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Boundedness for Vector-Valued Multilinear Singular Integral Operator on L^p Spaces with Variable Exponent

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Abstract. In this paper, we prove the boundedness for some vector-valued multilinear singular integral operators on L^p spaces with variable exponent by using a sharp estimate of the multilinear operators.

Mathematics subject classification: 42B20, 42B25.
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1 Introduction and Theorems

As the development of the Calderón-Zygmund singular integral operators and their commutators, multilinear singular integral operators have been well studied (see [4, 9, 16–19]). Let T be the Calderón-Zygmund singular integral operator. In [1–3], Cohen and Gosselin studied the L^p $(p > 1)$ boundedness of the multilinear singular integral operator T^A defined by

$$T^A(f)(x) = \int_{R^n} \frac{R_{m+1}(A; x, y)}{|x-y|^m} K(x, y) f(y) dy,$$

where

$$R_{m+1}(A; x, y) = A(x) - \sum_{|\alpha| \le m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha.$$

In [12], Hu and Yang proved a variant sharp estimate for the multilinear singular integral operators. In the last years, a theory of L^p spaces with variable exponent has been developed because of its connections with some questions in fluid dynamics, calculus of variations, differential equations and elasticity(see [5–8,15] and their references). Karlovich and Lerner have studied the boundedness of the commutators of singular integral operators on L^p spaces with variable exponent (see [13]). In this paper, we will study the boundedness properties for some vector-valued multilinear singular integral operators on L^p spaces with variable exponent, whose definition is the following.

Fix $\varepsilon > 0$. Let S and S' be Schwartz space and its dual and $T : S \to S'$ be a linear operator. If there exists a locally integrable function $K(x, y)$ on $R^n \times R^n \setminus \{(x, y) \in R^n \times R^n : x = y\}$ such that

$$T(f)(x) = \int_{R^n} K(x, y) f(y) dy$$

for every bounded and compactly supported function f, where K satisfies:

$$|K(x,y)| \le C|x-y|^{-n}$$

and

$$|K(y,x)-K(z,x)|+|K(x,y)-K(x,z)| \le C|y-z|^{\varepsilon}|x-z|^{-n-\varepsilon}$$

when $2|y-z| \le |x-z|$. Let m_j be positive integers $(j=1,\cdots,l)$, $m_1+\cdots+m_l=m$ and A_j be functions on R^n $(j=1,\cdots,l)$. For $1<s<\infty$, the vector-valued multilinear operator related to T is defined by

$$|T_A(f)(x)|_s = \left(\sum_{i=1}^{\infty} |T_A(f_i)(x)|^s\right)^{1/s},$$

where

$$T_A(f_i)(x) = \int_{R^n} \frac{\prod_{j=1}^{l} R_{m_j+1}(A_j;x,y)}{|x-y|^m} K(x,y) f_i(y)dy$$

and

$$R_{m_j+1}(A_j;x,y) = A_j(x) - \sum_{|\alpha|\le m_j} \frac{1}{\alpha!} D^{\alpha}A_j(y)(x-y)^{\alpha}.$$

We also denote

$$|T(f)(x)|_s = \left(\sum_{i=1}^{\infty} |T(f_i)(x)|^s\right)^{1/s} \text{ and } |f|_s = \left(\sum_{i=1}^{\infty} |f_i(x)|^s\right)^{1/s}.$$

Suppose that $|T|_s$ is weakly (L^1,L^1)-bounded.

Note that when $m=0$, $|T_A|_s$ is just the vector-valued multilinear commutator of T and A (see [19]). While when $m>0$, $|T_A|_s$ is non-trivial generalizations of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been studied by many authors (see [1–4, 9]). In [12], Hu and Yang proved a variant sharp estimate for the multilinear singular integral operators. In [18], Pérez and Trujillo-Gonzalez proved a sharp estimate for some multilinear commutator. The main purpose of this paper is to prove the boundedness for the vector-valued multilinear singular integral operators $|T_A|_s$ on L^p spaces with variable exponent. To do this, we first prove a sharp inequality for the vector-valued multilinear singular integral operators.

Now, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f and $\delta>0$, the sharp function of f is defined by

$$f_{\delta}^{\#}(x) = \sup_{Q\ni x}\left(\frac{1}{|Q|}\int_Q |f(y)-f_Q|^{\delta}dy\right)^{1/\delta},$$

where, and in what follows, $f_Q = |Q|^{-1}\int_Q f(x)dx$. It is well-known that (see [11,20])

$$f_{\delta}^{\#}(x) \approx \sup_{Q\ni x}\inf_{c\in C}\left(\frac{1}{|Q|}\int_Q |f(y)-c|^{\delta}dy\right)^{1/\delta}.$$

We write $f^{\#} = f^{\#}_{\delta}$ if $\delta = 1$. We say that f belongs to $BMO(R^n)$ if $f^{\#}$ belongs to $L^{\infty}(R^n)$ and define $||f||_{BMO} = ||f^{\#}||_{L^{\infty}}$. Let M be the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{Q \ni x} |Q|^{-1} \int_Q |f(y)|dy.$$

For $k \in N$, we denote by M^k the operator M iterated k times, i.e., $M^1(f)(x) = M(f)(x)$ and

$$M^k(f)(x) = M(M^{k-1}(f))(x) \text{ when } k \geq 2.$$

Let Φ be a Young function and $\tilde{\Phi}$ be the complementary associated to Φ. We denote the Φ-average for a function f by

$$||f||_{\Phi,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(y)|}{\lambda}\right)dy \leq 1\right\}$$

and the maximal function associated to Φ by

$$M_{\Phi}(f)(x) = \sup_{Q \ni x} ||f||_{\Phi,Q}.$$

The Young functions to be used in this paper are $\Phi(t) = t(1 + logt)^r$ and $\tilde{\Phi}(t) = exp(t^{1/r})$, the corresponding average and maximal functions denoted by $||\cdot||_{L(logL)^r,Q}$, $M_{L(logL)^r}$ and $||\cdot||_{expL^{1/r},Q}$, $M_{expL^{1/r}}$. Following [16-19], we know the generalized Hölder's inequality:

$$\frac{1}{|Q|}\int_Q |f(y)g(y)|dy \leq ||f||_{\Phi,Q}||g||_{\tilde{\Phi},Q}$$

and the following inequalities, for $r, r_j \geq 1, j = 1, \cdots, l$, with $1/r = 1/r_1 + \cdots + 1/r_l$, and any $x \in R^n$, $b \in BMO(R^n)$,

$$||f||_{L(logL)^{1/r},Q} \leq M_{L(logL)^{1/r}}(f) \leq CM_{L(logL)^l}(f) \leq CM^{l+1}(f),$$

$$||b - b_Q||_{expL^r,Q} \leq C||b||_{BMO},$$

$$|b_{2^{k+1}Q} - b_{2Q}| \leq Ck||b||_{BMO}.$$

The non-increasing rearrangement of a measurable function f on R^n is defined by

$$f^*(t) = \inf\{\lambda > 0 : |\{x \in R^n : |f(x)| > \lambda\}| \leq t\} \quad (0 < t < \infty).$$

For $\lambda \in (0, 1)$ and a measurable function f on R^n, the local sharp maximal function of f is defined by

$$M^{\#}_{\lambda}(f)(x) = \sup_{Q \ni x} \inf_{c \in C}((f - c)\chi_Q)^*(\lambda|Q|).$$

Let $p : R^n \to [1, \infty)$ be a measurable function. Denote by $L^{p(\cdot)}(R^n)$ the sets of all Lebesgue measurable functions f on R^n such that $m(\lambda f, p) < \infty$ for some $\lambda = \lambda(f) > 0$, where

$$m(f, p) = \int_{R^n} |f(x)|^{p(x)}dx.$$

The sets becomes Banach spaces with respect to the following norm

$$||f||_{L^{p(\cdot)}} = \inf\{\lambda > 0 : m(f/\lambda, p) \leq 1\}.$$

Denote by $M(R^n)$ the set of all measurable functions $p : R^n \to [1, \infty)$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(R^n)$ and the following holds

$$1 < p_- = ess \inf_{x \in R^n} p(x), \qquad ess \sup_{x \in R^n} p(x) = p_+ < \infty. \tag{1}$$

In recent years, the boundedness of classical operators on spaces $L^{p(\cdot)}(R^n)$ have attracted a great attention (see [5–8,15]). In this paper, we shall prove the following theorems.

Theorem 1. *Let $1 < s < \infty$ and $D^\alpha A_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \cdots, l$. Then there exists a constant $C > 0$ such that for any $f = \{f_i\} \in L_0^\infty(R^n)$, $0 < \delta < 1$ and $\tilde{x} \in R^n$,*

$$(|T_A(f)|_s)_\delta^{\#}(\tilde{x}) \leq C \prod_{j=1}^{l} \left(\sum_{|\alpha_j| = m_j} ||D^{\alpha_j} A_j||_{BMO} \right) M^{l+1}(|f|_s)(\tilde{x}).$$

Theorem 2. *Let $1 < s < \infty$, $p(\cdot) \in \mathrm{M}(R^n)$ and $D^\alpha A_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \cdots, l$. Then $|T_A|_s$ is bounded on $L^{p(\cdot)}(R^n)$, that is*

$$|||T_A(f)|_s||_{L^{p(\cdot)}} \leq C \prod_{j=1}^{l} \left(\sum_{|\alpha_j| = m_j} ||D^{\alpha_j} A_j||_{BMO} \right) |||f|_s||_{L^{p(\cdot)}}.$$

Remark 1. Let T be the Calderón-Zygmund operator (see [4, 11, 20]). Then Theorem 1 and Theorem 2 hold for T.

2 Some Lemmas

We begin with some preliminary lemmas.

Lemma 1 (see [3]). *Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for all α with $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(A; x, y)| \leq C|x - y|^m \sum_{|\alpha| = m} \left(\frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}$ is the cube centered at x and having side length $5\sqrt{n}|x - y|$.

Lemma 2 (see [11, p. 485]). *Let $0 < p < q < \infty$. We define that, for any function $f \geq 0$ and $1/r = 1/p - 1/q$,*

$$||f||_{WL^q} = \sup_{\lambda>0} \lambda|\{x \in R^n : f(x) > \lambda\}|^{1/q}, \; N_{p,q}(f) = \sup_E ||f\chi_E||_{L^p}/||\chi_E||_{L^r},$$

where the sup is taken for all measurable sets E with $0 < |E| < \infty$. Then

$$||f||_{WL^q} \leq N_{p,q}(f) \leq (q/(q-p))^{1/p}||f||_{WL^q}.$$

Lemma 3 (see [18]). *Let $r_j \geq 1$ for $j = 1, \cdots, l$ and we denote $1/r = 1/r_1 + \cdots + 1/r_l$. Then*

$$\frac{1}{|Q|}\int_Q |f_1(x)\cdots f_l(x)g(x)|dx \leq ||f||_{expL^{r_1},Q}\cdots ||f||_{expL^{r_l},Q}||g||_{L(logL)^{1/r},Q}.$$

Lemma 4 (see [13]). *Let $p : R^n \to [1, \infty)$ be a measurable function satisfying (1). Then $L_0^\infty(R^n)$ is dense in $L^{p(\cdot)}(R^n)$.*

Lemma 5 (see [14]). *Let $f \in L^1_{loc}(R^n)$ and g be a measurable function satisfying*

$$|\{x \in R^n : |g(x)| > \alpha\}| < \infty \quad \textit{for all} \quad \alpha > 0.$$

Then

$$\int_{R^n} |f(x)g(x)|dx \leq C_n \int_{R^n} M^\#_{\lambda_n}(f)(x)M(g)(x)dx.$$

Lemma 6 (see [14]). *Let $\delta > 0$, $0 < \lambda < 1$ and $f \in L^\delta_{loc}(R^n)$. Then*

$$M^\#_\lambda(f)(x) \leq (1/\lambda)^{1/\delta} f^\#_\delta(x).$$

Lemma 7 (see [13]). *Let $p : R^n \to [1, \infty)$ be a measurable function satisfying (1). If $f \in L^{p(\cdot)}(R^n)$ and $g \in L^{p'(\cdot)}(R^n)$ with $p'(x) = p(x)/(p(x)-1)$, then fg is integrable on R^n and*

$$\int_{R^n} |f(x)g(x)|dx \leq C||f||_{L^{p(\cdot)}}||g||_{L^{p'(\cdot)}}.$$

Lemma 8 (see [13]). *Let $p : R^n \to [1, \infty)$ be a measurable function satisfying (1). Set*

$$||f||'_{L^{p(\cdot)}} = \sup\left\{\int_{R^n} |f(x)g(x)|dx : f \in L^{p(\cdot)}(R^n), g \in L^{p'(\cdot)}(R^n)\right\}.$$

Then $||f||_{L^{p(\cdot)}} \leq ||f||'_{L^{p(\cdot)}} \leq C||f||_{L^{p(\cdot)}}$.

3 Proof of Theorem

It suffices to prove that for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q ||T_A(f)(x)|_s - C_0|^\delta dx\right)^{1/\delta} \le C\prod_{j=1}^l \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j}A_j||_{BMO}\right) M^{l+1}(|f|_s)(\tilde{x}).$$

Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}_j(x) = A_j(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A_j)_{\tilde{Q}} x^\alpha$, then $R_m(A_j; x, y) = R_m(\tilde{A}_j; x, y)$ and $D^\alpha \tilde{A}_j = D^\alpha A_j - (D^\alpha A_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We split $f = g + h = \{g_i\} + \{h_i\}$ for $g_i = f_i\chi_{\tilde{Q}}$ and $h_i = f_i\chi_{R^n\setminus\tilde{Q}}$. Write

$$\begin{aligned}
&T_A(f_i)(x) = \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j; x, y)}{|x-y|^m} K(x, y) f_i(y)dy \\
&= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j; x, y)}{|x-y|^m} K(x, y) h_i(y)dy \\
&+ \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y)}{|x-y|^m} K(x, y) g_i(y)dy \\
&- \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!}\int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} D^{\alpha_1}\tilde{A}_1(y)K(x, y)g_i(y)dy \\
&- \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!}\int_{R^n} \frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} D^{\alpha_2}\tilde{A}_2(y)K(x, y)g_i(y)dy \\
&+ \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!}\int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{A}_1(y) D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m} K(x, y)g_i(y)dy,
\end{aligned}$$

then, by Minkowski' inequality,

$$\begin{aligned}
&\left[\frac{1}{|Q|}\int_Q \left||T_A(f)(x)|_s - |T_{\tilde{A}}(h)(x_0)|_s\right|^\delta dx\right]^{1/\delta} \\
\le\ & \left[\frac{1}{|Q|}\int_Q \left(\sum_{i=1}^\infty |T_A(f_i)(x) - T_{\tilde{A}}(h_i)(x_0)|^s\right)^{\delta/s} dx\right]^{1/\delta} \\
\le\ & \left[\frac{C}{|Q|}\int_Q \left(\sum_{i=1}^\infty \left|\int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y)}{|x-y|^m} K(x, y)g_i(y)dy\right|^s\right)^{\delta/s} dx\right]^{1/\delta} \\
+\ & [\frac{C}{|Q|}\int_Q (\sum_{i=1}^\infty |\sum_{|\alpha_1|=m_1} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m}
\end{aligned}$$

$$
\begin{aligned}
& \times D^{\alpha_1}\tilde{A}_1(y)K(x,y)g_i(y)dy|^s)^{\delta/s}dx]^{1/\delta} \\
+ \quad & [\frac{C}{|Q|}\int_Q(\sum_{i=1}^{\infty}|\sum_{|\alpha_2|=m_2}\int_{R^n}\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m} \\
& \times D^{\alpha_2}\tilde{A}_2(y)K(x,y)g_i(y)dy|^s)^{\delta/s}dx]^{1/\delta} \\
+ \quad & [\frac{C}{|Q|}\int_Q(\sum_{i=1}^{\infty}|\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m} \\
& \times K(x,y)g_i(y)dy|^s)^{\delta/s}dx]^{1/\delta} \\
+ \quad & \left[\frac{C}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}\left|T_{\tilde{A}}(h_i)(x)-T_{\tilde{A}}(h_i)(x_0)\right|^s\right)^{\delta/s}dx\right]^{1/\delta} \\
:= \quad & I_1+I_2+I_3+I_4+I_5.
\end{aligned}
$$

Now, let us estimate I_1, I_2, I_3, I_4 and I_5, respectively. First, for $x\in Q$ and $y\in\tilde{Q}$, by Lemma 1, we get

$$R_{m_j}(\tilde{A}_j;x,y)\le C|x-y|^{m_j}\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO},$$

thus, by Lemma 2 and the weak type (1,1) of $|T|_s$, we obtain

$$
\begin{aligned}
I_1 \quad & \le \quad C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)\left(\frac{1}{|Q|}\int_Q|T(g)(x)|_s^{\delta}dx\right)^{1/\delta} \\
& = \quad C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|Q|^{-1}\frac{|||T(g)|_s\chi_Q||_{L^\delta}}{|Q|^{1/\delta-1}} \\
& \le \quad C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|Q|^{-1}|||T(g)|_s||_{WL^1} \\
& \le \quad C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|Q|^{-1}|||g|_s||_{L^1} \\
& \le \quad C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)M(|f|_s)(\tilde{x}).
\end{aligned}
$$

For I_2, by Lemma 2 and generalized Hölder's inequality, we get

$$
\begin{aligned}
I_2 \quad & \le \quad C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_{R^n}|T(D^{\alpha_1}\tilde{A}_1g)(x)|_s^{\delta}dx\right)^{1/\delta} \\
& \le \quad C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}|Q|^{-1}|||T(D^{\alpha_1}\tilde{A}_1g)(x)|_s\chi_Q||_{WL^1}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}\frac{1}{|Q|}\int_{R^n}|D^{\alpha_1}\tilde{A}_1(x)||g(x)|_s dx\\
&\leq C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}||D^{\alpha_1}A_1-(D^{\alpha_1}A_1)_{\tilde{Q}}||_{expL,\tilde{Q}}|||f|_s||_{L(logL),\tilde{Q}}\\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)M^2(|f|_s)(\tilde{x}).
\end{aligned}
$$

For I_3, similarly to the proof of I_2, we get

$$
I_3\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)M^2(|f|_s)(\tilde{x}).
$$

Similarly, for I_4, taking $r, r_1, r_2\geq 1$ such that $1/r = 1/r_1 + 1/r_2$, we obtain, by Lemma 3,

$$
\begin{aligned}
I_4 &\leq C\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\left(\frac{1}{|Q|}\int_{R^n}|T(D^{\alpha_1}\tilde{A}_1D^{\alpha_2}\tilde{A}_2f_1)(x)|_s^\delta dx\right)^{1/\delta}\\
&\leq C\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}|Q|^{-1}|||T(D^{\alpha_1}\tilde{A}_1D^{\alpha_2}\tilde{A}_2g)|_s\chi_Q||_{WL^1}\\
&\leq C\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{|Q|}\int_{R^n}|D^{\alpha_1}\tilde{A}_1(x)D^{\alpha_2}\tilde{A}_2(x)||g(x)|_s dx\\
&\leq C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\prod_{j=1}^{2}\left|\left|D^{\alpha_j}A_j-(D^{\alpha_j}A_j)_{\tilde{Q}}\right|\right|_{expL^{r_j},\tilde{Q}}\cdot|||f|_s||_{L(logL)^{1/r},\tilde{Q}}\\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)M^3(|f|_s)(\tilde{x}).
\end{aligned}
$$

For I_5, we write

$$
\begin{aligned}
&T_{\tilde{A}}(h_i)(x)-T_{\tilde{A}}(h_i)(x_0)=\int_{R^n}\left(\frac{K(x,y)}{|x-y|^m}-\frac{K(x_0,y)}{|x_0-y|^m}\right)\prod_{j=1}^{2}R_{m_j}(\tilde{A}_j;x,y)h_i(y)dy\\
&+\int_{R^n}\Big(R_{m_1}(\tilde{A}_1;x,y)-R_{m_1}(\tilde{A}_1;x_0,y)\Big)\frac{R_{m_2}(\tilde{A}_2;x,y)}{|x_0-y|^m}K(x_0,y)h_i(y)dy\\
&+\int_{R^n}\Big(R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;x_0,y)\Big)\frac{R_{m_1}(\tilde{A}_1;x_0,y)}{|x_0-y|^m}K(x_0,y)h_i(y)dy\\
&-\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}[\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}K(x,y)\\
&-\frac{R_{m_2}(\tilde{A}_2;x_0,y)(x_0-y)^{\alpha_1}}{|x_0-y|^m}K(x_0,y)]D^{\alpha_1}\tilde{A}_1(y)h_i(y)dy
\end{aligned}
$$

$$
\begin{aligned}
& -\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}[\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}K(x,y) \\
& -\frac{R_{m_1}(\tilde{A}_1;x_0,y)(x_0-y)^{\alpha_2}}{|x_0-y|^m}K(x_0,y)]D^{\alpha_2}\tilde{A}_2(y)h_i(y)dy \\
& +\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m}K(x,y)-\frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m}K(x_0,y)\right] \\
& \times D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)h_i(y)dy \\
= & \ I_5^{(1)}+I_5^{(2)}+I_5^{(3)}+I_5^{(4)}+I_5^{(5)}+I_5^{(6)}.
\end{aligned}
$$

By Lemma 1 and the following inequality(see [20])

$$
|b_{Q_1}-b_{Q_2}|\le C\log(|Q_2|/|Q_1|)||b||_{BMO}\text{ for }Q_1\subset Q_2,
$$

we know that, for $x\in Q$ and $y\in 2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}$,

$$
\begin{aligned}
|R_{m_j}(\tilde{A}_j;x,y)| &\le C|x-y|^{m_j}\sum_{|\alpha_j|=m_j}(||D^{\alpha_j}A||_{BMO}+|(D^{\alpha_j}A)_{\tilde{Q}(x,y)}-(D^{\alpha_j}A)_{\tilde{Q}}|) \\
&\le Ck|x-y|^{m_j}\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A||_{BMO}.
\end{aligned}
$$

Note that $|x-y|\sim|x_0-y|$ for $x\in Q$ and $y\in R^n\setminus\tilde{Q}$, we obtain, by the conditions on K,

$$
\begin{aligned}
|I_5^{(1)}| &\le C\int_{R^n}\left(\frac{|x-x_0|}{|x_0-y|^{m+n+1}}+\frac{|x-x_0|^\varepsilon}{|x_0-y|^{m+n+\varepsilon}}\right)\prod_{j=1}^2 R_{m_j}(\tilde{A}_j;x,y)||h_i(y)|dy \\
&\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right) \\
&\quad\times\sum_{k=0}^\infty\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k^2\left(\frac{|x-x_0|}{|x_0-y|^{n+1}}+\frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon}}\right)|f_i(y)|dy \\
&\le C\prod_{j=1}^2\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)\sum_{k=1}^\infty k^2(2^{-k}+2^{-\varepsilon k})\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f_i(y)|dy,
\end{aligned}
$$

thus, by Minkowski' inequality,

$$
\begin{aligned}
\left(\sum_{i=1}^\infty\left|I_5^{(1)}\right|^s\right)^{1/s} &\le C\prod_{j=1}^2\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)\sum_{k=1}^\infty k^2(2^{-k}+2^{-\varepsilon k}) \\
&\quad\times\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|_s dy
\end{aligned}
$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M(|f|_s)(\tilde{x}).$$

For $I_5^{(2)}$, by the formula (see [3]):

$$R_{m_j}(\tilde{A};x,y)-R_{m_j}(\tilde{A};x_0,y)=\sum_{|\beta|<m_j}\frac{1}{\beta!}R_{m_j-|\beta|}(D^\beta\tilde{A};x,x_0)(x-y)^\beta$$

and Lemma 1, we have

$$|R_{m_j}(\tilde{A};x,y)-R_{m_j}(\tilde{A};x_0,y)|\leq C\sum_{|\beta|<m_j}\sum_{|\alpha|=m_j}|x-x_0|^{m_j-|\beta|}|x-y|^{|\beta|}||D^\alpha A||_{BMO},$$

thus

$$\begin{aligned}\left(\sum_{i=1}^{\infty}\left|I_5^{(2)}\right|^s\right)^{1/s} &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)\\ &\quad\times\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k\frac{|x-x_0|}{|x_0-y|^{n+1}}|f(y)|_s dy\\ &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)M(|f|_s)(\tilde{x}).\end{aligned}$$

Similarly,

$$\left(\sum_{i=1}^{\infty}\left|I_5^{(3)}\right|^s\right)^{1/s}\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)M(|f|_s)(\tilde{x}).$$

For $I_5^{(4)}$, similar to the proof of $I_5^{(1)}$, $I_5^{(2)}$ and I_2, we get

$$\begin{aligned}&\left(\sum_{i=1}^{\infty}\left|I_5^{(4)}\right|^s\right)^{1/s}\\ \leq\ & C\sum_{|\alpha_1|=m_1}\int_{R^n\setminus\tilde{Q}}\left|\frac{(x-y)^{\alpha_1}K(x,y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1}K(x_0,y)}{|x_0-y|^m}\right|\\ &\times|R_{m_2}(\tilde{A}_2;x,y)||D^{\alpha_1}\tilde{A}_1(y)||f(y)|_s dy\\ &+C\sum_{|\alpha_1|=m_1}\int_{R^n\setminus\tilde{Q}}|R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;x_0,y)|\\ &\times\frac{|(x_0-y)^{\alpha_1}K(x_0,y)|}{|x_0-y|^m}|D^{\alpha_1}\tilde{A}_1(y)||f(y)|_s dy\end{aligned}$$

$$\leq C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_1}\tilde{A}_1(y)||f(y)|_s dy$$

$$\leq C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{BMO}\sum_{|\alpha_1|=m_1}\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})$$
$$\times||D^{\alpha_1}A_1-(D^{\alpha_1}A_1)_{\tilde{Q}}||_{expL,2^k\tilde{Q}}|||f|_s||_{L(logL),2^k\tilde{Q}}$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)M^2(|f|_s)(\tilde{x}).$$

Similarly,

$$\left(\sum_{i=1}^{\infty}\left|I_5^{(5)}\right|^s\right)^{1/s}\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)M^2(|f|_s)(\tilde{x}).$$

For $I_5^{(6)}$, we obtain

$$\left(\sum_{i=1}^{\infty}\left|I_5^{(6)}\right|^s\right)^{1/s}$$
$$\leq C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\int_{R^n\setminus\tilde{Q}}\left|\frac{(x-y)^{\alpha_1+\alpha_2}K(x,y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1+\alpha_2}K(x_0,y)}{|x_0-y|^m}\right|$$
$$\times|D^{\alpha_1}\tilde{A}_1(y)||D^{\alpha_2}\tilde{A}_2(y)||f(y)|_s dy$$
$$\leq C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\sum_{k=1}^{\infty}(2^{-k}+2^{-\varepsilon k})\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_1}\tilde{A}_1(y)||D^{\alpha_2}\tilde{A}_2(y)||f(y)|_s dy$$
$$\leq C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\prod_{j=1}^{2}\left|\left|D^{\alpha_j}A_j-(D^{\alpha_j}A_j)_{\tilde{Q}}\right|\right|_{expL^{r_j},2^k\tilde{Q}}\cdot|||f|_s||_{L(logL)^{1/r},2^k\tilde{Q}}$$
$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)M^3(|f|_s)(\tilde{x}).$$

Thus

$$|I_5|\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)M^3(|f|_s)(\tilde{x}).$$

This completes the proof of Theorem 1.

By Lemmas 4–7, we get, for $f=\{f_i\}\in L_0^{\infty}(R^n)$ and $g\in L^{p'(\cdot)}(R^n)$,

$$\int_{R^n}|T_A(f)(x)|_s g(x)|dx \leq C\int_{R^n}M^{\#}_{\lambda_n}(T_A(f)|_s)(x)M(g)(x)|dx$$
$$\leq C\int_{R^n}(T_A(f)|_s)^{\#}_{\delta}(x)M(g)(x)dx$$

$$\begin{aligned} &\leq C\int_{R^n} M^{l+1}(|f|_s)(x)M(g)(x)dx \\ &\leq C||M^{l+1}(|f|_s)||_{L^{p(\cdot)}}||M(g)||_{L^{p'(\cdot)}} \\ &\leq C|||f|_s||_{L^{p(\cdot)}}||g||_{L^{p'(\cdot)}}, \end{aligned}$$

thus, by Lemma 8,

$$|||T_A(f)(x)|_s||_{L^{p(\cdot)}} \leq |||f|_s||_{L^{p(\cdot)}}.$$

This completes the proof of Theorem 2.

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On Frattini subloops and normalizers of commutative Moufang loops

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Abstract. Let L be a commutative Moufang loop (CML) with the multiplication group $\mathfrak{M}$, and let $\mathfrak{F}(L)$, $\mathfrak{F}(\mathfrak{M})$ be the Frattini subloop of L and Frattini subgroup of $\mathfrak{M}$. It is proved that $\mathfrak{F}(L) = L$ if and only if $\mathfrak{F}(\mathfrak{M}) = \mathfrak{M}$, and the structure of this CML is described. The notion of normalizer for subloops in CML is defined constructively. Using this it is proved that if $\mathfrak{F}(L) \neq L$, then L satisfies the normalizer condition and that any divisible subgroup of $\mathfrak{M}$ is an abelian group and serves as a direct factor for $\mathfrak{M}$.

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It is known that in many classes of algebras the Frattini subalgebras essentially determine the structure of these algebras. In this paper this dependance is considered in the class of commutative Moufang loops (CML) and their multiplication groups. Let L be a CML with the multiplication group $\mathfrak{M}$, let $\mathfrak{F}(L)$ and $\mathfrak{F}(\mathfrak{M})$ denote the Frattini subloop of L and the Frattini subgroup of $\mathfrak{M}$. It is proved that $\mathfrak{F}(L) = L$ if and only if $\mathfrak{F}(\mathfrak{M}) = \mathfrak{M}$, and the structure of this CML and groups is described. In particular, if L has the exponent 3, then $\mathfrak{F}(L) = L$ if and only if $L = L'$, where L' denotes the associator subloop of L (Theorem 1). The existence of CML with $L' = L$ is proved in [1].

The normalizer $N_L(H)$ is defined constructively for subloop H of commutative Moufang loop L which, in general, has the same role as a normalizer for subgroups. The normalizer $N_L(H)$ is the unique maximal subloop of L such that H is normal in $N_L(H)$. By analogy with the group theory the notion of CML with normalizer condition is defined: every proper subloop of CML differs from his normalizer. Using essentially Theorem 1, it is proved that if a CML L satisfies the inequality $\mathfrak{F}(L) \neq L$ then L satisfies the normalizer condition. It is proved also that for multiplication groups of CML an analogous situation does not take place. There exists a CML L with multiplication group $\mathfrak{M}$ such that $\mathfrak{F}(\mathfrak{M}) \neq \mathfrak{M}$, but $\mathfrak{M}$ does not satisfy the normalizer condition.

Again, using essentially Theorem 1 it is proved that every divisible subgroup of multiplication group $\mathfrak{M}$ of any CML is an abelian group and serves as a direct factor for $\mathfrak{M}$ (Theorem 2). A similar result for divisible subloops of CML is proved in [2]. We note that in general case Theorem 2 is not true. In [3, Theorem 2.7

and Example 2.2] there is an example of a divisible non-periodic and non-abelian ZA-group.

At last we note that the material of present article has been published earlier, in [4].

1 Preliminaries

Let us bring some notions and results on the loop theory from [5, 6].

The *multiplication group* $\mathfrak{M}(L)$ of a loop L is the group generated by all *translations* $L(x)$, $R(x)$, where $L(x)y = xy$, $R(x)y = yx$. The subgroup $\mathfrak{I}(L)$ of group $\mathfrak{M}(L)$, generated by all *inner mappings* $L(x, y) = L^{-1}(xy)L(x)L(y)$, $R(x, y) = R^{-1}(xy)R(y)R(x)$, $T(x) = L^{-1}(x)R(x)$ is called the *inner mapping group* of loop L. A subloop H of a loop L is called *normal* in L if $\mathfrak{I}(L)H = H$. The set of all elements $x \in L$ which commute and associate with all elements of L so that for all $a, b \in L$ $ax = xa$, $ab \cdot x = a \cdot bx$, $ax \cdot b = a \cdot xb$, $xa \cdot b = x \cdot ab$ is a normal subloop $Z(L)$ of L, its *centre*.

Lemma 1 (see [5, p. 63]). *Let H be a normal subloop of loop L with the multiplication group $\mathfrak{M}$. Then $\mathfrak{M}(L/H) \cong \mathfrak{M}/H^*$ where $H^* = \{\alpha \in \mathfrak{M} | (\alpha x)H = xH \quad \forall x \in L\}$. Conversely, every normal subgroup $\mathfrak{N}$ of $\mathfrak{M}$ determines a normal subloop $H = \mathfrak{N}1 = \{\alpha 1 | \alpha \in \mathfrak{N}\}$ of L and $\mathfrak{N} \subseteq H^*$.*

Proposition 1. *Let $(L, \cdot, 1)$ be a loop with centre $Z(L)$, let $\mathfrak{M}$ be its multiplication group with centre $Z(\mathfrak{M})$ and let $\tilde{Z}(L) = \{\varphi 1 | \varphi \in Z(\mathfrak{M})\}$, $\tilde{Z}(\mathfrak{M}) = \{L(\varphi 1) | \varphi \in Z(\mathfrak{M})\}$, $\overline{Z}(\mathfrak{M}) = \{L(a) | a \in Z(L)\}$. Then $\overline{Z}(L) = Z(L) \cong \overline{Z}(\mathfrak{M}) = \tilde{Z}(\mathfrak{M}) = Z(\mathfrak{M})$.*

Proof. Let $a \in Z(Q)$ and $x, y \in L$. Then $R(a) = L(a)$, $a \cdot a^{-1}x = x$, $L(a)L(a^{-1})x = x$ $L(a^{-1}) = L^{-1}(a)$ and $a \cdot xy = ax \cdot y$, $L(a)L(y)x = L(y)L(a)x$, $L(a)L(y) = L(y)L(a)$. Similarly, for $a^{-1} \in Z(L)$ we obtain that $L(a^{-1})R(y) = R(y)L(a^{-1})$. Then $(L(a^{-1})R(y))^{-1} = (R(y)L(a^{-1}))^{-1}$, $R^{-1}(y)L^{-1}(a^{-1}) = L^{-1}(a^{-1})R^{-1}(y)$, $R^{-1}(y)L(a) = L(a)R^{-1}(y)$. Analogously, from $yx \cdot a = y \cdot xa$ and $R(a) = L(a)$ we get $L(a)L(y) = L(y)L(a)$, $L(a)L^{-1}(y) = L(y)^{-1}L(a)$. Then from the definition of the group $\mathfrak{M}(L)$ it follows that $L(a) \in Z(\mathfrak{M})$. Similarly, for $a^{-1} \in Z(L)$ we get that $L(a^{-1}) = L^{-1}(a) \in Z(\mathfrak{M})$. We also have $L(a)L(b) = L(ab)$ for $a, b \in Z(L)$. Then the set $\overline{Z}(\mathfrak{M})$ is a subgroup of $\mathfrak{M}$, $\overline{Z}(\mathfrak{M}) \subseteq Z(\mathfrak{M})$ and the isomorphism $Z(L) \cong \overline{Z}(\mathfrak{M})$, defined by $u \to L(u)$, $u^{-1} \to L^{-1}(u)$, $u \in L$, follows from the equality $L(a)L(b) = L(ab)$.

If $\varphi \in Z(\mathfrak{M})$, then $\varphi L(x) = L(x)\varphi$, $\varphi L(x)y = L(x)\varphi y, \varphi(xy) = x \cdot \varphi y$ and $\varphi R(x) = R(x)\varphi$, $\varphi R(x)y = R(x)\varphi y$, $\varphi(yx) = \varphi y \cdot x$ for any $x, y \in L$. Hence $\varphi(xy) = x \cdot \varphi y$ and $\varphi(yx) = \varphi y \cdot x$. Let $y = 1$. Then $\varphi x = x \cdot \varphi 1$, $\varphi x = \varphi 1 \cdot x$, i.e. $x \cdot \varphi 1 = \varphi 1 \cdot x$. Now, using the equality $\varphi(xy) = x \cdot \varphi y$ we obtain that $xy \cdot \varphi 1 = \varphi(xy \cdot 1) = \varphi(xy) = x \cdot \varphi y = x \cdot \varphi(y \cdot 1) = x(y \cdot \varphi 1)$ and using the equality $\varphi(yx) = \varphi y \cdot x$ we obtain that $\varphi 1 \cdot xy = \varphi(1 \cdot xy) = \varphi(xy) = \varphi x \cdot y = \varphi(1 \cdot x)y = (\varphi 1 \cdot x)y$. Hence, if $\varphi \in Z(\mathfrak{M})$ then $\varphi 1 \in Z(L)$, i.e. $\tilde{Z}(L) \subseteq Z(L)$. Conversely, let $a \in Z(L)$. Then $L(a) \in Z(\mathfrak{M})$ and $a = L(a)1 \in \tilde{Z}(L)$. Hence $Z(L) \subseteq \tilde{Z}(L)$ Consequently, $Z(L) = \tilde{Z}(L)$ and therefore $\overline{Z}(\mathfrak{M}) = \tilde{Z}(\mathfrak{M})$.

Let $\mathfrak{I}(L)$ be the inner mapping group of $\mathfrak{M}$. In the proof of Lemma IV.1.2 from [5] it is shown that each element $\alpha \in \mathfrak{M}$ has the form $\alpha = L(\alpha 1)\theta$ where $\theta \in \mathfrak{I}(L)$; moreover $\alpha \in \mathfrak{I}(L)$ if and only if $L(\alpha 1) = e$ where e is the unit of $\mathfrak{M}$. Let $\mathfrak{J}(L) = \mathfrak{I}(L) \cap Z(\mathfrak{M})$. If $\alpha \in Z(\mathfrak{M})$ then, by the cases considered above, $L(\alpha 1) \in \overline{Z}(\mathfrak{M})$. Then $\theta \in \mathfrak{J}(L)$. The subgroups $\overline{Z}(\mathfrak{M}) \subseteq Z(\mathfrak{M})$ and $\mathfrak{J}(L) \subseteq Z(\mathfrak{M})$ are normal in $\mathfrak{M}$. As $\overline{Z}(\mathfrak{M}) \cap \mathfrak{J}(L) = \varepsilon$ then $Z(\mathfrak{M}) = \overline{Z}(\mathfrak{M}) \times \mathfrak{J}(L)$. By Lemma 1 $\mathfrak{J}(L)1 = 1$ is a normal subloop of L and $\mathfrak{J}(L) \subseteq 1^*$ where $1^* = \{\alpha \in \mathfrak{M} | (\alpha x)1 = x1 \quad \forall x \in L\}$. But $1^* = e$, hence $\mathfrak{J}(L) = e$ and $Z(\mathfrak{M}) = \overline{Z}(\mathfrak{M})$, as required. □

A system Σ of subloops of loop L will be called a *subnormal system* if:

1) it contains 1 and L;

2) it is linearly ordered by inclusion, i. e. for all A, B from Σ either $A \subseteq B$, or $B \subseteq A$;

3) it is closed with respect to the unions and intersections, in particular, together with each $A \neq L$ it contains the intersection $A^\sharp$ of all $H \in \Sigma$ with the condition $H \supset A$ and together with each $B \neq 1$ it contains the union $B^\flat$ of all $H \in \Sigma$ with the condition $H \subset B$;

4) it satisfies the condition: A is normal in $A^\natural$ for all $A \in \Sigma$, $A \neq L$.

A system Σ is called *ascending* (respect. *descending*) if $A^\sharp \neq A$ (respect. $B^\flat \neq B$) for all $A \in \Sigma$, $A \neq L$ (respect. $B \in \Sigma$, $B \neq 1$) and is called *normal* if the subloops $A \in \Sigma$ are normal in L.

A loop L may be called an *SD-loop* if it has a descending subnormal system Σ such that the quotient loops $A^\sharp/A$ are abelian groups for all $A \in \Sigma$, $A \neq L$. If a loop L has an ascending normal system such that $A^\sharp/A \subseteq Z(L/A)$ for all $A \in \Sigma$, $A \neq L$ then L is called a *ZA-loop*.

If the upper central series of the ZA-loop has a finite length, then the loop is called *centrally nilpotent*. The least such length is called the *class* of the central nilpotency. If the loop L is centrally nilpotent of class k then the upper central series of L has the form

$$1 = Z_0(L) \subset Z_1(L) \subset \cdots \subset Z_k(L) = L, \tag{1}$$

where $Z_1(L) = Z(L)$, $Z_{i+1}(L)/Z_i(L) = Z(L/Z_i(L))$.

A *commutative Moufang loop* (*CML*) is characterized by the identity $x^2 \cdot yz = xy \cdot xz$. The *associator* (a, b, c) of the elements a, b, c of the CML Q is defined by the equality $ab \cdot c = (a \cdot bc)(a, b, c)$. The identities

$$L(x, y)z = z(z, y, x), \tag{2}$$

$$(x, y, z) = (y, z, x) = (y^{-1}, x, z) = (y, x, z)^{-1}, \tag{3}$$

$$(xy, u, v) = (x, u, v)((x, u, v), x, y)(y, u, v)((y, u, v), y, x) \tag{4}$$

hold in the CML.

If A, B, C are subsets of CML L, (A, B, C) denotes the set of all associators (a, b, c), $a \in A$, $b \in B$, $c \in C$. If $A = B = C = L$, then the normal subloop $L' = (L, L, L)$ is called the *associator subloop* of CML L.

Lemma 2 (see [5])**.** *Let L be a CML with centre $Z(L)$ and let $a \in L$. Then $a^3 \in Z(L)$.*

Lemma 3. *If $Z_2(L) \neq Z_1(L)$ for a CML L then $L' \neq L$.*

Proof. If $z \in Z_2(L) \backslash Z_1(L)$ then $((x, y, z), u, v) = 1$ for all $x, y, u, v \in L$ and there exist elements $x_0, y_0 \in L$ such that $(x_0, y_0, z) \neq 1$. From (4) it follows that $(uv, y_0, z) = (u, y_0, z)(v, y_0, z)$, which shows that the mapping $\varphi : u \to (u, y_0, z)$ is a homomorphism of L into $Z_1(L)$. The centre $Z_1(L)$ is an associative subloop and as $(x_0, y_0, z) \neq 1$ then $L' \subseteq \ker\varphi$ and $L/\ker\varphi$ is non-unitary. Hence $L' \neq L$, as required. □

Lemma 4. *Let L be a CML with the multiplication group $\mathfrak{M}$, let L' be the associator subloop of L and let $\mathfrak{M}'$ be the commutator subgroup of $\mathfrak{M}$. Then $L' \subseteq \mathfrak{F}(L)$ and $\mathfrak{M}' \subseteq \mathfrak{F}(\mathfrak{M})$.*

Proof. The inclusion $L' \subseteq \mathfrak{F}(L)$ is proved in [7]. The group $\mathfrak{M}$ is locally nilpotent [2], then the proof of inclusion $\mathfrak{M}' \subseteq \mathfrak{F}(\mathfrak{M})$ can be found, for example, in [8]. □

2 Frattini subloops

If $S, T, \ldots$ are subsets of elements of a loop L, let $< S, T, \ldots >$ denote the subloop of L generated by $S, T, \ldots$. An element x of a loop L is a *non-generator* of L if, for every subset S of L, $< x, S >= L$ implies $< S >= L$. The non-generators of L form the *Frattini subloop*, $\mathfrak{F}(L)$, of L. If L has at least one maximal proper subloop, then $\mathfrak{F}(L)$ is the intersection of all maximal proper subloops of L. In the contrary case, $\mathfrak{F}(L) = L$ [5].

Lemma 5. *Let θ be a homomorphism of the loop L into a loop and let $\mathfrak{F}(L) = L$. Then $\mathfrak{F}(\theta L) = \theta L$.*

Proof. In [5] it is proved that if φ is a homomorphism of the loop L into a loop, then $\varphi(\mathfrak{F}(L)) \subseteq \mathfrak{F}(\varphi(L))$. In our case we have $\theta L = \theta(\mathfrak{F}(L)) \subseteq \mathfrak{F}(\theta L) \subseteq \theta L$. Hence $\mathfrak{F}(\theta L)) = \theta L$, as required. □

Lemma 6. *For a CML L with the multiplication group $\mathfrak{M}$ the following statements are equivalent: 1) $\mathfrak{F}(\mathfrak{M}) = \mathfrak{M}$; 2) $\mathfrak{F}(L) = L$.*

Proof. 1) ⇒ 2). Let $\mathfrak{F}(\mathfrak{M}) = \mathfrak{M}$ and we assume that $\mathfrak{F}(L) \neq L$. Then L has at least one maximal proper subloop H and $\mathfrak{F}(L)$ is the intersection of all such subloops. By Lemma 4 the associator subloop L' lies in $\mathfrak{F}(L)$. Hence H is a normal subloop of L and the quotient loop L/H is a cyclic group of prime order p. Then

$\mathfrak{M}(L/H)$ is a cyclic group of order p too, and by Lemma 1 $\mathfrak{M}/H^*$ is a cyclic group of order p. Consequently, H^* is a maximal proper subgroup of $\mathfrak{M}$. Then $\mathfrak{F}(\mathfrak{M}) \neq \mathfrak{M}$. Contradiction. Hence 1) implies 2).

Conversely, let $\mathfrak{F}(L) = L$ and we assume that $\mathfrak{F}(\mathfrak{M}) \neq \mathfrak{M}$. Let $\mathfrak{N}$ be a maximal proper subgroup of $\mathfrak{M}$. $\mathfrak{M}$ is a locally nilpotent group [2], then by Lemma 4 the commutator subgroup $\mathfrak{M}'$ lies in $\mathfrak{F}(\mathfrak{M})$. Then $\mathfrak{N}$ is a normal subgroup of $\mathfrak{M}$ and $\mathfrak{M}/\mathfrak{N}$ is a cyclic group of prime order p. By Lemma 1 $\mathfrak{N}1 = H$ is a normal subloop of L and $\mathfrak{N} \subseteq H^*$. Then from $\mathfrak{M}(L/H) \cong \mathfrak{M}/H^*$ it follows that L/H is a cyclic group of order p. Hence H is a maximal subloop of L. Then $\mathfrak{F}(L) \neq L$. Contradiction. Hence 2) implies 1). □

Let $M(H)$ denote the subgroup of multiplication group of CML L, generated by $\{L(x) | \forall x \in H\}$, where H is a subset of L.

Lemma 7 (see [9]). *Let L be a CML with the multiplication group $\mathfrak{M}(L)$ and the inner mapping group $\mathfrak{I}(L)$. Then $\mathfrak{M}(L)' =< \mathfrak{I}(L), M(L') >= (L')^\star = \overline{\mathfrak{I}(L)}$, where $(L')^\star = \{\alpha \in \mathfrak{M}(L) | \alpha x \cdot L' = xL' \quad \forall x \in L\}$, $\overline{\mathfrak{I}(L)}$ is the normal subgroup of $\mathfrak{M}(L)$, generated by $\mathfrak{I}(L)$.*

Proposition 2. *For a CML L with the multiplication group $\mathfrak{M}$ the following statements are equivalent: 1) $\mathfrak{F}(L) = L$ and L satisfies the identity $x^3 = 1$; 2) $L = L'$; 3) $\mathfrak{M} = \mathfrak{M}'$; 4) $\mathfrak{F}(L) = L$ and $Z(L) = \{1\}$; 5) $\mathfrak{F}(\mathfrak{M}) = \mathfrak{M}$ and $Z(\mathfrak{M}) = \{e\}$.*

Proof. 1) ⇔ 2). As $\mathfrak{F}(L) = L$ then by Lemma 5 $\mathfrak{F}(L/L') = L/L'$. In [10] it is proved that for an abelian group G $\mathfrak{F}(G) = G$ if and only if G is a divisible group. The abelian group L/L' satisfies the identity $x^3 = 1$. L/L' is a divisible group, then L/L' is a unitary group. Hence $L' = L$, i.e. 1) implies 2). Conversely, let $L' = L$. By [5] the associator subloop L' satisfies the identity $x^3 = 1$ and from the relations $L = L' \subseteq \mathfrak{F}(L) \subseteq L$ it follows that $\mathfrak{F}(L) = L$. Hence 2) implies 1).

1) ⇒ 3). By Lemma 6 $\mathfrak{F}(L) = L$ implies $\mathfrak{F}(\mathfrak{M}) = \mathfrak{M}$. Like in the previous case from here it follows that $\mathfrak{M}/\mathfrak{M}'$ is a divisible abelian group. By definition the group $\mathfrak{M}$ is generated by translations $L(x), x \in L$. Then from the identity $x^3 = 1$ for L and diassociativity of L it follows that the divisible abelian group $\mathfrak{M}/\mathfrak{M}'$ satisfies the identity $x^3 = 1$. Then $\mathfrak{M}/\mathfrak{M}'$ is a unitary group. Hence $\mathfrak{M}' = \mathfrak{M}$, i.e. 1) ⇒ 3).

Conversely, let $\mathfrak{M}' = \mathfrak{M}$. By Lemma 4 $\mathfrak{M}' \subseteq \mathfrak{F}(\mathfrak{M})$. Then from the relations $\mathfrak{M} = \mathfrak{M}' \subseteq \mathfrak{F}(\mathfrak{M}) \subseteq \mathfrak{M}$ it follows that $\mathfrak{F}(\mathfrak{M}) = \mathfrak{M}$. By Lemma 1 $\mathfrak{M}(L/L') \cong \mathfrak{M}/(L')^*$. $\mathfrak{M}(L/L')$ is an abelian group. Then $\mathfrak{M}' \subseteq (L')^*$ and from the relation $\mathfrak{M}' = \mathfrak{M}$ it follows that $\mathfrak{M}(L/L')$ is unitary group. Hence $L' = L$. Consequently, 3) implies 2).

2) ⇒ 4). We consider the homomorphism $\alpha : L \to L/Z(L)$. The elements of quotient loop have the form $aZ(L)$, $a \in L$. From $L = L'$ it follows that the element a is a product of associators (u, v, w), $u, v, w \in L$. From the equalities $(u, v, w)Z(L) = (uZ(L), v, w) = (u, v, w)$, $ab \cdot Z(L) = a \cdot bZ(L) = aZ(L) \cdot b$ it follows that if $aZ(L) = bZ(L)$ then $a = b$. But this means that α is an isomorphism. Then $Z(L) = \{1\}$. Consequently, 2) implies 4).

Conversely, let $Z(L) = \{1\}$. Then from Lemma 2 it follows that CML L satisfies the identity $x^3 = 1$. Hence 4) implies 1). Further, the equivalence of statements 4), 5) follows from Lemma 6 and Proposition 1, as required. □

Theorem 1. *For a CML L with the multiplication group $\mathfrak{M}$ the following statements are equivalent: 1) $\mathfrak{F}(L) = L$; 2) L is a direct product $L = L' \times L^3$, where L' is the associator subloop of L and $L^3 = \{x^3 | x \in L\}$ is a divisible abelian group; 3) $\mathfrak{F}(\mathfrak{M}) = \mathfrak{M}$; 4) $\mathfrak{M}$ is a direct product $\mathfrak{M} = \mathfrak{M}' \times \mathfrak{D}$, where $\mathfrak{M}'$ is the commutator subgroup of $\mathfrak{M}$ and $\mathfrak{D}$ is a divisible abelian group. In these cases $\mathfrak{F}(L') = L'$, $Z(L') = \{1\}$, $Z(L) = L^3$, $L^3 \cong \mathfrak{D}$, $Z(\mathfrak{M}) = \mathfrak{D}$, $\mathfrak{F}(\mathfrak{M}') = \mathfrak{M}' = M(L') = (L')^\star = \overline{\mathfrak{I}(L)}$, where $(L')^\star = \{\alpha \in \mathfrak{M}(L) | \alpha x \cdot L' = xL' \quad \forall x \in L\}$, $\overline{\mathfrak{I}(L)}$ is the normal subgroup of $\mathfrak{M}(L)$, generated by the inner mapping group $\mathfrak{I}(L)$, $Z(\mathfrak{M}') = \{e\}$.*

Proof. 1) $\Rightarrow$ 2). Using the diassociativity of CML it is easy to prove that L^3 is a subloop of L. By Lemma 2 $L^3 \subseteq Z(L)$. Then L^3 is a normal associative subloop of L. The quotient loop L/L^3 satisfies the identity $x^3 = 1$. By Lemma 5 from $\mathfrak{F}(L) = L$ it follows that $\mathfrak{F}(L/L^3) = L/L^3$. Then by Proposition 1 $L/L^3 = (L/L^3)'$. But $(L/L^3)' = L'L^3/L^3$. Then from $L/L^3 = L'L^3/L^3$ it follows that $L = L'L^3$. Hence $L/L^3 = L'L^3/L^3 \cong L'/(L' \cap L^3)$. We have $\mathfrak{F}(L'/(L' \cap L^3)) = L'/(L'/(L' \cap L^3))$ and $L' \cap L^3 \subseteq Z(L)$. Then by analogy with the proof of implication 1) $\Rightarrow$ 4) of Proposition 2 it is easy to prove that $L' \cap L^3 = \{1\}$. But $L = L'L^3$. Then $L = L' \times L^3$. Further, by Lemma 5 we get that $\mathfrak{F}(L') \cong \mathfrak{F}(L/L^3) = \mathfrak{F}(L)/L^3 = L/L^3 \cong L'$, $\mathfrak{F}(L') = L'$ and by Proposition 2 $Z(L') = \{1\}$. Analogously, $\mathfrak{F}(L^3) = L^3$. The subloop L^3 is associative. Then from $\mathfrak{F}(L^3) = L^3$ it follows that L^3 is a divisible abelian group [9]. Consequently, 1) implies 2) and $\mathfrak{F}(L') = L'$, $Z(L') = \{1\}$. Further, from $L = L\prime \times L^3$, $Z(L') = \{1\}$ it follows that $Z(L) = L^3$.

Conversely, let $L = L' \times L^3$ and let $L^3 \subseteq Z(L)$ be a divisible group. Then $\mathfrak{F}(L^3) = L^3$ and $L' = (L' \times L^3)' = (L')'$. By Proposition 2 $\mathfrak{F}(L') = L'$. Hence $L = \mathfrak{F}(L') \times \mathfrak{F}(L^3)$. $\mathfrak{F}(L')$ and $\mathfrak{F}(L^3)$ do not have maximal proper subloops. From here it is easy to see that L does not have a maximal proper subloop, either. Then $\mathfrak{F}(L) = L$. Hence 2) implies 1) and, consequently, the statements 1), 2) are equivalent.

The equivalence of statements 1), 3) follows from Lemma 6.

2) $\Leftrightarrow$ 4). Let $L = L' \times L^3$. From here it follows that any element $a \in L$ has the form $a = ud$, where $u \in L'$, $d \in L^3$. As by Lemma 2 $L^3 \subseteq Z(L)$, then $L(a) = L(u)L(d)$, therefore, $\mathfrak{M} = M(L')M(L^3)$. Any element $\alpha \in M(L^3)$ has the form $\alpha = L(v)$, where $v \in L^3$. Let $\alpha \in M(L') \cap M(L^3)$. Then $\alpha 1 \in L' \cap L^3 = \{1\}$, $\alpha 1 = 1$, $L(v)1 = 1$, $v = 1$, $L(v) = e$, $M(L') \cap M(L^3) = \{e\}$. Further, by Proposition 1 $Z(L) = L^3$ implies $Z(\mathfrak{M}) = M(L^3) \cong Z(L) \cong \mathfrak{D}$. $M(L^3)$ is a normal subgroup of $\mathfrak{M}$. Then from $\mathfrak{M} = M(L')M(L^3)$ it follows that $M(L')$ is also normal in $\mathfrak{M}$. Hence $\mathfrak{M} = M(L') \times \mathfrak{D}$. The quotient loop $\mathfrak{M}/M(L')$ is abelian. Then $\mathfrak{M}' \subseteq M(L')$. By Lemma 7 $M(L') \subseteq \mathfrak{M}'$. Hence $M(L') = \mathfrak{M}'$. Consequently, $\mathfrak{M} = \mathfrak{M}' \times \mathfrak{D}$, i.e. 2) implies 4). Conversely, if $\mathfrak{M} = \mathfrak{M}' \times \mathfrak{D}$ then $\mathfrak{M} = M(L') \times M(L^3)$, $\mathfrak{M}1 = M(L')1 \times M(L^3)1$. Hence, 4) implies 2).

Finally, the equality $\mathfrak{F}(\mathfrak{M}') = \mathfrak{M}'$ follows, by Lemma 5, from the relations $\mathfrak{F}(\mathfrak{M}) = \mathfrak{M}$, $\mathfrak{M}/M(L^3) \cong \mathfrak{M}'$, the equalities $\mathfrak{M}' = (L')^\star = \overline{\mathfrak{I}(L)}$ follow from

Lemma 7 and the equality $Z(\mathfrak{M}') = \{e\}$ follows from equalities $\mathfrak{M} = \mathfrak{M}' \times M(L^3)$, $Z(\mathfrak{M}) = M(L^3)$. This completes the proof of Theorem 1. □

3 Normalizer condition

Let M be a subset, H be a subgroup of group G. The subgroup $N_H(M) = \{h | h \in H, h^{-1}Mh = M\}$ is called *the normalizer* of the set M in the subgroup H [8]. Now, constructively, we define the notion of normalizer for the subloops of CML. Let H, K, where $H \subseteq K$ be subloops of CML L. We define inductively the sequences of sets $\{P_\alpha\}$ and $\{D_\alpha\}$ as follows:

i) $P_1 = \{x \in K | (H, H, x) \subseteq H\}$ and $D_1 = \{x \in K | (H, x, P_1) \subseteq H\}$;

ii) for any ordinal α, $P_{\alpha+1} = \{x \in K | (H, D_\alpha, x) \subseteq H\}$ and $D_{\alpha+1} = \{x \in K | (H, x, P_{\alpha+1}) \subseteq H\}$;

iii) if α is a limit ordinal, $P_\alpha = \bigcap_{\beta<\alpha} P_\beta$ and $D_\alpha = \bigcup_{\beta<\alpha} D_\beta$.

Further, we will also denote the conditions of item ii) by $(H, D_\alpha, \overline{P}_{\alpha+1})$ and $(H, \overline{D}_{\alpha+1}, P_{\alpha+1})$ respectively. H is a subloop of CML L, then from $(H, H, H) \subseteq H$, $(H, H, \overline{P})$ it follows that $H \subseteq P_1$, from $H \subseteq P_1$, $(H, \overline{D}_1, P_1)$, $(H, H, \overline{P}_1)$ it follows that $H \subseteq D_1$, from $(H, H, \overline{P}_1)$, $(H, D_1, \overline{P}_2)$, $H \subseteq D_1$ it follows that $P_1 \supseteq P_2$, from $(H, \overline{D}_1, P_2)$, $(H, \overline{D}_2, P_2)$, $P_1 \supseteq P_2$ it follows that $D_1 \subseteq D_2$. Further, let α be a non-limit ordinal and we suppose by inductive hypothesis that $D_\alpha \supseteq D_{\alpha+1}$ and $P_\alpha \subseteq P_{\alpha+1}$. Then from $D_\alpha \supseteq D_{\alpha+1}$, $(H, D_\alpha, \overline{P}_{\alpha+1})$, $(H, D_{\alpha+1}, \overline{P}_{\alpha+2})$ it follows that $P_{\alpha+1} \subseteq P_{\alpha+2}$ and from $P_{\alpha+1} \subseteq P_{\alpha+2}$, $(H, \overline{D}_{\alpha+2}, P_{\alpha\alpha+1})$, $(H, \overline{D}_{\alpha+2}, P_{\alpha\alpha+2})$ it follows that $D_{\alpha+1} \supseteq D_{\alpha+2}$. Hence, if consider also item iii), we get a sequence of subsets

$$P_1 \supseteq P_2 \supseteq \ldots \supseteq P_\alpha \supseteq \ldots$$

$$D_1 \subseteq D_2 \subseteq \ldots \subseteq D_\alpha \subseteq \ldots . \quad (5)$$

The construction process of subsets P_α, D_α from (5) shall end with an ordinal number, whose cardinality does not exceed the cardinality of CML K itself. We suppose that $P_{\alpha+1} = P_{\alpha+2} = \ldots$. From $(H, \overline{D}_{\alpha+1}, P_{\alpha+1})$, $(H, \overline{D}_{\alpha+2}, P_{\alpha+2})$ it follows that $D_{\alpha+1} = D_{\alpha+2}$. Then from $(H, D_{\alpha+1}, \overline{P}_{\alpha+2})$, $(H, \overline{D}_{\alpha+2}, P_{\alpha+2})$ it follows that $(H, \overline{D}_{\alpha+2}, \overline{P}_{\alpha+2})$. We remind that the inscriptions $\overline{D}_{\alpha+2}$, $\overline{P}_{\alpha+2}$ denote the biggest subsets $D_{\alpha+2}$ and $P_{\alpha+2}$ such that the relation $(H, D_{\alpha+2}, P_{\alpha+2}) \subseteq H$ holds true. H is a subloop of CML L, then from (3) it follows that $D_{\alpha+2} = P_{\alpha+2}$ and using (3), (4) it is easy to prove that $D_{\alpha+2}$ is a subloop of CML L. Hence $D_{\alpha+2}$ is the biggest (and the only) subloop of CML K where by (2) H is a normal subloop. By analogy with group theory the subloop D_α will be called *the normalizer* of subloop H in subloop K of CML L and will be denoted by $N_K(H)$. If the subgroup where the normalizer is taken from is not indicated, it means that it is taken from the entire CML L. Consequently, from the construction of normalizer follows

Proposition 3. *Let H, L, K, where $H \subseteq L \subseteq K$, be subloops of CML L and let H be a normal subloop of L. Then $L \subseteq N_K(H)$.*

The group theory contains studies of the group that satisfies the normalizer condition (see, for example,[8]). These are such groups, where every proper subgroup differs from its normalizer. A similar notion can be introduced for CML. We will say that a *CML satisfies the normalizer condition* or, in short, is an N-loop if every proper subloop differs from its normalizer.

The CML L will be a N-loop if and only if an ascending subnormal system $\{H_\alpha\}$ *passes through each subloop H of CML L.*

Really, we denote $H_0 = 1, H_1 = H$ (respect. $\mathfrak{N}_0 = e$, $\mathfrak{N}_1 = \mathfrak{N}$). Further, for non-limit α we take as H_α the normalizer of subloop $H_{\alpha-1}$, and for limit α H_α will be the union of all H_β for $\beta < \alpha$. This ascending subnormal system, obviously, reaches CML L itself. Conversely, if all subloops of CML L are contained in some ascending subnormal system, then all proper subloop will be normal in some bigger subloop, and, consequently, by Proposition 3, will differ from its normalizer.

Using this result it is easy to prove that *all subloops and all quotient loops of* N*-loop will be* N*-loops themselves.* Really, let A be a subloop of N-loop L, and let B be a subloop of L such that $B \subseteq A$. By the aforementioned, an ascending subnormal system $\{B_\alpha\}$ passes through B. Then $\{B_\alpha \cap A\}$ after removing the repetitions will be an ascending subnormal system of A, passing through B. Hence A will be an N-loop. The second statement is proved by analogy.

Theorem 2. *If a CML L with the Frattini subloop* $\mathfrak{F}(L)$ *satisfies the inequality* $\mathfrak{F}(L) \neq L$ *then it satisfies the normalizer condition.*

Proof. As $\mathfrak{F}(L) \neq L$ then the CML L has a maximal proper subloop. Let H be an arbitrary proper subloop of CML L. If H is a maximal subloop of L then by [5] H is normal in L. Hence $H \neq N_L(H) = L$. Let now the subloop H be a non-maximal subloop. By Zorn's Lemma let M be a maximal subloop of L with respect to the property $H \subseteq M$ and let $a \notin L \backslash M$. We suppose that $a^3 = 1$. Let $K =< H, a >$. M is a maximal proper subloop of L, then by [5] the subloop M is normal in L. Let φ be a restriction on K of homomorphism $L \to L/M$. Obviously, $\mathrm{Ker}\varphi = M \cap K$. As $a^3 = 1$ then $M \cap < a >= 1$. Hence $K \backslash < a >= H$ and then $M \cap K = H$. Consequently, H is a normal subloop of K, and as $H \neq K$ then by Proposition 1 $H \neq Z_L(H)$, as required.

Let now $a^3 \neq 1$. By Lemma 1 $a^3 \in Z(L)$, hence $< a^3 >$ is a normal subloop of L. Let $a^3 \in H$. We denote $L/ < a^3 >= \overline{L}$. From $a^3 \in M$, $a \notin M$ it follows that $\overline{M}$ is a maximal proper subloop of $\overline{L}$. Hence $\mathfrak{F}(\overline{L}) \neq \overline{L}$. Further, $\overline{a}^3 = \overline{1}$, then by the previous cases $\overline{H} \neq N(\overline{H})$. As $a^3 \in H$ and $a^3 \in N(H)$ then the inverse images of $\overline{H}$ and $N(\overline{H})$ will be H and $N(H)$ respectively. Hence from $\overline{H} \neq N(\overline{H})$ it follows that $H \neq N(H)$, as required.

If $a^3 \notin H$, then $H \neq H < a^3 >$. By (3) and Lemma 1 we get $(H, H < a^3 >, H < a^3 >) = (H, H, H) \subseteq H$. This means by Proposition 1 that $H < a^3 > \subseteq N(H)$. Hence $H \neq N(H)$. This completes the proof of Theorem 2. □

Any subloop of a ZA-loop is a ZA-loop. From Lemma 3 it follows that a non-associative commutative Moufang ZA-loop has a non-trivial associative quotient loop. Hence it differs from its associator subloop. Hence *any commutative Moufang ZA-loop is a SD-loop.*

Corollary 1. *For a CML L let $Z_2(L) \neq Z_1(L)$. In particular, let L be a ZA-loop or a SD-loop. Then the CML L satisfies the normalizer condition.*

Proof. We suppose that $\mathfrak{F}(L) = L$. Then by Theorem 1 $L = L' \times Z(L)$, $\mathfrak{F}(L') = L'$, $Z(L') = \{1\}$. From here it follows that $Z_2(L) = Z_1(L)$. Contradiction. Hence $\mathfrak{F}(L) \neq L$ and by Theorem 2 the CML L satisfies the normalizer condition, as required. □

In [5] it is proved that CML L is centrally nilpotent of class n if and only if the group $\mathfrak{M}$ is nilpotent of class $2n-1$. Then Corollary 1 for $\mathfrak{M}$ follows from the known result about subnormal subgroups of nilpotent group (see, for example,[8]).

Proposition 4. *If L is a centrally nilpotent CML of class n, then for any subloop $H \subseteq L$ (respect. subgroup $\mathfrak{N}$ of group $\mathfrak{M}$) the sequence of consecutive normalizers reaches L (respect. $\mathfrak{M}$) not later that after n (respect. $2n-1$) steps.*

Proof. Let (1) be the upper central series of CML L. We denote $H_0 = H$, $H_{i+1} = N_L(H_i)$. It is sufficient to check that $Z_i(L) \subseteq H_i$. For $i = 0$, this is obvious. We suppose that $Z_i(L) \subseteq H_i$. From the relation $Z_{i+1}(L)/Z_i(L) = Z(L/Z_i(L)$ it follows that $(Z_{i+1}(L), L, L) \subseteq Z_i(L)$. In particular, $(Z_i(L), Z_{i+1}(L), Z_{i+1}(L)) \subseteq Z_i(L)$. As $Z_i(L) \subseteq H_i$, then $(H_i, Z_{i+1}(L), Z_{i+1}(L)) \subseteq H_i$. But this means that $Z_{i+1}(L)$ normalizes H_i. Hence $Z_{i+1}(L) \subseteq H_{i+1}$. This completes the proof of Proposition 4. □

Remark. Theorem 1 (see, also, Theorem 3) reveals a strong analogy between the Frattini subloops of CML and the Frattini subgroups of the multiplication groups of CML. However for the multiplication group of CML the statement, analogous to Theorem 2, is not true. In [5] there is an example of CML G of exponent 3, such that $G' \neq G$ and $Z(G) = 1$. By Proposition 2 $\mathfrak{F}(G) \neq G$. Then by Proposition 1 $Z(\mathfrak{M}) = e$ and by Lemma 6 $\mathfrak{F}(M) \neq \mathfrak{M}$, where $\mathfrak{M}(G)$ denotes the multiplication group of G. In [6] J. D. H. Smith showed that no group with trivial centre and satisfying the normalizer condition can be the multiplication group of a quasigroup. Hence the multiplication group $\mathfrak{M}$ satisfies the inequality $\mathfrak{F}(\mathfrak{M}) \neq \mathfrak{M}$ but it does not satisfy the normalizer condition.

4 Divisible subgroups of multiplication group

We remind ([8] (respect.[2]) that the group (respect. CML) G is called *divisible* or *complete* (by terminology of [3] *radically complete*) if the equality $x^n = a$ has at least one solution in G, for any number $n > 0$ and any element $a \in G$.

Theorem 3. *Any divisible subgroup $\mathfrak{N}$ of a multiplication group $\mathfrak{M}$ of a CML L is an abelian group and serves as a direct factor for $\mathfrak{M}$, i.e. $\mathfrak{M} = \mathfrak{N} \times \mathfrak{C}$ for a certain subgroup $\mathfrak{C}$ of $\mathfrak{M}$.*

Proof. If $\mathfrak{F}(\mathfrak{M}) = \mathfrak{M}$ then the statement follows from Theorem 1. Hence let $\mathfrak{F}(\mathfrak{M}) \neq \mathfrak{M}$. Then $\mathfrak{M}$ has a maximal proper subgroups. The group $\mathfrak{M}$ is locally nilpotent [2], then the maximal proper subgroups of $\mathfrak{M}$ are normal in $\mathfrak{M}$ [8]. Let $\mathfrak{H}$ be a maximal proper subgroup of $\mathfrak{M}$ such that $\varrho \notin \mathfrak{H}$ for some $\varrho \in \mathfrak{N}$. We will consider two cases: ϱ has a finite order and ϱ has an infinite order.

Let the element ϱ have a finite order n. Then the element $\alpha = \varrho^{n/p}$, where p is a prime divisor of n, has the order p. The subgroup $\mathfrak{N}$ is divisible. Then there exists a sequence $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_k, \ldots$ of elements in $\mathfrak{N}$ such that $\alpha_1^p = e$, $\alpha_{k+1}^p = \alpha_k$, where e is the unit of $\mathfrak{M}$. From here it follows that $\alpha_k^{p^k} = e$, $k = 1, 2, \ldots$.

We denote by $\mathfrak{C}$ the subgroup of $\mathfrak{N}$ generated by $\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots$. It is easy to prove that any element $\alpha \in \mathfrak{C}$ is a power of some generator α_k, i.e. $\alpha = \alpha_k^n$, and the cyclic groups $< \alpha_k >$ form a sequence

$$e \subset < \alpha_1 > \subset < \alpha_2 > \subset \ldots \subset < \alpha_k > \subset \ldots .$$

We prove that $\mathfrak{C} \cap \mathfrak{H} = e$. $< \alpha_1 >$ is a cyclic group of order p and $\alpha_1 \notin \mathfrak{H}$. Then $< \alpha_1 > \cap \mathfrak{H} = e$. We suppose that $< \alpha_k > \cap \mathfrak{H} = e$. We have $< \alpha_{k+1} > = \{\alpha_{k+1}, \alpha_{k+1}^p, \ldots, \alpha_{k+1}^{p^{k+1}-1}\} \cup < \alpha_k >$. We suppose that $\alpha_{k+1}^n \in \mathfrak{H}$ $(n = 1, 2, \ldots, p^{k+1} - 1)$. Then $(\alpha_{k+1}^n)^p \in \mathfrak{H}$. But $(\alpha_{k+1}^n)^p = (\alpha_{k+1}^p)^n = \alpha_k^n$. Hence $\alpha_k^n \in \mathfrak{H}$. But this contradicts the supposition $< \alpha_k > \cap \mathfrak{H} = e$. Hence $< \alpha_{k+1} > \cap \mathfrak{H} = e$ and, consequently, $\mathfrak{C} \cap \mathfrak{H} = e$. Let $\mathfrak{M}'$ denote the commutator subgroup of group $\mathfrak{M}$. By Lemma 4 $\mathfrak{M}' \subseteq \mathfrak{H}$. Then $\mathfrak{M}' \cap \mathfrak{C} = e$, $\mathfrak{C}' = e$, hence $\mathfrak{C}$ is an abelian group. More concretely, $\mathfrak{C}$ is isomorphic to a quasicyclic p-group. Further, the subgroup $\mathfrak{H}$ as maximal in $\mathfrak{M}$ is normal in $\mathfrak{M}$. Then from $\mathfrak{M} = \mathfrak{H}\mathfrak{C}$, $\mathfrak{H} \cap \mathfrak{C} = e$ it follows that $\mathfrak{C}$ is normal in $\mathfrak{M}$. Hence $\mathfrak{M} = \mathfrak{H} \times \mathfrak{C}$.

Let now $\varrho \in \mathfrak{N}$ be an element of infinite order. If $Z(\mathfrak{M})$ denotes the centre of $\mathfrak{M}$ then $\mathfrak{M}/Z(\mathfrak{M})$ is a locally finite 3-group [5]. Hence $\varrho^n \in Z(\mathfrak{M})$ for some n. Let $\mathfrak{H}$ be a maximal subloop of $\mathfrak{M}$ such that $\varrho^n \notin \mathfrak{H}$. $\mathfrak{N}$ is a divisible group. Then there exists a sequence $\varrho^n = \alpha_1, \alpha_2, \ldots, \alpha_k, \ldots$ of elements in $\mathfrak{N}$ such that $\alpha_{k+1}^{k+1} = \alpha_k$, $k = 1, 2, \ldots$. We denote by $\mathfrak{Q}$ the subgroup of $\mathfrak{N}$, generated by $\varrho^n = \alpha_1, \alpha_2, \ldots, \alpha_k, \ldots$. As $\alpha_1 \in Z(\mathfrak{M})$ then it is easy to see that $\mathfrak{Q} \subseteq Z(\mathfrak{M})$. Hence the subgroup $\mathfrak{Q}$ is normal in $\mathfrak{M}$. The subgroup $\mathfrak{Q}$ is without torsion. In [4] it is proved that the commutator subgroup of the multiplication group of any CML is a locally finite 3-group. Then $\mathfrak{Q} \cap \mathfrak{M}' = e$. From here it follows that $\mathfrak{Q}' = e$, i.e. $\mathfrak{Q}$ is an abelian group. More concretely, $\mathfrak{Q}$ is isomorphic to the additive group of rationales.

Thus, in both cases in group $\mathfrak{M}$ there exists an abelian normal subgroup $\mathfrak{D} \subseteq \mathfrak{N}$, which is isomorphic to quasicyclic p-group or additive group of rationales such that $\mathfrak{M} = \mathfrak{H} \times \mathfrak{D}$. We will use this procedure of separating the divisible subgroup from $\mathfrak{N}$ as direct factor for defining the subgroups $\mathfrak{M}_\beta$, $\mathfrak{A}_\beta$ of group $\mathfrak{M}_{\beta-1}$.

Let $\mathfrak{M}_0 = \mathfrak{M}$, $\mathfrak{M}_1 = \mathfrak{H}$, $\mathfrak{D}_1 = \mathfrak{D}$. For a non-limit ordinal β inductively we define $\mathfrak{M}_{\beta-1} = \mathfrak{M}_\beta \times \mathfrak{D}_\beta$. We denote $\mathfrak{A}_\beta = \mathfrak{D}_1 \times \mathfrak{D}_2 \times \ldots \times \mathfrak{D}_\beta$. As $\mathfrak{D}_1, \mathfrak{D}_2, \ldots, \mathfrak{D}_\beta \subseteq \mathfrak{N}$ then $\mathfrak{A}_\beta \subseteq \mathfrak{N}$. Further we consider the sequences of subgroups

$$\mathfrak{A}_1 \subset \mathfrak{A}_2 \subset \ldots \subset \mathfrak{A}_\beta \subset \ldots ,$$

$$\mathfrak{M}_1 \supset \mathfrak{M}_2 \supset \ldots \supset \mathfrak{M}_\beta \supset \ldots, \beta < \alpha,$$

where $\mathfrak{M}_{\beta-1} = \mathfrak{M}_\beta \times \mathfrak{D}_\beta$ if β is a non-limit ordinal and $\mathfrak{A}_\beta = \cup_{\gamma<\beta}\mathfrak{A}_\gamma$, $\mathfrak{M}_\beta = \cap_{\gamma<\beta}\mathfrak{M}_\gamma$ if β is a limit ordinal.

It is clear that $\mathfrak{M}_\beta$, $\mathfrak{A}_\beta$ are normal subgroups of $\mathfrak{M}$. We prove that $\mathfrak{M} = \mathfrak{M}_\beta \times \mathfrak{A}_\beta$ for any β. If β is a non-limit ordinal, then by induction $\mathfrak{M} = \mathfrak{M}_1 \times \mathfrak{D}_1 = \mathfrak{M}_1 \times \mathfrak{A}_1 = \mathfrak{M}_{\beta-1} \times \mathfrak{A}_{\beta-1} = \mathfrak{M}_\beta \times \mathfrak{D}_\beta \times \mathfrak{A}_{\beta-1} = \mathfrak{M}_\beta \times \mathfrak{A}_\beta$. Hence $\mathfrak{M} = \mathfrak{M}_\beta \times \mathfrak{A}_\beta$.

Let now β be a limit ordinal and let $e \neq \lambda \in \mathfrak{M}_\beta \cap \mathfrak{A}_\beta$. Then there exists a non-ordinal $\delta < \beta$ such that $\lambda \in \mathfrak{A}_\delta$. From $\lambda \in \mathfrak{M}_\beta = \cap_{\gamma<\beta}\mathfrak{M}_\gamma$ it follows that $\lambda \in \mathfrak{M}_\gamma$ for all $\gamma < \beta$. But $\delta < \beta$. Then $\lambda \in \mathfrak{M}_\delta \cap \mathfrak{A}_\delta$. Contradiction. Hence $\mathfrak{M}_\beta \cap \mathfrak{A}_\beta = e$ and we may consider the direct product $\mathfrak{M}_\beta \times \mathfrak{A}_\beta$.

Let $\lambda \in \mathfrak{M}\backslash(\mathfrak{M}_\beta \times \mathfrak{A}_\beta)$. Then $\lambda \notin \mathfrak{M}_\beta$, $\lambda \notin \mathfrak{A}_\beta$, i.e. $\lambda \notin \cap_{\gamma<\beta}\mathfrak{M}_\gamma$, $\lambda \notin \cup_{\gamma<\beta}A_\gamma$. Hence $\lambda \notin \mathfrak{M}_\gamma$ for all $\gamma < \beta$ and from $\lambda \in \mathfrak{M}$, $\mathfrak{M} = \mathfrak{M}_\gamma \times \mathfrak{A}_\gamma$ it follows that $\lambda \in \cap_{\gamma<\beta}\mathfrak{A}_\gamma = \mathfrak{A}_\beta$. We wet a contradiction. Hence $\mathfrak{M} = \mathfrak{M}_\beta \times \mathfrak{A}_\beta$ for all β.

The process of inductive construction of $\mathfrak{A}_\alpha$ will end on the first number γ for which $\mathfrak{A}_\gamma = \mathfrak{N}$. Consequently, $\mathfrak{M} = \mathfrak{M}_\gamma \times \mathfrak{N}$. This completes the proof of Theorem 3. □

By Theorem 3 any multiplication group $\mathfrak{M}$ of CML contains a maximal divisible associative subloop $\mathfrak{D}$ and $\mathfrak{M} = \mathfrak{D} \times \mathfrak{R}$, where obviously $\mathfrak{R}$ is a *reduced CML*, meaning that it has no non-unitary divisible subgroups. Consequently, we obtain

Corollary 2. *Any multiplication group $\mathfrak{M}$ of CML L is a direct product of a divisible abelian subgroup $\mathfrak{D}$ and a reduced subgroup $\mathfrak{R}$. The subgroup $\mathfrak{D}$ is uniquely defined, the subgroup $\mathfrak{R}$ is defined up to isomorphism.*

Proof. Let us prove the last statement. As $\mathfrak{D}$ is the maximal divisible subgroup of the multiplication group $\mathfrak{M}$, then it is invariant with respect to the endomorphisms of the group $\mathfrak{M}$. Let $\mathfrak{M} = \mathfrak{D}' \times \mathfrak{R}'$, where $\mathfrak{D}'$ is a divisible subgroup, and $\mathfrak{R}'$ is a reduced subgroup of the group $\mathfrak{M}$. We denote by φ, ψ the endomorphisms $\varphi : \mathfrak{M} \to \mathfrak{D}', \psi : \mathfrak{M} \to \mathfrak{R}'$. As $\mathfrak{D}$ is invariant with respect to the endomorphisms of the group $\mathfrak{M}$, then $\varphi\mathfrak{D}$ and $\psi\mathfrak{D}$ are subgroups of the group $\mathfrak{M}$. It follows from the inclusions $\varphi\mathfrak{D} \subseteq \mathfrak{D}'$ and $\psi\mathfrak{D} \subseteq \mathfrak{R}'$ that $\varphi\mathfrak{D} \cap \psi\mathfrak{D} = 1$. By Theorem 3 $\mathfrak{D}$ is a abelian group, therefore $\varphi\mathfrak{D}, \psi\mathfrak{D}$ are normal in $\mathfrak{D}$. Then $d = \varphi d \cdot \psi d$ $(d \in \mathfrak{D})$ gives $\mathfrak{D} = \varphi\mathfrak{D} \cdot \psi\mathfrak{D}$, so $\mathfrak{D} = \varphi\mathfrak{D} \times \psi\mathfrak{D}$. Obviously, $\varphi\mathfrak{D} \subseteq \mathfrak{D} \cap \mathfrak{D}', \psi\mathfrak{D} \subseteq \mathfrak{D} \cap \mathfrak{R}'$, then $\varphi\mathfrak{D} = \mathfrak{D} \cap \mathfrak{D}', \psi\mathfrak{D} = \mathfrak{D} \cap \mathfrak{R}'$. Hence $\mathfrak{D} = (\mathfrak{D} \cap \mathfrak{D}') \times (\mathfrak{D} \cap \mathfrak{R}')$. But $\mathfrak{D} \cap \mathfrak{R}' = 1$ as a direct factor of a divisible group of a reduced group. Therefore, $\mathfrak{D} \cap \mathfrak{D}' \subseteq \mathfrak{D}, \mathfrak{D} \subseteq \mathfrak{D}'$, i.e. $\mathfrak{D} = \mathfrak{D}'$. This completes the proof of Corollary 2. □

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On a Method for Estimation of Risk Premiums Loaded by a Fraction of the Variance of the Risk

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Abstract. In this paper we have obtained linear approximations which are unbiased estimates for the expected value part, respectively for the variance part and finally for the fluctuation part of the loading from the variance premium, using the greatest accuracy theory. The article provides a means to approximate the separate parts of the variance loaded premium by linear non-homogeneous credibility estimators. Apart from the purpose of this paper, which is to simply add "credibility" like estimators for the separate parts of the variance premium, we have presented some basic theorems from statistics and some basic results on finding estimators with minimal mean squared error from probability theory. The fact that it is based on complicated mathematics, involving conditional expectations, needs not bother the user more than it does when he applies statistical tools like, discriminating analysis and scoring models.

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Introduction

It is an original paper which describes techniques for estimating premiums for risks, containing a fraction of the variance of the risk as a loading on the net risk premium. An approach "in this sense" is to consider the variance premium. The problem under discussion is to get linear approximations, which are unbiased estimates for the expected value part, variance part, fluctuation part, i.e. for the separate parts of the variance premium, using the classical model of Bühlmann and the credibility for the Esscher premiums. The present article contains a method to estimate risk premiums loaded by a fraction of the variance of the risk, as opposed to the net premiums studied thus far in the credibility theory.

The first section shows that the Esscher premium approaches the variance principle and that this premium is derived as an optimal estimator minimizing a suitable loss function. In the first section it is shown that the Esscher premium can be used as an approximation to the variance loaded premium, by truncating the development of a power series. Also, the approach of the problem of Esscher premium, followed in the first section is to consider the best linear credibility estimator which minimizes the exponentially weighted squared error loss function. The second section analyses and presents the linear non-homogeneous credibility estimators for the separate parts of the variance premium.

It turns out that the linear credibility approximations for each of the parts in the variance premium to coincide with the unbiased estimates for the expected value part, the variance part and the fluctuation part from the variance premium.

The approach of the problem of loaded premiums, followed in the second section is to simply add credibility - like estimators for the separate parts of the variance premium.

1 Techniques for estimating premiums for risks, containing a fraction of the variance of the risk as a loading on the net risk premium

1.1 The classical model of Bühlmann

Consider a portfolio of contracts $j = 1, \ldots, k$ satisfying the constraints (B_1) and (B_2). The index contract j is a random vector consisting of the structural variables θ_j and the observable variables: $X_{j1}, \ldots X_{jt}$, where $j = 1, \ldots, k$.

(B_1) $E[X_{jr}|\theta_j] = \mu(\theta_j)$ - the net premium for a contract with risk parameter θ_j-,Cov$[\underline{X}_j|\theta_j] = \sigma^2(\theta_j)I^{(t,t)}$, $j = 1, \ldots, k$, and:

(B_2) the contracts $j = 1, \ldots, k$ are independent, the variables $\theta_1, \ldots, \theta_k$ are identically distributed, and the observations X_{jr} have finite variance, then the optimal non-homogeneous linear estimators $\hat{\mu}(\theta_j)$ for $\mu(\theta_j)$, $j = 1, \ldots, k$, in the least squares sense read: $\hat{\mu}(\theta_j) = (1-z)m + zM_j$, where $M_j = \dfrac{1}{t}\displaystyle\sum_{s=1}^{t} X_{js}$ denotes the individual estimator for $\mu(\theta_j)$. The resulting credibility factor z which appears in the credibility adjusted estimator $\hat{\mu}(\theta_j)$ is found as: $z = at/(s^2 + at)$, with the structural parameters m, a and s^2 as defined by the following formulae:

$$m = E[X_{jr}] = E[\mu(\theta_j)],\ a = \mathrm{Var}[\mu(\theta_j)],\ s^2 = E[\sigma^2(\theta_j)],\ j = 1, \ldots, k.$$

Here the identity or unit matrix I denotes a matrix with unities on the diagonal and zeros elsewhere.

1.2 The credibility for the Esscher premiums

Minimizing weighted mean squared error

When X and Y are two random variables, and Y must be estimated using a function $g(X)$ of X, the choice yielding the minimal weighted mean squared error $E[(Y - g(X))^2 e^{hY}]$ is the quantity:

$$E[Ye^{hY}|X]/E[e^{hY}|X].$$

Indeed:

$$E[(Y - g(X))^2 e^{hY}] = E\{E[(Y - g(X))^2 e^{hY}|X]\} =$$
$$= \int E[(Y - g(x))^2 e^{hY}|X = x] \cdot dF_X(x).$$

For a fixed x, the integrand can be written as: $E[(Z-p)^2 e^{hZ}]$, with $p = g(x)$ and Z distributed as Y, given $X = x$ ($Z \overset{(P)}{\equiv} [Y|(X = x)]$). This quadratic form in p is minimized taking $p = E[Ze^{hZ}]/E[e^{hZ}]$ or what is the same $g(x) = E[Ye^{hY}|X = = x]/E[e^{hY}|X = x]$.

Indeed:

$$\varphi(p) \overset{not}{=} E[(Z-p)^2 e^{hZ}] = E(Z^2 e^{hZ}) + p^2 E(e^{hZ}) - 2pE(Ze^{hZ}),$$

so $\varphi(p)$ is the following quadratic form in $p : E[(Z-p)^2 e^{hZ}]$. We have to solve the following minimization problem: $\underset{p}{\text{Min}}\varphi(p)$. Since this problem is the minimum of a positive definite quadratic form, it suffices to find a solution with the first derivative equal to zero. Taking the first derivative with respect to p, we get the equation: $2pE(e^{hZ}) - 2E(Ze^{hZ}) = 0$. So: $p = E(Ze^{hZ})/E(e^{hZ})$, because: $\varphi''(p) = 2E(e^{hZ}) > 0$. If the integrand is chosen minimal for each x, the integral over all x is minimized, too.

Definition. The quantity $E[Ye^{hY}|X]/E[e^{hY}|X]$, denoted by $H[Y|X]$ and which minimizes the weighted mean squared error $E[(Y - g(X))^2 e^{hY}]$ in the above theoretical result, entitled "Minimizing weighted mean squared error" is called the Esscher premium for Y, given X.

Applying the formula $H[Y|X] = E[Ye^{hY}|X]/E[e^{hY}|X]$ to $Y = X_{t+1}, j$ and $X = \underline{X}_j = (X_{j1}, \ldots, X_{jt})'$, we see that the best risk premium - in the sense of minimal weighted mean squared error - to charge for period $(t+1)$ is the Esscher premium for $X_{t+1,j}$, given $\underline{X}_j = (X_{j1}, X_{j2}, \ldots, X_{jt})'$:

$$H[X_{t+1,j}|\underline{X}_j] \overset{not}{=} g(X_j) = E[X_{t+1,j}e^{hX_{t+1,j}}|\underline{X}_j]/E[e^{hX_{t+1,j}}|\underline{X}_j]. \tag{1.1}$$

Apart from the optimal credibility result (1.1) for this situation we can obtain the Esscher premium as an optimal estimator minimizing a suitable loss function.

The linear credibility formula for exponentially weighted squared error loss function requires not just the knowledge of a few natural structure parameters, but it is necessary that for the structure function some values of the moment generating function are known.

This is why the less refined approach followed in Section 2, is more useful in practice.

2 The credibility for the variance premiums

For a small h, the optimal credibility estimated for the variance loaded premium can be approximated as:

$$\begin{aligned} g(X_j) &\cong (E[X_{t+1,j}|\underline{X}_j] + hE[X^2_{t+1,j}|\underline{X}_j] + O(h^2))/(1 + hE[X_{t+1,j}|\underline{X}_j] + O(h^2)) \cong \\ &\approx (E[X_{t+1,j}|\underline{X}_j] + hE[X^2_{t+1,j}|\underline{X}_j] + O(h^2))(1 - hE[X_{t+1,j}|\underline{X}_j] + O(h^2)) = \\ &= E[X_{t+1,j}|\underline{X}_j] + h\text{Var}[X_{t+1,j}|\underline{X}_j] + O(h^2) \cong E[X_{t+1,j}|\underline{X}_j] + \\ &+ h\text{Var}[X_{t+1,j}|\underline{X}_j] = E[\mu(\theta_j)|\underline{X}_j] + h\{E[\sigma^2(\theta_j)|\underline{X}_j] + \text{Var}[\mu(\theta_j)|\underline{X}_j]\} \end{aligned} \tag{2.1}$$

approximating numerator and denominator of $g(X_j)$ up to the first order in h.

The purpose of this section is to get linear approximations for each of the terms in the right-hand side. We will derive unbiased estimates for the:

$$\begin{cases} \text{expected value part} & E[\mu(\theta_j)|\underline{X}_j] & (2.2) \\ \text{variance part} & E[\sigma^2(\theta_j)|\underline{X}_j] & (2.3) \\ \text{fluctuation part} & \text{Var}[\mu(\theta_j)|\underline{X}_j] & (2.4) \end{cases}$$

Remark. Another problem appears if we want to find an estimate for the random variable:

$$p(\theta) := \mu(\theta) + \alpha\sigma^2(\theta).$$

Minimizing the squared error would lead to the following credibility estimator:

$$E[p(\theta)|\underline{X}] = E[\mu(\theta)|\underline{X}] + \alpha E[\sigma^2(\theta)|\underline{X}], \tag{2.5}$$

without the fluctuation part, because there is the following basic result on finding estimators with minimal mean squared error.

Minimizing mean squared error for conditional distributions

When X and Y are random variables, the function $g(\cdot)$ of X estimating Y with minimal mean squared error is:

$$g^*(X) = E[Y|X].$$

Applying this theorem to $Y = p(\theta)$ and $X = \underline{X} = (X_1, \dots, X_t)'$ we obtain that the verification of the equality (2.5) is readily performed.

One might argue that this premium is more reasonable, since the policyholder, having himself a fixed though unknown risk parameter, should not pay for the uncertainty concerning his own risk parameter, only for the variation of his claims.

2.1 The main results of this paper

Here and as follows we present **the main results** leaving the detailed calculation to the reader.

A) An approximation for the expected value part:

The expected value part has been dealt with in Subsection 1.1. We recall the result:

$$\hat{\mu}(\theta_j) = (1 - z)m + zM_j \tag{2.6}$$

where

$$z = at/(s^2 + at), \quad M_j = \frac{1}{t}\sum_{r=1}^{t} X_{jr}, \quad a = \text{Var}[\mu(\theta_j)], \quad s^2 = E[\sigma^2(\theta_j)], \ (j = \overline{1,k})$$

(for more details, see [5] or [6]). One could approximate the expected value part by its best linear non-homogeneous credibility estimator (2.6).

B) An approximation of the fluctuation part:

Next we consider the fluctuation part:

$$\mathrm{Var}[\mu(\theta_j)|\underline{X}_j] \stackrel{def}{=} E\{(\mu(\theta_j) - E[\mu(\theta_j)|\underline{X}_j])^2|\underline{X}_j\}. \tag{2.7}$$

It is difficult to estimate this expression because of the appearance of $E[\mu(\theta_j)|\underline{X}_j]$.

However, one could approximate this expectation by its best linear non-homogeneous credibility estimator (2.6), and try to estimate:

$$E\{[\mu(\theta_j) - (1-z)m - zM_j]^2|\underline{X}_j\} \tag{2.8}$$

(see (2.6)), where $z = at/(s^2 + at)$.

To obtain **an approximation for the fluctuation part**, this quantity is averaged once more over the entire collective (the averaging is representative for the conditioned variance, because:

$$\mathrm{Var}[\mu(\theta_j)|\underline{X}_j] \stackrel{(2.7)}{=} E\{(\mu(\theta_j) - E[\mu(\theta_j)|\underline{X}_j])^2|\underline{X}_j\}$$
$$\stackrel{(2.6)}{=} E\{[\mu(\theta_j) - (1-z)m - zM_j]^2|\underline{X}_j\}) :$$

$$\begin{aligned}
&E[E\{[\mu(\theta_j) - (1-z)m - zM_j]^2|\underline{X}_j\}] = E[E\{[\mu(\theta_j) - m - z(M_j - m)]^2|\underline{X}_j\}] = \\
&= E\{[\mu(\theta_j) - m - z(M_j - m)]^2\} = E[\mu^2(\theta_j) + m^2 + z^2(M_j - m)^2 - 2m\mu(\theta_j) + \\
&+2mz(M_j - m) - 2\mu(\theta_j)z(M_j - m)] = E[\mu^2(\theta_j)] + m^2 + z^2E[(M_j - m)^2] - \\
&-2mE[\mu(\theta_j)] + 2mzE[(M_j - m)] - 2zE[\mu(\theta_j)(M_j - m)] = E[\mu^2(\theta_j)] + m^2 + \\
&+z^2E\{[M_j - E(M_j)]^2\} - 2m\cdot m - 2zE\{[\mu(\theta_j) - m][M_j - m]\} = E[\mu^2(\theta_j)] - \\
&-m^2 + z^2\mathrm{Var}(M_j) - 2zE\{(\mu(\theta_j) - E[\mu(\theta_j)])[M_j - E(M_j)]\} = E[\mu^2(\theta_j)] - \\
&-E^2[\mu(\theta_j)] + z^2\mathrm{Var}(M_j) - 2z\mathrm{Cov}[\mu(\theta_j), M_j] = \mathrm{Var}[\mu(\theta_j)] + z^2\mathrm{Var}(M_j) - \\
&-2z\mathrm{Cov}[\mu(\theta_j), M_j] = \mathrm{Var}[\mu(\theta_j)] - 2z\mathrm{Cov}[\mu(\theta_j), M_j] + z^2\mathrm{Var}(M_j) = \\
&= a - 2za + z^2(a + s^2/t) = a(1 - 2z + z^2) + z^2s^2/t = a(1-z)^2 + z^2s^2/t,
\end{aligned} \tag{2.9}$$

because:

$$E(M_j) = m \tag{2.10}$$

$$E[\mu(\theta_j)] = E[E(X_{jr}|\theta_j)] = E(X_{jr}) \tag{2.11}$$

$$\mathrm{Var}[\mu(\theta_j)] = a \tag{2.12}$$

(see the definition of the structure parameter a).

$$\mathrm{Cov}[\mu(\theta_j), M_j] = \frac{1}{t}\sum_{r=1}^{t}\mathrm{Cov}[\mu(\theta_j), X_{jr}] = \frac{1}{t}\sum_{r=1}^{t}a = \frac{1}{t}ta = a, \tag{2.13}$$

$$\mathrm{Var}(M_j) = \mathrm{Cov}(M_j, M_j) = \frac{1}{t^2}\sum_{r,r'} \mathrm{Cov}(X_{jr}, X_{jr'}) = \frac{1}{t^2}\sum_{r,r'}(a + \delta_{rr'}s^2) =$$
$$= \frac{1}{t^2}\sum_{r}\left[(a+\delta_{rr}s^2) + \sum_{r',r'\neq r}(a+\delta_{rr'}s^2)\right] = \frac{1}{t^2}\sum_{r=1}^{t}[(a+s^2)+(t-1)a] = \tag{2.14}$$
$$= \frac{1}{t^2}\sum_{r=1}^{t}(s^2+at) = \frac{t(s^2+at)}{t^2} = \frac{s^2+at}{t} = a + \frac{s^2}{t}.$$

But inserting the value of the credibility factor z in the right hand side of (2.9) shows that it equals $(1-z)a$, so:

$$\mathrm{Var}[\mu(\theta_j)|\underline{X}_j] \cong E[E\{[\mu(\theta_j) - (1-z)m - zM_j]^2|\underline{X}_j\}] =$$
$$= a(1-z)^2 + \frac{z^2s^2}{t} = a\left(1 - \frac{at}{at+s^2}\right)^2 + \frac{s^2}{t}\cdot\frac{a^2t^2}{(at+s^2)^2} = \tag{2.15}$$
$$= \frac{as^4 + a^2s^2t}{(at+s^2)^2} = \frac{as^2(s^2+at)}{(at+s^2)^2} = \frac{as^2}{at+s^2} = a(1-z) = (1-z)a,$$

C) An approximation for the variance part:

For **the variance part**, there is in analogy with the expected value part, $E[\sigma^2(\theta_j)|\underline{X}_j]$ that is approximated as a non-homogeneous linear combination:

$$E[\sigma^2(\theta_j)|\underline{X}_j] \cong c_0 + c_1S_j^2 \tag{2.16}$$

where

$$S_j^2 = \sum_{s=1}^{t}(X_{js} - \overline{X}_j)^2/(t-1). \tag{2.17}$$

The following distance will be minimized:

$$E\{[\sigma^2(\theta_j) - c_0 - c_1S_j^2]^2\}. \tag{2.18}$$

So, for each $j = \overline{1,k}$ we have to solve the following minimization problem:

$$\underset{c_0,c_1}{\mathrm{Min}}\, E\{[\sigma^2(\theta_j) - c_0 - c_1S_j^2]^2\}. \tag{2.19}$$

As (2.19) is the minimum of a positive definitive quadratic form, it is enough to find a solution with all partial derivates equal to zero. Taking the partial derivative with respect to c_0 results in:

$$c_0 = E[\sigma^2(\theta_j)](1-c_1), \tag{2.20}$$

because if:

$$f(c_0,c_1) \overset{not.}{=} E\{[\sigma^2(\theta_j) - c_0 - c_1S_j^2]^2\} = E\{[\sigma^2(\theta_j)]^2\} + c_0^2 + c_1^2E[(S_j^2)^2] -$$
$$-2c_0E[\sigma^2(\theta_j)] + 2c_0c_1E(S_j^2) - 2c_1E[\sigma^2(\theta_j)S_j^2],$$

then $\frac{\partial f}{\partial c_0} = 0$ implies: $2c_0 - 2E[\sigma^2(\theta_j)] + 2c_1 E(S_j^2) = 0$, that is the verification of the equality (2.20) that is readily performed (see (2.24)). Inserting the result (2.20) in (2.19) we obtain:

$$\operatorname*{Min}_{c_1} E\{[\sigma^2(\theta_j) - E(\sigma^2(\theta_j))(1-c_1) - c_1 S_j^2]^2\}. \tag{2.21}$$

Taking the derivative with respect to c_1, gives:

$$\mathrm{Cov}[\sigma^2(\theta_j), S_j^2] = c_1 \mathrm{Cov}[S_j^2, S_j^2], \tag{2.22}$$

because if:

$$f(c_1) \overset{not.}{=} E\{[\sigma^2(\theta_j) - E(\sigma^2(\theta_j))(1-c_1) - c_1 S_j^2]^2\} = E\{[\sigma^2(\theta_j)]^2 + E^2[\sigma^2(\theta_j)]\cdot$$
$$\cdot(1-c_1)^2 + c_1^2(S_j^2)^2 - 2\sigma^2(\theta_j)E[\sigma^2(\theta_j)](1-c_1) - 2\sigma^2(\theta_j)c_1 S_j^2 +$$
$$+2E[\sigma^2(\theta_j)](1-c_1)c_1 S_j^2\} = E\{[\sigma^2(\theta_j)]^2\} + E^2[\sigma^2(\theta_j)](1-c_1)^2 + c_1^2 E[(S_j^2)^2] -$$
$$-2E^2[\sigma^2(\theta_j)](1-c_1) - 2c_1 E[\sigma^2(\theta_j)S_j^2] + 2E[\sigma^2(\theta_j)](1-c_1)c_1 E(S_j^2),$$

then $\frac{\partial f}{\partial c_1} = 0$ implies:

$$-2E^2[\sigma^2(\theta_j)](1-c_1) + 2c_1 E[(S_j^2)^2] + 2E^2[\sigma^2(\theta_j)] - 2E[\sigma^2(\theta_j)S_j^2] +$$
$$+2E[\sigma^2(\theta_j)]E(S_j^2)\cdot(1-2c_1) = 0,$$

that is:

$$\begin{aligned} &-E^2[\sigma^2(\theta_j)] + c_1 E^2[\sigma^2(\theta_j)] + c_1 E[(S_j^2)^2] + E^2[\sigma^2(\theta_j)] - E[\sigma^2(\theta_j)S_j^2] + \\ &+E[\sigma^2(\theta_j)]E(S_j^2) - 2E[\sigma^2(\theta_j)]E(S_j^2)c_1 = 0. \end{aligned} \tag{2.23}$$

But

$$\sigma^2(\theta_j) = E(S_j^2|\theta_j) \quad \text{and so} \quad E[\sigma^2(\theta_j)] = E[E(S_j^2|\theta_j)] = E(S_j^2). \tag{2.24}$$

Now after plugging (2.24) in (2.23) we obtain:

$$E[\sigma^2(\theta_j)S_j^2] - E[\sigma^2(\theta_j)]E(S_j^2) = c_1\{E[(S_j^2)^2] - E^2(S_j^2)\},$$

that is

$$\mathrm{Cov}[\sigma^2(\theta_j), S_j^2] = c_1 \mathrm{Cov}(S_j^2, S_j^2), \quad \text{or} \quad \mathrm{Cov}[\sigma^2(\theta_j), S_j^2] = c_1 \mathrm{Var}(S_j^2)$$

and so the verification of the equality (2.22) is readily performed. But:

$$\begin{aligned} &\mathrm{Cov}[\sigma^2(\theta_j), S_j^2] = E\{\mathrm{Cov}[\sigma^2(\theta_j), S_j^2|\theta_j]\} + \mathrm{Cov}\{E[\sigma^2(\theta_j)|\theta_j], E(S_j^2|\theta_j)\} = \\ &= E\{E[\sigma^2(\theta_j)S_j^2|\theta_j] - E[\sigma^2(\theta_j)|\theta_j]E(S_j^2|\theta_j)\} + \mathrm{Cov}[\sigma^2(\theta_j), \sigma^2(\theta_j)] = \\ &= E[\sigma^2(\theta_j)E(S_j^2|\theta_j) - \sigma^2(\theta_j)\sigma^2(\theta_j)] + \mathrm{Var}[\sigma^2(\theta_j)] = E[\sigma^2(\theta_j)\sigma^2(\theta_j) - \\ &-\sigma^2(\theta_j)\sigma^2(\theta_j)] + \mathrm{Var}[\sigma^2(\theta_j)] = \mathrm{Var}[\sigma^2(\theta_j)], \end{aligned} \tag{2.25}$$

$$\mathrm{Var}(S_j^2) = \mathrm{Var}[E(S_j^2|\theta_j)] + E[\mathrm{Var}(S_j^2|\theta_j)] = \mathrm{Var}[\sigma^2(\theta_j)] + E[\mathrm{Var}(S_j^2|\theta_j)]. \quad (2.26)$$

Inserting (2.25) and (2.26) in (2.22), the value of c_1 follows as:

$$c_1 = \mathrm{Var}[\sigma^2(\theta_j)]/\{\mathrm{Var}[\sigma^2(\theta_j)] + E[\mathrm{Var}(S_j^2|\theta_j)]\}. \quad (2.27)$$

We have

$$\mathrm{Var}(S_j^2|\theta_j) = 2\sigma^4(\theta_j)/(t-1) + O(t^{-2}) \cong 2\sigma^4(\theta_j)/(t-1), \quad (2.28)$$

for large values of t, and under the assumption of normality we get:

$$\mu_4(\theta_j) = 3\sigma^4(\theta_j), \quad (2.29)$$

because from statistics we recall some basic theorems:

(I) Suppose that X is a random variable, with Normal (μ, σ^2) distribution and in addition for all $r \in \mathbf{N}$:

$$\mu_{2r} = E[(X-\mu)^{2r}],$$

then:

$$\mu_{2r} = \frac{(2r)!}{2^r r!}\sigma^{2r}.$$

(II) Suppose that $X_1, X_2, \ldots, X_n$ are independent random variables with the same expectations μ and the variance σ^2, and in addition for each r:

$$\mu_4 = E[(X_r - \mu)^4].$$

Let $\tilde{S}^2$ be defined as: $\tilde{S}^2 = \dfrac{1}{n-1}\displaystyle\sum_{i=1}^{n}(X_i - \bar{X})^2$ the sample variance, where $\bar{X}$ is the sample mean of n i. i. d. random variables $X_1, X_2, \ldots, X_n$, that is:

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Then the following relation is valid:

$$\mathrm{Var}\,(\tilde{S}^2) = \frac{1}{n}\left(\mu_4 - \frac{n-3}{n-1}\sigma^4\right).$$

Here $(X_{js}|\theta_j)$, $s = \overline{1,t}$ are n i. i. d. random variables, with: $E(X_{js}|\theta_j) = \mu(\theta_j)$, $\mathrm{Var}\,(X_{js}|\theta_j) = \sigma^2(\theta_j)$ and $E\{[X_{js} - E(X_{js}|\theta_j)]^4|\theta_j\} = \mu_4(\theta_j)$ for all $s = \overline{1,t}$.

Let j be fixed. Under the assumption of normality we get: $(X_{js}|\theta_j) \in N(\mu(\theta_j), \sigma^2(\theta_j))$ for all $s = \overline{1,t}$. Applying result **(I)** to $X = (X_{js}|\theta_j)$, for all $s = \overline{1,t}$ and $r = 2$ we have:

$$\mu_4(\theta_j) = \frac{(2\cdot 2)!}{2^2\cdot 2!}(\sigma^2(\theta_j))^2 = 3\sigma^4(\theta_j).$$

So the verification of the equality (2.29) is readily performed. Applying result **(II)** to $\tilde{S}^2 = (S_j^2|\theta_j)$ we obtain:

$$\mathrm{Var}\,(S_j^2|\theta_j) = \frac{1}{t}\left[\mu_4(\theta_j) - \frac{t-3}{t-1}\sigma^4(\theta_j)\right] = \frac{1}{t}\left[3\sigma^4(\theta_j) - \frac{t-3}{t-1}\sigma^4(\theta_j)\right] =$$

$$= \frac{1}{t} \cdot \frac{3(t-1)-t+3}{t-1} \cdot \sigma^4(\theta_j) = \frac{2t}{t(t-1)}\sigma^4(\theta_j) = \frac{2\sigma^4(\theta_j)}{t-1}.$$

So the verification of the equality (2.28) is already performed.

Let $a^* = \mathrm{Var}[\sigma^2(\theta_j)]$, $s^{2*} = E[\sigma^4(\theta_j)]$, then we obtain for large values of t the following approximation:

$$c_1 \overset{(2.27)}{=} a^*/\{a^*+E[\mathrm{Var}(S_j^2|\theta_j)]\} \overset{(2.28)}{=} a^*/\{a^*+2E[\sigma^4(\theta_j)]/(t-1)\} = a^*/\{a^*+2s^{2^*}/(t-1)\}.$$

Consequently one obtains the following linear estimator for the variance part of the loading, i.e. the conditional expectation of $\sigma^2(\theta_j)$:

$$E[\sigma^2(\theta_j)|\underline{X}_j] \cong (1-c_1)E[\sigma^2(\theta_j)] + c_1 S_j^2$$

(see (2.16) and (2.20)).

3 Conclusions

In this paper we have obtained linear approximations which are unbiased estimates for the expected value part (i.e. for the conditional expectation of $\mu(\theta_j)$), respectively for the variance part (i.e. for the conditional expectation of $\sigma^2(\theta_j)$) and finally for the fluctuation part (the conditional variance of $\mu(\theta_j)$) of the loading from the variance premium, using the greatest accuracy theory.

The present article contains a method to estimate risk premiums loaded by a fraction of the variance of the risk, as opposed to the net premiums studied thus far in the credibility theory.

The first section shows that the Esscher premium approaches the variance principle and that this premium is derived as an optimal estimator minimizing a suitable loss function. So, in the first section it is shown that it can be used as an approximation to the variance loaded premium, by truncating a series expansion. Also, the approach of the problem of Esscher premium, followed in the first section is to consider the best linear credibility estimator which minimizes the exponentially weighted squared error loss function.

The second section analyses and presents the linear non-homogeneous credibility estimators for the separate parts of the variance premium. It happens that the linear credibility approximations for each of the parts in the variance premium to coincide with the unbiased estimates for the expected value part, the variance part and the fluctuation part from the variance premium. The approach of the problem of loaded premiums, followed in the second section is to simply add "credibility" like estimators for the separate parts of the variance premium.

So, the problem under discussion is to get linear approximations, which are unbiased estimates for the expected value part, variance part, fluctuation part, i.e. for the separate parts of the variance premium, using the classical model of Bühlmann and the credibility for the Esscher premiums.

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Properties of covers in the lattice of group topologies for nilpotent groups

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Abstract. A nilpotent group $\widehat{G}$ and two group topologies $\widehat{\tau}''$ and $\widehat{\tau}*$ on $\widehat{G}$ are constructed such that $\widehat{\tau}*$ is a coatom in the lattice of all group topologies of the group $\widehat{G}$ and such that between $\inf\{\widehat{\tau}'', \widehat{\tau}_d\}$ and $\inf\{\widehat{\tau}'', \widehat{\tau}*\}$ there exists an infinite chain of group topologies.

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1 Introduction

As is known, in any modular lattice, the lengths of any finite unrefinable chains with the same ends are equal. Moreover, the lengths of finite unrefinable chains do not become greater if we take the infimum or the supremum in these lattices.

The lattice of all group topologies for a nilpotent group need not be modular [1]. However, as is shown in [2], in the lattice of all group topologies on a nilpotent group, the lengths of any finite unrefinable chains which have the same ends are equal. Moreover, in the same article it is shown that the lengths of any finite unrefinable chains do not become greater if we take the supremum.

Given the above, it was natural to expect that the lengths of any finite unrefinable chains do not become greater if in the lattice of all group topologies for a nilpotent group we take the infimum. However, as shown in this article, it is not the case.

To present the further results we need the following known result (see [3], page 203):

Theorem 1. *Let $\mathcal{B}$ be a collection of subsets of a group $G(\cdot)$ such that the following conditions are satisfied:*

1) $e \in V$ for any $V \in \mathcal{B}$, where e is the unity element in the group $G(\cdot)$;

2) for any $V_1, V_2 \in \mathcal{B}$ there exists $V_3 \in \mathcal{B}$ such that $V_3 \subseteq V_1 \bigcap V_2$;

3) for any $V_1 \in \mathcal{B}$ there exists $V_2 \in \mathcal{B}$ such that $V_2 \cdot V_2 \subseteq V_1$;

4) for any $V_1 \in \mathcal{B}$ there exists $V_2 \in \mathcal{B}$ such that $V_2^{-1} \subseteq V_1$;

5) for any $V_1 \in \mathcal{B}$ and any element $g \in G$ there exists $V_2 \in \mathcal{B}$ such that $g \cdot V_2 \cdot g^{-1} \subseteq V_1$.

Then there exists a unique group topology τ on the group $G(\cdot)$ for which the collection $\mathcal{B}$ is a basis of neighborhoods of the unity element e [1] *(see [3], page 26).*

From Theorem 1 follows easily:

Corollary 2. Let group topologies τ_1 and τ_2 be defined on a group $G(\cdot)$. If $\mathcal{B}_1$ and $\mathcal{B}_2$ are bases of neighborhoods of the unity element in topological groups (G, τ_1) and (G, τ_2), respectively, then the collection $\mathcal{B} = \{U \bigcap V \mid U \in \mathcal{B}_1, V \in \mathcal{B}_2\}$ is a basis of neighborhoods of the unity element in the topological group (G, τ), where $\tau = \sup\{\tau_1, \tau_2\}$.

2 Basic results

To state basic results we need the following notations:

Notations 3.

3.1. $\mathbb{N}$ is the set of all natural numbers, $\mathbb{Z}$ is the set of all integers and $\mathbb{R}(+,\cdot)$ is the field of real numbers;

3.2. G is the set of all matrices $\begin{pmatrix} 1 & a_{1,2} & a_{1,3} \\ 0 & 1 & a_{2,3} \\ 0 & 0 & 1 \end{pmatrix}$ of the dimension 3×3 over the field $\mathbb{R}$ of real numbers such that $a_{i,i} = 1$ for $1 \leq i \leq 3$ and $a_{i,j} = 0$ for $1 \leq j < i \leq 3$,

$$G' = \left\{ \begin{pmatrix} 1 & a_{1,2} & a_{1,3} \\ 0 & 1 & a_{2,3} \\ 0 & 0 & 1 \end{pmatrix} \in G \mid a_{1,3} = a_{2,3} = 0 \right\};$$

$$G'' = \left\{ \begin{pmatrix} 1 & a_{1,2} & a_{1,3} \\ 0 & 1 & a_{2,3} \\ 0 & 0 & 1 \end{pmatrix} \in G \mid a_{1,2} = a_{1,3} = 0 \right\};$$

$$G(A) = \left\{ \begin{pmatrix} 1 & a_{1,2} & a_{1,3} \\ 0 & 1 & a_{2,3} \\ 0 & 0 & 1 \end{pmatrix} \in G \mid a_{1,2} = 0 \text{ and } a_{1,3} \in A \right\}$$ for any subgroup $A(+)$ of the group $\mathbb{R}(+)$;

3.3. $G_i(\cdot) = G(\cdot)$, $G'_i(\cdot) = G'(\cdot)$ and $G''_i(\cdot) = G''(\cdot)$ for every natural number i;

3.4. $G_i(A) = G(A)$ for every natural number i and any subgroup $A(+)$ of the group $\mathbb{R}(+)$;

3.5. $\widehat{G} = \sum\limits_{i=1}^{\infty} G_i$, $\widehat{G'} = \sum\limits_{i=1}^{\infty} G'_i$ and $\widehat{G''} = \sum\limits_{i=1}^{\infty} G''_i$;

3.6. $\widetilde{V_n} = \{\widetilde{g} \in \widetilde{G} \mid pr_i(\widetilde{g}) = e_i \text{ if } i \leq n\}$ for any $n \in \mathbb{N}$;

3.7. $\widehat{G}_k(A) = \{\widehat{g} \in \widehat{G} | pr_k(\widehat{g}) \in G''_k(A) \text{ and } pr_j(\widehat{g}) = \{e\} \text{ if } j \neq k\}$, where $k \in \mathbb{N}$ and $A(+)$ is a subgroup of the group $\mathbb{R}(+)$;

3.8. $\widehat{G}(A, S) = \{\widehat{g} \in \widehat{G} \mid pr_i(\widehat{g}) \in G_i(A) \text{ if } i \in S \text{ and } pr_j(\widehat{g}) \in G''_j \text{ if } j \notin S\}$, where $A(+)$ is a subgroup of the group $\mathbb{R}(+)$ and $S \subseteq \mathbb{N}$;

[1] As usual, the set V is called a neighborhood of an element a in the topological space (X, τ) if $a \in U \subseteq V$ for some $U \in \tau$.

3.9. τ_i is discrete in the group G_i and $\widetilde{\tau}$ is Tikhonov topology of the direct product $\widetilde{G} = \prod\limits_{i=1}^{\infty}(G_i, \tau_i)$.

Remark 4. It is easy to see that G with the usual operation of matrix multiplication is a group.

Since $\begin{pmatrix}1 & a & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{pmatrix}^{-1} = \begin{pmatrix}1 & -a & a\cdot c - b\\ 0 & 1 & -c\\ 0 & 0 & 1\end{pmatrix}$, then

$$\begin{pmatrix}1 & a & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & x & y\\ 0 & 1 & z\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & a & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{pmatrix}^{-1}\cdot\begin{pmatrix}1 & x & y\\ 0 & 1 & z\\ 0 & 0 & 1\end{pmatrix}^{-1} = \begin{pmatrix}1 & 0 & -y\cdot c + a\cdot z\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},$$

and as the center of the group G contains any matrix of the form $\begin{pmatrix}1 & 0 & d\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$ for $d \in \mathbb{R}$, then $G(\cdot)$ is a nilpotent group and its nilpotency index is 2.

In addition, since $\begin{pmatrix}1 & a & a_{1,3}\\ 0 & 1 & a_{2,3}\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 0 & b_{1,3}\\ 0 & 1 & b_{2,3}\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & a & a_{1,3}+b_{1,3}\\ 0 & 1 & a_{2,3}+b_{2,3}\\ 0 & 0 & 1\end{pmatrix}$, then $G'(\cdot)$, $G''(\cdot)$ and $G(A)(\cdot)$ are subgroups of the group $G(\cdot)$ for any subgroup $A(+)$ of the additive group of the field $\mathbb{R}(+,\cdot)$.

Proposition 5. *For the group $\widehat{G}$ the following statements are true:*

Statement 5.1. *The collection $\mathcal{B}' = \{\widetilde{V}_i \bigcap \widehat{G}' | i \in \mathbb{N}\}$ satisfies the conditions of Theorem 1, and hence, it is a basis of neighborhoods of the unity element for a group topology $\widehat{\tau}'$ on the group $\widehat{G}$;*

Statement 5.2. *The collection $\mathcal{B}'' = \{\widetilde{V}_i \bigcap \widehat{G}'' | i \in \mathbb{N}\}$ satisfies the conditions of Theorem 1, and hence, is a basis of neighborhoods of the unity element for a group topology $\widehat{\tau}''$ on the group $\widehat{G}$;*

Statement 5.3. *If A is a subgroup of the group $\mathbb{R}(+)$ of the field $\mathbb{R}(+,\cdot)$, and $\mathcal{F}$ is the Frechet filter*[2] *on the set $\mathbb{N}$, then the collection $\mathcal{B}(A,\mathcal{F}) = \{\widehat{G}(A,F) \bigcap \widetilde{V_n} | F \in \mathcal{F}, n \in \mathbb{N}\}$ satisfies all the conditions of Theorem 1, and hence, it is a basis of neighborhoods of the unity element for a group topology $\widehat{\tau}(A,\mathcal{F})$ on the group $\widehat{G}$.*

Proof. Since $\widehat{G}(A, F \bigcap S) \subseteq \widehat{G}(A,F) \bigcap \widehat{G}(A,S)$ for any subgroup $A(+)$ of the group $\mathbb{R}(+)$ and any subsets $S \subseteq \mathbb{N}$ and $F \subseteq \mathbb{N}$ for which $\widetilde{V}_i \subseteq \widetilde{V}_k$ if $k \leq i$, then any of the mentioned collections satisfies condition 2 of Theorem 1.

In addition, taking into consideration the definitions of sets $\widetilde{V}_n$, $\widehat{G}'$, $\widehat{G}''$, and $\widehat{G}(A,F)$ we obtain that any set from the collection $\mathcal{B}' \bigcup \mathcal{B}'' \bigcup \mathcal{B}(A,\mathcal{F})$ is a subgroup of the group $\widehat{G}(\cdot)$, and hence, any collection $\mathcal{B}'$, $\mathcal{B}''$, and $\mathcal{B}(A,\mathcal{F})$ satisfies conditions 1, 3 and 4 of Theorem 1.

[2] i.e. $\mathbb{N} \setminus \{1, \dots, k\} \in \mathcal{F}$ for every $k \in \mathbb{N}$.

To complete the proof of the theorem it remains to verify that for any of the mentioned collections condition 5 of Theorem 1 is also satisfied.

Let $\widehat{g} \in \widehat{G}$, then there exists a natural number n such that $pr_i(\widehat{g}) = e_i$ for $i > m$.

If $\widetilde{V_k} \bigcap \widehat{G}' \in \mathcal{B}'$ and $m = max\{k, n\}$, then $\widehat{g} \cdot \widehat{a} \cdot \widehat{g}^{-1} = \widehat{a}$ for any $\widehat{a} \in \widetilde{V}_m \bigcap \widehat{G}'$, and hence,

$$\widehat{g} \cdot (\widetilde{V}_m \bigcap \widehat{G}') \cdot \widehat{g}^{-1} = \widetilde{V}_m \bigcap \widehat{G}' \subseteq \widetilde{V_k} \bigcap \widehat{G}',$$

i.e. condition 5 of Theorem 1 holds for the collection $\mathcal{B}'$.

Analogously, if $\widetilde{V_k} \bigcap \widehat{G}'' \in \mathcal{B}''$ and $M = max\{k, n\}$, then $\widehat{g} \cdot \widehat{a} \cdot \widehat{g}^{-1} = \widehat{a}$ for any $\widehat{a} \in \widetilde{V}_m \bigcap \widehat{G}''$, and hence,

$$\widehat{g} \cdot (\widetilde{V}_m \bigcap \widehat{G}'') \cdot \widehat{g}^{-1} = \widetilde{V}_m \bigcap \widehat{G}'' \subseteq \widetilde{V_k} \bigcap \widehat{G}'',$$

i.e. condition 5 of Theorem 1 holds for the collection $\mathcal{B}''$.

If $\widehat{V}(A, F) \bigcap \widetilde{V}_k \in \mathcal{B}(A\mathcal{F})$ and $m = max\{n, k\}$, then $\widehat{V}(A, F) \bigcap \widetilde{V}_m \subseteq \widehat{V}(A, F) \bigcap \widetilde{V}_k$ and $\widehat{g} \cdot \widehat{a} \cdot \widehat{g}^{-1} = \widehat{a}$ for any $\widehat{a} \in \widehat{V}(A, F) \bigcap \widetilde{V}_m$, and hence,

$$\widehat{g} \cdot (\widehat{V}(A, F) \bigcap \widetilde{V_m}) \cdot \widehat{g}^{-1} = \widehat{V}(A, F) \bigcap \widetilde{V_m} \subseteq \widehat{V}(A, F) \bigcap \widetilde{V_k}$$

i.e. condition 5 of Theorem 1 holds for the collection $\mathcal{B}(A, \mathcal{F})$.

By this, the proposition is completely proved. □

Proposition 6. *Let $\widehat{\tau}'$ and $\widehat{\tau}''$ be group topologies on the group $\widehat{G}$, defined in Proposition 5, and $n \in \mathbb{N}$. If τ is a non-discrete group topology on the group $\widehat{G}$ such that $\tau \geq \widehat{\tau}'$, then for any neighborhood W of the unity element $\widehat{e}$ in the topological group $(\widehat{G}, \inf\{\tau, \widehat{\tau}''\})$ there exists a natural number $k \geq n$ such that (see 3.7) $\widehat{G}_k(\mathbb{R}) \subseteq W$.*

Proof. Let W be a neighborhood of the unity element in the topological group $(\widehat{G}, \inf\{\tau, \widehat{\tau}''\})$, and let W_1 be a neighborhood of the unity element in the topological group $(\widehat{G}, \inf\{\tau, \widehat{\tau}''\})$ such that $W_1 \cdot (W_1 \cdot W_1 \cdot (W_1)^{-1} \cdot (W_1)^{-1}) \subseteq W$.

Then W_1 is a neighborhood of the unity element in each of the topological groups $(\widehat{G}, \tau)$ and $(\widehat{G}, \widehat{\tau}'')$, and hence, there exists a natural number $n_0 \in \mathbb{N}$ such that $n_0 \geq n$ and $\widetilde{V}_{n_0} \bigcap \widehat{G}'' \subseteq W_1$.

Since $\tau \geq \widehat{\tau}'$, then $\widehat{G}' \bigcap \widetilde{V}_{n_0}$ is a neighborhood of the unity element in the topological group $(\widehat{G}, \tau)$. Then, $\widehat{G}' \bigcap \widetilde{V}_{n_0} \bigcap W_1$ is a neighborhood of the unity element in the topological group $(\widehat{G}, \tau)$.

Since τ is a non-discrete topology, then $\widehat{G}' \bigcap \widetilde{V}_{n_0} \bigcap W_1 \neq \{0\}$. If $0 \neq \widehat{g}_0 \in \widehat{G}' \bigcap \widetilde{V}_{n_0} \bigcap W_1 \neq \{0\}$, then there exists a natural number $k \geq n_0 \geq n$ such that $pr_k(\widehat{g}_0) \neq 0$.

Since $\widehat{g}_0 \in \widehat{G}'$, then $pr_k(\widehat{g}_0) = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $a \neq 0$.

For any numbers $r, x \in \mathbb{R}$ consider matrices $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & (a^{-1}) \cdot x \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then (see Remark 4)

$$\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a^{-1} \cdot x \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a^{-1} \cdot x \\ 0 & 0 & 1 \end{pmatrix}^{-1} \right).$$

For any numbers $r, x \in \mathbb{R}$ we consider elements $\widehat{g}_r, \widehat{g}_x \in \widehat{G}$ such that

$$pr_k(\widehat{g}_r) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix} \text{ and } pr_i(\widehat{g}) = e_i \text{ for } i \neq k,$$

$$pr_k(\widehat{g}_x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a^{-1} \cdot x \\ 0 & 0 & 1 \end{pmatrix} \text{ and } pr_i(\widehat{g}) = e_i \text{ for } i \neq k.$$

Since $k \geq n_0$, then $\widehat{g}_r \in \widetilde{V}_{n_0} \bigcap \widehat{G}'' \subseteq W_1$ and $\widehat{g}_x \in \widetilde{V}_{n_0} \bigcap \widehat{G}'' \subseteq W_1$ for any numbers $r, x \in \mathbb{R}$. Then $\widehat{g}_r \cdot (\widehat{g}_0 \cdot \widehat{g}_x \cdot \widehat{g}_0^{-1} \cdot \widehat{g}_x^{-1}) \in W_1 \cdot W_1 \cdot W_1 \cdot (W_1)^{-1} \cdot (W_1)^{-1} \subseteq W$ for any numbers $r, x \in \mathbb{R}$, and hence, $\widehat{G}_k(\mathbb{R}) = \{\widehat{g}_r \cdot \widehat{g}_0 \cdot \widehat{g}_x \cdot \widehat{g}_0^{-1} \cdot \widehat{g}_x^{-1} | r, x \in \mathbb{R}\} \subseteq W$.

As $pr_k(\widehat{g}_r \cdot \widehat{g}_0 \cdot \widehat{g}_x \cdot \widehat{g}_0^{-1} \cdot \widehat{g}_x^{-1}) =$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a^{-1} \cdot x \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a^{-1} \cdot x \\ 0 & 0 & 1 \end{pmatrix}^{-1} \right) =$$

$\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix}$ for any $r, x \in \mathbb{R}$, and $pr_i(\widehat{g}_i \cdot \widehat{g}_0 \cdot \widehat{g}_x \cdot \widehat{g}_0^{-1} \cdot \widehat{g}_x^{-1}) = e_i$ for any $R, x \in \mathbb{R}$ and for any $i \neq k$, then $pr_k(\widehat{A}_k) = \left\{ \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix} | r \text{ and } x \in \mathbb{R} \right\} = G_k(\mathbb{R})$ and $pr_i(\widehat{A}_k) = \{e_i\}$ for $i \neq k$.

By this, the proposition is completely proved. □

Theorem 7. *Let $\widehat{\tau}'$ and $\widehat{\tau}''$ be group topologies on the group $\widehat{G}$, defined in Proposition 5, and $\mathcal{F}$ be the Frechet filter. Then the following statements are true:*

Statement 7.1. *If τ is a group topology on the group $\widehat{G}$ such that $\tau \geq \widehat{\tau}'$, then*

$$\sup\{\widehat{\tau}(A,\mathcal{F}),\inf\{\widehat{\tau}'',\tau\}\} > \sup\{\widehat{\tau}(B,\mathcal{F}),\inf\{\widehat{\tau}'',\tau\}\}.$$

for any subgroups $A \subset B$ of the group $\mathbb{R}(+)$.

Statement 7.2. *If $\widehat{\tau}_d$ is the discrete topology on the group $\widehat{G}$, and $\widehat{\tau}*$ is a coatom in the lattice of all group topologies on the group $\widehat{G}$ such that $\widehat{\tau}* \geq \widehat{\tau}'$, then between the topologies $\inf\{\widehat{\tau}_d,\widehat{\tau}''\}$ and $\inf\{\widehat{\tau}*,\widehat{\tau}''\}$, there exists a chain of group topologies on the group $\widehat{G}$ which is infinitely decreasing and infinitely increasing.*

Proof. Proof of Statement 7.1. Since $A \subset B$, then (see the notation at the beginning of this article) $\widehat{V}(A,S) \subseteq \widehat{V}(B,S)$ for any $S \in \mathcal{F}$. Then (see Proposition 5) $\widehat{\tau}(A,\mathcal{F}) \geq \widehat{\tau}(B,\mathcal{F})$, and hence, $\sup\{\widehat{\tau}(A,\mathcal{F}),\inf\{\widehat{\tau}'',\tau\}\} \geq \sup\{\widehat{\tau}(B,\mathcal{F}),\inf\{\widehat{\tau}'',\tau\}\}$.

We will show that $\sup\{\widehat{\tau}(A,\mathcal{F}),\inf\{\widehat{\tau}'',\tau\}\} > \sup\{\widehat{\tau}(B,\mathcal{F}),\inf\{\widehat{\tau}'',\tau\}\}$.

Assume the contrary, i.e. that

$$\sup\{\widehat{\tau}(A,\mathcal{F}),\inf\{\widehat{\tau}'',\tau\}\} = \sup\{\widehat{\tau}(B,\mathcal{F}),\inf\{\widehat{\tau}'',\tau\}\},$$

and let $S_0 \in \mathcal{F}$. Then $\widehat{V}(A,S_0)$ is a neighborhood of the unity element in the topological group $(\widehat{G},\widehat{\tau}(A,\mathcal{F}))$, and hence, $\widehat{V}(A,S_0)$ is a neighborhood of the unity element in the topological group $(\widehat{G},\sup\{\widehat{\tau}(A,\mathcal{F}),\inf\{\widehat{\tau}'',\tau\}\}) = (\widehat{G},\sup\{\widehat{\tau}(B,\mathcal{F}),\inf\{\widehat{\tau}'',\tau\}\})$. Then there exists a neighborhood W of the unity element in the topological group $(\widehat{G},\inf\{\widehat{\tau}'',\tau\})$ such that $W\bigcap(\widehat{V}(B,S_1)\bigcap\widetilde{V}_n) \subseteq \widehat{V}(A,S_0)$ for some $S_1 \in \mathcal{F}$ and a natural number $n \in \mathbb{N}$.

Since $\mathcal{F}$ is the Frechet filter, then there exists a natural number $m \in \mathbb{N}$ such that $\{i \in \mathbb{N} \mid i > m\} \subseteq S_0\bigcap S_1$.

By Proposition 6, there exists a natural number $k \geq \max\{n,m\}$ such that $\widehat{G}_k(\mathbb{R}) \subseteq W$, and hence, $\widehat{G}_k(B) \subseteq \widehat{G}_k(\mathbb{R}) \subseteq W$.

As $k \in \{i \in \mathbb{N} \mid i > m\} \subseteq S_1$ then $\widehat{G}_k(B) \subseteq \widehat{V}(B,S_1)$, and as $k \geq n$ then $\widehat{G}_k(B) \subseteq \widetilde{V}_n$. Then $\widehat{G}_k(B) \subseteq W\bigcap\widehat{V}(B,S_1)\bigcap\widetilde{V}_n \subseteq \widehat{V}(A,S_0)$.

Since $k \in \{i \in \mathbb{N} \mid i > m\} \subseteq S_0$, then (see 3.7) $G_k(B) = pr_k(\widehat{G}_k(B)) \subseteq pr_k(\widehat{V}(A,S_0)) = G_k(A)$, but this contradicts that $B \nsubseteq A$.

By this, Statement 7.1 is proved.

Proof of Statement 7.2. There exists a chain $\{A_i \mid i \in \mathbb{Z}\}$ of subgroups A_i of the group $\mathbb{R}(+)$ such that $A_i \subseteq A_{i+1}$ for any $i \in \mathbb{Z}$, i.e. this chain of subgroups is infinitely decreasing and infinitely increasing.

For any subgroup A_i let consider the topology $\widehat{\tau}(A_i,\mathcal{F})$. Since $\widehat{\tau}* \geq \widehat{\tau}'$, then by Statement 7.1,

$$\sup\{\widehat{\tau}(A_i,\mathcal{F}),\inf\{\widehat{\tau}'',\widehat{\tau}*\}\} > \sup\{\widehat{\tau}(A_{i+1},\mathcal{F}),\inf\{\widehat{\tau}'',\widehat{\tau}*\}\},$$

and hence, the chain of group topologies $\{\sup\{\widehat{\tau}(A_i,\mathcal{F}),\inf\{\widehat{\tau}'',\widehat{\tau}*\}\} \mid i \in \mathbb{Z}\}$ is infinitely decreasing and infinitely increasing.

To complete the proof of the theorem it remains to verify that

$$\inf\{\widehat{\tau*},\widehat{\tau''}\} \leq \sup\{\widehat{\tau}(A_i,\mathcal{F}),\inf\{\widehat{\tau}*,\widehat{\tau}''\}\} \leq \inf\{\widehat{\tau}_d,\widehat{\tau}''\}$$

for any subgroup $A_i(+)$ of the group $\mathbb{R}(+)$, where $i \in \mathbb{Z}$.

In fact, from the definition of the sets $G(A)$ and G'' (see 3.2) it follows that $G(0) = G''$, and hence, $G_k(0) = G''_k$ for any $k \in \mathbb{N}$. Then $\widehat{G}(\{0\}, S) \bigcap \widetilde{V}_n = \widehat{G}'' \bigcap \widetilde{V}_n$ for any subset $S \subseteq \mathbb{N}$ and any $n \in \mathbb{N}$, and hence, the collection $\{\widehat{G}(\{0\}, S) \bigcap \widetilde{V}_n \mid n \in \mathbb{N}\}$ is a basis of neighborhoods of the unity element in the topological group $(\widehat{G}, \widehat{\tau}'')$.

Since $\widehat{\tau}_d$ is the discrete topology on the group $\widehat{G}$, then $\inf\{\widehat{\tau}'', \widehat{\tau}_d\} = \widehat{\tau}''$, and hence, the set $\{\widehat{G}(\{0\}, S) \bigcap \widetilde{V}_n \mid n \in \mathbb{N}\}$ is a basis of neighborhoods of the unity element in the topological group $(\widehat{G}, \inf\{\widehat{\tau}'', \widehat{\tau}_d\})$. Then $\widehat{\tau}(\{0\}, \mathcal{F}) \le \inf\{\widehat{\tau}'', \widehat{\tau}_d\}$.

So, we have proved that $\sup\{\widehat{\tau}(A_i, \mathcal{F}), \inf\{\tau*, \widehat{\tau}''\}\} \le \inf\{\tau_d, \widehat{\tau}''\}$. Since $\{0\} \subseteq A_i$ for any $i \in \mathbb{Z}$, then

$$\inf\{\widehat{\tau}*, \widehat{\tau}''\} \le \sup\{\widehat{\tau}(A_i, \mathcal{F}), \inf\{\widehat{\tau}*, \widehat{\tau}''\}\} \le \inf\{\widehat{\tau}_d, \widehat{\tau}''\}$$

for any subgroup $A_i(+)$ of the group $\mathbb{R}(+)$.

By this, the theorem is proved. □

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A Note on the Affine Subspaces of Three-Dimensional Lie Algebras

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Abstract. A classification of full-rank affine subspaces of (real) three-dimensional Lie algebras is presented. In the context of invariant control affine systems, this is exactly a classification of all full-rank systems evolving on three-dimensional Lie groups.

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1 Introduction

In this note we exhibit a classification, under $\mathfrak{L}$-equivalence, of full-rank affine subspaces of (real) three-dimensional Lie algebras. Two affine subspaces are $\mathfrak{L}$-equivalent, provided there exists a Lie algebra automorphism mapping one to the other. This classification is presented in three parts (see Theorems 1, 2, and 3). Proofs are omitted. However, a full treatment of each part will appear elsewhere [4–6]. Tables detailing these results are included as an appendix.

It turns out that two left-invariant control affine systems are detached feedback equivalent if (and only if) their traces are $\mathfrak{L}$-equivalent. Therefore, a classification under $\mathfrak{L}$-equivalence induces one under detached feedback equivalence.

2 Three-dimensional Lie algebras

The classification of three-dimensional Lie algebras is well known. The classification over $\mathbb{C}$ was done by S. Lie (1893), whereas the standard enumeration of the real cases is that of L. Bianchi (1918). In more recent times, a different (method of) classification was introduced by C. Behr (1968) and others (see [12–14] and the references therein). This is customarily referred to as the *Bianchi-Behr classification*, or even the "Bianchi-Schücking-Behr classification". Accordingly, any real three-dimensional Lie algebra is isomorphic to one of eleven types (in fact, there are nine algebras and two parametrised infinite families of algebras). In terms of an (appropriate) ordered basis (E_1, E_2, E_3), the commutation operation is given by

$$[E_2, E_3] = n_1 E_1 - a E_2$$
$$[E_3, E_1] = a E_1 + n_2 E_2$$

$$[E_1, E_2] = n_3 E_3.$$

The (Bianchi-Behr) structure parameters a, n_1, n_2, n_3 for each type are given in Table 1.

Type	Notation	a	n_1	n_2	n_3	Representatives
I	$3\mathfrak{g}_1$	0	0	0	0	$\mathbb{R}^3$
II	$\mathfrak{g}_{3.1}$	0	1	0	0	$\mathfrak{h}_3$
$III = VI_{-1}$	$\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$	1	1	-1	0	$\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R}$
IV	$\mathfrak{g}_{3.2}$	1	1	0	0	
V	$\mathfrak{g}_{3.3}$	1	0	0	0	
VI_0	$\mathfrak{g}_{3.4}^0$	0	1	-1	0	$\mathfrak{se}(1,1)$
$VI_h, \ {}^{h<0}_{h\neq -1}$	$\mathfrak{g}_{3.4}^h$	$\sqrt{-h}$	1	-1	0	
VII_0	$\mathfrak{g}_{3.5}^0$	0	1	1	0	$\mathfrak{se}(2)$
$VII_h, \ h>0$	$\mathfrak{g}_{3.5}^h$	$\sqrt{h}$	1	1	0	
$VIII$	$\mathfrak{g}_{3.6}$	0	1	1	-1	$\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{so}(2,1)$
IX	$\mathfrak{g}_{3.7}$	0	1	1	1	$\mathfrak{su}(2)$, $\mathfrak{so}(3)$

Table 1. Bianchi-Behr classification

We note that for the two infinite families, VI_h and VII_h, each value of the parameter h yields a distinct (i. e., non-isomorphic) Lie algebra. Furthermore, for the purposes of this paper, type $III = VI_{-1}$ will be considered as part of VI_h.

3 Affine subspaces and classification

An affine subspace Γ of a Lie algebra $\mathfrak{g}$ is written as

$$\Gamma = A + \Gamma^0 = A + \langle B_1, B_2, \ldots, B_\ell \rangle$$

where $A, B_1, \ldots, B_\ell \in \mathfrak{g}$. Let Γ_1 and Γ_2 be two affine subspaces of $\mathfrak{g}$. We say that Γ_1 and Γ_2 are *$\mathfrak{L}$-equivalent* if there exists a Lie algebra automorphism $\psi \in \mathsf{Aut}(\mathfrak{g})$ such that $\psi \cdot \Gamma_1 = \Gamma_2$. $\mathfrak{L}$-equivalence is a genuine equivalence relation. An affine subspace Γ is said to have *full rank* if it generates the whole Lie algebra (i.e., the smallest Lie algebra containing Γ is $\mathfrak{g}$). Note that the full-rank property is invariant under $\mathfrak{L}$-equivalence.

Clearly, if Γ_1 and Γ_2 are $\mathfrak{L}$-equivalent, then they are necessarily of the same dimension. Furthermore, $0 \in \Gamma_1$ if and only if $0 \in \Gamma_2$. We shall find it convenient to refer to an ℓ-dimensional affine subspace Γ as an $(\ell, 0)$-affine subspace when $0 \in \Gamma$ (i.e., Γ is a vector subspace) and as an $(\ell, 1)$-affine subspace, otherwise.

Remark 1. No $(1,0)$-affine subspace has full rank. A $(1,1)$-affine subspace has full rank if and only if A, B_1, and $[A, B_1]$ are linearly independent. A $(2,0)$-affine subspace has full rank if and only if B_1, B_2, and $[B_1, B_2]$ are linearly independent. Any $(2,1)$-affine subspace or $(3,0)$-affine subspace has full rank.

There is only one affine subspace whose dimension coincides with that of the Lie algebra $\mathfrak{g}$, namely the space itself. From the standpoint of classification, this case is trivial and hence will not be covered explicitly.

Let us fix a three-dimensional Lie algebra $\mathfrak{g}$ (together with an ordered basis). In order to classify the affine subspaces of $\mathfrak{g}$, one requires the (group of) automorphisms of $\mathfrak{g}$. These are well known (see, e. g., [7, 8, 14]); a summary is given in Table 2. For each type of Lie algebra, one constructs class representatives (by considering the action of automorphisms on a typical affine subspace). Finally, one verifies that none of these representatives are equivalent.

Type	Commutators	Automorphisms
II	$[E_2,E_3]=E_1$ $[E_3,E_1]=0$ $[E_1,E_2]=0$	$\begin{bmatrix} yw-vz & x & u \\ 0 & y & v \\ 0 & z & w \end{bmatrix}$; $yw \neq vz$
IV	$[E_2,E_3]=E_1-E_2$ $[E_3,E_1]=E_1$ $[E_1,E_2]=0$	$\begin{bmatrix} u & x & y \\ 0 & u & z \\ 0 & 0 & 1 \end{bmatrix}$; $u \neq 0$
V	$[E_2,E_3]=-E_2$ $[E_3,E_1]=E_1$ $[E_1,E_2]=0$	$\begin{bmatrix} x & y & z \\ u & v & w \\ 0 & 0 & 1 \end{bmatrix}$; $xv \neq yu$
VI_0	$[E_2,E_3]=E_1$ $[E_3,E_1]=-E_2$ $[E_1,E_2]=0$	$\begin{bmatrix} x & y & u \\ y & x & v \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} x & y & u \\ -y & -x & v \\ 0 & 0 & -1 \end{bmatrix}$; $x^2 \neq y^2$
VI_h	$[E_2,E_3]=E_1-aE_2$ $[E_3,E_1]=aE_1-E_2$ $[E_1,E_2]=0$	$\begin{bmatrix} x & y & u \\ y & x & v \\ 0 & 0 & 1 \end{bmatrix}$; $x^2 \neq y^2$
VII_0	$[E_2,E_3]=E_1$ $[E_3,E_1]=E_2$ $[E_1,E_2]=0$	$\begin{bmatrix} x & y & u \\ -y & x & v \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} x & y & u \\ y & -x & v \\ 0 & 0 & -1 \end{bmatrix}$; $x^2 \neq -y^2$
VII_h	$[E_2,E_3]=E_1-aE_2$ $[E_3,E_1]=aE_1+E_2$ $[E_1,E_2]=0$	$\begin{bmatrix} x & y & u \\ -y & x & v \\ 0 & 0 & 1 \end{bmatrix}$; $x^2 \neq -y^2$
$VIII$	$[E_2,E_3]=E_1$ $[E_3,E_1]=E_2$ $[E_1,E_2]=-E_3$	$M^\top JM=J$ $J=\mathrm{diag}(1,1,-1)$ $\det M=1$
IX	$[E_2,E_3]=E_1$ $[E_3,E_1]=E_2$ $[E_1,E_2]=E_3$	$M^\top M=I$ $I=\mathrm{diag}(1,1,1)$ $\det M=1$

Table 2. Automorphisms of three-dimensional Lie algebras

Type	Notation	(ℓ,ε)	Equivalence representative	Parameter
II	$\mathfrak{g}_{3.1}$	$(1,1)$	$E_2+\langle E_3\rangle$	
		$(2,0)$	$\langle E_2,E_3\rangle$	
		$(2,1)$	$E_1+\langle E_2,E_3\rangle$	
			$E_3+\langle E_1,E_2\rangle$	
IV	$\mathfrak{g}_{3.2}$	$(1,1)$	$E_2+\langle E_3\rangle$	$\alpha\neq 0$
			$\alpha E_3+\langle E_2\rangle$	
		$(2,0)$	$\langle E_2,E_3\rangle$	
		$(2,1)$	$E_1+\langle E_2,E_3\rangle$	
			$E_2+\langle E_3,E_1\rangle$	
			$\alpha E_3+\langle E_1,E_2\rangle$	
V	$\mathfrak{g}_{3.3}$	$(1,1)$	$\emptyset$	$\alpha\neq 0$
		$(2,0)$	$\emptyset$	
		$(2,1)$	$E_1+\langle E_2,E_3\rangle$	
			$\alpha E_3+\langle E_1,E_2\rangle$	
VI_0	$\mathfrak{g}_{3.4}^0$	$(1,1)$	$E_2+\langle E_3\rangle$	$\alpha>0$
			$\alpha E_3+\langle E_2\rangle$	
		$(2,0)$	$\langle E_2,E_3\rangle$	
		$(2,1)$	$E_1+\langle E_2,E_3\rangle$	
			$E_1+\langle E_1+E_2,E_3\rangle$	
			$\alpha E_3+\langle E_1,E_2\rangle$	
VI_h	$\mathfrak{g}_{3.4}^h$	$(1,1)$	$E_2+\langle E_3\rangle$	$\alpha\neq 0$
			$\alpha E_3+\langle E_2\rangle$	
		$(2,0)$	$\langle E_2,E_3\rangle$	
		$(2,1)$	$E_1+\langle E_2,E_3\rangle$	
			$E_1+\langle E_1+E_2,E_3\rangle$	
			$E_1+\langle E_1-E_2,E_3\rangle$	
			$\alpha E_3+\langle E_1,E_2\rangle$	
VII_0	$\mathfrak{g}_{3.5}^0$	$(1,1)$	$E_2+\langle E_3\rangle$	$\alpha>0$
			$\alpha E_3+\langle E_2\rangle$	
		$(2,0)$	$\langle E_2,E_3\rangle$	
		$(2,1)$	$E_1+\langle E_2,E_3\rangle$	
			$\alpha E_3+\langle E_1,E_2\rangle$	
VII_h	$\mathfrak{g}_{3.5}^h$	$(1,1)$	$E_2+\langle E_3\rangle$	$\alpha\neq 0$
			$\alpha E_3+\langle E_2\rangle$	
		$(2,0)$	$\langle E_2,E_3\rangle$	
		$(2,1)$	$E_1+\langle E_2,E_3\rangle$	
			$\alpha E_3+\langle E_1,E_2\rangle$	

Table 3. Affine subspaces (types II to VII, solvable)

We present our results for the solvable Lie algebras (types I to VII) in the following two theorems; a summary is given in Table 3. The classification of type I is trivial and is therefore omitted.

Theorem 1. *Any full-rank affine subspace of* $\mathfrak{g}_{3.1}$ *(type* II*) is* $\mathfrak{L}$*-equivalent to exactly one of* $E_2 + \langle E_3\rangle$, $\langle E_2, E_3\rangle$, $E_1 + \langle E_2, E_3\rangle$, *and* $E_3 + \langle E_1, E_2\rangle$. *Any full-rank affine subspace of* $\mathfrak{g}_{3.2}$ *(type* IV*) is* $\mathfrak{L}$*-equivalent to exactly one of* $E_2 + \langle E_3\rangle$, $\alpha E_3 + \langle E_2\rangle$, $\langle E_2, E_3\rangle$, $E_1 + \langle E_2, E_3\rangle$, $E_2 + \langle E_3, E_1\rangle$, *and* $\alpha E_3 + \langle E_1, E_2\rangle$. *Any full-rank affine subspace of* $\mathfrak{g}_{3.3}$ *(type* V*) is* $\mathfrak{L}$*-equivalent to exactly one of* $E_1 + \langle E_2, E_3\rangle$, *and* $\alpha E_3 + \langle E_1, E_2\rangle$. *Here* $\alpha \neq 0$ *parametrises families of class representatives, each different value corresponding to a distinct non-equivalent representative.*

The automorphisms of $\mathfrak{g}^0_{3.4}$, $\mathfrak{g}^h_{3.4}$ (including $h = -1$), $\mathfrak{g}^0_{3.5}$, and $\mathfrak{g}^h_{3.5}$ are very similar. Due to this similarity, we treat these types separately.

Theorem 2. *Any full-rank affine subspace of* $\mathfrak{g}^0_{3.5}$ *or* $\mathfrak{g}^h_{3.5}$ *(type* VII_0 *or* VII_h*, respectively) is* $\mathfrak{L}$*-equivalent (with respect to the different ordered bases) to exactly one of* $E_2 + \langle E_3\rangle$, $\alpha E_3 + \langle E_2\rangle$, $\langle E_2, E_3\rangle$, $E_1 + \langle E_2, E_3\rangle$, *and* $\alpha E_3 + \langle E_1, E_2\rangle$, *where* $\alpha > 0$ *for* $\mathfrak{g}^0_{3.5}$ *and* $\alpha \neq 0$ *for* $\mathfrak{g}^h_{3.5}$. *Any full-rank affine subspace of* $\mathfrak{g}^0_{3.4}$ *(type* VI_0*) is* $\mathfrak{L}$*-equivalent to exactly one of the above formal list for* $\mathfrak{g}^0_{3.5}$ *or* $E_1 + \langle E_1 + E_2, E_3\rangle$. *Any full rank-affine subspace of* $\mathfrak{g}^h_{3.4}$ *(type* VI_h*) is* $\mathfrak{L}$*-equivalent to exactly one of the above formal list for* $\mathfrak{g}^h_{3.5}$*, or one of* $E_1 + \langle E_1 + E_2, E_3\rangle$ *and* $E_1 + \langle E_1 - E_2, E_3\rangle$. *Here* α *parametrises families of class representatives, each different value corresponding to a distinct non-equivalent representative.*

Remark 2. The Lie algebras of types II, III, IV, V, VI_0, and VI_h are completely solvable, whereas those of types VII_0 and VII_h are not.

Type	Notation	(ℓ, ε)	Equivalence representative	Parameter
$VIII$	$\mathfrak{g}_{3.6}$	$(1,1)$	$E_3 + \langle E_2 + E_3\rangle$ $\alpha E_2 + \langle E_3\rangle$ $\alpha E_1 + \langle E_2\rangle$ $\alpha E_3 + \langle E_2\rangle$	$\alpha > 0$
		$(2,0)$	$\langle E_1, E_2\rangle$ $\langle E_2, E_3\rangle$	
		$(2,1)$	$E_3 + \langle E_1, E_2 + E_3\rangle$ $\alpha E_1 + \langle E_2, E_3\rangle$ $\alpha E_3 + \langle E_1, E_2\rangle$	
IX	$\mathfrak{g}_{3.7}$	$(1,1)$	$\alpha E_1 + \langle E_2\rangle$	$\alpha > 0$
		$(2,0)$	$\langle E_1, E_2\rangle$	
		$(2,1)$	$\alpha E_1 + \langle E_2, E_3\rangle$	

Table 4. Affine subspaces (types $VIII$ and IX, semisimple)

Now, consider the case of the semisimple algebras (types $VIII$ and IX). In each of the two cases, we employ a bilinear product ω (the Lorentz product and dot product, respectively) that is preserved by automorphisms. Most of the affine subspaces can then be characterised as being tangent to a level set (submanifold) $\{A \in \mathfrak{g} : \omega(A, A) = \alpha\}$. We present our classification in the following theorem; a summary is given in Table 4.

Theorem 3. *Any full-rank affine subspace of* $\mathfrak{g}_{3.6}$ *(type* $VIII$*) is* $\mathfrak{L}$*-equivalent to exactly one of* $E_3 + \langle E_2 + E_3 \rangle$, $\alpha E_2 + \langle E_3 \rangle$, $\alpha E_1 + \langle E_2 \rangle$, $\alpha E_3 + \langle E_2 \rangle$, $\langle E_1, E_2 \rangle$, $\langle E_2, E_3 \rangle$, $E_3 + \langle E_1, E_2 + E_3 \rangle$, $\alpha E_1 + \langle E_2, E_3 \rangle$, *and* $\alpha E_3 + \langle E_1, E_2 \rangle$. *Any full-rank affine subspace of* $\mathfrak{g}_{3.7}$ *(type* IX*) is* $\mathfrak{L}$*-equivalent to exactly one of* $\alpha E_1 + \langle E_2 \rangle$, $\langle E_1, E_2 \rangle$, *and* $\alpha E_1 + \langle E_2, E_3 \rangle$. *Here* $\alpha > 0$ *parametrises families of class representatives, each different value corresponding to a distinct non-equivalent representative.*

4 Control affine systems and classification

A left-invariant control affine system Σ is a control system of the form

$$\dot{g} = g\,\Xi\,(\mathbf{1}, u) = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \qquad g \in \mathsf{G},\, u \in \mathbb{R}^\ell.$$

Here G is a (real, finite-dimensional) Lie group with Lie algebra $\mathfrak{g}$. Also, the parametrisation map $\Xi(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}$ is an injective affine map (i. e., $B_1, \ldots, B_\ell$ are linearly independent). The "product" $g\,\Xi\,(\mathbf{1}, u)$ is to be understood as $T_\mathbf{1} L_g \cdot \Xi\,(\mathbf{1}, u)$, where $L_g : \mathsf{G} \to \mathsf{G}$, $h \mapsto gh$ is the left translation by g. Note that the dynamics $\Xi : \mathsf{G} \times \mathbb{R}^\ell \to T\mathsf{G}$ are invariant under left translations, i. e., $\Xi\,(g, u) = g\,\Xi\,(\mathbf{1}, u)$. We shall denote such a system by $\Sigma = (\mathsf{G}, \Xi)$ (cf. [2]).

The admissible controls are piecewise-continuous maps $u(\cdot) : [0, T] \to \mathbb{R}^\ell$. A *trajectory* for an admissible control $u(\cdot) : [0, T] \to \mathbb{R}^\ell$ is an absolutely continuous curve $g(\cdot) : [0, T] \to \mathsf{G}$ such that $\dot{g}(t) = g(t)\,\Xi\,(\mathbf{1}, u(t))$ for almost every $t \in [0, T]$. We say that a system Σ is *controllable* if for any $g_0, g_1 \in \mathsf{G}$, there exists a trajectory $g(\cdot) : [0, T] \to \mathsf{G}$ such that $g(0) = g_0$ and $g(T) = g_1$. For more details about (invariant) control systems see, e. g., [1, 10, 11, 15, 16].

The image set $\Gamma = \operatorname{im} \Xi(\mathbf{1}, \cdot)$, called the *trace* of Σ, is an affine subspace of $\mathfrak{g}$. Specifically, $\Gamma = A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell \rangle$. A system Σ is called *homogeneous* if $A \in \Gamma^0$, and *inhomogeneous* otherwise. Furthermore, Σ is said to have *full rank* if its trace (as an affine subspace) has full rank. Henceforth, we assume that all systems under consideration have full rank. (The full-rank condition is a necessary condition for a system Σ to be controllable.)

An important equivalence relation for invariant control systems is that of detached feedback equivalence. Two systems are detached feedback equivalent if there exists a "detached" feedback transformation which transforms the first system to the second (see [3, 9]). Two detached feedback equivalent control systems have the same set of trajectories (up to a diffeomorphism in the state space) which are parametrised differently by admissible controls. More precisely, let $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}', \Xi')$ be left-invariant control affine systems. Σ and Σ' are called *locally detached feedback equivalent* (shortly DF_{loc}-equivalent) if there exist open neighbourhoods N and N' of identity (in G and G', respectively) and a diffeomorphism $\Phi : N \times \mathbb{R}^\ell \to N' \times \mathbb{R}^{\ell'}$, $(g, u) \mapsto (\phi(g), \varphi(u))$ such that $\phi(\mathbf{1}) = \mathbf{1}$ and $T_g\phi \cdot \Xi\,(g, u) = \Xi'\,(\phi(g), \varphi(u))$ for $g \in N$ and $u \in \mathbb{R}^\ell$. It turns out that Σ and Σ' are DF_{loc}-equivalent if and only if there exists a Lie algebra isomorphism $\psi : \mathfrak{g} \to \mathfrak{g}'$ such that $\psi \cdot \Gamma = \Gamma'$ (see [3]).

For the purpose of classification, we may assume that Σ and Σ' have the same Lie algebra $\mathfrak{g}$. Then Σ and Σ' are DF_{loc}-equivalent if and only if their traces Γ and Γ' are $\mathfrak{L}$-equivalent. This reduces the problem of classifying under DF_{loc}-equivalence to that of classifying under $\mathfrak{L}$-equivalence. Suppose $\{\Gamma_i : i \in I\}$ is an exhaustive collection of (non-equivalent) class representatives (i.e., any affine subspace is $\mathfrak{L}$-equivalent to exactly one Γ_i). For each $i \in I$, we can easily find a system $\Sigma_i = (\mathsf{G}, \Xi_i)$ with trace Γ_i. Then any system Σ is DF_{loc}-equivalent to exactly one Σ_i.

Example. The Heisenberg group

$$\mathsf{H}_3 = \left\{ \begin{bmatrix} 1 & y & x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \,:\, x, y, z \in \mathbb{R} \right\}$$

is a (nilpotent) three-dimensional Lie group. Its Lie algebra $\mathfrak{h}_3$ has (ordered) basis

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The commutator relations are $[E_2, E_3] = E_1$, $[E_3, E_1] = 0$, and $[E_1, E_2] = 0$. Thus $\mathfrak{h}_3 \cong \mathfrak{g}_{3.1}$. Hence, any system $\Sigma = (\mathsf{H}_3, \Xi)$ is DF_{loc}-equivalent to exactly one $\Sigma_i = (\mathsf{H}_3, \Xi_i)$, where

$$\begin{aligned} &\Xi_1(g, u) = g(E_2 + uE_3) && \Xi_2(g, u) = g(u_1E_2 + u_2E_3) \\ &\Xi_3(g, u) = g(E_1 + u_1E_2 + u_2E_3) && \Xi_4(g, u) = g(E_3 + u_1E_1 + u_2E_3). \end{aligned}$$

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Solving the games generated by the informational extended strategies of the players

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Abstract. In this article we study the non-informational extended games which are generated by the two-directional informational flow extended strategies of the players. The theorem on the existence of the Nash equilibrium profiles in this type of games is also proved.

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Let

$$\Gamma = \langle I; X_p, p \in I; H_p : X \to \mathbb{R} \rangle \tag{1}$$

be the strategic form or normal form of the static noncooperative games with complete and imperfect information[1] where $I = \{1, 2, ..., n\}$ is the set of players, X_p is a set of available alternatives of the player $p \in I$, $H_p : X_p \to R$ is the payoff function of the player $p \in I$ and $X = \prod_{p \in I} X_i$ is the set of strategy profiles for the game. In [1] the author studied informational extensions of the games (1), generated by a one-way directional informational flow, denoted by $j \overset{\text{inf}}{\to} i$, which means: the player i, and only he, knows exactly what value of the strategy will be chosen by the player j, and two-directional informational flow, denoted by $i \overset{\text{inf}}{\leftrightarrows} j$, which means[2]: at any time simultaneously player i knows exactly what value of the strategy will be chosen by the player j and player j knows exactly what value of the strategy will be chosen by the player i. We mention that the game is static, in other words, the order of the chosen strategies is not significant. The players do not known the informational type of the other players, so the player i (respectively j) does not know that the player j (respectively i) knows what value of the strategies will be chosen. In the general case [2, 3] the set of the informational extended strategies of the player i (respectively j) is the set of the functions $\Theta_i = \{\theta_i : X_j \to X_i\}$ (respectively $\Theta_j = \{\theta_j : X_i \to X_j\}$)

[1]So the players know exactly their and of the other players payoff functions and they know the sets of strategies. Players do not know what kind of the strategy will be chosen by the players.

[2]Notation $i \overset{\text{inf}}{\leftrightarrows} j$ means the following: "the information about the concrete chosen value of the strategies of player i will be transmitted to the player j" and vice versa "the information about the concrete chosen value of the strategies of player j will be transmitted to the player i".

such that $\forall x_j \in X_j$, $\theta_i(x_j) \in X_i$ (respectively $\forall x_i \in X_i$, $\theta_j(x_i) \in X_j$). Following [1], if in the game Γ the sets of strategies $X_i = \left\{x_i^1, ..., x_i^l, ..., x_i^{|X_i|}\right\}$ and $X_j = \left\{x_j^1, ..., x_j^l, ..., x_j^{|X_j|}\right\}$ of the players i and j are at most countable, H_p is the discrete payoff function of the player $p \in I$, the sets of the informational extended strategies can be represented as $\Theta_i = \{\theta_i^\alpha : X_j \to X_i^\alpha, \alpha = 1, ..., \varkappa_i\}$ and respectively $\Theta_j = \left\{\theta_j^\beta : X_i \to X_j^\beta, \beta = 1, ..., \varkappa_j\right\}$, where

$$X_i^\alpha = \left\{\left(x_i^{\alpha_1}, x_i^{\alpha_2}, ..., x_i^{\alpha_l}, ..., x_i^{\alpha|X_j|}\right) : x_i^{\alpha_l} \in X_i, \forall l = \overline{1, |X_j|}\right\} \subseteq X_i,$$

$$X_j^\beta = \left\{\left(x_j^{\beta_1}, x_j^{\beta_2}, ..., x_j^{\beta_k}, ..., x_j^{\beta|X_j|}\right) : x_j^{\beta_k} \in X_i, \forall k = \overline{1, |X_j|}\right\} \subseteq X_j$$

for any $\alpha = 1, ..., \varkappa_i = |X_i|^{|X_j|}$, $\beta = 1, ..., \varkappa_j = |X_j|^{|X_i|}$. Thereby, the informational extended strategies of the player i are functions $\theta_i^\alpha : X_j \to X_i^\alpha$ such that for all $x_j^l \in X_j$ there is $x_i^{\alpha_l} \in X_i$ such that $\theta_i^\alpha\left(x_j^l\right) = x_i^{\alpha_l}$ and it means the following: the player i will choose the non-informational extended strategy $x_i^{\alpha_l} \in X_i^\alpha$ in case the player j will choose the non-informational extended strategy $x_j^l \in X_j$. Accordingly the informational extended strategies of the player j are functions $\theta_j^\beta : X_i \to X_j^\beta$ such that for all $x_i^k \in X_i$ there is $x_j^{\beta_k} \in X_j$ such that $\theta_j^\beta\left(x_i^k\right) = x_j^{\beta_k}$ and it means the following: the player j will choose the non-informational extended strategy $x_j^{\beta_k} \in X_j^\beta$ in case the player i will choose the non-informational extended strategy $x_i^k \in X_i$. Under the assumption that the players want maximize their payoffs we define the payoff functions of the player as follows:

$$\mathcal{H}_p\left(\theta_i^\alpha, \theta_j^\beta, x_{[-ij]}\right) = \begin{cases} \max\limits_{(x_i,x_j)\in\left[gr\theta_i^\alpha \bigcap gr\theta_j^\beta\right]} H_p\left(x_i, x_j, x_{[-ij]}\right) & \text{if } X\left(\theta_i^\alpha, \theta_j^\beta\right) \neq \varnothing, \\ -\infty & \text{if } X\left(\theta_i^\alpha, \theta_j^\beta\right) = \varnothing. \end{cases}$$

Here $X\left(\theta_i^\alpha, \theta_j^\beta\right) \subseteq X$ is the set of the strategy profiles of the players in the game (1) "generated" by the informational extended strategies θ_i^α and θ_j^β, $gr\theta_i^\alpha$, $gr\theta_j^\beta$ denotes the graph of the function θ_i^α and θ_j^β, $x_{[-ij]} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_{j-1}, x_{j+1}, ..., x_n)$. And finally, the normal form of the two-directional $i \overset{\inf}{\leftrightarrows} j$ informational extended game will be $\Gamma\left(i \overset{\inf}{\leftrightarrows} j\right) = \left\langle I, \Theta_i, \Theta_j, \{X\}_{p\in I\setminus\{i,j\}}, \{\mathcal{H}_p\}_{p\in I}\right\rangle$. Also by [1] for the bimatricial game $H_1 = \begin{pmatrix} 3 & 5 & 4 \\ 6 & 7 & 2 \end{pmatrix}$, $H_2 = \begin{pmatrix} 0 & 5 & 1 \\ 4 & 3 & 2 \end{pmatrix}$ the normal form of the $1 \overset{\inf}{\leftrightarrows} 2$ informational extended game will be the bimatricial game with the following payoff matrices for the player 1

$$\mathcal{H}_1 = \begin{pmatrix} 3 & 5 & 4 & 3 & 5 & 3 & 4 & 5 & 4 \\ 6 & 7 & 2 & 7 & 6 & 2 & 6 & 2 & 7 \\ 3 & 5 & 2 & 3 & 5 & 3 & -\infty & 3 & -\infty \\ 3 & 7 & 4 & 7 & -\infty & 3 & 4 & -\infty & 7 \\ 6 & 5 & 4 & -\infty & 7 & -\infty & 3 & 5 & 4 \\ 3 & 7 & 2 & 7 & -\infty & 3 & -\infty & 2 & 7 \\ 6 & 7 & 4 & -\infty & 6 & 2 & 6 & 5 & -\infty \\ 6 & 7 & 4 & 7 & 6 & -\infty & 6 & -\infty & 7 \end{pmatrix}$$

and for the player 2 correspondingly

$$\mathcal{H}_2 = \begin{pmatrix} 0 & 5 & 1 & 0 & 5 & 0 & 1 & 5 & 1 \\ 4 & 3 & 2 & 3 & 4 & 2 & 4 & 2 & 3 \\ 0 & 5 & 2 & 0 & 5 & 2 & -\infty & 5 & -\infty \\ 0 & 3 & 1 & 3 & -\infty & 0 & 1 & -\infty & 3 \\ 4 & 5 & 1 & -\infty & 5 & -\infty & 0 & 5 & 1 \\ 0 & 3 & 2 & 3 & -\infty & 2 & -\infty & 2 & 3 \\ 4 & 5 & 2 & -\infty & 5 & 2 & 4 & 5 & -\infty \\ 4 & 3 & 1 & 3 & 4 & -\infty & 4 & -\infty & 3 \end{pmatrix}.$$

Below the correspondence between Nash equilibrium profiles in the $\Gamma\left(1 \overset{\text{inf}}{\leftrightarrows} 2\right)$ game and profiles in the Γ game is shown:

$$\left(\theta_1^1, \theta_2^8\right) \Rightarrow (1,2)\,;\ \left(\theta_1^2, \theta_2^1\right) \Rightarrow (2,1)\,;\ \left(\theta_1^2, \theta_2^7\right) \Rightarrow (2,1)\,;\ \left(\theta_1^4, \theta_2^2\right) \Rightarrow (2,2)\,;$$
$$\left(\theta_1^4, \theta_2^4\right) \Rightarrow (2,2);\ \left(\theta_1^4, \theta_2^9\right) \Rightarrow (2,2);\ \left(\theta_1^5, \theta_2^5\right) \Rightarrow (2,1);\ \left(\theta_1^5, \theta_2^8\right) \Rightarrow (1,2);$$
$$\left(\theta_1^6, \theta_2^2\right) \Rightarrow (2,2)\,;\ \left(\theta_1^6, \theta_2^4\right) \Rightarrow (2,2);\ \left(\theta_1^6, \theta_2^9\right) \Rightarrow (2,2);\ \left(\theta_1^7, \theta_2^2\right) \Rightarrow (1,2)\,;$$
$$\left(\theta_1^7, \theta_2^8\right) \Rightarrow (1,2);\ \left(\theta_1^8, \theta_2^1\right) \Rightarrow (2,1)\,;\ \left(\theta_1^8, \theta_2^7\right) \Rightarrow (2,1).$$

Here the informational extended strategy of the player 1 is the function with the following values:

$\theta_1^1(j) = 1\ \forall j = 1,2,3;\ \theta_1^2(j) = 2\ \forall j = 1,2,3;\ \theta_1^3(1) = \theta_1^3(2) = 1, \theta_1^3(3) = 2;$ $\theta_1^4(1) = \theta_1^4(3) = 1,\ \theta_1^4(2) = 2;\ \theta_1^5(2) = \theta_1^5(3) = 1,\ \theta_1^5(1) = 2;\ \theta_1^6(2) = \theta_1^6(3) = 2,\ \theta_1^6(1) = 1;\ \theta_1^7(1) = \theta_1^7(3) = 2,\ \theta_1^7(2) = 1;\ \theta_1^8(1) = \theta_1^8(2) = 2,$ $\theta_1^8(3) = 1$

and correspondingly for the player 2:

$\theta_2^1(i) = 1\ \forall i = 1,2;\ \theta_2^2(i) = 2\ \forall i = 1,2;\ \theta_2^3(i) = 3\ \forall i = 1,2;\ \theta_2^4(1) = 1,$ $\theta_2^4(2) = 2;\ \theta_2^5(2) = 1,\ \theta_2^5(1) = 2;\ \theta_2^6(1) = 1,\ \theta_2^6(2) = 3;\ \theta_2^7(2) = 1,$ $\theta_2^7(1) = 3;\ \theta_2^8(1) = 2,\ \theta_2^8(2) = 3;\ \theta_2^9(1) = 3,\ \theta_2^9(2) = 2.$

Remark 1. We have to mention the following: the informational extended game (considered in this article) is not a dynamic game (in terms of the choice of the strategy, but not in terms of the strategies structure) because the strategies are chosen simultaneously.

In this article we study the case when the informational strategies of the players have already been chosen and so appears the necessity to study the informational non-extended game generated by the chosen informational extended strategies. These games differ in: a) the sets of the strategies that are the subsets of the sets of strategies in the initial non-extended informational game; b) how the payoff functions of the players will be constructed.

Let the payoff functions of the players be defined as $\widetilde{H}_p : \prod_{p\in I} X_p \to R$ where for all $x_i \in X_i$, $x_j \in X_j$, $x_{[-ij]} \in X_{[-ij]}$ we have $\widetilde{H}_p\left(x_i, x_j, x_{[-ij]}\right) \equiv H_p\left(\theta_i(x_j), \theta_j\left(x_i\right), x_{[-ij]}\right)$.

Definition 1. The game with the following normal form

$$\Gamma\left(\theta_i, \theta_j\right) = \left\langle I, \left\{X_p\right\}_{p\in I}, \left\{\widetilde{H}_p\right\}_{p\in I}\right\rangle \tag{2}$$

will be called informational non-extended game generated by the informational extended strategies θ_i and θ_j.

The game $\Gamma\left(\theta_i, \theta_j\right)$ is played as follows: independently and simultaneously each player $p \in I$ chooses the informational non-extended strategy $x_p \in X_p$, after that the players i and j calculate the value of the informational extended strategies $\theta_i(x_j)$ and $\theta_j\left(x_j\right)$, after that each player calculates the payoff values $H_p\left(\theta_i(x_j), \theta_j\left(x_j\right), x_{[-ij]}\right)$, and with this the game is finished. To all strategy profiles $\left(x_i, x_j, x_{[-ij]}\right)$ in the game (2) the following realization $\left(\theta_i(x_j), \theta_j\left(x_i\right), x_{[-ij]}\right)$ in terms of the informational extended strategies will correspond.

We introduce the following definition of the Nash equilibrium profile for normal form game $\Gamma\left(\theta_i, \theta_j\right)$.

Definition 2. The strategy profile $\left(x_i^*, x_j^*, x_{-ij}^*\right) \in X$ is called the Nash equilibrium of the game $\Gamma\left(\theta_i, \theta_j\right)$ if and only if the following conditions are satisfied:

$$\begin{cases} \widetilde{H}_i\left(x_i^*, x_j^*, x_{[-ij]}^*\right) \geqslant \widetilde{H}_i\left(x_i, x_j^*, x_{[-ij]}^*\right) \text{ for all } x_i \in X_i, \\ \widetilde{H}_j\left(x_i^*, x_i^*, x_{[-ij]}^*\right) \geqslant \widetilde{H}_j\left(x_i^*, x_j, x_{[-ij]}^*\right) \text{ for all } x_j \in X_j, \\ \widetilde{H}_p\left(x_i^*, x_i^*, x_p^*\right) \geqslant \widetilde{H}_p\left(x_i^*, x_j^*, x_p\right) \text{ for all } x_p \in X_p \text{ and for all } p \in I\backslash\{i, j\}. \end{cases}$$

We denote by $NE\left[\Gamma\left(\theta_i, \theta_j\right)\right]$ the set of Nash equilibrium profiles of the game $\Gamma\left(\theta_i, \theta_j\right)$. According to Definition 1 we have that $\left(x_i^*, x_j^*, x_{-ij}^*\right) \in NE\left[\Gamma\left(\theta_i, \theta_j\right)\right]$ if and only if

$$\begin{cases} H_i\left(\theta_i(x_j^*), \theta_j\left(x_i^*\right), x_{[-ij]}^*\right) \geqslant H_i\left(\theta_i(x_j^*), \theta_j\left(x_i\right), x_{[-ij]}^*\right) \text{ for all } x_i \in X_i, \\ H_j\left(\theta_i(x_j^*), \theta_j\left(x_i^*\right), x_{[-ij]}^*\right) \geqslant H_j\left(\theta_i(x_j), \theta_j\left(x_i^*\right), x_{[-ij]}^*\right) \text{ for all } x_j \in X_j, \\ H_p\left(\theta_i(x_j^*), \theta_j\left(x_i^*\right), x_{[-ij]}^*\right) \geqslant H_p\left(\theta_i(x_j^*), \theta_j\left(x_i^*\right), x_p\right) \text{ for all } x_p \in X_p \ p \in I\backslash\{i,j\}. \end{cases}$$

Another, and some times more convenient way of defining Nash equilibrium is via the best response correspondences $Br_p : \prod\limits_{k\in I\backslash\{p\}} X_k \to 2^{X_p}$ such that:

- for player i :

$$\begin{aligned} Br_i\left(x_{[-i]}\right) &= \left\{x_i \in X_i : H_i\left(\theta_i(x_j), \theta_j\left(x_i\right), x_{[-ij]}\right) \geq H_i\left(\theta_i(x_j), \theta_j\left(x_i'\right), \right.\right. \\ &\left.\left. x_{[-ij]}\right) \text{ for all } x_i' \in X_i\right\}; \end{aligned}$$

- for player j :

$$\begin{aligned} Br_j\left(x_{[-j]}\right) &= \left\{x_j \in X_j : H_j\left(\theta_i(x_j), \theta_j\left(x_i\right), x_{[-ij]}\right) \geq H_j\left(\theta_i(x_j'), \theta_j\left(x_i\right), \right.\right. \\ &\left.\left. x_{[-ij]}\right) \text{ for all } x_j' \in X_j\right\}; \end{aligned}$$

- for player $p \neq i, j$:

$$\begin{aligned} Br_p\left(x_{[-p]}\right) &= \left\{x_p \in X_p : H_p\left(\theta_i(x_j), \theta_j\left(x_i\right), x_p, x_{[-ijp]}\right) \geq H_p\left(\theta_i(x_j), \right.\right. \\ &\left.\left. \theta_j\left(x_i\right), x_p', x_{[-ijp]}\right) \text{ for all } x_p' \in X_p\right\}. \end{aligned}$$

Here 2^{X_p} denotes the set of all subsets of the set X_p and $x_{[-ijp]}$ denotes the strategies profiles without the strategies of the players i, j and p. If the payoff functions $H_p(\cdot), \left(p = \overline{1,n}\right)$ are continuous on the compact $\prod\limits_{p\in I} X_p$ and the functions $\theta_i : X_j \to X_i$, $\theta_j : X_i \to X_j$ are continuous on the compact X_j (correspondingly X_i) then the functions $\widetilde{H}_p, p = \overline{1,n}$ are continuous on the compact $\prod\limits_{p\in I} X_p$ as composite functions. Then according to the Weierstrass theorem we can write

$$\begin{aligned} Br_i\left(x_{[-i]}\right) &= Arg \max_{x_i\in X_i} H_i\left(\theta_i(x_j), \theta_j\left(x_i\right), x_{[-ij]}\right), \\ Br_j\left(x_{[-j]}\right) &= Arg \max_{x_j\in X_j} H_j\left(\theta_i(x_j), \theta_j\left(x_i\right), x_{[-ij]}\right), \\ Br_p\left(x_{[-p]}\right) &= Arg \max_{x_p\in X_p} H_p\left(\theta_i(x_j), \theta_j\left(x_i\right), x_p, x_{[-ijp]}\right) \end{aligned}$$

for all $p \in I$, $p \neq i, j$. In this case $\left(x_i^*, x_j^*, x_{-ij}^*\right) \in NE\left[\Gamma\left(\theta_i, \theta_j\right)\right]$ if and only if

$$\begin{cases} x_i^* \in Br_i\left(x_{[-i]}^*\right), \\ x_j^* \in Br_j\left(x_{[-j]}^*\right), \\ x_p^* \in Br_p\left(x_{[-p]}^*\right) \ \forall p \neq i, j. \end{cases}$$

Construct the point to set mapping $Br : X \to 2^X$ so that for all $x \in X$,

$$Br(x) = \left(Br_1\left(x_{[-1]}\right), ..., Br_i\left(x_{[-i]}\right), ..., Br_n\left(x_{[-n]}\right)\right) \subseteq X.$$

Then

$$\left(x_i^*, x_i^*, x_{[-ij]}^*\right) \in NE\left[\Gamma\left(\theta_i, \theta_j\right)\right]$$

if and only if

$$\left(x_i^*, x_i^*, x_{[-ij]}^*\right) \in Br\left(x_i^*, x_i^*, x_{[-ij]}^*\right),$$

that is $\left(x_i^*, x_i^*, x_{[-ij]}^*\right)$ is the fixed point of the mapping Br.

We shall analyze in more details the case of the bimatricial game. So we consider the following normal form game

$$\Gamma = \left\langle I = \{1, ..., n\}, J = \{1, ..., m\}, H_1 = \|a_{ij}\|_{i \in I}^{j \in J}, H_2 = \|b_{ij}\|_{i \in I}^{j \in J} \right\rangle.$$

For this game we construct the game according to Definition 1. The informational extended strategies are $\theta_1 : J \to I$ and $\theta_2 : I \to J$, the payoff matrices are $\widetilde{H}_1 = \|\widetilde{a}_{ij}\|_{i \in I}^{j \in J}$, $\widetilde{H}_2 = \left\|\widetilde{b}_{ij}\right\|_{i \in I}^{j \in J}$ where $\widetilde{a}_{ij} = a_{\theta_1(j)\theta_2(i)}$ and $\widetilde{b}_{ij} = b_{\theta_1(j)\theta_2(i)}$ for all $i \in I$, $j \in J$. So we will obtain the following normal form game

$$\Gamma\left(\theta_1, \theta_2\right) = \left\langle I = \{1, ..., n\}, J = \{1, ..., m\}, \widetilde{H}_1 = \|\widetilde{a}_{ij}\|_{i \in I}^{j \in J}, \ \widetilde{H}_2 = \left\|\widetilde{b}_{ij}\right\|_{i \in I}^{j \in J} \right\rangle \equiv$$
$$\equiv \left\langle I = \{1, ..., n\}, J = \{1, ..., m\}, \widetilde{H}_1 = \left\|a_{\theta_1(j)\theta_2(i)}\right\|_{i \in I}^{j \in J}, \ \widetilde{H}_2 = \left\|b_{\theta_1(j)\theta_2(i)}\right\|_{i \in I}^{j \in J} \right\rangle.$$

The strategy profile $(i^e, j^e) \in NE\left(\Gamma\left(\theta_1, \theta_2\right)\right)$ if and only if

$$\begin{cases} \widetilde{a}_{i^e j^e} \geqslant \widetilde{a}_{i j^e} \text{ for all } i \in I, \\ \widetilde{b}_{i^e j^e} \geqslant \widetilde{b}_{i^e j} \text{ for all } j \in J, \end{cases}$$

and according to Definition 1 we have that

$$\begin{cases} a_{\theta_1(j^e)\theta_2(i^e)} \geqslant a_{\theta_1(j^e)\theta_2(i)} \text{ for all } i \in I, \\ b_{\theta_1(j^e)\theta_2(i^e)} \geqslant b_{\theta_1(j)\theta_2(i^e)} \text{ for all } j \in J. \end{cases}$$

From the set of all informational extended strategies of the players i and j we will highlight the following class of "best responses" strategies

$$\widetilde{\Theta}_i = \left\{ \widetilde{\theta}_i : X_j \to X_i \mid \forall x_j \in X_j, \ \widetilde{\theta}_i(x_j) = \arg\max_{x_i \in X_i} H_i\left(x_i, x_j, x_{[-ij]}\right) \right\}, \quad (3)$$

$$\widetilde{\Theta}_j = \left\{ \widetilde{\theta}_j : X_i \to X_j \mid \forall x_i \in X_i, \ \widetilde{\theta}_j(x_i) = \arg\max_{x_j \in X_j} H_j\left(x_i, x_j, x_{[-ij]}\right) \right\}. \quad (4)$$

We consider now the following examples of the informational non-extended game $\Gamma\left(\theta_i, \theta_j\right)$ generated by the strategies type (3)-(4) of the players.

Example 1. Let us consider the two person game for which $X = [0,1]$, $Y = [0,1]$, $H_1(x,y) = \frac{3}{2}xy - x^2$, $H_2(x,y) = \frac{3}{2}xy - y^2$ are the sets of strategies and the payoff functions of the players. We construct the normal form game $\Gamma(\theta_i, \theta_j)$ generated by the informational extended strategies type (3)-(4) and determine the Nash equilibrium solution.

Solution. We determine the equilibrium profile using the "best response" mapping. We derive the best response (reaction) function for each player given the other players strategy. Because the problem is symmetric, first we will show only for player 1 and then apply the result to the case for player 2. First order condition will give us that $Br_1(y) = Arg \max\limits_{x\in[0,1]} H_1(x,y) = \left\{x \in [0,1]/x = \frac{3}{4}y\right\}$ where $Br_1(y)$ is the best response correspondence for player 1. Similarly, the best response corresponding to player 2 is $Br_2(x) = Arg \max\limits_{y\in[0,1]} H_2(x,y) = \left\{y \in [0,1]/y = \frac{3}{4}x\right\}$. So the solution of the problem $\begin{cases} x^* \in Br_1(y^*) \\ y^* \in Br_2(x^*) \end{cases}$ is $x^* = y^* = 0$. Consider the informational non-extended game generated by the informational extended strategies of the two-directional informational flow type $1 \overset{\text{inf}}{\leftrightarrows} 2$. As informational extended strategies we will use the functions $\theta_1 : Y \to X$ where $\forall y \in Y$, $\theta_1(y) = \arg\max\limits_{x\in X} H_1(x,y)$, respectively $\theta_2 : X \to Y$ where $\forall x \in X$, $\theta_2(x) = \arg\max\limits_{y\in Y} H_2(x,y)$. Using the necessary condition of optimality we obtain that $\theta_1(y) = \frac{3}{4}y \ \forall y \in [0,1]$ and $\theta_2(x) = \frac{3}{4}x \ \forall x \in [0,1]$. Thus $\widetilde{H}_1(x,y) = \frac{3}{2}\theta_1(y)\theta_2(x) - (\theta_1(y))^2 = \frac{3}{2}\left(\frac{3}{4}y\right)\left(\frac{3}{4}x\right) - \left(\frac{3}{4}y\right)^2 = \frac{27}{32}xy - \frac{9}{16}y^2$ and $\widetilde{H}_2(x,y) = \frac{3}{2}\theta_1(y)\theta_2(x) - (\theta_2(x))^2 = \frac{3}{2}\left(\frac{3}{4}y\right)\left(\frac{3}{4}x\right) - \left(\frac{3}{4}x\right)^2 = \frac{27}{32}xy - \frac{9}{16}x^2$. So we obtain the following normal form game $\Gamma(\theta_1, \theta_2) = \left\langle X, Y, \widetilde{H}_1, \widetilde{H}_2 \right\rangle$. Determine the equilibrium profile of the game $\Gamma(\theta_1, \theta_2)$. According to the definition $(x^*, y^*) \in NE[\Gamma(\theta_1, \theta_2)]$ if and only if $\begin{cases} \widetilde{H}_1(x^*, y^*) \geqslant \widetilde{H}_1(x, y^*) \ \forall x \in X, \\ \widetilde{H}_2(x^*, y^*) \geqslant \widetilde{H}_2(x^*, y) \ \forall y \in Y. \end{cases}$ So we have

$$\begin{cases} \frac{3}{2}\theta_1(y^*)\theta_2(x^*) - (\theta_1(y^*))^2 \geqslant \frac{3}{2}\theta_1(y^*)\theta_2(x) - (\theta_1(y^*))^2 \ \ \forall x \in X, \\ \frac{3}{2}\theta_1(y^*)\theta_2(x^*) - (\theta_1(y^*))^2 \geqslant \frac{3}{2}\theta_1(y)\theta_2(x^*) - (\theta_2(x^*))^2 \ \ \forall y \in Y, \end{cases}$$

and finally

$$\begin{cases} \frac{27}{32}x^*y^* - \frac{9}{16}y^{*2} = \max\limits_{x\in[0,1]} \left\{\frac{27}{32}xy^* - \frac{9}{16}(y^*)^2\right\}, \\ \frac{27}{32}x^*y^* - \frac{9}{16}y^{*2} = \max\limits_{y\in[0,1]} \left\{\frac{27}{32}x^*y - \frac{9}{16}(x^*)^2\right\}. \end{cases}$$

Thus the Nash equilibrium profile is $(x^*, y^*) = (1,1)$, that is $NE[\Gamma(\theta_1, \theta_2)] = \{(1,1)\}$ while $NE[\Gamma] = \{(0,0)\}$.

Example 2. We consider the following bimatricial game $H_1 = \begin{pmatrix} 3 & 5 & 4 \\ 6 & 7 & 2 \end{pmatrix}$, $H_2 =$

$\begin{pmatrix} 0 & 5 & 1 \\ 4 & 3 & 2 \end{pmatrix}$. We construct the normal form game generated by the informational extended strategies of type (3)-(4) and we determine the Nash equilibrium profiles.

Solution. The strategies of the type (3)-(4) in the game $\Gamma\left(2 \overset{\inf}{\leftrightarrows} 1\right)$ are $i^*(j) = \arg\max_i a_{ij} = \begin{cases} 1 \text{ if } j = 3, \\ 2 \text{ if } j = 1, 2 \end{cases}$ and $j^*(i) = \arg\max_j b_{ij} = \begin{cases} 2 \text{ if } i = 1, \\ 1 \text{ if } i = 2. \end{cases}$ We construct the game $\Gamma(i^*, j^*)$ according to Definition 1. In the table below the correspondence between the strategies profile in the informational non-extended game (initial game) and the strategies profile generated by the informational extended strategies i^* and j^* is presented

(i,j)	$(1,1)$	$(1,2)$	$(1,3)$	$(2,1)$	$(2,2)$	$(2,3)$
$(i^*(j), j^*(i))$	$(2,2)$	$(2,2)$	$(1,2)$	$(2,1)$	$(2,1)$	$(1,1)$

Then the payoff matrices will be $\widetilde{H}_1 = \left\| a_{i^*(j)j^*(i)} \right\|_{i\in I}^{j\in J} = \begin{pmatrix} 7 & 7 & 5 \\ 6 & 6 & 3 \end{pmatrix}$,
$\widetilde{H}_2 = \left\| b_{i^*(j)j^*(i)} \right\|_{i\in I}^{j\in J} = \begin{pmatrix} 3 & 3 & 5 \\ 4 & 4 & 0 \end{pmatrix}$ and so we have the following normal form game

$$\Gamma(i^*, , j^*) = \left\langle I = \{1,2\}, J = \{1,2,3\},\ \widetilde{H}_1 = \begin{pmatrix} 7 & 7 & 5 \\ 6 & 6 & 3 \end{pmatrix},\ \widetilde{H}_2 = \begin{pmatrix} 3 & 3 & 5 \\ 4 & 4 & 0 \end{pmatrix} \right\rangle.$$

The game is done in the following way. In case the players choose the informational extended strategies i^* and j^*, then the game (for players 1 and 2) like "if-then" starts, i.e. "if the player 1 chooses the line 1, then the player 2, knowing this, chooses the column 2 and simultaneously, if the player 2 chooses the column 1, then the player 1, knowing this, chooses the line 2 etc. We note that the equilibrium profile in the game $\Gamma(i^*, j^*)$ is $(i^e, j^e) = (1,3)$ and $\widetilde{H}_1(1,3) = 5$, $\widetilde{H}_2(1,3) = 5$. To this profile corresponds the following profile in the informational extended strategy $(i^*(j^e), j^*(i^e)) = (i^*(3), j^*(1)) = (1,2)$ for which we have that $H_1(1,2) = 5$, $H_2(1,2) = 5$. According to the definition of the Nash equilibrium profile (i^e, j^e) we have that

$$\begin{cases} a_{i^*(j^e)j^*(i^e)} \geqslant a_{i^*(j^e)j^*(i)} \text{ for all } i = 1,2, \\ b_{i^*(j^e)j^*(i^e)} \geqslant b_{i^*(j)j^*(i^e)} \text{ for all } j = 1,2,3, \end{cases}$$

from which we deduce the following relations:

$$\left\{ \widetilde{a}_{13} = 5 = a_{i^*(3)j^*(1)} > \widetilde{a}_{23} = 3 = a_{i^*(3)j^*(2)} = a_{11} = 3, \right.$$

$$\begin{cases} \widetilde{b}_{13} = 5 = b_{i^*(3)j^*(1)} > \widetilde{b}_{11} = 3 = b_{i^*(1)j^*(1)} = b_{22} = 3, \\ \widetilde{b}_{13} = 5 = b_{i^*(3)j^*(1)} > \widetilde{b}_{12} = 3 = b_{i^*(2)j^*(1)} = b_{22} = 3. \end{cases}$$

So we have shown that $NE\left[\Gamma(i^*, j^*)\right] = \{(1,3)\}$.

Remark 2. If the normal form $\Gamma(i^*, j^*)$ has already been constructed, then the equilibrium profile is determined using the matrices $\widetilde{H}_1$ and $\widetilde{H}_2$, otherwise, using the elements $a_{i^*(j)j^*(i)}$, $b_{i^*(j)j^*(i)}$ of the matrices of the game Γ.

We begin by proving Nash's Theorem about the existence of a strategy equilibrium profile in the normal form game $\Gamma(\theta_i, \theta_j)$ first giving some remarks about the Kakutani's fixed point theorem. **Kakutani's theorem states** [4]: *Let S be a non-empty, compact and convex subset of the Euclidean space R^n. Let $\varphi : S \to 2^S$ be a set-valued function on S with a closed graph and the property that $\varphi(x)$ is non-empty and convex for all $x \in S$. Then φ has a fixed point.*

The Kakutani fixed point theorem is a fixed-point theorem for point-to-set mapping. It provides sufficient conditions for a point-to-set mapping defined on a convex, compact subset of a Euclidean space to have a fixed point, i.e. a point which is mapped to a set containing it. The Kakutani fixed point theorem is a generalization of Brouwer fixed point theorem. The Brouwer fixed point theorem is a fundamental result in topology which proves the existence of fixed points for continuous functions defined on compact, convex subsets of Euclidean spaces. Kakutani theorem extends this to point-to-set mapping.

Mathematician **John Nash** used the Kakutani fixed point theorem to prove a major result in game theory. Stated informally, the theorem implies the existence of a Nash equilibrium in every finite game with mixed strategies for any number of players. In this case, S is the set of tuples of strategies chosen by each player in a game. The function $\varphi(x)$ gives a new tuple where each player's strategy is his best response to other players' strategies at x. Since there may be a number of responses which are equally good, φ is set-valued rather than single-valued. Then the Nash equilibrium of the game is defined as a fixed point of φ, i.e. a tuple of strategies where each player's strategy is a best response to the strategies of the other players. Kakutani's theorem ensures that this fixed point exists.

Let us prove the following theorem.

Theorem. *Let* $\Gamma(\theta_i, \theta_j) = \left\langle I, \{X_p\}_{p \in I}, \left\{\widetilde{H}_p\right\}_{p \in I} \right\rangle$ *be the normal form of the informational non-extended game generated by the informational extended strategies using the* $i \overset{\inf}{\leftrightarrows} j$ *type flow of information, where for all* $x_i \in X_i$, $x_j \in X_j$, $x_{[-ij]} \in X_{[-ij]}$ *we have* $\widetilde{H}_p\left(x_i, x_j, x_{[-ij]}\right) \equiv H_p\left(\theta_i(x_j), \theta_j(x_i), x_{[-ij]}\right)$. *Let this game satisfy the following conditions:*

1) *the X_p is a non-empty compact and convex subset of the finite-dimensional Euclidean space for all $p \in I$;*

2) *the functions θ_i (correspondingly θ_j) are continuous on X_j (correspondingly on X_i) and the functions H_p are continuous on X for all $p \in I$.*

3) *the functions θ_i (correspondingly θ_j) are quasi-concave on X_j (correspondingly on X_i), the functions H_p are quasi-concave on X_p, $p \in I \backslash \{i, j\}$ and monotonically increasing on $X_i \times X_j$.*

Then $NE\left[\Gamma\left(\theta_i,\theta_j\right)\right]\neq\emptyset$.

Proof. If we define the following correspondence (point-to-set mapping) $Br: X\rightarrow X$ such that $Br(x)=\left(Br_1\left(x_{[-1]}\right),...,Br_i\left(x_{[-i]}\right),...,Br_n\left(x_{[-n]}\right)\right)$ then if $x^*\in Br(x^*)$, then $x_i^*\in Br_i\left(x^*_{[-i]}\right)$ for all $i\in I$ and hence $x^*\in NE$. To prove this theorem we can show that: a) the X is a non-empty compact and convex subset of the Euclidean finite-dimensional space and b) the set-valued mapping $Br: X\rightarrow X$ has a closed graph, that is, if $\{x^k,y^k\}\rightarrow\{x,y\}$ with $y^k\in Br(x^n)$, then $y\in Br(x)$, and the set $Br(x)$ is nonempty, convex and compact for all $x\in X$. According to the Tikhonov's theorem: *a product of a family of compact topological spaces* $X=\prod\limits_{p\in I}X_p$ *is compact,* the item a) is fulfilled. For all $x_{[-i]}$ the set $Br_i\left(x_{[-i]}\right)$ is non-empty because according to conditions 1) and 2) $\widetilde{H}_i$ is continuous and X_i is compact (Weierstrass's theorem). According to condition 3) $Br_i\left(x_{[-i]}\right)$ is also convex because $\widetilde{H}_i$ is quasi-concave on X_i. Hence the set $Br(x)$is nonempty convex and compact for all $x\in X$. The mapping Br has a closed graph because each function $\widetilde{H}_p$ is continuous on X for all $p\in I$. Hence by Kakutani's theorem, the set-valued mapping Br has a fixed point. As we have noted, any fixed point is a Nash equilibrium. □

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Stability analysis of Pareto optimal portfolio of multicriteria investment maximin problem in the Hölder metric

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Abstract. We analyzed the stability of a Pareto-optimal portfolio of the multicriteria discrete variant of Markowitz's investment problem with Wald's maximin efficiency criteria. We obtained lower and upper bounds for the stability radius of such portfolio in the case of the Hölder metric l_p, $1 \le p \le \infty$, in the three-dimensional space of problem parameters. We also show the attainability of bounds in particular cases.

Mathematics subject classification: 90C09, 90C29, 90C31, 90C47.
Keywords and phrases: Multicriteria optimization, investment portfolio, Wald's maximin efficiency criteria, Pareto-optimal portfolio, stability radius.

In paper [1] we obtained lower and upper attainable bounds for the stability radius of a Pareto-optimal portfolio of the multicriteria Boolean investment problem with Savage's minimax risk criteria in the case of the Chebyshev metric l_1 in the three-dimensional space of problem parameters. In the present paper we obtained the results of similar nature for the stability radius of the multicriteria investment problem with Wald's maximin efficiency criteria and any Hölder metric l_p, $1 \le p \le \infty$, in the spaces of criteria, portfolio and market states.

1 Problem statement and definitions

We consider the multicriteria discrete variant of Markowitz's investment managing problem [2]. To this end, we introduce the following notations:

$N_n = \{1, 2, \dots, n\}$ be a set of investment alternative projects (assets);

N_m be a set of market states (conditions, scenarios);

N_s be a set of project efficiency measures;

$x = (x_1, x_2, \dots, x_n)^T \in X \subseteq \mathbf{E}^n$ be an investment portfolio, where $|X| \ge 2$, $\mathbf{E} = \{0, 1\}$,

$$x_j = \begin{cases} 1 & \text{if the project } j \text{ is implemented,} \\ 0 & \text{otherwise;} \end{cases}$$

e_{ijk} be an assessment of efficiency of measure $k \in N_s$ of investment project $j \in N_n$ in the situation when the market is in state $i \in N_m$;

$E = [e_{ijk}]$ be a three-dimensional $m \times n \times s$ matrix with elements from $\mathbf{R}$.

Note that there are several approaches to evaluate efficiency of investment projects (NPV, NFV, PI et al.), which take into account risk and uncertainty in different ways (see e.g. [3–5]). That way it is worth to consider a decision making problem with multiple criteria (several measures of project efficiency).

Let the following vector objective function

$$f(x,E) = (f_1(x,E_1), f_2(x,E_2), \dots, f_s(x,E_s)),$$

be given on a set of investment portfolios X whose components are Wald's maximin criteria [6]

$$f_k(x,E_k) = \min_{i\in N_m} E_{ik}x = \min_{i\in N_m} \sum_{j\in N_n} e_{ijk}x_j \to \max_{x\in X}, \qquad k\in N_s,$$

where $E_k \in \mathbf{R}^{m\times n}$ is the k-th cut of matrix $E = [e_{ijk}] \in \mathbf{R}^{m\times n\times s}$, $E_{ik} = (e_{i1k}, e_{i2k}, \dots, e_{ink})$ is the i-th row of that cut. Thus, the investor, following Wald's criteria, takes extreme caution and optimizes portfolio efficiency $E_{ik}x$ (for the k-th criteria), assuming that the market was in the worst state, namely the efficiency is minimal. Obviously such pessimistic approach in the market state estimation is justified when we are talking about the guaranteed result.

A multicriteria *investment Boolean problem* $Z^s(E)$, $s \in \mathbf{N}$, with Wald's criteria means the problem of searching the set of *Pareto-optimal investment portfolios* (the Pareto set)

$$P^s(E) = \{x\in X:\ \nexists x'\in X\ \ (g(x',x,E)\geq 0_{(s)}\ \ \&\ \ g(x',x,E)\neq 0_{(s)})\},$$

where

$$g(x',x,E) = (g_1(x',x,E_1), g_2(x',x,E_2),\ \dots, g_s(x',x,E_s)),$$

$$g_k(x',x,E_k) = f_k(x',E_k) - f_k(x,E_k) = \max_{i\in N_m}\ \min_{i'\in N_m}\ (E_{i'k}x' - E_{ik}x),\ k\in N_s,$$

$$0_{(s)} = (0,0,\dots,0)\in \mathbf{R}^s.$$

It is easy to see, in the particular case for $m = 1$ our multicriteria investment problem $Z^s(E)$ becomes the multicriteria problem of linear Boolean programming

$$Z^s_B(E):\quad Ex \to \max_{x\in X}, \tag{1}$$

where $X \subseteq \mathbf{E}^n$, $E = [e_{1jk}] \in \mathbf{R}^{1\times n\times s}$ is the matrix with rows $E_k = (e_{11k},\ e_{12k}, \dots$ $\dots, e_{1nk}) \in \mathbf{R}^n$, $k \in N_s$. Such case can be interpreted as the situation when the investor has not got another alternative market state.

For any positive integer $d \geq 2$ in the real space $\mathbf{R}^d$ we introduce *the Hölder metric* l_p, $1 \leq p \leq \infty$, where the norm of $a = (a_1, a_2, \dots, a_d) \in \mathbf{R}^d$ is defined by the formula

$$\|a\|_p = \begin{cases} \left(\sum_{j\in N_d} |a_j|^p\right)^{1/p} & \text{if } 1\leq p<\infty, \\ \max\{|a_j|:\ j\in N_d\} & \text{if } p=\infty, \end{cases}$$

and by the norm of a matrix means the norm of the vector composed of all matrix elements. Hence for matrix $E \in \mathbf{R}^{m\times n\times s}$ and any metric l_p, $1 \le p \le \infty$, we get the equalities

$$\|E\|_p = \|(\|E_1\|_p, \|E_2\|_p, \dots, \|E_s\|_p)\|_p, \tag{2}$$

$$\|E_k\|_p = \|(\|E_{1k}\|_p, \|E_{2k}\|_p, \dots, \|E_{mk}\|_p)\|_p, \quad k \in N_s. \tag{3}$$

Thus for $p < \infty$ the equations

$$\|E\|_p = (\sum_{k\in N_s} \|E_k\|_p^p)^{1/p}, \tag{4}$$

$$\|z\|_p = \|z\|_1^{1/p} \ \text{ for } \ z \in \{-1, 0, 1\}^n \tag{5}$$

hold. In addition, from (2) and (3) it follows that

$$\|E_{ik}\|_p \le \|E_k\|_p \le \|E\|_p, \quad i \in N_m,\ k \in N_s. \tag{6}$$

It is known, that the metric l_p defined in the space $\mathbf{R}^d$ includes the metric l_q in the dual space $(\mathbf{R}^d)^*$, and p, q, as it is well known, are related by the formula

$$\frac{1}{p} + \frac{1}{q} = 1, \qquad 1 < p < \infty. \tag{7}$$

In addition, as usual, we set $q = 1$ if $p = \infty$ and $q = \infty$ if $p = 1$. Thus, in what follows, we assume that the domain of variation of p and q is the interval $[1, \infty]$, while p, q obey the above conditions, moreover, we assume $1/p = 0$ for $p = \infty$.

Using (6) and the Hölder inequality

$$ab \le \|a\|_p \|b\|_q,$$

where $a = (a_1, a_2, \dots, a_n) \in \mathbf{R}^n$, $b = (b_1, b_2, \dots, b_n)^T \in \mathbf{R}^n$, it is easy to see that for x^0, $x \in X$ and $1 < p \le \infty$ the following inequalities hold:

$$E_{i'k}x^0 - E_{ik}x \ge -\|E_k\|_p(\|x^0\|_q + \|x\|_q), \qquad i,\ i' \in N_m, \quad k \in N_s, \tag{8}$$

and for $p = 1$:

$$E_{i'k}x^0 - E_{ik}x \ge -\|E_k\|_1, \qquad i,\ i' \in N_m, \quad k \in N_s. \tag{9}$$

In addition, for any $p \in [1, \infty]$ the following equality is obvious:

$$\|a\|_p = m^{1/p}\alpha \tag{10}$$

if any component of $a \in \mathbf{R}^m$ is the number $\alpha > 0$.

As usual [1, 7–9], *the stability radius of the investment portfolio* $x^0 \in P^s(E)$ in the Hölder metric l_p is defined as follows:

$$\rho^s(x^0, p, m) = \begin{cases} \sup\ \Xi_p & \text{if } \ \Xi_p \ne \emptyset, \\ 0 & \text{if } \ \Xi_p = \emptyset, \end{cases}$$

where

$$\Xi_p = \{\varepsilon > 0 : \ \forall E' \in \Omega_p(\varepsilon) \quad (x^0 \in P^s(E + E'))\},$$

$$\Omega_p(\varepsilon) = \{E' \in \mathbf{R}^{m\times n\times s} : \ \|E'\|_p < \varepsilon\}.$$

Here $\Omega(\varepsilon)$ is the set of perturbing matrixes, and $P^s(E + E')$ is the Pareto set of the perturbed problem $Z^s(E + E')$.

Thus, the stability radius defines an extreme level of problem initial data perturbations (elements of matrix E) preserving Pareto-optimality of the portfolio.

2 Lemmas

For the vector $a = (a_1, a_2, \dots, a_s) \in \mathbf{R}^s$ we introduce the positive cutoff function:

$$a^+ = [a]^+ = (a_1^+, a_2^+, \dots, a_s^+),$$

where $a_k^+ = [a_k]^+ = \max\{0, a_k\}$, $k \in N_s$.

Lemma 1. *Let* $\varphi_1 > 0$, $x^0 \neq x$,

$$\|g^+(x^0, x, E)\|_1 \geq \varphi_1. \tag{11}$$

Then

$$\forall E' \in \Omega_1(\varphi_1) \quad \exists l \in N_s \qquad (g_l(x^0, x, E_l + E_l') > 0). \tag{12}$$

Proof. Suppose, to the contrary, that there exists the perturbing matrix $E^0 \in \Omega_1(\varphi_1)$ such that the inequalities

$$g_k(x^0, x, E_k + E_k^0) \leq 0, \quad k \in N_s \tag{13}$$

hold.

Then, involving (9), we derive

$$0 \geq g_k(x^0, x, E_k + E_k^0) = \max_{i\in N_m} \min_{i'\in N_m} (E_{i'k}x^0 - E_{ik}x + E_{i'k}^0 x^0 - E_{ik}^0 x) \geq$$

$$\geq g_k(x^0, x, E_k) - \|E_k^0\|_1,$$

i. e. $g_k^+(x^0, x, E_k) \leq \|E_k^0\|_1$, $k \in N_s$. Hence, taking into account $E^0 \in \Omega_1(\varphi_1)$ it follows that the inequality

$$\|g^+(x^0, x, E)\|_1 = \sum_{k\in N_s} g_k^+(x^0, x, E_k) \leq \sum_{k\in N_s} \|E_k^0\|_1 = \|E^0\|_1 < \varphi_1$$

holds.

This inequality contradicts the condition (11) of Lemma 1. □

Lemma 2. *Let* $1 < p \leq \infty$, $\varphi_2 > 0$, $x^0 \neq x$,

$$\|g^+(x^0, x, E)\|_p \geq \varphi_2(\|x^0\|_q + \|x\|_q). \tag{14}$$

Then

$$\forall E' \in \Omega_p(\varphi_2) \quad \exists l \in N_s \qquad (g_l(x^0, x, E_l + E_l') > 0). \tag{15}$$

Proof. We again suppose, to the contrary, that there exists the perturbing matrix $E^0 \in \Omega_p(\varphi_2)$ with the conditions (13) and for any index $k \in N_s$ in view of (8) we find

$$0 \geq g_k(x^0, x, E_k + E_k^0) = \max_{i \in N_m} \min_{i' \in N_m} (E_{i'k}x^0 - E_{ik}x + E_{i'k}^0 x^0 - E_{ik}^0 x) \geq$$

$$\geq g_k(x^0, x, E_k) - \|E_k^0\|_p (\|x^0\|_q + \|x\|_q),$$

i.e.

$$g_k^+(x^0, x, E_k) \leq \|E_k^0\|_p (\|x^0\|_q + \|x\|_q), \quad k \in N_s.$$

Thus, taking into account (4) and $E^0 \in \Omega_p(\varphi_2)$ for $p < \infty$ we have

$$\|g^+(x^0, x, E)\|_p = \Big(\sum_{k \in N_s} (g_k^+(x^0, x, E_k))^p \Big)^{1/p} \leq$$

$$\leq \Big(\sum_{k \in N_s} \|E_k^0\|_p^p \Big)^{1/p} (\|x^0\|_q + \|x\|_q) = \|E^0\|_p (\|x^0\|_q + \|x\|_q) < \varphi_2 (\|x^0\|_q + \|x\|_q),$$

and for $p = \infty$ we derive

$$\|g^+(x^0, x, E)\|_\infty = \max_{k \in N_s} \ g_k^+(x^0, x, E_k) \leq \max_{k \in N_s} \ \|E_k^0\|_\infty (\|x^0\|_1 + \|x\|_1) =$$

$$= \|E^0\|_\infty (\|x^0\|_1 + \|x\|_1) < \varphi_2 (\|x^0\|_1 + \|x\|_1).$$

This inequality is contrary to the condition (14). □

By contradiction we can easily prove the following lemma.

Lemma 3. *Let $x^0 \in P^s(E)$, $\gamma > 0$ and $1 \leq p \leq \infty$. If for any portfolio $x \in X \setminus \{x^0\}$ and any perturbing matrix $E' \in \Omega_p(\gamma)$ there exists $l \in N_s$ such that the inequality $g_l(x^0, x, E_l + E'_l) > 0$ is true, then the portfolio x^0 is a Pareto-optimal portfolio of the perturbing problem $Z^s(E + E')$, i.e. $x^0 \in P^s(E + E')$ for $E' \in \Omega_p(\gamma)$.*

Lemma 4. *Let $1 \leq p \leq \infty$, $x^0 \neq x$, $\delta = (\delta_1, \delta_2, \ldots, \delta_s)$, $\delta_k > 0$, $k \in N_s$,*

$$\delta_k \|x^0 - x\|_q > g_k^+(x^0, x, E_k), \qquad k \in N_s. \tag{16}$$

Then for any number $\varepsilon > m^{1/p} \|\delta\|_p$ there exists a matrix $E^0 \in \Omega_p(\varepsilon)$ such that $x^0 \notin P^s(E + E^0)$.

Proof. Using components of δ (see (16)), we define elements of the perturbing matrix $E^0 = [e_{ijk}^0] \in \mathbf{R}^{m \times n \times s}$ as follows:

$$e_{ijk}^0 = \delta_k \, \frac{x_j - x_j^0}{\|x^0 - x\|_p}, \quad i \in N_m, \quad j \in N_n, \quad k \in N_s.$$

Because all rows E^0_{ik}, $i \in N_m$, of the cut $E^0_k \in \mathbf{R}^{m\times n}$ are equal, then denoting such rows as A_k, we have

$$A_k = \delta_k \frac{(x-x^0)^T}{\|x^0 - x\|_p}, \quad k \in N_s. \tag{17}$$

Thus $\|E^0_{ik}\|_p = \|A_k\|_p = \delta_k$, $i \in N_m$, $k \in N_s$. Hence, according to (2), (3) and (10) we find

$$\|E^0_k\|_p = m^{1/p}\delta_k, \quad k \in N_s,$$

$$\|E^0\|_p = m^{1/p}\|\delta\|_p,$$

and, therefore, $E^0 \in \Omega_p(\varepsilon)$ for any $\varepsilon > m^{1/p}\|\delta\|$. Here $1/p = 0$ is for $p = \infty$.

Further we prove that for any $p \in [1, \infty]$ and $k \in N_s$ the equality

$$A_k(x^0 - x) = -\delta_k\|x^0 - x\|_q \tag{18}$$

holds. Actually, for $p = \infty$ we have (in view of (17))

$$A_k(x^0 - x) = -\delta_k\|x^0 - x\|_1, \quad k \in N_s,$$

and for $1 \le p < \infty$, considering (5), (7) and (17), we get the following chain of equalities

$$A_k(x^0 - x) = -\delta_k \frac{\|x^0 - x\|_1}{\|x^0 - x\|_p} =$$

$$= -\delta_k \frac{\|x^0 - x\|_1}{\|x^0 - x\|_1^{1/p}} = -\delta_k\|x^0 - x\|_1^{1/q} = -\delta_k\|x^0 - x\|_q, \quad k \in N_s.$$

At last, using (16) and (18), we conclude that for any index $k \in N_s$ the relations

$$g_k(x^0, x, E_k + E^0_k) = \min_{i\in N_m}(E_{ik} + A_k)x^0 - \min_{i\in N_m}(E_{ik} + A_k)x =$$

$$= g_k(x^0, x, E_k) + A_k(x^0 - x) \le g^+_k(x^0, x, E_k) - \delta_k\|x^0 - x\|_q < 0$$

hold.

Hence, $x^0 \notin P^s(E + E^0)$. □

3 Stability radius bounds

For a Pareto-optimal portfolio x^0 of the problem $Z^s(E)$ denote

$$\varphi_1 = \varphi_1(x^0, p, m) = \min_{x\in X\setminus\{x^0\}} \|g^+(x^0, x, E)\|_p,$$

$$\varphi_2 = \varphi_2(x^0, p, m) = \min_{x\in X\setminus\{x^0\}} \frac{\|g^+(x^0, x, E)\|_p}{\|x^0\|_q + \|x\|_q},$$

$$\psi = \psi(x^0, p, m) = \min_{x\in X\setminus\{x^0\}} \frac{\|g^+(x^0, x, E)\|_p}{\|x^0 - x\|_q}.$$

Evidently, $\psi \ge 0$, $\varphi_i \ge 0$, $i \in N_2$, herewith $\varphi_1(x^0, 1, m) = \psi(x^0, 1, m)$ and $\varphi_2(x^0, p, m) \le \psi(x^0, p, m)$ for $1 < p \le \infty$.

Theorem. *For any m, $s \in \mathbf{N}$ and $1 \le p \le \infty$ the stability radius $\rho^s(x^0, p, m)$ of the investment portfolio $x^0 \in P^s(E)$ in the Hölder metric l_p has the following lower and upper bounds*

$$m^{1/p}\psi(x^0, p, m) \ge \rho^s(x^0, p, m) \ge \begin{cases} \varphi_1(x^0, p, m), & \text{if } p = 1, \\ \varphi_2(x^0, p, m), & \text{if } 1 < p \le \infty. \end{cases} \tag{19}$$

Proof. Let $x^0 \in P^s(E)$. First we will prove the validity of lower bounds (19). Without loss of generality we assume that $\varphi_i > 0$, $i \in N_2$ (otherwise, the inequalities $\rho \ge \varphi_i$, $i \in N_2$, are obvious). We shall consider separately the two possible cases.

Case 1: $p = 1$. According to the definition of $\varphi_1 = \varphi_1(x^0, 1, m)$ for any portfolio $x \ne x^0$ the inequality

$$\|g^+(x^0, x, E)\|_1 \ge \varphi_1,$$

holds. Therefore, due to Lemma 1 the formula (12) is valid. Then, according to Lemma 3 the portfolio $x^0 \in P^s(E + E')$ for any perturbing matrix $E' \in \Omega_1(\varphi_1)$. Thus, $\rho^s(x^0, 1, m) \ge \varphi_1(x^0, 1, m)$.

Case 2: $1 < p \le \infty$. According to the definition of $\varphi_2 = \varphi_2(x^0, p, m)$ the inequalities hold

$$\|g^+(x^0, x, E)\|_p \ge \varphi_2(\|x^0\|_q + \|x\|_q), \quad x \in X \setminus \{x^0\}.$$

Applying Lemma 2 yields the conclusion that for any portfolio $x \ne x^0$ the formula (15) holds. Hence from Lemma 3 it follows that the portfolio $x^0 \in P^s(E + E')$ for $E' \in \Omega_p(\varphi_2)$. Therefore, $\rho^s(x^0, p, m) \ge \varphi_2(x^0, p, m)$.

Further we will prove the validity of the upper bound (19) for any number $p \in [1, \infty]$. Let $\varepsilon > m^{1/p}\psi > 0$, and a portfolio $x^* \ne x^0$ be such that

$$\|g^+(x^0, x^*, E)\|_p = \psi\|x^0 - x^*\|_q.$$

Then, taking into account the continuous dependence of the norm of a vector on its coordinates we find a vector $\delta \in \mathbf{R}^s$ with positive components, which satisfy inequalities (16) shach that $\varepsilon/m^{1/p} > \|\delta\|_p > \psi$. Hence, due to Lemma 4 there exists a perturbing matrix $E^0 \in \Omega_p(\varepsilon)$ such that the portfolio $x^0 \in P^s(E)$ is not a Pareto-optimal portfolio of the perturbed problem $Z^s(E + E^0)$. Thus, we proved that for any number $\varepsilon > m^{1/p}\psi$ the inequality $\rho^s(x^0, p, m) < \varepsilon$ holds, i.e. the inequality $\rho^s(x^0, p, m) \le m^{1/p}\psi(x^0, p, m)$ is true for any number $p \in [1, \infty]$. □

4 Corollary

All of the following corollaries from Theorem are obvious and are valid for any number of criteria $s \in \mathbf{N}$.

Corollary 1. *For any $m \in \mathbf{N}$ the following bounds are true:*

$$m\varphi_1(x^0, 1, m) \ge \rho^s(x^0, 1, m) \ge \varphi_1(x^0, 1, m).$$

Hence we get the following well-known result, which shows that lower and upper bounds (19) are attainable for $p = m = 1$.

Corollary 2 [7, 10]. *The following formula holds:*

$$\rho^s(x^0, 1, 1) = \varphi_1(x^0, 1, 1) = \min_{x \in X \setminus \{x^0\}} \|[E(x^0 - x)]^+\|_1.$$

Corollary 3. *For any $m \in \mathbf{N}$ the following bounds are true:*

$$\psi(x^0, \infty, m) = \min_{x \in X \setminus \{x^0\}} \max_{k \in N_s} \max_{i \in N_m} \min_{i' \in N_m} \frac{E_{i'k}x^0 - E_{ik}x}{\|x^0 - x\|_1} \geq \rho^s(x^0, \infty, m) \geq$$
$$\geq \min_{x \in X \setminus \{x^0\}} \max_{k \in N_s} \max_{i \in N_m} \min_{i' \in N_m} \frac{E_{i'k}x^0 - E_{ik}x}{\|x^0\|_1 + \|x\|_1} = \varphi_2(x^0, \infty, m). \quad (20)$$

In paper [1] we proved the attainability of such bounds for the stability radius of the Pareto-optimal portfolio of the multicriteria investment problem with Savage's minimax criteria in the metric l_∞. Using the developed there techniques it is easy to prove, that lower and upper bounds (20), obtained here, are also attainable. In addition, the next statement follows from Corollary 3 and shows that lower and upper bound are attainable for $p = \infty$.

Corollary 4. *If for any portfolio $x \in X \setminus \{x^0\}$ the inequality $\|x^0\|_1 + \|x\|_1 = \|x^0 - x\|_1$ holds, then for index $m \in \mathbf{N}$ the following formula is true:*

$$\rho^s(x^0, \infty, m) = \varphi_2(x^0, \infty, m) = \psi(x^0, \infty, m).$$

Note that earlier in paper [7] (see also [8, 9]) the formula of the stability radius of the Pareto-optimal solution x^0 of the multicriteria linear Boolean programming problem $Z^s_B(E)$ (see (1)) in the Hölder metric was obtained:

$$\rho^s(x^0, p, 1) = \psi(x^0, p, 1) = \min_{x \in X \setminus \{x^0\}} \frac{\|[E(x^0 - x)]^+\|_p}{\|x^0 - x\|_q}, \qquad 1 \leq p \leq \infty.$$

This result shows that upper bound (19) is attainable in the linear case ($m = 1$).

Corollary 5. *For any parameters $m \in \mathbf{N}$ and $p \in [1, \infty]$ the stability radius $\rho^s(x^0, p, m) > 0$ if and only if*

$$\min_{x \in X \setminus \{x^0\}} \max_{k \in N_s} g_k^+(x^0, x, E_k) > 0.$$

Remark. Due to equivalence of any two metrics in finite dimensional linear spaces (see e. g. [11]), Corollary 5 is also valid not only for the Hölder metric l_p, but for another metrics in the space $\mathbf{R}^{m \times n \times s}$ of perturbing parameters of $Z^s(E)$.

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Invariant transformations of loop transversals. 2. The case of isotopy

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Abstract. The investigation of special transformations of loop transversals is continued. These transformations correspond to arbitrary isotopies of loop transversal operations (witch correspond to the considered loop transversals). Isotopies of loop transversal operations with the same unit are investigated.

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1 Introduction

This article is a continuation of the research of some special class of loop transversal transformations, begun in [5]. Transformations from the studied class correspond to arbitrary isotopies of transversal operations (which correspond to the considered loop transversals). We find a new class of loop transversal transformations which preserve the property to be a loop transversal. This investigation (as it was mentioned in [5]) is important for solving some other problems – for example, it can be used in the classification of G-loops.

2 Necessary definitions and statements

All necessary definitions and statements can be found in [5], §2. We remind the most important ones.

Definition 1. Let G be a group and H be its subgroup. Let $\{H_i\}_{i\in E}$ be the set of all left (right) cosets in G to H, and we assume $H_1 = H$. A set $T = \{t_i\}_{i\in E}$ of representativities of the left (right) cosets (by one from each coset H_i and $t_1 = e \in H$) is called a **left (right) transversal** in G to H.

On any left transversal T in a group G to its subgroup H it is possible to define the following operation (*transversal operation*) :

$$x \overset{(T)}{\cdot} y = z \quad \overset{def}{\Longleftrightarrow} \quad t_x t_y = t_z h, \quad h \in H,$$

Definition 2. If a system $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a loop, then such left transversal $T = \{t_x\}_{x\in E}$ is called a **loop transversal**.

Further we are going to use the following permutation representation $\widehat{G}$ of a group G by the left cosets of its subgroup H (see [2,3]):

$$\widehat{g}(x) = y \quad \stackrel{def}{\Longleftrightarrow} \quad gt_xH = t_yH.$$

For simplicity we assume that

$$Core_G(H) = \bigcap_{g\in G} gHg^{-1} = \{e\},$$

then this representation is exact (see Lemma 6 in [3]), and we have $\widehat{G} \cong G$. Notice that $\widehat{H} = St_1(\widehat{G})$.

Lemma 1 (see [3], Lemma 4). *Let $T = \{t_x\}_{x\in E}$ be a left transversal in G to H. Then the following statements are true:*

1. $\widehat{h}(1) = 1 \ \forall h \epsilon H$;
2. $\forall x, y \in E$:

   $\widehat{t_x}(y) = x \overset{(T)}{\cdot} y = \widehat{L}_x(y), \qquad \widehat{t_1}(x) = \widehat{t_x}(1) = x,$

   $\widehat{t_x^{-1}}(y) = x \overset{(T)}{\diagdown} y = \widehat{L}_x^{-1}(y), \qquad \widehat{t_x^{-1}}(1) = x \overset{(T)}{\diagdown} 1, \qquad \widehat{t_x^{-1}}(x) = 1,$

   where "$\overset{(T)}{\diagdown}$" *is a left division for the operation* $\langle E, \overset{(T)}{\cdot}, 1\rangle$ *(i.e.* $x \overset{(T)}{\diagdown} y = z \Longleftrightarrow x \overset{(T)}{\cdot} z = y$*).*

Lemma 2 (see [3], Lemma 7). *Let $T = \{t_x\}_{x\in E}$ and $P = \{p_x\}_{x\in E}$ be left transversals in G to H. Then there is a set of elements $\{h_{(x)}\}_{x\in E}$ from H such that:*

1. $p_x = t_x h_{(x)} \ \forall x \in E$;
2. $x \overset{(P)}{\cdot} y = x \overset{(T)}{\cdot} \hat{h}_{(x)}(y)$.

This set $\{h_{(x)}\}_{x\in E}$ is called (see [4]) a **derivation set** for the transversal T (and for the transversal operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$).

Definition 3 (see [1]). A triple of permutations $\Phi = (\alpha, \beta, \gamma)$ (α, β, γ are permutations on a set E) is called an **isotopy** of the operation $\langle E, \cdot\rangle$ on the operation $\langle E, \circ\rangle$ if

$$\gamma(x \circ y) = \alpha(x) \cdot \beta(y) \quad \forall x, y \in E.$$

If $\Phi = (\gamma, \gamma, \gamma)$, then such an isotopy is called an **isomorphism**. If $\Phi = (\alpha, \beta, id)$, then such an isotopy is called a **principal isotopy**.

According to Lemma 1.2 from [1] we have

Lemma 3. *If a loop $\langle E, \cdot, e_1\rangle$ is isotopic to a loop $\langle E, \circ, e_2\rangle$, then it is isomorphic to some principal isotope of a loop $\langle E, \circ\rangle$ (and this principal isotopy has the form $T_0 = (R_b^{-1}, L_a^{-1}, id), a \cdot b = e_2$).*

Remark 1. If a loop $\langle E, \cdot, 1\rangle$ is principally isotopic to a loop $\langle E, \circ, 1\rangle$, then this principal isotopy has the form $T_0 = (R_{a\diagdown 1}^{-1}, L_a^{-1}, id)$ for some $a \in E$ ($a^{-1} = a\diagdown 1$ is the right inverse element to a in the loop $\langle E, \cdot, 1\rangle$).

3 Transformations of loop transversals which correspond to an isotopy of their transversal operations

Let $T = \{t_x\}_{x\in E}$ and $P = \{p_x\}_{x\in E}$ be two loop transversals in a group G to its subgroup H, and $\langle E, \overset{(T)}{\cdot}, 1\rangle$, $\langle E, \overset{(P)}{\cdot}, 1\rangle$ be their transversal operations. Fix one of these loop transversals, for example $T = \{t_x\}_{x\in E}$.

As follows from Lemma 3, to investigate loop transversals transformations which correspond to an isotopy of operations $\langle E, \overset{(T)}{\cdot}, 1\rangle$ and $\langle E, \overset{(P)}{\cdot}, 1\rangle$ it is enough to study the case of principal isotopy $T_0 = (R_{a\diagdown 1}^{-1}, L_a^{-1}, id)$ (because the transformations which corresponds to an isomorphism of transversal operations were studied earlier in [5]).

Theorem 1. *Let loops $\langle E, \overset{(T)}{\cdot}, 1\rangle$ and $\langle E, \overset{(P)}{\cdot}, 1\rangle$ be principally isotopic and this principal isotopy has the form $T_0 = (R_b^{-1}, L_a^{-1}, id)$ for some $a \in E$ (note that $a, b \in E$, $a \overset{(T)}{\cdot} b = 1$). Then*

$$\widehat{P} = \widehat{T} \cdot \widehat{t}_a^{-1}.$$

Proof. Let the conditions of Theorem hold. Then

$$x \overset{(P)}{\cdot} y = R_b^{-1}(x) \overset{(T)}{\cdot} L_a^{-1}(y)$$

for some $a, b \in E$, $a \overset{(T)}{\cdot} b = 1$, and L_a, R_b are left and right translations in the loop $\langle E, \overset{(T)}{\cdot}, 1\rangle$. Then the left translation $\mathbf{L}_x$ in the loop $\langle E, \overset{(P)}{\cdot}, 1\rangle$ has the form:

$$\mathbf{L}_x(y) = x \overset{(P)}{\cdot} y = R_b^{-1}(x) \overset{(T)}{\cdot} L_a^{-1}(y) = L_{R_b^{-1}(x)} L_a^{-1}(y), \quad \forall x, y \in E,$$

that is

$$\mathbf{L}_x = L_{R_b^{-1}(x)} L_a^{-1} \quad \forall x \in E. \tag{1}$$

By Lemma 1 (item 2) we have

$$\{\mathbf{L}_x\}_{x\in E} \equiv \{\widehat{p}_x\}_{x\in E} = \widehat{P}$$

and

$$\{L_x\}_{x\in E} \equiv \left\{\widehat{t}_x\right\}_{x\in E} = \widehat{T}.$$

Since R_b^{-1} is a permutation on the set E for every $b \in E$, then it follows from (1): $\widehat{P} = \widehat{T} \cdot \widehat{t}_a^{-1}$ for some $a \in E$. □

Lemma 4. *Let loops* $\langle E, \overset{(T)}{\cdot}, 1\rangle$ *and* $\langle E, \overset{(P)}{\cdot}, 1\rangle$ *be isotopic. Then the following statement holds:*

$$\widehat{P} = \widehat{h}_0 \widehat{T} \widehat{t}_a^{-1} \widehat{h}_0^{-1}$$

for some $h_0 \in \widehat{H}$ *and some* $a \in E$.

Proof. Let loops $\langle E, \overset{(T)}{\cdot}, 1\rangle$ and $\langle E, \overset{(P)}{\cdot}, 1\rangle$ be isotopic. Then according to Lemma 3, their isotopy can be represented in the form of composition of a principal isotopy and an isomorphism:

$$(\alpha, \beta, \gamma) = (R_b^{-1}, L_a^{-1}, id) \circ (\gamma, \gamma, \gamma),$$

where $\gamma(1) = 1, \quad a \overset{(T)}{\cdot} b = 1$. Now our statement is a simple corollary from Theorem 1 and Lemma 7 of [5]. □

Theorem 2. *Let* $T = \{t_x\}_{x \in E}$ *be a fixed loop transversal in* G *to* H, *and* $a \in E$ *be an arbitrary element of the set* E. *Define the following set* $P = \{p_{x'}\}_{x' \in E}$ *of permutations:*

$$\widehat{p}_{x'} \overset{def}{=} \widehat{t}_x \widehat{t}_a^{-1} \quad \forall x \in E.$$

Then

1. $P = \{p_{x'}\}_{x' \in E}$ *is a left transversal in* G *to* H;
2. *A transversal operation* $\langle E, \overset{(P)}{\cdot}, 1\rangle$ *is principally isotopic to the operation* $\langle E, \overset{(T)}{\cdot}, 1\rangle$, *and the principal isotopy* S *has the following form:* $S = (R_{a \diagdown 1}^{-1}, L_a^{-1}, id)$;
3. P *is a loop transversal in* G *to* H.

Proof. 1. We have

$$x' = \widehat{p}_{x'}(1) = \widehat{t}_x \widehat{t}_a^{-1}(1) = \widehat{t}_x(a \diagdown 1) = x \overset{(T)}{\cdot} (a \diagdown 1) = R_{a \diagdown 1}(x).$$

Since $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a loop, then $R_{a \diagdown 1}$ is a permutation on the set E for every $a \in E$. Therefore the element x' runs over all the set E. So there is at least one element of P (element $p_{x'}$) in each left coset $H_{x'}$. It means that P is a left transversal in G to H. Moreover, $\quad e = t_a t_a^{-1} \in P$.

2. Let us study the following set of elements:

$$\widehat{p}_{x'} = \widehat{t}_x \widehat{t}_a^{-1}, \quad x \in E,$$

where a is an arbitrary fixed element of the set E. As we have seen,

$$x' = x \overset{(T)}{\cdot} (a \overset{(T)}{\diagdown} 1). \tag{2}$$

For the transversal operation $\langle E, \overset{(P)}{\cdot}, 1\rangle$ we have (by the definition):

$$p_{x'}p_{y'} = p_{x' \overset{(P)}{\cdot} y'} h^*, \quad h^* \in H.$$

Then by Lemma 1 and the definition of transversal operation we have

$$\begin{aligned} x' \overset{(P)}{\cdot} y' &= \widehat{p}_{x' \overset{(P)}{\cdot} y'} \widehat{h}^*(1) = \widehat{p}_{x'}\widehat{p}_{y'}(1) = \\ &= \widehat{t}_x \widehat{t}_a^{-1} \widehat{t}_y \widehat{t}_a^{-1}(1) = x \overset{(T)}{\cdot} \left[a \overset{(T)}{\diagdown} (y \overset{(T)}{\cdot} (a \overset{(T)}{\diagdown} 1)) \right]. \end{aligned} \tag{3}$$

Using (2) in (3), we obtain

$$\left[x \overset{(T)}{\cdot} (a \overset{(T)}{\diagdown} 1) \right] \overset{(P)}{\cdot} \left[y \overset{(T)}{\cdot} (a \overset{(T)}{\diagdown} 1) \right] = x \overset{(T)}{\cdot} \left[a \overset{(T)}{\diagdown} (y \overset{(T)}{\cdot} (a \overset{(T)}{\diagdown} 1)) \right]. \tag{4}$$

We replace:

$$\begin{cases} x = u \overset{(T)}{/} (a \overset{(T)}{\diagdown} 1) \\ y = v \overset{(T)}{/} (a \overset{(T)}{\diagdown} 1) \end{cases} \Longleftrightarrow \begin{cases} u = x \overset{(T)}{\cdot} (a \overset{(T)}{\diagdown} 1) = R_{a \diagdown 1}(x) \\ v = y \overset{(T)}{\cdot} (a \overset{(T)}{\diagdown} 1) = R_{a \diagdown 1}(y). \end{cases}$$

Since $R_{a \diagdown 1}$ is a permutation for every $a \in E$ in the loop $\langle E, \overset{(T)}{\cdot}, 1\rangle$, then u and v run over all the set E. Then we have from (4):

$$\begin{aligned} u \overset{(P)}{\cdot} v &= (u \overset{(T)}{/} (a \diagdown 1)) \overset{(T)}{\cdot} \left[a \overset{(T)}{\diagdown} ((v \overset{(T)}{/} (a \diagdown 1)) \overset{(T)}{\cdot} (a \diagdown 1)) \right] = \\ &= (u \overset{(T)}{/} (a \diagdown 1)) \overset{(T)}{\cdot} (a \diagdown v) = R^{-1}_{a \diagdown 1}(u) \overset{(T)}{\cdot} L_a^{-1}(v). \end{aligned}$$

From the last equality it follows that the operation $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is principally isotopic to the operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$ and this principal isotopy has the following form: $S = (R^{-1}_{a \diagdown 1}, L_a^{-1}, id)$.

3. According to item 2 the operation $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is a principal isotope of the loop operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$, and this principal isotopy has the form $S = (R^{-1}_{a \diagdown 1}, L_a^{-1}, id)$. It is well known that any isotope of a loop is a quasigroup, so the operation $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is a quasigroup. Moreover, the element 1 is a unit element of this quasigroup, that is the operation $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is a loop. It means that the transversal $P = \{p_x\}_{x \in E}$ is a loop transversal. □

Lemma 5. *Let* $T = \{t_x\}_{x\in E}$ *and* $P = \{p_x\}_{x\in E}$ *be two transversals in* G *to* H *which correspond to principally isotopic transversal operations. Let* $p_x = t_x h_{(x)}$ *and* $\{h_{(x)}\}_{x\in E}$ *be a derivation set. Then*

$$h_{(x)} = t_x^{-1} t_{x \overset{(T)}{/} (a\diagdown 1)} t_a^{-1}$$

for some $a \in E$.

Proof. According to Theorem 2 (item 2) we have for every $x \in E$:

$$\widehat{p}_{x \overset{(T)}{\cdot} (a\diagdown 1)} = \widehat{t}_x \widehat{t}_a^{-1}$$

for some element $a \in E$. Let us replace $u = x \overset{(T)}{\cdot} (a\diagdown 1)$, so $x = u \overset{(T)}{/} (a\diagdown 1)$. Then

$$p_u = t_{u \overset{(T)}{/} (a\diagdown 1)} t_a^{-1}, \quad \forall u \in E$$

On the other hand,

$$p_u = t_u h_{(u)}, \quad \forall u \in E.$$

So

$$t_u h_{(u)} = t_{u/(a\diagdown 1)} t_a^{-1},$$

and our Lemma is proved. □

Lemma 6. *Let* $T = \{t_x\}_{x\in E}$ *be a fixed loop transversal in* G *to* H*, and* $a \in E$ *be some element of the set* E*. Define the following set* $S = \{s_{x'}\}_{x'\in E}$ *of elements:*

$$s_{x'} \overset{def}{=} t_a t_x t_a^{-1} \quad \forall x \in E.$$

Then:

1. $S = \{s_{x'}\}_{x'\in E}$ *is a left transversal in* G *to* H*;*
2. *A transversal operation* $\langle E, \overset{(S)}{\cdot}, 1\rangle$ *is isotopic to the operation* $\langle E, \overset{(T)}{\cdot}, 1\rangle$*, and the isotopy* S *has the following form:* $S = (\beta\alpha, \alpha, \beta^{-1})$*, where* $\alpha = L_a^{-1}$*,* $\beta = R_{a\diagdown 1}^{-1}$*;*
3. S *is a loop transversal in* G *to* H*.*

Proof. 1. We have:

$$x' = \widehat{s}_{x'}(1) = \widehat{t}_a \widehat{t}_x \widehat{t}_a^{-1}(1) = \widehat{t}_a \widehat{t}_x (a\diagdown 1) = a \overset{(T)}{\cdot} (x \overset{(T)}{\cdot} (a\diagdown 1)) = L_a R_{a\diagdown 1}(x).$$

Since $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a loop, then L_a and $R_{a\diagdown 1}$ are permutations on the set E for every $a \in E$. Therefore an element x' runs over all the set E. So every left coset $H_{x'}$

contains an element of S (element $s_{x'}$). So $S = \{s_{x'}\}_{x' \in E}$ is a left transversal in G to H. Moreover, $e = t_a e t_a^{-1} = t_a t_1 t_a^{-1} \in E$.

2. Let us examine the following set of elements

$$s_{x'} = t_a t_x t_a^{-1}, \quad x \in E,$$

where a is an element of the set E. As we have seen,

$$x' = a \overset{(T)}{\cdot} (x \overset{(T)}{\cdot} (a \diagdown 1)). \tag{5}$$

For the transversal operation $\langle E, \overset{(S)}{\cdot}, 1\rangle$ we have

$$s_{x'} s_{y'} = s_{x' \overset{(S)}{\cdot} y'} h^*, \quad h^* \in H.$$

Then

$$\begin{aligned} x' \overset{(S)}{\cdot} y' &= \widehat{s}_{x' \overset{(S)}{\cdot} y'} \widehat{h}^*(1) = \widehat{s}_{x'} \widehat{s}_{y'}(1) = (\widehat{t}_a \widehat{t}_x \widehat{t}_a^{-1})(\widehat{t}_a \widehat{t}_y \widehat{t}_a^{-1})(1) = \\ &= \widehat{t}_a \widehat{t}_x \widehat{t}_y \widehat{t}_a^{-1}(1) = \widehat{t}_a \widehat{t}_x \widehat{t}_y (a \diagdown 1) = a \overset{(T)}{\cdot} (x \overset{(T)}{\cdot} (y \overset{(T)}{\cdot} (a \diagdown 1))). \end{aligned}$$

By (5) from the last equality we obtain:

$$\left[a \overset{(T)}{\cdot} (x \overset{(T)}{\cdot} (a \diagdown 1))\right] \overset{(S)}{\cdot} \left[a \overset{(T)}{\cdot} (y \overset{(T)}{\cdot} (a \diagdown 1))\right] = a \overset{(T)}{\cdot} (x \overset{(T)}{\cdot} (y \overset{(T)}{\cdot} (a \diagdown 1))). \tag{6}$$

We replace:

$$\left\{ \begin{array}{l} a \overset{(T)}{\cdot} (x \overset{(T)}{\cdot} (a \diagdown 1)) = u \\ a \overset{(T)}{\cdot} (y \overset{(T)}{\cdot} (a \diagdown 1)) = v \end{array} \right. \Longleftrightarrow \left\{ \begin{array}{l} x = (a \diagdown u)/(a \diagdown 1) \\ y = (a \diagdown v)/(a \diagdown 1) \end{array} \right. \Longleftrightarrow \left\{ \begin{array}{l} u = L_a R_{a \diagdown 1}(x) \\ v = L_a R_{a \diagdown 1}(y), \end{array} \right.$$

that is the elements u, v run over all the set E. Then from (6) we obtain:

$$\begin{aligned} u \overset{(S)}{\cdot} v &= a \overset{(T)}{\cdot} \left[((a \diagdown u)/(a \diagdown 1)) \overset{(T)}{\cdot} \left[((a \diagdown v)/(a \diagdown 1)) \overset{(T)}{\cdot} (a \diagdown 1)\right]\right] = \\ &= a \overset{(T)}{\cdot} \left[((a \diagdown u)/(a \diagdown 1)) \overset{(T)}{\cdot} (a \diagdown v)\right] = L_a \left[(R_{a \diagdown 1}^{-1} L_a^{-1}(u)) \overset{(T)}{\cdot} (L_a^{-1}(v))\right] \end{aligned}$$

and

$$L_a^{-1}(u \overset{(S)}{\cdot} v) = R_{a \diagdown 1}^{-1} L_a^{-1}(u) \overset{(T)}{\cdot} L_a^{-1}(v). \tag{7}$$

It means that the operations $\langle E, \overset{(S)}{\cdot}, 1\rangle$ and $\langle E, \overset{(T)}{\cdot}, 1\rangle$ are isotopic and the isotopy S has the form $S = (\beta\alpha, \alpha, \alpha)$, where $\alpha = L_a^{-1}$, $\beta = R_{a \diagdown 1}^{-1}$.

3. By item 2 the operation $\langle E, \overset{(S)}{\cdot}, 1\rangle$ is an isotope of the loop operation and this isotopy has the form $S = (\beta\alpha, \alpha, \alpha)$, where $\alpha = L_a^{-1}$, $\beta = R_{a \diagdown 1}^{-1}$. It is well known

that any isotope of a loop is a quasigroup, so the operation $\langle E, \overset{(S)}{\cdot}, 1\rangle$ is a quasigroup. Moreover,

$$s_{1'} = t_a t_1 t_a^{-1} = t_a \cdot e \cdot t_a^{-1} = e = t_1,$$

that is the element 1 is a unit element of this quasigroup. So the operation $\langle E, \overset{(S)}{\cdot}, 1\rangle$ is a loop and $S' = \{s_x\}_{x\in E}$ is a loop transversal. □

Lemma 7. *Let $T = \{t_x\}_{x\in E}$ be a fixed loop transversal in G to H and $a \in E$ be an arbitrary element in E. Define the following set $M = \{m_{x'}\}_{x'\in E}$ of elements:*

$$m_{x'} \overset{def}{=} t_a^{-1} t_x, \quad \forall x \in E.$$

Then:

1. *$M = \{m_{x'}\}_{x'\in E}$ is a left transversal in G to H.*
2. *The transversal operation $\langle E, \overset{(M)}{\cdot}, 1\rangle$ is isotopic to the operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$ and the isotopy Q has the following form: $Q = (L_a, id, L_a)$.*
3. *M is a loop transversal in G to H.*

Proof. 1. We have

$$x' = \widehat{m}_{x'}(1) = \widehat{t}_a^{-1}\widehat{t}_x(1) = a\diagdown x = L_a^{-1}(x). \tag{8}$$

Since $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a loop then L_a^{-1} is a permutation on the set E for every $a \in E$. So the element x' runs over all the set E, and M is a loop transversal in G to H.

2. Let us examine the following set of elements:

$$m_{x'} \overset{def}{=} t_a^{-1} t_x, \quad x \in E$$

where a is some element in E. As we have seen above, $x' = a\diagdown x$. For the transversal operation $\langle E, \overset{(M)}{\cdot}, 1\rangle$ we have

$$m_{x'} m_{y'} = m_{x' \overset{(M)}{\cdot} y'} h^*, \quad h^* \in H.$$

Then

$$\begin{aligned} x' \overset{(M)}{\cdot} y' &= \widehat{m}_{x' \overset{(M)}{\cdot} y'} \widehat{h}^*(1) = \widehat{m}_{x'}\widehat{m}_{y'}(1) = (\widehat{t}_a^{-1}\widehat{t}_x)(\widehat{t}_a^{-1}\widehat{t}_y)(1) = \\ &= \widehat{t}_a^{-1}\hat{t}_x(a\diagdown y) = a\diagdown \left[x \overset{(T)}{\cdot} (a\diagdown y)\right]. \end{aligned}$$

By (8) we obtain:

$$(a\diagdown x) \overset{(M)}{\cdot} (a\diagdown y) = a\diagdown \left[x \overset{(T)}{\cdot} (a\diagdown y)\right]. \tag{9}$$

We use the change of variables:

$$\begin{cases} a\diagdown x = u \\ a\diagdown y = v \end{cases} \Longleftrightarrow \begin{cases} x = a \overset{(T)}{\cdot} u \\ y = a \overset{(T)}{\cdot} v \end{cases} \Longleftrightarrow \begin{cases} u = L_a^{-1}(x) \\ v = L_a^{-1}(y) \end{cases}$$

So elements u, v run over all the set E. Then we have

$$u \overset{(M)}{\cdot} v = a\diagdown \left[(a \overset{(T)}{\cdot} u) \overset{(T)}{\cdot} (a\diagdown(a \overset{(T)}{\cdot} v)) \right] = a\diagdown \left[(a \overset{(T)}{\cdot} u) \overset{(T)}{\cdot} v \right],$$

that is

$$L_a(u \overset{(M)}{\cdot} v) = L_a(u) \overset{(T)}{\cdot} v.$$

It is an isotopy of the type (L_a, id, L_a).

3. Similar to the item 3 of Lemma 5 and Lemma 6. □

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