## Constantin

Sergeevich
Sibirsky
(1928-1990)
This issue is a tribute in honor of his 85th birthday


# New developments based on the mathematical legacy of C.S.Sibirschi 

Dana Schlomiuk


#### Abstract

In this article we survey new developments which occurred during the past ten years on planar polynomial differential equations, developments based on the theory of algebraic


 invariants founded by C.S. Sibirschi for such systems.In 2003 on the occasion of the 75th birthday of C. S. Sibirschi, my article entitled "The mathematical legacy of C.S.Sibirsky, basis for future work" appeared in the Bulletin of the Academy of Sciences of Moldova [29]. Ten years have since passed and it is now time to cast a glance over the work based on Sibisrchi's mathematical legacy which has been done in these years. On the occasion of the 85th anniversary of his birthday this year, there cannot be a better way of honoring his memory than by showing that the field founded by him, the invariant theory of polynomial differential equations, is an active area of research today and that many new results were obtained during these past ten years in this area.

Planar polynomial differential equations are systems of the form:

$$
\begin{equation*}
\frac{d x}{d t}=p(x, y), \quad \frac{d y}{d t}=q(x, y) \tag{S}
\end{equation*}
$$

where $p, q \in \mathbb{R}[x, y]$, i.e. $p, q$ are polynomials in $x, y$ over $\mathbb{R}$. We call degree of a system (1) the integer $n=\max (\operatorname{deg} p, \operatorname{deg} q)$. In particular we call quadratic, respectively cubic, a differential system (1) with $n=2$, respectively $n=3$, and we denote by QS the class of all quadratic systems.

Problems on polynomial differential systems are usually easy to state but extremely difficult to solve. Thus of the three famous classical problems on these differential systems which have been open for over a hundred years, Hilbert's 16th problem (1900,[18]), Poincaré's problem of the center (1885,[23]) and Poincaré's problem of algebraic integrability (1891,[24,25]), only the problem of the center was solved and this only for linear and quadratic differential systems or some very particular cases of higher degrees. These problems in their general context are daunting at this stage and for this reason let us recall the following words from Hilbert's address at the Paris International Congress of Mathematicians in 1900:
"In dealing with mathematical problems, specialization plays, as I believe, a still more important part than generalization. Perhaps in most cases where we seek in vain the answer to a question, the cause of the failure lies in the fact that problems simpler and easier than the one in hand have been not at all or incompletely solved."

Considering the three classical problems mentioned above, stated in the context of general polynomial differential equations, the simplest case is clearly the quadratic one for which the problem of the center was already solved.

At this stage however, the global study of the quadratic class is still a very hard problem. There are several reasons which support this statement. One of them is the elusive nature of limit cycles. Indeed, unlike singularities, limit cycles are usually very hard if not impossible to pin down and the history of their study for the quadratic class includes some notorious errors. Another reason is the large number of parameters involved in the study of this class. Indeed, planar quadratic differential systems depend on 12 parameters, the coefficients of the systems. On QS acts the group of affine transformations and time homotheties and due to this action, the study of QS ultimately depends on five parameters. To obtain the bifurcation diagram for this class thus means that we have to work in this 5 -dimensional topological space which is not $\mathbb{R}^{5}$ but a much more complicated space.

The third reason for the difficulties in this study lies in the necessity to perform complicated calculations. Indeed, consider for example the study of the bifurcation hypersurfaces of singularities of quadratic systems. These bifurcation hypersurfaces are algebraic but they sit in a 12 -dimensional space or in a 5 -dimensional space if we use the group action. Some of these hypersurfaces are of a high degree. Finding the singularities of these hypersurfaces means solving systems of polynomial equations of high degrees. Also we need to know the intersection points of these hypersurfaces and even more, namely how they intersect, in other words their intersection numbers. Of course, studying the singularities is just the beginning. What comes afterwards is not less complicated, namely the study of the analytic (non-algebraic) bifurcation hypersurfaces. This is mainly done by numerical analysis.

In our work on the quadratic class (see for example [2, 20]), the computations were done by using Mathematica, Maple or the program P4 (see [14]). There are
also other computer programs such as Macauley 2, CoCoa and Singular. These high level programming languages are used for Commutative Algebra and Algebraic Geometry but some begin to be used also for Dynamical Systems. One of the ingredients occurring in these specialized programs is the theory of Gröbner bases and the Buchberger's algorithm for computing them. In the future it would be wise to appropriate these programs for problems on polynomial differential systems. However, faced with the challenges mentioned above it seems that the computer programs we have, come still short of expectations.

This however does not mean that we should give up. Indeed, fortunately we can point out some achievements in the direction of computations. An example is the successful computer program P4 (see [14]) allowing us to construct phase portraits and determine the nature of singularities for individual polynomial differential systems.

The first subclass studied globally was the family of all of quadratic systems with a center. The phase portraits for this class were obtained by N. Vulpe (see [39]) followed by the bifurcation diagram of this class (see [22, 28, 41]).

The next subclass studied globally, using global geometric concepts was the class QW3 of quadratic systems with a third order weak focus (see [20]). This family depends on two parameters. Systems in QW3 are important for Hilbert's 16th problem since weak foci of third order produce up to a maximum of three limit cycles, close to the foci, in quadratic perturbations. The work on QW3 in [20] was based on the theorem saying that no limit cycle could surround a weak focus of third order of a quadratic system (see [19]) and on work (see [1]) done with usual techniques which do not involve global geometric concepts. In our study [20], global topological invariants were used for the classification.

A family which is again important for Hilbert's 16th problem is the class QW2 of quadratic differential systems with a weak focus of second order. Indeed, a quadratic system with a second order weak focus could produce up to a maximum of two limit cycles close to the focus in quadratic perturbations of the system. The study of this family was more challenging since this is a three parameter subclass of QS, modulo the group action. In this study both topological and polynomial invariants were used in the classification.

So far no global studies of families of quadratic systems which depend on four or five parameters were done, using global concepts and in particular topological and polynomial invariants.

However a large number of articles on classification problems for quadratic families of systems were done, but not from a global geometric viewpoint. These studies are tied down to fixed normal forms and cannot readily be applied to other presentations of the systems. They employ usual techniques which are not global, and in particular they do not use topological and polynomial invariants. This is a major drawback for several reasons. Firstly because the results cannot be applied in other contexts, for different presentations (normal forms) of the systems. Secondly, in a study several normal forms could occur but no mention is made of how the results obtained in one specific normal form relate to those obtained in another normal
form. There is no global viewpoint tying up the results in a single whole so as to lead us to a global understanding of the phenomena occurring in the specific family.

In recent years progress has however been made and the mathematical tools developed by the school of Sibirschi, the theory of polynomial invariants, played a major role. They are important because they allow us to study a family in its full parameter space independent of the several particular normal forms in which the systems are presented and which are necessary for their study. In particular they allow us to pass easily from one normal form to another and thus glue results obtained with respect to several such normal forms in a single global picture. Examples or works where this approach was taken are: [30,32-37]. The family of Lotka-Volterra differential systems is important since these systems occur in many areas of applied mathematics. The two studies [36] and [37] of this family not only produced the only complete and correct list of phase portraits of this family known in the literature but also characterized each one of the phase portraits in terms of invariant conditions with respect to the affine group and time homotheties. This was possible since the topological study done in [37] was based on the study of all possible configurations of invariants straight lines of this family which was done in [36]. The configuration of invariant lines is a concept introduced by the authors and this notion turned out to be a powerful geometric classification tool for this family. This last study was achieved because of the series of articles [30,32-35] where the classification of all quadratic systems possessing invariant lines of total multiplicity at least four was achieved.

We mentioned above subclasses of $\mathbf{Q S}$ for which we have obtained the topological classification and in some cases also the characterization of phase portraits in terms of invariant conditions.

We now turn to work done on classifying the whole class QS according to a specific feature such as for example according to their singularities. Recently the study of the whole class QS according to the global geometric configurations of singularities at infinity was completed (see [3]). This work was based on [31]. In [7], Artés, Llibre and Vulpe classified QS according to their finite singularities. They did not distinguish among the strong or weak foci, or among weak foci of various orders, or among the strong or weak saddles. Hence this work needs to be augmented so as to include all these distinctions which are important in the production of limit cycles. This is going to be done within the larger frame of classifying QS with respect to the global geometric configurations of both finite and infinite singularities. Work in this direction has already begun. Thus in the two articles (see $[4,5]$ ) the global geometric configurations of both finite and infinite singularities were given for quadratic systems having the total multiplicity $m_{f}$ of finite singularities less than three. Work is now in progress for the remaining two cases, i.e. $m_{f}=3$ and $m_{f}=4$.

We pass now to the second classical problem, namely the problem of the center. Invariant conditions with respect to the general linear group $G L(2, \mathbb{R})$ were given first by Sibirschi [38] for having quadratic systems with a center, when the center is placed at the origin. Then the invariant conditions with respect to the group Aff $(2, \mathbb{R})$ of affine transformations were determined by Boularas, Sibirschi and Vulpe [12] for
having quadratic systems with a center (or two centers) arbitrarily located on the phase plane.

Romanovski and Shafer wrote a book (see [27]) on the problem of the center which takes a computational approach. The work of Sibirschi is cited in their book which contains a chapter on invariants of the rotational group.

Three years ago the complete characterization of all weak singularities (foci, centers and saddles) via invariant theory, for the family of quadratic systems was done by Vulpe [40]. In this paper necessary and sufficient conditions for a real quadratic system to possess a fixed number of weak singularities of a specific order are given. The conditions are stated in terms of affine invariant polynomials in the 12 -dimensional space of the coefficients. These results play an important role in the determination of global geometric configurations of singularities mentioned above.

The third classical problem mentioned at the beginning is Poincaré's problem of algebraic integrability. This problem, stated by Poincaré in [25], asks for giving necessary and sufficient conditions for a planar polynomial system (1) to have a rational first integral. Such a system generates a foliation with singularities on the plane such that all its leaves are algebraic. This is a special case of the theory of invariant algebraic curves of polynomial differential equations developed by Darboux. Poincaré was very enthusiastic about this theory as it can be seen from the following lines of Poincaré which appeared in [24]:
"La question de l'intégrabilité algégrique des équations différentielles du premier ordre et du premier degré n'a pas attiré l'attention des géomètres autant qu'elle méritait. la voie a été ouverte, il y a vingt ans, par un admirable travail de M. Darboux;..."

In recent years the theory of Darboux has flourished and numerous new results were obtained on algebraic curves of differential equations. The theory of polynomial invariants has begun to intervene in some of the publications on this theme. We only mention here some of them.

The problem of characterizing in terms of invariant polynomials the class of quadratic systems which possess a polynomial first integral was completely solved in [8].

The problem of determining necessary and sufficient conditions for quadratic systems to possess a rational first integral of degree two was completely solved in terms of polynomial invariants in [6] where the first integral is a quotient of invariant polynomials.

In [11] the algebraic theory of invariants of differential equations is applied to construct the first integrals for the family of real polynomial differential systems of the form $x^{\prime}=c x+d y+x C_{r}(x, y), y^{\prime}=e x+f y+y C_{r}(x, y)$, where $C_{r}(x, y)$ is a real homogeneous polynomial of degree $r \geq 1$.

Within the mathematical school created by Sibirschi we observe a new direction of studies on applications of Lie algebras to the study of differential systems. For example we have [26] where such applications are developed. Using this theory a set of new results for various families of systems of differential equations where obtained. We mention here the articles $[13,16,17,21]$.

In conclusion we can safely say that during the ten years which have passed since the publication of [29], a wealth of new material appeared in print in which the invariant theory of planar polynomial differential systems founded by C.S. Sibirschi has played a major role. This field of studies is alive and other works are now in progress.

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## References

[1] Artés J. C., Llibre J. Quadratic vector fields with a weak focus of third order, Publ. Mat., 1997, 41, 7-39.
[2] Artés J. C., Llibre J., Schlomiuk D. The geometry of quadratic dufferential systems with a weak focus of second order, International J. of Bifurcation and Chaos, 2006, 16, 3127-3194.
[3] Artés J. C., Llibre J., Schlomiuk D., Vulpe N. From topological to geometric equivalence in the classification of singularities at infinity for quadratic vector fields, Rocky Mountain J. of Mathematics, to appear.
[4] Artés J. C., Llibre J., Schlomiuk D., Vulpe N. Geometric configurations of singularities for quadratic differential systems with total finite multiplicity lower than 2, Bull. Acad. Ştiinţe. Repub. Moldova, Mat., 2013, No. 1(71), 1-34. (??)
[5] Artés J. C., Llibre J., Schlomiuk D., Vulpe N. Configurations of singularities for quadratic differential systems with total finite multiplicity $m_{f}=2$, CRM Preprint no. 3325, Montreal, March 2013, 49 p.
[6] Artés J. C., Llibre J., Vulpe N. Quadratic systems with a rational first integral of degree 2: a complete classification in the coefficient space $\mathbb{R}^{12}$, Rendiconti del Circolo Matematico di Palermo, 2007, Serie II, T. LVI, 417-444.
[7] Artés J. C., Llibre J., Vulpe N. Singular points of quadratic systems: a complete classification in the coefficient space $\mathbb{R}^{12}$, International Journal of Bifurcation Theory and Chaos, 2008, 18, No. 2, 313-362.
[8] Artés J. C., Llibre J., Vulpe N. Quadratic systems with a polynomial first integral: a complete classification in the coefficient space $\mathbb{R}^{12}$, J. Differential Equations, 2009, 246, 3535-3558.
[9] Artés J. C., Llibre J., Vulpe N. Quadratic systems with an integrable saddle: A complete classification in the coefficient space $\mathbb{R}^{12}$, Nonlinear Analysis. Theory, Methods and Applications, 2012, 75, 5416-5447.
[10] Artés J. C., Llibre J., Vulpe N. Complete geometric invariant study of two classes of quadratic systems, Electronic J. Differential Equations, 2012, 2012, No. 09, 1-35.
[11] Baltag V., Calin Iu. The transvectants and the integrals for Darboux systems of differential equations. Bull. Acad. Ştiinţe. Repub. Moldova, Mat., 2008, No. 1(56), 4-18.
[12] Boularas Driss, Vulpe N., Sibirschi C. Solution of a problem of the center "in the large" for a general quadratic differential system, Differential Equations, 1989, 25, no. 11, 1294-1299.
[13] Diaconescu O. V. Multi-dimensional Darboux type differential systems with quadratic nonlinearities. Bull. Acad. Ştiinţe. Repub. Moldova, Mat., 2007, No. 1(53), 95-100.
[14] Dumortier F., Llibre J., Artés J. C. Qualitative Theory of Planar Differential Systems, Springer, 2006, 298 p.
[15] Gherstega N., Popa M. Lie algebras of the operators and three-dimensional polynomial systems, Bull. Acad. Ştiinţe. Repub. Moldova, Mat., 2005, No. 2(48), 51-64.
[16] Gherstega N., Popa M., Pricop V. The generators of the algebras of invariants for differential system with homogeneous nonlinearities of odd degree, Bull. Acad. Ştiinţe. Repub. Moldova, Mat., 2012, No. 2(69), 43-58.
[17] Gherstega N., Orlov V., Vulpe N. A complete classification of quadratic differential systems according to the dimensions of $\operatorname{Aff}(\mathbf{2}, \mathbb{R})$-orbits, Bull. Acad. Ştiinţe. Repub. Moldova, Mat., 2009, No. 2(60), 29-54.
[18] Hilbert D. Mathematical Ptoblems, Lecture delivered before the International Congress of Mathematicians at Paris in 1900, Translated for Bulletin of the AMS, 1902, 8, 437-479.
[19] Li C. Non-existence of limit cycles around a weak focus of order three for any quadratic systems, Chinese Ann. Math., Ser. B, 1986, 7, 174-190.
[20] Llibre J., Schlomiuk D. The geometry of quadratic differential systems with a weak focus of third order, Canadian J. Math., 2004, 56, 310-343.
[21] Orlov V.M. Classification of $G L(2, \mathbb{R})$-orbit's dimensions for the differential system with cubic nonlinearities., Bull. Acad. Ştiinţe. Repub. Moldova, Mat., 2008, No. 3(58), 116-118.
[22] Pal J., Schlomiuk D. Summing up the dynamics of quadratic Hamiltonian systems with a center, Canadian J. Math., 1997, 49, 583-599.
[23] Poincaré H. 'Mémoire sur les courbes définies par les équations différentielles, J. de Math. Pures Appl. (4) 1, 1885, 167-244; Oeuvres de Henri Poincaré, vol.1, Gauthier-Villars, Paris 1951, 95-114.
[24] Poincaré H. Sur l'intégration algégrique des équations différentielles, C.R. Acad. Sci. Paris, 1891, 112, 761-764.
[25] Poincaré H. Sur l'intégration algégrique des équations différentielles du premier ordre et du premier degré, Rend. Circ. Mat. Palermo, 1891, 5, 169-191.
[26] Popa M. Algebraic methods for differential systems, Editura the Flower Power, Universitatea din Piteşti, Seria Matematică Aplicată şi Industrială, 2004, Nr. 15 (in Romanian).
[27] Romanovski V., Shafer D. The Center and Cyclicity problems: A Computational Algebra Approach, Birkhäuser, 2011.
[28] Schlomiuk D. Algebraic particular integrals, integrability and the problem of the center, Trans. Amer. Math. Soc., 1993, 338, 799-841.
[29] Schlomiuk D. The mathematical legacy of C.S. Sibirsky, basis for future work, Bull. Acad. Ştiinţe. Repub. Moldova, Mat., 2003, No. 1(41), 3-6.
[30] Schlomiuk D., Vulpe N. Planar quadratic differential systems with invariant straight lines of at least five total multiplicity, Qualitative Theory of Dynamical Systems, 2004, 5, 135-194.
[31] Schlomiuk D., Vulpe N. Geometry of quadratic differential systems in the neighborhood of infinity, J. Differential Equations, 2005, 215, 357-400.
[32] Schlomiuk D., Vulpe N. Integrals and phase portraits of planar quadratic differential systems with invariant lines of at least five total multiplicity, Rocky Mountain Journal of Mathematics, 2008, 38, No. 6, 1-60.
[33] Schlomiuk D., Vulpe N. Planar quadratic differential systems with invariant straight lines of total multiplicity four, Nonlinear Anal., 2008, 68, No. 4, 681-715.
[34] Schlomiuk D., Vulpe N. Integrals and phase portraits of planar quadratic differential systems with invariant lines of total multiplicity four, Bull. Acad. Ştiinţe. Repub. Moldova, Mat., 2008, No. 1(56), 27-83.
[35] Schlomiuk D., Vulpe N. The full study of planar quadratic differential systems possessing a line of singularities at infinity, Journal of Dynamics and Diff. Equations, 2008, 20, No. 4, 737-775.
[36] Schlomiuk D., Vulpe N. Global classification of the planar Lotka-Volterra differential systems according to their configurations of invariant straight lines, Journal of Fixed Point Theory and Applications, 2010, 8, No. 1, 177-245.
[37] Schlomiuk D., Vulpe N. Global topological classification of the planar Lotka-Volterra differential systems, Electron. J.Differential Equations, 2012, 2012, No. 64, 1-69.
[38] Sibirskii K. S. Algebraic Invariants of Differential Equations and Matrices, Kishinev, Shtiintsa, 1976 (in Russian).
[39] Vulpe N. N. Affine-invariant conditions for the topological discrimination of quadratic systems with a center, Differential Equations, 1983, 19, 273-280.
[40] Vulpe N. Characterization of the finite weak singularities of quadratic systems via invariant theory. Nonlinear Analysis. Theory, Methods and Applications, 2011, 74, No. 4, 6553-6582.
[41] ŻOモĄDEK H. Quadratic systems with center and their perturbations, J. Differential Equations, 1994, 109, 223-273.

Dana Schlomiuk
Département de Mathématiques et de Statistiques
Université de Montréal
E-mail: dasch@dms.umontreal.ca

# Asymptotic Stability of Infinite-Dimensional Nonautonomous Dynamical Systems 

David Cheban


#### Abstract

This paper is dedicated to the study of the problem of asymptotic stability for general non-autonomous dynamical systems (both with continuous and discrete time). We study the relation between different types of attractions and asymptotic stability in the framework of general non-autonomous dynamical systems. Specially we investigate the case of almost periodic systems, i.e., when the base (driving system) is almost periodic. We apply the obtained results we apply to different classes of non-autonomous evolution equations: Ordinary Differential Equations, Functional Differential Equations (both with finite retard and neutral type) and Semi-Linear Parabolic Equations.

Mathematics subject classification: 34D05, 34D20, 34D23, 34D45, 34K20, 34K58, 37B25, 37B55, 37C55, 37C60, 37C75, 39A11, 39C10, 39C55. Keywords and phrases: Global attractor; non-autonomous dynamical system; asymptotic stability,almost periodic motions, semi-linear parabolic equation.


## 1 Introduction

The aim of this paper is the study the problem of asymptotic stability (both local and global) for non-autonomous differential systems. We study this problem in the framework of general non-autonomous dynamical systems (NDS). We formulate and prove our results for general (abstract) non-autonomous dynamical systems. We apply the obtained results to the study the problem of asymptotic stability for ordinary differential equations (ODEs), functional-differential equations (FDEs) and semi-linear parabolic equations (SLPEs).

Let $\mathbb{R}:=(-\infty,+\infty), E$ be a Banach space with the norm $|\cdot|, W$ be an open subset of $E$ containing the origin, $C(\mathbb{R} \times W, E)$ be the space of all continuous functions $f: \mathbb{R} \times W \mapsto E$ equipped with compact open topology.

Consider a differential equation

$$
\begin{equation*}
u^{\prime}=f(t, u), \tag{1}
\end{equation*}
$$

where $f \in C(\mathbb{R} \times W, E)$. Denote by $(C(\mathbb{R} \times W, E), \mathbb{R}, \sigma)$ the shift dynamical system [7,14] on the space $C\left(\mathbb{R} \times W, \mathbb{R}^{n}\right)$ (dynamical system of translations or Bebutov's dynamical system), i.e. $\sigma(\tau, f):=f_{\tau}$ for any $\tau \in \mathbb{R}$ and $f \in C(\mathbb{R} \times W$, $E)$, where $f_{\tau}(t, x):=f(t+\tau, x)$ for any $(t, x) \in \mathbb{R} \times W$.

Below we will use the following conditions:

[^0](A): for any $\left(t_{0}, x_{0}\right) \in \mathbb{R}_{+} \times W$ equation (1) admits a unique solution $x\left(t ; t_{0}, x_{0}\right)$ with initial data $\left(t_{0}, x_{0}\right)$ defined on $\mathbb{R}_{+}:=[0,+\infty)$, i.e. $x\left(t_{0} ; t_{0}, x_{0}\right)=x_{0}$;
(B): the right hand side $f$ is positively compact if the set $\Sigma_{f}^{+}:=\left\{f_{\tau}: \tau \in \mathbb{R}_{+}\right\}$is a relatively compact subset of $C(\mathbb{R} \times W, E)$;
(C): the equation
\[

$$
\begin{equation*}
v^{\prime}=g(t, v), \quad g \in \Omega_{f} \tag{2}
\end{equation*}
$$

\]

is called a limiting equation for (1), where $\Omega_{f}$ is the $\omega$-limit set of $f$ with respect to the shift dynamical system $(C(\mathbb{R} \times W, E), \mathbb{R}, \sigma)$, i.e. $\Omega_{f}:=\{g$ : there exists a sequence $\left\{\tau_{k}\right\} \rightarrow+\infty$ such that $f_{\tau_{k}} \rightarrow g$ as $\left.k \rightarrow \infty\right\}$;
(D): equation (1) (or its right hand side $f$ ) is regular if for all $p \in H^{+}(f)$ the equation

$$
x^{\prime}=p(t, x)
$$

admits a unique solution $\varphi\left(t, x_{0}, p\right)$ defined on $\mathbb{R}_{+}$with initial condition $\varphi\left(0, x_{0}, p\right)=x_{0}$, where $H^{+}(f):=\overline{\left\{f_{\tau}: \tau \in \mathbb{R}_{+}\right\}}$and by bar the closure in the space $C(\mathbb{R} \times W, E)$ is denoted;
(E): equation (1) admits a null (trivial) solution, i.e. $f(t, 0)=0$ for all $t \in \mathbb{R}_{+}$.

The null solution of equation (1) is said to be:

1. uniformly stable if for any positive number $\varepsilon$ there exists a number $\delta=\delta(\varepsilon)$ $(\delta \in(0, \varepsilon))$ such that $|u|<\delta$ implies $\left|\varphi\left(t, u, f_{\tau}\right)\right|<\varepsilon$ for any $t, \tau \in \mathbb{R}_{+}$;
2. uniformly attracting, if there exists a positive number $a$

$$
\lim _{t \rightarrow+\infty}\left|\varphi\left(t, u, f_{\tau}\right)\right|=0
$$

uniformly with respect to $|u| \leq a$ and $\tau \in \mathbb{R}_{+}$;
3. uniformly asymptotically stable if it is uniformly stable and uniformly attracting;
4. globally asymptotically stable if it is asymptotically stable and

$$
\lim _{t \rightarrow+\infty}|\varphi(t, v, g)|=0
$$

for any $(v, g) \in E \times H^{+}(f)$, where $\varphi(t, v, g)$ is a unique solution of equation (2) with initial data $\varphi(0, v, g)=v$.

The main results are contained in the following three theorems. The firs two (Theorems 1 and 2) are related to equation (1) and the third (Theorem 3) to equation
(1) with almost periodic right hand side $f$.

Let $E$ be a Banach space with the norm $|\cdot|$.

Theorem 1. Let $f \in C(\mathbb{R} \times E, E)$. Assume that the following conditions are fulfilled:

1. the function $f$ is regular;
2. the set $H^{+}(f)$ is compact;
3. $f(t, 0)=0$ for all $t \in \mathbb{R}_{+}$;
4. the cocycle $\varphi$ generated by equation (1) is locally compact, i.e. for every point $u \in E$ there exists a neighborhood $U$ of the point $u$ and a positive number $l$ such that the set $\varphi\left(l, U, H^{+}(f)\right)$ is relatively compact.

Then the null solution of equation (1) is globally asymptotically stable if and only if the following conditions hold:
1.

$$
\lim _{t \rightarrow+\infty} \sup _{v \in K, g \in \Omega_{f}}|\varphi(t, v, g)|=0
$$

for every compact subset $K$ of $E$;
2. for every $v \in E$ and $g \in H^{+}(f)$ the solution $\varphi(t, v, g)$ of equation (2) is relatively compact on $\mathbb{R}_{+}$.

Theorem 1 generalizes a statement (Theorem 2.6) established in the work [2] for finite-dimensional equation (1) (see also [13, Ch.I] and the bibliography therein).

Theorem 2. Let $f \in C(\mathbb{R} \times E, E)$. Assume that the following conditions are fulfilled:

1. the function $f$ is regular;
2. the set $H^{+}(f)$ is compact;
3. $f(t, 0)=0$ for all $t \in \mathbb{R}_{+}$;
4. the cocycle $\varphi$ generated by equation (1) is completely continuous, i.e., for every bounded subset $M \subseteq E$ there exists a positive number $l$ such that the set $\varphi\left(l, M, H^{+}(f)\right)$ is relatively compact.

Then the null solution of equation (1) is globally asymptotically stable if and only if the following conditions hold:
a) for every $g \in \Omega_{f}$ limiting equation (2) does not admit nontrivial bounded on $\mathbb{R}$ solutions;
b) for every $v \in E$ and $g \in H^{+}(f)$ the solution $\varphi(t, v, g)$ of equation (2) is bounded on $\mathbb{R}_{+}$.

Recall that a function $f \in C(\mathbb{R} \times W, E)$ is called almost periodic (respectively, almost recurrent) in $t \in \mathbb{R}$ uniformly in $u$ on every compact subset $K$ of $W$ if for an arbitrary number $\varepsilon>0$ and compact subset $K \subseteq W$ there exists a positive number $L=L(K, \varepsilon)$ such that on every segment $[a, a+L](a \in \mathbb{R})$ of the length $L$ there exists at least one number $\tau$ such that

$$
\max _{u \in K,|t| \leq 1 / \varepsilon}|f(t+s+\tau, u)-f(t+s)|<\varepsilon
$$

(respectively,

$$
\left.\max _{u \in K,|t| \leq 1 / \varepsilon}|f(t+\tau, u)-f(t, u)|<\varepsilon\right)
$$

for all $s \in \mathbb{R}$. If the function $f \in C(\mathbb{R} \times W, E)$ is almost recurrent and $H(f):=$ $\left\{f_{\tau}: \tau \in \mathbb{R}\right\}$ is compact, then $f$ is called recurrent (in $t \in \mathbb{R}$ uniformly in $u$ on every compact subset $K$ of $W$ ).

Theorem 3. Suppose that the following conditions are fulfilled:

1. the function $f \in C(\mathbb{R} \times W, E)$ is recurrent in $t \in \mathbb{R}$ uniformly in $u$ on every compact subset of $W$;
2. $f(t, 0)=0$ for all $t \in \mathbb{R}_{+}$;
3. the function $f$ is regular;
4. the cocycle $\varphi$ associated by equation (1) is asymptotically compact;
5. the null solution of equation (1) is uniformly stable;
6. there exists a positive number a such that

$$
\lim _{t \rightarrow+\infty}|\varphi(t, u, f)|=0
$$

for any $|u| \leq a$.
Then the null solution of equation (1) is asymptotically stable.
Remark 1. For finite-dimensional equation (1) with almost periodic hand right side $f$ Theorem 3 was established by Z. Artstein [3] (see also [1,12] and [13, Ch.I]).

We establish also analogical results for the functional-differential equations and for semi-linear parabolic equations.

The paper is organized as follows.
In Section 2 we collect some notions (global attractor, stability, asymptotic stability, uniform asymptotic stability, minimal set, recurrence, shift dynamical systems, cocycles, non-autonomous dynamical systems, etc) and facts from the theory of dynamical systems which will be needed in this paper.

Section 3 is devoted to the analysis of different types of stabilities for nonautonomous dynamical systems (NDSs). We prove that from the uniform attractivity the uniform asymptotic stability follows. It is proved that for an asymptotically
compact dynamical system the asymptotic stability and the uniform asymptotic stability are equivalent. We formulate and prove some tests of asymptotic stability (global asymptotic stability) for infinite-dimensional NDSs (Theorem 6, Theorem 7 and Theorem 8).

In Section 4 we present some results about NDSs with minimal base (driving system). The main result of this Section (Theorem 11) gives a sufficient condition of global asymptotic stability for this type of systems.

Finally, Section 5 contains a series of applications of our general results from Sections 3-4 for Ordinary Differential Equations (Theorem 12, Theorem 13 and Theorem 14), Functional-Differential Equations (both Functional-Differential Equations with finite delay (Theorem 17, Theorem 18 and Theorem 19) and Neutral Functional-Differential Equations (Theorem 20)) and Semi-Linear Parabolic Equations (Theorem 21, Theorem 22 and Theorem 23).

## 2 Some Notions and Facts from Dynamical Systems

### 2.1 Stable and asymptotically stable sets. Global attractors and Levinson center

Let $(X, \rho)$ be a complete metric space with the metric $\rho, \mathbb{R}(\mathbb{Z})$ be the group of real (integer) numbers, $\mathbb{R}_{+}\left(\mathbb{Z}_{+}\right)$be the semi-group of nonnegative real (integer) numbers, $\mathbb{S}$ be one of the two sets $\mathbb{R}$ or $\mathbb{Z}$ and $\mathbb{T} \subseteq \mathbb{S}$ be one of the sub-semigroups $\mathbb{R}_{+}$(respectively, $\mathbb{Z}_{+}$) or $\mathbb{R}($ respectively, $\mathbb{Z})$.

A triplet $(X, \mathbb{T}, \pi)$, where $\pi: \mathbb{T} \times X \rightarrow X$ is a continuous mapping satisfying the following conditions: $\pi(0, x)=x$ and $\pi(s, \pi(t, x))=\pi(s+t, x)$ is called a dynamical system. If $\mathbb{T}=\mathbb{R}\left(\mathbb{R}_{+}\right)$or $\mathbb{Z}\left(\mathbb{Z}_{+}\right)$, then $(X, \mathbb{T}, \pi)$ is called a group (semi-group) dynamical system. In the case when $\mathbb{T}=\mathbb{R}_{+}$or $\mathbb{R}$ the dynamical system $(X, \mathbb{T}, \pi)$ is called a flow, but if $\mathbb{T} \subseteq \mathbb{Z}$, then $(X, \mathbb{T}, \pi)$ is called a cascade (discrete flow).

The function $\pi(\cdot, x): \mathbb{T} \rightarrow X$ is called a motion passing through the point $x$ at moment $t=0$ and the set $\Sigma_{x}:=\pi(\mathbb{T}, x)$ is called a trajectory of this motion.

A nonempty set $M \subseteq X$ is called positively invariant (respectively, negatively invariant, invariant) with respect to dynamical system $(X, \mathbb{T}, \pi)$ or, simply, positively invariant (respectively, negatively invariant, invariant) if $\pi(t, M) \subseteq M$ $(M \subseteq \pi(t, M), \pi(t, M)=M)$ for every $t \in \mathbb{T}_{+}:=\{t \in \mathbb{T}: t \geq 0\}$.

A closed positively invariant set (respectively, invariant set) which does not contain own closed positively invariant (respectively, invariant) subset is called minimal.

Let $M \subseteq X$. The set

$$
\Omega(M):=\bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} \pi(\tau, M)}
$$

is called $\omega$-limit for $M$. If the set $M$ consists of single point $x$, i.e. $M=\{x\}$, then $\Omega(\{x\}):=\omega_{x}$ is called the $\omega$-limits set of the point $x$.

The set $W^{s}(\Lambda)$, defined by the equality

$$
W^{s}(\Lambda):=\left\{x \in X \mid \lim _{t \rightarrow+\infty} \rho(\pi(t, x), \Lambda)=0\right\}
$$

is called a stable manifold (or domain of attraction) of the set $\Lambda \subseteq X$.
The set $M$ is called:

- orbitally stable, if for every $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that $\rho(x, M)<$ $\delta$ implies $\rho(\pi(t, x), M)<\varepsilon$ for all $t \geq 0$;
- attracting if there exists $\gamma>0$ such that $B(M, \gamma) \subset W^{s}(M)$, where $B(M, \gamma):=$ $\{x \in X: \rho(x, M)<\gamma\} ;$
- asymptotically stable if it is orbital stable and attracting;
- global asymptotic stable, if it is asymptotically stable and $W^{s}(M)=X$;
- uniformly attracting if there exists $\gamma>0$ such that

$$
\lim _{t \rightarrow+\infty} \sup _{x \in B(M, \gamma)} \rho(\pi(t, x), M)=0
$$

The system $(X, \mathbb{T}, \pi)$ is called:

- point dissipative if there exists a nonempty compact subset $K \subseteq X$ such that for every $x \in X$

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} \rho(\pi(t, x), K)=0 \tag{3}
\end{equation*}
$$

- compactly dissipative if equality (3) takes place uniformly in $x$ on the compact subsets from $X$;
- locally dissipative if for any point $p \in X$ there exists $\delta_{p}>0$ such that equality (3) takes place uniformly in $x \in B\left(p, \delta_{p}\right)$;
- bounded dissipative if equality (3) holds uniformly in $x$ on every bounded subset of $X$;
- locally completely continuous (compact) if for any point $p \in X$ there are two positive numbers $\delta_{p}$ and $l_{p}$ such that the set $\pi\left(l_{p}, B\left(p, \delta_{p}\right)\right)$ is relatively compact.

Let $(X, \mathbb{T}, \pi)$ be compactly dissipative and $K$ be a compact set attracting every compact subset of $X$. Let us set

$$
\begin{equation*}
J=\Omega(K) \tag{4}
\end{equation*}
$$

It can be shown [7, Ch.I] that the set $J$ defined by equality (4) does not depend on the choice of the attractor $K$, but it is characterized only by the properties of the dynamical system $(X, \mathbb{T}, \pi)$ itself. The set $J$ is called the Levinson center of the compactly dissipative dynamical system $(X, \mathbb{T}, \pi)$.
Lemma 1 (see [8]). Let $(X, \mathbb{T}, \pi)$ be a dynamical system and $x \in X$ be a point with relatively compact semi-trajectory $\Sigma_{x}^{+}:=\{\pi(t, x): t \geq 0\}$. Then the following statements hold:

1. the dynamical system $(X, \mathbb{T}, \pi)$ induces on $H^{+}(x):=\overline{\Sigma_{x}^{+}}$a dynamical system $\left(H^{+}(x), \mathbb{T}_{+}, \pi\right)$;
2. the dynamical system $\left(H^{+}(x), \mathbb{T}_{+}, \pi\right)$ is compactly dissipative;
3. the Levinson center $J_{H^{+}(x)}$ of $\left(H^{+}(x), \mathbb{T}_{+}, \pi\right)$ coincides with the $\omega$-limit set $\omega_{x}$ of the point $x$.

### 2.2 Almost periodic and recurrent points (motions)

Given $\varepsilon>0$, a number $\tau \in \mathbb{T}$ is called an $\varepsilon$-shift (respectively, an $\varepsilon$-almost period) of $x$ if $\rho(\pi(\tau, x), x)<\varepsilon$ (respectively, $\rho(\pi(\tau+t, x), \pi(t, x))<\varepsilon$ for all $t \in \mathbb{T})$.

A point $x \in X$ is called almost recurrent (respectively, Bohr almost periodic) if for any $\varepsilon>0$ there exists a positive number $l$ such that in any segment of length $l$ there is an $\varepsilon$-shift (respectively, an $\varepsilon$-almost period) of the point $x \in X$.

If the point $x \in X$ is almost recurrent and the set $H(x):=\overline{\{\pi(t, x) \mid t \in \mathbb{T}\}}$ is compact, then $x$ is called recurrent.

Remark 2. Suppose that the phase space $X$ of dynamical system $(X, \mathbb{T}, \pi)$ is not a metric-space, but it is a pseudo metric space with the family of pseudo-metrics $\mathcal{P}$. For $\varepsilon>0$ and $\rho \in \mathcal{P}$ a number $\tau$ is called $(\varepsilon, \rho)$-shift (respectively, $(\varepsilon, \rho)$-almost period) of $x \in X$, if $\rho(\pi(\tau, x), x)<\varepsilon$ (respectively, $\rho(\pi(t+\tau, x), \pi(t, x))<\varepsilon$ for all $t \in \mathbb{T}$ ). Now it is easy to modify the notion of almost recurrence (respectively, almost periodicity, recurrence) for a pseudo-metric space.

### 2.3 Bebutov's dynamical system

Let $X, W$ be two metric spaces. Denote by $C(\mathbb{T} \times W, X)$ the space of all continuous mappings $f: \mathbb{T} \times W \mapsto X$ equipped with the compact-open topology and by $\sigma$ the mapping from $\mathbb{T} \times C(\mathbb{T} \times W, X)$ into $C(\mathbb{T} \times W, X)$ defined by the equality $\sigma(\tau, f):=f_{\tau}$ for all $\tau \in \mathbb{T}$ and $f \in C(\mathbb{T} \times W, X)$, where $f_{\tau}$ is the $\tau$-translation (shift) of $f$ with respect to variable $t$, i.e. $f_{\tau}(t, x)=f(t+\tau, x)$ for all $(t, x) \in \mathbb{T} \times W$. Then $[7$, Ch.I], [15, Ch.I] the triplet $(C(\mathbb{T} \times W, X), \mathbb{T}, \sigma)$ is a dynamical system on $C(\mathbb{T} \times W, X)$ which is called a shift dynamical system (dynamical system of translations or Bebutov's dynamical system).

A function $f \in C(\mathbb{T} \times W, X)$ is said to be almost periodic (respectively, recurrent in $t \in \mathbb{T}$ uniformly in $x \in W$ on every compact subset of $W)$ if $f \in C(\mathbb{T} \times W, X)$ is an almost periodic (respectively, recurrent) point of the Bebutov's dynamical system $(C(\mathbb{T} \times W, X), \mathbb{T}, \sigma)$.

### 2.4 Cocycles

Let $\mathbb{T}_{1} \subseteq \mathbb{T}_{2} \subseteq \mathbb{S}$ be two sub-semigroups of $\mathbb{S}$ and $\left(Y, \mathbb{T}_{2}, \sigma\right)$ be a dynamical system on the metric space $Y$. Recall that a triplet $\left\langle W, \varphi,\left(Y, T_{2}, \sigma\right)\right\rangle$ (or shortly $\varphi$ ), where $W$ is a metric space and $\varphi$ is a mapping from $\mathbb{T}_{1} \times W \times Y$ into $W$, is said to be a cocycle over $\left(Y, \mathbb{T}_{2}, \sigma\right)$ with the fiber $W$ if the following conditions are fulfilled:

1. $\varphi(0, u, y)=u$ for all $u \in W$ and $y \in Y$;
2. $\varphi(t+\tau, u, y)=\varphi(t, \varphi(\tau, u, y), \sigma(\tau, y))$ for all $t, \tau \in \mathbb{T}_{1}, u \in W$ and $y \in Y$;
3. the mapping $\varphi: \mathbb{T}_{1} \times W \times Y \mapsto W$ is continuous.

Example 1. Consider differential equation (1) with regular right hand side $f \in$ $C\left(\mathbb{R} \times W, \mathbb{R}^{n}\right)$, where $W \subseteq \mathbb{R}^{n}$. Denote by $\left(H^{+}(f), \mathbb{R}_{+}, \sigma\right)$ a semi-group shift dynamical system on $H^{+}(f)$ induced by Bebutov's dynamical system $\left(C\left(\mathbb{R} \times W, \mathbb{R}^{n}\right), \mathbb{R}, \sigma\right)$, where $H^{+}(f):=\overline{\left\{f_{\tau}: \tau \in \mathbb{R}_{+}\right\}}$. Let $\varphi(t, u, g)$ be a unique solution of the equation

$$
y^{\prime}=g(t, y), \quad\left(g \in H^{+}(f)\right),
$$

then from the general properties of the solutions of non-autonomous equations it follows that the following statements hold:

1. $\varphi(0, u, g)=u$ for all $u \in W$ and $g \in H^{+}(f)$;
2. $\varphi(t+\tau, u, g)=\varphi\left(t, \varphi(\tau, u, g), g_{\tau}\right)$ for all $t, \tau \in \mathbb{R}_{+}, u \in W$ and $g \in H^{+}(f)$;
3. the mapping $\varphi: \mathbb{R}_{+} \times W \times H^{+}(f) \mapsto W$ is continuous.

From above it follows that the triplet $\left\langle W, \varphi,\left(H^{+}(f), \mathbb{R}_{+}, \sigma\right)\right\rangle$ is a cocycle over $\left(H^{+}(f), \mathbb{R}_{+}, \sigma\right)$ with the fiber $W \subseteq \mathbb{R}^{n}$. Thus, every non-autonomous equation (1) with regular $f$ naturally generates a cocycle which plays a very important role in the qualitative study of equation (1).

Suppose that $W \subseteq E$, where $E$ is a Banach space with the norm $|\cdot|, 0 \in W$ ( 0 is the null element of $E$ ) and the cocycle $\left\langle W, \varphi,\left(Y, T_{2}, \sigma\right)\right\rangle$ admits a trivial (null) motion/solution, i.e., $\varphi(t, 0, y)=0$ for all $t \in \mathbb{T}_{1}$ and $y \in Y$.

The trivial motion/solution of cocycle $\varphi$ is said to be:

1. uniformly stable, if for any positive number $\varepsilon$ there exists a number $\delta=\delta(\varepsilon)$ $(\delta \in(0, \varepsilon))$ such that $|u|<\delta$ implies $|\varphi(t, u, y)|<\varepsilon$ for all $t \geq 0$ and $y \in Y$;
2. uniformly attracting if there exists a positive number $a$ such that

$$
\lim _{t \rightarrow+\infty}|\varphi(t, u, y)|=0
$$

uniformly with respect to $|u| \leq a$ and $y \in Y$;
3. uniformly asymptotically stable if it is uniformly stable and uniformly attracting.

### 2.5 Nonautonomous Dynamical Systems (NDS)

Recall [7] that a triplet $\left\langle\left(X, \mathbb{T}_{1}, \pi\right),\left(Y, \mathbb{T}_{2}, \sigma\right), h\right\rangle$ is said to be a nonautonomous dynamical system (NDS), where ( $X, \mathbb{T}_{1}, \pi$ ) (respectively, $\left(Y, \mathbb{T}_{2}, \sigma\right)$ ) is a dynamical system on $X$ (respectively, $Y$ ) and $h$ is a homomorphism from $\left(X, \mathbb{T}_{1}, \pi\right)$ onto $\left(Y, \mathbb{T}_{2}, \sigma\right)$.

Below we will give some examples of nonautonomous dynamical systems which play a very important role in the study of nonautonomous differential equations.

Example 2. (NDS generated by cocycle.) Note that every cocycle $\langle W, \varphi,(Y$, $\left.\left.\mathbb{T}_{2}, \sigma\right)\right\rangle$ naturally generates a NDS. In fact, let $X:=W \times Y$ and $\left(X, \mathbb{T}_{1}, \pi\right)$ be a skewproduct dynamical system on $X$ (i.e. $\pi(t, x):=(\varphi(t, u, y), \sigma(t, y))$ for all $t \in \mathbb{T}_{1}$ and $x:=(u, y) \in X)$. Then the triplet $\left\langle\left(X, \mathbb{T}_{1}, \pi\right),\left(Y, \mathbb{T}_{2}, \sigma\right), h\right\rangle$, where $h:=p r_{2}: X \mapsto Y$ is the second projection (i.e. $h(u, y)=y$ for all $u \in W$ and $y \in Y$ ), is a NDS.
Remark 3. There are Examples of NDS which are not generated by cocycles (see, for instance, [8]).

Let $(X, h, Y)$ be a vector bundle [11]. Denote by $\theta_{y}$ the null element of the vectorial space $X_{y}:=\{x \in X: h(x)=y\}$ and $\Theta:=\left\{\theta_{y}: y \in Y\right\}$ the null section of $(X, h, Y)$.

A vectorial bundle $(X, h, Y)$ is said to be locally trivial with fiber $F$ if for every point $y \in Y$ there exists a neighborhood $U$ of the point $y$ ( $U$ is an open subset of $Y$ containing $y$ ) such that $h^{-1}(U)$ and $U \times F$ are homeomorphic, i.e. there exists a homeomorphism $\alpha: h^{-1}(U) \mapsto U \times F$ (trivialization).

Lemma 2 (see [8]). Let $(X, h, Y)$ be a vector bundle and $\Theta$ be its null section. Suppose that the following conditions hold:

1. the space $Y$ is compact;
2. the vectorial bundle $(X, h, Y)$ is locally trivial.

Then the trivial section $\Theta$ is compact.
Consider a NDS $\left\langle\left(X, \mathbb{T}_{1}, \pi\right),\left(Y, \mathbb{T}_{2}, \sigma\right), h\right\rangle$ on the vector bundle $(X, h, Y)$. Everywhere in this paper we suppose that the null section $\Theta$ of $(X, h, Y)$ is a positively invariant set, i.e. $\pi(t, \theta) \in \Theta$ for all $\theta \in \Theta$ and $t \geq 0\left(t \in \mathbb{T}_{1}\right)$.

The null (trivial) section $\Theta$ of $\operatorname{NDS}\left\langle\left(X, \mathbb{T}_{1}, \pi\right),\left(Y, \mathbb{T}_{2}, \sigma\right), h\right\rangle$ is said to be:

1. uniformly stable if for every $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon)>0$ such that $|x|<\delta$ implies $|\pi(t, x)|<\varepsilon$ for all $t \geq 0\left(t \in \mathbb{T}_{1}\right)$;
2. attracting if there exists a number $\nu>0$ such that $B(\Theta, \nu) \subseteq W^{s}(\Theta)$, where $B(\Theta, \nu):=\{x \in X:|x|<\nu\} ;$
3. uniformly attracting if there exists a number $\nu>0$ such that

$$
\lim _{t \rightarrow+\infty} \sup \{|\pi(t, x)|:|x| \leq \nu\}=0
$$

4. asymptotically stable (respectively, uniformly asymptotically stable) if $\Theta$ is uniformly stable and attracting (respectively, uniformly attracting);
5. globally asymptotically (respectively, uniformly asymptotically) stable if $\Theta$ is asymptotically (respectively, uniformly asymptotically) stable and $W^{s}(\Theta)=X$.

## 3 Some Tests of Global Asymptotical Stability of NDS

Let $\left(Y, \mathbb{T}_{2}, \sigma\right)$ be a compactly dissipative dynamical system, $J_{Y}$ its Levinson center and $\left\langle\left(X, \mathbb{T}_{1}, \pi\right),\left(Y, \mathbb{T}_{2}, \sigma\right), h\right\rangle$ be a NDS. Denote by $\tilde{X}:=h^{-1}\left(J_{Y}\right)=\{x \in$ $\left.X: h(x)=y \in J_{Y}\right\}$, then evidently the following statements are fulfilled:

1. $\tilde{X}$ is closed;
2. $\pi(t, \tilde{X}) \subseteq \tilde{X}$ for all $t \in \mathbb{T}_{1}$ and, consequently, on the set $\tilde{X}$ a dynamical system $\left.\left(\tilde{X}, \mathbb{T}_{1}, \pi\right)\right)$ is induced by $\left(X, \mathbb{T}_{1}, \pi\right)$;
3. the triplet $\left\langle\left(\tilde{X}, \mathbb{T}_{1}, \pi\right),\left(J_{Y}, \mathbb{T}_{2}, \sigma\right), h\right\rangle$ is a NDS.

A dynamical system $\left(X, \mathbb{T}_{1}, \pi\right)$ is said to be:

1. completely continuous (compact) if for every bounded subset $B \subseteq X$ there exists a number $l=l(B)>0$ such that the set $\pi(l, M)$ is relatively compact, where $\pi(l, M):=\{\pi(l, x): x \in M\} ;$
2. locally completely continuous (locally compact) if for every point $p \in X$ there exit positive numbers $l=l(p)$ and $\delta=\delta(p)$ such that the set $\pi(l, B(p, \delta))$ is relatively compact, where $B(p, \delta):=\{x \in X: \rho(x, p)<\delta\} ;$
3. asymptotically compact if for any positively invariant subset $M \subseteq X$ there exists a compact subset $K \subseteq X$ such that $\lim _{t \rightarrow+\infty} \beta(\pi(t, M), K)=0$, where $\beta(A, B):=\sup _{a \in A} \rho(a, B)$ and $\rho(a, B):=\inf _{b \in B} \rho(a, b)$.

Remark 4. 1. The dynamical system $\left(X, \mathbb{T}_{1}, \pi\right)$ is completely continuous if one of the following conditions is fulfilled:

1. the space $X$ possesses the property of Heine-Borel, i.e. every bounded set $B \subseteq X$ is relatively compact;
2. for some $t_{0} \in T_{1}$ the mapping $\pi^{t_{0}}: X \mapsto X$, defined by the equality $\pi^{t_{0}}(x):=$ $\pi\left(t_{0}, x\right)(\forall x \in X)$ is completely continuous, i.e. for any bounded subset $B$ of $X$ the set $\pi^{t_{0}}(B)$ is relatively compact.
3. Every completely continuous dynamical system $\left(X, \mathbb{T}_{1}, \pi\right)$ is locally completely continuous and asymptotically compact.
4. Let $(X, \mathbb{T}, \pi)$ be a dynamical system associated by cocycle $\langle(W, \varphi,(Y, \mathbb{T}, \sigma)\rangle$ and $Y$ be a compact space. Then $(X, \mathbb{T}, \pi)$ is asymptotically compact if and only
if for every bounded sequence $\left\{u_{n}\right\} \subseteq W,\left\{y_{n}\right\} \subseteq Y$ and $t_{n} \rightarrow+\infty$ the sequence $\left\{\varphi\left(t_{n}, u_{n}, y_{n}\right)\right\}$ is relatively compact if it is bounded. In this case the cocycle $\varphi$ is called asymptotically compact.

Theorem 4 (see [8]). Let $\left\langle\left(X, \mathbb{T}_{1}, \pi\right),\left(Y, \mathbb{T}_{2}, \sigma\right), h\right\rangle$ be a NDS. Suppose that the following conditions are fulfilled:

1. $Y$ is compact;
2. the dynamical system $\left(X, \mathbb{T}_{1}, \pi\right)$ is locally compact;
3. the trivial section $\Theta$ of $(X, h, Y)$ is positively invariant;
4. the trivial section $\tilde{\Theta}$ of $N D S\left\langle\left(\tilde{X}, \mathbb{T}_{1}, \pi\right),\left(J_{Y}, \mathbb{T}_{2}, \sigma\right), h\right\rangle$ is uniformly attracting.

Then the trivial section $\Theta$ of non-autonomous dynamical system $\left\langle\left(X, \mathbb{T}_{1}, \pi\right)\right.$, $\left.\left(Y, \mathbb{T}_{2}, \sigma\right), h\right\rangle$ is uniformly stable.

Remark 5. Theorem 4 remains true:

1. if we replace the condition of uniform attraction of $\Theta$ by the following one: there exists a positive number $\tilde{\alpha}$ such that for any compact subset $K \subseteq B[\tilde{\Theta}, \tilde{\alpha}]$ we have

$$
\lim _{t \rightarrow+\infty} \sup \{|\pi(t, x)|: x \in K\}=0
$$

where $B[M, r]:=\{x \in X: \rho(x, M) \leq r\} ;$
2. if we replace the condition of local compactness for $\left(X, \mathbb{T}_{1}, \pi\right)$ by the following: there are positive numbers $\alpha$ and $l$ such that the set $\pi(l, B(\Theta, \alpha))$ is relatively compact, where $B(M, r):=\{x \in X: \rho(x, M)<r\}$.

Corollary 1 (see [8]). Under the conditions of Theorem 4 the trivial section $\Theta$ of $N D S\left\langle\left(X, \mathbb{T}_{1}, \pi\right),\left(Y, \mathbb{T}_{2}, \sigma\right), h\right\rangle$ is uniformly asymptotically stable.

Theorem 5. Let $\left.\left\langle\left(X, \mathbb{T}_{1}, \pi\right),\left(Y, \mathbb{T}_{2}, \sigma\right), h\right)\right\rangle$ be a NDS and the following conditions hold:

1. the trivial section $\Theta$ of $(X, h, Y)$ is positively invariant;
2. $Y$ is compact.

Then the following statements are equivalent:
a) $\left.\left\langle\left(X, \mathbb{T}_{1}, \pi\right),\left(Y, \mathbb{T}_{2}, \sigma\right), h\right)\right\rangle$ is compactly dissipative and its Levinson center $J_{X}$ is included in $\Theta$;
b) the trivial section $\Theta$ is globally asymptotically stable;
c) the equality

$$
\lim _{t \rightarrow+\infty}|\pi(t, x)|=0
$$

holds for all $x \in X$ uniformly in $x$ on every compact subset $M$ of $X$.

Proof. Suppose that condition a. is fulfilled. We will show that $\Theta$ is globally asymptotically stable. Under condition a. it is sufficient to show that $\Theta$ is stable. If we suppose that it is not true, then there are $\varepsilon_{0}>0,0<\delta_{n} \rightarrow 0,\left|x_{n}\right|<\delta_{n}$ and $t_{n} \rightarrow+\infty$ such that

$$
\begin{equation*}
\left|\pi\left(t_{n}, x_{n}\right)\right| \geq \varepsilon_{0} \tag{5}
\end{equation*}
$$

By Lemma 2 the set $\Theta$ is compact, then the sequence $\left\{x_{n}\right\}$ is relatively compact. Since $\left(X, \mathbb{T}_{1}, \pi\right)$ is compactly dissipative, then the sequence $\left\{\pi\left(t_{n}, x_{n}\right)\right\}$ is relatively compact. Thus, without loss of generality, we can suppose that the sequence $\left\{\pi\left(t_{n}, x_{n}\right)\right\}$ is convergent. Denote by $\bar{x}:=\lim _{n \rightarrow \infty} \pi\left(t_{n}, x_{n}\right)$. Then $\bar{x} \in J_{X} \subseteq \Theta$ and, consequently, $|\bar{x}|=0$. On the other hand, passing to limit in (5) as $n \rightarrow \infty$ we obtain $0=|\bar{x}| \geq \varepsilon_{0}$. The obtained contradiction proves our statement.

Now we will prove that condition b) implies a). Indeed, according to Theorem 3.6 [6] the set $\Theta$ is orbitally stable. By Theorem 1.13 [7, Ch.I] the dynamical system $\left(X, \mathbb{T}_{1}, \pi\right)$ is compactly dissipative and its Levinson center $J_{X}$ is included in $\Theta$.

Suppose that condition c). is fulfilled. We will show that c. implies a). Let $M$ be an arbitrary compact subset of $X$, then by condition c). we have the following equality

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} \sup _{x \in M} \rho(\pi(t, x), \Theta)=0 \tag{6}
\end{equation*}
$$

In fact

$$
\rho(\pi(t, x), \Theta) \leq \rho\left(\pi(t, x), \theta_{h(\pi(t, x))}\right)=|\pi(t, x)| \leq \max _{x \in M}|\pi(t, x)| \rightarrow 0
$$

as $t \rightarrow+\infty$. Since the sets $M$ and $\Theta$ are compact, then by Lemma 1.3 [7, Ch.I] we have:

1. the set $\Sigma_{M}^{+}:=\bigcup\{\pi(t, x): t \geq 0, x \in M\}$ is relatively compact;

2 . the set $\Omega(M)$ is nonempty, compact and invariant;
3.

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} \sup _{x \in M} \rho(\pi(t, x), \Omega(M))=0 . \tag{7}
\end{equation*}
$$

From (6) and (7) we obtain $\Omega(M) \subseteq \Theta$ for any compact subset $M$ of $X$, i.e. the compact subset $\Theta$ attracts every compact subset $M$ of $X$. This means that the dynamical system $\left(X, \mathbb{T}_{1}, \pi\right)$ is compactly dissipative and, evidently, its Levinson center $J_{X}$ is included in $\Theta$, i.e. c) implies a).

Finally we will establish the implication a) $\Rightarrow$ c). Suppose that it is not true, then there are a compact subset $M_{0} \subseteq X$, a sequence $\left\{x_{n}\right\} \subseteq M_{0}, t_{n} \rightarrow+\infty$ and $\varepsilon_{0}>0$ such that

$$
\begin{equation*}
\left|\pi\left(t_{n}, x_{n}\right)\right| \geq \varepsilon_{0} \tag{8}
\end{equation*}
$$

Since $\left(X, \mathbb{T}_{1}, \pi\right)$ is compactly dissipative and $Y$ is compact, then without loss of generality, we can consider that the sequences $\left\{\pi\left(t_{n}, x_{n}\right)\right\}$ and $\left\{\sigma\left(t_{n}, y_{n}\right)\right\}$ are convergent, where $y_{n}:=h\left(x_{n}\right)$. Denote by $\bar{y}=\lim _{n \rightarrow \infty} \sigma\left(t_{n}, y_{n}\right)$ and $\bar{x}=\lim _{n \rightarrow \infty} \pi\left(t_{n}, x_{n}\right)$,
then $\bar{x} \in J_{X}$ and $h(\bar{x})=\bar{y}$. Since $J_{X} \subseteq \Theta$, then $|\bar{x}|=0$. Taking into account the last equality and passing to limit in (8) as $n \rightarrow \infty$ we will have $\varepsilon_{0} \leq 0$. The obtained contradiction proves our statement. Theorem is proved.

A continuous mapping $\gamma: \mathbb{S} \mapsto X$ is called an entire motion (trajectory) of the semi-group dynamical system $(X, \mathbb{T}, \pi)$ passing through the point $x$ if $\gamma(0)=x$ and $\pi(t, \gamma(s))=\gamma(t+s)$ for all $t \in \mathbb{T}$ and $s \in \mathbb{S}$.

Denote by $\mathcal{F}_{x}(\pi)$ the set of all entire trajectories of $(X, \mathbb{T}, \pi)$ passing through the point $x$ and $\mathcal{F}(\pi):=\bigcup_{x \in X} \mathcal{F}_{x}(\pi)$.

Theorem 6. Let $Y$ be a compact metric space and $\left(X, \mathbb{T}_{1}, \pi\right)$ be asymptotically compact. The following statements hold:

1. if the trivial section $\Theta$ of $\left.\left\langle\left(X, \mathbb{T}_{1}, \pi\right),\left(Y, \mathbb{T}_{2}, \sigma\right), h\right)\right\rangle$ is globally asymptotically stable, then:
a) every motion of $\left(X, \mathbb{T}_{1}, \pi\right)$ is bounded on $\mathbb{T}_{1}^{+}$, i.e. $\sup _{t \in \mathbb{T}_{1}^{+}}|\pi(t, x)|<+\infty$ for all $x \in X$, where $\mathbb{T}_{1}^{+}:=\left\{t \in \mathbb{T}_{1}: t \geq 0\right\} ;$
b) the dynamical system $\left(X, \mathbb{T}_{1}, \pi\right)$ does not have nontrivial entire bounded on $\mathbb{S}$ motions;
2. if $\left(X, \mathbb{T}_{1}, \pi\right)$ is locally compact, then under conditions a) and b) the trivial section $\Theta$ of $\left.N D S\left\langle\left(X, \mathbb{T}_{1}, \pi\right),\left(Y, \mathbb{T}_{2}, \sigma\right), h\right)\right\rangle$ is globally asymptotically stable.

Proof. Let $Y$ be compact, $\left(X, \mathbb{T}_{1}, \pi\right)$ be asymptotically compact and the trivial section $\Theta$ of $\left.\left\langle\left(X, \mathbb{T}_{1}, \pi\right),\left(Y, \mathbb{T}_{2}, \sigma\right), h\right)\right\rangle$ be globally asymptotically stable. According to Theorem 5 the dynamical system $\left(X, \mathbb{T}_{1}, \pi\right)$ is compactly dissipative and its Levinson center $J_{X}$ is included in $\Theta$. Hence, every positive semi-trajectory $\Sigma_{x}^{+}:=\{\pi(t, x): \quad t \geq 0\}$ is relatively compact and, in particular, it is bounded. Let now $\gamma \in \mathcal{F}(\pi)$ be an arbitrary entire trajectory of dynamical system $\left(X, \mathbb{T}_{1}, \pi\right)$ bounded on $\mathbb{S}$. Since the dynamical system $\left(X, \mathbb{T}_{1}, \pi\right)$ is asymptotically compact, then $\gamma(\mathbb{S})$ is relatively compact. Taking into account that the Levinson center $J_{X}$ is a maximal compact invariant set of dynamical system $\left(X, \mathbb{T}_{1}, \pi\right)$, then $\gamma(\mathbb{S}) \subseteq J_{X} \subseteq \Theta$. Thus the first statement of the theorem is proved.

Now we will establish the second statement of the theorem. From condition a) and asymptotical compactness of $\left(X, \mathbb{T}_{1}, \pi\right)$ it follows that every semi-trajectory $\Sigma_{x}^{+}$ is relatively compact and, consequently, every $\omega$-limit set $\omega_{x}(x \in X)$ is non-empty, compact and invariant. Note that $\omega_{x} \subseteq \Theta$. In fact, let $x \in X$ and $p \in \omega_{x}$ be an arbitrary point from $\omega_{x}$. Since the set $\omega_{x}$ is compact and invariant, then there exists an entire trajectory $\gamma \in \mathcal{F}_{x}$ such that $\gamma(\mathbb{S}) \subseteq \omega_{x}$. According to condition b. we have $\gamma(0)=p \in \gamma(\mathbb{S}) \subseteq \Theta$. Thus we established the inclusion $\Omega_{X}:=\overline{\bigcup\left\{\omega_{x}: x \in X\right\}} \subseteq \Theta$. This means that the dynamical system $\left(X, \mathbb{T}_{1}, \pi\right)$ is point dissipative. By Theorem 1.10 [7, Ch.I] it is also compactly dissipative. Let $J_{X}$ be its Levinson center and $x \in J_{X}$. Since $J_{X}$ is a compact invariant set of dynamical system $\left(X, \mathbb{T}_{1}, \pi\right)$, then
there exists an entire motion $\gamma \in \mathcal{F}_{x}$ such that $\gamma(\mathbb{S}) \subseteq J_{X}$. According to condition b. we obtain $x \in \gamma(\mathbb{S}) \subseteq \Theta$ and, consequently, $J_{X} \subseteq \Theta$. Now to finish the proof of Theorem it is sufficient to apply Theorem 5.

Remark 6. 1. Under the conditions of Theorem 6 condition a) is equivalent to the following one: $\lim _{t \rightarrow+\infty}|\pi(t, x)|=0$ for all $x \in X$.-
2. It is not difficult to check that Theorem 6 remains true if we replace condition b) by the following one:
$\left.b^{\prime}\right)$ the dynamical system $\left(\tilde{X}, \mathbb{T}_{1}, \pi\right)$ does not have nontrivial entire bounded on $\mathbb{S}$ motions.

The second statement of Remark 6 directly follows from Theorem 6. In fact if $\gamma \in \mathcal{F}(\pi)$ is a bounded on $\mathbb{S}$ motion of $\left(X, \mathbb{T}_{1}, \pi\right)$, then under the conditions of Theorem 6 the set $\gamma(S)$ is relatively compact and, consequently, $\nu:=h \circ \gamma$; (i.e. $\nu(s):=h(\gamma(s)) \forall s \in \mathbb{S})$ is an entire trajectory with relatively compact rank $\nu(\mathbb{S})$. This means that $\nu(\mathbb{S}) \subseteq J_{Y}$ and, consequently, $\gamma(\mathbb{S}) \subseteq \tilde{X}$.

From Theorem 6 and Remark 4 follows the following statement follows immediately.

Corollary 2. Let $\left\langle\left(X, \mathbb{T}_{1}, \pi\right),\left(Y, \mathbb{T}_{2}, \sigma\right), h\right\rangle$ be a NDS and the following conditions hold:

1. $Y$ is compact;
2. the dynamical system $\left(X, \mathbb{T}_{1}, \pi\right)$ is completely continuous.

Then the trivial section $\Theta$ is globally asymptotically stable if and only if conditions a) and b) of Theorem 6 hold.

Remark 7. Corollary 2 was established in [4] in the particular case when $(X, h, Y)$ is finite-dimensional and $Y$ is a compact and invariant set.

Theorem 7. Let $\left\langle\left(X, \mathbb{T}_{1}, \pi\right),\left(Y, \mathbb{T}_{2}, \sigma\right), h\right\rangle$ be a $N D S$ and $Y$ be compact. The trivial section $\Theta$ of $\left\langle\left(X, \mathbb{T}_{1}, \pi\right),\left(Y, \mathbb{T}_{2}, \sigma\right), h\right\rangle$ is globally asymptotically stable if and only if the following conditions hold:

1. the trivial section $\tilde{\Theta}$ of $\left\langle\left(\tilde{X}, \mathbb{T}_{1}, \pi\right),\left(J_{Y}, \mathbb{T}_{2}, \sigma\right), h\right\rangle$ is globally asymptotically stable;
2. for any compact subset $K \subseteq X$ the set $\Sigma_{K}^{+}$is relatively compact.

Proof. Necessity. Suppose that the trivial section $\Theta$ of $\left\langle\left(X, \mathbb{T}_{1}, \pi\right),\left(Y, \mathbb{T}_{2}, \sigma\right), h\right\rangle$ is globally asymptotically stable, then by Theorem 5 the dynamical system $\left(X, \mathbb{T}_{1}, \pi\right)$ is compactly dissipative and its Levinson center $J_{X}$ is contained in $\Theta$. Since the Levinson center $J_{Y}$ of $\left(Y, \mathbb{T}_{2}, \sigma\right)$ is its maximal compact invariant set, then the set $\tilde{\Theta}$ is also invariant and, consequently, $J_{X}=\tilde{\Theta}$. Taking into account that $\Theta \supseteq \tilde{\Theta}=$ $J_{X}$, then it is easy to check that $\tilde{\Theta}$ is a globally asymptotically stable set of NDS
$\left\langle\left(\tilde{X}, \mathbb{T}_{1}, \pi\right),\left(J_{Y}, \mathbb{T}_{2}, \sigma\right), h\right\rangle$. To finish the proof of the first statement it is sufficient to note that since the dynamical system $\left(X, \mathbb{T}_{1}, \pi\right)$ is compactly dissipative, then by Theorem 1.5 [7, Ch.I] for every compact subset $K \subseteq X$ the set $\Sigma_{K}^{+}$is relatively compact.

Sufficiency. Let the trivial section $\tilde{\Theta}$ of $\operatorname{NDS}\left\langle\left(\tilde{X}, \mathbb{T}_{1}, \pi\right),\left(J_{Y}, \mathbb{T}_{2}, \sigma\right), h\right\rangle$ be globally asymptotically stable. By Theorem 5 the dynamical system $\left(\tilde{X}, \mathbb{T}_{1}, \pi\right)$ is compactly dissipative and its Levinson center $J_{\tilde{X}}$ is included in $\tilde{\Theta}$. Reasoning as in the proof of the first statement of Theorem and taking into account the invariance of the set $J_{Y}$ we conclude that $J_{\tilde{X}}=\tilde{\Theta}$. Now we will establish that the dynamical system $\left(X, \mathbb{T}_{1}, \pi\right)$ is also compactly dissipative. To prove this statement, according to Theorem 1.15 [7, Ch.I], it is sufficient to establish that $\left(X, \mathbb{T}_{1}, \pi\right)$ is point dissipative. Let $x$ be an arbitrary point of $X$, since the positive semi-trajectory $\Sigma_{x}^{+}$of $x$ is relatively compact, then its $\omega$-limit set $\omega_{x}$ is a non-empty, compact, invariant set, and

$$
\lim _{t \rightarrow+\infty} \rho\left(\pi(t, x), \omega_{x}\right)=0
$$

Note that $h\left(\omega_{x}\right) \subseteq J_{Y}$, since $J_{Y}$ is a maximal compact invariant set of $\left(Y, \mathbb{T}_{2}, \sigma\right)$, and, consequently, $\omega_{x} \subseteq \tilde{X}$. On the other hand $\tilde{\Theta}$ is a maximal compact invariant set of $\left(\tilde{X}, \mathbb{T}_{2}, \sigma\right)$, hence $\omega_{x} \subseteq \tilde{\Theta}$. Thus $\Omega_{X}:=\overline{\left\{\omega_{x}: x \in X\right\}}$ is a compact set, i.e. the dynamical system $\left(X, \mathbb{T}_{1}, \pi\right)$ is point dissipative and, consequently, it is compactly dissipative, too. Let now $J_{X}$ be its Levinson center, then $h\left(J_{X}\right) \subseteq J_{Y}$ and, consequently, $J_{X} \subseteq \tilde{X}$. On the other hand, $J_{\tilde{X}}=\tilde{\Theta}$ is a maximal compact set of ( $\tilde{X}, \mathbb{T}_{1}, \pi$ ) and, consequently, $J_{X} \subseteq \tilde{\Theta}$. Now we will prove that the set $\Theta$ is uniformly stable. Suppose that it is not true, then there are $\delta_{n} \rightarrow 0\left(\delta_{n}>0\right)$, $\left\{x_{n}\right\} \subseteq X$ and $t_{n} \rightarrow+\infty$ such that

$$
\begin{equation*}
\left|x_{n}\right|<\delta_{n} \text { and }\left|\pi\left(t_{n}, x_{n}\right)\right| \geq \varepsilon_{0} \tag{9}
\end{equation*}
$$

for any $n \in \mathbb{N}$. By Lemma $2 \Theta$ is a compact set and the dynamical system $\left(X, \mathbb{T}_{1}, \pi\right)$ is compactly dissipative, then without loss of generality, we can suppose that the sequences $\left\{x_{n}\right\}$ and $\left\{\pi\left(t_{n}, x_{n}\right)\right\}$ are convergent. Denote by $x_{0}$ (respectively, by $\bar{x}_{0}$ ) the limit of $\left\{x_{n}\right\}$ (respectively, $\left\{\pi\left(t_{n}, x_{n}\right)\right\}$ ). Then by (9) we have $x_{0} \in \Theta$ and $|\bar{x}| \geq \varepsilon_{0}>0$. On the other hand $\bar{x} \in J_{X} \subseteq \tilde{\Theta}$ and, consequently, $|\bar{x}|=0$. The obtained contradiction proves our statement. Let now $x$ be an arbitrary point from $X$, then $\lim _{t \rightarrow+\infty}|\pi(t, x)|=0$. In fact if we suppose the contrary, then there exist $x_{0} \in X, \varepsilon_{0}>0$, and $t_{n} \rightarrow+\infty$ such that

$$
\begin{equation*}
\left|\pi\left(t_{n}, x_{0}\right)\right| \geq \varepsilon_{0} \tag{10}
\end{equation*}
$$

for any $n \in \mathbb{N}$. Since the semi-trajectory $\Sigma_{x_{0}}^{+}$of $x_{0}$ is relatively compact, then we can suppose that the sequence $\left\{\pi\left(t_{n}, x_{0}\right)\right\}$ is convergent. Let $\bar{x}_{0}$ be its limit, then from (10) we have $\left|\bar{x}_{0}\right| \geq \varepsilon_{0}>0$. On the other hand, $\bar{x}_{0} \in \omega_{x_{0}} \subseteq J_{X} \subseteq \tilde{\Theta}$ and, consequently, $\left|\bar{x}_{0}\right|=0$. The obtained contradiction completes the proof of the global asymptotic stability of trivial section $\Theta$. Theorem is proved.

Theorem 8. Let $\left\langle\left(X, \mathbb{T}_{1}, \pi\right),\left(Y, \mathbb{T}_{2}, \sigma\right), h\right\rangle$ be a NDS, $Y$ be compact and $\left(X, \mathbb{T}_{1}, \pi\right)$ be locally compact. The trivial section $\Theta$ of $\left\langle\left(X, \mathbb{T}_{1}, \pi\right),\left(Y, \mathbb{T}_{2}, \sigma\right), h\right\rangle$ is globally asymptotically stable if and only if the following conditions hold:

1. the trivial section $\tilde{\Theta}$ of $\left\langle\left(\tilde{X}, \mathbb{T}_{1}, \pi\right),\left(J_{Y}, \mathbb{T}_{2}, \sigma\right)\right.$, $\left.h\right\rangle$ is globally asymptotically stable;
2. for any $x \in X$ the set $\Sigma_{x}^{+}$is relatively compact.

Proof. The necessity of Theorem follows from Theorem 7. To prove the sufficiency, according to Theorem 7 , it is enough to show that the set $\Sigma_{K}^{+}$is relatively compact for any compact subset $K \subseteq X$. To this end we note (reasoning as in the proof of Theorem 7) that the dynamical system $\left(X, \mathbb{T}_{1}, \pi\right)$ is point dissipative. Since dynamical system $\left(X, \mathbb{T}_{1}, \pi\right)$ is locally compact, then by Theorem $1.10[7, \mathrm{Ch} . \mathrm{I}]$ this system is also compactly dissipative. Due to Theorem 1.15 [7, Ch.I] for any compact subset $K \subseteq X$ the set $\Sigma_{K}^{+}$is relatively compact.
Corollary 3. Let $\left\langle\left(X, \mathbb{T}_{1}, \pi\right),\left(Y, \mathbb{T}_{2}, \sigma\right), h\right\rangle$ be a NDS, $Y$ be compact and $\left(X, \mathbb{T}_{1}, \pi\right)$ be completely continuous. The trivial section $\Theta$ of $\left\langle\left(X, \mathbb{T}_{1}, \pi\right),\left(Y, \mathbb{T}_{2}, \sigma\right), h\right\rangle$ is globally asymptotically stable if and only if the following conditions hold:

1. the trivial section $\tilde{\Theta}$ of $\left\langle\left(\tilde{X}, \mathbb{T}_{1}, \pi\right),\left(J_{Y}, \mathbb{T}_{2}, \sigma\right), h\right\rangle$ is globally asymptotically stable;
2. for any $x \in X$ the set $\Sigma_{x}^{+}$is bounded.

Proof. This statement follows directly from Theorem 8. To this end it is sufficient to note that every completely continuous dynamical system is locally compact and every bounded semi-trajectory $\Sigma_{x}^{+}$is relatively compact if ( $X, \mathbb{T}_{1}, \pi$ ) is completely continuous.

Lemma 3. Suppose that the following conditions hold:

1. $\left\langle\left(X, \mathbb{T}_{1}, \pi\right),\left(Y, \mathbb{T}_{2}, \sigma\right), h\right\rangle$ is a NDS;
2. $Y$ is compact;
3. the trivial section $\Theta$ of $(X, h, Y)$ is positively invariant.

Then the following two statements are equivalent:

1. $\Theta$ is uniformly stable;
2. $\Theta$ is orbitally stable with respect to $\left(X, \mathbb{T}_{1}, \pi\right)$.

Proof. Let $\Theta$ be uniformly stable, then it is orbitally stable with respect to $\left(X, \mathbb{T}_{1}, \pi\right)$. If we suppose that it is not true, then there are $\varepsilon_{0}>0,0<\delta_{n} \rightarrow 0,\left\{x_{n}\right\}$ and $t_{n} \rightarrow+\infty$ such that

$$
\begin{equation*}
\rho\left(x_{n}, \Theta\right)<\delta_{n} \text { and } \rho\left(\pi\left(t_{n}, x_{n}\right), \Theta\right) \geq \varepsilon_{0} . \tag{11}
\end{equation*}
$$

Since $\Theta$ is compact then, without loss of generality, we can suppose that the sequence $\left\{x_{n}\right\}$ is convergent. Denote its limit by $x_{0}$, then $y_{0}=\lim _{n \rightarrow \infty} y_{n}$, where $y_{n}:=h\left(x_{n}\right)$. Denote by $\delta_{0}=\delta\left(\varepsilon_{0} / 2\right)$ a positive number chosen for $\varepsilon_{0} / 2$ from the uniform stability of $\Theta$, i.e. $|x|<\delta_{0}$ implies $|\pi(t, x)|<\varepsilon_{0} / 2$ for all $t \geq 0\left(t \in \mathbb{T}_{1}\right)$. Since $\left|x_{n}\right|=$ $\rho\left(x_{n}, \theta_{y_{n}}\right) \leq \rho\left(x_{n}, \theta_{y_{0}}\right)+\rho\left(\theta_{y_{0}}, \theta_{y_{n}}\right) \rightarrow 0$ as $n \rightarrow \infty$. Thus, there exists a number $n_{0} \in \mathbb{N}$ such that $\left|x_{n}\right|<\delta_{0}$ for all $n \geq n_{0}$ and, consequently, we obtain

$$
\begin{equation*}
\left|\pi\left(t_{n}, x_{n}\right)\right|<\varepsilon_{0} / 2 \tag{12}
\end{equation*}
$$

On the other hand from (11) we receive

$$
\begin{equation*}
\left|\pi\left(t_{n}, x_{n}\right)\right| \geq \rho\left(\pi\left(t_{n}, x_{n}\right), \Theta\right) \geq \varepsilon_{0} \tag{13}
\end{equation*}
$$

The inequalities (12) and (13) are contradictory. The obtained contradiction proves our statement.

Now we will show that from the orbital stability of $\Theta$ it follows that it is uniformly stable. This statement may be proved using the same reasoning as in the proof of Theorem 5.

Let $M \subset X$. Denote by $D^{+}(M):=\bigcap_{\varepsilon>0} \overline{\bigcup\{\pi(t, B(M, \varepsilon)) \mid t \geq 0\}}$.
Theorem 9. Let $\left\langle\left(X, \mathbb{T}_{1}, \pi\right),\left(Y, \mathbb{T}_{2}, \sigma\right), h\right\rangle$ be a non-autonomous dynamical system, $Y$ be a compact metric space, $(X, h, Y)$ be a finite-dimensional vector bundle and $\Theta$ be its null section. If $\Theta$ is uniformly stable, then the following properties are equivalent:

1. for every $\varepsilon>0$ and $x \in X$ there exists a number $\tau=\tau(\varepsilon, x)>0$ such that $|\pi(\tau, x)|<\varepsilon$;
2. for every $\varepsilon>0$ and $x \in X$ there exists a number $l=l(\varepsilon, x)>0$ such that $|\pi(t, x)|<\varepsilon$ for any $t \geq l$;
3. the dynamical system $\left(X, \mathbb{T}_{1}, \pi\right)$ is point dissipative and $\Omega_{X} \subseteq \Theta$;
4. $\omega_{x} \bigcap \Theta \neq \emptyset$ for any $x \in X$;
5. for any $\varepsilon>0$ and $r>0$ there exists $L=L(\varepsilon, r)>0$ such that

$$
\begin{equation*}
|\pi(t, x)|<\varepsilon \text { for any } t \geq L(\varepsilon, r) \text { and }|x| \leq r \tag{14}
\end{equation*}
$$

Proof. It is easy to check that, under the conditions of Theorem, the following implications $2 . \Longleftrightarrow 3 . \Rightarrow 4 . \Longleftrightarrow 1$. hold. Now we will establish the implication 4. $\Rightarrow 3$. To this end we note that by Lemma 3 the set $\Theta$ is orbitally stable and, consequently, $D^{+}(\Theta)=\Theta$. According to Theorem 1.13 [7, Ch.I] the dynamical system $\left(X, \mathbb{T}_{1}, \pi\right)$ is compactly dissipative and its Levinson center $J_{X}$ is included in $D^{+}(\Theta)$. Thus we obtain $J_{X} \subseteq \Theta$. Since $\left(X, \mathbb{T}_{1}, \pi\right)$ is point dissipative and $\Omega_{X} \subseteq J_{X}$ we obtain the necessary statement.

To finish the proof of Theorem it is sufficient, for example, to show that $3 . \Longleftrightarrow 5$. The implication $5 . \Rightarrow 3$. is evident. According to condition 3. the dynamical system $\left(X, \mathbb{T}_{1}, \pi\right)$ is point dissipative and $\Omega_{X} \subseteq \Theta$. By Theorem 1.10 [7, Ch.I] the dynamical system $\left(X, \mathbb{T}_{1}, \pi\right)$ is compactly dissipative and $J_{X}=D^{+}\left(\Omega_{X}\right) \subseteq \Theta$, since the set $\Theta$ is uniformly stable. Since the Levinson center $J_{X}$ attracts every compact subset of $J_{X}$ we have (14). Indeed if we suppose that it is not true, then there are $\varepsilon_{0}>0$, $r_{0}>0,\left\{x_{n}\right\}$ and $t_{n} \rightarrow+\infty$ such that

$$
\begin{equation*}
\left|x_{n}\right| \leq r_{0} \quad \text { and } \quad\left|\pi\left(t_{n}, x_{n}\right)\right| \geq \varepsilon_{0} \tag{15}
\end{equation*}
$$

for any $n \in \mathbb{N}$. Since $Y$ is compact, $(X, h, Y)$ is finite-dimensional and $\left(X, \mathbb{T}_{1}, \pi\right)$ is compact dissipative, then we can suppose that the sequence $\left\{\pi\left(t_{n}, x_{n}\right)\right\}$ is convergent. Denote by $\bar{x}$ its limit, then passing to limit in (15) we obtain $|\bar{x}| \geq \varepsilon_{0}>0$. On the other hand $\bar{x} \in J_{X} \subseteq \Theta$ and, consequently, $|\bar{x}|=0$. The obtained contradiction completes the proof of Theorem.

Remark 8. 1. Note that Theorem 9 remains true also for the infinite-dimensional case too (i.e. $(X, h, Y)$ is infinite-dimensional) if we suppose that the dynamical system $\left(X, \mathbb{T}_{1}, \pi\right)$ is completely continuous.
2. Theorem 9 remains true if we replace the uniform stability of the set $\Theta$ by the uniform stability of $\tilde{\Theta}=h^{-1}\left(J_{Y}\right) \cap \Theta$.

## 4 Asymptotic Stability of NDS with Minimal Base

In this section we suppose that the complete metric space $Y$ is compact and the dynamical system $\left(Y, \mathbb{T}_{2}, \sigma\right)$ is minimal, i.e. every trajectory $\Sigma_{y}:=\left\{\sigma(t, y): t \in \mathbb{T}_{2}\right\}$ is dense in $Y$ (this means that $H(y)=Y$ for any $y \in Y$, where $H(y):=\bar{\Sigma}_{y}$ ).
Theorem 10. Suppose that the following conditions are fulfilled:

1. the trivial section $\Theta$ is uniformly stable with respect to $N D S\left\langle\left(X, \mathbb{T}_{1}, \pi\right)\right.$, $\left.\left(Y, \mathbb{T}_{2}, \sigma\right), h\right\rangle$;
2. $L^{+}(X)=X$, where $L^{+}(X):=\left\{x \in X: \Sigma_{x}^{+}\right.$is relatively compact $\}$;
3. there exists a point $y_{0} \in Y$ such that $X_{y_{0}}^{s}=X_{y_{0}}$, where $X_{y}:=\{x \in X: h(x)=$ $y\}$ and $X_{y}^{s}:=\left\{x \in X_{y}: \lim _{t \rightarrow+\infty}|\pi(t, x)|=0\right\}$.
Then $X_{y}^{s}=X_{y}$ for any $y \in Y$.
Proof. Suppose that there exists $\tilde{y} \in Y$ such that $X_{\tilde{y}}^{s} \neq X_{\tilde{y}}$ and let $\tilde{x} \in X_{\tilde{y}} \backslash X_{\tilde{y}}^{s}$. Since $\Sigma_{\tilde{x}}^{+}$is relatively compact, then the $\omega$-limit set $\omega_{\tilde{x}}$ of the point $\tilde{x}$ is a nonempty compact and invariant set. According to the choice of the point $\tilde{x}$ there exists at least one point $\bar{x} \in \omega_{\tilde{x}}$ such that $|\bar{x}| \neq 0$. Let $\gamma \in \mathcal{F}_{\bar{x}}(\pi)$ be an entire trajectory of $\left(X, \mathbb{T}_{1}, \pi\right)$ passing through the point $\bar{x}$ at initial moment with the condition $\gamma(\mathbb{S}) \subseteq$ $\omega_{\tilde{x}}$. We will show that

$$
\begin{equation*}
\alpha:=\inf _{s \leq 0}|\gamma(s)|>0 . \tag{16}
\end{equation*}
$$

If we suppose that (16) is not true, then there exists a sequence $s_{n} \rightarrow-\infty$ such that $\left|\gamma\left(s_{n}\right)\right| \rightarrow 0$ as $n \rightarrow \infty$. Since $\Theta$ is uniformly stable then for all $0<\varepsilon<|\bar{x}| / 2$ there exists a positive number $\delta=\delta(\varepsilon)$ such that $|x|<\delta$ implies the inequality $|\pi(t, x)|<\varepsilon$ for all $t \geq 0$. Let $n_{0} \in \mathbb{N}$ be a sufficiently large number (such that $\left|\gamma\left(s_{n}\right)\right|<\delta$ for all $\left.n \geq n_{0}\right)$, then we have $|\bar{x}|=\left|\pi\left(-s_{n_{0}}, \gamma\left(s_{n_{0}}\right)\right)\right|<\varepsilon<|\bar{x}| / 2$. The obtained contradiction proves our statement. Denote by $\nu$ the entire trajectory of the dynamical system $\left(Y, \mathbb{T}_{2}, \sigma\right)$ defined by the equality $\nu:=h \circ \gamma$, i.e. $\nu(s)=h(\gamma(s))$ for all $s \in \mathbb{S}$, then $\nu \in \mathcal{F}_{\bar{y}}(\sigma)$, where $\bar{y}:=h(\bar{x})$. Since $Y$ is minimal, then there exists a sequence $\left\{\tau_{n}\right\}$ from $\mathbb{S}$ such that $\tau_{n} \rightarrow-\infty$ and $\nu\left(\tau_{n}\right) \rightarrow y_{0}$. Under the conditions of Theorem, without loss of generality, we may suppose that the functional sequences $\left\{\gamma\left(t+\tau_{n}\right)\right\}_{t \in \mathbb{S}}$ and $\left\{\nu\left(t+\tau_{n}\right)\right\}_{t \in \mathbb{S}}$ are convergent (uniformly with respect to $t$ on every compact subset of $\mathbb{S}$ ). Let $\tilde{\gamma}$ (respectively, $\tilde{\nu}$ ) be the limit of the sequence $\left\{\gamma\left(t+\tau_{n}\right)\right\}_{t \in \mathbb{S}}$ (respectively, $\left.\left\{\nu\left(t+\tau_{n}\right)\right\}_{t \in \mathbb{S}}\right)$. Then it is clear that $\tilde{\gamma} \in \mathcal{F}_{\tilde{\gamma}(0)}(\pi), \tilde{\gamma}(\mathbb{S})$ $\subseteq \alpha_{\gamma}:=\left\{z:\right.$ there exists a sequence $s_{n} \rightarrow-\infty$ such that $\left.\gamma\left(s_{n}\right) \rightarrow z\right\}$ and $|\tilde{\gamma}(s)| \geq \alpha$ for all $s \in \mathbb{S}$. On the other hand $\tilde{\gamma}(t)=\pi(t, \tilde{\gamma}(0))$ for any $t \geq 0, \tilde{\gamma}(0) \in X_{y_{0}}$ and, consequently, $\lim _{t \rightarrow+\infty}|\pi(t, \tilde{\gamma}(0))|=0$. The obtained contradiction completes the proof of Theorem.

Lemma 4. Suppose that the trivial section $\Theta$ is uniformly stable with respect to NDS $\left\langle\left(X, \mathbb{T}_{1}, \pi\right),\left(Y, \mathbb{T}_{2}, \sigma\right), h\right\rangle$. Let $y_{0} \in Y$ be an arbitrary point, then the following conditions are equivalent:

1. $X_{y_{0}}^{s}=X_{y_{0}}$;
2. for every $x \in X_{y_{0}}$ the semi-trajectory $\Sigma_{x}^{+}$is relatively compact and $\omega_{x} \subseteq \Theta$;
3. $\omega_{x} \bigcap \Theta \neq \emptyset$ for any $x \in X_{y_{0}}$;
4. for arbitrary $\varepsilon>0$ and $x \in X_{y_{0}}$ there exists a positive number $\tau=\tau(x, \varepsilon)$ such that $|\pi(\tau, x)|<\varepsilon$.

Proof. Note that the implications $1 . \Longrightarrow 2 . \Longrightarrow 3 . \Longrightarrow 4$. are evident. To finish the proof of Lemma it is sufficient to show that 4 . implies 1 .. Indeed, let $\varepsilon>0$ be an arbitrary positive number, $x \in X, \varepsilon_{k}:=1 / k(k \in \mathbb{N})$, and $\tau_{k}$ be a positive number such that $\left|\pi\left(\tau_{k}, x\right)\right|<1 / k$. Denote by $\delta(\varepsilon)$ the positive number from the uniform stability of $\Theta$ for $\varepsilon$ (i.e. $|x|<\delta$ implies $|\pi(t, x)|<\varepsilon$ for any $t \geq 0$ ), then for the sufficiently large $k(1 / k<\delta)$ we have $\left|\pi\left(t+\tau_{k}, x\right)\right|<\varepsilon$ for any $t \geq 0$. Thus for $\varepsilon>0$ there exists $l(\varepsilon, x)>0$ such that $|\pi(t, x)|<\varepsilon$ for any $t \geq l(\varepsilon, x)$, i.e. $x \in X_{y_{0}}^{s}$.

Remark 9. 1. The implications $1 . \Longrightarrow 2 . \Longrightarrow 3 . \Longrightarrow 4$. are true without assumption of uniform stability of $\Theta$.
2. Lemma 4 remains true without compactness and minimality of $Y$.

From Theorem 10 and Lemma 4 we have the following statement.
Corollary 4. Suppose that the following conditions are fulfilled:

1. the trivial section $\Theta$ is uniformly stable with respect to $N D S\left\langle\left(X, \mathbb{T}_{1}, \pi\right)\right.$, $\left.\left(Y, \mathbb{T}_{2}, \sigma\right), h\right\rangle ;$
2. $L^{+}(X)=X$;
3. there exists a point $y_{0} \in Y$ such that one of the conditions 1.-4. of Lemma 4 is fulfilled.

Then $X_{y}^{s}=X_{y}$ for all $y \in Y$.
Below we give a local version of Theorem 10.
Theorem 11. Suppose that the following conditions are fulfilled:

1. the dynamical system $\left(X, \mathbb{T}_{1}, \pi\right)$ is asymptotically compact;
2. the trivial section $\Theta$ is uniformly stable with respect to $N D S\left\langle\left(X, \mathbb{T}_{1}, \pi\right)\right.$, $\left.\left(Y, \mathbb{T}_{2}, \sigma\right), h\right\rangle ;$
3. there exist positive number $\delta_{0}$ and point $y_{0} \in Y$ such that $B\left(\theta_{y_{0}}, \delta_{0}\right) \subset X_{y_{0}}^{s}$, where $B\left(\theta_{y}, r\right):=\left\{x \in X_{y}:|x|<r\right\}$.

Then the trivial section $\Theta$ is asymptotically stable, i.e. there exists a positive number $\beta$ such that $B(\Theta, \beta) \subset X^{s}$, where $B(\Theta, \beta):=\bigcup\left\{B\left(\theta_{y}, \beta\right): y \in Y\right\}$ and $X^{s}:=\bigcup\left\{X_{y}^{s}: y \in Y\right\}$.

Proof. Since $\Theta$ is uniformly stable, then there exists a positive number $\delta_{1}$ such that $|\pi(t, x)| \leq \delta_{0}$ for any $t \geq 0$ and $x \in X$ with $|x| \leq \delta_{1}$. Let now $\beta:=\min \left\{\delta_{0}, \delta_{1}\right\}$. We will show that $B(\Theta, \beta) \subset X^{s}$. If we suppose that it is not so, then using the same reasoning as in the proof of Theorem 10 we obtain a contradiction which proves our statement.

Remark 10. All results of Sections 3-4 remain true if:

1. we replace the positive invariance of the trivial section $\Theta$ by the following condition: there exists a compact positively invariant set $M \subseteq X$ such that $M_{y}:=$ $\{x \in M: h(x)=y\}$ consists of a single point for any $y \in Y$;
2. we the compact metric space $Y$ by an arbitrary compact regular topological space.

## 5 Some Applications

### 5.1 Ordinary differential equations

Consider a differential equation

$$
\begin{equation*}
u^{\prime}=f(t, u) \tag{17}
\end{equation*}
$$

where $f \in C(\mathbb{R} \times W, E)$.

Applying general results from Sections 3-4 we will obtain a series of results for equation (17). Below we formulate some of them.

Denote by $\Omega_{f}:=\left\{g \in H^{+}(f)\right.$ : there exists a sequence $\tau_{n} \rightarrow+\infty$ such that $\left.g=\lim _{n \rightarrow \infty} f_{\tau_{n}}\right\}$ the $\omega$-limit set of $f$.

A trivial solution of equation (17) is called uniformly attracting (respectively, eventually uniformly attracting [2]) if for every compact subset $K \subset E$ and for every $\varepsilon>0$ there exists $L=L(K, \varepsilon)>0$ (respectively, there exist $\gamma=\gamma(K)>0$ and $L=L(K, \varepsilon)>0)$ such that

$$
x_{0} \in K, t \geq t_{0}+L \text { implies }\left|x_{f}\left(t ; t_{0}, x_{0}\right)\right|<\varepsilon
$$

(respectively,

$$
\left.x_{0} \in K, t_{0} \geq \gamma, \quad t \geq t_{0}+L \text { implies }\left|x_{f}\left(t ; t_{0}, x_{0}\right)\right|<\varepsilon\right),
$$

where by $x_{f}\left(t ; t_{0}, x_{0}\right)$ a unique solution $x(t)$ of equation (17) with initial data $x\left(t_{0}\right)=$ $x_{0}$ is denoted.

The solutions of equation (17) are said to be uniformly bounded [2] if for any $\alpha>0$ there exists $\beta=\beta(\alpha)>0$ such that

$$
\left|x_{0}\right| \leq \alpha, \quad t_{0} \in \mathbb{R}_{+}, t \geq t_{0} \Rightarrow\left|x_{f}\left(t ; t_{0}, x_{0}\right)\right| \leq \beta
$$

Lemma 5. Suppose that the following conditions are fulfilled:

1. $f \in C(\mathbb{R} \times E, E)$;
2. the function $f$ is regular;
3. the set $H^{+}(f)$ is compact;
4. $f(t, 0)=0$ for any $t \in \mathbb{R}_{+}$.

Let $\varphi$ be a cocycle, generated by equation (17) (see Example 1), then the following statements hold:

1. if the trivial solution of equation (17) is uniformly attracting, then the trivial solution/motion of the cocycle $\varphi$ is uniformly attracting;
2. if the trivial solution of equation (17) is eventually uniformly attracting, then the trivial solution/motion of the cocycle $\varphi$ possesses the following property:

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} \max _{x \in K, g \in \Omega_{f}}|\varphi(t, x, g)|=0 \tag{18}
\end{equation*}
$$

for any compact subset $K$ of $E$;
3. if the solutions of equation (17) are uniformly bounded, then the solutions/motions of the cocycle $\varphi$ are uniformly bounded, i.e. for any $\alpha>0$ there exists $\beta=\beta(\alpha)>0$ such that $|x| \leq \alpha$ implies $|\varphi(t, x, g)| \leq \beta$ for any $t \in \mathbb{R}_{+}$and $g \in H^{+}(f)$.

Proof. The first statement of Lemma is well known [14, Ch.VIII].
To prove the second statement we note that $\varphi\left(t, x, f_{t_{0}}\right)=x\left(t+t_{0} ; t_{0}, x\right)$ for any $t, t_{0} \in \mathbb{R}_{+}$and $x \in E$. Let now $K$ be an arbitrary compact subset of $E$ and $\varepsilon>0$ be an arbitrary positive number. Denote by $\gamma=\gamma(K)$ and $L=L(K, \varepsilon)$ positive numbers from eventually uniform attractivity of null solution for equation (17). Let now $x \in K$ and $g \in \Omega_{f}$, then there exists a sequence $t_{n} \rightarrow+\infty$ such that $f_{t_{n}} \rightarrow g$ (in the space $C(\mathbb{R} \times E, E)$ ) and, consequently, $t_{n} \geq \gamma$ for sufficiently large $n$. Note that

$$
\begin{gather*}
|\varphi(t, x, g)|=\lim _{n \rightarrow+\infty}\left|\varphi\left(t, x, f_{t_{n}}\right)\right|=\lim _{n \rightarrow+\infty}\left|x_{f}\left(t+t_{n} ; t_{n}, x\right)\right| \leq \varepsilon  \tag{19}\\
\text { for all } t \geq L(K, \varepsilon) .
\end{gather*}
$$

From (19) evidently follows (18).
Finally we will prove the third statement. Let $\alpha>0$ and $\beta=\beta(\alpha)>0$ is taken from the uniform boundedness of the solutions of (17). Let $|x| \leq \alpha, g \in H^{+}(f)$ and $t \in \mathbb{R}_{+}$, then there exists a sequence $\left\{t_{n}\right\} \subseteq \mathbb{R}_{+}$such that $g=\lim _{t \rightarrow+\infty} f_{t_{n}}$. Note that

$$
|\varphi(t, x, g)|=\lim _{n \rightarrow \infty}\left|\varphi\left(t, x, f_{t_{n}}\right)\right|=\lim _{n \rightarrow \infty}\left|x_{f}\left(t+t_{n} ; t_{n}, x\right)\right| \leq \beta(\alpha) .
$$

Lemma is completely proved.
Theorem 12. Let $f \in C(\mathbb{R} \times E, E)$. Assume that the following conditions are fulfilled:

1. the function $f$ is regular;
2. the set $H^{+}(f)$ is compact;
3. $f(t, 0)=0$ for any $t \in \mathbb{R}_{+}$;
4. the cocycle $\varphi$ generated by equation (17) is locally compact, i.e. for every point $u \in E$ there exists a neighborhood $U$ of the point $u$ and a positive number $l$ such that the set $\varphi\left(l, U, H^{+}(f)\right)$ is relatively compact.

Then the null solution of equation (17) is globally asymptotically stable if and only if the following conditions hold:
1.

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} \sup _{v \in K, g \in \Omega_{f}}|\varphi(t, v, g)|=0 \tag{20}
\end{equation*}
$$

for every compact subset $K$ of $E$;
2. for every $v \in E$ and $g \in H^{+}(f)$ the solution $\varphi(t, v, g)$ of equation (2) is relatively compact on $\mathbb{R}_{+}$.

Proof. Consider the dynamical system $\left(H^{+}(f), \mathbb{R}_{+}, \sigma\right)$. Since the space $H^{+}(f)$ is compact, then $\left(H^{+}(f), \mathbb{R}_{+}, \sigma\right)$ is compactly dissipative and its Levinson center (maximal compact invariant set) $J_{H^{+}(f)}$ by Lemma 1 coincides with $\omega$-limit set $\Omega_{f}$ of $f$. Let $Y:=H^{+}(f)$ and $\left(Y, \mathbb{R}_{+}, \sigma\right)$ be the shift dynamical system on $Y$. Denote by $X:=W \times Y$ and $\left(X, \mathbb{R}_{+}, \pi\right)$ the skew-product dynamical system generates by $\left(Y, \mathbb{R}_{+}, \sigma\right)$ and cocycle $\varphi$, i.e. $\pi(t,(v, g)):=(\varphi(t, v, g), \sigma(t, g))$ for all $t \in \mathbb{R}_{+}$and $(v, g) \in X$. Now consider a non-autonomous dynamical system $\left\langle\left(X, \mathbb{R}_{+}, \pi\right),\left(Y, \mathbb{R}_{+}, \pi\right), h\right\rangle\left(h:=p r_{2}\right)$ associated with equation (17). It is easy to check that under the conditions of Theorem 12 this NDS possesses the following properties:

1. by Lemma 1 the dynamical system $\left(Y, \mathbb{R}_{+}, \sigma\right)$ is compactly dissipative and its Levinson center $J_{Y}$ coincides with $\Omega_{f}$;
2. the null section $\Theta$ of $\left\langle\left(X, \mathbb{R}_{+}, \pi\right),\left(Y, \mathbb{R}_{+}, \pi\right), h\right\rangle$ coincides with $\{0\} \times Y$;
3. $\Theta$ is a positively invariant subset of $\left(X, \mathbb{R}_{+}, \pi\right)$;
4. according to (20) the null section $\tilde{\Theta}$ of $\operatorname{NDS}\left\langle\left(\tilde{X}, \mathbb{R}_{+}, \pi\right),\left(J_{Y}, \mathbb{R}_{+}, \sigma\right), h\right\rangle$ is uniformly attracting because $|\pi(t, x)|=|\varphi(t, v, g)|$ for any $t \in \mathbb{R}_{+}$and $x:=(v, g) \in X ;$
5. every trajectory $\Sigma_{(u, g)}^{+}\left((u, g) \in E \times H^{+}(f)\right)$ of the skew-product dynamical system $\left(X, \mathbb{R}_{+}, \pi\right)$, generated by equation (17), is relatively compact.

Now to finish the proof it is sufficient to apply Theorem 5 and Theorem 8.
Corollary 5. Let $f \in C(\mathbb{R} \times E, E)$. Assume that the following conditions are fulfilled:

1. the function $f$ is regular;
2. the set $H^{+}(f)$ is compact;
3. $f(t, 0)=0$ for any $t \in \mathbb{R}_{+}$;
4. the cocycle $\varphi$ generated by equation (17) is locally compact;
5. the null solution of equation (17) is eventually uniformly attracting;
6. for every $v \in E$ and $g \in H^{+}(f)$ the solution $\varphi(t, v, g)$ of equation (2) is relatively compact on $\mathbb{R}_{+}$.

Then the null solution of equation (17) is globally asymptotically stable.
Proof. This statement follows from Theorem 12. Indeed, according to Lemma 5 from the uniform eventual attraction of the null solution of equation (17) follows condition (20). Now to finish the proof of this statement it is sufficient to apply Theorem 12.

Remark 11. 1. For finite-dimensional equation (17) Corollary 5 generalizes a statement (Theorem 2.6) established in the work [2] (see also [13, Ch.I] and the bibliography therein).
2. If the cocycle $\varphi$ associated with equation (17) is asymptotically compact (in particular if it is completely continuous), then Theorem 12 remains true if we replace condition (ii) by the following one: for any $v \in E$ and $g \in H^{+}(f)$ the solution $\varphi(t, v, g)$ is bounded on $\mathbb{R}_{+}$.

Theorem 13. Let $f \in C(\mathbb{R} \times E, E)$. Assume that the following conditions are fulfilled:

1. the function $f$ is regular;
2. the set $H^{+}(f)$ is compact;
3. $f(t, 0)=0$ for any $t \in \mathbb{R}_{+}$;
4. the cocycle $\varphi$ generated by equation (17) is completely continuous, i.e. for every bounded subset $M \subseteq E$ there exists a positive number $l$ such that the set $\varphi\left(l, M, H^{+}(f)\right)$ is relatively compact.
Then the null solution of equation (17) is globally asymptotically stable if and only if the following conditions hold:
a) for every $g \in \Omega_{f}$ limiting equation (2) does not have nontrivial bounded on $\mathbb{R}$ solutions;
b) for every $v \in E$ and $g \in H^{+}(f)$ the solution $\varphi(t, v, g)$ of equation (2) is bounded on $\mathbb{R}_{+}$.

Proof. This statement follows from Corollary 2 and can be proved using the same arguments as in the proof of Theorem 12.

Remark 12. Theorem 13 remains true if we replace the completely continuity by the following two conditions:

1. the cocycle $\varphi$ is asymptotically compact:
2. the cocycle $\varphi$ is locally completely continuous.

Theorem 14. Suppose that the following conditions are fulfilled:

1. the function $f \in C(\mathbb{R} \times W, E)$ is recurrent in $t \in \mathbb{R}$ uniformly in $u$ on every compact subset from $W$;
2. $f(t, 0)=0$ for any $t \in \mathbb{R}_{+}$;
3. the function $f$ is regular;
4. the cocycle $\varphi$ associated with equation (17) is asymptotically compact;
5. the null solution of equation (17) is uniformly stable;
6. there exists a positive number a such that

$$
\lim _{t \rightarrow+\infty}|\varphi(t, u, f)|=0
$$

for any $|u| \leq a$.
Then the null solution of equation (17) is asymptotically stable.
Proof. This statement follows directly from Theorem 11 using the same arguments as in the proof of Theorem 12.

Remark 13. For finite-dimensional equation (17) with almost periodic right hand side $f$ Theorem 14 was established by Z. Artstein [3] (see also [1,12] and [13, Ch.I]).

### 5.2 Difference equations

Consider a difference equation

$$
\begin{equation*}
u(t+1)=f(t, u(t)), \tag{21}
\end{equation*}
$$

where $f \in C(\mathbb{Z} \times W, E)$.
Along with equation (21) we consider the family of equations

$$
\begin{equation*}
v(t+1)=g(t, v(t)), \tag{22}
\end{equation*}
$$

where $g \in H^{+}(f):=\overline{\left\{f_{\tau}: \tau \in \mathbb{Z}_{+}\right\}}$. Let $\varphi(t, v, g)$ be a unique solution of equation (22) with initial data $\varphi(0, v, g)=v$. Denote by $\left(H^{+}(f), \mathbb{Z}_{+}, \sigma\right)$ the shift dynamical system on $H^{+}(f)$, then the triplet $\left\langle W, \varphi,\left(H^{+}(f), \mathbb{Z}_{+}, \sigma\right)\right\rangle$ is a cocycle (with discrete time) over $\left(H^{+}(f), \mathbb{Z}_{+}, \sigma\right)$ with the fibre $W$.

Applying the results from Sections $3-4$ we will obtain a series of results for difference equation (21). Below we formulate two of them.

Theorem 15. Let $f \in C(\mathbb{Z} \times W, E)$. Assume that the following conditions are fulfilled:

1. the set $H^{+}(f)$ is compact;
2. $f(t, 0)=0$ for any $t \in \mathbb{Z}_{+}$;
3. there exists a neighborhood $U$ of 0 and a positive number $l$ such that $\varphi\left(l, U, H^{+}(f)\right)$ is relatively compact;
4. there exists a positive number a such that

$$
\lim _{t \rightarrow+\infty} \sup _{|v| \leq a, g \in \Omega_{f}}|\varphi(t, v, g)|=0 .
$$

Then the null solution of equation (21) is uniformly asymptotically stable.
Theorem 16. Let $f \in C(\mathbb{Z} \times W, E)$. Assume that the following conditions are fulfilled:

1. the function $f \in C\left(\mathbb{Z}_{+} \times W, E\right)$ is recurrent in $t \in \mathbb{Z}_{+}$uniformly in $u$ on every compact subset of $W$;
2. $f(t, 0)=0$ for any $t \in \mathbb{Z}_{+}$;
3. the null solution of equation (21) is uniformly stable;
4. there exists a positive number a such that

$$
\lim _{t \rightarrow+\infty}|\varphi(t, u, f)|=0
$$

for any $|u| \leq a$.
Then the null solution of equation (21) is uniformly asymptotically stable.

### 5.3 Functional-differential equations

We will apply now the abstract theory developed in the previous Sections to the analysis of a class of functional differential equations.

### 5.3.1 Functional-differential equations (FDEs) with finite delay

Let us first recall some notions and notations from [9]. Let $r>0, C\left([a, b], \mathbb{R}^{n}\right)$ be the Banach space of all continuous functions $\varphi:[a, b] \rightarrow \mathbb{R}^{n}$ equipped with the sup-norm. If $[a, b]=[-r, 0]$, then we set $\mathcal{C}:=C\left([-r, 0], \mathbb{R}^{n}\right)$. Let $\sigma \in \mathbb{R}, A \geq 0$ and $u \in C\left([\sigma-r, \sigma+A], \mathbb{R}^{n}\right)$. We will define $u_{t} \in \mathcal{C}$ for any $t \in[\sigma, \sigma+A]$ by the equality $u_{t}(\theta):=u(t+\theta),-r \leq \theta \leq 0$. Consider a functional differential equation

$$
\begin{equation*}
\dot{u}=f\left(t, u_{t}\right), \tag{23}
\end{equation*}
$$

where $f: \mathbb{R} \times \mathcal{C} \rightarrow \mathbb{R}^{n}$ is continuous.
Denote by $C\left(\mathbb{R} \times \mathcal{C}, \mathbb{R}^{n}\right)$ the space of all continuous mappings $f: \mathbb{R} \times \mathcal{C} \mapsto \mathbb{R}^{n}$ equipped with the compact open topology. On the space $C\left(\mathbb{R} \times \mathcal{C}, \mathbb{R}^{n}\right)$ is defined (see, for example, $[7, \mathrm{ChI}]$ and $[15, \mathrm{ChI}])$ a shift dynamical system $\left(C\left(\mathbb{R} \times \mathcal{C}, \mathbb{R}^{n}\right), \mathbb{R}, \sigma\right)$, where $\sigma(\tau, f):=f_{\tau}$ for any $f \in C\left(\mathbb{R} \times \mathcal{C}, \mathbb{R}^{n}\right)$ and $\tau \in \mathbb{R}$ and $f_{\tau}$ is $\tau$-translation of $f$, i.e. $f_{\tau}(t, \phi):=f(t+\tau, \phi)$ for any $(t, \phi) \in \mathbb{R} \times \mathcal{C}$. Let us set $H^{+}(f):=\overline{\left\{f_{s}: s \in \mathbb{R}_{+}\right\}}$.

Along with equation (23) let us consider the family of equations

$$
\begin{equation*}
\dot{v}=g\left(t, v_{t}\right) \tag{24}
\end{equation*}
$$

where $g \in H^{+}(f)$.
Below, in this subsection, we suppose that equation (23) is regular.

Remark 14. 1. Denote by $\tilde{\varphi}(t, u, f)$ the solution of equation (23) defined on $\mathbb{R}_{+}$ (respectively, on $\mathbb{R}$ ) with the initial condition $\varphi(0, u, f)=u \in \mathcal{C}$, i.e. $\varphi(s, u, f)$ $=u(s)$ for any $s \in[-r, 0]$. By $\varphi(t, u, f)$ we will denote below the trajectory of equation (23), corresponding to the solution $\tilde{\varphi}(t, u, f)$, i.e. the mapping from $\mathbb{R}_{+}$ (respectively, $\mathbb{R}$ ) into $\mathcal{C}$, defined by $\varphi(t, u, f)(s):=\tilde{\varphi}(t+s, u, f)$ for any $t \in \mathbb{R}_{+}$ (respectively, $t \in \mathbb{R}$ ) and $s \in[-r, 0]$.
2. Due to item 1. of this remark, below we will use the notions of "solution" and "trajectory" for equation (23) as synonymous concepts.

It is well known $[5,14]$ that the mapping $\varphi: \mathbb{R}_{+} \times \mathcal{C} \times H^{+}(f) \mapsto \mathbb{R}^{n}$ possesses the following properties:

1. $\varphi(0, v, g)=v$ for any $v \in \mathcal{C}$ and $g \in H^{+}(f)$;
2. $\varphi(t+\tau, v, g)=\varphi(t, \varphi(\tau, v, g), \sigma(\tau, g))$ for any $t, \tau \in \mathbb{R}_{+}, v \in \mathcal{C}$ and $g \in H^{+}(f)$;
3. the mapping $\varphi$ is continuous.

Thus, a triplet $\left\langle\mathcal{C}, \varphi,\left(H^{+}(f), \mathbb{R}_{+}, \sigma\right)\right\rangle$ is a cocycle which is associated to equation (23). Applying the results from Sections $3-4$ we will obtain a series of results for functional differential equation (23). Below we formulate some of them.

Lemma 6 (see [8]). Suppose that the following conditions hold:

1. the function $f \in C\left(\mathbb{R} \times \mathcal{C}, \mathbb{R}^{n}\right)$ is regular;
2. the set $H^{+}(f)$ is compact;
3. the function $f$ is completely continuous, i.e. the set $f\left(\mathbb{R}_{+} \times A\right)$ is bounded for any bounded subset $A \subseteq \mathcal{C}$.

Then the cocycle $\varphi$ associated with (23) is completely continuous, i.e. for any bounded subset $A \subseteq W$ there exists a positive number $l=l(A)$ such that the set $\varphi\left(l, A, H^{+}(f)\right)$ is relatively compact in $\mathcal{C}$.

Theorem 17. Let $f \in C\left(\mathbb{R} \times \mathcal{C}, \mathbb{R}^{n}\right)$. Assume that the following conditions are fulfilled:

1. the function $f$ is regular and completely continuous;
2. the set $H^{+}(f)$ is compact;
3. $f(t, 0)=0$ for any $t \in \mathbb{R}_{+}$.

Then the null solution of equation (23) is globally asymptotically stable if and only if the following conditions hold:
a) for every $a>0$

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} \sup _{|v| \leq a, g \in \Omega_{f}}|\varphi(t, v, g)|=0 ; \tag{25}
\end{equation*}
$$

b) for every $v \in \mathcal{C}$ and $g \in H^{+}(f)$ the solution $\varphi(t, v, g)$ of equation (24) is bounded on $\mathbb{R}_{+}$.

Proof. Consider the dynamical system $\left(H^{+}(f), \mathbb{R}_{+}, \sigma\right)$. Since the space $H^{+}(f)$ is compact, then $\left(H^{+}(f), \mathbb{R}_{+}, \sigma\right)$ is compactly dissipative and by Lemma 1 its Levinson center $J_{H^{+}(f)}$ coincides with the $\omega$-limit set $\Omega_{f}$ of $f$. Let $Y:=H^{+}(f)$ and $\left(Y, \mathbb{R}_{+}, \sigma\right)$ be the shift dynamical system on $Y$. Denote $X:=\mathcal{C} \times Y$ and $\left(X, \mathbb{R}_{+}, \pi\right)$ the skewproduct dynamical system generates by $\left(Y, \mathbb{R}_{+}, \sigma\right)$ and cocycle $\varphi$. Now consider a $\operatorname{NDS}\left\langle\left(X, \mathbb{R}_{+}, \pi\right),\left(Y, \mathbb{R}_{+}, \pi\right), h\right\rangle\left(h:=p r_{2}\right)$ associated with equation (23). It is easy to verify that this NDS has the following properties:

1. the dynamical system $\left(Y, \mathbb{R}_{+}, \sigma\right)$ is compact dissipative and its Levinson center $J_{Y}$ coincides with $\Omega_{f}$;
2. the null section $\Theta$ of $\left\langle\left(X, \mathbb{R}_{+}, \pi\right),\left(Y, \mathbb{R}_{+}, \pi\right), h\right\rangle$ coincides with $\{0\} \times Y$;
3. $\Theta$ is a positively invariant subset of $\left(X, \mathbb{R}_{+}, \pi\right)$;
4. according to (25) the null section $\tilde{\Theta}$ of $\operatorname{NDS}\left\langle\left(\tilde{X}, \mathbb{R}_{+}, \pi\right),\left(J_{Y}, \mathbb{R}_{+}, \sigma\right), h\right\rangle$ is uniformly attracting because $|\pi(t, x)|=|\varphi(t, v, g)|$ for any $t \in \mathbb{R}_{+}$and $x:=(v, g) \in X ;$
5. according to Lemma 6 the dynamical system $\left(X, \mathbb{R}_{+}, \pi\right)$ is completely continuous;
6. every positive semi-trajectory $\Sigma_{x}^{+}$of the skew-product dynamical system ( $X$, $\left.\mathbb{R}_{+}, \pi\right)$ is relatively compact.

Now to finish the proof it is sufficient to apply Corollary 3.
Theorem 18. Let $f \in C\left(\mathbb{R} \times \mathcal{C}, \mathbb{R}^{n}\right)$. Assume that the following conditions are fulfilled:

1. the function $f$ is regular and completely continuous;
2. the set $H^{+}(f)$ is compact;
3. $f(t, 0)=0$ for any $t \in \mathbb{R}_{+}$.

Then the null solution of equation (23) is globally asymptotically stable if and only if the following conditions hold:
a. for every $g \in \Omega_{f}$ limiting equation (24) does not have nontrivial bounded on $\mathbb{R}$ solutions;
b. for every $v \in \mathcal{C}$ and $g \in H^{+}(f)$ the solution $\varphi(t, v, g)$ of equation (24) is bounded on $\mathbb{R}_{+}$.

Proof. This statement follows from Corollary 2 and can be proved using the same arguments as in the proof of Theorem 17.

Theorem 19. Suppose that the following conditions are fulfilled:

1. the function $f \in C(\mathbb{R} \times \mathcal{C}, \mathcal{C})$ is recurrent in $t \in \mathbb{R}$ uniformly in $u$ on every compact subset of $\mathcal{C}$;
2. $f(t, 0)=0$ for any $t \in \mathbb{R}_{+}$;
3. the function $f$ is regular and completely continuous;
4. the null solution of equation (23) is uniformly stable;
5. there exists a positive number a such that

$$
\lim _{t \rightarrow+\infty} \sup _{|u| \leq a}|\varphi(t, u, f)|=0 .
$$

Then the null solution of equation (23) is asymptotically stable.
Proof. This statement follows directly from Theorem 11 using the same arguments as in the proof of Theorem 17.

### 5.3.2 Neutral functional-differential equations

Now consider the neutral functional-differential equation

$$
\begin{equation*}
\frac{d}{d t} D u_{t}=f\left(t, u_{t}\right), \tag{26}
\end{equation*}
$$

where $f \in C(\mathbb{R} \times \mathcal{C}, \mathcal{C})$ is a regular function and the operator $D: \mathcal{C} \mapsto \mathbb{R}^{n}$ is atomic at zero [9, p.67]. Like (23), equation (26) generates a $\operatorname{NDS}\left\langle\left(X, \mathbb{R}_{+}, \pi\right),\left(Y, \mathbb{R}_{+}, \sigma\right), h\right\rangle$, where $X:=\mathcal{C} \times Y, Y:=H^{+}(f)$, and $\pi:=(\varphi, \sigma)$.

An operator $D$ is said to be stable if the zero solution of difference equation $D y_{t}=0$ is uniformly asymptotically stable (see, for example, $[9$, p.337]).

Lemma 7. Let $H^{+}(f)$ be compact. If the function $f \in C\left(\mathbb{R} \times \mathcal{C}, \mathbb{R}^{n}\right)$ is completely continuous, then the $\left.\operatorname{NDS}\left(X, \mathbb{R}_{+}, \pi\right),(Y, \mathbb{R}, \sigma), h\right\rangle$ generated by equation (26) is asymptotically compact.

Proof. This statement can be proved by slight modification of the proof of Theorem 12.6.3 and Lemma 12.6.1 from [9, Ch.XII] and taking into account that $Y=H^{+}(A)$ is compact.

Theorem 20. Suppose that the following conditions are fulfilled:

1. the function $f \in C(\mathbb{R} \times \mathcal{C}, \mathcal{C})$ is recurrent in $t \in \mathbb{R}$ uniformly in $u$ on every compact subset of $\mathcal{C}$;
2. $f(t, 0)=0$ for any $t \in \mathbb{R}_{+}$;
3. the function $f$ is regular and completely continuous;
4. the null solution of equation (26) is uniformly stable;
5. there exists a positive number a such that

$$
\begin{equation*}
\lim _{t \rightarrow+\infty}|\varphi(t, u, f)|=0 \tag{27}
\end{equation*}
$$

for any $|u| \leq a$.
Then the null solution of equation (26) is asymptotically stable, i.e., there exists a positive number $\delta$ such that $\lim _{t \rightarrow+\infty}|\varphi(t, v, g)|=0$ for any $|v|<\delta$ and $g \in H^{+}(f)$.

Proof. Let $\left.\left(X, \mathbb{R}_{+}, \pi\right),(Y, \mathbb{R}, \sigma), h\right\rangle$ be a NDS generated by equation (26). It is easy to check that under the conditions of Theorem 20 the following statements hold:

1. the dynamical system $\left(Y, \mathbb{R}_{+}, \sigma\right)$ is compactly dissipative and its Levinson center $J_{Y}$ coincides with $Y=H^{+}(f)=\Omega_{f}$;
2. the null section $\Theta$ of $\left\langle\left(X, \mathbb{R}_{+}, \pi\right),\left(Y, \mathbb{R}_{+}, \pi\right), h\right\rangle$ coincides with $\{0\} \times Y$;
3. $\Theta$ is a positively invariant subset of $\left(X, \mathbb{R}_{+}, \pi\right)$;
4. according to (27) we have $B\left(0_{f}, a\right) \subset X_{f}^{s}$, where $0_{f}:=(0, f), 0$ is the null element of $\mathcal{C}$ and $B\left(x_{0}, a\right):=\left\{x \in \mathcal{C}:\left|x-x_{0}\right|<a\right\} ;$
5. according to Lemma 7 the dynamical system $\left(X, \mathbb{R}_{+}, \pi\right)$ is asymptotically compact.

Now to finish the proof of Theorem it is sufficient to apply Theorem 11.

### 5.4 Semi-linear parabolic equations

Let $E$ be a Banach space, and let $A: D(A) \rightarrow E$ be a linear closed operator with the dense domain $D(A) \subseteq E$.

An operator $A$ is called [10] sectorial if for some $\phi \in(0, \pi / 2)$, some $M \geq 1$, and some real $a$, the sector

$$
S_{a, \phi}:=\{\lambda: \phi \leq|\arg (\lambda-a)| \leq \pi, \lambda \neq a\}
$$

lies in the resolvent set $\rho(A)$ and $\left\|(I \lambda-A)^{-1}\right\| \leq M|\lambda-a|^{-1}$ for all $\lambda \in S_{a, \varphi}$.
If $A$ is a sectorial operator, then there exists an $a_{1} \geq 0$ such that $\operatorname{Re} \sigma\left(A+a_{1} I\right)$ $>0(\sigma(A):=\mathbb{C} \backslash \rho(A))$. Let $A_{1}=A+a_{1} I$. For $0<\alpha<1$, one defines the operator [10]

$$
A_{1}^{-\alpha}:=\frac{\sin \pi \alpha}{\pi} \int_{0}^{+\infty} \lambda^{-\alpha}\left(\lambda I+A_{1}\right)^{-1} d \lambda
$$

which is linear, bounded, and one-to-one. Set $E^{\alpha}:=D\left(A_{1}^{\alpha}\right)$, and let us equip the space $E^{\alpha}$ with the graph norm $|x|_{\alpha}:=\left|A_{1}^{\alpha} x\right|(x \in E), E^{0}:=E$, and $E^{1}:=D(A)$.

Then $E^{\alpha}$ is a Banach space with the norm $|\cdot|_{\alpha}$ and is densely and continuously embedded in $E$.

Consider the differential equation

$$
\begin{equation*}
x^{\prime}+A x=F(t, x) \tag{28}
\end{equation*}
$$

where $F \in C\left(\mathbb{R} \times E^{\alpha}, E\right)$ and $C\left(\mathbb{R} \times E^{\alpha}, E\right)$ is the space of all the continuous functions equipped with compact open topology.

Along with equation (28), consider the family of equations

$$
\begin{equation*}
y^{\prime}+A y=G(t, y) \tag{29}
\end{equation*}
$$

where $G \in H^{+}(F):=\overline{\left\{F_{\tau}: \tau \in \mathbb{R}_{+}\right\}}$.
Recall that a function $F$ is said to be regular if for every $(v, G) \in E^{\alpha} \times H^{+}(F)$ equation (29) admits a unique solution [10, Ch.III] $\varphi(t, v, G)$ with initial data $\varphi(0, v, G)=v$ and the mapping $\varphi: \mathbb{R}_{+} \times E^{\alpha} \times H^{+}(F) \mapsto E^{\alpha}$ is continuous.

Regularity conditions for $F$ are given in Theorems 3.3.3, 3.3.4, 3.3.6, and 3.4.1 in [10, Ch.III].

Assuming that $F$ is regular, a non-autonomous dynamical system can be associated in a natural way with equation (28). Namely, we set $Y:=H^{+}(F)$ and by $\left(Y, \mathbb{R}_{+}, \sigma\right)$ denote the dynamical system of translations on $Y$. Further, let $X:=E^{\alpha} \times Y$, and let $\left(X, \mathbb{R}_{+}, \pi\right)$ be the dynamical system on $X$ defined by the relation $\pi^{\tau}(v, G)=\left\langle\varphi(\tau, v, G), G_{\tau}\right\rangle$. Finally, by setting $h=p r_{2}: X \rightarrow Y$, we obtain the non-autonomous system $\left\langle\left(X, \mathbb{R}_{+}, \pi\right),\left(Y, \mathbb{R}_{+}, \sigma\right), h\right\rangle$ determined by equation (28).

Applying results from Sections $3-4$ we obtain a series of results for evolution equation (28). Now we will formulate some of them.

Recall that a function $F \in C\left(\mathbb{R} \times E^{\alpha}, E\right)$ is said to be locally Hölder continuous in $t$ and locally Lipschitz in $x$ if for every $\left(t_{0}, x_{0}\right) \in \mathbb{R} \times E^{\alpha}$ there exists a neighborhood $V\left(\left(t_{0}, x_{0}\right) \in V\right)$ and positive numbers $L$ and $\theta$ such that

$$
\left|F\left(t_{1}, x_{1}\right)-F\left(t_{2}, x_{2}\right)\right| \leq L\left(\left|t_{1}-t_{2}\right|^{\theta}+\left|x_{1}-x_{2}\right|_{\alpha}\right)
$$

for any $\left(t_{i}, x_{i}\right) \in V(i=1,2)$.
Lemma 8. Suppose that the following conditions are fulfilled:

1. $A$ is a sectorial operator;
2. the resolvent of operator $A$ is compact;
3. $0 \leq \alpha<1$ and $F \in C\left(\mathbb{R} \times E^{\alpha}, E\right)$;
4. the function $F$ is locally Hölder continuous in $t$ and locally Lipschitz in $x$.

Under the conditions listed above if the function $F$ is regular and the set $H^{+}(F)$ is compact, then the cocycle $\varphi$ associated with equation (28) is completely continuous.

Proof. This statement can be proved with the slight modification of the proof of Theorem 3.3.6 [10, Ch.III].

Theorem 21. Assume that the following conditions are fulfilled:

1. the function $F$ is regular;
2. the set $H^{+}(F)$ is compact;
3. $F(t, 0)=0$ for any $t \in \mathbb{R}_{+}$;
4. $0 \leq \alpha<1$ and $F \in C\left(\mathbb{R} \times E^{\alpha}, E\right)$;
5. the function $F$ is locally Hölder continuous in $t$ and locally Lipschitz in $x$.

Then the null solution of equation (28) is globally asymptotically stable if and only if the following conditions hold:
1.

$$
\lim _{t \rightarrow+\infty} \sup _{|v| \leq a, g \in \Omega_{f}}|\varphi(t, v, G)|=0
$$

for every $a>0$;
2. for every $v \in E^{\alpha}$ and $G \in H^{+}(F)$ the solution $\varphi(t, v, G)$ of equation (28) is bounded on $\mathbb{R}_{+}$.

Proof. Consider the dynamical system $\left(H^{+}(F), \mathbb{R}_{+}, \sigma\right)$. Since the space $H^{+}(F)$ is compact, then $\left(H^{+}(f), \mathbb{R}_{+}, \sigma\right)$ is compactly dissipative and its Levinson center $J_{H^{+}(F)}$ coincides with the $\omega$-limit set $\Omega_{F}$ of $F$. Let $Y:=H^{+}(F)$ and $\left(Y, \mathbb{R}_{+}, \sigma\right)$ be the shift dynamical system on $Y$. Denote by $X:=E^{\alpha} \times Y$ and $\left(X, \mathbb{R}_{+}, \pi\right)$ the skew-product dynamical system generated by $\left(Y, \mathbb{R}_{+}, \sigma\right)$ and cocycle $\varphi$. Consider a non-autonomous dynamical system $\left\langle\left(X, \mathbb{R}_{+}, \pi\right),\left(Y, \mathbb{R}_{+}, \pi\right), h\right\rangle\left(h:=p r_{2}\right)$ associated with equation (28). It is easy to verify that for this NDS the following properties hold:

1. the dynamical system $\left(Y, \mathbb{R}_{+}, \sigma\right)$ is compactly dissipative and by Lemma 1 its Levinson center $J_{Y}$ coincides with $\Omega_{F}$;
2. the null section $\Theta$ of $\left\langle\left(X, \mathbb{R}_{+}, \pi\right),\left(Y, \mathbb{R}_{+}, \pi\right), h\right\rangle$ coincides with $\{0\} \times Y$;
3. $\Theta$ is a positively invariant subset of $\left(X, \mathbb{R}_{+}, \pi\right)$;
4. according to (30) the null section $\tilde{\Theta}$ of $\operatorname{NDS}\left\langle\left(\tilde{X}, \mathbb{R}_{+}, \pi\right),\left(J_{Y}, \mathbb{R}_{+}, \sigma\right), h\right\rangle$ is uniformly attracting because $|\pi(t, x)|=|\varphi(t, v, g)|$ for any $t \in \mathbb{R}_{+}$and $x:=(v, g) \in X ;$
5. by Lemma 8 the cocycle $\varphi$ (and, consequently, the skew-product dynamical system $\left(X, \mathbb{R}_{+}, \pi\right)$ too $)$ is completely continuous;
6. every positive semi-trajectory $\Sigma_{x}^{+}$of the skew-product dynamical system ( $X$, $\left.\mathbb{R}_{+}, \pi\right)$ is relatively compact.

Now to finish the proof it is sufficient to apply Theorem 8 .

Theorem 22. Assume that the following conditions are fulfilled:

1. the function $F$ is regular;
2. the set $H^{+}(F)$ is compact;
3. $F(t, 0)=0$ for any $t \in \mathbb{R}_{+}$;
4. $0 \leq \alpha<1$ and $F \in C\left(\mathbb{R} \times E^{\alpha}, E\right)$;
5. the function $F$ is locally Hölder continuous in $t$ and locally Lipschitz in $x$.

Then the null solution of equation (28) is globally asymptotically stable if and only if the following conditions hold:
a. for every $G \in \Omega_{F}$ limiting equation (29) does not have nontrivial bounded on $\mathbb{R}$ solutions;
b. for every $v \in \mathcal{C}$ and $G \in H^{+}(F)$ the solution $\varphi(t, v, g)$ of equation (29) is bounded on $\mathbb{R}_{+}$.

Proof. This statement can be proved using the same arguments as in the proof of Theorem 21 plus applying Corollary 2.

Theorem 23. Suppose that the following conditions are fulfilled:

1. $0 \leq \alpha<1$ and $F \in C\left(\mathbb{R} \times E^{\alpha}, X\right)$;
2. the function $F$ is locally Hölder continuous in $t$ and locally Lipschitz in $x$;
3. the function $F$ is recurrent in $t \in \mathbb{R}$ uniformly in $u$ on every compact subset of $W \subseteq E^{\alpha}$;
4. $F(t, 0)=0$ for any $t \in \mathbb{R}_{+}$;
5. the function $F$ is regular;
6. the null solution of equation (28) is uniformly stable;
7. there exists a positive number a such that

$$
\lim _{t \rightarrow+\infty}|\varphi(t, u, F)|=0
$$

for any $|u| \leq a$.
Then the null solution of equation (28) is asymptotically stable.
Proof. This statement follows directly from Theorem 11 using the same arguments as in the proof of Theorem 21.

## References

[1] Armando D'Anna. Total Stability Properties for an Almost Periodic Equation by Means of Limiting Equations. Funkciolaj Ekvacioj, 1984, 27, 201-209.
[2] Armando D'Anna, Alfonso Maio and Vinicio Moauro. Global stability properties by means of limiting equations. Nonlinear Analysis, 1980, 4, 2, 407-410.
[3] Artstein Zvi. Uniform asymptotic stability via the limiting equations. Journal of Differential Equations, 1978, 27 (2), 172-189.
[4] Boularas Driss and Cheban David. Asymptototic Stability of Switching Systems. Electronic Journal of Differential Equations, 2010, 21, 1-18.
[5] Bronsteyn I. U. Extensions of Minimal Transformation Group, Noordhoff, 1979.
[6] Caraballo Tomas and Cheban David. On the Structure of the Global Attractor for Nonautonomous Dynamical Systems with Weak Convergence. Communications in Pure and Applied Analysis, 2012, 11, 2, 809-828.
[7] Cheban D. N. Global Attractors of Non-Autonomous Dissipstive Dynamical Systems. Interdisciplinary Mathematical Sciences 1. River Edge, NJ: World Scientific, 2004, xxiii +502 pp .
[8] Cheban D. N. Sell's Conjecture for Non-Autonomous Dynamical Systems. Nonlinear Analysis: TMA, 2012, 75, 7, 3393-3406.
[9] Hale J. K. Theory of Functional-Differential Equations. Springer-Verlag, New York-Heidelberg-Berlin, 1977 [Russian translation: Theory of Functional-Differential Equations. Mir, Moscow, 1984].
[10] Henry D. Geometric Theory of Semilinear Parabolic Equations. Lecture Notes in Mathematics, 840, Springer-Verlag, New York 1981.
[11] Husemoller D. Fibre Bundles. Springer-Verlag, Berlin-Heidelberg-New York, 1994.
[12] Junji Kato and Taro Yoshizawa. Remarks on Global Properties in Limiting Equations. Funkciolaj Ekvacioj, 1981, 24, 363-371.
[13] Martyniuk A. A., Kato D. and Shestakov A. A. Stability of Motion : Method of Limit Equations. Kiev, Naukova Dumka, 1990. (in Russian) [English translation in Gordon and Breach Publishers, Luxembourg, 1996].
[14] Sell G. R. Topological Dynamics and Ordinary Differential Equations. Van NostrandReinhold, London, 1971.
[15] Shcherbakov B. A. Topologic Dynamics and Poisson Stability of Solutions of Differential Equations. Ştiinţa, Chişinău, 1972, 231 pp. (in Russian).

David Cheban
Received August 29, 2012
State University of Moldova
Faculty of Mathematics and Informatics
Department of Fundamental Mathematics
A. Mateevich Street 60

MD-2009 Chişinău, Moldova
E-mail: cheban@usm.md, davidcheban@yahoo.com

# Applications of algebraic methods in solving the center-focus problem 

M. N. Popa, V.V. Pricop


#### Abstract

The nonlinear differential system $\dot{x}=\sum_{i=0}^{\ell} P_{m_{i}}(x, y), \dot{y}=\sum_{i=0}^{\ell} Q_{m_{i}}(x, y)$ is considered, where $P_{m_{i}}$ and $Q_{m_{i}}$ are homogeneous polynomials of degree $m_{i} \geq 1$ in $x$ and $y, m_{0}=1$. The set $\left\{1, m_{i}\right\}_{i=1}^{\ell}$ consists of a finite number $(l<\infty)$ of distinct integer numbers. It is shown that the maximal number of algebraically independent focal quantities that take part in solving the center-focus problem for the given differential system with $m_{0}=1$, having at the origin of coordinates a singular point of the second type (center or focus), does not exceed $\varrho=2\left(\sum_{i=1}^{\ell} m_{i}+\ell\right)+3$. We make an assumption that the number $\omega$ of essential conditions for center which solve the center-focus problem for this differential system does not exceed $\varrho$, i. e. $\omega \leq \varrho$.


Mathematics subject classification: 34C14, 34C40.
Keywords and phrases: Differential systems, the center-focus problem, focal quantities, Sibirsky graded algebras, Hilbert series, Krull dimension, Lie algebras of operators.

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## 1 Introduction

The nonlinear differential system

$$
\begin{equation*}
\frac{d x}{d t}=\sum_{i=0}^{\ell} P_{m_{i}}(x, y), \frac{d y}{d t}=\sum_{i=0}^{\ell} Q_{m_{i}}(x, y) \tag{1}
\end{equation*}
$$

is considered, where $P_{m_{i}}$ and $Q_{m_{i}}$ are homogeneous polynomials of degree $m_{i} \geq 1$ in $x$ and $y, m_{0}=1$. The set $\left\{1, m_{1}, m_{2}, \ldots, m_{\ell}\right\}$ consists of a finite number $(l<\infty)$ of distinct natural numbers. The coefficients and variables in polynomials $P_{m_{i}}$ and $Q_{m_{i}}$ take values from the field of the real numbers $\mathbb{R}$.

It is known that if the roots of characteristic equation of the singular point $O(0,0)$ of the system (1) are imaginary, then the singular point $O$ is a center (surrounded by closed trajectories) or a focus (surrounded by spirals) [1,5]. In this case the origin of coordinates is a singular point of the second type.

Hereafter we denote the system (1) by $s\left(1, m_{1}, m_{2}, \ldots, m_{\ell}\right)$.
The center-focus problem can be formulated as follows: Let for the system $s\left(1, m_{1}, m_{2}, \ldots, m_{\ell}\right)$ the origin of coordinates be a singular point of the second type (center or focus). Find the conditions which distinguish center from focus. This problem was posed by A. Poincaré $[1,2]$. The basic results were obtained by A. M. Lyapunov [5]. It was shown that the conditions for center are the vanishing of an infinite sequence of polynomials (focal quantities)

$$
\begin{equation*}
L_{1}, L_{2}, \ldots, L_{k}, \ldots \tag{2}
\end{equation*}
$$

in the coefficients of right side of the system (1). If at least one of the quantities (2) is not zero, then the origin of coordinates for the system (1) is a focus. These conditions are necessary and sufficient.

In the case of the system (1) from Hilbert's theorem on the finiteness of basis of polynomial ideals it follows that in the mentioned sequence (2) only a finite number of conditions for center are essential, the rest are consequences of them. Then the center-focus problem for the system (1) takes the following formulation: How many polynomials (essential conditions for center)

$$
\begin{equation*}
L_{n_{1}}, L_{n_{2}}, \ldots, L_{n_{\omega}}, \ldots\left(n_{i} \in\{1,2, \ldots, k, \ldots\} ; i=\overline{1, \omega} ; \omega<\infty\right) \tag{3}
\end{equation*}
$$

from (2) must be equal to zero in order that all other polynomials (2) would vanish?
The problem of determining the number $\omega$ of essential conditions for center (3) is complicated. It is completely solved for the systems $s(1,2)$ and $s(1,3)$ [8,11], for which we have respectively $\omega=3$ and 5 . Until now $\omega$ has not been known for the system $s(1,2,3)$. There exists only a Zolâdek hypothesis, which is based mostly on intuition, that for the system $s(1,2,3)$ the number $\omega \leq 13$. To the present day this hypothesis has not been disproved. But in [12] it is proved that for the system $s(1,2,3) 12$ focal quantities are not enough for solving the center-focus problem in the complex plane.

It is natural to ask why there is still no answer about the value of $\omega$ from (3) for any system $s\left(1, m_{1}, m_{2}, \ldots, m_{\ell}\right)$ ?

We can explain this failure as follows: searching for a finite $\omega$ from (3), till now the researchers have used basically a known approach, i. e. with the help of certain calculations they constructed the explicit form of the first focal quantities from (2), without knowing a priori the number $\omega$. Sometimes the existence of some geometric properties for the system (1) was assumed, for example, the existence of integral straight lines, conics and other curves. Then with their help the attempts were made to show that the vanishing of the available quantities implies the vanishing of other members of the sequence (2), often there was only a vague idea about their expressions.

This approach gave quite unsatisfactory results. One of the reasons is due to the enormous computing for focal quantities, which can not be overcome using supercomputers even for the system $s(1,2,3)$, not to mention more complicated systems $s\left(1, m_{1}, m_{2}, \ldots, m_{\ell}\right)$. Therefore, it is clear that the results obtained in this direction refer more to the systems (1) of special forms.

From what has been said above the following conclusion can be drawn: solving the center-focus problem is equivalent to finding the essential conditions for center (3), that requires knowledge of the number $\omega$, the finiteness of which follows from Hilbert's theorem on the finiteness of basis of polynomial ideals.

Therefore, the problem of finding the number $\omega<\infty$ or obtaining for it an argued numerical upper bound (even as a hypothesis), which is still absent, is a very important condition of the complete solving of the center-focus problem for the system (1).

The last affirmation can be considered as a generalized center-focus problem for the systems $s\left(1, m_{1}, m_{2}, \ldots, m_{\ell}\right)$, and obtaining an answer to it will be qualified as perhaps one of the sufficient conditions in solving the mentioned problem.

## 2 Graded algebras of comitants of the system (1)

In $[5,6,7]$ the type of center-affine polynomial comitant with respect to the centeraffine group $G L(2, \mathbb{R})$ for any differential system $s\left(m_{0}, m_{1}, m_{2}, \ldots, m_{\ell}\right)$ was determined, and it is denoted as follows:

$$
\begin{equation*}
(d)=\left(\delta, d_{0}, d_{1}, \ldots, d_{\ell}\right) \tag{4}
\end{equation*}
$$

where $\delta$ is the degree of homogeneity of this comitant in phase variables $x, y$, and $d_{i}(i=\overline{1, l})$ is the degree of homogeneity of the same comitant in the coefficients of the polynomials $P_{m_{i}}(x, y), Q_{m_{i}}(x, y)$ from the right side of the system (1).

In [7] the following affirmations were proved:
Proposition 1. The set of center-affine comitants of the system (1) of the same type (4) forms a finite linear space $V_{m_{0}, m_{1}, m_{2}, \ldots, m_{\ell}}^{(d)}$, i.e. it has a finite maximal system of linearly independent comitants of the given type (linear basis), all the rest are linearly expressed trough them.

Proposition 2. In order that any homogeneous polynomial of the type (4) in phase variables and coefficients of the system (1) would be a center-affine comitant of this system, it is necessary and sufficient that it be an unimodular comitant (invariant polynomial with respect to the unimodular group $S L(2, \mathbb{R})$ ) of the same type (4) for the given system.
Proposition 3. [6] For any center-affine comitant of differential system of the type (4) the following equality holds:

$$
\begin{equation*}
2 g=\sum_{i=0}^{\ell} d_{i}\left(m_{i}-1\right)-\delta, \tag{5}
\end{equation*}
$$

where $g$ is usually called the weight of given comitant, and it is an integer number.
Following Propositions 1-2 and according to [7] we denote the space of unimodular comitants of the type (4) for the system $s\left(1, m_{1}, m_{2}, \ldots, m_{\ell}\right)$ by

$$
S_{1, m_{1}, m_{2}, \ldots, m_{\ell}}^{(d)} \cong V_{1, m_{1}, m_{2}, \ldots, m_{\ell}}^{(d)} .
$$

Let us consider the linear space

$$
\begin{equation*}
S_{1, m_{1}, m_{2}, \ldots, m_{\ell}}=\sum_{(d)} S_{1, m_{1}, m_{2}, \ldots, m_{\ell}}^{(d)} \tag{6}
\end{equation*}
$$

which is a graded algebra of comitants of the system $s\left(1, m_{1}, m_{2}, \ldots, m_{\ell}\right)$, where its components satisfy the inclusion

$$
S_{1, m_{1}, m_{2}, \ldots, m_{\ell}}^{(d)} S_{1, m_{1}, m_{2}, \ldots, m_{\ell}}^{(e)} \subseteq S_{1, m_{1}, m_{2}, \ldots, m_{\ell}}^{(d+e)}, S_{1, m_{1}, m_{2}, \ldots, m_{\ell}}^{(0)}=\mathbb{R} .
$$

We denote by $S I_{1, m_{1}, m_{2}, \ldots, m_{\ell}}$ a graded algebra of unimodular invariants (comitants that do not depend on the phase variables $x, y)$ of the system $s\left(1, m_{1}, m_{2}, \ldots, m_{\ell}\right)$, which satisfies the inclusion

$$
\begin{equation*}
S I_{1, m_{1}, m_{2}, \ldots, m_{\ell}} \subset S_{1, m_{1}, m_{2}, \ldots, m_{\ell}} \tag{7}
\end{equation*}
$$

As for the first time the comitants and invariants for systems of the form (1) were introduced by K. S. Sibirsky [14], hereafter we will refer to these and similar algebras as Sibirsky algebras.

## 3 Krull dimension for Sibirsky graded algebras

From the theory of invariants and tensors [5,13] it results that the Sibirsky graded algebras $S_{1, m_{1}, m_{2}, \ldots, m_{\ell}}$ and $S I_{1, m_{1}, m_{2}, \ldots, m_{\ell}}$ are commutative and finitely determined algebras. If for these algebras we introduce a unified notation $A$, then the last affirmation can be written as

$$
\begin{equation*}
A=<a_{1}, a_{2}, \ldots, a_{m} \mid f_{1}=0, f_{2}=0, \ldots, f_{n}=0>(m, n<\infty) \tag{8}
\end{equation*}
$$

where $a_{i}$ are generators for this algebra, and $f_{j}$ are defining relations (syzygies).

It is known from [7] that for the simplest differential system $s(0,1)$ of the form

$$
\begin{equation*}
\dot{x}=a+c x+d y, \dot{y}=b+e x+f y \tag{9}
\end{equation*}
$$

the finitely defined graded algebras of comitants $S_{0,1}$ and invariants $S I_{0,1}$ can be written respectively

$$
\begin{gather*}
S_{0,1}=<i_{1}, i_{2}, i_{3}, k_{1}, k_{2}, k_{3} \mid\left(i_{1} k_{1}-k_{3}\right)^{2}+k_{3}^{2}-i_{2} k_{1}^{2}-2 i_{3} k_{2}=0> \\
S I_{0,1}=<i_{1}, i_{2}, i_{3}> \tag{10}
\end{gather*}
$$

where

$$
\begin{gather*}
i_{1}=c+f, i_{2}=c^{2}+2 d e+f^{2}, i_{3}=-e a^{2}+(c-f) a b+d b^{2} \\
k_{1}=-b x+a y, k_{2}=-e x^{2}+(c-f) x y+d y^{2}  \tag{11}\\
k_{3}=-(e a+f b) x+(c a+d b) y
\end{gather*}
$$

We note that using the system (9) the whole theory of center-affine (unimodular) comitants and invariants for two-dimensional polynomial differential systems can be illustrated.
Definition 1. [15] Elements $a_{1}, a_{2}, \ldots, a_{r}$ of the algebra $A$ are called algebraically independent if for any non-trivial polynomial $F$ in these $r$ elements the following inequality holds:

$$
F\left(a_{1}, a_{2}, \ldots, a_{r}\right) \neq 0
$$

Definition 2. The maximal number of algebraically independent elements of an graded algebra $A$ is called the Krull dimension of this algebra and is denoted by $\varrho(A)$.

It is known [7] that for an algebra $A$ of the form (8) the equality $n=m-$ $-\varrho(A)$ holds. However, this equality is not very effective because it is impossible to determine the numbers $m$ and $n$ for most algebras of invariants and comitants for systems of the form (1).

In the classical theory of invariants [16] a set of elements $a_{1}, a_{2}, \ldots, a_{\varrho(A)}$ from $A$ which define the Krull dimension of the algebra $A$ is called an algebraic basis. This means that for any $a \in A\left(a \neq a_{j}\right)$ there exists a natural number $p$ such that the following identity holds:

$$
\begin{equation*}
P_{0} a^{p}+P_{1} a^{p-1}+\ldots+P_{p}=0 \tag{12}
\end{equation*}
$$

where $P_{k}(k=\overline{0, p})$ are polynomials in $a_{j}(j=\overline{1, \varrho(A)})$. We note that in general $P_{0} \not \equiv 1$.

If for any $a \in A$ in (12) we have $P_{0} \equiv 1$, then this basis is called integer algebraic basis. The existence a basis was shown by D. Hilbert (see [16]). We denote the number of its elements by $\varrho^{\prime}(A)$.

We note that in general the numbers of elements in the mentioned bases does not always coincide. For example, from $[7]$ we have that for the system $s(4)$ the Krull dimension $\varrho\left(S I_{4}\right)=7$, but from [17] for the same system we obtain that the number of elements in the integer algebraic basis of the same algebra is $\varrho^{\prime}\left(S I_{4}\right)=9$,
i. e. $\varrho\left(S I_{4}\right)<\varrho^{\prime}\left(S I_{4}\right)$. From [7] we have that for the system $s(0,1)$ the equality $\varrho\left(S_{0,1}\right)=\varrho^{\prime}\left(S_{0,1}\right)=5$ holds, and $\varrho\left(S I_{0,1}\right)=\varrho^{\prime}\left(S I_{0,1}\right)=3$. Also from [5,6,7] and [18] it follows that for the systems $s(2)$ and $s(3)$ we have $\varrho\left(S I_{2}\right)=\varrho^{\prime}\left(S I_{2}\right)=3$, $\varrho\left(S I_{3}\right)=\varrho^{\prime}\left(S I_{3}\right)=5$. From [7] and [19] we find that for the system $s(1,2)$ the equalities $\varrho\left(S I_{1,2}\right)=\varrho^{\prime}\left(S I_{1,2}\right)=7$ are valid. However, for the system $s(1,2,3)$ according to $[7,20]$ we have that $\varrho\left(S I_{1,2,3}\right)=15$, but $\varrho^{\prime}\left(S I_{1,2,3}\right)=21$.

The mentioned examples lead us to the relation

$$
\varrho(A) \leq \varrho^{\prime}(A) .
$$

This inequality accentuates that the integer algebraic basis contains an algebraic basis of an algebra $A$. The proof of this fact can be easily obtained by an indirect proof.
Remark 1. The main property of an integer algebraic basis of an algebra $A$ of invariants is that it is the minimum number of elements of the algebra $A$ such that if they are equal to zero, all elements of the algebra $A$ vanish.

Hereafter we need some evident affirmations:
Proposition 4. If $B$ is a graded subalgebra of an algebra $A$, then between the Krull dimensions of these algebras the following inequality holds:

$$
\varrho(B) \leq \varrho(A)
$$

It is evident
Proposition 5. If the Krull dimension of an algebra $A$ is $\varrho(A)$, then on any variety $V=\{a=0, b<0\}$ with fixed $a, b \in A$ (b has no effect on the mentioned variety) in the algebra $A$ there are not more than $\varrho(A)$ algebraically independent elements (possibly no more than $\varrho(A)$ elements which form an integer algebraic basis) of this algebra.

## 4 Hilbert series for Sibirsky graded algebras $S_{1, m_{1}, m_{2}, \ldots, m_{\ell}}$ and $S I_{1, m_{1}, m_{2}, \ldots, m_{\ell}}$

According to Proposition 1 for the spaces of the algebra $S_{1, m_{1}, m_{2}, \ldots, m_{\ell}}$ from (5) we have $\operatorname{dim}_{\mathbb{R}} S_{1, m_{1}, m_{2}, \ldots, m_{\ell}}^{(d)}<\infty$. Then, following [7], by the generalized Hilbert series of the algebra $S_{1, m_{1}, m_{2}, \ldots, m_{\ell}}$ we mean a formal series

$$
\begin{equation*}
H\left(S_{1, m_{1}, m_{2}, \ldots, m_{\ell}} ; u, z_{0}, z_{1}, \ldots, z_{\ell}\right)=\sum_{(d)} \operatorname{dim}_{\mathbb{R}} S_{1, m_{1}, m_{2}, \ldots, m_{\ell}}^{(d)} u z_{0}^{\delta} z_{0}^{d_{0}} z_{1}^{d_{1}} \ldots z_{\ell}^{d_{\ell}} \tag{13}
\end{equation*}
$$

which is said to reflect a $u, z$-graduation of the considered algebra.
From the definition of the algebra of invariants $S I_{1, m_{1}, m_{2}, \ldots, m_{\ell}}$ and (13) it follows that

$$
\begin{equation*}
H\left(S I_{1, m_{1}, m_{2}, \ldots, m_{\ell}}, z_{0}, z_{1}, \ldots, z_{\ell}\right)=H\left(S_{1, m_{1}, m_{2}, \ldots, m_{\ell}}, 0, z_{0}, z_{1}, \ldots, z_{\ell}\right), \tag{14}
\end{equation*}
$$

and the common Hilbert series will be written respectively

$$
\begin{gather*}
H_{S_{1, m_{1}, m_{2}}, \ldots, m_{\ell}}(u)=H\left(S_{1, m_{1}, m_{2}, \ldots, m_{\ell}}, u, u, u, \ldots, u\right), \\
H_{S I_{1, m_{1}, m_{2}}, \ldots, m_{\ell}}(z)=H\left(S I_{1, m_{1}, m_{2}, \ldots, m_{\ell}}, z, z, \ldots, z\right) . \tag{15}
\end{gather*}
$$

The last series contain meaningfull information about asymptotic character of the behavior of the considered algebras.

The method of construction of the generalized Hilbert series (13)-(15) for the algebras $S_{1, m_{1}, m_{2}, \ldots, m_{\ell}}$ and $S I_{1, m_{1}, m_{2}, \ldots, m_{\ell}}$ was developed in [7].

For example, the generalized Hilbert series for the algebras $S_{0,1}$ and $S I_{0,1}$ of unimodular comitants and invariants of the system $s(0,1)$ have, respectively, the forms

$$
\begin{gathered}
H\left(S_{0,1}, u, z_{0}, z_{1}\right)=\frac{1+u z_{0} z_{1}}{\left(1-u z_{0}\right)\left(1-z_{1}\right)\left(1-z_{1}^{2}\right)\left(1-z_{0}^{2} z_{1}\right)\left(1-u^{2} z_{1}\right)} \\
H\left(S I_{0,1}, z_{0}, z_{1}\right)=\frac{1}{\left(1-z_{1}\right)\left(1-z_{1}^{2}\right)\left(1-z_{0}^{2} z_{1}\right)}
\end{gathered}
$$

and the corresponding common Hilbert series will be written as

$$
H_{S_{0,1}}(u)=\frac{1-u+u^{2}}{(1-u)^{2}\left(1-u^{2}\right)\left(1-u^{3}\right)^{2}}, \quad H_{S I_{0,1}}(z)=\frac{1}{(1-z)\left(1-z^{2}\right)\left(1-z^{3}\right)}
$$

Remark 2. We note, following [21], that the Krull dimension $\varrho\left(S_{1, m_{1}, m_{2}, \ldots, m_{\ell}}\right)$ (respectively $\varrho\left(S I_{1, m_{1}, m_{2}, \ldots, m_{\ell}}\right)$ ) of the graded algebra $S_{1, m_{1}, m_{2}, \ldots, m_{\ell}}$ (respectively $\left.S I_{1, m_{1}, m_{2}, \ldots, m_{\ell}}\right)$ is equal to the multiplicity of the pole of the common Hilbert series $H_{S_{1, m_{1}, m_{2}, \ldots, m_{\ell}}}(u)$ (respectively $H_{S I_{1, m_{1}, m_{2}, \ldots, m_{\ell}}}(z)$ ) at the unit.

For example, considering the above mentioned common Hilbert series $H_{S_{0,1}}(u)$ and $H_{S I_{0,1}}(z)$ for the Krull dimension of the algebras $S_{0,1}$ and $S I_{0,1}$ we obtain $\varrho\left(S_{0,1}\right)=5$ and $\varrho\left(S I_{0,1}\right)=3$, respectively.

In other cases, when there is no explicit form of the common Hilbert series, but the power series expansion is known, then we can use the following
Remark 3. Accept that the comparison of series with non-negative coefficients is performed coefficient-wise ( $\left.\sum a_{n} t^{n} \leq \sum b_{n} t^{n} \Leftrightarrow a_{n} \leq b_{n} ; \forall n\right)$. Taking this into account, if for commutative graded algebras $A$ and $B$ we have

$$
\begin{equation*}
H_{A}(t) \leq H_{B}(t) \tag{16}
\end{equation*}
$$

then for their Krull dimensions we also have $\varrho(A) \leq \varrho(B)$.
It is also evident that if for the common Hilbert series of a commutative graded algebra $A$ we have

$$
\begin{equation*}
H_{A}(t) \leq \frac{C}{(1-t)^{m}} \tag{17}
\end{equation*}
$$

where $C$ is a fixed constant, then we obtain $\varrho(A) \leq m$.
The extended theory and bibliography about Hilbert series for graded algebras can be found in [22].

## 5 Lie algebras of operators admitted by polynomial differential systems

It is shown in [7] that any differential system $s\left(m_{0}, m_{1}, m_{2}, \ldots, m_{\ell}\right)$ from (1) admits a four-dimensional reductive Lie algebra $L_{4}$, which consists of operators

$$
\begin{equation*}
X_{1}=x \frac{\partial}{\partial x}+D_{1}, X_{2}=y \frac{\partial}{\partial x}+D_{2}, X_{3}=x \frac{\partial}{\partial y}+D_{3}, X_{4}=y \frac{\partial}{\partial y}+D_{4} \tag{18}
\end{equation*}
$$

where the differential operators $D_{1}, D_{2}, D_{3}, D_{4}$ are operators of the representation of the center-affine group $G L(2, \mathbb{R})$ in the space of the coefficients of the polynomials $P_{m_{i}}$ and $Q_{m_{i}}(i=\overline{1, \ell})$ of the system (1).

In [7] it is proved
Theorem 1. For a polynomial $k$ in the coefficients of the system $s\left(m_{0}, m_{1}, m_{2}, \ldots, m_{\ell}\right)$ from (1) and phase variables $x, y$ to be a center-affine comitant of this system with the weight $g$, it is necessary and sufficient that it satisfies the equations

$$
X_{1}(k)=X_{4}(k)=-g k, X_{2}(k)=X_{3}(k)=0
$$

With the help of this theorem and properties of rational absolute center-affine comitants of the system (1) from [7], following the classical theory of these invariants [16], it can be shown that for the number of elements in an algebraic basis of centeraffine comitants of the system $s\left(m_{0}, m_{1}, m_{2}, \ldots, m_{\ell}\right)$ the following formula holds:

$$
\begin{equation*}
\varrho=2\left(\sum_{i=0}^{\ell} m_{i}+\ell\right)+1 \tag{19}
\end{equation*}
$$

In the theory of center-affine comitants of polynomial differential systems [6] it is shown that if $S$ is a semi-invariant in the center-affine comitant $k$, then

$$
\begin{equation*}
k=S x^{\delta}-D_{3}(S) x^{\delta-1} y+\frac{1}{2!} D_{3}^{2}(S) x^{\delta-2} y^{2}+\ldots+\frac{(-1)^{\delta}}{\delta!} D_{3}^{\delta}(S) y^{\delta} \tag{20}
\end{equation*}
$$

where $D_{3}$ is defined in [7].
Remark 4. [6] With the help of this equality it can be shown that the center-affine comitants $k_{1}, k_{2}, \ldots, k_{\varrho\left(S_{\left.m_{0}, m_{1}, m_{2}, \ldots, m_{\ell}\right)}\right)} \in S_{m_{0}, m_{1}, m_{2}, \ldots, m_{\ell}}$ which belong to the system $s\left(m_{0}, m_{1}, m_{2}, \ldots, m_{\ell}\right)$ are algebraically independent if and only if their semi-invariants are algebraically independent.

## 6 An invariant variety in the center-focus problem of the system $s\left(1, m_{1}, m_{2}, \ldots, m_{\ell}\right)$

The center-focus problem for systems of the form (1) has the following classical formulation: for an infinite system of polynomials

$$
\begin{equation*}
\left\{\left(x^{2}+y^{2}\right)^{k}\right\}_{k=1}^{\infty} \tag{21}
\end{equation*}
$$

there exists a function

$$
\begin{equation*}
U(x, y)=x^{2}+y^{2}+\sum_{k=3}^{\infty} f_{k}(x, y) \tag{22}
\end{equation*}
$$

where $f_{k}(x, y)$ are homogeneous polynomials of degree $k$ in $x, y$, and such constants

$$
\begin{equation*}
L_{1}, L_{2}, \ldots, L_{k}, \ldots \tag{2}
\end{equation*}
$$

that the identity

$$
\begin{equation*}
\frac{d U}{d t}=\sum_{k=1}^{\infty} L_{k}\left(x^{2}+y^{2}\right)^{k+1} \tag{23}
\end{equation*}
$$

(with respect to $x$ and $y$ ) holds along the trajectories of the system

$$
\begin{equation*}
\dot{x}=y+\sum_{i=1}^{\ell} P_{m_{i}}(x, y), \dot{y}=-x+\sum_{i=1}^{\ell} Q_{m_{i}}(x, y) . \tag{24}
\end{equation*}
$$

The constants (2) are polynomials in coefficients of the system (24), and are called focal quantities.

We note that the algebra $S_{1, m_{1}, m_{2}, \ldots, m_{\ell}}$ for any differential system $s\left(1, m_{1}, m_{2}, \ldots, m_{\ell}\right)$, written in the form

$$
\begin{equation*}
\dot{x}=c x+d y+\sum_{i=1}^{\ell} P_{m_{i}}(x, y), \dot{y}=e x+f y+\sum_{i=1}^{\ell} Q_{m_{i}}(x, y) \tag{25}
\end{equation*}
$$

contains among its generators the polynomials

$$
\begin{equation*}
i_{1}=c+f, i_{2}=c^{2}+2 d e+f^{2}, k_{2}=-e x^{2}+(c-f) x y+d y^{2} \tag{26}
\end{equation*}
$$

which are given already in (11).
Remark 5. We note that the set

$$
\begin{equation*}
\mathcal{V}=\left\{i_{1}=c+f=0, \operatorname{Discr}\left(k_{2}\right)=2 i_{2}-i_{1}^{2}<0\right\} \tag{27}
\end{equation*}
$$

is a Sibirsky invariant variety for center and focus for the system (25), because the comitant $k_{2}$ from (26) through a real center-affine transformation of the plane $x O y$ can be brought to the form

$$
\begin{equation*}
x^{2}+y^{2} \tag{28}
\end{equation*}
$$

and the system (25) can be brought to the form (24) [5], for which the roots of the characteristic equation are imaginary, i.e. the origin of coordinates for this system is a singular point of the second type (center or focus).

Considering Remark 5 we have
Remark 6. Taking into account the comitant $k_{2}$ from (26) and the fact that its expression through a real center-affine transformation on the invariant variety $\mathcal{V}$ can be brought to the form (28), then formally this variety for the system (25) can be written as

$$
\begin{equation*}
\mathcal{V}=\{f=-c\} \cup\{c=0, d=-e=1\} \tag{29}
\end{equation*}
$$

## 7 Null focal pseudo-quantity of the system (25) and relations between the quantities $G_{k}$ and the focal quantities $L_{k}$ of the system (24)

Let us consider for the system (25) the identity

$$
\begin{equation*}
\left[c x+d y+\sum_{i=1}^{\ell} P_{m_{i}}(x, y)\right] \frac{\partial U}{\partial x}+\left[e x+f y+\sum_{i=1}^{\ell} Q_{m_{i}}(x, y)\right] \frac{\partial U}{\partial y}=\sum_{k=1}^{\infty} G_{k} k_{2}^{k+1} \tag{30}
\end{equation*}
$$

where

$$
\begin{equation*}
U(x, y)=k_{2}+\sum_{r=3}^{\infty} F_{r}(x, y) \tag{31}
\end{equation*}
$$

$\left(k_{2} \not \equiv 0\right.$ from (26)), which splits by powers of $x$ and $y$ into an infinite number of algebraic equations, where the variables are the coefficients of the homogeneous polynomials $F_{r}(x, y)$ of degree $r$ in $x, y$, and also the quantities $G_{1}, G_{2}, \ldots, G_{k}, \ldots$.

For any system (25) from the identity (30) with $k_{2}$ from (26) we find that the first three equations have the following form:

$$
x^{2}: e(c+f)=0, x y:(c-f)(c+f)=0, y^{2}: d(c+f)=0
$$

These equalities are equivalent to one of two sets of the conditions: 1) $c+f=0$; 2) $e=c-f=d=0$. Since $k_{2} \not \equiv 0$, then, according to (26), these conditions are equivalent to the condition $c+f=0$, which is contained in the variety $\mathcal{V}$ from (27).

In this way from Definition 3 and formulation of the center-focus problem for the system (24) we conclude: for $L_{k}$ from (2) and $G_{k}$ from (30) the following equalities take place:

$$
\begin{equation*}
L_{k}=G_{k} \mid \mathcal{V}(k=1,2, \ldots) \tag{32}
\end{equation*}
$$

where $\mathcal{V}$ is from (27).
Hereafter some concretizations for these equalities will be done.
From the above mentioned follows
Remark 7. The identity (30) with function (31) on the variety $\mathcal{V}$ from (27) guarantees that the system (25) has at the origin of coordinates a singular point of the second type (center or focus).

We denote the expression $c+f$, which is contained in the variety $\mathcal{V}$ from (27), by

$$
\begin{equation*}
G_{0} \equiv i_{1}=c+f \tag{33}
\end{equation*}
$$

and will call it the null focal pseudo-quantity. We note that $G_{0}$ from (33) is a centeraffine (unimodular) invariant of the system $s\left(1, m_{1}, m_{2}, \ldots, m_{\ell}\right)$ of the type

$$
(0,1, \underbrace{0, \ldots, 0}_{\ell}) .
$$

To get a more clear idea about the quantities $G_{1}, G_{2}, \ldots, G_{k}, \ldots$ from the identity (30) with the function (31), we write the remaining equations, in which this identity is
splitted by powers $x^{3}, x^{2} y, x y^{2}, y^{3}, \ldots$ without taking into consideration the equality $i_{1}=c+f=0$ on the variety $\mathcal{V}$.

To explain the further way of implementation of this scenario, we consider the identity (30) with unknown constants $G_{1}, G_{2}, \ldots$ for the example of the simplest differential system $s(1,2)$ with the quadratic nonlinearities

$$
\begin{align*}
& \dot{x}=c x+d y+g x^{2}+2 h x y+k y^{2} \\
& \dot{y}=e x+f y+l x^{2}+2 m x y+n y^{2} \tag{34}
\end{align*}
$$

with the finitely defined graded algebra of unimodular comitants $S_{1,2}[7]$. For this algebra we write the function (31) as

$$
\begin{gather*}
U(x, y)=k_{2}+a_{0} x^{3}+3 a_{1} x^{2} y+3 a_{2} x y^{2}+a_{3} y^{3}+b_{0} x^{4}+4 b_{1} x^{3} y+ \\
+6 b_{2} x^{2} y^{2}+4 b_{3} x y^{3}+b_{4} y^{4}+c_{0} x^{5}+5 c_{1} x^{4} y+10 c_{2} x^{3} y^{2}+ \\
+10 c_{3} x^{2} y^{3}+5 c_{4} x y^{4}+c_{5} y^{5}+d_{0} x^{6}+6 d_{1} x^{5} y+15 d_{2} x^{4} y^{2}+ \\
+20 d_{3} x^{3} y^{3}+15 d_{4} x^{2} y^{4}+6 d_{5} x y^{5}+d_{6} y^{6}+e_{0} x^{7}+7 e_{1} x^{6} y+  \tag{35}\\
+21 e_{2} x^{5} y^{2}+35 e_{3} x^{4} y^{3}+21 e_{5} x^{2} y^{5}+7 e_{6} x y^{6}+e_{7} y^{7}+f_{0} x^{8}+ \\
+8 f_{1} x^{7} y+28 f_{2} x^{6} y^{2}+56 f_{3} x^{5} y^{3}+70 f_{4} y^{4}+56 f_{5} x^{3} y^{5}+ \\
\quad+28 f_{6} x^{2} y^{6}+8 f_{7} x y^{7}+f_{8} y^{8}+\ldots,
\end{gather*}
$$

where $k_{2}$ is from (26) and $a_{0}, a_{1}, \ldots, f_{7}, f_{8}, \ldots$ are unknown constants. Then without taking into consideration the variety $\mathcal{V}$, the identity (30) along the trajectories of the system (34) with the function (35) splits into the following systems of equations

$$
\begin{align*}
x^{3}: & 3 c a_{0}+3 e a_{1}=2 e g-(c-f) l, \\
x^{2} y: & 3 d a_{0}+3(2 c+f) a_{1}+6 e a_{2}=(f-c)(g+2 m)-2 d l+4 e h, \\
x y^{2}: & 6 d a_{1}+3(2 f+c) a_{2}+3 e a_{3}=(f-c)(2 h+n)+2 e k-4 d m,  \tag{36}\\
y^{3}: & 3 d a_{2}+3 f a_{3}=(f-c) k-2 d n ; \\
x^{4}: & 4 c b_{0}+4 e b_{1}-e^{2} G_{1}=-3 g a_{0}-3 l a_{1}, \\
x^{3} y: & 4 d b_{0}+4(f+3 c) b_{1}+12 e b_{2}+2 e(c-f) G_{1}=-6 h a_{0}- \\
& -6(g+m) a_{1}-6 l a_{2}, \\
x^{2} y^{2}: & 12 d b_{1}+12(c+f) b_{2}+12 e b_{3}+\left[2 d e-(c-f)^{2}\right] G_{1}= \\
& =-3 k a_{0}-3(4 h+n) a_{1}-3(g+4 m) a_{2}-3 l a_{3},  \tag{37}\\
x y^{3}: & 12 b d_{2}+4(3 f+c) b_{3}+4 e b_{4}+2 d(f-g) G_{1}=-6 k a_{1}- \\
& -6(h+n) a_{2}-6 m a_{3}, \\
y^{4}: & 4 d b_{3}+4 f b_{4}-d^{2} G_{1}=-3 k a_{2}-3 n a_{3} ; \\
x^{5}: & 5 c c_{0}+5 e c_{1}=-4 g b_{0}-4 l b_{1}, \\
x^{4} y: & 5 d c_{0}+5(4 c+f) c_{1}+20 e c_{2}=-8 h b_{0}-4(3 g+2 m) b_{1}- \\
& -12 l b_{2},
\end{align*}
$$

$$
\begin{align*}
& x^{3} y^{2}: 20 d c_{1}+10(3 c+2 f) c_{2}+30 e c_{3}=-4 k b_{0}-4(6 h+n) b_{1}- \\
& -12(g+2 m) b_{2}-12 l b_{3}, \\
& x^{2} y^{3}: 30 d c_{2}+10(2 c+3 f) c_{3}+20 e c_{4}=-12 k b_{1}-12(2 h+ \\
& +n) b_{2}-4(g+6 m) b_{3}-4 l b_{4},  \tag{38}\\
& x y^{4}: 20 d c_{3}+5(c+4 f) c_{4}+5 e c_{5}=-12 k b_{2}-4(2 h+3 n) b_{3}- \\
& -8 m b_{4}, \\
& y^{5}: 5 d c_{4}+5 f c_{5}=-4 k b_{3}-4 n b_{4} ; \\
& x^{6}: 6 c d_{0}+6 e d_{1}+e^{3} G_{2}=-5 g c_{0}-5 l c_{1}, \\
& x^{5} y: 6 d d_{0}+6(5 c+f) d_{1}+30 e d_{2}+3 e^{2}(f-c) G_{2}=-10 h c_{0}- \\
& -10(2 g+m) c_{1}-20 l c_{2}, \\
& x^{4} y^{2}: 30 d d_{1}+30(2 c+f) d_{2}+60 e d_{3}+3 e\left[(c-f)^{2}-d e\right] G_{2}= \\
& =-5 k c_{0}-5(8 h+n) c_{1}-10(3 g+4 m) c_{2}-30 l c_{3}, \\
& x^{3} y^{3}: 60 d d_{2}+60(c+f) d_{3}+60 e d_{4}+(f-c)\left[(c-f)^{2}-\right.  \tag{39}\\
& -6 d e] G_{2}=-20 k c_{1}-20(3 h+n) c_{2}-20(g+3 m) c_{3}-20 l c_{4}, \\
& x^{2} y^{4}: 60 d d_{3}+30(c+2 f) d_{4}+30 e d_{5}+3 d\left[d e-(c-f)^{2}\right] G_{2}= \\
& =-30 k c_{2}-10(4 h+3 n) c_{3}-5(g+8 m) c_{4}-5 l c_{5}, \\
& x y^{5}: 30 d d_{4}+6(c+5 f) d_{5}+6 e d_{6}+3 d^{2}(f-c) G_{2}=-20 k c_{3}- \\
& -10(h+2 n) c_{4}-10 m c_{5}, \\
& y^{6}: 6 d d_{5}+6 f d_{6}-d^{3} G_{2}=-5 k c_{4}-5 n c_{5} ; \\
& x^{7}: 7 c e_{0}+7 e e_{1}=-6 g d_{0}-6 l d_{1}, \\
& x^{6} y: 7 d e_{0}+7(6 c+f) e_{1}+42 e e_{2}=-12 h d_{0}-6(5 g+2 m) d_{1}- \\
& -30 \text { ld }_{2} \text {, } \\
& x^{5} y^{2}: 42 d e_{1}+7(15 c+6 f) e_{2}+105 e e_{3}=-6 k d_{0}-6(10 h+ \\
& +n) d_{1}-60(g+m) d_{2}-60 l d_{3}, \\
& x^{4} y^{3}: 105 d e_{2}+5(28 c+21 f) e_{3}+140 e e_{4}=-30 k d_{1}-30(4 h+ \\
& +n) d_{2}-60(g+2 m) d_{3}-60{ }^{2} d_{4}, \\
& x^{3} y^{4}: 140 d e_{3}+35(3 c+4 f) e_{4}+105 e e_{5}=-60 k d_{2}-60(2 h+  \tag{40}\\
& +n) d_{3}-30(g+4 m) d_{4}-30 l d_{5}, \\
& x^{2} y^{5}: 105 d e_{4}+7(6 c+15 f) e_{5}+42 e e_{6}=-60 k d_{3}-60(h+ \\
& +n) d_{4}-6(g+10 m) d_{5}-6 l d_{6}, \\
& x y^{6}: 42 d e_{5}+7(c+6 f) e_{6}+7 e e_{7}=-30 k d_{4}-6(2 h+5 n) d_{5}- \\
& -12 m d_{6}, \\
& y^{7}: 7 d e_{6}+7 f e_{7}=-6 k d_{5}-6 n d_{6} ; \\
& x^{8}: 8 c f_{0}+8 e f_{1}-e^{4} G_{3}=-7 g e_{0}-7 l e_{1},
\end{align*}
$$

$$
\begin{align*}
x^{7} y: & 8 d f_{0}+8(7 c+f) f_{1}+56 e f_{2}+4 e^{3}(c-f) G_{3}= \\
& =-14 h e_{0}-14(3 g+m) e_{1}-42 l e_{2}, \\
x^{6} y^{2}: & 56 d f_{1}+56(3 c+f) f_{2}+168 e f_{3}+2 e^{2}\left[2 d e-3(c-f)^{2}\right] G_{3}= \\
& =-7 k e_{0}-7(12 h+n) e_{1}-21(5 g+4 m) e_{2}-105 l e_{3}, \\
x^{5} y^{3}: & 168 d f_{2}+56(5 c+3 f) f_{3}+280 e f_{4}+4 e(f-c)[3 d e-(c- \\
& \left.-f)^{2}\right] G_{3}=-42 k e_{1}-42(5 h+n) e_{2}-70(2 g+3 m) e_{3}-140 l e_{4}, \\
x^{4} y^{4}: & 280 d f_{3}+280(c+f) f_{4}+280 e f_{5}+\left[12 d e(c-f)^{2}-6 d^{2} e^{2}-\right. \\
& \left.-(c-f)^{4}\right] G_{3}=-105 k e_{2}-35(8 h+3 n) e_{3}-35(3 g+  \tag{41}\\
& +8 m) e_{4}-105 l e_{5}, \\
x^{3} y^{5}: & 280 d f_{4}+56(3 c+5 f) f_{5}+168 e f_{6}+4 d(f-c)\left[(c-f)^{2}-\right. \\
& -3 d e] G_{3}=-140 k e_{3}-70(3 h+2 n) e_{4}-42(g+5 m) e_{5}-42 l e_{6}, \\
x^{2} y^{6}: & 168 d f_{5}+56(c+3 f) f_{6}+56 e f_{7}+2 d^{2}\left[2 d e-3(c-f)^{2}\right] G_{3}=- \\
& -105 k e_{4}-21(4 h+5 n) e_{5}-7(g+12 m) e_{6}-7 l e_{7}, \\
x y^{7}: & 56 d f_{6}+8(c+7 f) f_{7}+8 e f_{8}+4 d^{3}(f-c) G_{3}=-42 k e_{5}- \\
& -14(h+3 n) e_{6}-14 m e_{7}, \\
y^{8}: & 78 d f_{7}+8 f f_{8}-d^{4} G_{3}=-7 k e_{6}-7 n e_{7} .
\end{align*}
$$

It is evident that the linear systems of equations (36)-(41) in variables $a_{0}, a_{1}, a_{2}, a_{3}$, $b_{0}, b_{1}, \ldots, b_{4}, c_{0}, c_{1}, \ldots, c_{5}, d_{0}, d_{1}, \ldots, d_{6}, e_{0}, e_{1}, \ldots, e_{7}, f_{0}, f_{1}, \ldots, f_{8}, \ldots, G_{1}, G_{2}, G_{3}, \ldots$ can be extended by adding, after the last equation from (41), an infinite number of equations, obtained from the equality of coefficients of $x^{\alpha} y^{\beta}$ for $\alpha+\beta>8$ in the identity (30) along the trajectories of the system (34).

## 8 Determining the quantities $G_{1}, G_{2}, G_{3}$ from the systems (36)-(41) and the corresponding focal quantities

To obtain the quantity $G_{1}$ we write the equations (36)-(37) in the matrix form

$$
\begin{equation*}
A_{1} B_{1}=C_{1}, \tag{42}
\end{equation*}
$$

where

$$
A_{1}=\left(\begin{array}{cccccccccc}
3 c & 3 e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 d & 3(2 c+f) & 6 e & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 6 d & 3(2 c+f) & 3 e & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 d & 3 f & 0 & 0 & 0 & 0 & 0 & 0 \\
3 g & 3 l & 0 & 0 & 4 c & 4 e & 0 & 0 & 0 & -e^{2} \\
6 h & 6(g+m) & 6 l & 0 & 4 d & 4(f+3 c) & 12 e & 0 & 0 & 2 e(c-f) \\
3 k & 3(4 h+n) & 3(g+4 m) & 3 l & 0 & 12 d & 12(c+f) & 12 e & 0 & 2 d e-(g)^{2} \\
0 & 6 k & 6(h+n) & 6 m & 0 & 0 & 12 d & 4(3 f+c) & 4 l & 2 d(f-c) \\
0 & 0 & 3 k & 3 n & 0 & 0 & 0 & 4 d & 4 f & -d^{2}
\end{array}\right),
$$

$$
B_{1}=\left(\begin{array}{c}
a_{0}  \tag{43}\\
a_{1} \\
a_{2} \\
a_{3} \\
b_{0} \\
b_{1} \\
b_{2} \\
b_{3} \\
b_{4} \\
G_{1}
\end{array}\right), \quad C_{1}=\left(\begin{array}{c}
2 e g+(f-c) l \\
(f-c)(g+2 m)-2 d l+4 e h \\
(f-c)(2 h+n)+3 e k-4 d m \\
(f-c) k-2 d n \\
0 \\
0 \\
0 \\
0 \\
0
\end{array}\right)
$$

Sice the dimension of the matrix $A_{1}$ is $9 \times 10$, clearly we have at least one free parameter. Therefore choosing as a free parameter $b_{i}(i \in\{0,1, \ldots, 4\})$ with the help of Cramer's rule from the system (42) for each fixed $i$ we obtain

$$
\begin{equation*}
G_{1}=\frac{G_{1, i}+B_{1, i} b_{i}}{\sigma_{1, i}} \tag{44}
\end{equation*}
$$

where $G_{1, i}, B_{1, i}, \sigma_{1, i}$ are polynomials in the coefficients of the system (34), and $b_{i}$ are undetermined coefficients of the function $U(x, y)$ from (35).

By studying the matrices (43) of the system (42) we conclude that $G_{1, i}$ from (44) are homogeneous polynomials of degree 8 with respect to the linear part, and of degree 2 with respect to the quadratic part of the system (34).

Because $G_{1, i}$ from (44) are homogeneous polynomials in the coefficients of the system (34), then, according to $[6,23]$, for $i=0,1,2,3,4$ we can determine respectively and isobarity

$$
(3,-1),(2,0),(1,1),(0,2),(-1,3)
$$

According to the formula (5) (for the system (34) and the theory of invariants of differential systems $[5,6]$ ) it suggests that the numerators of the fractions (44) can be coefficients in comitants of the weight -1 of the type $(4,8,2)$. This means that according to (20) with the help of the Lie differential operator $D_{3}$ for the system (34) from [7] and the numerator of the fraction (44) we obtain a redefined system of four linear non-homogeneous differential equations

$$
\begin{align*}
& D_{3}\left(G_{1,0}+B_{1,0} b_{0}\right)=G_{1,1}+B_{1,1} b_{1}, \quad D_{3}\left(G_{1,1}+B_{1,1} b_{1}\right)=-G_{1,2}-B_{1,2} b_{2}, \\
& -D_{3}\left(G_{1,2}+B_{1,2} b_{2}\right)=G_{1,3}+B_{1,3} b_{3}, \quad D_{3}\left(G_{1,3}+B_{1,3} b_{3}\right)=-G_{1,4}-B_{1,4} b_{4} \tag{45}
\end{align*}
$$

with five unknowns $b_{0}, b_{1}, b_{2}, b_{3}, b_{4}$. We can note that a particular solution to this system is $b_{0}=b_{1}=b_{2}=b_{3}=b_{4}=0$, for which the polynomial

$$
\begin{equation*}
f_{4}^{\prime}(x, y)=G_{1,0} x^{4}+4 G_{1,1} x^{3} y+2 G_{1,2} x^{2} y^{2}+4 G_{1,3} x y^{3}+G_{1,4} y^{4} \tag{46}
\end{equation*}
$$

is a center-affine comitant of the system (34). This fact is also confirmed by Theorem 1 with the operators $X_{1}-X_{4}$ from [7] for the system (34), for which

$$
X_{1}\left(f_{4}^{\prime}\right)=X_{4}\left(f_{4}^{\prime}\right)=f_{4}^{\prime}, \quad X_{2}\left(f_{4}^{\prime}\right)=X_{3}\left(f_{4}^{\prime}\right)=0
$$

Similarly, one can see that another particular solution for the system (45) is given by the following expressions:

$$
\begin{aligned}
b_{0} & =\frac{-e\left(g^{2}+2 h l+m^{2}\right)}{3 c^{2}-4 d e+10 c f+3 f^{2}} \\
b_{1} & =\frac{(c-f)\left(g^{2}+2 h l+m^{2}\right)-2 e(g h+k l+h m+m n)}{4\left(3 c^{2}-4 d e+10 c f+3 f^{2}\right)} \\
b_{2} & =\frac{2(c-f)(g h+k l+h m+m n)-e\left(h^{2}+2 k m+n^{2}\right)+d\left(g^{2}+2 h l+m^{2}\right)}{6\left(3 c^{2}-4 d e+10 c f+3 f^{2}\right)} \\
b_{3} & =\frac{(c-f)\left(h^{2}+2 k m+n^{2}\right)+2 d(g h+k l+h m+m n)}{4\left(3 c^{2}-4 d e+10 c f+3 f^{2}\right)} \\
b_{4} & =\frac{d\left(h^{2}+2 k m+n^{2}\right)}{3 c^{2}-4 d e+10 c f+3 f^{2}}
\end{aligned}
$$

whose denominators are different from zero on the variety $\mathcal{V}$ from (27). They define the center-affine comitant

$$
\begin{align*}
& f_{4}^{\prime \prime}(x, y)=\left(G_{1,0}+B_{1,0} b_{0}\right) x^{4}+4\left(G_{1,1}+B_{1,1} b_{1}\right) x^{3} y+2\left(G_{1,2}+\right. \\
& \left.\quad+B_{1,2} b_{2}\right) x^{2} y^{2}+4\left(G_{1,3}+B_{1,3} b_{3}\right) x y^{3}+\left(G_{1,4}+B_{1,4} b_{4}\right) y^{4} \tag{47}
\end{align*}
$$

It is evident that the differential system (45) has an infinite number of solutions $b_{0}, b_{1}, b_{2}, b_{3}, b_{4}$, which define center-affine comitants of the type (47).

In view of the above, the comitants (46)-(47) belong to the space

$$
S_{1,2}^{(4,8,2)}
$$

Remark that the comitants (46)-(47) on the variety $\mathcal{V}$ from (27) for the system (34) have the following form:

$$
\begin{equation*}
\left.f_{4}^{\prime}(x, y)\right|_{\mathcal{V}}=\left.f_{4}^{\prime \prime}(x, y)\right|_{\mathcal{V}}=-8 L_{1}\left(x^{2}+y^{2}\right) \tag{48}
\end{equation*}
$$

where

$$
L_{1}=\frac{1}{2}[g(l-h)-k(h+n)+m(l+n)]
$$

is the first focal quantity of the system (34) on the invariant variety $\mathcal{V}$ (see [4, p. 110]).
Similarly to the previous case, for determining the quantity $G_{2}$ we write the equations (36)-(39) in the matrix form

$$
\begin{equation*}
A_{2} B_{2}=C_{2} \tag{49}
\end{equation*}
$$

from which we find

$$
\begin{equation*}
G_{2}=\frac{G_{2, i, j}+B_{2, i, j} b_{i}+D_{2, i, j} d_{j}}{\sigma_{2, i, j}},(i=\overline{0,4}, j=\overline{0,6}) \tag{50}
\end{equation*}
$$

By studying the matrix equality (49) we obtain that $\operatorname{deg} G_{2, i, j}=24$, and using the system (36)-(39) we obtain that $G_{2, i, j}$ from (50) has the type ( $0,20,4$ ), i. e. $G_{2, i, j}$
are homogeneous polynomials of degree 20 in coefficients of the linear part and of degree 4 in coefficients of the quadratic part of the system $s(1,2)$ from (34).

Computing the expressions $G_{2, i, j}$ for each $i=\overline{0,4}$ and $j=\overline{0,6}$, according to $[6,23]$, we obtain for their isobarity the following table:

Table 1

| $G_{2, i, j}$ | $d_{0}$ | $d_{1}$ | $d_{2}$ | $d_{3}$ | $d_{4}$ | $d_{5}$ | $d_{6}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $b_{0}$ | $(7,-3)$ | $(6,-2)$ | $(5,-1)$ | $(4,0)$ | $(3,1)$ | $(2,2)$ | $(1,3)$ |
| $b_{1}$ | $(6,-2)$ | $(5,-1)$ | $(4,0)$ | $(3,1)$ | $(2,2)$ | $(1,3)$ | $(0,4)$ |
| $b_{2}$ | $(5,-1)$ | $(4,0)$ | $(3,1)$ | $(2,2)$ | $(1,3)$ | $(0,4)$ | $(-1,5)$ |
| $b_{3}$ | $(4,0)$ | $(3,1)$ | $(2,2)$ | $(1,3)$ | $(0,4)$ | $(-1,5)$ | $(-2,6)$ |
| $b_{4}$ | $(3,1)$ | $(2,2)$ | $(1,3)$ | $(0,4)$ | $(-1,5)$ | $(-2,6)$ | $(-3,7)$ |

By studying the isobarity of $G_{2, i, j}$ top-down for each line of this table, according to the theory of invariants of differential systems [5,6], we find that the numerators of the fraction (50) can be coefficients in center-affine comitants with the corresponding weights $-3,-2,-1,0,1$. Using these weights and the formula (5) for the system (34), as well as the fact that $G_{2, i, j}$ have the type $(0,20,4)$, we obtain that the mentioned comitants correspond to the types

$$
\begin{equation*}
(10,20,4),(8,20,4),(6,20,4),(4,20,4),(2,20,4) \tag{51}
\end{equation*}
$$

As the quantity $G_{2}$ in (30) is the coefficient in front of the homogeneity of degree 6 in the phase variables $x$ and $y$, then it is logical to choose from (51) the type

$$
\begin{equation*}
(6,20,4) \tag{52}
\end{equation*}
$$

which corresponds to the expression $G_{2,2, j}(j=\overline{0,6})$ in Table 1.
This means that according to (20) using the Lie differential operator $D_{3}$ for the system (34) from [7] and the numerator of the fraction (50) for fixed $i=2$, we obtain one redefined system of six linear non-homogeneous differential equations

$$
\begin{gather*}
D_{3}\left(G_{2,2,0}+B_{2,2,0} b_{0}+D_{2,2,0} d_{0}\right)=-\left(G_{2,2,1}+B_{2,2,1} b_{1}+D_{2,2,1} d_{1}\right), \\
-D_{3}\left(G_{2,2,1}+B_{2,2,1} b_{1}+D_{2,2,1} d_{1}\right)=G_{2,2,2}+B_{2,2,2} b_{2}+D_{2,2,2} d_{2}, \\
D_{3}\left(G_{2,2,2}+B_{2,2,2} b_{2}+D_{2,2,2} d_{2}\right)=-\left(G_{2,2,3}+B_{2,2,3} b_{3}+D_{2,2,3} d_{3}\right), \\
-D_{3}\left(G_{2,2,3}+B_{2,2,3} b_{3}+D_{2,2,3} d_{3}\right)=G_{2,2,4}+B_{2,2,4} b_{4}+D_{2,2,4} d_{4},  \tag{53}\\
D_{3}\left(G_{2,2,4}+B_{2,2,4} b_{4}+D_{2,2,4} d_{4}\right)=-\left(G_{2,2,5}+B_{2,2,5} b_{5}+D_{2,2,5} d_{5}\right), \\
-D_{3}\left(G_{2,2,5}+B_{2,2,5} b_{5}+D_{2,2,5} d_{5}\right)=G_{2,2,6}+B_{2,2,6} b_{6}+D_{2,2,6} d_{6},
\end{gather*}
$$

with eight unknowns $b_{2}, d_{0}, d_{1}, \ldots, d_{6}$. From these six equations it results that the expressions contained in them can be coefficients in comitants of the type $(6,20,4)$. Observe that obtaining an explicit form for solutions of the system (53) is a difficult task. We will show the importance of homogeneities of $G_{2,2, j}$ from (50) in obtaining the focal quantities for the system (34) on the invariant variety $\mathcal{V}$ for center and
focus from (27). According to (52) the system (53) defines center-affine comitants belonging to the space

$$
\begin{equation*}
S_{1,2}^{(6,20,4)} \tag{54}
\end{equation*}
$$

According to (20) and (53) such a comitant, belonging to this space, can be written as

$$
\begin{gathered}
f_{6}^{\prime}(x, y)=\left(G_{2,2,0}+B_{2,2,0} b_{2}+D_{2,2,0} d_{0}\right) x^{6}-\left(G_{2,2,1}+B_{2,2,1} b_{2}+D_{2,2,1} d_{1}\right) x^{5} y+ \\
+\frac{1}{2!}\left(G_{2,2,2}+B_{2,2,2} b_{2}+D_{2,2,2} d_{2}\right) x^{4} y^{2}-\frac{1}{3!}\left(G_{2,2,3}+B_{2,2,3} b_{2}+D_{2,2,3} d_{3}\right) x^{3} y^{3}+ \\
+\frac{1}{4!}\left(G_{2,2,4}+B_{2,2,4} b_{2}+D_{2,2,4} d_{4}\right) x^{2} y^{4}-\frac{1}{5!}\left(G_{2,2,5}+B_{2,2,5} b_{2}+D_{2,2,5} d_{5}\right) x y^{5}+ \\
+\frac{1}{6!}\left(G_{2,2,6}+B_{2,2,6} b_{2}+D_{2,2,6} d_{6}\right) y^{6}
\end{gathered}
$$

Observe that on the variety $\mathcal{V}$ from (27) for the system (34) the expressions $G_{2,2, j}(j=$ $\overline{0,6}$ ) have the following expressions:

$$
\begin{gather*}
G_{2,2,0}\left|\mathcal{V}=G_{2,2,2}\right| \mathcal{V}=G_{2,2,4}\left|\mathcal{V}=G_{2,2,6}\right| \mathcal{V}=-2304 L_{2}  \tag{55}\\
G_{2,2,1}\left|\mathcal{V}=G_{2,2,3}\right| \mathcal{V}=G_{2,2,5} \mid \mathcal{V}=0
\end{gather*}
$$

where

$$
\begin{gathered}
24 L_{2}=62 g^{3} h-2 g h^{3}+95 g^{2} h k-2 h^{3} k+38 g h k^{2}+5 h k^{3}-62 g^{3} l+ \\
+27 g h^{2} l-39 g^{2} k l+29 h^{2} k l-15 g k^{2} l-8 g h l^{2}+15 h k l^{2}-5 g l^{3}+ \\
+53 g^{2} h m+66 g h k m+13 h k^{2} m-127 g^{2} l m-6 h^{2} l m-68 g k l m- \\
-15 k^{2} l m-13 h l^{2} m-5 l^{3} m+6 g h m^{2}+6 h k m^{2}-63 g l m^{2}-29 k l m^{2}+ \\
+2 l m^{3}+6 g^{3} n+61 g h^{2} n+72 g^{2} k n+63 h^{2} k n+33 g k^{2} n+5 k^{3} n- \\
-10 g h l n+68 h k l n-33 g l^{2} n+15 k l^{2} n-72 g^{2} m n-6 h^{2} m n+ \\
+10 g k m n+8 k^{2} m n-66 h l m n-38 l^{2} m n-61 g m^{2} n-27 k m^{2} n+ \\
+2 m^{3} n+72 g h n^{2}+127 h k n^{2}-72 g l n^{2}+39 k n^{2}-53 h m n^{2}- \\
-95 l m n^{2}-6 g n^{3}+62 k n^{3}-62 m n^{3}
\end{gathered}
$$

is the second focal quantity of the system (34) on the invariant variety $\mathcal{V}$ for center and focus (see [4, p. 110]).

Now we concentrate our attention to the construction of the quantity $G_{3}$ which is in front of the homogeneity of degree 8 in $x$ and $y$ in (50). Writing the system (36)-(41) in the matrix form

$$
A_{3} B_{3}=C_{3}
$$

we obtain

$$
\begin{equation*}
G_{3}=\frac{G_{3, i, j, k}+B_{3, i, j, k} b_{i}+D_{3, i, j, k} d_{j}+F_{3, i, j, k} f_{k}}{\sigma_{3, i, j, k}}, \quad(i=\overline{0,4} ; j=\overline{0,6} ; k=\overline{0,8}) \tag{56}
\end{equation*}
$$

Similarly to the previous case, we choose a comitant of the weight -1 of the system $s(1,2)$ from (34) which contains as semi-invariant the expression $G_{3,2, j, k}+$ $+B_{3,2, j, k} b_{2}+D_{3,2, j, k} d_{j}+F_{3,2, j, k} f_{k}(k=\overline{0,8})$, and we find that it belongs to the space

$$
S_{1,2}^{(8,37,6)}
$$

## 9 General type of comitants which have as coefficients expressions with generalized focal pseudo-quantities of the system (34)

Let's consider the extension of the system (36)-(41) obtained from the identity (30) for the system (34) and the function (35) which contains the quantity $G_{k}$, which we write in a matrix form as follows $A_{k} B_{k}=C_{k}$. We denote by $m_{G_{k}}$ the number of equations and by $n_{G_{k}}$ the number of unknowns of this system. Observe that these numbers can be written as

$$
\begin{gathered}
m_{G_{k}}=\underbrace{4+5}_{G_{1}}+\underbrace{6+7}_{G_{2}}+\underbrace{8+9}_{G_{3}}+\cdots+\underbrace{(2 k+2)+(2 k+3)}_{G_{k}},(k=1,2,3, \ldots), \\
n_{G_{k}}=\underbrace{4+6}_{G_{1}}+\underbrace{6+8}_{G_{2}}+\underbrace{8+10}_{G_{3}}+\cdots+\underbrace{(2 k+2)+(2 k+4)}_{G_{k}} .
\end{gathered}
$$

Hence we obtain

$$
\begin{equation*}
m_{G_{k}}=k(2 k+7), n_{G_{k}}=m_{G_{k}}+k>m_{G_{k}} . \tag{57}
\end{equation*}
$$

Similarly to the previous cases, from this system we have

$$
\begin{equation*}
G_{k}=\frac{G_{k, i_{1}, i_{2}, \ldots, i_{k}}+B_{k, i_{1}, i_{2}, \ldots, i_{k}} b_{i_{1}}+\cdots+Z_{k, i_{1}, i_{2}, \ldots, i_{k}} z_{i_{k}}}{\sigma_{k, i_{1}, i_{2}, \ldots, i_{k}}} \tag{58}
\end{equation*}
$$

Now it is important to determine the degree of the polynomial $G_{k, i_{1}, i_{2}, \ldots, i_{k}}$ in coefficients of the differential system (34).

Observe that the degree of non-zero polynomial coefficient of $G_{i}(i=\overline{1, k)}$ in coefficients of the system (34) in the matrix of Cramer's determinant of the order $m_{G_{k}}$, when the column corresponding to the last quantity $G_{k}$ is replaced with the column corresponding to free members, forms the following diagram (the last quantity $G_{k}$ has the degree 2 according to the substitution):

$$
\begin{array}{ccccc}
G_{1}, & G_{2}, & G_{3}, \ldots, & G_{k-1}, & G_{k} . \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
2 & 3 & 4 & k & 2
\end{array}
$$

Then the degree of the polynomial $G_{k, i_{1}, i_{2}, \ldots, i_{k}}$ in coefficients of the system (34), denoted by $N_{G_{k}}$, can be written as

$$
N_{G_{k}}=m_{G_{k}}-k+\frac{k(k+1)}{2}+1,
$$

hence according to (57) we have

$$
\begin{equation*}
N_{G_{k}}=\frac{1}{2}\left(5 k^{2}+13 k+2\right) . \tag{59}
\end{equation*}
$$

It is the degree of homogeneity of $G_{k, i_{1}, i_{2}, \ldots, i_{k}}$ in coefficients of the linear and the quadratic parts of the differential system (34) which is contained in a polynomial of the type $(d)=\left(\delta, d_{1}, d_{2}\right)$. Since $\delta=2(k+1)$ and $d_{2}=2 k$, then $d_{1}=N_{G_{k}}-2 k$. So we obtain that a comitant of the weight -1 of the system $s(1,2)$ from (34), containing the semi-invariant $G_{k, i_{1}, i_{2}, \ldots, i_{k}}+B_{k, i_{1}, i_{2}, \ldots, i_{k}} b_{i_{1}}+\cdots+Z_{k, i_{1}, i_{2}, \ldots, i_{k}} z_{i_{k}}$, which corresponds to the quantity $G_{k}$ for $k=1,2,3, \ldots$, belongs to the type

$$
\begin{equation*}
\left(2(k+1), \frac{1}{2}\left(5 k^{2}+9 k+2\right), 2 k\right) \tag{60}
\end{equation*}
$$

where $2(k+1)$ is the degree of homogeneity of the comitant in phase variables $x, y$; $\frac{1}{2}\left(5 k^{2}+9 k+2\right)$ is the degree of homogeneity of the comitant in coefficients of the linear part $c, d, e, f$ and $2 k$ is the degree of homogeneity of the comitant in coefficients of the quadratic part of the system (34).

Hereafter the expressions $G_{k, i_{1}, i_{2}, \ldots, i_{k}}$, which determine the types of comitants (60) corresponding to the quantity $G_{k}(k=1,2,3, \ldots)$ will be called the defining focal quantities. The comitants of the type (60) for $k=1,2,3, \ldots$ which contains as the coefficients expressions with the generalized focal pseudo-quantities

$$
G_{k, i_{1}, i_{2}, \ldots, i_{k}}+B_{k, i_{1}, i_{2}, \ldots, i_{k}} b_{i_{1}}+\cdots+Z_{k, i_{1}, i_{2}, \ldots, i_{k}} z_{i_{k}} .
$$

will be called the comitants associated to generalized focal pseudo-quantities.
For $G_{0}$ from (32), which for the system $s(1,2)$ from (34) has the type $(0,1,0)$, we retain the name a null focal pseudo-quantity.

The space of comitants of the system $s(1,2)$ from (34), corresponding to the type (60), will be denoted by

$$
\begin{equation*}
S_{1,2}^{\left(2(k+1), \frac{1}{2}\left(5 k^{2}+9 k+2\right), 2 k\right)} . \tag{61}
\end{equation*}
$$

## 10 Comitants which have as coefficients expressions with generalized focal pseudo-quantities of the system $s(1,2,3)$

Let us consider the system $s(1,2,3)$ of the form

$$
\begin{align*}
& \dot{x}=c x+d y+g x^{2}+2 h x y+k x^{2}+p x^{3}+3 q x^{2} y+3 r x y^{2}+s y^{3}, \\
& \dot{y}=e x+f y+l x^{2}+2 m x y+n y^{2}+t x^{3}+3 u x^{2} y+3 c x y^{2}+w y^{3} \tag{62}
\end{align*}
$$

with finitely determined Sibirsky graded algebra of unimodular comitants $S_{1,2,3}$ [7], for which the function (31) will be write in the form (35), where $K_{2}$ is from (26) and $a_{0}, a_{1}, \ldots, f_{7}, f_{8}, \ldots, G_{1}, G_{2}, \ldots$ are unknowns. Similarly as in the Sections 6 and 7 for
determining the quantity $G_{1}$ we write the equations in which splits the identity (30) in the case of the system (62) in the matrix form

$$
\begin{equation*}
\widetilde{A}_{1} \widetilde{B}_{1}=\widetilde{C}_{1} \tag{63}
\end{equation*}
$$

where

$$
\begin{align*}
& \widetilde{A}_{1}=\left(\begin{array}{cccccccccc}
3 c & 3 e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 d & 6 c+3 f & 6 e & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 6 d & 3 c+6 f & 3 e & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 d & 3 f & 0 & 0 & 0 & 0 & 0 & 0 \\
3 g & 3 l & 0 & 0 & 4 c & 4 e & 0 & 0 & 0 & -e^{2} \\
6 h & 6 g+6 m & 6 l & 0 & 4 d & 12 c+4 f & 12 e & 0 & 0 & 2 c e-2 e f \\
3 k & 12 h+3 n & 3 g+12 m & 3 l & 0 & 12 d & 12 c+12 f & 12 e & 0 & -c^{2}+2 d e+2 c f-f^{2} \\
0 & 6 k & 6 h+6 n & 6 m & 0 & 0 & 12 d & 4 c+12 f & 4 e & -2 c d+2 d f f \\
0 & 0 & 3 k & 3 n & 0 & 0 & 0 & 4 d & 4 f & -d^{2}
\end{array}\right), \\
& \widetilde{B}_{1}=\left(\begin{array}{c}
a_{0} \\
a_{1} \\
a_{2} \\
a_{3} \\
b_{0} \\
b_{1} \\
b_{2} \\
b_{3} \\
b_{4} \\
G_{1}
\end{array}\right), \quad \widetilde{C}_{1}=\left(\begin{array}{c}
2 e g-c l+f l \\
-c g+f g+4 e h-2 d l-2 c m+2 f m \\
-2 c h+2 f h+2 e k-4 d m-c n+f n \\
-c k+f k-2 d n \\
2 e p-c t+f t \\
-c p+f p+6 e q-2 d t-3 c u+3 f u \\
-3 c q+3 f q+6 e r-6 d u-3 c v+3 f v \\
-3 c r+3 f r+2 e s-6 d v-c w+f w \\
-c s+f s-2 d w
\end{array}\right) \tag{64}
\end{align*}
$$

For each fixed $i \in\{0,1, \ldots, 4\}$ using the Cramer's rule from the system (63) we find

$$
\begin{equation*}
\widetilde{G}_{1}=\frac{\widetilde{G}_{1, i}+\widetilde{B}_{1, i} b_{i}}{\widetilde{\sigma}_{1, i}} \tag{65}
\end{equation*}
$$

where $\widetilde{G}_{1, i}, \widetilde{B}_{1, i}, \widetilde{\sigma}_{1, i}$ are polynomials in the coefficients of the system (62) and $b_{i}$ are undetermined coefficients of the function $U(x, y)$ from (35).

By studying the matrices (63)-(64) of the system (62) we conclude that the focal pseudo-quantity $\widetilde{G}_{1, i}$ for fixed $i$ from (65) can be write as

$$
\begin{equation*}
\widetilde{G}_{1, i}=\widetilde{G}_{1, i}^{\prime}+\widetilde{G}_{1, i}^{\prime \prime}, \quad(i=0,1,2,3,4) \tag{66}
\end{equation*}
$$

where $\widetilde{G}_{1, i}^{\prime}$ (respectively $\widetilde{G}_{1, i}^{\prime \prime}$ ) are homogeneous polynomials of degree 8 (respectively $9)$ in coefficients of the linear part and of degree 2 in the coefficients of the quadratic part (respectively of degree 1 in the coefficients of the cubic part) of the differential system (62).

Using here the operators (18) of Lie algebra $L_{4}$ from [7] for the system (62) we construct the corresponding operators which we denote respectively by $\mathcal{X}_{1}, \mathcal{X}_{2}, \mathcal{X}_{3}, \mathcal{X}_{4}$. Applying these operators under the expressions from (66) we find

$$
\begin{aligned}
& \mathcal{X}_{1}\left(\tilde{f}_{4}^{\prime}\right)=\mathcal{X}_{4}\left(\tilde{f}_{4}^{\prime}\right)=\widetilde{f}_{4}^{\prime}, \quad \mathcal{X}_{2}\left(\tilde{f}_{4}^{\prime}\right)=\mathcal{X}_{3}\left(\tilde{f}_{4}^{\prime}\right)=0 \\
& \mathcal{X}_{1}\left(\widetilde{f}_{4}^{\prime \prime}\right)=\mathcal{X}_{4}\left(\tilde{f}_{4}^{\prime \prime}\right)=\widetilde{f}_{4}^{\prime \prime}, \quad \mathcal{X}_{2}\left(\tilde{f}_{4}^{\prime \prime}\right)=\mathcal{X}_{3}\left(\widetilde{f}_{4}^{\prime \prime}\right)=0
\end{aligned}
$$

where

$$
\begin{align*}
& \widetilde{f}_{4}^{\prime}(x, y)=\widetilde{G}_{1,0}^{\prime} x^{4}-4 \widetilde{G}_{1,1}^{\prime} x^{3} y+2 \widetilde{G}_{1,2}^{\prime} x^{2} y^{2}+4 \widetilde{G}_{1,3}^{\prime} x y^{3}-\widetilde{G}_{1,4}^{\prime} y^{4} \\
& \widetilde{f}_{4}^{\prime \prime}(x, y)=\widetilde{G}_{1,0}^{\prime \prime} x^{4}-4 \widetilde{G}_{1,1}^{\prime \prime} x^{3} y+2 \widetilde{G}_{1,2}^{\prime \prime} x^{2} y^{2}+4 \widetilde{G}_{1,3}^{\prime \prime} x y^{3}-\widetilde{G}_{1,4}^{\prime \prime} y^{4} \tag{67}
\end{align*}
$$

are comitants of the weight -1 of the system (62) and $\widetilde{G}_{1, i}^{\prime}, \widetilde{G}_{1, i}^{\prime \prime}$ are from (65).
According to the above mentioned and (4) the given comitants (67) belongs respectively to the linear spaces

$$
\begin{equation*}
S_{1,2,3}^{(4,8,2,0)}, S_{1,2,3}^{(4,9,0,1)} \tag{68}
\end{equation*}
$$

which are components of Sibirsky graded algebra of comitants $S_{1,2,3}$ for the system (62).

Taking into account (65) for $b_{i}=0(i=\overline{0,4})$ on the variety $\mathcal{V}$ from (27) and also (66), (67) we find out that the first focal quantity $L_{1}$ of the system (62) is related to the comitants (67) as follows

$$
\left.\left[\widetilde{f}_{4}^{\prime}(x, y)+\widetilde{f}_{4}^{\prime \prime}(x, y)\right]\right|_{\mathcal{V}}=8 L_{1}\left(x^{2}+y^{2}\right)^{2}
$$

where

$$
L_{1}=\frac{1}{4}\{[g(l-h)-k(h+n)+m(l+n)]-3[p+r+u+v]\}
$$

Similarly to the previous case, for determining the quantity $G_{2}$ for the system (62), from the identity (30) we obtain the following equation in the matrix form

$$
\begin{equation*}
\widetilde{A}_{2} \widetilde{B}_{2}=\widetilde{C}_{2} \tag{69}
\end{equation*}
$$

For each fixed $i \in\{0,1, \ldots, 4\}, j \in\{0,1,2, \ldots, 6\}$ we find the expression

$$
\begin{equation*}
\widetilde{G}_{2}=\frac{\widetilde{G}_{2, i, j}+\widetilde{B}_{2, i, j} b_{i}+\widetilde{D}_{2, i, j} d_{j}}{\widetilde{\sigma}_{2, i, j}} \tag{70}
\end{equation*}
$$

By studying the matrix equality (69) we find that the focal pseudo-quantity from (70) can be written in the form of homogeneity of degree 24 that can be represented in the form

$$
\begin{equation*}
\widetilde{G}_{2, i, j}=\widetilde{G}_{2, i, j}^{\prime}+\widetilde{G}_{2, i, j}^{\prime \prime}+\widetilde{G}_{2, i, j}^{\prime \prime \prime} \tag{71}
\end{equation*}
$$

where $\widetilde{G}_{2, i, j}^{\prime}, \quad \widetilde{G}_{2, i, j}^{\prime \prime}$ and $\widetilde{G}_{2, i, j}^{\prime \prime \prime}$, are homogeneity of the type (4) respectively of the form $(0,20,4,0),(0,21,2,1)$ and $(0,22,0,2)$. We note that on the variety $\mathcal{V}$ from (27) for the system (62) the quantities $\widetilde{G}_{2,2, j}(j=\overline{0,6})$ have the expressions

$$
\widetilde{G}_{2,2, j}\left|\mathcal{V}=2304 L_{2}, \quad(j=0,2,4,6), \quad \widetilde{G}_{2,2, j}\right| \mathcal{V}=0, \quad(j=1,3,5)
$$

On the other hand, the second focal quantity $L_{2}$ of the system (62) can be written with the terms from (71) as follows

$$
24 L_{2}=\widetilde{G}_{2,2, j}^{\prime}\left|\mathcal{V}+\widetilde{G}_{2,2, j}^{\prime \prime}\right| \mathcal{V}+\widetilde{G}_{2,2, j}^{\prime \prime \prime} \mid \mathcal{V},(j=0,2,4,6)
$$

where

$$
\begin{aligned}
& \widetilde{G}_{2,2, j}^{\prime} \mid \mathcal{V}=4\left(62 g^{3} h-2 g h^{3}+95 g^{2} h k-2 h^{3} k+38 g h k^{2}+5 h k^{3}-62 g^{3} l+27 g h^{2} l-\right. \\
& -39 g^{2} k l+29 h^{2} k l-15 g k^{2} l-8 g h l^{2}+15 h k l^{2}-5 g l^{3}+53 g^{2} h m+66 g h k m+ \\
& \quad+13 h k^{2} m-127 g^{2} l m-6 h^{2} l m-68 g k l m-15 k^{2} l m-13 h l^{2} m-5 l^{3} m+ \\
& +6 g h m^{2}+6 h k m^{2}-63 g l m^{2}-29 k l m^{2}+2 l m^{3}+6 g^{3} n+61 g h^{2} n+72 g^{2} k n+ \\
& +63 h^{2} k n+33 g k^{2} n+5 k^{3} n-10 g h l n+68 h k l n-33 g l^{2} n+15 k l^{2} n-72 g^{2} m n- \\
& -6 h^{2} m n+10 g k m n+8 k^{2} m n-66 h l m n-38 l^{2} m n-61 g m^{2} n-27 k m^{2} n+ \\
& +2 m^{3} n+72 g h n^{2}+127 h k n^{2}-72 g l n^{2}+39 k l n^{2}-53 h m n^{2}-95 l m n^{2}- \\
& \left.\quad-6 g n^{3}+62 k n^{3}-62 m n^{3}\right), \\
& \widetilde{G}_{2,2, j}^{\prime \prime} \mid \mathcal{V}=-2\left(186 g^{2} p+10 h^{2} p+117 g k p+45 k^{2} p+59 h l p+15 l^{2} p+159 g m p+\right. \\
& +75 k m p+18 m^{2} p+143 h n p+89 l n p+196 n^{2} p-69 g h q-57 h k q+69 g l q+ \\
& +12 k l q+9 l m q+60 g n q+3 k n q+21 m n q+168 g^{2} r-6 h^{2} r+69 g k r+15 k^{2} r+ \\
& +87 h l r+45 l^{2} r+123 g m r+39 k m r+18 m^{2} r+171 h n r+129 l n r+222 n^{2} r- \\
& \quad-13 g h s-17 h k s-15 g l s-16 h m s-15 l m s-16 g n s-17 k n s-19 m n s- \\
& \quad-19 g h t-15 h k t-17 g l t-16 h m t-17 l m t-16 g n t-15 k n t-13 m n t+ \\
& +222 g^{2} u+18 h^{2} u+129 g k u+45 k^{2} u+39 h l u+15 l^{2} u+171 g m u+87 k m u- \\
& -6 m^{2} u+123 h n u+69 l n u+168 n^{2} u+21 g h v+9 h k v+3 g l v+12 k l v-57 l m v+ \\
& +60 g n v+69 k n v-69 m n v+196 g^{2} w+18 h^{2} w+89 g k w+15 k^{2} w+75 h l w+ \\
& \left.+45 l^{2} w+143 g m w+59 k m w+10 m^{2} w+159 h n w+117 l n w+186 n^{2} w\right), \\
& \widetilde{G}_{2,2, j}^{\prime \prime \prime} \mid \mathcal{V}=-9(11 p q+15 q r-5 p s-r s+p t+5 r t+3 q u-5 s u+t u-7 p v-3 r v- \\
& \quad-15 u v+7 q w-s w+5 t w-11 v w) .
\end{aligned}
$$

Similarly to the technique described in the Sections 6 and 7 we choose a comitant of the weight -1 of the system $s(1,2,3)$ from (62) which contains as a semi-invariant the expression $\widetilde{G}_{2, i, j}+\widetilde{B}_{2, i, j} b_{i}+\widetilde{D}_{2, i, j} d_{j}$. According to the decomposition (71) and the types shown below we find that this comitant is a sum of comitants belonging to the spaces

$$
\begin{equation*}
S_{1,2,3}^{(6,20,4,0)}, S_{1,2,3}^{(6,21,2,1)}, S_{1,2,3}^{(6,22,0,2)} \tag{72}
\end{equation*}
$$

Following this process with the help of the matrix equation

$$
\widetilde{A}_{3} \widetilde{B}_{3}=\widetilde{C}_{3}
$$

for each fixed $i \in\{0,1, \ldots, 4\}, j \in\{0,1, \ldots, 6\}, k \in\{0,1, \ldots, 8\}$ we obtain

$$
\begin{equation*}
\widetilde{G}_{3}=\frac{\widetilde{G}_{3, i, j, k}+\widetilde{B}_{3, i, j, k} b_{i}+\widetilde{D}_{3, i, j, k} d_{j}+\widetilde{F}_{3, i, j, k} f_{j}}{\widetilde{\sigma}_{3, i, j, k}} \tag{73}
\end{equation*}
$$

Similarly to the previous case we find that the focal pseudo-quantity $\widetilde{G}_{3, i, j, k}$ splits into a sum of four terms of the same degree 43 in the coefficients of the system (62),
which according to (4) belongs to the types $(0,37,6,0),(0,38,4,1),(0,39,2,2)$ and $(0,40,0,3)$. Then it results that the comitant of the weight -1 having as a semiinvariant one of the expressions (73) consists of the sum of comitants of the system (62) which belongs to the spaces

$$
\begin{equation*}
S_{1,2,3}^{(8,37,6,0)}, S_{1,2,3}^{(8,38,4,1)}, S_{1,2,3}^{(8,39,2,2)}, S_{1,2,3}^{(8,40,0,3)} \tag{74}
\end{equation*}
$$

Following this process we obtain the sequence of linear spaces $(68),(72),(74)$ etc. of comitants of the system (62). It remain to underline that the generalized focal pseudo-quantities corresponding to $G_{k}$ of the given system is exactly a sum of coefficients of these comitants.

It is not difficult to deduce that the generic formula of the types of the comitants in which the generalized focal pseudo-quantities corresponding to $G_{k}$ splits as a sum, has the form:

$$
\left(2(k+1), \frac{1}{2}\left(5 k^{2}+9 k+2\right)+i, 2(k-i), i\right),(i=\overline{0, k})
$$

## 11 Graded algebra of comitants whose spaces contain comitants associated to generalized focal pseudo-quantities of the system (34) and (62)

Thus we obtain for the system (34) the set of spaces of center-affine (unimodular) comitants

$$
\begin{equation*}
\mathbb{R}=S_{1,2}^{(0,0,0)}, S_{1,2}^{(0,1,0)}, S_{1,2}^{(4,8,2)}, S_{1,2}^{(6,20,4)}, \ldots, S_{1,2}^{\left(2(k+1), \frac{1}{2}\left(5 k^{2}+9 k+2\right), 2 k\right)}, \ldots \subset S_{1,2} \tag{75}
\end{equation*}
$$

were $S_{1,2}$ is Sibirsky graded algebra of the system (34).
Let's consider the graded algebra $S_{1,2}^{\prime}$, generated by the space $S_{1,2}^{\left(\delta^{\prime}, d_{1}^{\prime}, d_{2}^{\prime}\right)}$ from (75), which can be written as

$$
\begin{equation*}
S_{1,2}^{\prime}=\bigoplus_{\left(d^{\prime}\right)} S_{1,2}^{\left(d^{\prime}\right)} \tag{76}
\end{equation*}
$$

Here $S_{1,2}^{\left(d^{\prime}\right)}$ denote linear spaces, contained in $S_{1,2}^{\left(\delta^{\prime}, d_{1}^{\prime}, d_{2}^{\prime}\right)}$ for all $\left(d^{\prime}\right)$, as well as the spaces from $S_{1,2}$ which contains all possible products of spaces (75).

Since the algebra $S_{1,2}^{\prime}$ is a graded subalgebra of the algebra $S_{1,2}$ for the system (34), according to Proposition 4, we obtain that for the Krull dimensions of these algebras the following inequality takes place:

$$
\begin{equation*}
\varrho\left(S_{1,2}^{\prime}\right) \leq \varrho\left(S_{1,2}\right) \tag{77}
\end{equation*}
$$

Taking into account this inequality, and the fact that from [7] we have $\varrho\left(S_{1,2}\right)=9$, according to Definition 2, we have
Lemma 1. The maximal number of algebraically independent generalized focal pseudo-quantities in the center-focus problem for the system (34) does not exceed 9.

According to the equalities (32), (48), (55) etc. and the conclusion, resulting from Proposition 5, that the number of algebraically independent focal quantities $L_{k}(k=\overline{0, \infty})$ can not exceed the maximal number of algebraically independent generalized focal pseudo-quantities, using Lemma 1, we have
Theorem 2. The maximal number of algebraically independent focal quantities of the system (34) on the variety $\mathcal{V}$ from (27) or, equivalently, from (29), that take part in solving the center-focus problem, does not exceed 9.

With the help of Hilbert series of the algebras $S_{1,2}, S_{1,2}^{\prime}, S I_{1,2}$ [23] and Remark 3 it can be shown that the predicted upper bound of algebraically independent focal quantities of the system (34) on the variety $\mathcal{V}$ from (27) ((29)) can be much less than 9 , and can be equal to 7 or, may be, even 5 .

We note that the similar studies that for the system $s(1,2)$ from (34) were realized in the works $[27,28,31]$ for the systems $s(1,3), s(1,4), s(1,5)$ respectively. This scenario is confirmed for the system $s(1,2,3)$ by studies in the case 9 which allow to form the algebra $S_{1,2,3}^{\prime}$ with the same properties as the algebra $S_{1,2}^{\prime}$.

## 12 Main results

Similarly to the considered cases, for any system $s\left(1, m_{1}, m_{2}, \ldots, m_{\ell}\right)$ from (1) we have that the algebras similar to the obtained in the above mentioned examples satisfy the inclusion condition

$$
S_{1, m_{1}, m_{2}, \ldots, m_{\ell}}^{\prime} \subset S_{1, m_{1}, m_{2}, \ldots, m_{\ell}}
$$

hence according to Proposition 5, for their Krull dimensions we have

$$
\begin{equation*}
\varrho\left(S_{1, m_{1}, m_{2}, \ldots, m_{\ell}}^{\prime}\right) \leq \varrho\left(S_{1, m_{1}, m_{2}, \ldots, m_{\ell}}\right) \tag{78}
\end{equation*}
$$

By the formula (19) we obtain

$$
\begin{equation*}
\varrho\left(S_{1, m_{1}, m_{2}, \ldots, m_{\ell}}\right)=2\left(\sum_{i=1}^{\ell} m_{i}+\ell\right)+3 \tag{79}
\end{equation*}
$$

Analogously to Lemma 1 and other considered examples, with the help of (78) and (79) it can be shown that the following lemma is true:

Lemma 2. The maximal number of algebraically independent generalized focal pseudo-quantities in the center-focus problem for the system (1) does not exceed the number from (79).
Remark 8. According to the Remarks 5 and 6 and formulation of center-focus problem given in Section 5, as well as the identities (32) we can say that the generalized focal pseudo-quantities, being semi-invariants in above mentioned comitants, have as projections on the variety $\mathcal{V}$ from $(27)((29))$ the focal quantities $L_{k}(k=1,2, \ldots)$.

From identity (30) and Lyapunov's function (35) it results that for any system $s\left(1, m_{1}, m_{2}, \ldots, m_{\ell}\right)$ we can write the identities of the type (58) for quantities $G_{k}$ $(k=1,2, \ldots)$, which have as numerators the generalized focal pseudo-quantities.

Using these quantities and the operator $D_{3}$ from (18) of system $s\left(1, m_{1}, m_{2}, \ldots, m_{\ell}\right)$ we can determine comitants of the given system, having as coefficients the above mentioned focal pseudo-quantities.

According to the Remark 8 we conclude that the following statement take place: Theorem 3. The maximal number of algebraically independent focal quantities of the system (1) on the variety $\mathcal{V}$ from (27) or, equivalently, from (29), that take part in solving the center-focus problem does not exceed the number from (79).

We recall that in the introduction it was told that for the systems $s(1,2)$ and $s(1,3)$ the number of essential conditions for center $\omega=3$ and 5 , respectively, but for the system $s(1,2,3)$ there is an assumption that $\omega \leq 13$.

From Theorem 3 we obtain that the maximal number of algebraically independent focal quantities for the system $s(1,2)$ does not exceed 9 , for $s(1,3)$ does not exceed 11 , and for $s(1,2,3)$ does not exceed 17 .

These arguments and Proposition 5 with $\mathcal{V}$ from (27) or, equivalently, from (29), and the defined above algebra $S_{1, m_{1}, m_{2}, \ldots, m_{\ell}}^{\prime}$ suggest that is true
The main hypothesis. The number $\omega$ of essential conditions for center from (3) which solve the center-focus problem for the system (1), having at the origin of coordinates a singular point of the second type, does not exceed the number from (79). Remark 9. The equality (79) shows that the quantity $\varrho$ is equal to the number of coefficients of the right parts of the system (1) minus one.

Besides [23], the authors have published their vision of the center-focus problem in the theses [24-33].

## References

[1] Mathematical Encyclopedia Center and focus problem. http://dic.academic.ru/dic.nsf/ encmathematics/6088/ЦЕНТРA (in Russian).
[2] Poincaré H. Mémoire sur les courbes définies par une équation différentielle. J. Math. Pures et Appl.: (Sér. 3) 7, (1881) 375-422; (Sér. 3) 8 (1882) 251-296; (Sér. 4) 1 (1885) 167-244; (Sér. 4) 2 (1886) 151-217.
[3] Liapunoff A. Problème général de la stabilité du mouvement. Annales de la Faculté des Sciences de Touluose Sér 29 (1907) 204-477. Reproduction in Annals of Mathematics Studies 17, Princeton: Princeton University Press, 1947, reprinted 1965, Kraus Reprint Corporation, New York.
[4] Sadovskil A. P. Polynomial ideals and manifolds. BSU, Minsk, 2008 (in Russian).
[5] Sibirsky K. S. Introduction to the algebraic theory of invariants of differential equations. Translated from Russian. Nonlinear Science: Theory and applications, Manchester University Press, Manchester, 1988.
[6] Vulpe N. I. Polynomial bases of comitants of differential systems and their application in qualitative theory. Kishinev, Shtiintsa, 1986 (in Russian).
[7] Popa M. N. Algebraic methods for differential systems. Editura the Flower Power, Universitatea din Piteşti, Seria Matematică Aplicată şi Industrială, 2004, 15 (in Romanian).
[8] Bautin N. N. On the number of limit cycles which appear with the variation of coefficients from equilibrium position of focus or center type. Math. Sb. 1952, 30(72) 1952, 181-196; Amer. Math. Soc. Transl., 1954, 100, 397-413.
[9] Dulac H. Determination et integration d'une certaine classe d'equations differentielles ayant pour point singulier un center. Bull. Sciences Math. Ser., 1908, 2, 32(1), 230-252.
[10] Sibirsky K. S. On the number of limit cycles in the neighborhood of a singular point. Differentsialnye Uravnenya, 1965, 1, 51-56 (in Russian).
[11] Zolâdek H. On certain generalization of the Bautin's theorem. Nonlinearity, 1994, 7, 273-279.
[12] Graf V. Bothmer H. C., Kröker J. Focal Values of Plane Cubic Centers. Qual. Theory Dyn. Syst., 2010, 9, 319-324.
[13] Gurevich G. B. Foundation of the Theory of Algebraic Invariants. Noordoff, Groningen, 1964.
[14] Sibirsky K.S. Method of invariants in the qualitative theory of differential equations. Kishinev, RIO AN Moldavian SSR, 1968 (in Russian).
[15] Arzhantsev I. V. Graded Algebras and 14-th Hilbert Problem. Moscow, Publishers MCCME, 2009, 63 p. (in Russian).
[16] Alekseev V. G. The theory of rational invariants of binary forms. Iuriev, 1899 (in Russian).
[17] Macari P. M., Popa M. N., Vulpe N. I. The integer algebraic basis of center-affine invariants of homogeneous cubic differential system. Bul. Acad. Ştiinţe Repub. Moldova, Mat., 1996, No. 1(20), 48-55 (in Russian).
[18] Vulpe N. I. The integer algebraic basis of center-affine invariants of homogeneous cubic differential system. Mat. issled., vyp. 55, Kishinev, Shtiintsa, 1980, 37-45 (in Russian).
[19] Sibirsky K. S., Lunchevich V.A. Quadratic null systems of differential equations. Differentsialnye Uravnenya, 1989, 25(6), 1056-1058 (in Russian).
[20] Baltag V. A. The topological classification of cubic null systems of differential equations. Preprint, Chişinău, ASM, Institute of Mathematics, 1989, 59 p. (in Russian).
[21] Springer T. A. Invariant theory. Springer-Verlag, Berlin-Heidelberg-New York, 1977.
[22] Ufnarovskij V.A. Combinatorial and Asymptotic Methods in Algebra. Encyclopedia of Mathematical Sciences VI, Springer, 1995.
[23] Popa M. N., Pricop V. V. Applications of algebras to the center-focus problem. Preprint, Chişinău, ASM, Institute of Mathematics and Computer Science, No. 0007, 2011, 59 p. http://www.math.md/files/download/epublications/PreprintPopaMPricopV.pdf
[24] Popa M., Pricop V. About Generating function on the problem of center and focus. Scientific Conference dedicated to the 80-th anniversary of the foundation of the Tiraspol State University and of the Faculty of Physics, Mathematics and Information Tehnologies „Actual Problems of Mathematics and Informatics", Chişinău, September 24-25, 2010, Communications, pp. 126-127.
[25] Popa M., Pricop V. Combinatorial and asymptotic aspects to the center-focus problem. The 18 th Conference on Applied and Industrial Mathematics-CAIM 2010, October 14-17, 2010, Iaşi, Romania, Abstracts, p. 74.
[26] Popa M. N., Pricop V. V. Applications of generating functions and Hilbert series to the center-focus problem. The $8^{\text {th }}$ International Algebraic Conference in Ukraine, Lugansk Taras Shevchenko National University, July 5-12, 2011, Lugansk, Ukraine, Book of abstracts, p. 10.
[27] Popa M. N., Pricop V.V. About the maximal number of algebraically independent focal pseudo-quantities of the system $s(1,3)$. „Mathematics and IT: Research and Education (MITRE-2011)", Chişinău, August 22-25, 2011, Abstracts, p. 94.
[28] Pricop V.V. The differential system $s(1,4)$ and algebraically independent focal pseudoquantities. The 19th Conference on Applied and Industrial Mathematics-CAIM 2011, Abstracts, September 22-25, 2011, Iaşi, Romania, p. 15-16.
[29] Popa M. N., Pricop V. V. The Krull dimension in solving the center-focus problem for polynomial differential systems. The III-th School-Conference „Lie algebras, algebraic groups and invariant theory" dedicated to the 75-th anniversary of E. B. Vinberg, Tolyatti, Russian Federation, June 25-30, 2012, Abstracts, p. 41-43.
[30] Popa M. N. Applications of algebraic methods to the center-focus problem. The 20th Conference on Applied and Industrial Mathematics dedicated to academician Mitrofan M. CiobanuCAIM 2012, Chişinău, August 22-25, 2012, Communications, p. 184-186.
[31] Pricop V.V. The numerical upper bound of the number of algebraically independent focal pseudo-quantities of the differential system $s(1,5)$. The 20th Conference on Applied and Industrial Mathematics dedicated to academician Mitrofan M. Ciobanu-CAIM 2012, Chişinău, August 22-25, 2012, Communications, p. 190-192.
[32] Popa M. N., Pricop V. V. The Hilbert series and Lie algebras in solving the center-focus problem. International Mathematical Conference on occasion to the 70th year anniversary of Professor Vladimir Kirichenko, June 13-19, 2012, Mykolayiv V. O. Sukhomlynsky National University, Mykolayiv, Ukraine, Book of abstracts, p. 114.
[33] M. N. Popa, V. V. Pricop. About a solution of the center-focus problem. XV International Scientific Conference on Differential Equations "Erugin readings - 2013", Yanka Kupala State University of Grodno, May 13-16, 2013, Grodno, Republic of Belarus, Abstracts, Part I, p. 69-70.
M. N. Popa, V. V. Pricop

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Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
5, Academiei str., Chişinău, MD-2028
Moldova
E-mail: popam@math.md; pricopv@mail.md

# Geometric configurations of singularities for quadratic differential systems with total finite multiplicity lower than 2 

J. C. Artés, J. Llibre, D. Schlomiuk, N. Vulpe


#### Abstract

In [3] we classified globally the configurations of singularities at infinity of quadratic differential systems, with respect to the geometric equivalence relation. The global classification of configurations of finite singularities was done in [2] modulo the coarser topological equivalence relation for which no distinctions are made between a focus and a node and neither are they made between a strong and a weak focus or between foci of different orders. These distinctions are however important in the production of limit cycles close to the foci in perturbations of the systems. The notion of geometric equivalence relation of configurations of singularities allows us to incorporates all these important purely algebraic features. This equivalence relation is also finer than the qualitative equivalence relation introduced in [20]. In this article we initiate the joint classification of configurations of singularities, finite and infinite, using the finer geometric equivalence relation, for the subclass of quadratic differential systems possessing finite singularities of total multiplicity $m_{f} \leq 1$. We obtain 84 geometrically distinct configurations of singularities for this family. We also give here the global bifurcation diagram, with respect to the geometric equivalence relation, of configurations of singularities, both finite and infinite, for this class of systems. This bifurcation set is algebraic. The bifurcation diagram is done in the 12-dimensional space of parameters and it is expressed in terms of polynomial invariants. The results can therefore be applied for any family of quadratic systems, given in any normal form. Determining the configurations of singularities for any family of quadratic systems, becomes thus a simple task using computer algebra calculations.


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## 1 Introduction and statement of main results

We consider here differential systems of the form

$$
\begin{equation*}
\frac{d x}{d t}=p(x, y), \quad \frac{d y}{d t}=q(x, y) \tag{1}
\end{equation*}
$$

where $p, q \in \mathbb{R}[x, y]$, i.e. $p, q$ are polynomials in $x, y$ over $\mathbb{R}$. We call degree of a system (1) the integer $m=\max (\operatorname{deg} p, \operatorname{deg} q$ ). In particular we call quadratic a differential system (1) with $m=2$. We denote here by QS the whole class of real quadratic differential systems.

[^2]The study of the class QS has proved to be quite a challenge since hard problems formulated more than a century ago, are still open for this class. The complete characterization of the phase portraits for real quadratic vector fields is not known, and attempting to topologically classify these systems, which occur rather often in applications, is a very complex task. This is partly due to the elusive nature of limit cycles and partly to the rather large number of parameters involved. This family of systems depends on twelve parameters but due to the group action of real affine transformations and time homotheties, the class ultimately depends on five parameters, still a rather large number of parameters. For the moment only subclasses depending on at most three parameters were studied globally, including global bifurcation diagrams (for example [2]). On the other hand we can restrict the study of the whole quadratic class by focusing on specific global features of the systems in this family. We may thus focus on the global study of singularities and their bifurcation diagram. The singularities are of two kinds: finite and infinite. The infinite singularities are obtained by compactifying the differential systems on the sphere or on the Poincaré disk as they are defined in Section 6.1 (see also [17]).

The global study of quadratic vector fields in the neighborhood of infinity was initiated by Coll in [13] where he characterizes all the possible phase portraits in a neighborhood of infinity. Later Nikolaev and Vulpe in [23] classified topologically the singularities at infinity in terms of invariant polynomials. Schlomiuk and Vulpe used geometrical concepts defined in [30], and also introduced some new geometrical concepts in [31] in order to simplify the invariant polynomials and the classification. To reduce the number of phase portraits in half, in both cases the topological equivalence relation was taken to mean the existence of a homeomorphism carrying orbits to orbits and preserving or reversing the orientation. In [4] the authors classified topologically (adding also the distinction between nodes and foci) the whole quadratic class, according to configurations of their finite singularities.

In the topological classification no distinction was made among the various types of foci or saddles, strong or weak of various orders. However these distinctions, of algebraic nature, are very important in the study of perturbations of systems possessing such singularities. Indeed, the maximum number of limit cycles which can be produced close to the weak foci in perturbations depends on the orders of the foci.

The distinction among weak saddles is also important since for example when a loop is formed using two separatrices of one weak saddle, the maximum number of limit cycles that can be obtained close to the loop in perturbations is the order of the weak saddle (see, for example,[26]).

There are also three kinds of simple nodes as we can see in Figure 1 below where the local phase portraits around the singularities are given.

In the three phase portraits of Figure 1 the corresponding three singularities are stable nodes. These portraits are topologically equivalent but the solution curves do not arrive at the nodes in the same way. In the first case, any two distinct nontrivial phase curves arrive at the node with distinct slopes. Such a node is called a star node. In the second picture all non-trivial solution curves excepting two of


Figure 1. Different types of nodes
them arrive at the node with the same slope but the two exception curves arrive at the node with a different slope. This is the generic node with two directions. In the third phase portrait all phase curves arrive at the node with the same slope.

We recall that the first and the third types of nodes could produce foci in perturbations because their eigenvalues are equal. The linear part of the first is diagonal and the one of the third is not. We can distinguish algebraically among the three types of nodes. Here algebraic means that the linearization matrices at these nodes and their eigenvalues, distinguish the nodes in Figure 1. The first type of nodes is also involved in the existence of invariant straight lines of differential systems. For example it can be shown that if a quadratic differential system has two finite star nodes then necessarily the system possesses invariant straight lines of total multiplicity 6 , see [32].

Furthermore, a generic node may or may not have the two exceptional curves lying on the line at infinite. This leads to two situations which geometrically are different. Indeed, in the case when the two exceptional curves lie on the line at infinity, all the other phase curves have a common asymptote while in the case the two exceptional curves lie in the affine plane, all other phase curves are tangent to the line at infinity. From the geometric viewpoint these two situations are different. Polynomial vector fields should not be viewed just as particular cases of analytic vector fields. They are also algebraic and geometric objects in their own right and as such the algebraic and geometric behavior of their phase curves matters. For this reason we split the generic nodes at infinity in two types.

The distinctions among the nilpotent and linearly zero singularities finite or infinite can also be refined, as it will be seen in Section 4. Such singularities are usually called degenerate singularities so here too we call them degenerate.

The geometric equivalence relation for finite or infinite singularities, introduced in [3], takes into account such distinctions. This equivalence relation is finer than the qualitative equivalence relation introduced by Jiang and Llibre in [20] since it distinguishes among the foci of different orders and among the various types of nodes. This equivalence relation also induces a finer distinction among the more complicated degenerate singularities.

To distinguish among the foci (or saddles) of various orders we use the algebraic concept of Poincaré-Lyapounov constants. We call strong focus (or strong saddle) a focus with non-zero trace of the linearization matrix at this point. Such a focus (or saddle) will be considered to have the order zero. A focus (or saddle) with trace zero
is called a weak focus (weak saddle). For details on Poincaré-Lyapounov constants and weak foci we refer to [21].

The finer distinctions of singularities are also algebraic in nature through the Lyapounov-Poincaré constants. In fact the whole bifurcation diagram of the global configurations of singularities, finite and infinite, in quadratic vector fields and more generally in polynomial vector fields can be obtained by using only algebraic means, among them, the algebraic tool of polynomial invariants.

Algebraic information may not be significant for the local (topological) phase portrait around a singularity. For example, topologically there is no distinction between a focus and a node or between a weak and a strong focus. However, as indicated before, algebraic information plays a fundamental role in the study of perturbations of systems possessing such singularities.

In [14] Coppel wrote: "Ideally one might hope to characterize the phase portraits of quadratic systems by means of algebraic inequalities on the coefficients. However, attempts in this direction have met with very limited success..."

This proved to be impossible to realize. Indeed, Dumortier and Fiddelaers [16] and Roussarie [27] exhibited examples of families of quadratic vector fields which have non-algebraic bifurcation sets. However, the following is a legitimate question:

How far can we go in the global theory of quadratic (or more generally polynomial) vector fields by using mainly algebraic means?

For certain subclasses of quadratic vector fields the full description of the phase portraits as well as of the bifurcation diagrams can be obtained using only algebraic tools. Examples of such classes are:

- the quadratic vector fields possessing a center [24,28, 40, 43];
- the quadratic Hamiltonian vector fields [1,5];
- the quadratic vector fields with invariant straight lines of total multiplicity at least four [32, 33];
- the planar quadratic differential systems possessing a line of singularities at infinity [34];
- the quadratic vector fields possessing an integrable saddle [6];
- the family of Lotka-Volterra systems [35,36], once we assume Bautin's analytic result saying that such systems have no limit cycles.

In the case of other subclasses of the quadratic class QS, such as the subclass of systems with a weak focus of order 3 or 2 (see $[2,21]$ ) the bifurcation diagrams were obtained by using an interplay of algebraic, analytic and numerical methods. These subclasses were of dimensions 2 and 3 modulo the action of the affine group and time rescaling. So far no 4 -dimensional subclasses of $\mathbf{Q S}$ were studied globally so as to produce also bifurcation diagrams and such problems are very difficult due to the number of parameters as well as the increased complexities of these classes.

Although we now know that in trying to understand these systems, there is a limit to the power of algebraic methods, these methods have not been used far enough. For example the global classification of singularities, finite and infinite, using the geometric equivalence relation, which is finer than the qualitative equivalence relation, can be done by using only algebraic methods. The first step in this direction was done in [3] where the study of the whole class QS, according to the configurations of the singularities at infinity was obtained by using only algebraic methods. This classification was done with respect to the geometric equivalence relation. Our work in [3] can be extended by incorporating also the finite singularities. In this way we can obtain the global geometric classification of all possible configurations of singularities, finite and infinite, of quadratic differential systems, by purely algebraic means.

Our goal in this work is to take the first step in this direction by joining the results for infinite singularities in [3] with finite singularities of total multiplicity $m_{f} \leq 1$, of quadratic differential systems.

We extend here below the notion of configuration of singularities defined in [3] only for infinite singularities, to all singularities, both finite and infinite. We distinguish two cases.

1) If we have a finite number of infinite singular points and a finite number of finite singularities, we call configuration of singularities, finite and infinite, the set of all these singularities each endowed with its own multiplicity together with their local phase portraits endowed with additional geometric structure involving the concepts of tangent, order and blow-up equivalences defined in Section 4 and using the notations described in Section 5.
2) If the line at infinity $Z=0$ is filled up with singularities, in each one of the charts at infinity $X \neq 0$ and $Y \neq 0$, the system is degenerate and we need to do a rescaling of an appropriate degree of the system, so that the degeneracy be removed. The resulting systems have only a finite number of singularities on the line $Z=0$. In this case we call configuration of singularities, finite and infinite, the union of the set of all points at infinity (they are all singularities) with the set of finite singularities - taking care of singling out the singularities of the "reduced" system at infinity -, taken together with the local phase portraits of finite singularities endowed with additional geometric structure as above and of the infinite singularities of the reduced system.

We continue to use here ISPs as a shorthand for "infinite singular points".
We obtain the following
Main Theorem. (A) The configurations of singularities, finite and infinite, of all quadratic vector fields with finite singularities of total multiplicity $\boldsymbol{m}_{\boldsymbol{f}} \leq \mathbf{1}$ are classified in Diagrams 1 and 2 according to the geometric equivalence relation. We have 84 geometrically distinct configurations of singularities, finite and infinite. More precisely 32 configurations with $\boldsymbol{m}_{\boldsymbol{f}}=\mathbf{0}$ and 52 with $\boldsymbol{m}_{\boldsymbol{f}}=\mathbf{1}$.
(B) For $\boldsymbol{m}_{\boldsymbol{f}}=\mathbf{1}$ we have only two configurations with a center but 5 configurations with a finite integrable saddle, and the maximum order of a weak focus (or of a weak saddle) is one.
(C) For $\boldsymbol{m}_{\boldsymbol{f}}=1$ we have: 4 configurations with a weak focus of order one but only 2 configurations with a weak finite saddle of order one; 6 configurations with a strong focus but 7 configurations with a strong finite saddle.
(D) Necessary and sufficient conditions for each one of the 84 different equivalence classes can be assembled from Diagrams 1 and 2 in terms of 30 invariant polynomials with respect to the action of the affine group and time rescaling, given in Section 7.
(E) The Diagrams 1 and 2 actually contain the global bifurcation diagram in the 12-dimensional space of parameters, of the global configurations of singularities, finite and infinite, of this family $\left(\boldsymbol{m}_{\boldsymbol{f}} \leq \mathbf{1}\right)$ of quadratic differential systems.
$(\boldsymbol{F})$ The phase portraits in the neighborhood of the line at infinity corresponding to $\boldsymbol{m}_{\boldsymbol{f}}=\mathbf{0}$ and to $\boldsymbol{m}_{\boldsymbol{f}}=\mathbf{1}$ are given in Figure 1. More precisely we have:
$\boldsymbol{m}_{\boldsymbol{f}}=\mathbf{0}$ : Configs - 3; 4; 5; 30; 18; 28; 17; 13; 8; 24; 11; 15; 36; 35; 32; 46 ;
$\boldsymbol{m}_{\boldsymbol{f}}=1$ : Configs - 2; 6; 31; 20; 14; 26; 25; 9; 23; 16; 12; 21; 39; 37; 33; 38; 45.

We note that the case $m_{f}=1$ was considered in [37], were all 52 possible geometrically distinct configurations of singularities are given but without proof. The complete proof is done here below.

The invariants and comitants of differential equations used for proving our main results are obtained following the theory of algebraic invariants of polynomial differential systems, developed by Sibirsky and his disciples (see for instance [7,12, 25, 38, 41]).

## 2 Some geometrical concepts

In this section we use the same concepts we considered in [3] such as orbit $\gamma$ tangent to a semi-line $L$ at $p$, well defined angle at $p$, characteristic orbit at a singular point $p$, characteristic angle at a singular point, characteristic direction at $p$. Since these are basic concepts for the notion of geometric equivalence relation we recall here their definitions.

We assume that we have an isolated singularity $p$. Suppose that in a neighborhood $U$ of $p$ there is no other singularity. Consider an orbit $\gamma$ in $U$ defined by a solution $\Gamma(t)=(x(t), y(t))$ such that $\lim _{t \rightarrow+\infty} \Gamma(t)=p\left(\right.$ or $\left.\lim _{t \rightarrow-\infty} \Gamma(t)=p\right)$. For a fixed $t$ consider the unit vector $C(t)=(\overrightarrow{\Gamma(t)-p}) /\|\overrightarrow{\Gamma(t)-p}\|$. Let $L$ be a semiline ending at $p$. We shall say that the orbit $\gamma$ is tangent to a semi-line $L$ at $p$ if $\lim _{t \rightarrow+\infty} C(t)$ (or $\left.\lim _{t \rightarrow-\infty} C(t)\right)$ exists and $L$ contains this limit point on the unit circle centered at $p$. In this case we call a well defined angle of $\Gamma$ at $p$ the angle between the positive $x$-axis and the semi-line $L$ measured in the counterclockwise sense. We may also say that the solution curve $\Gamma(t)$ tends to $p$ with a well defined angle. A characteristic orbit at a singular point $p$ is the orbit of a solution curve $\Gamma(t)$ which tends to $p$ with a well defined angle. We call a characteristic angle at the singular point $p$ a well defined angle of a solution curve $\Gamma(t)$. The line through $p$ extending the semi-line $L$ is called a characteristic direction.


DIAGRAM 1. Global configurations: case $\mu_{0}=\mu_{1}=\mu_{2}=\mu_{3}=0, \mu_{4} \neq 0$


DIAGRAM 2. Global configurations: case $\mu_{0}=\mu_{1}=\mu_{2}=0, \mu_{3} \neq 0$


Diagram 2 (continued). Global configurations: case $\mu_{0}=\mu_{1}=\mu_{2}=0$, $\mu_{3} \neq 0$

If a singular point has an infinite number of characteristic directions, we will call it a star-like point.

It is known that the neighborhood of any isolated singular point of a polynomial vector field which is not a focus, a center or a star-like point, is formed by a


Figure 2. Topologically distinct local configurations of ISPs ([31, 34])
finite number of sectors which could only be of three types: parabolic, hyperbolic and elliptic (see [17]). It is also known that any degenerate singular point can be desingularized by means of a finite number of changes of variables, called blow-up's, into elementary singular points (for more details see Section 3 or [17]).

Consider the three singular points given in Figure 3. All three are topologically equivalent and their neighborhoods can be described as having two elliptic sectors
and two parabolic ones. But we can easily detect some geometric features which distinguish them. For example ( $a$ ) and (b) have three characteristic directions and (c) has only two. Moreover in (a) the solution curves of the parabolic sectors are tangent to only one characteristic direction and in (b) they are tangent to two characteristic directions. All these properties can be determined algebraically.


Figure 3. Some topologically equivalent singular points

The usual definition of a sector is of a topological nature and it is local with respect to a neighborhood around the singular point. We work with a new notion, namely of geometric local sector, introduced in [3] (we will improve that definition in this paper) which distinguishes the systems of Figure 3 as well as the nodes in Figure 1. This notion is characterized by algebraic means.

We consider first the case of an elemental star-node $p$. This is a very special case because this has an infinite number of characteristic directions. Literally speaking we have no parabolic sectors here although each orbits is tangent to a half-line at $p$. We shall consider that this node has just one geometric local parabolic sector which is the complement of $\{p\}$ in an open neighborhood of $p$.

We introduce an equivalence relation for the orbits of solutions $\Gamma(t)$ tending to a singular point $p$ when $t$ tends to either $+\infty$ or to $-\infty$. We say that two such orbits are equivalent if and only if after the complete desingularization, these orbits lifted to the final stage are tangent to the same half-line at the same singular point, or end as an orbit of a star-node on the same half-plane. We will call borsec a representative of an equivalence class, with the exception of the case when in the desingularized picture the characteristic direction is the same as the direction of the blow-up, and in addition the singular point in the desingularization picture is a two directions node or a saddle-node.

We call geometric local sector of a singular point $p$ with respect to a neighborhood $V$ as a region in $V$ delimited by two consecutive borsecs.

A semi-elemental saddle-node can be topologically described as a singular point having two hyperbolic sectors and a single parabolic one. But if we add a borsec which is an orbit of the parabolic sector (any orbit in that sector could be this borsec), then the description consists of two hyperbolic sectors and two parabolic ones. This distinction will be significant when trying to describe a singular point like the one in Figure 4 which is an intricate singularity, topologically a saddle-node but different from a semi-elemental saddle-node. Indeed, in an elemental saddlenode in the parabolic sector all orbits are tangent to just one half-line at $p$, while in


Figure 4. Local phase portrait of a saddle-node

Figure 4 some of the orbits of the parabolic sector are tangent to one half-line at $p$ while others are tangent to a different half-line at $p$.

Generically a geometric local sector is defined by two consecutive borsecs tangent to two distinct half-lines at the singular point $p$ with two different well defined angles. If this sector is parabolic, then the solutions can arrive at the singular point $p$ with one of the two half-lines at $p$ on the characteristic direction lines at $p$ and this is a geometrical information than can be revealed with the blow-up.

There is also the possibility that two borsecs defining a geometric local sector are tangent to the same half-line at the singular point. Such a sector will be called a cusp-like sector which can either be hyperbolic, elliptic or parabolic respectively denoted by $H_{\curlywedge}, E_{\curlywedge}$ and $P_{\curlywedge}$.

In the case of parabolic sectors we want to include the information as to whether the orbits arrive tangent to one or to the other borsec. We distinguish the two cases writing by $\widehat{P}$ if they arrive tangent to the borsec limiting the previous sector in clockwise sense or $\overparen{P}$ if they arrive tangent to the borsec limiting the next sector. In the case of a cusp-like parabolic sector, all orbits must arrive with only one well determined angle, but the distinction between $\widehat{P}$ and $\widehat{P}$ is still valid because it occurs at some stage of the desingularization and this can be algebraically determined. Thus complicated degenerate singular points like the two we see in Figure 5 may be described as $\widehat{P E} \overparen{P} H H H$ (case (a)) and $E \widehat{P}_{\curlywedge} H H \widehat{P}_{\curlywedge} E$ (case (b)), respectively.


Figure 5. Two phase portraits of degenerate singular points

## 3 The blow-up technique

To draw the phase portrait around an elementary hyperbolic singularity of a smooth planar vector field we just need to use the Hartman-Grobman theorem. For an elementary non-hyperbolic singularity the system can be brought by an affine change of coordinates and time rescaling to the form $d x / d t=-y+\ldots, d y / d t=x+\ldots$ and it is well known that in this case the singularity is either a center or a focus. One way to see this is by the Poincaré-Lyapounov theory. In the quadratic case we can actually determine using the Poincaré-Lyapounov constants if it is a focus or a center and then the local phase portrait is known (see $[28,40]$ ). For higher order systems we have the center-focus problem: we can only say that the phase portrait around the singularity is of a center or of a focus but we cannot determine with certainty which one of the two it is.

In the case of a more complicated singularity, such as a degenerate one, we need to use the blow-up technique. This is a well known technique but since it plays such a crucial role in this work, we shall briefly describe it here. We are using this technique in a slightly modified (actually simplified) way to lighten the calculations. This slightly modified way is in complete agreement with the usual blow-up procedure.

The idea behind the blow-up technique is to replace a singular point $p$ by a circle or by a line on which the "composite" degenerate singularity decomposes (ideally) into a finite number of simpler singularities $p_{i}$. For this idea to work we need to construct a new surface, on which we have a diffeomorphic copy of our vector field on $\mathbb{R}^{2} \backslash\{p\}$ or at least on the complement of a line passing through $p$, and whose associated foliation with singularities extends also to the circle (or to a line) which replaces the point $p$ on the new surface.

One way to do this is to use polar coordinates. Clearly we may assume that the singularity is placed at the origin. Consider the map $\phi: \mathbb{S}^{1} \times \mathbb{R} \longrightarrow \mathbb{R}^{2}$ defined by $\phi(\theta, r) \mapsto(r \cos \theta, r \sin \theta)$. Restrictions of this map $\phi$ on $\mathbb{S}^{1} \times(0, \infty)$ and on $\mathbb{S}^{1} \times(-\infty, 0)$ are diffeomeorphisms, mapping the upper, respectively lower part of the cylinder on $\mathbb{R}^{2} \backslash\{(0,0)\}$. But $\phi^{-1}(0,0)$ is the circle $\mathbb{S}^{1} \times\{0\}$. This application defines a diffeomorphic vector field on the upper part of the cylinder $\mathbb{S}^{1} \times \mathbb{R}$. In fact this is the passing to polar coordinates. The resulting smooth vector field extends to the whole cylinder just by allowing $r$ to be negative or zero. This full vector field on the cylinder has either a finite number of singularities on the circle (this occurs when the initial singular point is nilpotent) or the circle is filled up with singularities (when we start with a point for which the linear part of the system at this point vanishes). In this latter case we need to work with the reduced system obtained by dividing the right hand side of the equations by a factor $r^{s}$ with an adequate $s$ to obtain a finite number of singularities. Since $\mathbb{R}^{2} \backslash\{(0,0)\}$ is diffeomorphic to the upper part of the cylinder we only need to consider $r>0$ for which this factor $r^{s}$ is also positive. Removing this factor does not affect the nature of the orbits and their orientation. The map $\phi$ collapses the circle on the cylinder (and hence the singularities located on this circle) to the origin of coordinates in the plane. In case
the phase portraits around the singularities on the circle can be drawn then the inverse process of blowing down the upper side of the cylinder completed with the circle allows to draw the portrait around the origin of $\mathbb{R}^{2}$. In case the singularities on the circle are still degenerate, we need to repeat this process a finite number of times. This is guaranteed by the theorem of desingularization of singularities (see [10] and [15], or [20]).

The blow-up by polar coordinates is simple, leading to a simple surface (the cylinder), on which a diffeomorphic copy of our vector field on $\mathbb{R}^{2} \backslash\{(0,0)\}$ extends to a vector field on the full cylinder. The origin of the plane "blows-up" to the circle $\phi^{-1}(0,0)$ on which the singularity splits into several simpler singularities. The visualization of this blow-up is easy. But this process has the disadvantage of using the transcendental functions: cos and sin and in case several such blow-ups are needed this is computationally very inconvenient.

It would be more advantageous to use a construction involving rational functions. More difficult to visualize, this algebraic blow-up is computationally simpler, using only rational transformations. The blow-up in this case starts with a directional blow-up of a point of the plane, by this meaning that in this case to replace the point with a line sitting on a manifold playing the role of the cylinder in the preceding case.

Consider the algebraic surface $S$ in $\mathbb{R}^{3}$ defined by the equation $y=x z$. We may think of this surface as being here the analogue of the cylinder in the polar blow-up. Like the cylinder, $S$ is a differentiable manifold. Indeed, the projection $\pi_{1,2}: S \rightarrow \mathbb{R}^{2}, \pi_{1,2}(x, x z, z)=(x, x z)$, is a global chart for this manifold. We observe that the line $L_{z}=\{(0,0, z) \mid z \in \mathbb{R}\}$ (the $z$-axis in $\mathbb{R}^{3}$ ), lies on $S$. The projection $\pi_{1,2}$ collapses the $z$-axis to the point $(0,0)$. The line $L_{z}$ may be thought here as the analogue of the circle in the polar blow-up construction. The restriction

$$
\psi=\left.\pi_{1,2}\right|_{S \backslash L_{z}}: S \backslash L_{z} \longrightarrow \mathbb{R}^{2} \backslash\{x=0\}
$$

of $\pi_{1,2}$ to $S \backslash L_{z}$ is a diffeomorphism with inverse $\psi^{-1}(x, y)=(x, y, y / x)$ transferring our vector field restricted to the open set $x \neq 0$ of the plane $(x, y)$ to a diffeomorphic vector field on $S \backslash L_{z}$. The map $\pi_{1,3} \circ \psi^{-1}$ carries our vector field on the plane $(x, y)$, restricted to $x \neq 0$, to a diffeomorphic vector field on the open set $x \neq 0$ of the plane $(x, z)$. This is actually the vector field on $S \backslash L_{z}$ calculated in the chart given by $\pi_{1,3}$.

We now compute this vector field on the plane $(x, z)$. We start with a polynomial differential system of the form (1) with a degenerate singular point at the origin $(0,0)$. We have $p(x, y)=p_{1}(x, y)+\ldots+p_{n}(x, y)$ and $q(x, y)=q_{1}(x, y)+\ldots+q_{n}(x, y)$ where $p_{i}(x, y)$ and $q_{i}(x, y)$ (for $i=1, \ldots, n$ ) are the sums of the homogeneous terms involving $x^{r} y^{l}$ with $r+l=i$ of $p$ and $q$. We call the starting degree of (1) the positive integer $m$ such that $\left(p_{m}(x, y), q_{m}(x, y)\right) \neq(0,0)$ but $\left(p_{i}(x, y), q_{i}(x, y)\right)=(0,0)$ for $i=0,1, \ldots, m-1$.

This differential system when transferred on $S$ and calculated in the chart $\pi_{1,3}$
by using $y=x z$ becomes:

$$
\begin{gathered}
d x / d t=x^{m}\left(p_{m}(1, z)+\ldots+x^{n-m} p_{n}(1, z)\right) \\
d z / d t=x^{m-1}\left[q_{m}(1, z)+\ldots+x^{n-m} q_{n}(1, z)-z x\left(p_{m}(1, z)+\ldots+x^{n-m} p_{n}(1, z)\right)\right]
\end{gathered}
$$

because $d y / d t=d(x z) / d t=z d x / d t+x d z / d t$. This system is defined over the whole plane $(x, z)$ and when $m>1$ the line $x=0$ (the $z$-axis in the plane $(x, z)$ ) is filled up with singularities. If $m=1$ then $p_{1}(x, y)$ and $q_{1}(x, y)$ cannot be both identically zero. If $q_{1}(x, y) \equiv 0$ then $q_{1}(1, z) \equiv 0$, and again we must have the z-axis filled up with singularities. But if $q_{1}(x, y)=a x+b y$ is not identically zero, then $(a, b) \neq(0,0)$. If $b \neq 0$ then $q_{1}(1, z)=a+b z$ and $(0,-a / b)$ is the unique singular point on the $z$-axis. If however $b=0$ then $q_{1}(x, y)=a x$ and hence $q_{1}(1, z)=a \neq 0$, and we have no singular point on the $z$-axis. So for a nilpotent point with $m=1$ we either get an infinite number of singularities, or a unique singularity, or no singularity on the $z$-axis.

Just like in the polar blow-up when we eliminated the common factor $r^{s}$, here we eliminate the common factor $x^{m-1}$ (or $x^{m}$ in case $q_{m}(x, y) \equiv 0$ but $p_{m}$ is not identically zero). But in doing so we need to take some precautions which we explain below. Consider the system above and its associated "reduced" system

$$
\begin{gather*}
d x / d t=x\left[p_{m}(1, z)+\ldots+p_{n}(1, z)\right] \\
d z / d t=q_{m}(1, z)+\ldots+x^{n-m} q_{n}(1, z)-z\left[p_{m}(1, z)+\ldots+x^{n-m} p_{n}(1, z)\right] \tag{2}
\end{gather*}
$$

obtained by removing the common factor $x^{m-1}$ on the right side of the equations. We observe that for $x>0$ the two systems have the same orbits and their orbits have the same orientations, but the orbits are described by the solutions of the two systems with different speeds so we have a time change (rescaling). If $m$ is even then $m-1$ is odd, and hence $x^{m-1}$ is negative for $x<0$ and the orbits of the two systems for $x<0$ are described by the solutions of the two differential systems with opposite orientations. We need to take care of this when at the end we blow down the line to the point $(0,0)$. At the points on the $z$-axis $(x=0)$ for which $q_{m}(1, z)=0$ we have singularities. The finite number of singularities obtained in this way for the reduced system is analogous to the finite number of singularities on the circle we obtained in the reduced system in the polar blow-up. Thus the singular point at the origin is blown-up to a finite number of singularities on the $z$-axis of the plane $(x, z)$. We call this the directional blow-up in the direction of $y$-axis of the plane $(x, y)$.

In this blow-up construction the $y$-axis was excluded. Indeed, the surface $S$ does not contain the $y$-axis and we have a copy of our vector field on $S$ only for the complement in the plane $(x, y)$ of the $y$-axis, i.e. only on the open set $x \neq 0$. However, by doing an analogous blow-up in the direction of $x$-axis, the $y$-axis can be included. The two blow-ups can then be glued so as to obtain a complete blow-up on a Möbius band which will in this case be the full analogue of the cylinder in the polar blow-up. The circle at the center of the Möbius band is then viewed as the space $P_{1}(\mathbb{R})$ of all directions in the plane $(x, y)$. To see here the need of this twisting on the Möbius band we observe that the map $\pi_{1,3} \circ \psi^{-1}$ sends the left side of the $y$-axis
of the $(x, y)$ plane to the left side of the $z$-axis of the $(x, z)$ plane. While sending the semi-line $y=0$ and $x<0$ to the semi-line $z=0$ and $x<0$ this map flips the second and third quadrant in the $(x, z)$ plane. Indeed, the second (respectively third) quadrant in the $(x, y)$ plane are sent to the third (respectively second) quadrant in the $(x, z)$ plane. In this work we use a procedure, a sort of shortcut, to be explained further below which enables to manage without the Möbius band.

The equation giving the singular points on the $z$-axis in the $(x, z)$ plane according to $(2)$ is $z p_{m}(1, z)-q_{m}(1, z)=0$ and going back to the $(x, y)$ coordinates by replacing $z=y / x$ (for $x \neq 0$ ) we get the equation $y p_{m}(x, y)-x q_{m}(x, y)=0$.

The polynomial $P C D(x, y)=y p_{m}(x, y)-x q_{m}(x, y)$, where $m$ is the starting degree of a system of the form (1), is called the Polynomial of Characteristic Directions of $(1)$. In case $P C D(x, y) \not \equiv 0$ the factorization of $P C D(x, y)$ gives the characteristic directions at the origin. So, in order to be sure that the $y$-axis is not a characteristic direction we only need to show that $x$ is not a factor of $P C D(x, y)$. In case it is, we need to do a linear change of variables which moves this direction out of the vertical axis and does not place any other characteristic direction on this axis. If all the directions are characteristic, i.e. $P C D(x, y) \equiv 0$, then the degenerate point will be star-like and at least two blow-ups must be done to obtain the desingularization. Anyway, in quadratic systems there are no degenerate star-like singular points. So, the number of characteristic directions is finite and there exists the possibility to do such a linear change. We will use changes of the type $(x, y) \rightarrow(x+k y, y)$ where $k$ is some number (usually 1 ). It seems natural to call this linear change a $k$-twist as the $y$-axis gets twisted with some angle depending on $k$. It is obvious that the phase portrait of the degenerate point which is studied cannot depend on the set of $k$ 's used in the desingularization process.

Once we are sure that we have no characteristic direction on the $y$-axis we do the directional blow-up $(x, y)=(x, x z)$. This change sends the $x$-axis of the $(x, y)$ plane to the $X$-axis of the $(x, z)$ plane and replaces the singular point $(0,0)$ with a whole vertical axis in the $(x, z)$ plane. The old orbits which arrived at $(0,0)$ with a well defined slope $s$ now arrive at the singular point $(0, s)$ of the new system. Studying these new singular points, one can determine the local behavior around them and their separatrices which after the blow-down describe the behavior of the orbits around the original singular point up to geometrical equivalence (for definition see next section). Often one needs to do a tree of blow-up's (combined with some translation and/or twists) if some of the singular points which appear on $x=0$ after the first blow-up are also degenerate.

## 4 Equivalence relations for singularities of planar polynomial vector fields

We first recall the topological equivalence relation as it is used in most of the literature. Two singularities $p_{1}$ and $p_{2}$ are topologically equivalent if there exist open neighborhoods $N_{1}$ and $N_{2}$ of these points and a homeomorphism $\Psi: N_{1} \rightarrow N_{2}$ carrying orbits to orbits and preserving their orientations. To reduce the number
of cases, by topological equivalence we shall mean here that the homeomorphism $\Psi$ preserves or reverses the orientation. We observe that this second notion which is usually used in the literature on classification problems of polynomial vector fields (see $[2,20]$ ), does not conserve stability.

In [20] Jiang and Llibre introduced another equivalence relation for singularities, which is finer than the topological equivalence:

We say that $p_{1}$ and $p_{2}$ are qualitatively equivalent if i) they are topologically equivalent through a local homeomorphism $\Psi$, and ii) two orbits are tangent to the same straight line at $p_{1}$ if and only if the corresponding two orbits are also tangent to the same straight line at $p_{2}$.

We say that two simple finite nodes, with the respective eigenvalues $\lambda_{1}, \lambda_{2}$ and $\sigma_{1}, \sigma_{2}$, of a planar polynomial vector field are tangent equivalent if and only if they satisfy one of the following three conditions: a) $\left(\lambda_{1}-\lambda_{2}\right)\left(\sigma_{1}-\sigma_{2}\right) \neq 0$; b) $\lambda_{1}-\lambda_{2}=$ $0=\sigma_{1}-\sigma_{2}$ and both linearization matrices at the two singularities are diagonal; c ) $\lambda_{1}-\lambda_{2}=0=\sigma_{1}-\sigma_{2}$ and the corresponding linearization matrices are not diagonal.

We say that two infinite simple nodes $P_{1}$ and $P_{2}$ are tangent equivalent if and only if their corresponding singularities on the sphere are tangent equivalent and in addition, in case they are generic nodes, we have $\left(\left|\lambda_{1}\right|-\left|\lambda_{2}\right|\right)\left(\left|\sigma_{1}\right|-\left|\sigma_{2}\right|\right)>0$ where $\lambda_{1}$ and $\sigma_{1}$ are the eigenvalues of the eigenvectors tangent to the line at infinity.

Finite and infinite singular points may either be real or complex. In case we have a complex singular point we will specify this with the symbols © and © for finite and infinite points respectively. We point out that the sum of the multiplicities of all singular points of a quadratic system with a finite number of singular points, is always 7 (here of course we refer to the compactification on the complex projective plane $P_{2}(\mathbb{C})$ of the foliation with singularities associated to the complexification of the vector field, see Section 6.1). The sum of the multiplicities of the infinite singular points is always at least 3 , more precisely it is always 3 plus the sum of the multiplicities of the finite points disappeared at infinity.

We use here the following terminology for singularities:
We call elemental a singular point with its both eigenvalues not zero;
We call semi-elemental a singular point with exactly one of its eigenvalues equal to zero;

We call nilpotent a singular point with both its eigenvalues zero but with its Jacobian matrix at that point not identically zero;

We call intricate a singular point with its Jacobian matrix identically zero.
The intricate singularities are usually called in the literature linearly zero. We use here the term intricate to indicate the rather complicated behavior of phase curves around such a singularity.

Roughly speaking a singular point $p$ of an analytic differential system $\chi$ is a multiple singularity of multiplicity $m$ if $p$ generates $m$ singularities, as close to $p$ as we
wish, in analytic perturbations $\chi_{\varepsilon}$ of this system and $m$ is the maximal such number. In polynomial differential systems of fixed degree $n$ we have several possibilities for obtaining multiple singularities. i) A finite singular point splits into several finite singularities in $n$-degree polynomial perturbations. ii) An infinite singular point splits into some finite and some infinite singularities in $n$-degree polynomial perturbations. iii) An infinite singularity splits only in infinite singular points of the systems in n-degree perturbations. To all these cases we can give a precise mathematical meaning using the notion of intersection multiplicity at a point $p$ of two algebraic curves (see $[29,30]$ ).

We will say that two foci (or saddles) are order equivalent if their corresponding orders coincide.

Semi-elemental saddle-nodes are always topologically equivalent.
To define the notion of geometric equivalence relation of singularities we first define for nilpotent and intricate singular points, the notion of blow-up equivalence. We start by having a degenerate singular point $p_{1}$ at the origin of the plane of coordinates $\left(x_{0}, y_{0}\right)$, such that $p_{1}$ has a positive number of characteristic directions. We define an $\varepsilon$-twist as a $k$-twist with $k$ small enough so that no characteristic direction (or special characteristic direction in the case of a star point) with negative slope is moved to positive slope. Then if $x_{0}=0$ is a characteristic direction, we do an $\varepsilon$-twist. After the blow-up $\left(x_{0}, y_{0}\right)=\left(x_{1}, y_{1} x_{1}\right)$ the singular point is replaced by the straight line $x_{1}=0$ in the plane $\left(x_{1}, y_{1}\right)$. The neighborhood of the straight line $x_{1}=0$ in the projective plane obtained identifying the opposite infinite points of the Poincaré disk is a Möbius band $M_{1}$.

The straight line $x_{1}=0$ will be invariant and may be formed by a continuum of singular points. In that case, with a time change, this degeneracy may be removed and the $y_{1}$-axis will remain invariant.

Now we have a number $k_{1}$ of singularities located on the affine axis $x_{1}=0$. We do not include the infinite singular point which is the origin of the local chart $U_{2}$ at infinity $(Y \neq 0)$ because we already know that it does not play any role in understanding the local phase portrait of the singularity $p_{1}$. We can then list the $k_{1}$ singularities as $p_{1,1}, p_{1,2}, \ldots, p_{1, k_{1}}$ with decreasing order of the $y_{1}$ coordinate. The $p_{1, i}$ is adjacent to $p_{1, i+1}$ in the usual sense and $p_{1, k_{1}}$ is also adjacent to $p_{1,1}$ on the Möbius band.

Assume now that we have a degenerate singular point $p_{1}$ at the origin of the plane ( $x_{0}, y_{0}$ ) with an infinite number of characteristic directions. Then if $x_{0}=0$ is a special characteristic direction, we do an $\varepsilon$-twist. After the blow-up $\left(x_{0}, y_{0}\right)=$ $\left(x_{1}, y_{1} x_{1}\right)$ the singular point is replaced by the straight line $x_{1}=0$ in the plane $\left(x_{1}, y_{1}\right)$. The neighborhood of the straight line $x_{1}=0$ in the projective plane obtained identifying the opposite infinite points of the Poincaré disk is a Möbius band $M_{1}$.

The straight line $x_{1}=0$ will be invariant and formed by a continuum of singular points. In that case, with a time change, this degeneracy may be removed and the $y_{1}$-axis will no longer be invariant.

Now we have a set of cardinality $k_{1}$ formed by singularities located on the axis
$x_{1}=0$ plus contact points of the flow with the axis $x_{1}=0$. Again we do not include the infinite singular point at the origin of the local chart $U_{2}$ at infinity $(Y \neq 0)$ because we already know that it does not play any role in understanding the local phase portrait of the singularity $p_{1}$. We list again the $k_{1}$ points as $p_{1,1}, p_{1,2}, \ldots, p_{1, k_{1}}$ with decreasing order of the $y_{1}$ coordinate. The $p_{1, i}$ is adjacent to $p_{1, i+1}$ in the usual sense and $p_{1, k_{1}}$ is also adjacent to $p_{1,1}$ by the Möbius band.

Let $p_{2}$ be a degenerate singularity of another polynomial vector field and suppose that it is located at the origin of the plane ( $\bar{x}_{0}, \bar{y}_{0}$ ).

The next definition works whether the singular points are star-like or not.
We say that $p_{1}$ and $p_{2}$ are one step blow-up equivalent if modulus a rotation with center $p_{2}$ (before the blow-up) and a reflection (if needed) we have:
(i) the cardinality $k_{1}$ from $p_{1}$ equals the cardinality $k_{2}$ from $p_{2}$;
(ii) we can construct a homeomorphism $\phi_{p_{1}}^{1}: M_{1} \rightarrow M_{2}$ such that $\phi_{p_{1}}^{1}\left(\left\{x_{1}=\right.\right.$ $0\})=\left\{\bar{x}_{1}=0\right\}, \phi_{p_{1}}^{1}$ sends the points $p_{1, i}$ to $p_{2, i}$ and the phase portrait in a neighborhood $U$ of the axis $x_{1}=0$ is topologically equivalent to the phase portrait on $\phi_{p_{1}}^{1}(U)$;
(iii) $\phi_{p_{1}}^{1}$ sends an elemental (respectively semi-elemental, nilpotent or intricate) singular point to an elemental (respectively semi-elemental, nilpotent or intricate) singular point;
(iv) $\phi_{p_{1}}^{1}$ sends a contact point to a contact point.

Assuming $p_{1, j}$ and $\phi_{p_{1}}^{1}\left(p_{1, j}\right)=p_{2, j}$ are both intricate or both nilpotent, then the process of desingularization (blow-up) must be continued.

We do exactly the same study we did before for $p_{1}$ and $p_{2}$ now for $p_{1, j}$ and $p_{2, j}$. We move them to the respective origins of the planes $\left(x_{1}, y_{1}\right)$ and $\left(\bar{x}_{1}, \bar{y}_{1}\right)$ and we determine whether they are one step blow-up equivalent or not.

If successive degenerate singular points appear from desingularization of $p_{1}$ we do the same kind of changes that we did for $p_{1, j}$ and apply the corresponding definition of one step blow-up equivalence. This is repeated until after a finite number of blow-up's all the singular points that appear are elemental or semi-elemental.

We say that two singularities $p_{1}$ and $p_{2}$, both nilpotent or both intricate, of two polynomial vector fields $\chi_{1}$ and $\chi_{2}$, are blow-up equivalent if and only if
(i) they are one step blow-up equivalent;
(ii) at each level $j$ in the process of desingularization of $p_{1}$ and of $p_{2}$, two singularities which are related via the corresponding homeomorphism are one step blow-up equivalent.

Definition 1. Two singularities $p_{1}$ and $p_{2}$ of two polynomial vector fields are locally geometrically equivalent if and only if they are topologically equivalent, they have the same multiplicity and one of the following conditions is satisfied:

- $p_{1}$ and $p_{2}$ are order equivalent foci (or saddles);
- $p_{1}$ and $p_{2}$ are tangent equivalent simple nodes;
- $p_{1}$ and $p_{2}$ are both centers;
- $p_{1}$ and $p_{2}$ are both semi-elemental singularities;
- $p_{1}$ and $p_{2}$ are blow-up equivalent nilpotent or intricate singularities.

We say that two infinite isolated singularities $P_{1}$ and $P_{2}$ of two polynomial vector fields are blow-up equivalent if they are blow-up equivalent finite singularities in the corresponding infinite local charts and the number, type and ordering of sectors on each side of the line at infinity of $P_{1}$ coincide with those of $P_{2}$.

Definition 2. Let $\chi_{1}$ and $\chi_{2}$ be two polynomial vector fields each having a finite number of singularities. We say that $\chi_{1}$ and $\chi_{2}$ have geometricallyequivalent configurations of singularities if and only if we have a bijection $\vartheta$ carrying the singularities of $\chi_{1}$ to singularities of $\chi_{2}$ and for every singularity $p$ of $\chi_{1}, \vartheta(p)$ is geometricallyequivalent with $p$.

## 5 Notations for singularities of polynomial differential systems

In this work we encounter all the possibilities we have for the geometric features of both the finite and the infinite singularities in the whole quadratic class as well as the way they assemble in systems of this class. Since we want to describe precisely these geometric features and in order to facilitate understanding, it is important to have a clear, compact and congenial notation which conveys easily the information. The notation we use, even though it is used here to describe finite and infinite singular points of quadratic systems, can easily be extended to general polynomial systems.

We describe the finite and infinite singularities, denoting the first ones with lower case letters and the second with capital letters. When describing in a sequence both finite and infinite singular points, we will always place first the finite ones and only later the infinite ones, separating them by a semicolon ${ }^{\prime} ;$ '.

Elemental points: We use the letters ' $s$ ', $S$ ' for "saddles"; ' $n$ ', ' $N$ ' for "nodes"; ' $f$ ' for "foci"; ' $c$ ' for "centers" and © (respectively © (c) for complex finite (respectively infinite) singularities. In order to augment the level of precision we will distinguish the finite nodes as follows:

- ' $n$ ' for a node with two distinct eigenvalues (generic node);
- ' $n$ ' (a one-direction node) for a node with two identical eigenvalues whose Jacobian matrix is not diagonal;
- ' $n$ ') (a star-node) for a node with two identical eigenvalues whose Jacobian matrix is diagonal.

Moreover, in the case of an elemental infinite generic node, we want to distinguish whether the eigenvalue associated to the eigenvector directed towards the affine plane is, in absolute value, greater or lower than the eigenvalue associated to the eigenvector tangent to the line at infinity. This is relevant if we consider the geometric behavior of the phase curves around the node (see page 74). We will denote them as ' $N$ ' and ' $N$ ' respectively.

Finite elemental foci and saddles are classified as strong or weak foci, respectively strong or weak saddles. When the trace of the Jacobian matrix evaluated at those singular points is not zero, we call them strong saddles and strong foci and we maintain the standard notations ' $s$ ' and ' $f$.' But when the trace is zero, except for centers and saddles of infinite order (i.e. saddles with all their Poincaré-Lyapounov constants equal to zero), it is known that the foci and saddles, in the quadratic case, may have up to 3 orders. We denote them by ' $s$ (i)' and ' $f^{(i)}$ ' where $i=1,2,3$ is the order. In addition we have the centers which we denote by ' $c$ ' and saddles of infinite order (integrable saddles) which we denote by ' $\$$ '.

Foci and centers cannot appear as singular points at infinity and hence there is no need to introduce their order in this case. In the case of saddles, we can have weak saddles at infinity but the maximum order of weak singularities in cubic systems is not yet known. For this reason, a complete study of weak saddles at infinity cannot be done at this stage. Due to this, in this work we shall not even distinguish between a saddle and a weak saddle at infinity.

All non-elemental singular points are multiple points, in the sense that there are perturbations which have at least two elemental singular points as close as we wish to the multiple point. For finite singular points we denote with a subindex their multiplicity as in ' $\bar{S}_{(5)}$ ' or in ' $\widehat{e s}(3)$ ' (the notation ' $\rightarrow$ ' indicates that the saddle is semi-elemental and ' $\widehat{e s}_{(3)}$ ' indicates that the singular point is nilpotent). In order to describe the various kinds of multiplicity for infinite singular points we use the concepts and notations introduced in [31]. Thus we denote by ' $\left.\begin{array}{l}a \\ b \\ b\end{array}\right)$...' the maximum number $a$ (respectively $b$ ) of finite (respectively infinite) singularities which can be obtained by perturbation of the multiple point. For example ${ }^{( }\binom{1}{1} S N$ ' means a saddlenode at infinity produced by the collision of one finite singularity with an infinite one; ' $\binom{0}{3} S$ ' means a saddle produced by the collision of 3 infinite singularities.

Semi-elemental points: They can either be nodes, saddles or saddle-nodes, finite or infinite. We will denote the semi-elemental ones always with an overline, for example ' $\overline{s n}$ ', ' $\bar{s}$ ' and ' $\bar{n}$ ' with the corresponding multiplicity. In the case of infinite points we will put ${ }^{\text {( })}$ on top of the parenthesis with multiplicities.

Moreover, in cases that will be explained later (see page 94), an infinite saddlenode may be denoted by ${ }^{\binom{1}{1}} N S^{\prime}$ instead of $\binom{\overline{1}}{1} S N$ '. Semi-elemental nodes could never be ' $n$ ' ' or ' $n$ ' since their eigenvalues are always different. In the case of an infinite semi-elemental node, the type of collision determines whether the point is


Nilpotent points: They can either be saddles, nodes, saddle-nodes, ellipticsaddles, cusps, foci or centers. The first four of these could be at infinity. We denote
the nilpotent singular points with a hat ${ }^{〔}$, as in $\widehat{e s}_{(3)}$ for a finite nilpotent ellipticsaddle of multiplicity 3 and $\widehat{c p}_{(2)}$ for a finite nilpotent cusp point of multiplicity 2 . In the case of nilpotent infinite points, we will put the '^, on top of the parenthesis with multiplicity, for example $\binom{1}{2} P E P-H$ (the meaning of $P E P-H$ will be explained in the next paragraph). The relative position of the sectors of an infinite nilpotent point, with respect to the line at infinity, can produce topologically different phase portraits. This forces to use a notation for these points similar to the notation which we will use for the intricate points.

Intricate points: It is known that the neighborhood of any singular point of a polynomial vector field (except for foci and centers) is formed by a finite number of sectors which could only be of three types: parabolic, hyperbolic and elliptic (see [17]). Then, a reasonable way to describe intricate and nilpotent points is to use a sequence formed by the types of their sectors. The description we give is the one which appears in the clockwise direction (starting anywhere) once the blowdown of the desingularization is done. Thus in non-degenerate quadratic systems, we have just seven possibilities for finite intricate singular points of multiplicity four (see [4]) which are the following ones:

- a) $p h p p h p_{(4)}$;
- b) $p h p h_{(4)}$;
- c) $h h_{(4)}$;
- d) $h h h h h h_{(4)}$;
- e) $\operatorname{peppep}_{(4)}$;
- f) $\operatorname{pepe}_{(4)}$;
- g) $e e_{(4)}$.

We use lower case letters because of the finite nature of the singularities and add the subindex (4) since they are all of multiplicity 4.

For infinite intricate and nilpotent singular points, we insert a dash (hyphen) between the sectors to split those which appear on one side or the other of the equator of the sphere. In this way we will distinguish between $\binom{2}{2} P H P-P H P$ and $\binom{2}{2} P P H-P P H$.

Whenever we have an infinite nilpotent or intricate singular point, we will always start with a sector bordering the infinity (to avoid using two dashes). When one needs to describe a configuration of singular points at infinity, then the relative positions of the points, is relevant in some cases. In [3] this situation only occurs once for systems having two semi-elemental saddle-nodes at infinity and a third singular point which is elemental. In this case we need to write $N S$ instead of $S N$ for one of the semi-elemental points in order to have coherence of the positions of the parabolic (nodal) sector of one point with respect to the hyperbolic (saddle) of the
other semi-elemental point. More concretely, Figure 3 from [31] (which corresponds to Config. 3 in Figure 1) must be described as $\overline{\binom{1}{1}} S N, \overline{\binom{1}{1}} S N, N$ since the elemental node lies always between the hyperbolic sectors of one saddle-node and the parabolic ones of the other. However, Figure 4 from [31] (which corresponds to Config. 4 in Figure 1) must be described as $\overline{\binom{1}{1}} S N, \overline{\binom{1}{1}} N S, N$ since the hyperbolic sectors of each saddle-node lie between the elemental node and the parabolic sectors of the other saddle-node. These two configurations have exactly the same description of singular points but their relative position produces geometrically (and topologically) different portraits.

For the description of the topological phase portraits around the isolated singular points the information described above is sufficient. However we are interested in additional geometrical features such as the number of characteristic directions which figure in the final global picture of the desingularization. In order to add this information we need to introduce more notation. If two borsecs (the limiting orbits of a sector) arrive at the singular point with the same slope and direction, then the sector will be denoted by $H_{\curlywedge}, E_{\curlywedge}$ or $P_{\curlywedge}$. The index in this notation refers to the cusplike form of limiting trajectories of the sectors. Moreover, in the case of parabolic sectors we want to make precise whether the orbits arrive tangent to one borsec or to the other. We distinguish the two cases by $\widehat{P}$ if they arrive tangent to the borsec limiting the previous sector in clockwise sense or $\overparen{P}$ if they arrive tangent to the borsec limiting the next sector. Clearly, a parabolic sector denoted by $P^{*}$ would correspond to a sector in which orbits arrive with all possible slopes between the those of the borsecs. In the case of a cusp-like parabolic sector, all orbits must arrive with only one slope, but the distinction between $\overparen{P}$ and $\overparen{P}$ is still valid if we consider the different desingularizations we obtain from them. Thus, complicated intricate singular points like the two we see in Figure 5 may be described as $\binom{4}{2} \widehat{P E} \overparen{P}-H H H$ (case (a)) and ( $\left.\begin{array}{l}4 \\ 3\end{array}\right) E \hat{P}_{\curlywedge} H-H \tilde{P}_{\curlywedge} E$ (case (b)), respectively.

The lack of finite singular points will be encapsulated in the notation $\emptyset$. In the cases we need to point out the lack of an infinite singular point, we will use the symbol $\emptyset$.

Finally there is also the possibility that we have an infinite number of finite or of infinite singular points. In the first case, this means that the polynomials defining the differential system are not coprime. Their common factor may produce a line or conic with real coefficients filled up with singular points.

Line at infinity filled up with singularities: It is known that any such system has in a sufficiently small neighborhood of infinity one of 6 topologically distinct phase portraits (see [34]). The way to determine these portraits is by studying the reduced systems on the infinite local charts after removing the degeneracy of the systems within these charts. In case a singular point still remains on the line at infinity we study such a point. In [34] the tangential behavior of the solution curves was not considered in the case of a node. If after the removal of the degeneracy in the local charts at infinity a node remains, this could either be of the type $N^{d}, N$ and $N^{\star}$ (this last case does not occur in quadratic systems as it was shown in [3]).

Since no eigenvector of such a node $N$ (for quadratic systems) will have the direction of the line at infinity we do not need to distinguish $N^{f}$ and $N^{\infty}$. Other types of singular points at infinity of quadratic systems, after removal of the degeneracy, can be saddles, centers, semi-elemental saddle-nodes or nilpotent elliptic-saddles. We also have the possibility of no singularities after the removal of the degeneracy. To convey the way these singularities were obtained as well as their nature, we use the notation $[\infty ; \emptyset],[\infty ; N],\left[\infty ; N^{d}\right],[\infty ; S],[\infty ; C],\left[\infty ;\binom{\overline{1}}{0} S N\right]$ or $\left[\infty ;\binom{\widehat{3}}{0} E S\right]$.

Degenerate systems: We will denote with the symbol $\ominus$ the case when the polynomials defining the system have a common factor. This symbol stands for the most generic of these cases which corresponds to a real line filled up with singular points. The degeneracy can also be produced by a common quadratic factor which defines a conic. It is well known that by an affine transformation any conic over $\mathbb{R}$ can be brought to one of the following forms: $x^{2}+y^{2}-1=0$ (real ellipse), $x^{2}+y^{2}+1=0$ (complex ellipse), $x^{2}-y^{2}=1$ (hyperbola), $y-x^{2}=0$ (parabola), $x^{2}-y^{2}=0$ (pair of intersecting real lines), $x^{2}+y^{2}=0$ (pair of intersecting complex lines), $x^{2}-1=0$ (pair of parallel real lines), $x^{2}+1=0$ (pair of parallel complex lines), $x^{2}=0$ (double line).

We will indicate each case by the following symbols:

- $\ominus[\mid]$ for a real straight line;
- $\ominus[0]$ for a real ellipse;
- $\ominus[\subset]$ for a complex ellipse;
- $\ominus[)(]$ for an hyperbola;
- $\ominus[\cup]$ for a parabola;
- $\ominus[\times]$ for two real straight lines intersecting at a finite point;
$\bullet \ominus[\cdot]$ for two complex straight lines which intersect at a real finite point.
- $\ominus[\|]$ for two real parallel lines;
- $\ominus\left[\|^{c}\right]$ for two complex parallel lines;
- $\ominus[\mid 2]$ for a double real straight line.

Moreover, we also want to determine whether after removing the common factor of the polynomials, singular points remain on the curve defined by this common factor. If the reduced system has no finite singularity on this curve, we will use the symbol $\emptyset$ to describe this situation. If some singular points remain we will use the corresponding notation of their types. As an example we complete the notation above as follows:

- $(\ominus[] ; \emptyset)$ denotes the presence of a real straight line filled up with singular points such that the reduced system has no singularity on this line;
- $(\ominus[]] ; f)$ denotes the presence of the same straight line such that the reduced system has a strong focus on this line;
- ( $\Theta[\cup] ; \emptyset)$ denotes the presence of a parabola filled up with singularities such that no singular point of the reduced system is situated on this parabola.

Degenerate systems with non-isolated singular points at infinity, which are however isolated on the line at infinity: The existence of a common factor of the polynomials defining the differential system also affects the infinite singular points. We point out that the projective completion of a real affine line filled up with singular points has a point on the line at infinity which will then be also a non-isolated singularity.

In order to describe correctly the singularities at infinity, we must mention also this kind of phenomena and describe what happens to such points at infinity after the removal of the common factor. To show the existence of the common factor we will use the same symbol $\ominus$ as before, and for the type of degeneracy we use the symbols introduced above. We will use the symbol $\emptyset$ to denote the non-existence of real infinite singular points after the removal of the degeneracy. We will use the corresponding capital letters to describe the singularities which remain there. We take note that a simple straight line, two parallel lines (real or complex), one double line or one parabola defined by the common factor (all taken over the reals) imply the existence of one real non-isolated singular point at infinity in the original degenerate system. However a hyperbola and two real straight lines intersecting at a finite point imply the presence of two real non-isolated singular points at infinity in the original degenerate system. Finally, a complex ellipse and two complex straight lines which intersect at a real finite point imply the presence of two complex nonisolated singular points at infinity in the original degenerate system. Thus, in the reduced system these points may disappear as singularities and in case they remain, they must be described. For the first four cases mentioned above we will give the description of the corresponding infinite point. In the next four cases we will give the description of the corresponding two singular points. According to our notation, we will use capital letters to denote them since they are on the line at infinity. We give below some examples:

- $N^{f}, S,(\ominus[\| ; \emptyset)$ means that the system has a node at infinity such that an infinite number of orbits arrive tangent to the eigenvector in the affine part, a saddle, and one non-isolated singular point which belongs to a real affine straight line filled up with singularities, and that the reduced linear system has no infinite singular points in that position;
- $\left.S,(\ominus[]] ; N^{*}\right)$ means that the system has a saddle at infinity, and one nonisolated singular point which belongs to a real affine straight line filled up with singularities, and that the reduced linear system has a star node in that position;
- $S,(\ominus[)(] ; \emptyset, \emptyset)$ means that the system has a saddle at infinity, and two nonisolated singular points which belong to a hyperbola filled up with singularities, and that the reduced constant system has no singularities in those positions;
- $\left(\ominus[\times] ; N^{*}, \emptyset\right)$ means that the system has two non-isolated singular points at infinity which belong to two real intersecting straight lines filled up with singularities, and that the reduced constant system has a star node in one of those positions and no singularities in the other;
- $S,(\ominus[\circ] ; \emptyset, \emptyset)$ means that the system has a saddle at infinity, and two nonisolated (complex) singular points which are located on the complexification of a real ellipse which has no real points at infinity, and the reduced constant system has no singularities in those positions.

When there is a non-isolated infinite singular point such that the reduced system has a singularity at that position, it may happen that one or several characteristic directions at this point, directed towards the affine plane, could coincide with a tangent line to the curve of singularities at this point. This situation could produce many different geometrical (or even topological) combinations but in the quadratic case we only have a few of them for which we introduce a coherent notation. This notation can be further developed for higher degree systems. In quadratic systems we only need to distinguish among some situations in which, after the removal of the degeneracy, a characteristic direction of the infinite singular point may coincide or may not coincide with a tangent line to the curve of singularities at this point. We show in Figure 6 two cases that need to be distinguished (case (a) and (b)). Here we will use a numerical subscript which denotes the cardinal number $\mathcal{K}$ of the union of the set of characteristic directions, together with the set of tangent lines to the curve of singularities at this point, all of them considered in a neighborhood of the point at infinity on the Poincaré sphere. The singularities at infinity of examples (a) and (b) of Figure 6 would then be denoted by $S$, $\left.(\ominus[]] ; N_{3}^{\infty}\right)$ (case (a)) and $S,\left(\ominus[\mid] ; N_{2}^{\infty}\right)($ case (b)).


Figure 6.

## Degenerate systems with the line at infinity filled up with singularities:

For a quadratic system this implies that the polynomials must have a common linear factor and there are only two possible phase portraits, which can be seen in Figure

6 (the portraits $(c)$ and $(d)$ ). In order to be consistent with our notation and considering generalization to higher degree systems, we describe the two cases in a way coherent with what we have done up to now.

The case $(c)$ is denoted by $\left[\infty ;\left(\ominus[\|] ; \emptyset_{3}\right)\right]$ which means:

- the line at infinity is filled up with singular points;
- the reduced quadratic system has on one of the infinite local charts a nonisolated singular point on the line at infinity due to the affine line of degeneracy;
- once the original system at infinity is reduced to a linear one by removing the common factor, the infinity continues to be filled up with singular points;
- once the system on a local chart around the singularity which is common to both lines filled up with singular points, is reduced by completely removing the degeneracy, there is no singular point on that intersection;
- the cardinal number $\mathcal{K}$ is 3 . This means that apart from the line of singularities and the line at infinity, we have another characteristic direction pointing towards the affine plane.

The second case is denoted by $\left[\infty ;\left(\ominus[\|] ; \emptyset_{2}\right)\right]$, which means exactly the same items as above with the exception that cardinal number $\mathcal{K}$ is 2 . That is, beyond the line of singularities and the line at infinity, we have no other characteristic direction.

## 6 Assembling multiplicities for global configurations of singularities at infinity using divisors

The singular points at infinity belong to compactifications of planar polynomial differential systems, defined on the affine plane. We begin this section by briefly recalling these compactifications.

### 6.1 Compactifications associated to planar polynomial differential systems

### 6.1.1 Compactification on the sphere and on the Poincaré disk

Planar polynomial differential systems (1) can be compactified on the sphere. For this we consider the affine plane of coordinates $(x, y)$ as being the plane $Z=1$ in $\mathbb{R}^{3}$ with the origin located at $(0,0,1)$, the $x$-axis parallel with the $X$-axis in $\mathbb{R}^{3}$, and the $y$-axis parallel to the $Y$-axis. We use central projection to project this plane on the sphere as follows: for each point $(x, y, 1)$ we consider the line joining the origin with $(x, y, 1)$. This line intersects the sphere in two points $P_{1}=(X, Y, Z)$ and $P_{2}=(-X,-Y,-Z)$ where $(X, Y, Z)=\left(1 / \sqrt{x^{2}+y^{2}+1}\right)(x, y, 1)$. The applications $(x, y) \mapsto P_{1}$ and $(x, y) \mapsto P_{2}$ are bianalytic and associate to a vector field on the plane $(x, y)$ an analytic vector field $\Psi$ on the upper hemisphere and also an analytic
vector field $\Psi^{\prime}$ on the lower hemisphere. A theorem stated by Poincaré and proved in [18] says that there exists an analytic vector field $\Theta$ on the whole sphere which simultaneously extends the vector fields on the two hemispheres. By the Poincaré compactification on the sphere of a planar polynomial vector field we mean the restriction $\bar{\Psi}$ of the vector field $\Theta$ to the union of the upper hemisphere with the equator. For more details we refer to [21]. The vertical projection of $\bar{\Psi}$ on the plane $Z=0$ gives rise to an analytic vector field $\Phi$ on the unit disk of this plane. By the compactification on the Poincaré disk of a planar polynomial vector field we understand the vector field $\Phi$. By a singular point at infinity of a planar polynomial vector field we mean a singular point of the vector field $\bar{\Psi}$ which is located on the equator of the sphere, respectively a singular point of the vector field $\Phi$ located on the circumference of the Poincaré disk.

### 6.1.2 Compactification on the projective plane

To a polynomial system (1) we can associate a differential equation $\omega_{1}=$ $q(x, y) d x-p(x, y) d y=0$. Assuming the differential system (1) is with real coefficients, we may associate to it a foliation with singularities on the real, respectively complex, projective plane as indicated below. The equation $\omega_{1}=0$ defines a foliation with singularities on the real or complex plane depending if we consider the equation as being defined over the real or complex affine plane. It is known that we can compactify these foliations with singularities on the real respectively complex projective plane. In the study of real planar polynomial vector fields, their associated complex vector fields and their singularities play an important role. In particular such a vector field could have complex, non-real singularities, by this meaning singularities of the associated complex vector field. We briefly recall below how these foliations with singularities are defined.

The application $\Upsilon: \mathbb{K}^{2} \longrightarrow P_{2}(\mathbb{K})$ defined by $(x, y) \mapsto[x: y: 1]$ is an injection of the plane $\mathbb{K}^{2}$ over the field $\mathbb{K}$ into the projective plane $P_{2}(\mathbb{K})$ whose image is the set of $[X: Y: Z]$ with $Z \neq 0$. If $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$ this application is an analytic injection. If $Z \neq 0$ then $(\Upsilon)^{-1}([X: Y: Z])=(x, y)$ where $(x, y)=(X / Z, Y / Z)$. We obtain a $\operatorname{map} i: \mathbb{K}^{3} \backslash\{Z=0\} \longrightarrow \mathbb{K}^{2}$ defined by $[X: Y: Z] \mapsto(X / Z, Y / Z)$.

Considering that $d x=d(X / Z)=(Z d X-X d Z) / Z^{2}$ and $d y=(Z d Y-$ $Y d Z) / Z^{2}$, the pull-back of the form $\omega_{1}$ via the map $i$ yields the form $i *\left(\omega_{1}\right)=$ $q(X / Z, Y / Z)(Z d X-X d Z) / Z^{2}-p(X / Z, Y / Z)(Z d Y-Y d Z) / Z^{2}$ which has poles on $Z=0$. Then the form $\omega=Z^{m+2} i *\left(\omega_{1}\right)$ on $K^{3} \backslash\{Z=0\}, K$ being $\mathbb{R}$ or $\mathbb{C}$ and $m$ being the degree of systems (1) yields the equation $\omega=0$ :

$$
A(X, Y, Z) d X+B(X, Y, Z) d Y+C(X, Y, Z) d Z=0
$$

on $K^{3} \backslash\{Z=0\}$ where $A, B, C$ are homogeneous polynomials over $K$ with $A(X, Y, Z)=Z Q(X, Y, Z), Q(X, Y, Z)=Z^{m} q(X / Z, Y / Z), B(X, Y, Z)=Z P(X, Y, Z)$, $P(X, Y, Z)=Z^{m} p(X / Z, Y / Z)$ and $C(X, Y, Z)=Y P(X, Y, Z)-X Q(X, Y, Z)$.

The equation $A d X+B d Y+C d Z=0$ defines a foliation $F$ with singularities on the projective plane over $K$ with $K$ either $\mathbb{R}$ or $\mathbb{C}$. The points at infinity of the
foliation defined by $\omega_{1}=0$ on the affine plane are the points $[X: Y: 0$ ] and the line $Z=0$ is called the line at infinity of the foliation with singularities generated by $\omega_{1}=0$.

The singular points of the foliation $F$ are the solutions of the three equations $A=0, B=0, C=0$. In view of the definitions of $A, B, C$ it is clear that the singular points at infinity are the points of intersection of $Z=0$ with $C=0$.

### 6.2 Assembling data on infinite singularities in divisors of the line at infinity

In the previous sections we have seen that there are two types of multiplicities for a singular point $p$ at infinity: one expresses the maximum number $m$ of infinite singularities which can split from $p$, in small perturbations of the system and the other expresses the maximum number $m^{\prime}$ of finite singularities which can split from $p$, in small perturbations of the system. In Section 2 we mentioned that we shall use a column $\left(m, m^{\prime}\right)^{t}$ to indicate this situation.

We are interested in the global picture which includes all singularities at infinity. Therefore we need to assemble the data for individual singularities in a convenient, precise way. To do this we use for this situation the notion of cycle on an algebraic variety as indicated in [24] and which was used in [21] as well as in [31].

We briefly recall here the definition of this notion. Let $V$ be an irreducible algebraic variety over a field $\mathbb{K}$. A cycle of dimension $r$ or $r-c y c l e ~ o n ~ V ~ i s ~ a ~ f o r m a l ~$ sum $\sum_{W} n_{W} W$, where $W$ is a subvariety of $V$ of dimension $r$ which is not contained in the singular locus of $V, n_{W} \in \mathbb{Z}$, and only a finite number of the coefficients $n_{W}$ are non-zero. The degree $\operatorname{deg}(J)$ of a cycle $J$ is defined by $\sum_{W} n_{W}$. An $(n-1)$-cycle is called a divisor on $V$. These notions were used for classification purposes of planar quadratic differential systems in $[21,24,31]$.

To a system (1) we can associate two divisors on the line at infinity $Z=0$ of the complex projective plane: $D_{S}(P, Q ; Z)=\sum_{w} I_{w}(P, Q) w$ and $D_{S}(C, Z)=$ $\sum_{w} I_{w}(C, Z) w$ where $w \in\{Z=0\}$ and where by $I_{w}(F, G)$ we mean the intersection multiplicity at $w$ of the curves $F(X, Y, Z)=0$ and $G(X, Y, Z)=0$, with $F$ and $G$ homogeneous polynomials in $X, Y, Z$ over $\mathbb{C}$. For more details see [21].

Following [31] we assemble the above two divisors on the line at infinity into just one but with values in the ring $\mathbb{Z}^{2}$ :

$$
D_{S}=\sum_{\omega \in\{Z=0\}}\binom{I_{w}(P, Q)}{I_{w}(C, Z)} w .
$$

This divisor encodes the total number of singularities at infinity of a system (1) as well as the two kinds of multiplicities which each singularity has. The meaning of these two kinds of multiplicities are described in the definition of the two divisors $D_{S}(P, Q ; Z)$ and $D_{S}(C, Z)$ on the line at infinity.

## 7 Invariant polynomials and preliminary results

Consider real quadratic systems of the form:

$$
\begin{align*}
& \frac{d x}{d t}=p_{0}+p_{1}(x, y)+p_{2}(x, y) \equiv P(x, y) \\
& \frac{d y}{d t}=q_{0}+q_{1}(x, y)+q_{2}(x, y) \equiv Q(x, y) \tag{3}
\end{align*}
$$

with homogeneous polynomials $p_{i}$ and $q_{i}(i=0,1,2)$ of degree $i$ in $x, y$ :

$$
\begin{gathered}
p_{0}=a_{00}, \quad p_{1}(x, y)=a_{10} x+a_{01} y, \quad p_{2}(x, y)=a_{20} x^{2}+2 a_{11} x y+a_{02} y^{2} \\
q_{0}=b_{00}, \quad q_{1}(x, y)=b_{10} x+b_{01} y, \quad q_{2}(x, y)=b_{20} x^{2}+2 b_{11} x y+b_{02} y^{2} .
\end{gathered}
$$

Let $\tilde{a}=\left(a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02}\right)$ be the 12 -tuple of the coefficients of systems (3) and denote $\mathbb{R}[\tilde{a}, x, y]=\mathbb{R}\left[a_{00}, \ldots, b_{02}, x, y\right]$.

### 7.1 Affine invariant polynomials associated to infinite singularities

It is known that on the set QS of all quadratic differential systems (3) acts the group $\operatorname{Aff}(2, \mathbb{R})$ of the affine transformations on the plane (cf.[31]). For every subgroup $G \subseteq A f f(2, \mathbb{R})$ we have an induced action of $G$ on $\mathbf{Q S}$. We can identify the set $\mathbf{Q S}$ of systems (3) with a subset of $\mathbb{R}^{12}$ via the map $\mathbf{Q S} \longrightarrow \mathbb{R}^{12}$ which associates to each system (3) the 12 -tuple $\left(a_{00}, \ldots, b_{02}\right)$ of its coefficients.

For the definitions of a $G L$-comitant and invariant as well as for the definitions of a $T$-comitant and a $C T$-comitant we refer the reader to the paper [31] (see also [38]). Here we shall only construct the necessary $T$-comitants and $C T$-comitants associated to configurations of infinite singularities (including multiplicities) of quadratic systems (3).

Consider the polynomial $\Phi_{\alpha, \beta}=\alpha P^{*}+\beta Q^{*} \in \mathbb{R}[\tilde{a}, X, Y, Z, \alpha, \beta]$, where $P^{*}=Z^{2} P(X / Z, Y / Z), Q^{*}=Z^{2} Q(X / Z, Y / Z), P, Q \in \mathbb{R}[\tilde{a}, x, y]$ and $\max \left(\operatorname{deg}_{(x, y)} P, \operatorname{deg}_{(x, y)} Q\right)=2$. Then

$$
\begin{aligned}
\Phi_{\alpha, \beta}= & s_{11}(\tilde{a}, \alpha, \beta) X^{2}+2 s_{12}(\tilde{a}, \alpha, \beta) X Y+s_{22}(\tilde{a}, \alpha, \beta) Y^{2}+2 s_{13}(\tilde{a}, \alpha, \beta) X Z \\
& +2 s_{23}(\tilde{a}, \alpha, \beta) Y Z+s_{33}(\tilde{a}, \alpha, \beta) Z^{2}
\end{aligned}
$$

and we denote

$$
\begin{aligned}
& \widetilde{D}(\tilde{a}, x, y)=4 \operatorname{det}\left\|s_{i j}(\tilde{a}, y,-x)\right\|_{i, j \in\{1,2,3\}} \\
& \widetilde{H}(\tilde{a}, x, y)=4 \operatorname{det}\left\|s_{i j}(\tilde{a}, y,-x)\right\|_{i, j \in\{1,2\}}
\end{aligned}
$$

We consider the polynomials

$$
\begin{align*}
& C_{i}(\tilde{a}, x, y)=y p_{i}(\tilde{a}, x, y)-x q_{i}(\tilde{a}, x, y), \\
& D_{i}(\tilde{a}, x, y)=\frac{\partial}{\partial x} p_{i}(\tilde{a}, x, y)+\frac{\partial}{\partial y} q_{i}(\tilde{a}, x, y), \tag{4}
\end{align*}
$$

in $\mathbb{R}[\tilde{a}, x, y]$ for $i=0,1,2$ and $i=1,2$ respectively. Using the so-called transvectant of order $k$ (see [19],[22]) of two polynomials $f, g \in \mathbb{R}[\tilde{a}, x, y]$

$$
(f, g)^{(k)}=\sum_{h=0}^{k}(-1)^{h}\binom{k}{h} \frac{\partial^{k} f}{\partial x^{k-h} \partial y^{h}} \frac{\partial^{k} g}{\partial x^{h} \partial y^{k-h}}
$$

we construct the following $G L$-comitants of the second degree with the coefficients of the initial system

$$
\begin{array}{lll}
T_{1}=\left(C_{0}, C_{1}\right)^{(1)}, & T_{2}=\left(C_{0}, C_{2}\right)^{(1)}, & T_{3}=\left(C_{0}, D_{2}\right)^{(1)}, \\
T_{4}=\left(C_{1}, C_{1}\right)^{(2)}, & T_{5}=\left(C_{1}, C_{2}\right)^{(1)}, & T_{6}=\left(C_{1}, C_{2}\right)^{(2)},  \tag{5}\\
T_{7}=\left(C_{1}, D_{2}\right)^{(1)}, & T_{8}=\left(C_{2}, C_{2}\right)^{(2)}, & T_{9}=\left(C_{2}, D_{2}\right)^{(1)} .
\end{array}
$$

Using these $G L$-comitants as well as the polynomials (4) we construct the additional invariant polynomials (see also [31])

$$
\begin{aligned}
& \widetilde{M}(\tilde{a}, x, y)=\left(C_{2}, C_{2}\right)^{(2)} \equiv 2 \operatorname{Hess}\left(C_{2}(\tilde{a}, x, y)\right) ; \\
& \eta(\tilde{a})=(\widetilde{M}, \widetilde{M})^{(2)} / 384 \equiv \operatorname{Discrim}\left(C_{2}(\tilde{a}, x, y)\right) ; \\
& \widetilde{K}(\tilde{a}, x, y)= \operatorname{Jacob}\left(p_{2}(\tilde{a}, x, y), q_{2}(\tilde{a}, x, y)\right) ; \\
& K_{1}(\tilde{a}, x, y)= p_{1}(\tilde{a}, x, y) q_{2}(\tilde{a}, x, y)-p_{2}(\tilde{a}, x, y) q_{1}(\tilde{a}, x, y) ; \\
& K_{2}(\tilde{a}, x, y)= 4\left(T_{2}, \widetilde{M}-2 \widetilde{K}\right)^{(1)}+3 D_{1}\left(C_{1}, \widetilde{M}-2 \widetilde{K}\right)^{(1)}- \\
&-(\widetilde{M}-2 \widetilde{K})\left(16 T_{3}-3 T_{4} / 2+3 D_{1}^{2}\right) ; \\
& K_{3}(\tilde{a}, x, y)= C_{2}^{2}\left(4 T_{3}+3 T_{4}\right)+C_{2}\left(3 C_{0} \widetilde{K}-2 C_{1} T_{7}\right)+2 K_{1}\left(3 K_{1}-C_{1} D_{2}\right) ; \\
& \tilde{L}(\tilde{a}, x, y)= 4 \widetilde{K}+8 \widetilde{H}-\widetilde{M} ; \\
& L_{1}(\tilde{a}, x, y)=\left(C_{2}, \widetilde{D}\right)^{(2)} ; \\
& \widetilde{R}(\tilde{a}, x, y)= \tilde{L}+8 \widetilde{K} ; \\
& \kappa(\widetilde{a})=(\widetilde{M}, \widetilde{K})^{(2)} / 4 ; \\
& \kappa_{1}(\widetilde{a})=\left(\widetilde{M}, C_{1}\right)^{(2)} ; \\
& \widetilde{N}(\tilde{a}, x, y)=\widetilde{K}(\tilde{a}, x, y)+\widetilde{H}(\tilde{a}, x, y) ; \\
& \theta_{6}(\tilde{a}, x, y)= C_{1} T_{8}-2 C_{2} T_{6} .
\end{aligned}
$$

The geometrical meaning of the invariant polynomials $C_{2}, \widetilde{M}$ and $\eta$ is revealed in the next lemma (see [31]).

Lemma 1. The form of the divisor $D_{S}(C, Z)$ for systems (3) is determined by the corresponding conditions indicated in Table 1, where we write $w_{1}^{c}+w_{2}^{c}+w_{3}$ if two of the points, i.e. $w_{1}^{c}, w_{2}^{c}$, are complex but not real. Moreover, for each form of the divisor $D_{S}(C, Z)$ given in Table 1 the quadratic systems (3) can be brought via a linear transformation to one of the following canonical systems $\left(\mathbf{S}_{I}\right)-\left(\mathbf{S}_{V}\right)$ corresponding to their behavior at infinity.

Table 1

| Case | Form of $D_{S}(C, Z)$ | Necessary and <br> sufficient conditions <br> on the comitants |
| :---: | :---: | :---: |
| 1 | $w_{1}+w_{2}+w_{3}$ | $\eta>0$ |
| 2 | $w_{1}^{c}+w_{2}^{c}+w_{3}$ | $\eta<0$ |
| 3 | $2 w_{1}+w_{2}$ | $\eta=0, \quad \widetilde{M} \neq 0$ |
| 4 | $3 w$ | $\widetilde{M}=0, \quad C_{2} \neq 0$ |
| 5 | $D_{S}(C, Z)$ undefined | $C_{2}=0$ |

$$
\begin{align*}
& \left\{\begin{aligned}
\dot{x} & =a+c x+d y+g x^{2}+(h-1) x y, \\
\dot{y} & =b+e x+f y+(g-1) x y+h y^{2} ;
\end{aligned}\right.  \tag{I}\\
& \begin{cases}\dot{x} & =a+c x+d y+g x^{2}+(h+1) x y, \\
\dot{y} & =b+e x+f y-x^{2}+g x y+h y^{2} ;\end{cases}  \tag{II}\\
& \left\{\begin{aligned}
\dot{x} & =a+c x+d y+g x^{2}+h x y, \\
\dot{y} & =b+e x+f y+(g-1) x y+h y^{2} ;
\end{aligned}\right.  \tag{III}\\
& \begin{cases}\dot{x} & =a+c x+d y+g x^{2}+h x y, \\
\dot{y} & =b+e x+f y-x^{2}+g x y+h y^{2},\end{cases}  \tag{IV}\\
& \left\{\begin{aligned}
\dot{x} & =a+c x+d y+x^{2}, \\
\dot{y} & =b+e x+f y+x y .
\end{aligned}\right. \tag{V}
\end{align*}
$$

### 7.2 Affine invariant polynomials associated to finite singularities

Consider the differential operator $\mathcal{L}=x \cdot \mathbf{L}_{2}-y \cdot \mathbf{L}_{1}$ acting on $\mathbb{R}[a, x, y]$ constructed in [9], where

$$
\begin{aligned}
& \mathbf{L}_{1}=2 a_{00} \frac{\partial}{\partial a_{10}}+a_{10} \frac{\partial}{\partial a_{20}}+\frac{1}{2} a_{01} \frac{\partial}{\partial a_{11}}+2 b_{00} \frac{\partial}{\partial b_{10}}+b_{10} \frac{\partial}{\partial b_{20}}+\frac{1}{2} b_{01} \frac{\partial}{\partial b_{11}}, \\
& \mathbf{L}_{2}=2 a_{00} \frac{\partial}{\partial a_{01}}+a_{01} \frac{\partial}{\partial a_{02}}+\frac{1}{2} a_{10} \frac{\partial}{\partial a_{11}}+2 b_{00} \frac{\partial}{\partial b_{01}}+b_{01} \frac{\partial}{\partial b_{02}}+\frac{1}{2} b_{10} \frac{\partial}{\partial b_{11}} .
\end{aligned}
$$

Using this operator and the affine invariant $\mu_{0}=\operatorname{Res}_{x}\left(p_{2}(\tilde{a}, x, y), q_{2}(\tilde{a}, x, y)\right) / y^{4}$ we construct the following polynomials

$$
\mu_{i}(\tilde{a}, x, y)=\frac{1}{i!} \mathcal{L}^{(i)}\left(\mu_{0}\right), i=1, . ., 4,
$$

where $\mathcal{L}^{(i)}\left(\mu_{0}\right)=\mathcal{L}\left(\mathcal{L}^{(i-1)}\left(\mu_{0}\right)\right)$.
These polynomials are in fact comitants of systems (3) with respect to the group $G L(2, \mathbb{R})$ (see [9]). Their geometrical meaning is revealed in Lemmas 2 and 3 below.

Lemma 2. ([8]) The total multiplicity of all finite singularities of a quadratic system (3) equals $k$ if and only if for every $i \in\{0,1, \ldots, k-1\}$ we have $\mu_{i}(\tilde{a}, x, y)=0$ in
$\mathbb{R}[x, y]$ and $\mu_{k}(\tilde{a}, x, y) \neq 0$. Moreover a system (3) is degenerate (i.e. $\operatorname{gcd}(P, Q) \neq$ constant) if and only if $\mu_{i}(\tilde{a}, x, y)=0$ in $\mathbb{R}[x, y]$ for every $i=0,1,2,3,4$.

Lemma 3. ([9]) The point $M_{0}(0,0)$ is a singular point of multiplicity $k(1 \leq k \leq 4)$ for a quadratic system (3) if and only if for every $i \in\{0,1, \ldots, k-1\}$ we have $\mu_{4-i}(\tilde{a}, x, y)=0$ in $\mathbb{R}[x, y]$ and $\mu_{4-k}(\tilde{a}, x, y) \neq 0$.

We denote

$$
\sigma(\tilde{a}, x, y)=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}=\sigma_{0}(\tilde{a})+\sigma_{1}(\tilde{a}, x, y)\left(\equiv D_{1}(\tilde{a})+D_{2}(\tilde{a}, x, y)\right)
$$

and observe that the polynomial $\sigma(\tilde{a}, x, y)$ is an affine comitant of systems (3). It is known that if $\left(x_{i}, y_{i}\right)$ is a singular point of a system (3) then for the trace of its respective linear matrix we have $\rho_{i}=\sigma\left(x_{i}, y_{i}\right)$.

Applying the differential operators $\mathcal{L}$ and $(*, *)^{(k)}$ (i.e. transvectant of index $k$ ) we shall define the following polynomial function which governs the values of the traces for finite singularities of systems (3).

Definition 3 ([39]). We call trace polynomial $\mathfrak{T}(w)$ over the ring $\mathbb{R}[\tilde{a}]$ the polynomial defined as follows:

$$
\begin{equation*}
\mathfrak{T}(w)=\sum_{i=0}^{4} \frac{1}{(i!)^{2}}\left(\sigma_{1}^{i}, \frac{1}{i!} \mathcal{L}^{(i)}\left(\mu_{0}\right)\right)^{(i)} w^{4-i}=\sum_{i=0}^{4} \mathcal{G}_{i}(\tilde{a}) w^{4-i} \tag{6}
\end{equation*}
$$

where the coefficients $\mathcal{G}_{i}(\tilde{a})=\frac{1}{(i!)^{2}}\left(\sigma_{1}^{i}, \mu_{i}\right)^{(i)} \in \mathbb{R}[\tilde{a}], \quad i=0,1,2,3,4\left(\mathcal{G}_{0}(\tilde{a}) \equiv \mu_{0}(\tilde{a})\right)$ are $G L$-invariants.

Using the polynomial $\mathfrak{T}(w)$ we could construct the following four affine invariants $\mathcal{T}_{4}, \mathcal{T}_{3}, \mathcal{I}_{2}, \mathcal{T}_{1}$, which are responsible for the weak singularities:

$$
\mathcal{T}_{4-i}(\tilde{a})=\left.\frac{1}{i!} \frac{d^{i} \mathfrak{T}}{d w^{i}}\right|_{w=\sigma_{0}} \in \mathbb{R}[\tilde{a}], \quad i=0,1,2,3 \quad\left(\mathcal{T}_{4} \equiv \mathfrak{T}\left(\sigma_{0}\right)\right) .
$$

The geometric meaning of these invariants is revealed by the next lemma (see [39]).

Lemma 4. Consider a non-degenerate system (3) and let $\boldsymbol{a} \in \mathbb{R}^{12}$ be its 12-tuple of coefficients. Denote by $\rho_{s}$ the trace of the linear part of this system at a finite singular point $M_{s}, 1 \leq s \leq 4$ (real or complex, simple or multiple). Then the following relations hold, respectively:
(i) For $\mu_{0}(\boldsymbol{a}) \neq 0$ (total multiplicity 4):

$$
\begin{align*}
& \mathcal{T}_{4}(\boldsymbol{a})=\mathcal{G}_{0}(\boldsymbol{a}) \rho_{1} \rho_{2} \rho_{3} \rho_{4}, \\
& \mathcal{T}_{3}(\boldsymbol{a})=\mathcal{G}_{0}(\boldsymbol{a})\left(\rho_{1} \rho_{2} \rho_{3}+\rho_{1} \rho_{2} \rho_{4}+\rho_{1} \rho_{3} \rho_{4}+\rho_{2} \rho_{3} \rho_{4}\right)  \tag{7}\\
& \mathcal{T}_{2}(\boldsymbol{a})=\mathcal{G}_{0}(\boldsymbol{a})\left(\rho_{1} \rho_{2}+\rho_{1} \rho_{3}+\rho_{1} \rho_{4}+\rho_{2} \rho_{3}+\rho_{2} \rho_{4}+\rho_{3} \rho_{4}\right) \\
& \mathcal{T}_{1}(\boldsymbol{a})=\mathcal{G}_{0}(\boldsymbol{a})\left(\rho_{1}+\rho_{2}+\rho_{3}+\rho_{4}\right)
\end{align*}
$$

(ii) For $\mu_{0}(\boldsymbol{a})=0, \mu_{1}(\boldsymbol{a}, x, y) \neq 0$ (total multiplicity 3):

$$
\begin{array}{ll}
\mathcal{T}_{4}(\boldsymbol{a})=\mathcal{G}_{1}(\boldsymbol{a}) \rho_{1} \rho_{2} \rho_{3}, & \mathcal{T}_{3}(\boldsymbol{a})=\mathcal{G}_{1}(\boldsymbol{a})\left(\rho_{1} \rho_{2}+\rho_{1} \rho_{3}+\rho_{2} \rho_{3}\right), \\
\mathcal{T}_{2}(\boldsymbol{a})=\mathcal{G}_{1}(\boldsymbol{a})\left(\rho_{1}+\rho_{2}+\rho_{3}\right), & \mathcal{T}_{1}(\boldsymbol{a})=\mathcal{G}_{1}(\boldsymbol{a}) ; \tag{8}
\end{array}
$$

(iii) For $\mu_{0}(\boldsymbol{a})=\mu_{1}(\boldsymbol{a}, x, y)=0, \mu_{2}(\boldsymbol{a}, x, y) \neq 0$ (total multiplicity 2):

$$
\begin{array}{ll}
\mathcal{T}_{4}(\boldsymbol{a})=\mathcal{G}_{2}(\boldsymbol{a}) \rho_{1} \rho_{2}, & \mathcal{T}_{3}(\boldsymbol{a})=\mathcal{G}_{2}(\boldsymbol{a})\left(\rho_{1}+\rho_{2}\right), \\
\mathcal{T}_{2}(\boldsymbol{a})=\mathcal{G}_{2}(\boldsymbol{a}), & \mathcal{T}_{1}(\boldsymbol{a})=0 \tag{9}
\end{array}
$$

(iv) For $\mu_{0}(\boldsymbol{a})=\mu_{1}(\boldsymbol{a}, x, y)=\mu_{2}(\boldsymbol{a}, x, y)=0, \mu_{3}(\boldsymbol{a}, x, y) \neq 0$ (one singularity):

$$
\begin{equation*}
\mathcal{T}_{4}(\boldsymbol{a})=\mathcal{G}_{3}(\boldsymbol{a}) \rho_{1}, \quad \mathcal{T}_{3}(\boldsymbol{a})=\mathcal{G}_{3}(\boldsymbol{a}), \quad \mathcal{T}_{2}(\boldsymbol{a})=\mathcal{T}_{1}(\boldsymbol{a})=0 \tag{10}
\end{equation*}
$$

In order to be able to calculate the values of the needed invariant polynomials directly for every canonical system we shall define here a family of $T$-comitants (see [31] for detailed definitions) expressed through $C_{i}(i=0,1,2)$ and $D_{j}(j=1,2)$ :

$$
\begin{aligned}
\hat{A}= & \left(C_{1}, T_{8}-2 T_{9}+D_{2}^{2}\right)^{(2)} / 144, \\
\widehat{D}= & \frac{1}{36}\left[2 C_{0}\left(T_{8}-8 T_{9}-2 D_{2}^{2}\right)+C_{1}\left(6 T_{7}-T_{6}-\left(C_{1}, T_{5}\right)^{(1)}+\right.\right. \\
& \left.+6 D_{1}\left(C_{1} D_{2}-T_{5}\right)-9 D_{1}^{2} C_{2}\right], \\
\widehat{E}= & {\left[D_{1}\left(2 T_{9}-T_{8}\right)-3\left(C_{1}, T_{9}\right)^{(1)}-D_{2}\left(3 T_{7}+D_{1} D_{2}\right)\right] / 72, } \\
\widehat{F}= & {\left[6 D_{1}^{2}\left(D_{2}^{2}-4 T_{9}\right)+4 D_{1} D_{2}\left(T_{6}+6 T_{7}\right)+48 C_{0}\left(D_{2}, T_{9}\right)^{(1)}-9 D_{2}^{2} T_{4}+288 D_{1} \widehat{E}\right.} \\
& \left.-24\left(C_{2}, \widehat{D}\right)^{(2)}+120\left(D_{2}, \widehat{D}\right)^{(1)}-36 C_{1}\left(D_{2}, T_{7}\right)^{(1)}+8 D_{1}\left(D_{2}, T_{5}\right)^{(1)}\right] / 144, \\
\widehat{B}= & \left\{16 D_{1}\left(D_{2}, T_{8}\right)^{(1)}\left(3 C_{1} D_{1}-2 C_{0} D_{2}+4 T_{2}\right)+32 C_{0}\left(D_{2}, T_{9}\right)^{(1)}\left(3 D_{1} D_{2}-\right.\right. \\
& \left.-5 T_{6}+9 T_{7}\right)+2\left(D_{2}, T_{9}\right)^{(1)}\left(27 C_{1} T_{4}-18 C_{1} D_{1}^{2}-32 D_{1} T_{2}+32\left(C_{0}, T_{5}\right)^{(1)}\right) \\
& +6\left(D_{2}, T_{7}\right)^{(1)}\left[8 C_{0}\left(T_{8}-12 T_{9}\right)-12 C_{1}\left(D_{1} D_{2}+T_{7}\right)+D_{1}\left(26 C_{2} D_{1}+32 T_{5}\right)+\right. \\
& \left.+C_{2}\left(9 T_{4}+96 T_{3}\right)\right]+6\left(D_{2}, T_{6}\right)^{(1)}\left[32 C_{0} T_{9}-C_{1}\left(12 T_{7}+52 D_{1} D_{2}\right)-32 C_{2} D_{1}^{2}\right] \\
& +48 D_{2}\left(D_{2}, T_{1}\right)^{(1)}\left(2 D_{2}^{2}-T_{8}\right) \\
& -32 D_{1} T_{8}\left(D_{2}, T_{2}\right)^{(1)}+9 D_{2}^{2} T_{4}\left(T_{6}-2 T_{7}\right)-16 D_{1}\left(C_{2}, T_{8}\right)^{(1)}\left(D_{1}^{2}+4 T_{3}\right) \\
& +12 D_{1}\left(C_{1}, T_{8}\right)^{(2)}\left(C_{1} D_{2}-2 C_{2} D_{1}\right)+6 D_{1} D_{2} T_{4}\left(T_{8}-7 D_{2}^{2}-42 T_{9}\right) \\
& +12 D_{1}\left(C_{1}, T_{8}\right)^{(1)}\left(T_{7}+2 D_{1} D_{2}\right)+96 D_{2}^{2}\left[D_{1}\left(C_{1}, T_{6}\right)^{(1)}+D_{2}\left(C_{0}, T_{6}\right)^{(1)}\right] \\
& -16 D_{1} D_{2} T_{3}\left(2 D_{2}^{2}+3 T_{8}\right)-4 D_{1}^{3} D_{2}\left(D_{2}^{2}+3 T_{8}+6 T_{9}\right)+6 D_{1}^{2} D_{2}^{2}\left(7 T_{6}+2 T_{7}\right) \\
& \left.-252 D_{1} D_{2} T_{4} T_{9}\right\} /\left(2^{8} 3^{3}\right), \\
\widehat{K}= & \left(T_{8}+4 T_{9}+4 D_{2}^{2}\right) / 72 \equiv \widetilde{K} / 4, \\
\widehat{H}= & \left(8 T_{9}-T_{8}+2 D_{2}^{2}\right) / 72 \equiv-\widetilde{H} / 4, \\
\widehat{M}= & T_{8} .
\end{aligned}
$$

These polynomials in addition to (4) and (5) will serve as bricks in constructing affine invariant polynomials for systems (3).

The following 42 affine invariants $A_{1}, \ldots, A_{42}$ form the minimal polynomial basis of affine invariants up to degree 12. This fact was proved in [11] by constructing $A_{1}, \ldots, A_{42}$ using the above bricks.

$$
\begin{array}{lr}
A_{1}=\hat{A}, & \left.\left.\left.\left.A_{22}=\frac{1}{1152}\left[C_{2}, \widehat{D}\right)^{(1)}, D_{2}\right)^{(1)}, D_{2}\right)^{(1)}, D_{2}\right)^{(1)} D_{2}\right)^{(1)}, \\
A_{2}=\left(C_{2}, \widehat{D}\right)^{(3)} / 12, & \left.A_{23}=[\widehat{F}, \widehat{H})^{(1)}, \widehat{K}\right)^{(2)} / 8, \\
\left.\left.A_{3}=\left(C_{2}, D_{2}\right)^{(1)}, D_{2}\right)^{(1)}, D_{2}\right)^{(1)} / 48, & \left.\left.A_{24}=\left[C_{2}, \widehat{D}\right)^{(2)}, \widehat{K}\right)^{(1)}, \widehat{H}\right)^{(2)} / 32, \\
A_{4}=(\widehat{H}, \widehat{H})^{(2)}, & \left.A_{25}=[\widehat{D}, \widehat{D})^{(2)}, \widehat{E}\right)^{(2)} / 16, \\
A_{5}=(\widehat{H}, \widehat{K})^{(2)} / 2, & A_{26}=(\widehat{B}, \widehat{D})^{(3)} / 36,
\end{array}
$$

$$
\begin{aligned}
& A_{6}=(\widehat{E}, \widehat{H})^{(2)} / 2 \text {, } \\
& \left.\left.\left.A_{7}=\left[C_{2}, \widehat{E}\right)^{(2)}, D_{2}\right)^{(1)} / 8, \quad A_{28}=\left[C_{2}, \widehat{K}\right)^{(2)}, \widehat{D}\right)^{(1)}, \widehat{E}\right)^{(2)} / 16, \\
& \left.\left.A_{8}=[\widehat{D}, \widehat{H})^{(2)}, D_{2}\right)^{(1)} / 8, \quad A_{29}=[\widehat{D}, \widehat{F})^{(1)}, \widehat{D}\right)^{(3)} / 96, \\
& \left.\left.\left.\left.A_{9}=\left[\widehat{D}, D_{2}\right)^{(1)}, D_{2}\right)^{(1)}, D_{2}\right)^{(1)} / 48, \quad A_{30}=\left[C_{2}, \widehat{D}\right)^{(2)}, \widehat{D}\right)^{(1)}, \widehat{D}\right)^{(3)} / 288 \text {, } \\
& \left.\left.\left.A_{10}=[\widehat{D}, \widehat{K})^{(2)}, D_{2}\right)^{(1)} / 8, \quad A_{31}=[\widehat{D}, \widehat{D})^{(2)}, \widehat{K}\right)^{(1)}, \widehat{H}\right)^{(2)} / 64, \\
& \left.\left.\left.A_{11}=(\widehat{F}, \widehat{K})^{(2)} / 4, \quad A_{32}=[\widehat{D}, \widehat{D})^{(2)}, D_{2}\right)^{(1)}, \widehat{H}\right)^{(1)}, D_{2}\right)^{(1)} / 64, \\
& \left.\left.\left.A_{12}=(\widehat{F}, \widehat{H})^{(2)} / 4, \quad A_{33}=\left[\widehat{D}, D_{2}\right)^{(1)}, \widehat{F}\right)^{(1)}, D_{2}\right)^{(1)}, D_{2}\right)^{(1)} / 128, \\
& \left.\left.\left.\left.\left.A_{13}=\left[C_{2}, \widehat{H}\right)^{(1)}, \widehat{H}\right)^{(2)}, D_{2}\right)^{(1)} / 24, \quad A_{34}=[\widehat{D}, \widehat{D})^{(2)}, D_{2}\right)^{(1)}, \widehat{K}\right)^{(1)}, D_{2}\right)^{(1)} / 64 \text {, } \\
& \left.\left.\left.A_{14}=\left(\widehat{B}, C_{2}\right)^{(3)} / 36, \quad A_{35}=[\widehat{D}, \widehat{D})^{(2)}, \widehat{E}\right)^{(1)}, D_{2}\right)^{(1)}, D_{2}\right)^{(1)} / 128, \\
& \left.\left.A_{15}=(\widehat{E}, \widehat{F})^{(2)} / 4, \quad A_{36}=[\widehat{D}, \widehat{E})^{(2)}, \widehat{D}\right)^{(1)}, \widehat{H}\right)^{(2)} / 16, \\
& \left.\left.\left.\left.A_{16}=\left[\widehat{E}, D_{2}\right)^{(1)}, C_{2}\right)^{(1)}, \widehat{K}\right)^{(2)} / 16, \quad A_{37}=[\widehat{D}, \widehat{D})^{(2)}, \widehat{D}\right)^{(1)}, \widehat{D}\right)^{(3)} / 576 \text {, } \\
& \left.\left.\left.\left.\left.A_{17}=[\widehat{D}, \widehat{D})^{(2)}, D_{2}\right)^{(1)}, D_{2}\right)^{(1)} / 64, \quad A_{38}=\left[C_{2}, \widehat{D}\right)^{(2)}, \widehat{D}\right)^{(2)}, \widehat{D}\right)^{(1)}, \widehat{H}\right)^{(2)} / 64 \text {, } \\
& \left.\left.\left.A_{18}=[\widehat{D}, \widehat{F})^{(2)}, D_{2}\right)^{(1)} / 16, \quad A_{39}=[\widehat{D}, \widehat{D})^{(2)}, \widehat{F}\right)^{(1)}, \widehat{H}\right)^{(2)} / 64, \\
& \left.\left.\left.A_{19}=[\widehat{D}, \widehat{D})^{(2)}, \widehat{H}\right)^{(2)} / 16, \quad A_{40}=[\widehat{D}, \widehat{D})^{(2)}, \widehat{F}\right)^{(1)}, \widehat{K}\right)^{(2)} / 64, \\
& \left.\left.\left.\left.A_{20}=\left[C_{2}, \widehat{D}\right)^{(2)}, \widehat{F}\right)^{(2)} / 16, \quad A_{41}=\left[C_{2}, \widehat{D}\right)^{(2)}, \widehat{D}\right)^{(2)}, \widehat{F}\right)^{(1)}, D_{2}\right)^{(1)} / 64, \\
& \left.\left.\left.A_{21}=[\widehat{D}, \widehat{D})^{(2)}, \widehat{K}\right)^{(2)} / 16, \quad A_{42}=[\widehat{D}, \widehat{F})^{(2)}, \widehat{F}\right)^{(1)}, D_{2}\right)^{(1)} / 16 .
\end{aligned}
$$

In the above list, the bracket "[" is used in order to avoid placing the otherwise necessary up to five parenthesizes "(".

Using the elements of the minimal polynomial basis given above we construct
the affine invariants

$$
\begin{aligned}
\mathcal{F}_{1}(\tilde{a})= & A_{2}, \\
\mathcal{F}_{2}(\tilde{a})= & -2 A_{1}^{2} A_{3}+2 A_{5}\left(5 A_{8}+3 A_{9}\right)+A_{3}\left(A_{8}-3 A_{10}+3 A_{11}+A_{12}\right)- \\
& -A_{4}\left(10 A_{8}-3 A_{9}+5 A_{10}+5 A_{11}+5 A_{12}\right), \\
\mathcal{F}_{3}(\tilde{a})= & -10 A_{1}^{2} A_{3}+2 A_{5}\left(A_{8}-A_{9}\right)-A_{4}\left(2 A_{8}+A_{9}+A_{10}+A_{11}+A_{12}\right)+ \\
& +A_{3}\left(5 A_{8}+A_{10}-A_{11}+5 A_{12}\right), \\
\mathcal{F}_{4}(\tilde{a})= & 20 A_{1}^{2} A_{2}-A_{2}\left(7 A_{8}-4 A_{9}+A_{10}+A_{11}+7 A_{12}\right)+A_{1}\left(6 A_{14}-22 A_{15}\right)- \\
& -4 A_{33}+4 A_{34}, \\
\mathcal{F}(\tilde{a})= & A_{7}, \\
\mathcal{B}(\tilde{a})= & -\left(3 A_{8}+2 A_{9}+A_{10}+A_{11}+A_{12}\right), \\
\mathcal{H}(\tilde{a})= & -\left(A_{4}+2 A_{5}\right),
\end{aligned}
$$

as well as the $C T$-comitants:

$$
\begin{aligned}
\mathcal{B}_{1}(\tilde{a})= & \left\{\left(T_{7}, D_{2}\right)^{(1)}\left[12 D_{1} T_{3}+2 D_{1}^{3}+9 D_{1} T_{4}+36\left(T_{1}, D_{2}\right)^{(1)}\right]\right. \\
& -2 D_{1}\left(T_{6}, D_{2}\right)^{(1)}\left[D_{1}^{2}+12 T_{3}\right]+D_{1}^{2}\left[D_{1}\left(T_{8}, C_{1}\right)^{(2)}+\right. \\
& \left.\left.+6\left(\left(T_{6}, C_{1}\right)^{(1)}, D_{2}\right)^{(1)}\right]\right\} / 144,
\end{aligned}
$$

$$
\begin{aligned}
& \mathcal{B}_{2}(\tilde{a})=\left\{\left(T_{7}, D_{2}\right)^{(1)}\left[8 T_{3}\left(T_{6}, D_{2}\right)^{(1)}-D_{1}^{2}\left(T_{8}, C_{1}\right)^{(2)}-4 D_{1}\left(\left(T_{6}, C_{1}\right)^{(1)}, D_{2}\right)^{(1)}\right]+\right. \\
&\left.+\left[\left(T_{7}, D_{2}\right)^{(1)}\right]^{2}\left(8 T_{3}-3 T_{4}+2 D_{1}^{2}\right)\right\} / 384, \\
& \mathcal{B}_{3}(\tilde{a}, x, y)=-D_{1}^{2}\left(4 D_{2}^{2}+T_{8}+4 T_{9}\right)+3 D_{1} D_{2}\left(T_{6}+4 T_{7}\right)-24 T_{3}\left(D_{2}^{2}-T_{9}\right), \\
& \mathcal{B}_{4}(\tilde{a}, x, y)=D_{1}\left(T_{5}+2 D_{2} C_{1}\right)-3 C_{2}\left(D_{1}^{2}+2 T_{3}\right) .
\end{aligned}
$$

We note that the invariant polynomials $\mathcal{T}_{i}, \mathcal{F}_{i}, \mathcal{B}_{i}(\mathrm{i}=1,2,3,4)$, and $\mathcal{B}, \mathcal{F}, \mathcal{H}$ and $\sigma$ are responsible for weak singularities of the family of quadratic systems (see [39, Main Theorem]).

Now we need also the invariant polynomials which are responsible for the types of the finite singularities. These were constructed in [4]. Here we need only the
following ones (we keep the notations from [4]):

$$
\begin{aligned}
W_{4}(\tilde{a})= & {\left[1512 A_{1}^{2}\left(A_{30}-2 A_{29}\right)-648 A_{15} A_{26}+72 A_{1} A_{2}\left(49 A_{25}+39 A_{26}\right)\right.} \\
& +6 A_{2}^{2}\left(23 A_{21}-1093 A_{19}\right)-87 A_{2}^{4}+4 A_{2}^{2}\left(61 A_{17}+52 A_{18}+11 A_{20}\right) \\
& -6 A_{37}\left(352 A_{3}+939 A_{4}-1578 A_{5}\right)-36 A_{8}\left(396 A_{29}+265 A_{30}\right) \\
& +72 A_{29}\left(17 A_{12}-38 A_{9}-109 A_{11}\right)+12 A_{30}\left(76 A_{9}-189 A_{10}-273 A_{11}\right. \\
& \left.-651 A_{12}\right)-648 A_{14}\left(23 A_{25}+5 A_{26}\right)-24 A_{18}\left(3 A_{20}+31 A_{17}\right) \\
& +36 A_{19}\left(63 A_{20}+478 A_{21}\right)+18 A_{21}\left(2 A_{20}+137 A_{21}\right)-4 A_{17}\left(158 A_{17}\right. \\
& \left.\left.+30 A_{20}+87 A_{21}\right)-18 A_{19}\left(238 A_{17}+669 A_{19}\right)\right] / 81, \\
W_{7}(\tilde{a})= & 12 A_{26}\left(A_{26}-2 A_{25}\right)+\left(2 A_{29}-A_{30}\right)\left(A_{2}^{2}-20 A_{17}-12 A_{18}+6 A_{19}+6 A_{21}\right) \\
& +48 A_{37}\left(A_{1}^{2}-A_{8}-A_{12}\right), \\
W_{8}(\tilde{a})= & 64 D_{1}\left[\left(\left(T_{6}, C_{1}\right)^{(1)}, D_{2}\right)^{(1)}\right]^{2}\left[16\left(C_{0}, T_{6}\right)^{(1)}-37\left(D_{2}, T_{1}\right)^{(1)}+12 D_{1} T_{3}\right] \\
& +4\left(108 D_{1}^{4}-3 T_{4}^{2}-128 T_{3} T_{4}+42 D_{1}^{2} T_{4}\right)\left[\left(\left(T_{6}, C_{1}\right)^{(1)}, D_{2}\right)^{(1)}\right]^{2} \\
& +36 D_{1}\left(\left(T_{6}, C_{1}\right)^{(1)}, D_{2}\right)^{(1)}\left[4 D_{1}\left(C_{0}, T_{6}\right)^{(1)}-D_{1}^{2}\left(4 T_{3}+T_{4}\right)\right. \\
& \left.+24 T_{3}^{2}\right]\left(C_{1}, T_{8}\right)^{(2)}+64\left[\left(\left(T_{6}, C_{1}\right)^{(1)}, D_{2}\right)^{(1)}\right]^{2}\left[27 T_{3}^{2}\right. \\
& \left.+16\left(\left(T_{6}, C_{1}\right)^{(1)}, C_{0}\right)^{(1)}\right]-54\left[8 D_{1}^{4}+D_{1}^{2} T_{4}-8 D_{1}\left(C_{0}, T_{6}\right)^{(1)}\right. \\
& \left.+8 D_{1}^{2} T_{3}+8 T_{3}^{2}\right]\left(\left(T_{6}, C_{1}\right)^{(1)}, T_{6}\right)^{(1)}\left(C_{1}, T_{8}\right)^{(2)}+108 D_{1} T_{3}\left[\left(C_{1}, T_{8}\right)^{(2)}\right]^{2} \times \\
& \times\left[D_{1} T_{3}-2\left(C_{0}, T_{6}\right)^{(1)}\right]+576\left(\left(T_{6}, C_{1}\right)^{(1)}, D_{2}\right)^{(1)} \times \\
& \times\left(\left(T_{6}, C_{1}\right)^{(1)}, T_{6}\right)^{(1)}\left[2\left(D_{2}, T_{1}\right)^{(1)}-5 D_{1} T_{3}\right] \\
& -27\left[\left(C_{1}, T_{8}\right)^{(2)}\right]^{2}\left[T_{4}^{4} / 8+\left(C_{0}, T_{1}\right)^{(1)}\right], \\
F_{4}(\tilde{a}, x, y)= & \mu_{3}(\tilde{a}, x, y), \\
F_{5}(\tilde{a}, x, y)= & T_{5}+2 C_{1} D_{2}-3 C_{2} D_{1}, \\
G_{3}(\tilde{a})= & A_{2} .
\end{aligned}
$$

Finally we need the invariant polynomials which are responsible for the existence of one (or two) star node(s) arbitrarily located on the phase plane of a system (3). We have the following lemma (see [42]):

Lemma 5. A quadratic system (3) possesses one star node if and only if one of the following sets of conditions hold:
(i) $U_{1} \neq 0, U_{2} \neq 0, U_{3}=Y_{1}=0$;
(ii) $U_{1}=U_{4}=U_{5}=U_{6}=0, \quad Y_{2} \neq 0$;
and it possesses two star nodes if and only if

$$
\text { (iii) } \quad U_{1}=U_{4}=U_{5}=0, \quad U_{6} \neq 0, \quad Y_{2}>0
$$

where

$$
\begin{aligned}
& U_{1}=\widetilde{N}, \quad U_{2}=\left(C_{1}, \widetilde{H}-\widetilde{K}\right)^{(1)}-2 D_{1} \widetilde{N}, \\
& U_{3}=3 \widetilde{D}\left(D_{2}^{2}-16 \widetilde{K}\right)+C_{2}\left[\left(C_{2}, \widetilde{D}\right)^{(2)}-5\left(D_{2}, \widetilde{D}\right)^{(1)}+6 \widetilde{F}\right], \\
& U_{4}=2 T_{5}+C_{1} D_{2}, \quad U_{5}=3 C_{1} D_{1}+4 T_{2}-2 C_{0} D_{1}, \\
& U_{6}=\widetilde{H}, \quad Y_{1}=A_{1}, \quad Y_{2}=2 D_{1}^{2}+8 T_{3}-T_{4} .
\end{aligned}
$$

We base our work here on results obtained in [3] and [4].

## 8 The proof of the Main Theorem

### 8.1 The family of systems without finite singularities

The total multiplicity $m_{f}$ of finite singularities of every system in this family is zero. In [3] we gave the full global geometric classification of the whole class of quadratic systems according to their singularities at infinity. Since only infinite singularities occur in this family ( $m_{f}=0$ ), we can extract from [3] the classification of the configurations of singularities of this family. In fact from [3] we obtain more. Indeed, we extract from [3] the part of the global bifurcation diagram of configurations of singularities at infinity of QS, the fragment covering the case we need here, i.e. $m_{f}=0$. We obtain the bifurcation diagram (see Diagram 1 ) of configurations of singularities of this class, done in the 12-parameter space of coefficients and obtained with the help of invariant polynomials. The proof for this diagram is completely covered in [3] and thus there is no need for a proof here. We shall only give here examples, one for each kind of distinct geometric configurations occurring in this family.

1) Systems with $\eta<0$;

- $\left.\overline{{ }_{1}^{4}}{ }_{1}^{2}\right) N$, ©, © : Example $\Rightarrow\left(\dot{x}=1+x y ; \dot{y}=-x^{2}\right)$;
- $N^{*},\binom{2}{1} \subsetneq,\binom{2}{1}$ © : Example $\Rightarrow\left(\dot{x}=1 ; \dot{y}=-x^{2}-y^{2}\right)$.

2) Systems with $\eta>0$;

- $\overline{\binom{4}{1}} N, S, N^{\infty}$ : Example $\Rightarrow\left(\dot{x}=-1+x y ; \dot{y}=1-x y+2 y^{2}\right)$;
- $\overline{\binom{4}{1}} S, N^{f}, N^{f}$ : Example $\Rightarrow\left(\dot{x}=1-x y ; \dot{y}=2-2 x y+y^{2}\right)$;
- $\overline{\binom{3}{1}} S N, \overline{\binom{1}{1}} S N, N^{d}$ : Example $\Rightarrow(\dot{x}=1+x-x y ; \dot{y}=1-x y)$;
- $\overline{\binom{3}{1}}$ SN, $\overline{\binom{1}{1}} N S, N^{d}$ : Example $\Rightarrow(\dot{x}=1-x+x y ; \dot{y}=1+x y)$;
- $\overline{\binom{2}{1}} S, \overline{\binom{2}{1}} N, N^{*}$ : Example $\Rightarrow(\dot{x}=1-x y ; \dot{y}=-x y)$.

3) Systems with $\eta=0, \widetilde{M} \neq 0$;

- $\overline{\binom{0}{2}} S N, \overline{\binom{4}{1}} N$ : Example $\Rightarrow\left(\dot{x}=1+x y ; \dot{y}=-1-x y+y^{2}\right)$;
- ( $\left.\begin{array}{l}4 \\ 2\end{array}\right) \overparen{P} H_{\curlywedge} \widehat{P}-\overparen{P} H_{\curlywedge} \overparen{P}, N^{f}$ : Example $\Rightarrow\left(\dot{x}=x^{2} / 4 ; \dot{y}=1-3 x y / 4\right)$;
- ( $\left.\begin{array}{l}4 \\ 2\end{array}\right) \widehat{P}_{P} \widehat{P}_{\curlywedge} H-H \widehat{P}_{\curlywedge} \widehat{P}, N^{f}:$ Example $\Rightarrow\left(\dot{x}=2 x^{2} / 3 ; \dot{y}=1-x y / 3\right)$;
- ( $\left.\begin{array}{l}4 \\ 2\end{array}\right) \widehat{P} H-H \overparen{P}, N^{f}:$ Example $\Rightarrow\left(\dot{x}=x^{2} / 2 ; \dot{y}=1-x y / 2\right)$;
- $\binom{4}{2} \widehat{P} \widehat{P}_{\curlywedge} E-E \widehat{P}_{\curlywedge} \widehat{P}, S:$ Example $\Rightarrow\left(\dot{x}=-x^{2} ; \dot{y}=1-2 x y\right)$;
- $\binom{4}{2} \widehat{P} \widehat{P}_{\curlywedge} H-H \widehat{P}_{\curlywedge} \widehat{P}, N^{\infty}:$ Example $\Rightarrow\left(\dot{x}=2 x^{2} ; \dot{y}=1+x y\right)$;
- $\widehat{\binom{4}{2}} \widetilde{P}_{\curlywedge} \overparen{P} H_{\curlywedge}-H, N^{*}$ : Example $\Rightarrow\left(\dot{x}=y+x^{2} ; \dot{y}=1\right)$;
- $\binom{4}{2} H-H, N^{d}$ : Example $\Rightarrow\left(\dot{x}=1+x^{2} ; \dot{y}=x\right)$;
- ( $\left.\begin{array}{l}4 \\ 2\end{array}\right) H-H, N^{*}$ : Example $\Rightarrow\left(\dot{x}=1+x^{2} ; \dot{y}=1\right)$;
- ( $\left.\begin{array}{l}4 \\ 2\end{array}\right) \widehat{P} E \hat{P}-H H H, N^{d}$ : Example $\Rightarrow\left(\dot{x}=-2+x^{2} ; \dot{y}=1+x\right)$;
- $\binom{4}{2} \overparen{P} \widehat{P} H-H \overparen{P} \widehat{P}, N^{d}:$ Example $\Rightarrow\left(\dot{x}=-1+x^{2} ; \dot{y}=2+x\right)$;
- ( $\left.\begin{array}{l}4 \\ 2\end{array}\right) \hat{P} \hat{P} H-H \hat{P} \hat{P}, N^{*}:$ Example $\Rightarrow\left(\dot{x}=-1+x^{2} ; \dot{y}=1\right)$;
- ( $\left.\begin{array}{l}4 \\ 2\end{array}\right) \widehat{P} \widehat{P}_{\curlywedge} H-H \hat{P}_{\curlywedge} \widehat{P}, N^{d}$ : Example $\Rightarrow\left(\dot{x}=x^{2} ; \dot{y}=1+x\right)$;
- ( $\left.\begin{array}{l}4 \\ 2\end{array}\right) \overparen{P} \widehat{P}_{\mathcal{\curlywedge}} H-H \widehat{P}_{\curlywedge} \overparen{P}, N^{*}$ : Example $\Rightarrow\left(\dot{x}=x^{2} ; \dot{y}=1\right)$;
- $\widehat{\binom{1}{2}} \tilde{P}_{\mathcal{\curlywedge}} E \widehat{P}_{\mathcal{\curlywedge}}-H, \overline{\binom{3}{1}} S N$ : Example $\Rightarrow(\dot{x}=y ; \dot{y}=1-x y)$;
- $\binom{3}{2} E \widehat{P}-\widehat{P} H, \overline{\binom{1}{1}} S N: \quad$ Example $\Rightarrow(\dot{x}=x ; \dot{y}=1-x y)$;
- $\binom{2}{2} E-E, \overline{\binom{2}{1}} S:$ Example $\Rightarrow(\dot{x}=-1 ; \dot{y}=1-x y)$;
- $\binom{2}{2} H-H, \overline{\binom{2}{1}} N:$ Example $\Rightarrow(\dot{x}=1 . \dot{y}=1-x y)$;

4) Systems with $\eta=\widetilde{M}=0$;

- $\binom{4}{3} E \hat{P}_{\curlywedge} H-H \widehat{P}_{\curlywedge} E:$ Example $\Rightarrow\left(\dot{x}=x^{2} ; \dot{y}=1-x^{2}+x y\right)$;
- $\binom{4}{3} \widehat{P} \widehat{P}_{\curlywedge} \widehat{P}-\widehat{P} \widehat{P}_{\curlywedge} \widehat{P}$ : Example $\Rightarrow\left(\dot{x}=x^{2} ; \dot{y}=-1-x^{2}+x y\right)$;
- $\binom{4}{3} \hat{P}_{\curlywedge} E E \hat{P}_{\curlywedge}-H H: \quad$ Example $\Rightarrow\left(\dot{x}=x ; \dot{y}=1-x^{2}\right)$;

- $\binom{4}{3} \widetilde{P}_{\curlywedge} E H_{\curlywedge}-\overparen{P}:$ Example $\Rightarrow\left(\dot{x}=1 ; \dot{y}=y-x^{2}\right)$;
- $\binom{4}{3} \widehat{P}_{\curlywedge} \widehat{P}-\widehat{P} \widehat{P}_{\mathcal{\jmath}}:$ Example $\Rightarrow\left(\dot{x}=1 ; \dot{y}=-x^{2}\right)$;
- $\left[\infty ;\binom{\widehat{3}}{0}\right.$ ES $]$ : Example $\Rightarrow\left(\dot{x}=x^{2} ; \dot{y}=1+x y\right)$.


### 8.2 The family of quadratic differential systems with only one finite singularity which in addition is elemental

In this subsection we consider all quadratic vector fields with total multiplicity $m_{f}$ of finite singularities equal to 1 . Since we have only one finite singular point, this point is of course real. To obtain the full global classification of configurations of singularities with respect to the geometric equivalence relation for this family, we need to: i) deepen the topological classification of all configurations of finite singularities done in [2] by using the finer geometric equivalence relation; ii) to integrate this with the geometric classification of infinite singularities done in [3]
and iii) to search for a minimal set of invariants which allow to obtain for this family the bifurcation diagram with respect to the geometric equivalence relation of configurations of singularities, finite and infinite, in the 12-dimensional space of parameters.

According to [39] in this case the conditions $\mu_{0}=\mu_{1}=\mu_{2}=0$ and $\mu_{3} \neq 0$ must be satisfied and according to [3] the following lemma is valid.
Lemma 6. The configurations of singularities at infinity of the family of quadratic systems possessing one elemental (real) finite singularity (i.e. $\mu_{0}=\mu_{1}=\mu_{2}=0$ and $\mu_{3} \neq 0$ ) are classified in DIAGRAM 3 according to the geometric equivalence relation. Necessary and sufficient conditions for each one of the 22 different equivalence classes can be assembled from this diagram in terms of 14 invariant polynomials with respect to the action of the affine group and time rescaling, given in Section 7.

According to [39] the family of quadratic systems with one elemental finite singularity could be brought via an affine transformation to one of the two canonical forms in [39], governed by invariant polynomial $\widetilde{K} \neq 0$. In what follows we consider two cases: $\widetilde{K} \neq 0$ and $\widetilde{K}=0$.

### 8.2.1 Systems with $\widetilde{K} \neq 0$

In this case by [39] via an affine transformation quadratic systems in this family could be brought to the systems

$$
\begin{align*}
& \dot{x}=c x+d y+(2 c+d) x^{2}+2 d x y \\
& \dot{y}=e x+f y+(2 e+f) x^{2}+2 f x y \tag{11}
\end{align*}
$$

possessing the singular points $M_{1}(0,0)$. For these systems calculations yield

$$
\begin{equation*}
\mu_{0}=\mu_{1}=\mu_{2}=0, \quad \mu_{3}=(c f-d e)^{2} x^{3}, \quad \kappa=256 d^{2}(d e-c f) . \tag{12}
\end{equation*}
$$

We remark that for the systems above we have $\mu_{3} \neq 0$ and therefore in what follows we assume that the condition $c f-d e \neq 0$ holds (i.e. the singular point $M_{1}(0,0)$ is elemental).
8.2.1.1 The case $\kappa \neq 0$. Then $d \neq 0$ and due to a time rescaling we may assume $d=1$. So we consider the 3 -parameter family of systems:

$$
\begin{align*}
& \dot{x}=c x+y+(2 c+1) x^{2}+2 x y, \\
& \dot{y}=e x+f y+(2 e+f) x^{2}+2 f x y, \quad c f-e \neq 0 \tag{13}
\end{align*}
$$

for which calculations yield

$$
\begin{gather*}
\mu_{0}=\mu_{1}=\mu_{2}=0, \quad \mu_{3}=(c f-e)^{2} x^{3}, \quad \widetilde{K}=8(c f-e) x^{2}, \\
\eta=4\left[(2 c+1+2 f)^{2}+16(e-c f)\right], \quad \kappa=256(e-c f), \\
\mathcal{T}_{4}=-8(c+f)(c f-e)^{2}, \quad \mathcal{T}_{3}=-8(c f-e)^{2}, \quad \mathcal{F}_{1}=6(e-c f),  \tag{14}\\
W_{4}=64(c f-e)^{4}\left[(c-f)^{2}+4 e\right]=64(c f-e)^{4}\left[(c+f)^{2}+4(e-c f)\right], \\
\widetilde{M}=-8\left[(1+2 c-2 f)^{2}+6(2 e+f)\right] x^{2}-16(1+2 c-2 f) x y-32 y^{2} .
\end{gather*}
$$



DIAGRAM 3. The case $\mu_{0}=\mu_{1}=\mu_{2}=0, \mu_{3} \neq 0$

Considering (14) we make the remark:
Remark 1. Assume that the condition $\kappa \neq 0$ holds. Then
(i) $\mathcal{I}_{3} \mathcal{F}_{1} \neq 0$ and $\operatorname{sign}(\widetilde{K})=-\operatorname{sign}(\kappa)$;
(ii) the condition $\kappa>0$ implies $\eta>0$ and $W_{4}>0$;
(iii) in the case $\mathcal{T}_{4}=0$ we have $W_{4} \neq 0$ and $\operatorname{sign}\left(W_{4}\right)=\operatorname{sign}(\kappa)$.

The first two statements follow obviously from (14). In the case $\mathcal{T}_{4}=0$ we get $f=-c$ and then $\kappa=256\left(c^{2}+e\right), \quad W_{4}=256\left(c^{2}+e\right)^{5}$ and this proves the last assertion.
8.2.1.1.1 The subcase $\boldsymbol{\kappa}<\mathbf{0}$. Then by Remark 1 we obtain $\widetilde{K}>0$.

1) The possibility $W_{4}<0$. In this case considering the condition $\widetilde{K}>0$, according to [4] (see Table 1, line 184) the finite singularity is a focus.
a) Assume first $\mathcal{T}_{4} \neq 0$. Then by [39] the focus is strong. As $\kappa<0$ according to Lemma 6 we get the following three global configurations of singularities:

- $f ;\binom{\overline{3}}{1} S N$, © ©, © : Example $\Rightarrow(c=0, e=-1, f=1) \quad($ if $\eta<0)$;
- $f ; \overline{\binom{3}{1}} S N, S, N^{\infty}$ : Example $\Rightarrow(c=0, e=-1, f=7 / 4) \quad($ if $\eta>0)$;
- $f ;\binom{\overline{0}}{2} S N,\left(\begin{array}{l}\binom{3}{1} \\ )\end{array} N\right.$ : Example $\Rightarrow(c=0, e=-1, f=3 / 2) \quad($ if $\eta=0)$.
b) Suppose now $\mathcal{T}_{4}=0$. Then $f=-c$ and since by Remark 1 we have $\mathcal{T}_{3} \mathcal{F}_{1} \neq$ 0 , then by [39] the finite singularity is a first order weak focus. Considering the types of the infinite singularities mentioned above we obtain the following three configurations
- $f^{(1)} ;\binom{\overline{3}}{1} S N$, © , © : Example $\Rightarrow(c=0, e=-1, f=0) \quad($ if $\eta<0)$;
- $f^{(1)} ; \overline{\binom{3}{1}} S N, S, N^{\infty}$ : Example $\Rightarrow(c=0, e=-1 / 18, f=0) \quad($ if $\eta>0)$;
- $f^{(1)} ; \overline{\binom{0}{2}} S N, \overline{\binom{3}{1}} S N: \quad$ Example $\Rightarrow(c=0, e=-1 / 16, f=0) \quad($ if $\eta=0)$.

2) The possibility $W_{4}>0$. Since $\widetilde{K}>0$, according to [4] systems (13) possess a node which is generic (due to $W_{4} \neq 0$ ). So considering Lemma 6 we have the configurations

- $n ;\binom{\overline{3}}{1} S N$, © © © © : Example $\Rightarrow(c=0, e=-1, f=-9 / 4) \quad($ if $\eta<0)$;
- $n ;\binom{\overline{3}}{1} S N, S, N^{\infty}$ : Example $\Rightarrow(c=0, e=-1, f=-3) \quad($ if $\eta>0)$;
- $n ;\binom{\overline{0}}{2} S N, \overline{\binom{3}{1}} S N$ : Example $\Rightarrow(c=0, e=-1, f=-5 / 2) \quad($ if $\eta=0)$.

3) The possibility $W_{4}=0$. Then the singular point $M_{1}(0,0)$ of systems (13) is a node with coinciding eigenvalues which could not be a star node (due to the respective linear matrix). Considering the types of the infinite singularities given by Lemma 6 we get the next three configurations

- $n^{d} ;\binom{\overline{3}}{1} S N$, © , © © : Example $\Rightarrow(c=0, e=-1 / 4, f=-1) \quad($ if $\eta<0)$;
- $n^{d} ; \overline{\binom{3}{1}} S N, S, N^{\infty}:$ Example $\Rightarrow(c=0, e=-1 / 4, f=1) \quad($ if $\eta>0)$;
- $n^{d} ; \overline{\binom{0}{2}} S N, \overline{\binom{3}{1}} S N:$ Example $\Rightarrow(c=0, e=-1 / 64, f=-1 / 4) \quad($ if $\eta=0)$.
8.2.1.1.2 The subcase $\boldsymbol{\kappa}>\mathbf{0}$. According to Remark 1 we obtain $\widetilde{K}<0$ and according to [4] (see Table 1, line 178) the finite singularity is a saddle. By Remark 1, in this case we have $\eta>0$ and considering Lemma 6 we have the unique configuration of infinite singularities $\left(\overline{\binom{3}{1}} S N, N^{f}, N^{f}\right.$.

1) Assume first $\mathcal{T}_{4} \neq 0$. In this case by [39] the saddle is strong and we arrive at the configuration

- $s ;\left(\overline{(3)} 1 \begin{array}{l}3 \\ 1\end{array}\right) S N, N^{f}, N^{f}:$ Example $\Rightarrow(c=0, e=1, f=1)$.

2) Suppose now $\mathcal{T}_{4}=0$. Then $f=-c$ and as by Remark 1 , we have $\mathcal{T}_{3} \mathcal{F}_{1} \neq 0$. Considering [39] we deduce that the finite singularity is a weak saddle of the first order. So we obtain the configuration

$$
\text { - } s^{(1)} ;\left(\begin{array}{l}
\binom{3}{1} \\
\end{array} N, N^{f}, N^{f}: \text { Example } \Rightarrow(c=0, e=1, f=0) .\right.
$$

8.2.1.2 The case $\boldsymbol{\kappa}=\mathbf{0}$. Then by (13) we have $d=0$ and considering the condition $\mu_{3}=c^{2} f^{2} x^{3} \neq 0$ we obtain $c f \neq 0$. So doing a time rescaling we may assume $f=1$ and we consider the 2-parameter family of systems:

$$
\begin{equation*}
\dot{x}=c x+2 c x^{2}, \quad \dot{y}=e x+y+(2 e+1) x^{2}+2 x y, \quad c \neq 0, \tag{15}
\end{equation*}
$$

for which calculations yield

$$
\begin{gather*}
\mu_{0}=\mu_{1}=\mu_{2}=0, \quad \mu_{3}=c^{2} x^{3}, \quad \widetilde{K}=8 c x^{2}, \quad \eta=\kappa=0, \quad \widetilde{M}=-32(c-1)^{2} x^{2}, \\
C_{2}=-(1+2 e) x^{3}+2(c-1) x^{2} y, \quad \sigma=1+c+2(1+2 c) x, \\
\mathcal{T}_{i}=0, i=1,2,3,4, \quad \mathcal{F}_{1}=\mathcal{H}=\mathcal{B}=\mathcal{B}_{1}=\mathcal{B}_{2}=0, \\
\mathcal{B}_{3}=-288 c^{3}(1+c) x^{2} \quad W_{4}=0, \quad \tilde{L}=32 c(c-1) x^{2} . \tag{16}
\end{gather*}
$$

Remark 2. We observe that the corresponding matrix for the singular point $M_{1}(0,0)$ is $\left(\begin{array}{cc}c & 0 \\ e & 1\end{array}\right)$ and hence this singular point is i) a saddle if $c<0$; ii) a node with two direction if $c>0$ and $c \neq 1$; iii) a node with one direction if $c=1$ and $e \neq 0$; iv) $a$ star node if $c=1$ and $e=0$.
8.2.1.2.1 The subcase $\widetilde{\boldsymbol{K}}<\mathbf{0}$. Then $c<0$ and by the remark above the finite singularity is a saddle. Considering (16) according to [39] the saddle is weak if and only if $\mathcal{B}_{3}=0$ (see the statement $e_{3}[\gamma]$ of Main Theorem. Moreover in this case we have an integrable saddle.

Since $c<0$ we have $\widetilde{M} \neq 0$. Then according to Lemma 6 at infinity we get the unique configuration of singularities given by $\binom{3}{2} \overparen{P} E \overparen{P}-\overparen{P} \overparen{P} H, N^{f}$. So we arrive at the next two global configurations of singularities

- $s ;\binom{3}{2} \overparen{P} E \overparen{P}-\overparen{P} \overparen{P} H, N^{f}: \quad$ Example $\Rightarrow(c=-2, e=0) \quad\left(\right.$ if $\left.\mathcal{B}_{3} \neq 0\right)$;
- $\$ ;\binom{3}{2} \widehat{P} E \widehat{P}-\widehat{P} \widehat{P} H, N^{f}:$ Example $\Rightarrow(c=-1, e=0) \quad\left(\right.$ if $\left.\mathcal{B}_{3}=0\right)$.
8.2.1.2.2 The subcase $\widetilde{\boldsymbol{K}}>\mathbf{0}$. Then $c>0$ and by Remark 2 the finite singularity is a node. We observe that due to $\mu_{3} \neq 0$ the condition $c=1$ is equivalent to $\tilde{L}=0$.

1) The possibility $\widetilde{L} \neq 0$. Then $\widetilde{M} \neq 0$ and by Remark 2 we have a generic node. On the other hand as $\widetilde{K}>0$ we obtain $\operatorname{sign}(\tilde{L})=\operatorname{sign}(c-1)$ and considering Lemma 6 we get the following two configurations

- $n ;\binom{3}{2} \stackrel{\overparen{ }}{P} H \stackrel{\Im}{P}-\stackrel{\overparen{P}}{\mathscr{P}} E, S: \quad$ Example $\Rightarrow(c=1 / 2, e=0) \quad($ if $\tilde{L}<0)$;
- $n ;\binom{3}{2} H \stackrel{\overparen{P}}{P} \stackrel{\overparen{P}}{P}-H H H, N^{\infty}: \quad$ Example $\Rightarrow(c=2, e=0) \quad($ if $\tilde{L}>0)$.

2) The possibility $\tilde{L}=0$. Then $c=1$ and this implies $\widetilde{M}=0$. By Remark 2 we have a node with coinciding eigenvalues. On the other hand for $c=1$ we obtain $C_{2}=-(1+2 e) x^{3}, U_{3}=-24 e x^{5}$.
a) Assume first $C_{2} \neq 0$. Then we have a single real infinite singularity of multiplicity six and according to Lemma 6 the type of this singularity depends on the sign of the invariant polynomial $K_{3}=6(1+2 e) x^{6}$, which is nonzero due to $C_{2} \neq 0$.

Thus taking into consideration Remark 2 and Lemma 6 we arrive at the next configurations

- $n^{d} ;\binom{3}{3} H \stackrel{\curvearrowright}{P} E-\stackrel{饣}{P} H H: \quad$ Example $\Rightarrow(c=1, e=-1) \quad\left(\right.$ if $\left.K_{3}<0\right)$;
- $n^{d} ;\binom{3}{3} H H \stackrel{\curvearrowright}{P}-\overparen{P} \stackrel{\curvearrowright}{P} \stackrel{\overparen{P}}{ }$ : Example $\Rightarrow(c=1, e=1) \quad\left(\right.$ if $\left.K_{3}>0, U_{3} \neq 0\right)$;
- $n^{*} ;\binom{3}{3} H H \overparen{P}-\overparen{ค} \overparen{\overparen{P}} \overparen{\overparen{P}}: \quad$ Example $\Rightarrow(c=1, e=0) \quad\left(\right.$ if $\left.K_{3}>0, U_{3}=0\right)$.
b) Suppose now $C_{2}=0$. Then $e=-1 / 2$ and we get the system

$$
\dot{x}=x(1+2 x), \quad \dot{y}=-x / 2+y+2 x y
$$

possessing a node $n^{d}$ and the infinite line filled up with singularities. Considering Lemma 6 we obtain the configuration


### 8.2.2 Systems with $\widetilde{K}=0$

In this case, according to [39] we consider the following family of systems

$$
\begin{align*}
& \dot{x}=x+d y \\
& \dot{y}=e x+f y+l x^{2}+2 m x y-d(d l-2 m) y^{2} \tag{17}
\end{align*}
$$

possessing the singular points $M_{1}(0,0)$. For these systems calculations yield

$$
\begin{equation*}
\eta=4 d^{2}(d l-m)^{2}(d l-2 m)^{2}, \quad \tilde{L}=8 d(2 m-d l)(x+d y)[l x-(d l-2 m) y] \tag{18}
\end{equation*}
$$

We consider two cases: $\eta \neq 0$ and $\eta=0$.
8.2.2.1 The case $\boldsymbol{\eta} \neq \mathbf{0}$. Then $d(d l-m)(d l-2 m) \neq 0$ and we may assume $d=l=1$ and $m=0$ due to the transformation

$$
x_{1}=(d l-2 m) x, \quad y_{1}=-\frac{(d l-2 m) m}{d l-m} x+\frac{d(d l-2 m)^{2}}{d l-m} y, \quad t_{1}=\frac{d l-m}{d l-2 m} t .
$$

So we consider the 2-parameter family of systems

$$
\begin{equation*}
\dot{x}=x+y, \quad \dot{y}=e x+f y+x^{2}-y^{2}, \tag{19}
\end{equation*}
$$

for which calculations yield

$$
\begin{gather*}
\mu_{0}=\mu_{1}=\mu_{2}=0, \quad \mu_{3}=(f-e)(x-y)(x+y)^{2}, \quad \widetilde{K}=\kappa=0 \\
\eta=4, \quad K_{1}=(x-y)(x+y)^{2}, \quad F_{4} F_{5}=6(f-e)(x-y)^{2}(x+y)^{4}, \\
G_{3}=2(e-f), \quad W_{4}=64(e-f)^{2}\left[(f-1)^{2}+4 e\right]  \tag{20}\\
\mathcal{T}_{4}=8(f-e)(1+f), \quad \mathcal{T}_{3}=8(f-e), \quad \mathcal{F}_{1}=2(e-f) .
\end{gather*}
$$

Remark 3. In the case $\eta \neq 0$ the condition $\mu_{3} \neq 0$ implies $\mathcal{T}_{3} \mathcal{F}_{1} F_{4} F_{5} G_{3} \neq 0$ and $\operatorname{sign}\left(\mu_{3} K_{1}\right)=\operatorname{sign}\left(F_{4} F_{5}\right)$.
8.2.2.1.1 The subcase $\boldsymbol{\mu}_{\mathbf{3}} \boldsymbol{K}_{\mathbf{1}}<\mathbf{0}$. By Remark 3 we have $F_{4} F_{5}<0$ and according to [4] (see Table 1, line 179) the finite singularity is a saddle. Clearly this saddle is weak if and only if $f=-1$ and this is equivalent to $\mathcal{T}_{4}=0$. On the other hand by Remark 3 we have $\mathcal{T}_{3} \mathcal{F}_{1} \neq 0$ and according to [39] the weak saddle could be only of the first order. So considering Lemma 6 we get the following two global configurations of singularities

- $s ;\binom{\overline{2}}{1} N, \overline{\binom{1}{1}} S N, N^{d}:$ Example $\Rightarrow(e=2, f=1) \quad\left(\right.$ if $\left.\mathcal{T}_{4} \neq 0\right)$;
- $\left.s^{(1)} ; \overline{\binom{2}{1}} N, \overline{(\overline{1}} 1\right) S N, N^{d}:$ Example $\Rightarrow(e=2, f=-1) \quad\left(\right.$ if $\left.\mathcal{T}_{4}=0\right)$.
8.2.2.1.2 The subcase $\boldsymbol{\mu}_{\mathbf{3}} \boldsymbol{K}_{\mathbf{1}}>\mathbf{0}$. In this case we have $F_{4} F_{5}>0$ and as $G_{3} \neq 0$ by [4] we have a focus or a center if $W_{4}<0$ and a node if $W_{4} \geq 0$.

1) The possibility $W_{4}<0$. Then we have a focus which is strong if $\mathcal{I}_{4} \neq 0$ and it is weak of the first order if $\mathcal{T}_{4}=0$ (due to [39] and $\mathcal{T}_{3} \mathcal{F}_{1} \neq 0$, see Remark 3). Considering Lemma 6 we arrive at the next two configurations

- $f ; \overline{\binom{2}{1}} S, \overline{(1)}\binom{1}{1} S N, N^{d}: \quad$ Example $\Rightarrow(e=-2, f=1) \quad\left(\right.$ if $\left.\mathcal{T}_{4} \neq 0\right)$;
- $f^{(1)} ; \overline{\binom{2}{1}} S, \overline{\binom{1}{1}} S N, N^{d}$ : Example $\Rightarrow(e=-2, f=-1) \quad\left(\right.$ if $\left.\mathcal{I}_{4}=0\right)$.

2) The possibility $W_{4}>0$. In this case we have a generic node (as $W_{4} \neq 0$ ) and hence we get

- $n ; \overline{\binom{2}{1}} S, \overline{\binom{1}{1}} S N, N^{d}:$ Example $\Rightarrow(e=0, f=2)$.

3) The possibility $W_{4}=0$. Then we have a node with coinciding eigenvalues and due to the linearization matrix at the singularity $M_{1}(0,0)$ this is a one-direction node, and we have the configuration

- $n^{d} ; \overline{\binom{2}{1}} S, \overline{\binom{1}{1}} S N, N^{d}$ : Example $\Rightarrow(e=-1 / 4, f=0)$.
8.2.2.2 The case $\boldsymbol{\eta}=\mathbf{0}$. Then $d(d l-m)(d l-2 m) \neq 0$ and we consider two subcases: $\tilde{L} \neq 0$ and $\tilde{L}=0$.
8.2.2.2.1 The subcase $\tilde{\boldsymbol{L}} \neq \mathbf{0}$. Considering (18) we obtain $d(d l-2 m) \neq 0$ and then the condition $\eta=0$ gives $m=d l$. In this case we have $\tilde{L}=8 d^{2} l^{2}(x+d y)^{2} \neq$ 0 and then via the rescaling $(x, y) \mapsto\left(x /(d l), y /\left(d^{2} l\right)\right)$ we obtain the following 2parameter family of systems:

$$
\begin{equation*}
\dot{x}=x+y, \quad \dot{y}=e x+f y+(x+y)^{2} . \tag{21}
\end{equation*}
$$

For these systems calculations yield

$$
\begin{gather*}
\mu_{0}=\mu_{1}=\mu_{2}=\eta=0, \quad \mu_{3}=(f-e)(x+y)^{3}, \quad \tilde{L}=8(x+y)^{2}=-\widetilde{M}, \\
\widetilde{K}=\widetilde{N}=\kappa=0, \quad K_{1}=(x+y)^{3}, \quad F_{4} F_{5}=6(f-e)(x+y)^{6},  \tag{22}\\
G_{3}=0, \quad W_{8}=2^{14} 3^{3}(e-f)^{4}\left[(f-1)^{2}+4 e\right], \quad \mathcal{T}_{i}=0, \quad i=1,2,3,4, \\
\sigma=1+f+2 x+2 y, \quad \mathcal{F}_{1}=\mathcal{H}=0, \quad \mathcal{B}_{1}=4(e-f)^{2}(1+f), \quad \mathcal{B}_{2}=4(e-f)^{3}
\end{gather*}
$$

and we again have $\operatorname{sign}\left(\mu_{3} K_{1}\right)=\operatorname{sign}\left(F_{4} F_{5}\right)$.

1) The possibility $\mu_{3} K_{1}<0$. Then we have $F_{4} F_{5}<0$ and according to [4] (see Table 1, line 179) the finite singularity is a saddle. Clearly this saddle is weak if and only if $f=-1$ and this is equivalent to $\mathcal{B}_{1}=0$. On the other hand considering (22) we obtain $\mathcal{B}_{2}>0$ and according to [39] the weak saddle is an integrable one. So considering Lemma 6 we get the following two configurations

- $s ; \widehat{\binom{3}{2}} \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge}-H, N^{d}:$ Example $\Rightarrow(e=2, f=1) \quad\left(\right.$ if $\left.\mathcal{B}_{1} \neq 0\right)$;
- $\$ ;\binom{\widehat{3}}{2} \widehat{P}_{\mathcal{\curlywedge}} E \widehat{P}_{\mathcal{\curlywedge}}-H, N^{d}$ : Example $\Rightarrow(e=2, f=-1) \quad\left(\right.$ if $\left.\mathcal{B}_{1}=0\right)$.

2) The possibility $\mu_{3} K_{1}>0$ In this case we have $F_{4} F_{5}>0$ and as $G_{3}=\widetilde{N}=0$ by [4] we have a focus or a center if $W_{8}<0$ and a node if $W_{8} \geq 0$.
a) The case $W_{8}<0$. Then we have a focus which is strong if $\mathcal{B}_{1} \neq 0$. Considering (22) we have $\mathcal{B}_{2}<0$ and according to [39] in the case $\mathcal{B}_{1}=0$ we have a center. So considering Lemma 6 we arrive at the configurations

- $f ; \widehat{\binom{3}{2}} H_{\curlywedge} H H_{\curlywedge}-H, N^{d}: \quad$ Example $\Rightarrow(e=-2, f=1) \quad\left(\right.$ if $\left.\mathcal{B}_{1} \neq 0\right)$;
- $c ;\binom{\widehat{3}}{2} H_{\curlywedge} H H_{\curlywedge}-H, N^{d}$ : Example $\Rightarrow(e=-2, f=-1) \quad\left(\right.$ if $\left.\mathcal{B}_{1}=0\right)$.
b) The case $W_{8}>0$. In this case we have a generic node (as the condition $W_{8} \neq 0$ implies $\left.\delta_{1}=(f-1)^{2}+4 e \neq 0\right)$ and hence we get the configuration
- $n ; \widehat{\binom{3}{2}} H_{\curlywedge} H H_{\curlywedge}-H, N^{d}$ : Example $\Rightarrow(e=0, f=2)$.
c) The case $W_{8}=0$. Then we have a node with coinciding eigenvalues and due to the matrix of the linearization of the system at the singularity $M_{1}(0,0)$, this is a one-direction node, providing the configuration
- $\left.n^{d} ; \widehat{(3)} \begin{array}{l}3 \\ 2\end{array}\right) H_{\curlywedge} H H_{\curlywedge}-H, N^{d}:$ Example $\Rightarrow(e=-1 / 4, f=0)$.
8.2.2.2.2 The subcase $\tilde{\boldsymbol{L}}=\mathbf{0}$. Considering (18) we obtain $d(d l-2 m)=0$ and as for systems (17) we have

$$
\widetilde{M}=-32 m^{2} x^{2}-8 d(d l-2 m)\left(3 l x^{2}-2 m x y+d^{2} l y^{2}-2 d m y^{2}\right)
$$

The condition above gives $\widetilde{M}=-32 m^{2} x^{2}$. We consider two possibilities: $\widetilde{M} \neq 0$ and $\widetilde{M}=0$.

1) The possibility $\widetilde{M} \neq 0$. Then $m \neq 0$ and as the condition $d(d l-2 m)=0$ holds, applying the transformation

$$
x_{1}=d l x, \quad y_{1}=d l(x+d y)
$$

when $d \neq 0$ (then $m=d l / 2 \neq 0$ due to $\widetilde{M} \neq 0)$, or the transformation

$$
x_{1}=2 m x, \quad y_{1}=l x /(2 m)+y
$$

when $d=0$, we arrive at the following family of systems

$$
\begin{align*}
& \dot{x}=\varepsilon_{1} x+\varepsilon_{2} y, \quad \varepsilon_{1} \varepsilon_{2}=0  \tag{23}\\
& \dot{y}=e x+f y+x y, \quad \varepsilon_{1}+\varepsilon_{2}=1
\end{align*}
$$

For these systems calculations yield

$$
\begin{gather*}
\mu_{0}=\mu_{1}=\mu_{2}=0, \quad \mu_{3}=\left(\varepsilon_{1} f-\varepsilon_{2} e\right) x y\left(\varepsilon_{1} x+\varepsilon_{2} y\right), \quad \widetilde{K}=\kappa=\tilde{L}=0 \\
\widetilde{N}=-x^{2}, \quad \eta=0, \quad \widetilde{M}=-8 x^{2}, \quad \kappa_{1}=-32 \varepsilon_{2}, \quad K_{1}=x y\left(\varepsilon_{1} x+\varepsilon_{2} y\right) \\
F_{4} F_{5}=6\left(\varepsilon_{1} f-\varepsilon_{2} e\right) x^{2} y^{2}\left(\varepsilon_{1} x+\varepsilon_{2} y\right)^{2}, \quad W_{7}=3 \varepsilon_{2} e^{2}\left(4 \varepsilon_{2} e+f^{2}\right) / 16  \tag{24}\\
G_{3}=0, \quad \mathcal{T}_{i}=0, \quad i=1,2,3,4, \quad \sigma=\varepsilon_{1}+f+x, \quad \mathcal{F}_{1}=\mathcal{H}=0 \\
\mathcal{B}_{1}=-\varepsilon_{2} e f, \quad \mathcal{B}_{2}=\varepsilon_{2} e / 4
\end{gather*}
$$

So we obtain again $\operatorname{sign}\left(\mu_{3} K_{1}\right)=\operatorname{sign}\left(F_{4} F_{5}\right)$ and we consider two cases: $\mu_{3} K_{1}<0$ and $\mu_{3} K_{1}>0$.
a) Assume first $\mu_{3} K_{1}<0$. Then we have $F_{4} F_{5}<0$ and according to [4] (see Table 1, line 179) the finite singularity is a saddle. Clearly this saddle is weak if and only if $\varepsilon_{1}+f=0$.
a) The case $\kappa_{1} \neq 0$. Then by (24) we have $\varepsilon_{2}=1, \varepsilon_{1}=0$ and the condition $\mu_{3} K_{1}<0$ yields $e>0$. So $\mathcal{B}_{2}>0$ and we have $\mathcal{B}_{1}=0$ if and only if $f=0$. In this case according to [39] we have an integrable saddle. Therefore considering the condition $\kappa_{1} \neq 0$ and Lemma 6 we get the following two global configurations of singularities

- $s ; \widehat{\binom{1}{2}} \stackrel{\overparen{P}}{\curlywedge} E \widehat{饣}_{\curlywedge}-H, \overline{\binom{2}{1}} N: \quad$ Example $\Rightarrow\left(\varepsilon_{2}=1, e=1, f=1\right) \quad\left(\right.$ if $\left.\mathcal{B}_{1} \neq 0\right) ;$
- $\$ ; \widehat{\binom{1}{2}} \stackrel{\overparen{P}}{\curlywedge} E \stackrel{\overparen{P}}{\curlywedge}-H, \overline{\binom{2}{1}} N: \quad$ Example $\Rightarrow\left(\varepsilon_{2}=1, e=1, f=0\right) \quad\left(\right.$ if $\left.\mathcal{B}_{1}=0\right)$.
$\beta$ ) The case $\kappa_{1}=0$. Then we have $\varepsilon_{2}=0, \varepsilon_{1}=1$ and the condition $\mu_{3} K_{1}<0$ yields $f<0$. We observe that in this case the saddle is a weak one if and only if $f+1=0$. On the other hand calculations yield

$$
\begin{equation*}
\mathcal{F}_{1}=\mathcal{H}=\mathcal{B}_{1}=\mathcal{B}_{2}=\mathcal{B}_{3}=0, \quad \mathcal{B}_{4}=6(1+f) x^{2} y \tag{25}
\end{equation*}
$$

So according to [39] in the case of weak saddle (i.e. $f=-1$ ) we have an integrable saddle. Therefore considering the condition $\kappa_{1}=0$ and Lemma 6 we obtain the configurations

- $s ;\binom{2}{2} \hat{P} E-\tilde{P} E, \overline{\binom{1}{1}} S N: \quad$ Example $\Rightarrow\left(\varepsilon_{2}=0, e=0, f=-2\right) \quad\left(\right.$ if $\left.\mathcal{B}_{4} \neq 0\right)$;
- $\$ ;\binom{2}{2} \widehat{P} E-\widehat{P} E, \overline{\binom{1}{1}} S N: \quad$ Example $\Rightarrow\left(\varepsilon_{2}=0, e=0, f=-1\right) \quad\left(\right.$ if $\left.\mathcal{B}_{4}=0\right)$.
b) Suppose now $\mu_{3} K_{1}>0$. In this case we have $F_{4} F_{5}>0$ and as $G_{3}=0$ and $\widetilde{N} \neq 0$, according to $[4]$ (see Table 1, lines $182,186,188$ ) we have a focus or a center if $W_{7}<0$ and a node if $W_{7} \geq 0$.
$\alpha)$ The case $W_{7}<0$. Then we have $\varepsilon_{2}=1, \varepsilon_{1}=0$ (i.e. $\kappa_{1} \neq 0$ ) and $e<0$. So the finite singularity is a focus and according to [39] we have a strong focus if $\mathcal{B}_{1} \neq 0$ and we have a center if $\mathcal{B}_{1}=0$.

Thus considering Lemma 6 we arrive at the following two configurations
$\bullet f ;\binom{1}{2} \widetilde{P}_{\curlywedge} E \widehat{P}_{\curlywedge}-H, \overline{\binom{2}{1}} S: \quad$ Example $\Rightarrow\left(\varepsilon_{2}=1, e=-1, f=1\right) \quad\left(\right.$ if $\kappa_{1} \neq 0$, $\mathcal{B}_{1} \neq 0$ );

- $c ; \widehat{\binom{1}{2}} \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge}-H, \overline{(\overline{2})} S: \quad$ Example $\Rightarrow\left(\varepsilon_{2}=1, e=-1, f=0\right) \quad\left(\right.$ if $\kappa_{1} \neq 0$, $\mathcal{B}_{1}=0$ ).
$\beta)$ The case $W_{7}>0$. Then we again have $\varepsilon_{2}=1, \varepsilon_{1}=0$ and hence $\kappa_{1} \neq 0$. So the singular point is a generic node and by Lemma 6 we get the configuration
- $n ; \widehat{\binom{1}{2}} \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge}-H, \overline{\binom{2}{1}} S: \quad$ Example $\Rightarrow\left(\varepsilon_{2}=1, e=-1, f=3\right)$.
$\gamma)$ The case $W_{7}=0$. Then by (24) we have $\varepsilon_{2} e\left(4 \varepsilon_{2} e+f^{2}\right)=0$ and we consider two subcases: $\kappa_{1} \neq 0$ and $\kappa_{1}=0$.
$\left.\gamma_{1}\right)$ The subcase $\kappa_{1} \neq 0$. Then we have $\varepsilon_{2}=1, \varepsilon_{1}=0$ and the condition $W_{7}=0$ gives $e=-f^{2} / 4$. Considering the linearization matrix of the singularity $M_{1}(0,0)$ we conclude that systems (23) possess a node $n^{d}$. So by Lemma 6 we have the configuration
- $n^{d} ; \widehat{\binom{1}{2}} \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge}-H,\left(\begin{array}{c}2_{1}^{2}\end{array}\right) S$ : Example $\Rightarrow\left(\varepsilon_{2}=1, e=-1, f=2\right)$.
$\gamma_{2}$ ) The subcase $\kappa_{1}=0$. In this case we have $\varepsilon_{2}=0, \varepsilon_{1}=1$ and the linearization matrix of the singularity $M_{1}(0,0)$ is $\left(\begin{array}{ll}1 & 0 \\ e & f\end{array}\right)$ with $f>0$ due to $\mu_{3} K_{1}>0$. So systems (23) possess i) a generic node if $f \neq 1$; ii) a one-direction node if $f=1$ and $e \neq 0$, and iii) a star node if $f=1$ and $e=0$. On the other hand for these systems in the considered case we have

$$
U_{7}=12(f-1) x^{4}, \quad U_{3}=-3 x^{4}[e f x+(1-f) y],
$$

and clearly these invariant polynomials govern the possibilities mentioned above. So considering the condition $\kappa_{1}=0$ and Lemma 6 we get the following three configurations

- $n ;\binom{2}{2} \widehat{P} H-\widehat{P} H, \overline{\binom{1}{1}} S N: \quad$ Example $\Rightarrow\left(\varepsilon_{2}=0, e=1, f=2\right) \quad\left(\right.$ if $\left.U_{7} \neq 0\right)$;
- $\left.n^{d} ;\binom{2}{2} \overparen{P} H-\overparen{P} H, \overline{(1} 1\right) S N: \quad$ Example $\Rightarrow\left(\varepsilon_{2}=0, e=1, f=1\right) \quad\left(\right.$ if $U_{7}=0$, $U_{3} \neq 0$ );
- $\left.n^{*} ;\binom{2}{2} \widehat{P} H-\widehat{P} H, \overline{(1)} 1 \begin{array}{l}1 \\ 1\end{array}\right) S N: \quad$ Example $\Rightarrow\left(\varepsilon_{2}=0, e=0, f=1\right) \quad\left(\right.$ if $U_{7}=0$, $\left.U_{3}=0\right)$.

2) The possibility $\widetilde{M}=0$. In this case $m=0$ and then the condition $\widetilde{M}=0$ yields $d l=0$. As $l \neq 0$ (due to $\mu_{3} \neq 0$ ) we get $d=0$ and then via the rescaling $(x, y) \mapsto(x / l, y / l)$ we may assume $l=1$. Therefore we obtain the family of systems

$$
\begin{equation*}
\dot{x}=x, \quad \dot{y}=e x+f y+x^{2} \tag{26}
\end{equation*}
$$

for which calculations yield

$$
\begin{gather*}
\mu_{0}=\mu_{1}=\mu_{2}=0, \quad \mu_{3}=f x^{3}, \quad \eta=\widetilde{M}=0, \quad C_{2}=-x^{3}, \quad K_{1}=x^{3} \\
\widetilde{K}=\kappa=\tilde{L}=\widetilde{N}=0, \quad G_{3}=W_{8}=0, \quad K_{3}=6(2-f) f x^{6}  \tag{27}\\
F_{4} F_{5}=6 f x^{6}, \quad \mathcal{T}_{i}=0, \quad i=1,2,3,4, \quad \sigma=1+f
\end{gather*}
$$

a) The case $\mu_{3} K_{1}<0$. Then we have $F_{4} F_{5}<0$ and according to [4] (see Table 1 , line 179) the finite singularity is a saddle. Clearly this saddle is weak if and only if $f=-1$ and this is equivalent to $\sigma=0$. However in the last case we get Hamiltonian systems and hence the weak saddle is an integrable one. So considering Lemma 6 we arrive at the configurations

- $s ;\binom{3}{3} \widehat{P}_{\curlywedge} E E \widehat{P}_{\curlywedge}-\widehat{P} \widehat{P}:$ Example $\Rightarrow(e=0, f=-2) \quad($ if $\sigma \neq 0)$;
- $\$ ;\binom{3}{3} \widehat{P}_{\curlywedge} E E \widehat{P}_{\curlywedge}-\widehat{P} \widehat{P}_{P}: \quad$ Example $\Rightarrow(e=0, f=-1) \quad($ if $\sigma=0)$.
b) The case $\mu_{3} K_{1}>0$. In this case we have $F_{4} F_{5}>0$ (i.e. $f>0$ ) and considering the matrix of the linearization at the singular point, we conclude that the singular point $M_{1}(0,0)$ is a node. Moreover, this node is: i) generic if $f \neq 1$; ii) one-direction node if $f=1$ and $e \neq 0$, and iii) it is a star node if $f=1$ and $e=0$. On the other hand for these systems in the considered case we have

$$
U_{4}=-6(f-1) x^{3},\left.\quad U_{5}\right|_{f=1}=-6 e x^{2}
$$

The behavior of the trajectories in the vicinity of the infinite singularity (which is of multiplicity six) according to Lemma 6 is governed by the invariant polynomial $K_{3}$. By (27) as $f>0$ we have $\operatorname{sign}\left(K_{3}\right)=\operatorname{sign}(2-f)$. Thus we arrive at the following five geometrically distinct global configurations of singularities

- $n ;\binom{3}{3} H_{\curlywedge} \widehat{P} \widehat{P} H_{\curlywedge}-\overparen{P} \widehat{P}: \quad$ Example $\Rightarrow(e=0, f=3) \quad\left(\right.$ if $\left.K_{3}<0\right)$;
- $n ;\binom{3}{3} H H-\widehat{P} \widehat{P}:$ Example $\Rightarrow(e=0, f=2) \quad\left(\right.$ if $\left.K_{3}=0\right)$;
- $n ;\binom{3}{3} \widehat{P}_{\curlywedge} \widehat{P} \widehat{P}_{P_{\curlywedge}}-H H: \quad$ Example $\Rightarrow(e=0, f=1 / 2) \quad\left(\right.$ if $\left.K_{3}>0, U_{4} \neq 0\right)$;
- $n^{d} ;\binom{3}{3} \widehat{P}_{\curlywedge} \widehat{P} \widehat{P}^{\widehat{P}_{\curlywedge}}-H H: \quad$ Example $\Rightarrow(e=1, f=1) \quad\left(\right.$ if $K_{3}>0, U_{4}=0$, $U_{5} \neq 0$ );
- $n^{*} ;\binom{3}{3} \widehat{P}_{\curlywedge} \widehat{P} \overparen{P}_{P} \widehat{P}_{\curlywedge}-H H: \quad$ Example $\Rightarrow(e=0, f=1) \quad\left(\right.$ if $K_{3}>0, U_{4}=0$, $U_{5}=0$ ).

As all the cases have been considered we have got 52 possible geometrically distinct global configurations of singularities of the family of quadratic systems with only one finite singularity which in addition is elemental.

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## References

[1] Artés J. C., Llibre J. Quadratic Hamiltonian vector fields, J. Differential Equations, 1994, 107, 80-95.
[2] Artés J. C., Llibre J., Schlomiuk D. The geometry of quadratic differential systems with a weak focus of second order, International J. of Bifurcation and Chaos, 2006, 16, 3127-3194.
[3] Artés J. C., Llibre J., Schlomiuk D., Vulpe N. From topological to geometric equivalence in the classification of singularities at infinity for quadratic vector fields, Rocky Mountain J. of Math. (accepted).
[4] Artés J. C., Llibre J., Vulpe N. I. Singular points of quadratic systems: A complete classification in the coefficient space $\mathbb{R}^{12}$, International J. of Bifurcation and Chaos, 2008, 18, 313-362.
[5] Artés J. C., Llibre J., Vulpe N. I. Complete geometric invariant study of two classes of quadratic systems, Electron. J. Differential Equations, 2012, 2012, No. 9, 1-35.
[6] Artés J. C., Llibre J., Vulpe N. Quadratic systems with an integrable saddle: A complete classification in the coefficient space $\mathbb{R}^{12}$, Nonlinear Analysis, 2012, 75, 5416-5447.
[7] Baltag V. A. Algebraic equations with invariant coefficients in qualitative study of the polynomial homogeneous differential systems, Bull. Acad. Ştiinţe. Repub. Moldova, Mat., 2003, No. 2(42), 31-46.
[8] Baltag V. A., Vulpe N. I. Affine-invariant conditions for determining the number and multiplicity of singular points of quadratic differential systems, Bull. Acad. Ştiinţe. Repub. Moldova, Mat., 1993, No. 1(11), 39-48 (in Russian).
[9] Baltag V. A., Vulpe N. I. Total multiplicity of all finite critical points of the polynomial differential system, Planar nonlinear dynamical systems (Delft, 1995), Differential Equations \& Dynam. Systems, 1997, 5, 455-471.
[10] Bendixson I. Sur les courbes définies par des équations différentielles, Acta Math., 1901, 24, 1-88.
[11] Bularas D., Calin Iu., Timochouk L., Vulpe N. T-comitants of quadratic systems: A study via the translation invariants, Delft University of Technology, Faculty of Technical Mathematics and Informatics, Report no. 96-90, 1996; (URL: ftp://ftp.its.tudelft.nl/publications/techreports/1996/DUT-TWI-96-90. ps.gz).
[12] Calin Iu. On rational bases of $G L(2, \mathbb{R})$-comitants of planar polynomial systems of differential equations, Bull. Acad. Ştiinţe. Repub. Moldova, Mat., 2003, No. 2(42), 69-86.
[13] Coll B. Qualitative study of some classes of vector fields in the plane, Ph. D., Universitat Autónoma de Barcelona, 1987, 5-34.
[14] Coppel W. A. A survey of quadratic systems, J. Differential Equations, 1966, 2, 293-304.
[15] Dumortier F. Singularities of vector fields on the plane, J. Differential Equations, 1977, 23, 53-106.
[16] Dumortier F., Fiddelaers P. Quadratic models for generic local 3-parameter bifurcations on the plane, Trans. Am. Math. Soc., 1991, 326, 101-126.
[17] Dumortier F., Llibre J., Artés J. C. Qualitative Theory of Planar Differential Systems, Universitext, Springer-Verlag, New York-Berlin, ISBN: 3-540-32893-9, 2008.
[18] Gonzalez Velasco E. A. Generic properties of polynomial vector fields at infinity, Trans. Amer. Math. Soc., 1969, 143, 201-222.
[19] Grace J. H., Young A. The algebra of invariants, New York, Stechert, 1941.
[20] Jiang Q., Llibre J. Qualitative classification of singular points, Qualitative Theory of Dynamical Systems, 2005, 6, 87-167.
[21] Llibre J., Schlomiuk D. Geometry of quadratic differential systems with a weak focus of third order, Canad. J. of Math., 2004, 56, 310-343.
[22] Olver P. J. Classical Invariant Theory, London Math. Soc. Student Texts 44, Cambridge University Press, 1999.
[23] Nikolaev I., Vulpe N. Topological classification of quadratic systems at infinity, J. London Math. Soc., 1997, 2, 473-488.
[24] Pal J., Schlomiuk D. Summing up the dynamics of quadratic Hamiltonian systems with a center, Canad. J. Math., 1997, 56, 583-599.
[25] Popa M. N. Applications of algebraic methods to differential systems, Romania, Piteşti Univers., The Flower Power Edit., 2004.
[26] Roussarie R. On the number of limit cycles which appear by perturbation of separatrix loop of planar vector fields, Bol. Soc. Bras. Mat., 1986, 17, No. 2, 67-101.
[27] Roussarie R. Smoothness property for bifurcation diagrams, Publicacions Matemàtiques, 1997, 56, 243-268.
[28] Schlomiuk D. Algebraic particular integrals, integrability and the problem of the center, Trans. Amer. Math. Soc., 1993, 338, 799-841.
[29] Schlomiuk D. Basic algebro-qeometric concepts in the study of planar polynomial vector fields, Publicacions Mathemàtiques, 1997, 41, 269--295.
[30] Schlomiuk D., Pal J. On the geometry in the neighborhood of infinity of quadratic differential phase portraits with a weak focus, Qualitative Theory of Dynamical Systems, 2001, 2, 1-43.
[31] Schlomiuk D., Vulpe N. I. Geometry of quadratic differential systems in the neighborhood of infinity, J. Differential Equations, 2005, 215, 357-400.
[32] Schlomiuk D., Vulpe N. I. Integrals and phase portraits of planar quadratic differential systems with invariant lines of at least five total multiplicity, Rocky Mountain Journal of Mathematics, 2008, 38, No. 6, 1-60.
[33] Schlomiuk D., Vulpe N. I. Integrals and phase portraits of planar quadratic differential systems with invariant lines of total multiplicity four, Bull. Acad. Ştiinţe. Repub. Moldova, Mat., 2008, No. 1(56), 27-83.
[34] Schlomiuk D., Vulpe N. I. The full study of planar quadratic differential systems possessing a line of singularities at infinity, J. Dynam. Differential Equations, 2008, 20, 737-775.
[35] Schlomiuk D., Vulpe N. I. Global classification of the planar Lotka-Volterra differential system according to their configurations of invariant straight lines, J. Fixed Point Theory Appl., 2010, 8, 177-245.
[36] Schlomiuk D., Vulpe N. I. The Global Topological classification of the Lotka-Volterra quadratic differential systems, Electron. J. Differential Equations, 2012, 2012, No. 64, 1-69.
[37] Schlomiuk D., Vulpe N. I. Applications of symbolic calculations and polynomial invariants to the classification of singularities of differential systems, CASC 2013, Lecture Notes in Computer Science 8136, Berlin, Springer, 2013, 340-354.
[38] Sibirski K.S. Introduction to the algebraic theory of invariants of differential equations, Translated from Russian. Nonlinear Science: Theory and Applications. Manchester University Press, Manchester, 1988.
[39] Vulpe N. Characterization of the finite weak singularities of quadratic systems via invariant theory. Nonlinear Analysis. Theory, Methods and Applications, 2011, 74, No. 4, 6553-6582.
[40] Vulpe N. I. Affine-invariant conditions for the topological discrimination of quadratic systems with a center, Differential Equations, 1983, 19, 273-280.
[41] Vulpe N. I. Polynomial bases of comitants of differential systems and their applications in qualitative theory, Kishinev, Shtiintsa, 1986 (in Russian).
[42] Vulpe N., Lupan M. Quadratic systems with dicritical points, Bull. Acad. Ştiinţe. Repub. Moldova, Mat., 1994, No. 3(16), 52-60.
[43] ŻOモA̧DEK H. Quadratic systems with center and their perturbations, J. Differential Equations, 1994, 109, 223-273.

Joan C. Artes, Jaume Llibre
Received February 12, 2013
Departament de Matemàtiques
Universitat Autònoma de Barcelona
Spain
E-mail: artes@mat.uab.cat; jllibre@mat.uab.cat
Dana Schlomiuk
Département de Mathématiques
et de Statistiques Université de Montréal
E-mail:dasch@dms.umontreal.ca
Nicolae Vulpe
Institute of Mathematics and Computer Science
Academy of Science of Moldova
E-mail: nvulpe@gmail.com

# The order of convexity for a general integral operator 

Laura Stanciu, Daniel Breaz


#### Abstract

In this paper, we consider the classes of the univalent functions denoted by $\mathcal{S H}(\beta), \mathcal{S P}$ and $\mathcal{S P}(\alpha, \beta)$. On these classes we study the order of convexity of the integral operator $\int_{0}^{z}\left(t e^{f(t)}\right)^{\gamma} d t$, where the function $f$ belongs to these classes.


Mathematics subject classification: 30C45, 30C75.
Keywords and phrases: Analytic function; Integral Operator; Starlike function; Convex function.

## 1 Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of all functions of the form

$$
f(z)=z+\sum_{k=2}^{\infty} a_{k} z^{k}
$$

which are analytic in the open unit disk

$$
\mathbb{U}=\{z \in \mathbb{C}: \quad|z|<1\}
$$

and satisfy the following usual normalization condition

$$
f(0)=f^{\prime}(0)-1=0 .
$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f$ which are univalent in $\mathbb{U}$.
A function $f \in \mathcal{A}$ is the starlike function of order $\alpha, 0 \leq \alpha<1$ if $f$ satisfies the inequality

$$
\operatorname{Re}\left(\frac{z f^{\prime}(z)}{f(z)}\right)>\alpha, \quad z \in \mathbb{U} .
$$

We denote this class by $\mathcal{S}^{*}(\alpha)$.
A function $f \in \mathcal{A}$ is a convex function of order $\alpha, 0 \leq \alpha<1$, if $f$ satisfies the inequality

$$
\operatorname{Re}\left(\frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}+1\right)>\alpha, \quad z \in \mathbb{U} .
$$

We denote this class by $\mathcal{K}(\alpha)$.
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In [4], J. Stankiewicz and A. Wisniowska introduced the class of univalent functions $\mathcal{S H}(\beta), \beta>0$, defined by

$$
\begin{equation*}
\left|\frac{z f^{\prime}(z)}{f(z)}-2 \beta(\sqrt{2}-1)\right|<\operatorname{Re}\left\{\sqrt{2} \frac{z f^{\prime}(z)}{f(z)}\right\}+2 \beta(\sqrt{2}-1) \tag{1}
\end{equation*}
$$

for all $z \in \mathbb{U}$.
Also, in [3], F. Ronning introduced the class of univalent functions $\mathcal{S P}$, defined by

$$
\begin{equation*}
\left|\frac{z f^{\prime}(z)}{f(z)}-1\right|<\operatorname{Re}\left(\frac{z f^{\prime}(z)}{f(z)}\right) \tag{2}
\end{equation*}
$$

for all $z \in \mathbb{U}$.
The geometric interpretation of the relation (2) is that the class $\mathcal{S P}$ is the class of all functions $f \in \mathcal{S}$ for which the expression $z f^{\prime}(z) / f(z), z \in \mathbb{U}$, takes all values in the parabolic region

$$
\begin{aligned}
\Omega & =\{\omega: \quad|\omega-1| \leq \operatorname{Re} \omega\} \\
& =\left\{\omega=u+i v: \quad v^{2} \leq 2 u-1\right\} .
\end{aligned}
$$

In [2], F. Ronning introduced the class of univalent functions $\mathcal{S P}(\alpha, \beta), \alpha>0$, $\beta \in[0,1)$, as the class of all functions $f \in \mathcal{S}$ which have the property

$$
\begin{equation*}
\left|\frac{z f^{\prime}(z)}{f(z)}-(\alpha+\beta)\right| \leq \operatorname{Re}\left(\frac{z f^{\prime}(z)}{f(z)}\right)+\alpha-\beta, \tag{3}
\end{equation*}
$$

for all $z \in \mathbb{U}$.
Geometric interpretation: $f \in \mathcal{S P}(\alpha, \beta)$ if and only if $z f^{\prime}(z) / f(z), z \in \mathbb{U}$, takes all values in the parabolic region

$$
\begin{aligned}
\Omega_{\alpha, \beta} & =\{\omega: \quad|\omega-(\alpha+\beta)| \leq \operatorname{Re} \omega+\alpha-\beta\} \\
& =\left\{\omega=u+i v: \quad v^{2} \leq 4 \alpha(u-\beta)\right\} .
\end{aligned}
$$

In the present paper, we will obtain the order of convexity of the following integral operator:

$$
\begin{equation*}
F(z)=\int_{0}^{z}\left(t e^{f(t)}\right)^{\gamma} d t \tag{4}
\end{equation*}
$$

where the function $f \in \mathcal{A}$ and $\gamma \in \mathbb{C}$.
Remark 1. The integral operator defined by (4) was introduced by Frasin and Ahmad in [1].

## 2 Main results

Theorem 1. Let $f \in \mathcal{A}$ be in the class $\mathcal{S H}(\beta), \beta>0$ and $f$ satisfies the condition $|f(z)| \leq M$, for $M$ a positive real number, $M \geq 1$ for all $z \in \mathbb{U}$. If $\operatorname{Re}\left(\frac{z f^{\prime}(z)}{f(z)}\right) \leq 1$, $z \in \mathbb{U}$, then the integral operator $F(z)$ defined by (4) is in $\mathcal{K}(\delta)$, where

$$
\delta=1-|\gamma|[(4 \beta(\sqrt{2}-1)+\sqrt{2}) M+1]
$$

and

$$
|\gamma|[(4 \beta(\sqrt{2}-1)+\sqrt{2}) M+1]<1, \quad \gamma \in \mathbb{C} .
$$

Proof. We calculate for $F(z)$ the derivatives of the first and second order. From (4) we obtain

$$
F^{\prime}(z)=\left(z e^{f(z)}\right)^{\gamma}
$$

and

$$
F^{\prime \prime}(z)=\gamma\left(z e^{f(z)}\right)^{\gamma-1}\left(e^{f(z)}+z f^{\prime}(z) e^{f(z)}\right)
$$

After the calculus, we obtain that

$$
\begin{align*}
\frac{z F^{\prime \prime}(z)}{F^{\prime}(z)} & =\gamma\left(1+z f^{\prime}(z)\right) \\
& =\gamma\left(\frac{z f^{\prime}(z)}{f(z)} f(z)+1\right) \tag{5}
\end{align*}
$$

It follows from (5) that

$$
\begin{align*}
\left|\frac{z F^{\prime \prime}(z)}{F^{\prime}(z)}\right| & \leq|\gamma|\left(\left|\frac{z f^{\prime}(z)}{f(z)}\right||f(z)|+1\right) \\
& \leq \gamma\left(\left(\left|\frac{z f^{\prime}(z)}{f(z)}-2 \beta(\sqrt{2}-1)\right|+2 \beta(\sqrt{2}-1)\right)|f(z)|+1\right) . \tag{6}
\end{align*}
$$

Because $f \in \mathcal{S H}(\beta), \beta>0$ and $|f(z)| \leq M, M \geq 1$ for all $z \in \mathbb{U}$, we apply in the condition (6) the inequality (1) and we obtain

$$
\begin{aligned}
\left|\frac{z F^{\prime \prime}(z)}{F^{\prime}(z)}\right| & \leq|\gamma|\left(\left(\operatorname{Re}\left\{\sqrt{2} \frac{z f^{\prime}(z)}{f(z)}\right\}+4 \beta(\sqrt{2}-1)\right) M+1\right) \\
& \leq|\gamma|\left(\left(\sqrt{2} \operatorname{Re}\left(\frac{z f^{\prime}(z)}{f(z)}\right)+4 \beta(\sqrt{2}-1)\right) M+1\right)
\end{aligned}
$$

From the hypothesis of Theorem 1 we have $\operatorname{Re}\left(\frac{z f^{\prime}(z)}{f(z)}\right) \leq 1$ and we obtain

$$
\left|\frac{z F^{\prime \prime}(z)}{F^{\prime}(z)}\right| \leq|\gamma|[(4 \beta(\sqrt{2}-1)+\sqrt{2}) M+1]=1-\delta
$$

which implies that the integral operator $F(z)$ defined by (4) is in the class $\mathcal{K}(\delta)$.

Theorem 2. Let the function $f \in \mathcal{S P}$, where $f$ satisfies the condition $|f(z)| \leq M$, for $M$ a positive real number, $M \geq 1, z \in \mathbb{U}$. If $\operatorname{Re}\left(\frac{f^{\prime}(z)}{f(z)}\right) \leq 1, z \in \mathbb{U}$, then the integral operator $F(z)$ defined by (4) is in $\mathcal{K}(\delta)$, where

$$
\delta=1-|\gamma|(2 M+1)
$$

and

$$
|\gamma|(2 M+1)<1, \quad \gamma \in \mathbb{C} .
$$

Proof. Following the same steps as in Theorem 1, we have

$$
\begin{equation*}
\frac{z F^{\prime \prime}(z)}{F^{\prime}(z)}=\gamma\left(\frac{z f^{\prime}(z)}{f(z)} f(z)+1\right) \tag{7}
\end{equation*}
$$

It follows from (7) that

$$
\begin{align*}
\left|\frac{z F^{\prime \prime}(z)}{F^{\prime}(z)}\right| & \leq|\gamma|\left(\left|\frac{z f^{\prime}(z)}{f(z)}\right||f(z)|+1\right) \\
& \leq \gamma\left(\left(\left|\frac{z f^{\prime}(z)}{f(z)}-1\right|+1\right)|f(z)|+1\right) . \tag{8}
\end{align*}
$$

Because $f \in \mathcal{S P}$ and $|f(z)| \leq M, M \geq 1$ for all $z \in \mathbb{U}$, we apply in the condition (8) the inequality (2) and we obtain

$$
\left|\frac{z F^{\prime \prime}(z)}{F^{\prime}(z)}\right| \leq|\gamma|\left(\left(\operatorname{Re}\left(\frac{z f^{\prime}(z)}{f(z)}\right)+1\right) M+1\right) .
$$

Because $\operatorname{Re}\left(\frac{z f^{\prime}(z)}{f(z)}\right) \leq 1$, we obtain that

$$
\left|\frac{z F^{\prime \prime}(z)}{F^{\prime}(z)}\right| \leq|\gamma|(2 M+1)=1-\delta
$$

which implies that the integral operator $F(z)$ defined by (4) is in the class $\mathcal{K}(\delta)$.
Theorem 3. Let the function $f \in \mathcal{S P}(\alpha, \beta), \alpha>0, \beta \in[0,1)$, where $f$ satisfies the condition $|f(z)| \leq M$, for $M$ a positive real number, $M \geq 1, z \in \mathbb{U}$. If $\operatorname{Re}\left(\frac{f^{\prime}(z)}{f(z)}\right) \leq 1$, $z \in \mathbb{U}$ then the integral operator $F(z)$ defined by (4) is in $\mathcal{K}(\delta)$, where

$$
\delta=1-|\gamma|[(1+2 \alpha) M+1)
$$

and

$$
|\gamma|[(1+2 \alpha) M+1)<1, \quad \gamma \in \mathbb{C} .
$$

Proof. From the proof of Theorem 1, we have

$$
\left|\frac{z F^{\prime \prime}(z)}{F^{\prime}(z)}\right| \leq|\gamma|\left(\left|\frac{z f^{\prime}(z)}{f(z)}\right||f(z)|+1\right)
$$

$$
\begin{equation*}
\leq|\gamma|\left(\left(\left|\frac{z f^{\prime}(z)}{f(z)}-(\alpha+\beta)\right|+(\alpha+\beta)\right)|f(z)|+1\right) . \tag{9}
\end{equation*}
$$

Because $f \in \mathcal{S P}(\alpha, \beta), \alpha>0, \beta \in[0,1)$ and $|f(z)| \leq M, M \geq 1$ for all $z \in \mathbb{U}$, we apply in the condition (9) the inequality (3) and we obtain

$$
\begin{aligned}
\left|\frac{z F^{\prime \prime}(z)}{F^{\prime}(z)}\right| & \leq|\gamma|\left(\left(\operatorname{Re}\left(\frac{z f^{\prime}(z)}{f(z)}\right)+\alpha-\beta+\alpha+\beta\right) M+1\right) \\
& \leq|\gamma|\left(\left(\operatorname{Re}\left(\frac{z f^{\prime}(z)}{f(z)}\right)+2 \alpha\right) M+1\right) .
\end{aligned}
$$

Because $\operatorname{Re}\left(\frac{z f^{\prime}(z)}{f(z)}\right) \leq 1, z \in \mathbb{U}$, we obtain that

$$
\left|\frac{z F^{\prime \prime}(z)}{F^{\prime}(z)}\right| \leq|\gamma|[(1+2 \alpha) M+1]=1-\delta
$$

which implies that the integral operator $F(z)$ defined by (4) is in the class $\mathcal{K}(\delta)$.
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## References

[1] Frasin B. A., Ahmad A. S. The order of convexity of two integral operators, Babeş Bolyai, Mathematica, 2010, LV, No. 2.
[2] Ronning F. Integral reprezentations of bounded starlike functions, Ann. Polon. Math., 1995, LX (3), 289-297.
[3] Ronning F. Uniformly convex functions and a corresponding class of starlike functions, Proc. Amer. Math. Soc., 1993, 118(1), 190-196.
[4] Stankiewicz J., Wisniowska A. Starlike functions associated with some hyperbola, Folia Scientiarum Universitatis Tehnicae Resoviensis 147, Matematyka, 1996, 19, 117-126.

Laura Stanciu
Received February 17, 2012
University of Piteşti
Department of Mathematics
Argeş, România.
E-mail: laura_stanciu_30@yahoo.com
Daniel Breaz
"1 Decembrie 1918" University of Alba Iulia
Department of Mathematics
Alba Iulia, Str. N. Iorga, 510000, No. 11-13, România.
E-mail: dbreaz@uab.ro

# Semilattice decompositions of trioids 

Anatolii V. Zhuchok


#### Abstract

We describe all semilattice congruences on an arbitrary trioid and define the least semilattice congruence on this trioid. We also show that every trioid is a semilattice of $s$-simple subtrioids.


Mathematics subject classification: 17A30, 20M10.
Keywords and phrases: Trioid, semilattice congruence, semilattice of subtrioids, dimonoid, semigroup.

## 1 Introduction

Trioids were introduced by J.-L. Loday and M. O. Ronco [1] for the study of ternary planar trees. Trialgebras, which are based on the notion of a trioid, have been studied in different papers (see, for example, [1-3]). It is well known that the notion of a trioid generalizes the notion of a dimonoid [4,5]. Dimonoids play a prominent role in problems from the theory of Leibniz algebras. Trioids were studied in some papers of the author (see, for example, [6-8]). Note that if the operations of a trioid coincide then it becomes a semigroup. So, trioids are a generalization of semigroups.

In this work we describe semilattice decompositions of trioids. In Section 2 we give necessary definitions, auxiliary results (Proposition 1 and Lemma 1) and describe some connections between trioids and dimonoids (Lemma 2). Yamada [9] described all semilattice congruences on an arbitrary semigroup and proved that every semigroup is a semilattice of $s$-simple semigroups. These results were generalized to dimonoids in [10]. In Section 3 we extend results from [10] to the case of trioids (Theorems 1 and 2).

## 2 Preliminaries

A nonempty set $T$ equipped with three binary associative operations $\dashv, \vdash$ and $\perp$ satisfying the following axioms:

$$
\begin{align*}
& (x \dashv y) \dashv z=x \dashv(y \vdash z),  \tag{T1}\\
& (x \vdash y) \dashv z=x \vdash(y \dashv z),  \tag{T2}\\
& (x \dashv y) \vdash z=x \vdash(y \vdash z),  \tag{T3}\\
& (x \dashv y) \dashv z=x \dashv(y \perp z), \tag{T4}
\end{align*}
$$

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$$
\begin{gather*}
(x \perp y) \dashv z=x \perp(y \dashv z),  \tag{T5}\\
(x \dashv y) \perp z=x \perp(y \vdash z),  \tag{T6}\\
(x \vdash y) \perp z=x \vdash(y \perp z),  \tag{T7}\\
(x \perp y) \vdash z=x \vdash(y \vdash z) \tag{T8}
\end{gather*}
$$

for all $x, y, z \in T$, is called a trioid. If the operations of a trioid coincide, then the trioid becomes a semigroup.

Recall that a nonempty set $T$ equipped with two binary associative operations $\dashv$ and $\vdash$ satisfying the axioms $(T 1)-(T 3)$ is called a dimonoid (see, for example, $[4,5])$.

Let $(T, \perp)$ be an arbitrary semigroup. Define operations $\dashv$ and $\vdash$ on $T$ by

$$
x \dashv y=x, \quad x \vdash y=y
$$

for all $x, y \in T$.
Proposition 1. ([8], Proposition 10). $(T, \dashv \vdash, \vdash, \perp)$ is a trioid.
The trioid $(T, \dashv, \vdash, \perp)$ will be denoted by $T_{l r}^{\perp}$.
Other examples of trioids can be found in [1, 6-8].
A commutative idempotent semigroup is called a semilattice.
Lemma 1. ([7], Lemma 1). The operations of a trioid $(T, \dashv, \vdash, \perp)$ coincide if $(T, \dashv)$ is a semilattice.

Let $X=\{1,2,3\}$. For every pair $(x, y) \in X \times X$ let $T^{(x, y)}=\left(T, *_{x}, *_{y}\right)$ be an ordered triple, where $T$ is a nonempty set and $*_{x}, *_{y}$ are binary operations on $T$. Let

$$
B=\{(1,1),(2,2),(3,3),(1,2)\} \subset X \times X
$$

The following lemma describes connections between trioids and dimonoids.
Lemma 2. For any trioid $\left(T, *_{1}, *_{2}, *_{3}\right)$ the algebra $T^{(x, y)},(x, y) \in X \times X$, is a dimonoid if $(x, y) \in B$. There exists some trioid $\left(T, *_{1}, *_{2}, *_{3}\right)$ for which the algebra $T^{(x, y)},(x, y) \in X^{2} \backslash B$, is not a dimonoid.

Proof. Let $\left(T, *_{1}, *_{2}, *_{3}\right)$ be a trioid. It is easy to see that the algebras $T^{(1,1)}, T^{(2,2)}$, $T^{(3,3)}$ and $T^{(1,2)}$ are dimonoids.

Now we shall prove the second part of the lemma.
Let $F[A]$ be the free semigroup on a set $A$ and $F[A]_{l r}^{\perp}$ be a triod (see Proposition 1) such that $\perp$ is the concatenation on $F[A]$. Assume $\left(T, *_{1}, *_{2}, *_{3}\right)=F[A]_{l r}^{\perp}$ and show that for any $(x, y) \in X^{2} \backslash B$ the algebra $T^{(x, y)}$ is not a dimonoid.

Let $w, u, \omega \in T^{(x, y)}$.
For $T^{(1,3)}$ check the axiom ( $T 3$ ):

$$
\left(w *_{1} u\right) *_{3} \omega=w *_{3} \omega=w \omega \neq w u \omega=w *_{3}\left(u *_{3} \omega\right) .
$$

As the axiom ( $T 3$ ) does not hold, then $T^{(1,3)}$ is not a dimonoid.
For $T^{(2,1)}, T^{(2,3)}, T^{(3,1)}$ and $T^{(3,2)}$ check the axiom ( $T 1$ ).
For $T^{(2,1)}$ we have

$$
\left(w *_{2} u\right) *_{2} \omega=\omega \neq u=w *_{2} u=w *_{2}\left(u *_{1} \omega\right) .
$$

For $T^{(2,3)}$ :

$$
\left(w *_{2} u\right) *_{2} \omega=\omega \neq u \omega=w *_{2}\left(u *_{3} \omega\right) .
$$

For $T^{(3,1)}$ :

$$
\left(w *_{3} u\right) *_{3} \omega=w u \omega \neq w u=w *_{3}\left(u *_{1} \omega\right) .
$$

For $T^{(3,2)}$ :

$$
\left(w *_{3} u\right) *_{3} \omega=w u \omega \neq w \omega=w *_{3}\left(u *_{2} \omega\right) .
$$

The axiom ( $T 1$ ) does not hold for all fourth cases, so $T^{(2,1)}, T^{(2,3)}, T^{(3,1)}$ and $T^{(3,2)}$ are not dimonoids.

The notion of a triband of subtrioids was introduced and investigated in [7]. Recall this definition.

A trioid $(T, \dashv, \vdash, \perp)$ is called an idempotent trioid or a triband if $x \dashv x=$ $x \vdash x=x \perp x=x$ for all $x \in T$. If $\varphi: S \rightarrow M$ is a homomorphism of trioids, then the corresponding congruence on $S$ will be denoted by $\Delta_{\varphi}$.

Let $S$ be an arbitrary trioid, $J$ be some idempotent trioid and

$$
\alpha: S \rightarrow J: x \mapsto x \alpha
$$

be a homomorphism. Then every class of the congruence $\Delta_{\alpha}$ is a subtrioid of the trioid $S$, and the trioid $S$ itself is a union of such trioids $S_{\xi}, \xi \in J$ that

$$
\begin{gathered}
x \alpha=\xi \Leftrightarrow x \in S_{\xi}=\Delta_{\alpha}^{x}=\left\{t \in S \mid(x, t) \in \Delta_{\alpha}\right\}, \\
S_{\xi} \dashv S_{\varepsilon} \subseteq S_{\xi \dashv \varepsilon}, \quad S_{\xi} \vdash S_{\varepsilon} \subseteq S_{\xi \vdash \varepsilon}, \quad S_{\xi} \perp S_{\varepsilon} \subseteq S_{\xi \perp \varepsilon}, \\
\xi \neq \varepsilon \Rightarrow S_{\xi} \bigcap S_{\varepsilon}=\varnothing .
\end{gathered}
$$

In this case we say that $S$ is decomposable into a triband of subtrioids (or $S$ is a triband $J$ of subtrioids $S_{\xi}, \xi \in J$ ). If $J$ is a band (=idempotent semigroup), then we say that $S$ is a band $J$ of subtrioids $S_{\xi}, \xi \in J$. If $J$ is a commutative band, then we say that $S$ is a semilattice $J$ of subtrioids $S_{\xi}, \xi \in J$.

Observe that the notion of a triband of subtrioids generalizes the notion of a diband of subdimonoids [5] and the notion of a band of semigroups [11].

Examples of trioids which are decomposed into a triband of subtrioids can be found in [7].

## 3 Main results

In this section we describe all semilattice congruences on an arbitrary trioid and define the least semilattice congruence on this trioid. We also show that every trioid is a semilattice of $s$-simple subtrioids.

Let $(T, \dashv, \vdash, \perp)$ be an arbitrary dimonoid. Yamada introduced the notion of a $P$-subsemigroup of an arbitrary semigroup (see [9]). We denote by $\Omega$ the collection of all $P$-subsemigroups of $(T, \dashv)$ and by $T_{\alpha}, T_{\beta}, \ldots$ the elements of $\Omega$.

If $\rho$ is a congruence on a trioid $(T, \dashv, \vdash, \perp)$ such that the operations of $(T, \dashv, \vdash, \perp) / \rho$ coincide and it is a semilattice, then we say that $\rho$ is a semilattice congruence.

For every subset $\Gamma$ of $\Omega$ define a relation $\Gamma_{\dashv}$ on $(T, \dashv, \vdash, \perp)$ by

$$
\begin{gathered}
a \Gamma_{\dashv} b \text { if and only if } \\
\left\{(x, y) \mid x \dashv a \dashv y \in T_{\alpha}\right\}=\left\{(x, y) \mid x \dashv b \dashv y \in T_{\alpha}\right\}
\end{gathered}
$$

for every $T_{\alpha} \in \Gamma$.
Theorem 1. The relation $\Gamma_{\dashv}$ on any trioid $(T, \dashv, \vdash, \perp)$ is a semilattice congruence. Conversely, any semilattice congruence on $(T, \dashv, \vdash, \perp)$ can be obtained by this way.

Proof. The fact that the relation $\Gamma_{\dashv}$ is a semilattice congruence on a dimonoid $(T, \dashv, \vdash)$ has been proved in [10]. Show that $\Gamma_{\dashv}$ is compatible concerning the operation $\perp$.

Let $a \Gamma_{\dashv} b, a, b, c \in T$. As $a \dashv c \Gamma \dashv b \dashv c$, then

$$
\left\{(x, y) \mid x \dashv(a \dashv c) \dashv y \in T_{\alpha}\right\}=\left\{(x, y) \mid x \dashv(b \dashv c) \dashv y \in T_{\alpha}\right\}
$$

for every $T_{\alpha} \in \Gamma$. By the associativity of the operation $\dashv$ and the axiom (T4) of a trioid we obtain

$$
\begin{aligned}
& x \dashv(a \dashv c) \dashv y=((x \dashv a) \dashv c) \dashv y= \\
& =(x \dashv(a \perp c)) \dashv y=x \dashv(a \perp c) \dashv y, \\
& x \dashv(b \dashv c) \dashv y=((x \dashv b) \dashv c) \dashv y= \\
& =(x \dashv(b \perp c)) \dashv y=x \dashv(b \perp c) \dashv y .
\end{aligned}
$$

So, $a \perp c \Gamma_{\dashv} b \perp c$. Analogously, we can prove that $c \perp a \Gamma_{\dashv} c \perp b$. Thus, $\Gamma_{\dashv}$ is a congruence on $(T, \dashv, \vdash, \perp)$.

As $(T, \dashv) / \Gamma_{\dashv}$ is a semilattice, then by Lemma 1 the operations of $(T, \dashv, \vdash, \perp) / \Gamma_{\dashv}$ coincide and so, it is a semilattice.

The converse statement follows from [9] (see also [10]).
Theorem 1 generalizes Yamada's theorem [9] about the structure of all semilattice congruences on an arbitrary semigroup and the description [10] of all semilattice congruences on an arbitrary dimonoid.

A trioid $(T, \dashv, \vdash, \perp)$ will be called $s$-simple if its least semilattice congruence coincides with the universal relation on $T$.

Theorem 2. The relation $\Omega_{\dashv}$ on any trioid $(T, \dashv, \vdash, \perp)$ is the least semilattice congruence. Every trioid $(T, \dashv, \vdash, \perp)$ is a semilattice of $s$-simple subtrioids.

Proof. By Theorem $1 \Omega_{\dashv}$ is a semilattice congruence. If $a \Omega_{\dashv} b, a, b \in T$, then it is easy to see that $a \Gamma_{\dashv} b$ for any $\Gamma \subseteq \Omega$. So, $\Omega_{\dashv} \subseteq \Gamma_{\dashv}$.

Now we shall prove the second statement of the theorem.
Since $\Omega_{\dashv}$ is a congruence on $(T, \dashv, \vdash, \perp)$ and $(T, \dashv, \vdash, \perp) / \Omega_{\dashv}$ is a semilattice, then

$$
(T, \dashv, \vdash, \perp) \rightarrow(T, \dashv, \vdash, \perp) / \Omega_{\dashv}: x \mapsto[x]
$$

is a homomorphism ( $[x]$ is a class of the congruence $\Omega_{\dashv}$ which contains $x$ ). From [10] it follows that every class $A$ of the congruence $\Omega_{\dashv}$ is an $s$-simple dimonoid concerning operations $\dashv$ and $\vdash$. Hence we obtain $s$-simplicity of the subtrioid $A$ of a trioid $(T, \dashv, \vdash, \perp)$.

Theorem 2 generalizes Yamada's theorem [9] about the structure of the least semilattice congruence on an arbitrary semigroup and the description [10] of the least semilattice congruence on an arbitrary dimonoid.

## References

[1] Loday J.-L., Ronco M. O. Trialgebras and families of polytopes, Contemp. Math., 2004, 346, 369-398.
[2] Novelli J.-C., Thibon J. Y. Construction of dendriform trialgebras, C. R., Math., Acad. Sci. Paris 342, 2006, 6, 365-369.
[3] Casas J. M. Trialgebras and Leibniz 3-algebras, Boleten dela Sociedad Matematica Mexicana, 2006, 12, No. 2, 165-178.
[4] Loday J.-L. Dialgebras. Dialgebras and related operads, Lect. Notes Math., Springer-Verlag, Berlin 1763, 2001, 7-66.
[5] Zhuснок A. V. Dimonoids, Algebra and Logic, 2011, 50, No. 4, 323-340.
[6] Zhuchoк A. V. Free trioids, Bulletin of University of Kyiv, Ser. Physics and Mathematics, 2010, 4, 23-26 (In Ukrainian).
[7] Zhuchoк A. V. Tribands of subtrioids, Proc. Inst. Applied Math. and Mech., 2010, 21, 98-106.
[8] Zhuchoк A. V. Some congruences on trioids, Journal of Mathematical Sciences, 2012, 187, No. 2, 138-145.
[9] Yamada M. On the greatest semilattice decomposition of a semigroup, Kodai Math. Sem. Rep., 1955, 7, 59-62.
[10] Zhuchoк A. V. The least semilattice congruence on a dimonoid, Bulletin of University of Kyiv, Ser. Physics and Mathematics, 2009, 3, 22-24 (In Ukrainian).
[11] Clifford A. H. Bands of semigroups, Proc. Amer. Math. Soc., 1954, 5, 499-504.

E-mail: zhuchok_a@mail.ru


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