Remark 2. **e)**, taking $f=R_1^j$, $g=[\![W(F_{3m-1}),\overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j},R_1)^{(1)}$ and $\varphi=R_1$, we obtain

$$=(3m+2)(3m-1)(3m+2)!\times$$
$$\times\sum_{j=0}^{[\frac{3m+1}{2}]}\frac{2^{j+1}}{(3m-2j+1)!\cdot\prod_{i=0}^{j}\Big((3m-2i+2)^2\cdot(R_1,R_1)^{(2)}\Big)}\times$$
$$\times\Bigg[\frac{2j}{3m+2}(R_1^j,R_1)^{(1)}\cdot[\![W(F_{3m-1}),\overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j},R_1)^{(1)}+$$
$$+\frac{3m-2j+2}{3m+2}R_1^j\cdot[\![W(F_{3m-1}),\overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j},R_1)^{(1)},R_1)^{(1)}\Bigg]+$$
$$+(3m-1)W(F_{3m-1})=$$

according to Remark 2. **b)**, the first term in square brackets is equal to zero, because $(R_1^j,R_1)^{(1)}=0$. For the second term, by applying Remark 3, taking $f=[\![W(F_{3m-1}),\overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j}$ and $\varphi=R_1$, we obtain

$$=(3m+2)(3m-1)(3m+2)!\times$$
$$\times\sum_{j=0}^{[\frac{3m+1}{2}]}\frac{2^{j+1}}{(3m-2j+1)!\cdot\prod_{i=0}^{j}\Big((3m-2i+2)^2\cdot(R_1,R_1)^{(2)}\Big)}\times$$
$$\times\Bigg[\frac{(3m-2j+1)(3m-2j+2)}{(3m-2j+2)(3m+2)}R_1^{j+1}\cdot[\![W(F_{3m-1}),\overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j},R_1)^{(2)}-$$

$$-\frac{3m-2j+2}{2(3m+2)}R_1^j\cdot(R_1,R_1)^{(2)}\cdot[\![W(F_{3m-1}),\overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j}\Bigg]+$$

$$+(3m-1)W(F_{3m-1})=$$

$$=(3m-1)(3m+2)!\times$$

$$\times\Bigg[\sum_{j=0}^{[\frac{3m+1}{2}]}\frac{2^{j+1}\cdot(3m-2j+1)\cdot R_1^{j+1}\cdot[\![W(F_{3m-1}),\overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j},R_1)^{(2)}}{(3m-2j+1)!\cdot\prod_{i=0}^{j}\Big((3m-2i+2)^2\cdot(R_1,R_1)^{(2)}\Big)}-$$

$$-\sum_{j=0}^{[\frac{3m+1}{2}]}\frac{2^{j+1}\cdot(3m-2j+2)\cdot R_1^j\cdot(R_1,R_1)^{(2)}\cdot[\![W(F_{3m-1}),\overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j}}{2(3m-2j+1)!\cdot\prod_{i=0}^{j}\Big((3m-2i+2)^2\cdot(R_1,R_1)^{(2)}\Big)}\Bigg]+$$

$$+(3m-1)W(F_{3m-1})=$$

because for $j=0$, the term obtained from the second sum is equal to $-(3m-1)W(F_{3m-1})$, we get

$$=(3m-1)(3m+2)!\times$$

$$\times\Bigg[\sum_{j=0}^{[\frac{3m+1}{2}]}\frac{2^{j+1}\cdot(3m-2j+1)\cdot R_1^{j+1}\cdot[\![W(F_{3m-1}),\overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j+1}}{(3m-2j+1)!\cdot\Big[(R_1,R_1)^{(2)}\Big]^{j+1}\cdot\prod_{i=0}^{j}(3m-2i+2)^2}-$$

$$-\sum_{j=1}^{[\frac{3m+1}{2}]}\frac{2^{j}\cdot(3m-2j+3)\cdot R_1^{j}\cdot[\![W(F_{3m-1}),\overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j}}{(3m-2j+2)!\cdot(3m-2j+3)\cdot\Big[(R_1,R_1)^{(2)}\Big]^{j}\cdot\prod_{i=0}^{j-1}(3m-2i+2)^2}\Bigg]=$$

by changing the sum index in the second sum, we obtain

$$=(3m-1)(3m+2)!\times$$

$$\times\Bigg[\sum_{j=0}^{[\frac{3m+1}{2}]}\frac{2^{j+1}\cdot(3m-2j+1)\cdot R_1^{j+1}\cdot[\![W(F_{3m-1}),\overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j+1}}{(3m-2j+1)!\cdot\Big[(R_1,R_1)^{(2)}\Big]^{j+1}\cdot\prod_{i=0}^{j}(3m-2i+2)^2}-$$

$$-\sum_{j=0}^{[\frac{3m+1}{2}]-1}\frac{2^{j+1}\cdot(3m-2j+1)\cdot R_1^{j+1}\cdot[\![W(F_{3m-1}),\overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{j+1}}{(3m-2j+1)!\cdot\Big[(R_1,R_1)^{(2)}\Big]^{j+1}\cdot\prod_{i=0}^{j}(3m-2i+2)^2}\Bigg]=$$

$$= (3m-1)(3m+2)!\times$$

$$\times\frac{2^{\left[\frac{3m+3}{2}\right]}\cdot\left(3m-2\left[\frac{3m+1}{2}\right]+1\right)\cdot R_1^{\left[\frac{3m+3}{2}\right]}\cdot [\![W(F_{3m-1}),\overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{\left[\frac{3m+3}{2}\right]}}{\left(3m-2\left[\frac{3m+1}{2}\right]+1\right)!\cdot\left[(R_1,R_1)^{(2)}\right]^{\left[\frac{3m+3}{2}\right]}\cdot\prod\limits_{i=0}^{\left[\frac{3m+1}{2}\right]}(3m-2i+2)^2}. \quad (16)$$

If m is an odd number, i.e. $m=2l-1,\ l\in\mathbb{N}^*$, the expression (16) is written in the form:

$$\frac{(6l-4)(6l-1)!\cdot 2^{3l}\cdot 0\cdot R_1^{3l}\cdot [\![W(F_{6l-4}),\overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{3l}}{\left[(R_1,R_1)^{(2)}\right]^{3l}\cdot\prod\limits_{i=0}^{3l-1}(6l-2i-1)^2},$$

where the transvectant

$$[\![W(F_{6l-4}),\overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{3l}$$

is equal to 0, because the degree of comitant $W(F_{6l-4})$ with respect to the coordinates of the vector $\mathbf{x}$ is equal to $6l-1$, but the total index of transvectants with R_1 is equal to $6l$.

If m is an even number, i.e. $m=2l,\ l\in\mathbb{N}^*$, the expression (16) is written in the form:

$$\frac{(6l-1)(6l+2)!\cdot 2^{3l+1}\cdot R_1^{3l+1}\cdot [\![W(F_{6l-1}),\overbrace{R_1)^{(2)},\ldots,R_1)^{(2)}}^{3l+1}}{\left[(R_1,R_1)^{(2)}\right]^{3l+1}\cdot\prod\limits_{i=0}^{3l}(6l-2i+2)^2}=G_{6l+2}\cdot R_1^{3l+1}, \quad (17)$$

where G_{6l+2} coincides with the expression (15). So, for establishing the Lyapunov quantities for the system (1) with the conditions $S_1=0, I_2\neq 0$, the formulas (14) and (15) can be used.

Notice that, when $m=2l-1,\ l\in\mathbb{N}^*$, the respective equations of the system (13) have a unique solution with respect to F_{3m+2}, i.e. in this case F_{3m+2} are determined unambiguously. In the case $m=2l,\ l\in\mathbb{N}^*$, the solutions of respective equations of the system (13) with respect to F_{3m+2} are determined up to a term of the form $CR_1^{\frac{3m+2}{2}}$, where C is an arbitrary real constant. This implies that Lyapunov quantities $G_{6l+2},\ l\in\mathbb{N}^*$, are not determined unambiguously.

Notice that the numerators in formulas (14) and (15) are expressed by transvectants constructed by using the comitants R_1, R_4 and S_4, but the denominators represent the powers of invariant $I_2=(R_1,R_1)^{(2)}$. Based on Remark 1, it follows that the numerators in formulas (14) and (15) are $GL(2,\mathbb{R})$-comitants for the system

(1). Since the $GL(2,\mathbb{R})$-comitants in (15) does not depend on the coordinates of the vector $\mathbf{x}$ it follows they are $GL(2,\mathbb{R})$-invariants for the system (1).

On the above analysis, it results that the system (1), with the conditions $S_1 = 0, I_2 \neq 0$ and all Lyapunov quantities (15) being equal to zero, admits first formal integral of the form:

$$F(x,y) = \sum_{m=0}^{\infty} F_{3m+2}(x,y),$$

where $F_2(x,y) = R_1$, but $F_{3m+2}(x,y),\ m \in \mathbb{N}^*$ are expressions (14).

3 The center conditions for the class of bidimensional polynomial systems of differential equations with nonlinearities of the fourth degree with $S_1 = 0,\ I_2 > 0,\ I_3 = I_4 = 0$

Let us consider the bidimensional polynomial system of differential equations with nonlinearities of the fourth degree (1).

By using the comitants R_i and S_i $(i = \overline{1,4})$ the system (1) can be written in the form (5).

We will consider the system (5) (or (1)) with the conditions $S_1 = 0$, $I_2 > 0$ which has a center or a weak focus at $(0,0)$.

Remark 4. *If $R_4 \cdot S_4 \equiv 0$ then the system* (5) *(or* (1)*) with $S_1 = 0$ and $I_2 > 0$ has a singular point of the center type at the origin of coordinates.*

Indeed, if $R_4 \equiv 0$, then the system (5) has the invariant algebraic curve

$$\Phi(x,y) = 32R_1 \cdot K_2 + 8I_2 \cdot K_1 - 5I_2^2 = 0$$

and the first integral

$$|\Phi|^{\frac{2}{3}} \cdot |R_1|^{-1} = c_1,$$

where c_1 is a real constant.

If $S_4 \equiv 0$, then the system (5) has the first integral:

$$5R_1 + 2R_4 = c_2,$$

where c_2 is a real constant.

For the system (1) with $S_1 = 0$, $I_2 > 0$ and $I_3 = I_4 = 0$ the $GL(2,\mathbb{R})$-invariant conditions for distinguishing between center and focus were established.

Theorem 1. *The system* (1) *with the conditions $S_1 = 0$, $I_2 > 0$ and $I_3 = I_4 = 0$ has the center at the origin of coordinates if and only if the following conditions are fulfilled*

$$G_8 = G_{26} = G_{32} = G_{38} = 0,$$

where G_8, G_{26}, G_{32} and G_{38} are Lyapunov quantities given in (15).

Moreover, the above conditions are equivalent to the following invariant ones:

$$J_5 = J_6 = 0.$$

Proof. *Necessity.* The system (1) (or (2)) with $S_1 = 0$, $I_2 > 0$ can be reduced by a centeraffine transformation and time scaling to the form

$$\begin{aligned}\frac{dx}{dt} &= y + \mathrm{g}x^4 + 4\mathrm{h}x^3y + 6\mathrm{k}x^2y^2 + 4\mathrm{l}xy^3 + \mathrm{m}y^4,\\ \frac{dy}{dt} &= -x + \mathrm{n}x^4 + 4\mathrm{p}x^3y + 6\mathrm{q}x^2y^2 + 4\mathrm{r}xy^3 + \mathrm{s}y^4.\end{aligned} \tag{18}$$

By a transformation of rotation, in the system (18) can be obtained the equality

$$\mathrm{h} + \mathrm{q} = 0. \tag{19}$$

By using the substitutions

$$\begin{aligned}&\mathrm{g} = \frac{4P + 5H}{5}, \quad \mathrm{h} = \frac{10K + 6Q}{10}, \quad \mathrm{k} = \frac{30L + 12R}{30}, \quad \mathrm{l} = \frac{5M + S}{5}, \quad \mathrm{m} = N,\\ &\mathrm{n} = -G, \quad \mathrm{p} = \frac{P - 5H}{5}, \quad \mathrm{q} = \frac{12Q - 30K}{30}, \quad \mathrm{r} = \frac{6R - 10L}{10}, \quad \mathrm{s} = \frac{4S - 5M}{5}\end{aligned}$$

and using (19), the system (18) can be reduced to the form

$$\begin{aligned}\frac{dx}{dt} &= y + \frac{5H + 4P}{5}x^4 + 4Kx^3y + \frac{30L + 12R}{5}x^2y^2 + \frac{20M + 4S}{5}xy^3 + Ny^4,\\ \frac{dy}{dt} &= -x - Gx^4 + \frac{4P - 20H}{5}x^3y - 6Kx^2y^2 + \frac{12R - 20L}{5}xy^3 + \frac{4S - 5M}{5}y^4,\end{aligned} \tag{20}$$

for which

$$\begin{aligned}&R_1 = x^2 + y^2,\\ &R_4 = Gx^5 + 5Hx^4y + 10Kx^3y^2 + 10Lx^2y^3 + 5Mxy^4 + Ny^5,\\ &S_4 = Px^3 + 3Rxy^2 + Sy^3,\\ &I_3 = (P + R)^2 + S^2,\\ &I_4 = (G + 2K + M)^2 + (H + 2L + N)^2.\end{aligned}$$

So, $I_3 = 0$ implies $S = 0$ and $R = -P$, and $I_4 = 0$, implies $G = -2K - M$ and $N = -2L - H$, i.e., the system (1) with $S_1 = 0$, $I_2 > 0$ and $I_3 = I_4 = 0$ can be reduced to the form

$$\begin{aligned}\frac{dx}{dt} &= y + \frac{5H + 4P}{5}x^4 + 4Kx^3y + \frac{30L - 12P}{5}x^2y^2 + 4Mxy^3 - (H + 2L)y^4,\\ \frac{dy}{dt} &= -x + (2K + M)x^4 + \frac{4P - 20H}{5}x^3y - 6Kx^2y^2 - \frac{12P + 20L}{5}xy^3 - My^4,\end{aligned} \tag{21}$$

for which

$$\begin{aligned}&R_4 = -(2K + M)x^5 + 5Hx^4y + 10Kx^3y^2 + 10Lx^2y^3 + 5Mxy^4 - (H + 2L)y^5,\\ &S_4 = Px^3 - 3Pxy^2.\end{aligned}$$

Applying the formulas (14) and (15) for the system (21) we obtain the following expresions for Lyapunov quantities $G_8,\ G_{14},\ G_{20}$:

$$\begin{aligned}
G_8 &= (K+M)P = J_5/4,\\
G_{14} &= J_5\left(405 I_2 J_1 - 2160 I_2 J_2 + 952 I_2 J_3 + 2025 J_4\right)/14400,\\
G_{20} &= J_5\big(2815560 I_2^2 J_1^2 - 19591875 I_2^2 J_1 J_2 + 63518400 I_2^2 J_2^2 + 8637786 I_2^2 J_1 J_3 -\\
&\quad 58484160 I_2^2 J_2 J_3 + 14084096 I_2^2 J_3^2 + 13454100 I_2 J_1 J_4 - 71938125 I_2 J_2 J_4 +\\
&\quad 29031030 I_2 J_3 J_4 - 3118500 J_4^2\big)/414720000.
\end{aligned}$$

Since the condition $G_8 = 0$ for the system (21) is equivalent to the $GL(2,\mathbb{R})$- invariant condition $J_5 = 0$, we obtain the first $GL(2,\mathbb{R})$- invariant necessary condition to have a center at the origin of coordinates of system (1) with $S_1 = 0,\ I_2 > 0$ and $I_3 = I_4 = 0$.

So we have that $G_8 = 0$ implies $G_{14} = G_{20} = 0$. Because for the system (21) $G_8 = (K+M)P$, then the condition $G_8 = 0$ implies $P = 0$ or $K + M = 0$.

If, $P = 0$, then the comitant $S_4 \equiv 0$. In this case, by Remark 4., the system has center at the origin of coordinates.

So, next we consider the situation when $K + M = 0$. In this case, the system (21) is reduced to the system:

$$\begin{aligned}
\frac{dx}{dt} &= y + \frac{5H+4P}{5}x^4 + 4Kx^3y + \frac{30L-12P}{5}x^2y^2 - 4Kxy^3 - (H+2L)y^4,\\
\frac{dy}{dt} &= -x + Kx^4 + \frac{4P-20H}{5}x^3y - 6Kx^2y^2 - \frac{12P+20L}{5}xy^3 + Ky^4. \qquad (22)
\end{aligned}$$

For the system (22) the Lyapunov quantities G_{26}, G_{32}, G_{38}, calculated by using the formulas (14) and (15), have the following form:

$$\begin{aligned}
G_{26} &= F_0F_1F_2F_3F_4/84000000\\
G_{32} &= G_{26}(922393092509 I_2J_1 - 7764307622400 I_2J_2 + 4866278972800 I_2J_3 +\\
&\quad 3192990020695 J_4)/3146766336000 +\\
&\quad 3F_0F_2F_3F_4(H+L)T_1/36700160000 +\\
&\quad F_0F_1F_3F_4(H+L)T_2/3369074688000 -\\
&\quad 221F_0F_1F_2F_4(H+L)T_3/23506452480000 -\\
&\quad 19F_0F_1F_2F_3(H+L)T_4/580123856076800,\\
G_{38} &= G_{26}\big(1260330988434177209628113 I_2^2J_1^2 - 15650227814700317619459000 I_2^2J_1J_2 +\\
&\quad 3961006936844834443936320 I_2^2J_1J_3 - 816812053926570075 2256\cdot 10^3 I_2^2J_2J_3 +\\
&\quad 2369232236068131016396800 I_2^2J_3^2 + 1024560662360577342447 3980 I_2J_1J_4 -\\
&\quad 5406135013075353898294500 I_2J_2J_3 + 1917900060775920639459 3600 I_2J_3J_4 +\\
&\quad 1999503569367527784282 2075 J_4^2\big)/833778038297581977600000 -\\
&\quad F_0F_2F_3F_4(H+L)\big(79683781250(H+L)^4 + 16596426225(H+L)^2T_1 -\\
&\quad 142466T_1^2\big)/465032131379200000 -\\
&\quad F_0F_1F_3F_4(H+L)\big(5162357307858086250(H+L)^4 +
\end{aligned}$$

$$56310112366375(H+L)^2T_2 - 29394738T_2^2\big)/2504577444987591563673600000+$$
$$F_0F_1F_2F_4(H+L)\big(242620599754476562500(H+L)^4+$$
$$11785658137723675(H+L)^2T_3 + 12640691034T_3^2\big)/8656725653336988057600000+$$
$$F_0F_1F_2F_3(H+L)\big(3648566975734071058 0038147(H+L)^4 - 1810577808T_4^2+$$
$$22352124982450552136(H+L)^2T_4)/153408002525451763169034240 0000,$$

where polynomials $F_i, i=\overline{0,4}$, and $T_j, j=\overline{1,4}$, have the forms

$$F_0 = K(-3H^2+16K^2+18HL-27L^2)P,$$
$$F_1 = 45H+45L+8P,$$
$$F_2 = 35H+35L+24P,$$
$$F_3 = 85H+85L+24P,$$
$$F_4 = 665H+665L+116P,$$
$$T_1 = -2051H^2+1584K^2-4894HL-1259L^2,$$
$$T_2 = -373481H^2+994704K^2-1244314HL+123871L^2,$$
$$T_3 = -105177H^2+36368K^2-228538HL-86993L^2,$$
$$T_4 = -215747339H^2+134963680K^2-498976518HL-148265499L^2.$$

If $F_0=0$, then the Lyapunov quantities G_{26}, G_{32} and G_{38} are equal to zero.

If $F_0 \neq 0$, then $G_{26}=0$ if and only if $F_1F_2F_3F_4=0$. If at least two of polynomials $F_i, i=\overline{1,4}$, are equal to zero, then $H+L=0$ and $P=0$ which implies $G_{32}=G_{38}=0$. Moreover, this implies also $F_0=0$.

We claim that even the equality with zero of only one of the polynomials $F_i, i=\overline{1,4}$, together with $G_{32}=G_{38}=0$ also implies $F_0=0$. For the vanishing of G_{26}, we consider the following four cases:

1. $F_1=45H+45L+8P=0$ with $F_2,F_3,F_4\neq 0$,

2. $F_2=35H+35L+24P=0$ with $F_1,F_3,F_4\neq 0$,

3. $F_3=85H+85L+24P=0$ with $F_1,F_2,F_4\neq 0$ and

4. $F_4=665H+665L+116P=0$ with $F_1,F_2,F_3\neq 0$.

Case 1. Let $F_1=45H+45L+8P=0$ and $F_2,F_3,F_4\neq 0$. In this case

$$G_{32} = 3F_0F_2F_3F_4(H+L)T_1/36700160000$$

and for the vanishing of G_{32} we have the following subcases:

1.1. $H+L=0$ and

1.2. $T_1=0$.

Subcase 1.1. If $H+L=0$ then together with the condition $F_1=45H+45L+8P=0$ it leads to $P=0$, which implies the comitant $S_4\equiv 0$. In this case the system has a center at the origin of coordinates.

Subcase 1.2. If $T_1=0$, then G_{38}, up to a numerical factor, has the form $G_{38}=F_0F_2F_3F_4(H+L)^5$. Notice that the Lyapunov quantity G_{38} can be nonzero and this implies that the condition $G_{38}=0$ is a necessary condition for the existence of a center at the origin of coordinates. The condition $G_{38} = 0$ implies $H + L = 0$ then together with the condition $F_1 = 45H + 45L + 8P = 0$ it leads to $P = 0$. In this case the system has a center at the origin of coordinates.

So, in this case for the existence of a center at the origin of coordinates of the phase plane of system (22) the vanishing of Lyapunov quantities G_{26}, G_{32} and G_{38} is necessary , which implies

$$F_0 = K(-3H^2 + 16K^2 + 18HL - 27L^2)P = 0.$$

This condition is equivalent with the following invariant condition

$$J_6 = 16K(-3H^2 + 16K^2 + 18HL - 27L^2)P^5 = 0.$$

Cases 2, 3 and 4 can be analyzed by the same way described above and it leads to the same result. So, we obtain that for the existence of a center at the origin of coordinates of the phase plane of system (21) the realization of the conditions:

$$G_8 = G_{26} = G_{32} = G_{38} = 0$$

is necessary, which leads to the invariant conditions:

$$J_5 = J_6 = 0.$$

Sufficiency. In proving the necessity, it was established that the condition

$$KP\left[(16K^2 - 3(H - 3L)^2\right] = 0 \tag{23}$$

is the necessary one for the existence of a center at the origin of coordinates for the system (22). Next we prove the sufficiency of this condition. Condition (23) is verified if one of the following equalities is fulfilled:

$$(i)\ P = 0; \quad (ii)\ K = 0; \quad (iii)\ K = \frac{\sqrt{3}}{4}(H - 3L); \quad (iv)\ K = -\frac{\sqrt{3}}{4}(H - 3L).$$

Case (i). If $P = 0$, then $S_4 \equiv 0$ and the point $(0;0)$ is a singular point of center type for the system (22). This case was analyzed above.

Case (ii). If $K = 0$, then in this case the system (22) takes the form:

$$\begin{aligned} \frac{dx}{dt} &= y + \frac{5H + 4P}{5}x^4 + \frac{30L - 12P}{5}x^2y^2 - (H + 2L)y^4, \\ \frac{dy}{dt} &= -x + \frac{4P - 20H}{5}x^3y - \frac{12P + 20L}{5}xy^3. \end{aligned} \tag{24}$$

For the system (24), the condition

$$\mathbf{Q}(-x;y)\mathbf{P}(x;y) = -\mathbf{P}(-x;y)\mathbf{Q}(x;y) \tag{25}$$

is fulfilled, i.e. the straight line defined by the equation $x = 0$ is a symmetry axis for the system (24). So, the point $(0;0)$ is a singular point of center type for the system (24), i.e. for the system (22) with $K = 0$.

Case (iii). If $K = \dfrac{\sqrt{3}}{4}(H - 3L)$, then the system (22) takes the form

$$\begin{aligned} \frac{dx}{dt} &= y + \frac{5H+4P}{5}x^4 + (\sqrt{3}H - 3\sqrt{3}L)x^3y + \frac{30L-12P}{5}x^2y^2 - \\ &\quad (\sqrt{3}H - 3\sqrt{3}L)xy^3 - (H+2L)y^4, \\ \frac{dy}{dt} &= -x + \frac{\sqrt{3}H - 3\sqrt{3}L}{4}x^4 + \frac{4P-20H}{5}x^3y - \frac{3\sqrt{3}H - 9\sqrt{3}L}{2}x^2y^2 - \\ &\quad \frac{12P+20L}{5}xy^3 + \frac{\sqrt{3}H - 3\sqrt{3}L}{4}y^4. \end{aligned} \tag{26}$$

The trajectories of the system (26) are symmetric with respect to the straight line defined by the equation $x - \sqrt{3}y = 0$. With the rotation of axes

$$x_1 = x\cos\alpha + y\sin\alpha, \quad y_1 = -x\sin\alpha + y\cos\alpha \tag{27}$$

with the angle $\alpha = -\dfrac{\pi}{3}$, the system (26) becomes as follows:

$$\begin{aligned} \frac{dx_1}{dt} &= y_1 - \frac{5H+45L+16P}{20}{x_1}^4 + \frac{-45H+75L+24P}{10}{x_1}^2{y_1}^2 + \frac{7H-L}{4}{y_1}^4, \\ \frac{dy_1}{dt} &= -x_1 + \frac{5H+45L-4P}{5}{x_1}^3y_1 + \frac{15H-25L+12P}{5}x_1{y_1}^3. \end{aligned} \tag{28}$$

For the system (28) the condition (25) is verified in coordinates of x_1 and y_1, i.e. the straight line defined by the equation $x_1 = 0$ is a symmetry axis for the system (28). Therefore, it follows that the straight line defined by the equation $x - \sqrt{3}y = 0$ is the symmetry axis for the system (26). So, the point $(0;0)$ is a singular point of center type for the system (26), or for the system (22) with $K = \dfrac{\sqrt{3}}{4}(H - 3L)$.

Case (iv). If $K = -\dfrac{\sqrt{3}}{4}(H - 3L)$, then the system (22) takes the form

$$\begin{aligned} \frac{dx}{dt} &= y + \frac{5H+4P}{5}x^4 - (\sqrt{3}H - 3\sqrt{3}L)x^3y + \frac{30L-12P}{5}x^2y^2 + \\ &\quad (\sqrt{3}H - 3\sqrt{3}L)xy^3 - (H+2L)y^4, \\ \frac{dy}{dt} &= -x - \frac{\sqrt{3}H - 3\sqrt{3}L}{4}x^4 + \frac{4P-20H}{5}x^3y + \frac{3\sqrt{3}H - 9\sqrt{3}L}{2}x^2y^2 - \\ &\quad \frac{12P+20L}{5}xy^3 - \frac{\sqrt{3}H - 3\sqrt{3}L}{4}y^4. \end{aligned} \tag{29}$$

The trajectories of system (29) are symmetric with respect to the straight line defined by the equation $x + \sqrt{3}y = 0$. With the rotation of axes (27) with the angle $\alpha = \dfrac{\pi}{3}$, the system (29) becomes like the system (28), for which the line defined by the equation $x_1 = 0$ is a symmetry axis. So, the point $(0;0)$ is a singular point of center type for the system (29), or for the system (22) with $K = -\dfrac{\sqrt{3}}{4}(H - 3L)$.

In such a way the conditions

$$G_8 = G_{26} = G_{32} = G_{38} = 0 \tag{30}$$

or the invariant conditions

$$J_5 = J_6 = 0 \tag{31}$$

are sufficient conditions for the existence of a singular point of center type at the origin of coordinates for the system (21). Because G_8, G_{26}, G_{32}, G_{38}, J_5, J_6 are $GL(2,\mathbb{R})$-invariants and the system (21) was obtained from system (1), with conditions $S_1 = 0$, $I_2 > 0$, $I_3 = I_4 = 0$, by linear transformation and time scaling, it follows that the conditions (30) and (31) are necessary and sufficient for the existence of a singular point of center type at the origin of coordinates for the system (1) with $S_1 = 0$, $I_2 > 0$ and $I_3 = I_4 = 0$.

Theorems 1 is proved.

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Iurie Calin
Institute of Mathematics and Computer Science
Academy of Sciences of Moldova &
Moldova State University, Chişinău
Moldova
E-mail: *iucalin@yahoo.com*

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Stanislav Ciubotaru
Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
E-mail: *stanislav.ciubotaru@yahoo.com*