BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(75), 2014, Pages 3–8
ISSN 1024–7696

On a class of weighted composition operators on Fock space

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Abstract. Let T_ϕ be the Toeplitz operator defined on the Fock space $L^2_a(\mathbb{C})$ with symbol $\phi \in L^\infty(\mathbb{C})$. Let for $\lambda \in \mathbb{C}$, $k_\lambda(z) = e^{\frac{\bar{\lambda}z}{2} - \frac{|\lambda|^2}{4}}$, the normalized reproducing kernel at λ for the Fock space $L^2_a(\mathbb{C})$ and $t_\alpha(z) = z - \alpha, z, \alpha \in \mathbb{C}$. Define the weighted composition operator W_α on $L^2_a(\mathbb{C})$ as $(W_\alpha f)(z) = k_\alpha(z)(f \circ t_\alpha)(z)$. In this paper we have shown that if M and H are two bounded linear operators from $L^2_a(\mathbb{C})$ into itself such that $MT_\psi H = T_{\psi \circ t_\alpha}$ for all $\psi \in L^\infty(\mathbb{C})$, then M and H must be constant multiples of the weighted composition operator W_α and its adjoint respectively.

Mathematics subject classification: 47B35, 32M15.
Keywords and phrases: Fock space; Toeplitz operators; weighted composition operators.

1 Introduction

For $x, y \in \mathbb{C}^N$(for some integer $N \geq 1$), we write $\bar{x}y = \sum_{n=1}^N \bar{x}_n y_n$ and $|x| = (\bar{x}x)^{\frac{1}{2}}$. Thus, $|x-y|$ is the usual Euclidean distance between x and y. The symbol dz denotes the Lebesgue measure in $\mathbb{C}^N$ for all $N \geq 1$. The Gaussian measure on $\mathbb{C}^N$ is, by definition, $d\mu(z) = (2\pi)^{-N} e^{-\frac{|z|^2}{2}} dz$. Denote $L^p(\mathbb{C}^N, d\mu)$ the usual Lebesgue spaces on $\mathbb{C}^N$ with respect to the measure μ; $L^\infty(\mathbb{C}^N, d\mu)$ shall be occasionally abbreviated to $L^\infty(\mathbb{C}^N) = L^\infty(\mathbb{C}^N, dz)$, since they happen to coincide [5]. Set, for $1 \leq p \leq \infty$,

$$L^p_a(\mathbb{C}^N) = \{f \in L^p(\mathbb{C}^N, d\mu) : f \text{ is an entire function on } \mathbb{C}^N\}.$$

The space $L^p_a(\mathbb{C}^N)$ is a closed subspace of $L^p(\mathbb{C}^N, d\mu)$, $L^\infty_a(\mathbb{C}^N) = H^\infty(\mathbb{C}^N)$. For $p = 2$, $L^2_a(\mathbb{C}^N)$ is a Hilbert space, called the Fock or Siegal-Bargmann space.

For a multiindex $n = (n_1, n_2, \cdots, n_N) \in \mathbb{N}^N$, the following abbreviations will be employed:

$$a_n = a_{n_1, n_2, \cdots, n_N},$$

$$z^n = z_1^{n_1} z_2^{n_2} \cdots z_N^{n_N} (\text{ for } z \in \mathbb{C}^N),$$

$$n! = n_1! n_2! \cdots n_N!,$$

$$2^n = 2^{n_1+n_2+\cdots+n_N}.$$

If f is an entire function, $f(z)=\sum_{n\in\mathbb{N}^N} f_n z^n$, then

$$\int_{\mathbb{C}^N} |f(z)|^2 d\mu(z) = \sum_{n\in\mathbb{N}^N} n!2^n|f_n|^2.$$

Consequently, $f\in L^2_a(\mathbb{C}^N)$ if and only if the last expression is finite. The inner product of f and $g(z)=\sum_{n\in\mathbb{N}^N} g_n z^n$, $f,g\in L^2_a(\mathbb{C}^N)$, is given by

$$\langle f,g\rangle = \sum_{n\in\mathbb{N}^N} n!2^n f_n\bar{g}_n.$$

The set $\{(n!2^n)^{-\frac{1}{2}}z^n\}_{n\in\mathbb{N}^N}$ is an orthonormal basis of $L^2_a(\mathbb{C}^N)$. The polynomials are dense in $L^2_a(\mathbb{C}^N)$. The space $L^2_a(\mathbb{C}^N)$ is a reproducing kernel space; the reproducing kernel at $\lambda\in\mathbb{C}^N$ is given by $g_\lambda(z)=e^{\frac{\bar{\lambda}z}{2}}$, and $\|g_\lambda\|_2=e^{\frac{|\lambda|^2}{4}}$. For $\phi\in L^\infty(\mathbb{C}^N,d\mu)=L^\infty(\mathbb{C}^N)$, the Toeplitz operator T_ϕ is defined from $L^2_a(\mathbb{C}^N)$ into itself as $T_\phi f=P(\phi f)$ where P is the orthogonal projection from $L^2(\mathbb{C}^N,d\mu)$ onto $L^2_a(\mathbb{C}^N)$. Further, for $\phi\in L^\infty(\mathbb{C}^N)$, define the Hankel operator H_ϕ from $L^2_a(\mathbb{C}^N)$ into $(L^2_a(\mathbb{C}^N))^\perp$ by $H_\phi f=(I-P)(\phi f)$. Here $(L^2_a(\mathbb{C}^N))^\perp$ denotes the orthogonal complement of $L^2_a(\mathbb{C}^N)$. Define for $\lambda\in\mathbb{C}^N$, $k_\lambda(z)=\frac{g_\lambda(z)}{\|g_\lambda\|}=e^{\frac{\bar{\lambda}z}{2}-\frac{|\lambda|^2}{4}}$, the normalized reproducing kernel at λ for the Fock space $L^2_a(\mathbb{C}^N)$. In this paper we shall only concentrate our attention on the Fock space $L^2_a(\mathbb{C})$. Notice that it has an orthonormal basis $\{e_n\}_{n=0}^\infty$ where

$$e_n(z)=(n!2^n)^{-\frac{1}{2}}z^n.$$

For $\alpha\in\mathbb{C}$, define W_α from $L^2_a(\mathbb{C})$ into itself by $(W_\alpha f)(z)=k_\alpha(z)f(z-\alpha)$. Note for $f\in L^2_a(\mathbb{C}), W^*_\alpha f=(f\circ t_{-\alpha})k_{-\alpha}=W_{-\alpha}f$ and therefore the operator W_α is a unitary operator on $L^2_a(\mathbb{C})$ for each $\alpha\in\mathbb{C}$ and the operator can be defined on $L^2(\mathbb{C})$.

2 The forward shift operator and Toeplitz algebra on Fock space

Let Z be the forward shift operator with respect to the basis $\{e_n\}_{n=0}^\infty$, and let $\Phi(z)=\frac{z}{|z|}=e^{i\arg z}$. Let $\mathcal{L}(L^2_a(\mathbb{C}))$ be the space of all bounded linear operators from $L^2_a(\mathbb{C})$ into itself and $\mathcal{LC}(L^2_a(\mathbb{C}))$ be the space of all compact operators in $\mathcal{L}(L^2_a(\mathbb{C}))$. For $M,T\in\mathcal{L}(L^2_a(\mathbb{C}))$, let $[M,T]=MT-TM$. Let

$$\mathcal{A}(T_\Phi)=\{T\in\mathcal{L}(L^2_a(\mathbb{C})):[T,T_\Phi]\in\mathcal{LC}(L^2_a(\mathbb{C}))\}$$

and

$$\mathcal{A}(Z)=\{T\in\mathcal{L}(L^2_a(\mathbb{C})):[T,Z]\in\mathcal{LC}(L^2_a(\mathbb{C}))\}.$$

Lemma 2.1. *The following hold.*

(i) *The operator T_Φ is a compact perturbation of Z and $\mathcal{A}(T_\Phi)=\mathcal{A}(Z)$.*

(ii) *The Toeplitz operator $T_\Psi\in\mathcal{A}(T_\Phi)$ for every $\Psi\in L^\infty(\mathbb{C})$.*

Proof. (i) Notice that

$$\begin{aligned} \langle T_\Phi z^n, z^m\rangle &= \int_{\mathbb{C}} \frac{z}{|z|} z^n \bar{z}^m d\mu(z) \\ &= \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} r^{n+m} e^{i(n-m+1)t} e^{-\frac{r^2}{2}} r dt dr. \end{aligned}$$

This is zero unless $m = n+1$, and in that case it equals

$$\int_0^\infty r^{2n+1} e^{-\frac{r^2}{2}} r dr = \int_0^\infty 2^{n+\frac{1}{2}} t^{n+\frac{1}{2}} e^{-t} dt = 2^{n+\frac{1}{2}} \Gamma\left(n+\frac{3}{2}\right),$$

where Γ is Euler's gamma function. Thus

$$\langle T_\Phi e_n, e_m\rangle = \begin{cases} 0 & \text{if} \quad m \neq n+1; \\ (n!2^n)^{-\frac{1}{2}} (m!2^m)^{-\frac{1}{2}} 2^{n+\frac{1}{2}} \Gamma(n+\frac{3}{2}) & \text{if} \quad m = n+1. \end{cases}$$

Consequently, $T_\Phi e_n = c_n e_{n+1}$, where $c_n = \frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)^{\frac{1}{2}}\Gamma(n+2)^{\frac{1}{2}}}$. Let $\operatorname{diag}(1-c_n)$ be the diagonal matrix whose nth diagonal entry is $1-c_n$. Now it follows that $Z - T_\Phi = Z \cdot \operatorname{diag}(1-c_n)$, and in order to verify our claim it suffices to show that $c_n \to 1$ as $n \to +\infty$. According to Stirling's formula [1],

$$\Gamma(x+1) \sim \sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x},$$

where "$\sim$" means that the ratio of the right-hand to the left-hand side approaches 1 as $x \to +\infty$. Substituting this into the expression for c_n produces

$$c_n \sim \frac{\left(n+\frac{1}{2}\right)^{n+1} e^{-n-\frac{1}{2}} \sqrt{2\pi}}{n^{\frac{n}{2}+\frac{1}{4}} e^{-\frac{n}{2}} (2\pi)^{\frac{1}{4}} (n+1)^{\frac{n}{2}+\frac{3}{4}} e^{-\frac{n}{2}-\frac{1}{2}} (2\pi)^{\frac{1}{4}}}.$$

The terms containing π cancel, as well as those containing e, and what remains is the product of

$$\left(\frac{n+\frac{1}{2}}{n}\right)^{\frac{n}{2}}, \left(\frac{n+\frac{1}{2}}{n+1}\right)^{\frac{n+1}{2}} \quad \text{and} \quad \frac{\left(n+\frac{1}{2}\right)^{\frac{1}{2}}}{n^{\frac{1}{4}}(n+1)^{\frac{1}{4}}},$$

which tend to $e^{\frac{1}{4}}, e^{-\frac{1}{4}}$ and 1, respectively. So, $c_n \to 1$ and the assertion (i) follows.

Now we shall prove (ii). The formulas

$$T_{\psi\theta} - T_\psi T_\theta = H^*_{\overline{\psi}} H_\theta, \tag{1}$$

$$T_\psi T_\theta - T_\theta T_\psi = H^*_{\overline{\theta}} H_\psi - H^*_{\overline{\psi}} H_\theta, \tag{2}$$

hold for arbitrary $\psi, \theta \in L^\infty(\mathbb{C})$. Owing to (2),

$$T_\psi T_\Phi - T_\Phi T_\psi = H^*_{\overline{\Phi}} H_\psi - H^*_{\overline{\psi}} H_\Phi$$

will be compact for arbitrary $\psi \in L^\infty(\mathbb{C})$ if $H_\Phi, H_{\overline{\Phi}}$ are compact. The latter is equivalent to $H_\Phi^* H_\Phi, H_{\overline{\Phi}}^* H_{\overline{\Phi}}$ are compact, respectively, and from (1) it follows that this is equivalent to $I - T_\Phi^* T_\Phi$ and $I - T_\Phi T_\Phi^*$ are compact, respectively. Owing to (i), the last two operators are compact perturbations of $I - Z^*Z = 0$ and $I - ZZ^* = \langle ., e_0\rangle e_0$, respectively and the result follows. $\square$

Let $\mathbb{T}$ denote the unit circle in the complex plane $\mathbb{C}$. Let $L^\infty(\mathbb{T})$ be the space of all essentially bounded measurable functions on $\mathbb{T}$ with the essential supremum norm. Let H^2 be the Hardy space on the unit circle $\mathbb{T}$. For $\phi \in L^\infty(\mathbb{T})$, the Toeplitz operator B_ϕ with symbol ϕ is the operator on H^2 sending $f \in H^2$ to $P_+(\phi f)$, where P_+ is the orthogonal projection of $L^2(\mathbb{T})$ onto H^2. It is easy to check that $B_z^* B_\phi B_z = B_\phi$ for any $\phi \in L^\infty(\mathbb{T})$. According to a classical result [3], the converse holds: if an operator $T \in \mathcal{L}(H^2)$ satisfies $B_z^* T B_z = T$, then $T = B_\phi$ for some $\phi \in L^\infty(\mathbb{T})$. This result serves as a starting point for the theory of symbols of operators. It is also shown in [3], that the only compact Toeplitz operator is the zero Toeplitz operator. If $\phi \in H^\infty(\mathbb{T})$ then $B_\phi \in \mathcal{L}(H^2)$ is called an analytic Toeplitz operator and $B_\phi^* = B_{\bar{\phi}}$ is called a coanalytic Toeplitz operator. Let

$$\begin{aligned} \mathcal{A}(B_z) &= \{T \in \mathcal{L}(H^2) : T - B_z^* T B_z \in \mathcal{LC}(H^2)\} \\ &= \{T \in \mathcal{L}(H^2) : [T, B_z] \in \mathcal{LC}(H^2)\}, \end{aligned}$$

the essential commutant of the forward shift operator B_z on H^2. It is known [2] that $\mathcal{A}(B_z)$ is a C^*-subalgebra of $\mathcal{L}(H^2)$ and $B_\phi \in \mathcal{A}(B_z)$ for all $\phi \in L^\infty(\mathbb{T})$.

Lemma 2.2. *There exists a unitary operator $U : H^2 \to L_a^2(\mathbb{C})$ such that the transformation $T \mapsto U^*TU$ is a C^*-isomorphism of $\mathcal{A}(Z)$ onto $\mathcal{A}(B_z)$.*

Proof. Define $U : H^2 \to L_a^2(\mathbb{C})$ by mapping the standard basis of H^2 onto the basis $\{e_n\}_{n\in\mathbb{N}}$ of $L_a^2(\mathbb{C})$,

$$U : z^n \in H^2 \mapsto \frac{z^n}{\sqrt{n!2^n}} \in L_a^2(\mathbb{C}).$$

This operator is unitary and the transformation $T \to U^*TU$ maps Z to B_z; hence,

$$\begin{aligned} T \in \mathcal{A}(Z) &\Leftrightarrow [T, Z] \in \mathcal{LC}(L_a^2(\mathbb{C})) \\ &\Leftrightarrow U^*TZU - U^*ZTU \in \mathcal{LC}(H^2) \\ &\Leftrightarrow (U^*TU)(U^*ZU) - (U^*ZU)(U^*TU) \in \mathcal{LC}(H^2) \\ &\Leftrightarrow (U^*TU)B_z - B_z(U^*TU) \in \mathcal{LC}(H^2) \\ &\Leftrightarrow U^*TU \in \mathcal{A}(B_z). \end{aligned}$$

The proof is complete. $\square$

3 Main result

We now prove the main result of the work.

Theorem 3.1. *Let $\alpha \in \mathbb{C}$ and define the translation operator on $\mathbb{C}$ as $t_\alpha(z) = z - \alpha$. Suppose M and H are two linear bounded operators from $L_a^2(\mathbb{C})$ into itself such that*

$MT_\psi H = T_{\psi\circ t_\alpha}$ for all $\psi \in L^\infty(\mathbb{C}, dz)$. Then $M = cW_\alpha$ and $H = \frac{1}{c}W_\alpha^*$ and $MH = I$, the identity operator on $L_a^2(\mathbb{C})$.

Proof. Notice that the Fock space $L_a^2(\mathbb{C})$ is an invariant subspace for W_α and $W_\alpha^* = W_{-\alpha}$ and therefore $PW_\alpha = W_\alpha P$. For $f \in L_a^2(\mathbb{C})$ and $\alpha \in \mathbb{C}$, we have

$$\begin{aligned} T_\psi W_\alpha f &= T_\psi \left[(f \circ t_\alpha)\, k_\alpha\right] \\ &= P\left(\psi\,(f \circ t_\alpha)\, k_\alpha\right) \\ &= P\left((\psi \circ t_{-\alpha} \circ t_\alpha)\,(f \circ t_\alpha) k_\alpha\right) \\ &= P\left[(((\psi \circ t_{-\alpha}) f) \circ t_\alpha) k_\alpha\right] \\ &= PW_\alpha \left[(\psi \circ t_{-\alpha}) f\right] \\ &= W_\alpha P \left[(\psi \circ t_{-\alpha}) f\right] \\ &= W_\alpha T_{\psi\circ t_{-\alpha}} f. \end{aligned}$$

Thus we get $W_\alpha^* T_\psi W_\alpha f = T_{\psi\circ t_{-\alpha}} f$, for $\alpha \in \mathbb{C}$. Now let $R_\alpha = W_\alpha^* M$ and $S_\alpha = HW_\alpha$. Since $MT_\psi H = T_{\psi\circ t_\alpha}$ it follows that $R_\alpha T_\psi S_\alpha = W_\alpha^* M T_\psi H W_\alpha = W_\alpha^* T_{\psi\circ t_\alpha} W_\alpha = T_\psi$ for all $\psi \in L^\infty(\mathbb{C})$. It is known [4] that the norm closure of the set of all Toeplitz operators in $\mathcal{L}(L_a^2(\mathbb{C}))$ contains $\mathcal{LC}(L_a^2(\mathbb{C}))$. In fact, if $\mathcal{T}_1 = \{T_\phi : \phi \in \mathcal{D}(\mathbb{C})\}$ then $\mathrm{clos}\mathcal{T}_1 = \mathcal{LC}(L_a^2(\mathbb{C}))$ where $\mathcal{D}(\mathbb{C})$ is the set of all infinitely differentiable functions on $\mathbb{C}$ whose supports are compact subsets of $\mathbb{C}$. Thus

$$\begin{aligned} R_\alpha T_\psi S_\alpha T_\Phi &= T_\psi T_\Phi = T_{\psi\Phi} + G \quad (\text{for some } G \in \mathcal{LC}(L_a^2(\mathbb{C}))) \\ &= R_\alpha T_{\psi\Phi} S_\alpha + G \\ &= R_\alpha (T_\psi T_\Phi - G) S_\alpha + G \\ &= R_\alpha (T_\psi T_\Phi - \lim_{n\to\infty} T_{\phi_n}) S_\alpha + G \quad (\text{where } G = \lim_{n\to\infty} T_{\phi_n}) \\ &= R_\alpha T_\psi T_\Phi S_\alpha - \lim_{n\to\infty} R_\alpha T_{\phi_n} S_\alpha + G \\ &= R_\alpha T_\psi T_\Phi S_\alpha - \lim_{n\to\infty} T_{\phi_n} + G \\ &= R_\alpha T_\psi T_\Phi S_\alpha - G + G \\ &= R_\alpha T_\psi T_\Phi S_\alpha. \end{aligned}$$

It follows therefore that $R_\alpha T_\psi (S_\alpha T_\Phi - T_\Phi S_\alpha) = 0$. We shall now show that $S_\alpha T_\Phi - T_\Phi S_\alpha = 0$. Suppose on the contrary that there is some $x \neq 0$ in $\mathrm{Ran}(S_\alpha T_\Phi - T_\Phi S_\alpha)$. Then, by the last relation, $R_\alpha T_\psi x = 0$ for all $\psi \in L^\infty(\mathbb{C})$, so the kernel of R_α contains the set $\{T_\psi x : \psi \in L^\infty(\mathbb{C})\}$. Consider some $y \in L_a^2(\mathbb{C})$ orthogonal to this set. Then $0 = \langle y, T_\psi x\rangle = \langle y, P(\psi x)\rangle = \int_{\mathbb{C}} y(z)\overline{\psi(z) x(z)}\, d\mu(z)$ for all $\psi \in L^\infty(\mathbb{C})$; because $\bar{x}y \in L^1(\mathbb{C}, d\mu)$, we conclude that $\bar{x}y = 0$, and this is only possible if at least one of the analytic functions x, y is identically zero. But $x \neq 0$ by assumption, so y must be zero, which means that our set is dense in $L_a^2(\mathbb{C})$. Because this set is contained in $\ker R_\alpha$, we have $R_\alpha = 0$, so $T_\psi = R_\alpha T_\psi S_\alpha = 0$ for all ψ – a contradiction. This proves that $S_\alpha T_\Phi - T_\Phi S_\alpha = 0$. Hence $S_\alpha T_\Phi^n = T_\Phi^n S_\alpha$ for all $n \in \mathbb{N}$. Therefore $S_\alpha (Z + \widetilde{K})^n = (Z + \widetilde{K})^n S_\alpha$ as $T_\Phi = Z + \widetilde{K}$ for some $\widetilde{K} \in \mathcal{LC}(L_a^2(\mathbb{C}))$. Hence, it follows that $S_\alpha Z^n - Z^n S_\alpha = K_n$ for some $K_n \in \mathcal{LC}(L_a^2(\mathbb{C}))$. Thus

$$(U^* S_\alpha U)(U^* Z^n U) - (U^* Z^n U)(U^* S_\alpha U) = C_n$$

for some $C_n \in \mathcal{LC}(H^2)$ for all $n \in \mathbb{N}$. Hence $U^*S_\alpha U$ lies in the essential commutant of all analytic Toeplitz operators in $\mathcal{L}(H^2)$. Thus $U^*S_\alpha U = B_\phi + K$ for some $\phi \in H^\infty(\mathbb{T})$ and $K \in \mathcal{LC}(H^2)$.

Similarly one can show that $U^*R_\alpha U = B_{\bar\theta} + K'$, for some $\theta \in H^\infty(\mathbb{T})$ and $K' \in \mathcal{LC}(H^2)$. This is because $R_\alpha T_\psi S_\alpha = T_\psi$ for all $\psi \in L^\infty(\mathbb{C})$ implies $S_\alpha^* T_\psi R_\alpha^* = T_\psi$ for all $\psi \in L^\infty(\mathbb{C})$. Now $(U^*R_\alpha U)(U^*S_\alpha U) = B_{\bar\theta\phi} + C$, for some $C \in \mathcal{LC}(H^2)$. Hence $I = (U^*R_\alpha S_\alpha U) = B_{\bar\theta\phi} + C$ and therefore $B_{1-\bar\theta\phi} = C$. This implies $1 - \bar\theta\phi = 0$ as the only compact Toeplitz operator in $\mathcal{L}(H^2)$ is the zero Toeplitz operator. Thus $C = 0$ and $\bar\theta = \frac{1}{\phi}$. This implies $\theta \in H^\infty(\mathbb{T})$ and $\bar\theta \in H^\infty(\mathbb{T})$. Thus $\bar\theta = d$ and $\phi = \frac{1}{d}$ for some constant d. Hence it follows that $U^*R_\alpha U = B_d + K' = dI + K'$ and $U^*S_\alpha U = B_{\frac{1}{d}} + K = \frac{1}{d}I + K$. Thus $I = (dI + K')(\frac{1}{d}I + K)$ and therefore

$$dK + \frac{K'}{d} + K'K = 0. \tag{3}$$

On the other hand, $U^*S_\alpha U = \frac{1}{d}I + K$ implies $S_\alpha = \frac{1}{d} + UKU^* = \frac{1}{d} + E$ where $E = UKU^* \in \mathcal{LC}(L_a^2(\mathbb{C}))$. Hence

$$Z^{*n} S_\alpha Z^n \to \frac{1}{d} \tag{4}$$

as $Z^{*n}EZ^n \to 0$ (see [2] for the proof) strongly. Further, since $S_\alpha Z^n - Z^n S_\alpha = K_n$ for some $K_n \in \mathcal{LC}(L_a^2(\mathbb{C}))$, hence

$$Z^{*n} S_\alpha Z^n - S_\alpha = J_n \tag{5}$$

for some $J_n = Z^{*n}K_n \in \mathcal{LC}(L_a^2(\mathbb{C}))$. Since $\{J_n\}$ converges strongly to 0, we obtain from (4) and (5) that $S_\alpha = \frac{1}{d}$. Hence $E = 0$ and therefore $K = 0$. It follows hence from (3) that $K' = 0$. Thus $U^*S_\alpha U = \frac{1}{d}$ and $U^*R_\alpha U = d$. Hence $S_\alpha = \frac{1}{d}$ and $R_\alpha = d$. Thus $M = W_\alpha R_\alpha = dW_\alpha$ and $H = S_\alpha W_\alpha^* = \frac{1}{d}W_\alpha^*$ and the theorem follows. □

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Received November 1, 2011

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(75), 2014, Pages 9–17
ISSN 1024–7696

Invariant Characteristics of Special Compositions in Weyl Spaces W_N

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Abstract. In the present paper invariant characteristics of geodesic, chebyshevian and quasi-chebyshevian compositions $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p}$ in Weyl spaces $W_N(n_1 + n_2 + \cdots + n_p = N)$ are found with the help of the prolonged covariant differentiation. The characteristics of the spaces W_N which contain such special compositions are found.

Mathematics subject classification: 53A40, 53A55.
Keywords and phrases: Weyl space, net, composition, prolonged covariant differentiation; chebyshevian, geodesic, quasi-shebyshevian compositions.

1 Preliminary

1. A prolonged covariant differentiation in W_N.

Let $W_N(g_{\alpha\beta}, T_\sigma)$ be Weyl space with a fundamental tensor $g_{\alpha\beta}$ and a complementary covector T_σ. Let us accept that the fundamental tensor $g_{\alpha\beta}$ is normed by the law (see [1], p.152)

$$\breve{g}_{\alpha\beta} = \lambda^2 g_{\alpha\beta} \ , \tag{1}$$

where λ is a function of the point. It is known (see [1], p.153) that after renormalization (1): the complementary covector T_σ transforms by the law $\breve{T}_\sigma = T_\sigma + \partial_\sigma ln\lambda$, which means T_σ is a normalizer; the reciprocal tensor $g^{\alpha\beta}$ to $g_{\alpha\beta}$ transforms by the law $g^{\alpha\beta} = \lambda^{-2} g^{\alpha\beta}$. The coefficients of the connectedness $\Gamma^\sigma_{\alpha\beta}$ of the Weyl space W_N have the presentation $\Gamma^\sigma_{\alpha\beta} = \frac{1}{2} g^{\sigma\nu}(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}) - (T_\alpha \delta^\sigma_\beta + T_\beta \delta^\sigma_\alpha - T_\nu g^{\nu\sigma} g_{\alpha\beta})$ (see [1], p.154).

Let N independent fields of directions $\underset{\sigma}{v}^\alpha$ $(\sigma, \alpha = 1, 2, \ldots, N)$ be given in W_N. Renorm the fields of directions $\underset{\sigma}{v}^\alpha$ by the condition [8]

$$g_{\alpha\beta} \underset{\sigma}{v}^\alpha \underset{\sigma}{v}^\beta = 1. \tag{2}$$

The reciprocal covectors $\overset{\sigma}{v}_\alpha$ are defined by the following equalities

$$\underset{\sigma}{v}^\alpha \ \overset{\sigma}{v}_\beta = \delta^\alpha_\beta \Longleftrightarrow \underset{\beta}{v}^\sigma \ \overset{\alpha}{v}_\sigma = \delta^\alpha_\beta. \tag{3}$$

The renormalization of the fundumental tensor accompanies with the following renorming $\underset{\sigma}{\breve{v}}^\alpha = \lambda^{-1} \underset{\sigma}{v}^\alpha$, $\ \overset{\sigma}{\breve{v}}\alpha = \lambda \overset{\sigma}{v}_\alpha$.

According to (see [1], p.152) the fundamental tensor $g_{\alpha\beta}$ and the complementary covector T_σ satisfy the equalities

$$\nabla_\sigma\ g_{\alpha\beta} = 2T_\sigma\ g_{\alpha\beta}\ ,\ \nabla_\sigma\ g^{\alpha\beta} = -2T_\sigma\ g^{\alpha\beta} \tag{4}$$

According to [7] the pseudo-quantities $A \in W_N$ which after renormalization of the fundamental tensor $g_{\alpha\beta}$ by the formula (1) transform by the law $\breve{A} = \lambda^k A$ are called satellites of $g_{\alpha\beta}$ with a weight $\{k\}$. Hence $g^{\alpha\beta}\{-2\}$, $\underset{\sigma}{v}^{\alpha}\{-1\}$ $\overset{\sigma}{v}_{\alpha}\{1\}$.

The existence of the normalizer T_σ allows to introduce a prolonged covariant differentiation of the satellites A $\{k\}$ of the tensor $g_{\alpha\beta}$ by the formula $\overset{\circ}{\nabla}_\sigma\ A = \nabla_\sigma A - kT_\sigma A$ [8]. According to [8,9] we have.

$$\overset{\circ}{\nabla}_\sigma\ g_{\alpha\beta} = 0\ ,\ \overset{\circ}{\nabla}_\sigma\ g^{\alpha\beta} = 0\ ,\ \overset{\circ}{\nabla}_\sigma\ \underset{\alpha}{v}^{\beta} = \nabla_\sigma\ \underset{\alpha}{v}^{\beta} + T_\sigma \underset{\alpha}{v}^{\beta},\ \overset{\circ}{\nabla}_\sigma\ \overset{\alpha}{v}_{\beta} = \nabla_\sigma\ \overset{\alpha}{v}_{\beta} - T_\sigma \overset{\alpha}{v}_{\beta}. \tag{5}$$

Ozdeger obtained significant results in the understanding the geometry of Weyl and Einstein-Weyl manifolds [11], using the prolonged covariant differentiation, introduced in [8].

2. Compositions in W_N.

Consider in the space W_N the composition $X_m \times X_{N-m}$ of two base manifolds X_m and X_{N-m}, i.e. their topological product. Two positions $P(X_m)$ and $P(X_{N-m})$ of these base manifolds pass through any point of the space $W_N(X_m \times X_{N-m})$ [2]. According to [2] and [3] any composition is completely defined with the field of the affinor a_α^β, satisfying the condition

$$a_\alpha^\sigma a_\sigma^\beta = \delta_\alpha^\beta. \tag{6}$$

According to [4] the projecting affinors $\overset{m}{a}{}_\alpha^\beta$, $\overset{N-m}{a}{}_\alpha^\beta$ are defined by the equalities $\overset{m}{a}{}_\alpha^\beta = \frac{1}{2}(\delta_\alpha^\beta + a_\alpha^\beta)$, $\overset{N-m}{a}{}_\alpha^\beta = \frac{1}{2}(\delta_\alpha^\beta - a_\alpha^\beta)$. For an arbitrary vector v^α we have $v^\alpha = \overset{m}{a}{}_\sigma^\alpha\ v^\sigma + \overset{N-m}{a}{}_\sigma^\alpha\ v^\sigma = \underset{m}{V}^\alpha + \underset{N-m}{V}^\alpha$, where $\underset{m}{V}^\alpha = \overset{m}{a}{}_\sigma^\alpha\ v^\sigma \in P(X_m)$, $\underset{N-m}{V}^\alpha = \overset{N-m}{a}{}_\sigma^\alpha\ v^\sigma \in P(X_{N-m})$. The partial projections or the full ones of an arbitrary tensor are defined analogously.

3. Derivative equations in W_N.

For the independent fields of directions $\underset{\sigma}{v}^\alpha$ $(\sigma, \alpha = 1, 2, \ldots, N)$ and their reciprocal covectors $\overset{\sigma}{v}_\alpha$, defined by (3), are fulfilled the following derivative equations [8,9]

$$\overset{\circ}{\nabla}_\sigma\ \underset{\alpha}{v}^\beta = \underset{\alpha}{\overset{\nu}{T}}_\sigma \underset{\nu}{v}^\beta\ ,\ \ \overset{\circ}{\nabla}_\sigma\ \overset{\alpha}{v}_\beta = -\underset{\nu}{\overset{\alpha}{T}}_\sigma \overset{\nu}{v}_\beta\ , \tag{7}$$

where $\underset{\alpha}{\overset{\beta}{T}}_\beta$ $\{0\}$. We obtain, using the integrability condition of (7), the next equality $\nabla_{[\alpha} \underset{\sigma}{\overset{\sigma}{T}}_{\beta]} + \underset{\nu}{\overset{\sigma}{T}}_{[\beta} \underset{\sigma}{\overset{\nu}{T}}_{\alpha]}$ =0 [8]. Let us denote by $(\underset{\beta}{v})$ the lines, defined from the field of directions $\underset{\beta}{v}^{\alpha}$ and by $(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v})$ the net, defined from the independent fields of directions $\underset{\sigma}{v}^{\alpha}$, $(\sigma = 1,2,\dots,N)$. It is known that the field of directions $\underset{\sigma}{v}^{\alpha}$ is parallelly translated along the lines $(\underset{\beta}{v})$ if and only if $\nabla_{\nu}\, \underset{\sigma}{v}^{\alpha}\underset{\beta}{v}^{\nu} = \mu \underset{\sigma}{v}^{\alpha}$, where μ is an arbitrary function of the point. According to (5) the last equality can be written in the form

$$\overset{\circ}{\nabla}_{\nu}\, \underset{\sigma}{v}^{\alpha}\underset{\beta}{v}^{\nu} = \mu \underset{\sigma}{v}^{\alpha}. \tag{8}$$

2 Coordinate net in W_N

Let us chose the net $(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v})$ as a coordinate one. From (2) and $g_{\alpha\beta}\underset{\sigma}{v}^{\alpha}\underset{\nu}{v}^{\beta} = cos\underset{\sigma\nu}{\omega}$ it follows that in the parameters of the coordinate net

$$\begin{gathered} g_{\alpha\beta} = \underset{\alpha}{f}\underset{\beta}{f} cos\underset{\alpha\beta}{\omega}, \\ \underset{1}{v}^{\alpha}(\tfrac{1}{\underset{1}{f}},0,0,\dots,0),\quad \underset{2}{v}^{\alpha}(0,\tfrac{1}{\underset{2}{f}},0,\dots,0),\quad \dots,\quad \underset{N}{v}^{\alpha}(0,0,0,\dots,\tfrac{1}{\underset{N}{f}}), \\ \overset{1}{v}_{\alpha}(\underset{1}{f},0,0,\dots,0),\quad \overset{2}{v}_{\alpha}(0,\underset{2}{f},0,\dots,0),\quad \dots,\quad \overset{N}{v}_{\alpha}(0,0,0,\dots,\underset{N}{f}), \end{gathered} \tag{9}$$

where $\underset{\alpha}{f} = \underset{\alpha}{f}(\overset{\sigma}{u})$, $\underset{\alpha}{f}\{1\}$, $\underset{\alpha\beta}{\omega} = \underset{\alpha\beta}{\omega}(\overset{\sigma}{u})$, $\underset{\alpha\beta}{\omega}\{0\}$, $\sigma = 1,2,\dots,N$.

Lemma 1. *When the net* $(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v})$ *is chosen as a coordinate one then there exist the following relations between the coefficients* $\underset{\alpha}{\overset{\beta}{T}}_{\sigma}$ *from the derivative equations (7) and the coefficients of the connection* $\Gamma^{\sigma}_{\alpha\beta}$

$$\underset{\alpha}{\overset{\beta}{T}}_{\sigma} = \frac{\underset{\beta}{f}}{\underset{\alpha}{f}}\Gamma^{\beta}_{\sigma\alpha}\ ,\quad \alpha\neq\beta\ ;\quad \underset{\alpha}{\overset{\alpha}{T}}_{\sigma} = \Gamma^{\alpha}_{\sigma\alpha} - \partial_{\sigma} ln(\underset{1}{f}\underset{2}{f}\dots\underset{N}{f}) + NT_{\sigma}\ . \tag{10}$$

Proof. Using (3), (5) and (7) we obtain

$$\underset{\alpha}{\overset{\beta}{T}}_{\sigma} = \partial_{\sigma}\underset{\alpha}{v}^{\nu}\overset{\beta}{v}_{\nu} + \Gamma^{\tau}_{\sigma\nu}\underset{\alpha}{v}^{\nu}\overset{\beta}{v}_{\tau} + T_{\sigma}\delta^{\beta}_{\alpha}\ . \tag{11}$$

After applying (9) in (11) we establish the validity of (10). □

3 Weyl spaces of compositions $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p}$

Let us introduce the notations:

$$
\begin{aligned}
&\alpha, \beta, \gamma, \delta, \sigma, \nu, \tau = 1, 2, \ldots, N; \ i_1, j_1, k_1, s_1 = 1, 2, \ldots, n_1; \\
&\bar{i}_1, \bar{j}_1, \bar{k}_1, \bar{s}_1 = n_1 + 1, n_1 + 2, \ldots, N; \\
&i_2, j_2, k_2, s_2 = n_1 + 1, n_1 + 2, \ldots, n_1 + n_2; \\
&\bar{i}_2, \bar{j}_2, \bar{k}_2, \bar{s}_2 = 1, 2, \ldots, n_1, n_1 + n_2 + 1, n_1 + n_2 + 2, \ldots, N; \\
&i_3, j_3, k_3, s_3 = n_1 + n_2 + 1, n_1 + n_2 + 2, \ldots, n_1 + n_2 + n_3; \\
&\bar{i}_3, \bar{j}_3, \bar{k}_3, \bar{s}_3 = 1, 2, \ldots, n_1 + n_2 + n_3 + 1, n_1 + n_2 + n_3 + 2, \ldots, N; \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&i_p, j_p, k_p, s_p = n_1 + n_2 + \cdots + n_{p-1} + 1, \\
&n_1 + n_1 + n_2 + \cdots + n_{p-1} + 2, \ldots, N; \\
&\bar{i}_p, \bar{j}_p, \bar{k}_p, \bar{s}_p = 1, 2, \ldots, n_1 + n_2 + \cdots + n_{p-1}\ .
\end{aligned}
\tag{12}
$$

Following [10] we shall consider the affinors

$$\overset{n_m}{a}{}_{\alpha}^{\beta} = \underset{i_m}{v}{}^{\beta} \overset{i_m}{v}{}_{\alpha} - \underset{\bar{i}_m}{v}{}^{\beta} \overset{\bar{i}_m}{v}{}_{\alpha} \quad \textit{for any} \quad m = 1, 2, \ldots, p. \tag{13}$$

The affinors (13) have weight $\{0\}$. According to (3) the affinors (13) satisfy (6), i.e. they define the following compositions $X_{n_1} \times X_{N-n_1}, X_{n_2} \times X_{N-n_2}, \ldots, X_{n_p} \times X_{N-n_p}$. Let us consider the composition $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p}$ and let us denote the positions of the manifolds $X_{n_1}, X_{n_2}, \ldots, X_{n_p}$, by $P(X_{n_1})$, $P(X_{n_2})$, $\ldots$, $P(X_{n_p})$, respectively.

The affinors

$$\overset{m}{a}{}_{\alpha}^{\beta} = \underset{i_m}{v}{}^{\beta} \overset{i_m}{v}{}_{\alpha}\ , \quad m = 1, 2, \ldots, p, \tag{14}$$

with weight $\{0\}$ will be called the projective affinors of the composition $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p}$.

From (3) and (14) follow $\overset{1}{a}{}_{\alpha}^{\beta} + \overset{2}{a}{}_{\alpha}^{\beta} + \cdots + \overset{p}{a}{}_{\alpha}^{\beta} = \delta_{\alpha}^{\beta}$, $\overset{m}{a}{}_{\alpha}^{\beta} \overset{m}{a}{}_{\sigma}^{\alpha} = \overset{m}{a}{}_{\sigma}^{\beta}$, $\overset{m}{a}{}_{\alpha}^{\beta} \overset{l}{a}{}_{\sigma}^{\alpha} = 0$, where $m, l = 1, 2, \ldots, p$, $m \neq l$. If v^{β} is an arbitrary vector, then $v^{\beta} = \overset{1}{a}{}_{\alpha}^{\beta} v^{\alpha} + \overset{2}{a}{}_{\alpha}^{\beta} v^{\alpha} + \cdots + \overset{p}{a}{}_{\alpha}^{\beta} v^{\alpha} = \underset{1}{V}{}^{\beta} + \underset{2}{V}{}^{\beta} + \cdots + \underset{p}{V}{}^{\beta}$, where $\underset{1}{V}{}^{\beta} = \overset{1}{a}{}_{\alpha}^{\beta} v^{\alpha} \in P(X_{n_1})$, $\underset{2}{V}{}^{\beta} = \overset{2}{a}{}_{\alpha}^{\beta} v^{\alpha} \in P(X_{n_2})$, $\ldots$, $\underset{p}{V}{}^{\beta} = \overset{p}{a}{}_{\alpha}^{\beta} v^{\alpha} \in P(X_{n_p})$.

With the help of the projective affinors (14) the fundamental tensor $g_{\alpha\beta}$ can be presented in the form $g_{\alpha\beta} = \overset{1}{G}_{\alpha\beta} + \overset{2}{G}_{\alpha\beta} + \cdots + \overset{p}{G}_{\alpha\beta} + 2\ \overset{12}{G}_{\alpha\beta} + 2\ \overset{13}{G}_{\alpha\beta} + \cdots + 2\ \overset{p-1p}{G}_{\alpha\beta}$, where $\overset{m}{G}_{\alpha\beta} = \overset{m}{a}{}_{\alpha}^{\sigma} \overset{m}{a}{}_{\beta}^{\nu} g_{\sigma\nu}$, $\overset{ml}{G}_{\alpha\beta} = \overset{m}{a}{}_{(\alpha}^{\sigma} \overset{l}{a}{}_{\beta)}^{\nu} g_{\sigma\nu}$ and $m, l = 1, 2, \ldots, p$, $m \neq l$.

The tensors $\overset{m}{G}_{\alpha\beta}$ are full projections of the fundamental tensor $g_{\alpha\beta}$ on the positions $P(X_{n_m})$ and they define metrics on these positions. Following [5] the tensors $\overset{m}{G}_{\alpha\beta}$ will be called positional fundamental tensors. They satisfy the equalities $\overset{m}{a}{}^{\sigma}_{\alpha}\overset{m}{G}_{\sigma\beta} = \overset{m}{a}{}^{\sigma}_{\beta}\overset{m}{G}_{\alpha\sigma} = \overset{m}{G}_{\alpha\beta}$, $\overset{m}{a}{}^{\sigma}_{\alpha}\overset{l}{G}_{\sigma\beta} = \overset{m}{a}{}^{\sigma}_{\beta}\overset{l}{G}_{\alpha\sigma} = 0$, when $m \neq l$. Following [5] the tensors $\overset{ml}{G}_{\alpha\beta}$ will be called hybridian tensors. They satisfy the equalities $\overset{m}{a}{}^{\sigma}_{\alpha}\overset{l}{a}{}^{\nu}_{\beta}\overset{ml}{G}_{\sigma\nu} = \frac{1}{2}\overset{m}{a}{}^{\sigma}_{\alpha}\overset{l}{a}{}^{\nu}_{\beta}\, g_{\sigma\nu}$, $\overset{m}{a}{}^{\sigma}_{\alpha}\overset{m}{a}{}^{\nu}_{\beta}\overset{ml}{G}_{\sigma\nu} = 0$.

4 Special compositions $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p}$ in W_N

Definition 1. The composition $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p} \in W_N$ will be called geodesic if for any $m = 1, 2, \dots, p$ the position $P(X_{n_m})$ is parallelly translated along any line of the manifold X_{n_m}.

Theorem 1. *The composition $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p} \in W_N$ is geodesic if and only if the coefficients from the derivative equations (7) satisfy the equalities*

$$\underset{i_m}{\overset{\bar{k}_m}{T}}{}_{\sigma}\; \underset{s_m}{v}{}^{\sigma} = 0\ , \ \ for\ \ any\ \ m = 1, 2, \dots, p. \tag{15}$$

Proof. According to (8) the composition $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p}$ is geodesic if and only if $\overset{\circ}{\nabla}_{\sigma}\, \underset{i_m}{v}{}^{\alpha}\, \underset{s_m}{v}{}^{\sigma} = \mu\, \underset{i_m}{v}{}^{\alpha}$ for any $m = 1, 2, \dots, p$. From (7) and the last equality we obtain $\underset{i_m}{\overset{\nu}{T}}{}_{\sigma}\; \underset{\nu}{v}{}^{\alpha}\; \underset{s_m}{v}{}^{\sigma} = \mu\, \underset{i_m}{v}{}^{\alpha}$. Now after contraction by $\overset{\tau}{v}_{\alpha}$ we find $\underset{i_m}{\overset{\tau}{T}}{}_{\sigma}\; \underset{s_m}{v}{}^{\sigma} = \mu\ \delta^{\tau}_{i_m}$, from where (15) follows. □

From (9), (10) and Theorem 1 follows the validity of the following statement:

Corollary 1. *If the composition $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p} \in W_N$ is geodesic then:*

i) In the parameters of the coordinate net the coefficients of the derivative equations (7) satisfy the equalities $\underset{i_m}{\overset{\bar{k}_m}{T}}{}_{s_m} = 0$ for any $m = 1, 2, \dots, p$;

ii) In the parameters of the coordinate net the coefficients of the connection satisfy the equalities $\Gamma^{\bar{k}_m}_{s_m i_m} = 0$ for any $m = 1, 2, \dots, p$.

If the composition $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p} \in W_N$ is geodesic and the net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{N}{v})$ is chosen as a coordinate one, then using Corollary 1, for the components of the tensor of the curvature $R_{\alpha\beta\gamma}{}^{\delta}$ we obtain $R_{i_m j_m k_m}{}^{\bar{s}_m} = 0$ for any $m = 1, 2, \dots, p$.

Definition 2. The composition $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p} \in W_N$ will be called chebyshevian if for any $m, l = 1, 2, \dots, p$ and $m \neq l$, the position $P(X_{n_m})$ is parallelly translated along any line of the manifold X_{n_l}.

Theorem 2. *The composition $X_{n_1}\times X_{n_2}\times\cdots\times X_{n_p}\in W_N$ is chebyshevian if and only if the coefficients from the derivative equations (7) satisfy the equalities*

$$\underset{i_m}{\overset{\overline{k}_m}{T}}{}_{\sigma}\ \underset{s_l}{v}^{\sigma}=0\ ,\ \ for\ \ any\ \ m,l=1,2,\dots,p,m\neq l. \tag{16}$$

Proof. According to (8) the composition $X_{n_1}\times X_{n_2}\times\cdots\times X_{n_p}$ is chebyshevian if and only if $\overset{\circ}{\nabla}_{\sigma}\ \underset{i_m}{v}^{\alpha}\underset{s_l}{v}^{\sigma}=\mu\underset{i_m}{v}^{\alpha}$ for any $m=1,2,\dots,p$. From (7) and the last equality we obtain (16). ☐

From (9), (10) and Theorem 2 follows the validity of the following statement:

Corollary 2. *If the composition $X_{n_1}\times X_{n_2}\times\cdots\times X_{n_p}\in W_N$ is chebyshevian then:*

i) In the parameters of the coordinate net the coefficients of the derivative equations (7) satisfy the equalities $\underset{i_m}{\overset{\overline{k}_m}{T}}{}_{s_l}=0$ for any $m,l=1,2,\dots,p,\ \ m\neq l$;

ii) In the parameters of the coordinate net the coefficients of the connection satisfy the equalities $\Gamma^{\overline{k}_m}_{s_l\ i_m}=0$ for any $m,l=1,2,\dots,p,\ \ m\neq l$.

Theorem 3. *If the composition $X_{n_1}\times X_{n_2}\times\cdots\times X_{n_p}\in W_N$ is chebyshevian then the space W_N is Riemannian and the metric tensor has in the chosen coordinate system the presentation*

$$g_{i_l i_m}=\underset{i_l}{f}(\overset{i_l}{u})\ \underset{i_m}{f}(\overset{i_m}{u})\ cos\ \underset{i_l i_m}{\omega}(\overset{i_l}{u},\overset{i_m}{u})\ . \tag{17}$$

Proof. Let the composition $X_{n_1}\times X_{n_2}\times\cdots\times X_{n_p}\in W_N$ be chebyshevian. We chose the net $(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v})$ as a coordinate one. Then from (4) and Corollary 2 we obtain

$$\partial_{i_m}g_{i_l i_r}=2T_{i_m}g_{i_l i_r},\ \text{for any } m,l,r=1,2,\dots,p,\ m\neq l,\ m\neq r. \tag{18}$$

From (18) it follows $T_\sigma=grad$, i.e. W_n is Riemannian. Let us renormalize the fundumental tensor $g_{\alpha\beta}$ such that $T_\sigma=0$, (see [1], p.157). Then the equalities (18) accept the form $\partial_{i_m}g_{i_l i_r}=0$, from where (17) follows. ☐

Let now the composition $X_{n_1}\times X_{n_2}\times\cdots\times X_{n_p}\in W_N$ be chebyshevian and X_{n_m} are one-dimensional manifolds. Then the composition defines a chebyshevian net $(\underset{1}{v},\underset{2}{v},\dots,\underset{N}{v})$. According to Theorem 3 W_N is Riemannian. Using (17) and changing the variables, we obtain for the metric tensor of the Riemannian space $g_{\alpha\beta}=cos\ \underset{\alpha\beta}{\omega}(\overset{\alpha}{u},\overset{\beta}{u})$.

Let us consider an orthogonal composition $X_{n_1}\times X_{n_2}\times\cdots\times X_{n_p}\in W_N$, which means that at any point of the space any two directions $\underset{m}{V}^{\alpha}\in P(X_{n_m})$ and $\underset{l}{V}^{\alpha}\in P(X_{n_l})$, when $m,l=1,2,\dots,p,\ m\neq l$, are orthogonal. In this case $g_{\alpha\beta}\underset{m}{V}^{\alpha}\underset{l}{V}^{\beta}=0$. Since $\underset{m}{V}^{\alpha}=\overset{m}{a}{}^{\alpha}_{\sigma}\ v^{\sigma}$, $\underset{l}{V}^{\alpha}=\overset{l}{a}{}^{\alpha}_{\sigma}\ v^{\sigma}$, then $g_{\alpha\beta}\underset{m}{V}^{\alpha}\underset{l}{V}^{\beta}=0\Longleftrightarrow g_{\alpha\beta}\overset{m}{a}{}^{\alpha}_{\sigma}\ \overset{l}{a}{}^{\beta}_{\nu}\ v^{\sigma}u^{\nu}=$

$g_{\alpha\beta}\overset{l}{a}{}^{\alpha}_{\sigma}\overset{m}{a}{}^{\beta}_{\nu}v^{\sigma}u^{\nu}=0$. Because v^{α} and u^{α} are arbitrary vector fields, then $g_{\alpha\beta}\overset{m}{a}{}^{\alpha}_{\sigma}\overset{l}{a}{}^{\beta}_{\nu} = g_{\alpha\beta}\overset{l}{a}{}^{\alpha}_{\sigma}\overset{m}{a}{}^{\beta}_{\nu}=0$, from where it follows $\overset{ml}{G}_{\alpha\beta}=0$. Hence $g_{\alpha\beta}=\overset{1}{G}_{\alpha\beta}+\overset{2}{G}_{\alpha\beta}+\cdots+\overset{p}{G}_{\alpha\beta}$.

Theorem 4. *The orthogonal composition $X_{n_1}\times X_{n_2}\times\cdots\times X_{n_p}\in W_N$ is chebyshevian if and only if it is geodesic one.*

Proof. Let the composition $X_{n_1}\times X_{n_2}\times\cdots\times X_{n_p}\in W_N$ be orthogonal. Then from $\underset{i_m}{v}{}^{\alpha}\in P(X_{n_m})$, $\underset{i_k}{v}{}^{\alpha}\in P(X_{n_k})$ it follows $g_{\alpha\beta}\underset{i_m}{v}{}^{\alpha}\underset{i_k}{v}{}^{\beta}=0$ for any $m,k=1,2,\dots,p,\ m\neq k$. After prolonged covariant differentiation of the last equality and taking into account (5) and (7) we find $g_{\alpha\beta}\overset{j_k}{\underset{i_m}{T}}{}_{\sigma}\underset{j_k}{v}{}^{\alpha}\underset{i_k}{v}{}^{\beta}+g_{\alpha\beta}\overset{j_m}{\underset{i_k}{T}}{}_{\sigma}\underset{i_m}{v}{}^{\alpha}\underset{j_m}{v}{}^{\beta}=0$. Now after contraction by $\underset{s_k}{v}{}^{\sigma}$ we obtain

$$g_{\alpha\beta}\overset{j_k}{\underset{i_m}{T}}{}_{\sigma}\underset{s_k}{v}{}^{\sigma}\underset{j_k}{v}{}^{\alpha}\underset{i_k}{v}{}^{\beta}+g_{\alpha\beta}\overset{j_m}{\underset{i_k}{T}}{}_{\sigma}\underset{s_k}{v}{}^{\sigma}\underset{i_m}{v}{}^{\alpha}\underset{j_m}{v}{}^{\beta}=0. \tag{19}$$

From (19), Theorem 1 and Theorem 2 the validity of the Theorem 4 follows.

The compositions $X_m\times X_{N-m}$ for which the positions $P(X_m)$ and $P(X_{N-m})$ are quasi-parallelly translated along any line of the manifold X_{N-m} and X_m, respectively are studied in [2,5,6].

Let us consider the composition $X_{n_1}\times X_{n_2}\times\cdots\times X_{n_p}\in W_N$. According to [2,5,6] and (7) the positions $P(X_{n_m})$ will be quasi-parallelly translated along any line of the manifold X_{n_k} if and only if

$$\overset{\circ}{\nabla}_{\sigma}\underset{i_m}{v}{}^{\alpha}\underset{j_k}{v}{}^{\sigma}=\lambda_{i_m}\underset{j_k}{v}{}^{\alpha}+\overset{s_m}{\underset{i_m}{T}}{}_{\sigma}\underset{s_m}{v}{}^{\alpha}\underset{j_k}{v}{}^{\sigma},\quad m\neq k. \tag{20}$$

The vector λ_{i_m} has the weight $\{-1\}$. □

Definition 3. The composition $X_{n_1}\times X_{n_2}\times\cdots\times X_{n_p}\in W_N$ will be called quasi-chebyshevian if for any $m,k=1,2,\dots,p,\ m\neq k$, the positions $P(X_{n_m})$ are quasi-parallelly translated along any line of the manifold X_{n_k} .

Theorem 5. *The composition $X_{n_1}\times X_{n_2}\times\cdots\times X_{n_p}\in W_N$ is quasi-chebyshevian if and only if the coefficients from the derivative equations (7) satisfy the equalities*

$$\overset{\overline{s}_m}{\underset{i_m}{T}}{}_{\sigma}\underset{j_k}{v}{}^{\sigma}=\lambda_{i_m}\delta^{\overline{s}_m}_{j_k},\ \ for\ \ any\ \ m,k=1,2,\dots,p,\ m\neq k. \tag{21}$$

Proof. According to (7) and (20) the composition $X_{n_1}\times X_{n_2}\times\cdots\times X_{n_p}\in W_N$ will be quasi-chebyshevian if and only if $\overset{\overline{s}_m}{\underset{i_m}{T}}{}_{\sigma}\underset{\overline{s}_m}{v}{}^{\alpha}\underset{j_k}{v}{}^{\sigma}=\lambda_{i_m}\underset{j_k}{v}{}^{\alpha}$. The last equalities are equivalent to (21). □

From (9), (10) and Theorem 5 follows the validity of the following statement:

Corollary 3. *If the composition $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p} \in W_N$ is quasi-chebyshevian then:*

i) In the parameters of the coordinate net the coefficients of the derivative equations (7) satisfy the equalities $\frac{1}{\underset{j_k}{f}} \overset{\overline{s}_m}{\underset{i_m}{T}}{}_{j_k} = \lambda_{i_m} \delta^{\overline{s}_m}_{j_k}$, for any $m, k = 1, 2, \dots, p$, $m \neq k$.

ii) In the parameters of the coordinate net the coefficients of the connection satisfy the equalities $\Gamma^{\overline{s}_m}_{j_k\ i_m} = \psi_{i_m} \delta^{\overline{s}_m}_{j_k}$ for any $m, k = 1, 2, \dots, p$, $m \neq k$, where the vector $\psi_{i_m} = \frac{\lambda_{i_m}}{f_{i_m}}$ has the weight $\{0\}$.

Following [2] the vector ψ_{i_m} will be called a vector of the quasi-parallel translation. If for any $m, k = 1, 2, \dots, p$ $\psi_{i_m} = 0$, then according to Theorem 2 the composition $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p} \in W_N$ will be chebyshevian.

Theorem 6. *The composition $X_{n_1} \times X_{n_2} \times \cdots \times X_{n_p} \in W_N$ is geodesic or chebyshevian, or quasi-chebyshevian if and only if the projecting affinors (14) satisfy for any $m, k = 1, 2, \dots, p$, $m \neq k$ the equalities*

$$\begin{aligned} &\overset{m}{a}{}^{\sigma}_{\alpha} \overset{m}{a}{}^{\nu}_{\delta} \overset{\circ}{\nabla}_{\sigma} \overset{m}{a}{}^{\beta}_{\nu} = 0, \\ &\overset{k}{a}{}^{\sigma}_{\alpha} \overset{m}{a}{}^{\nu}_{\delta} \overset{\circ}{\nabla}_{\sigma} \overset{m}{a}{}^{\beta}_{\nu} = 0, \\ &\overset{k}{a}{}^{\sigma}_{\alpha} \overset{m}{a}{}^{\nu}_{\delta} \overset{\circ}{\nabla}_{\sigma} \overset{m}{a}{}^{\beta}_{\nu} - \psi_{\sigma} \overset{m}{a}{}^{\sigma}_{\delta} \overset{k}{a}{}^{\beta}_{\alpha} = 0, \end{aligned} \tag{22}$$

respectively.

Proof. Let the net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{N}{v})$ be chosen as a coordinate one. In the parameters of this coordinate net we have $\overset{m}{a}{}^{\beta}_{\alpha} = \delta^{i_m}_{s_m}$, $\overset{k}{a}{}^{\beta}_{\alpha} = \delta^{i_k}_{s_k}$. For the components of the tensors $\overset{m}{a}{}^{\sigma}_{\alpha} \overset{m}{a}{}^{\nu}_{\delta} \overset{\circ}{\nabla}_{\sigma} \overset{m}{a}{}^{\beta}_{\nu}$, $\overset{k}{a}{}^{\sigma}_{\alpha} \overset{m}{a}{}^{\nu}_{\delta} \overset{\circ}{\nabla}_{\sigma} \overset{m}{a}{}^{\beta}_{\nu}$, $\overset{k}{a}{}^{\sigma}_{\alpha} \overset{m}{a}{}^{\nu}_{\delta} \overset{\circ}{\nabla}_{\sigma} \overset{m}{a}{}^{\beta}_{\nu} - \psi_{\sigma} \overset{m}{a}{}^{\sigma}_{\delta} \overset{k}{a}{}^{\beta}_{\alpha}$, which are diffrent from zero, we find

$$\begin{aligned} &\overset{m}{a}{}^{\sigma}_{i_m} \overset{m}{a}{}^{\nu}_{j_m} \overset{\circ}{\nabla}_{\sigma} \overset{m}{a}{}^{\overline{s}_m}_{\nu} = \Gamma^{\overline{s}_m}_{i_m j_m}, \\ &\overset{k}{a}{}^{\sigma}_{i_m} \overset{m}{a}{}^{\nu}_{j_k} \overset{\circ}{\nabla}_{\sigma} \overset{m}{a}{}^{\overline{s}_m}_{\nu} = \Gamma^{\overline{s}_m}_{i_m j_k}, \\ &\overset{k}{a}{}^{\sigma}_{i_k} \overset{m}{a}{}^{\nu}_{j_m} \overset{\circ}{\nabla}_{\sigma} \overset{m}{a}{}^{\overline{s}_m}_{\nu} - \psi_{\sigma} \overset{m}{a}{}^{\sigma}_{j_m} \overset{k}{a}{}^{\overline{s}_m}_{l_k} = \psi_{j_m} \delta^{\overline{s}_m}_{i_k}. \end{aligned} \tag{23}$$

From Corollaries 1, 2, 3 and (23) follows (22). □

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Received February 22, 2012

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BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(75), 2014, Pages 18–28
ISSN 1024–7696

Interpolating Bézier spline curves with local control

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Abstract. The paper presents a technique for construction of interpolating spline curves in linear spaces by means of blending parametric curves. A class of polynomials which satisfy special boundary conditions is used for blending. Properties of the polynomials are stated. An application of the technique to construction of interpolating Bézier spline curves with local control is considered. The presented interpolating Bézier spline curves can be used in on-line geometric applications or for fast sketching and prototyping of spline curves in geometric design.

Mathematics subject classification: 65D05, 65D07, 65D17.
Keywords and phrases: Blending parametric curves, interpolating curves, spline curves, Bezier curves.

1 Introduction

Blending curves is an important technique for smoothing corners of curves in computer-aided geometric design. Besides the technique can be applied to the design of parametric spline curves which have local shape control. Firstly the construction of spline curves by linear blending of parabolic arcs was proposed by Overhauser [5] and considered by Rogers and Adams [8]. The construction of spline curves by linear blending of circular arcs was considered by Zavjalov, Leus, Skorospelov [15], Wenz [10] and Liska, Shashkov, Swartz [3]. The construction of spline curves by trigonometric blending of circular arcs was considered by Szilvási-Nagy, Vendel [12], Séquin, Kiha Lee, Jane Yen [11]. Using linear blending of conics for the construction of spline curves was considered by Chuan Sun, Huanxi Zhao [1]. The paper presents an approach to the construction of interpolating spline curves by means of blending quadric Bezier curves using a class of polynomials which ensure a necessary continuity of the designed curves. The properties of the polynomials are stated. The presented approach can be considered as a generalization of the linear blending.

2 Polynomials approximating a jump function

The purpose is to determine polynomials which can be used for smooth deformation of parametric curves in linear spaces. To solve the problem define polynomials which smoothly approximate the jump function

$$\delta(u) = \begin{cases} 0, & 0 \le u < 1/2 \\ 1/2, & u = 1/2 \\ 1, & 1/2 < u \le 1 \end{cases}.$$

It can be seen that the jump function $\delta(u)$ is infinitely smooth at the boundaries but has a discontinuity at the middle of the domain. In order to avoid the discontinuity approximate the jump function $\delta(u)$ by means of Bernstein polynomials

$$b_{n,m}(u) = \frac{n!}{m!(n-m)!}(1-u)^{n-m}u^m,\ u \in [0,1].$$

For this purpose introduce the following knot sequences:

$$(\underbrace{0,0,\ldots,0}_{n},\underbrace{1,1,\ldots,1}_{n}),\ n \in \mathbb{N}$$

and define the polynomials

$$w_n(u) = \sum_{i=0}^{n-1} 0 \cdot b_{2n-1,i}(u) + \sum_{i=n}^{2n-1} 1 \cdot b_{2n-1,i}(u) = \sum_{i=n}^{2n-1} b_{2n-1,i}(u)$$

for $n \in \mathbb{N}$. It follows from this definition that the polynomials $w_n(u)$ have the following boundary values:

$$w_n(0) = 0,\ w_n(1) = 1 \tag{1}$$

and their derivatives satisfy the following boundary conditions:

$$w_n^{(m)}(0) = w_n^{(m)}(1) = 0 \tag{2}$$

for $m \in \{1, 2, ..., n-1\}$. The following polynomials:

$$w_1(u) = u,\ w_2(u) = 3(1-u)u^2 + u^3,\ w_3(u) = 10(1-u)^2u^3 + 5(1-u)u^4 + u^5$$

are usually used in geometric applications. The polynomials $w_n(u)$ have the following properties.

Property 1. The polynomials $w_n(u)$ satisfy the equation

$$w_n(u) + w_n(1-u) = 1.$$

Proof. This property follows from the property of Bernstein polynomials

$$\sum_{m=0}^{n} b_{n,m}(u) = 1,\ \forall n \in \mathbb{N}.$$

Property 2. The polynomials $w_n(u)$ are symmetric with respect to the point $u = 1/2$.

Proof. It follows from Property 1 that

$$w_n(1/2 + v) + w_n(1/2 - v) = 1, \ \forall v \in [-1/2, 1/2].$$

This means that polynomials $w_n(u)$ are symmetric with respect to the point $u = 1/2$.

Property 3.

$$\lim_{n\to\infty} \int_0^{1/2} w_n(u)du = 0.$$

Proof. It is obvious that the polynomials $w_n(u)$ can be represented by linear combinations of polynomials from the power polynomial basis u^n, u^{n+1}, ..., u^{2n-1} with coefficients linearly depending on n. Then the indefinite integral of the polynomial $w_n(u)$ is a linear combination of the polynomials u^{n+1}, u^{n+2}, ..., u^{2n} whose coefficients also linearly depend on n. Therefore the limit of the definite integrals equals zero.

It follows from Properties 2 and 3 that the polynomial $w_n(u)$ indefinitely close approaches the jump function $\delta(u)$ while its degree is rising.

Property 4. The polynomial $w_n(u)$ is a minimum of the functional

$$J_n(f) = \int_0^1 |f^{(n)}(u)|^2 du, \ \forall n \in \mathbb{N}$$

where the function $f(u)$, $u \in [0, 1]$, satisfies the following boundary conditions:

$$f(0) = 0, \ f(1) = 1, \ f^{(m)}(0) = f^{(m)}(1) = 0 \tag{3}$$

for $m \in \{1, 2, ..., n-1\}$.

Proof. Assume that a function $g(u)$ is a minimum of the functional $J_n(f)$. Consider the function

$$(g - w_n)(u) = g(u) - w_n(u).$$

Then

$$|(g - w_n)^{(n)}|^2 = |g^{(n)} - w_n^{(n)}|^2 = (g^{(n)})^2 - 2g^{(n)} w_n^{(n)} + (w_n^{(n)})^2.$$

or equivalently

$$|(g - w_n)^{(n)}|^2 = (g^{(n)})^2 - (w_n^{(n)})^2 - 2(g^{(n)} - w_n^{(n)})w_n^{(n)}.$$

It follows from the last equation that

$$J_n(g - w_n) = J_n(g) - J_n(w_n) - 2\int_0^1 (g^{(n)}(u) - w_n^{(n)}(u))w_n^{(n)}(u)du.$$

The last integral can be computed by parts as follows:

$$\int_0^1 (g^{(n)}(u) - w_n^{(n)}(u))w_n^{(n)}(u)du = \int_0^1 w_n^{(n)}(u)d(g^{(n-1)}(u) - w_n^{(n-1)}(u)) =$$
$$= w_n^{(n)}(u)(g^{(n-1)}(u) - w_n^{(n-1)}(u))\Big|_0^1 - \int_0^1 (g^{(n-1)}(u) - w_n^{(n-1)}(u))w_n^{(n+1)}(u)du =$$
$$= -\int_0^1 (g^{(n-1)}(u) - w_n^{(n-1)}(u))w_n^{(n+1)}(u)du$$

taking into account that

$$g^{(n-1)}(0) = w_n^{(n-1)}(0) = 0,\ g^{(n-1)}(1) = w_n^{(n-1)}(1) = 0.$$

Recurrently computing the obtained integrals by parts and taking into account that the function $w_n^{(2n-1)}(u)$ is a constant it is obtained that

$$\int_0^1 (g^{(n-1)}(u) - w_n^{(n-1)}(u))w_n^{(n+1)}(u)du =$$
$$= (-1)^n \int_0^1 (g^{'}(u) - w_n^{'}(u))w_n^{(2n-1)}(u)du =$$
$$= (-1)^n (g(u) - w_n(u))w_n^{(2n-1)}(u)\Big|_0^1 = 0$$

because

$$g(0) = w_n(0) = 0,\ g(1) = w_n(1) = 1.$$

Thus it is proven that

$$J_n(g - w_n) = J_n(g) - J_n(w_n).$$

The last equation can be rewritten as follows:

$$J_n(g) = J_n(w_n) + J_n(g - w_n).$$

It follows from the definition of the functional $J_n(f)$ that

$$J_n(g - w_n) \geq 0.$$

Therefore

$$J_n(w_n) \leq J_n(g).$$

But the function $g(u)$ is a minimum of the functional $J_n(f)$ by assumption, therefore

$$g(u) = w_n(u).$$

Thus it is proven that the polynomial $w_n(u)$ is a minimum of the functional $J_n(f)$.

Now prove that this minimum is unique. Suppose the opposite. Let there exist such a function $g(u)$ which satisfies the condition

$$J_n(g) = J_n(w_n).$$

It follows from this equation that

$$J_n(g - w_n) = 0,$$

which is equivalent to

$$g^{(n)}(u) = w_n^{(n)}(u),\ \forall u \in [0, 1].$$

It follows from the last equation that

$$g(u) = w_n(u) + \sum_{i=0}^{n-1} a_i u^i.$$

But the coefficients a_i are equal to zero $\forall i \in \{0, 1, ..., n-1\}$ taking into account boundary conditions which must be satisfied by the function $g(u)$. Therefore

$$g(u) = w_n(u).$$

Thus the property is proven.

The functional $J_n(f)$ can be considered as energy of $n-th$ derivative of the function which satisfies boundary conditions (3). Property 4 shows that the polynomial $w_n(u)$ is a minimum of the functional $J_n(f)$.

The polynomials $w_n(u)$ were firstly introduced by the author [7, 8] for the construction of spline curves by blending of circular arcs and linear segments. Then the polynomials were used by the author for the construction of spline curves on smooth manifolds [9]. The polynomials were also used by Jakubiak, Leite and Rodrigues [3] for smooth spline generation on Riemannian manifolds and by Hartmann [2] for parametric G^n blending of curves and surfaces. Wiltsche [14] proposed Bézier representation of the considered polynomials.

3 Polynomial blending of parametric curves

Consider two parametric curves $\mathbf{p}_1(u)$ and $\mathbf{p}_2(u)$, $u \in [0, 1]$, which have the same boundary points, that is

$$\mathbf{p}_1(0) = \mathbf{p}_2(0),\ \mathbf{p}_1(1) = \mathbf{p}_2(1). \tag{4}$$

The problem is to construct a parametric curve $\mathbf{p}(u)$, $u \in [0, 1]$, which has the boundaries

$$\mathbf{p}(0) = \mathbf{p}_1(0),\ \mathbf{p}(1) = \mathbf{p}_2(1) \tag{5}$$

and derivatives of the parametric curve $\mathbf{p}(u)$ satisfy the following boundary conditions:

$$\mathbf{p}^{(m)}(0) = \mathbf{p}_1^{(m)}(0),\ \mathbf{p}^{(m)}(1) = \mathbf{p}_2^{(m)}(1),\ \forall m \in \{1, 2, ..., n\}, \tag{6}$$

where $n \in \mathbb{N}$. In topology a parametric curve $\mathbf{p}(u)$ which satisfies conditions (5) is called a deformation of the parametric curve $\mathbf{p}_1(u)$ into the parametric curve $\mathbf{p}_2(u)$.

In this case the parametric curves $\mathbf{p}_1(u)$ and $\mathbf{p}_2(u)$ are called homotopic. In geometric design the parametric curve $\mathbf{p}(u)$ must satisfy additional boundary conditions (6) and in this case $\mathbf{p}(u)$ is called a parametric curve blending the parametric curves $\mathbf{p}_1(u)$ and $\mathbf{p}_2(u)$.

Using the polynomials $w_n(u)$ define a blending parametric curve $\mathbf{p}(u)$ as follows:

$$\mathbf{p}(u) = (1 - w_n(u))\mathbf{p}_1(u) + w_n(u)\mathbf{p}_2(u), \; u \in [0, 1]. \tag{7}$$

It follows form the definition of the polynomials $w_n(u)$ that the parametric curve $\mathbf{p}(u)$ satisfies conditions (5) because

$$\mathbf{p}(0) = (1 - w_n(0))\mathbf{p}_1(0) + w_n(0)\mathbf{p}_2(0) = \mathbf{p}_1(0) \tag{8}$$

and

$$\mathbf{p}(1) = (1 - w_n(1))\mathbf{p}_1(1) + w_n(1)\mathbf{p}_2(1) = \mathbf{p}_2(1). \tag{9}$$

Derivatives of the parametric curve $\mathbf{p}(u)$ can be computed as follows:

$$\mathbf{p}^{(m)}(u) = \sum_{i=0}^{m} \frac{m!}{i!(m-i)!}((1 - w_n(u))^{(i)}\mathbf{p}_1^{(m-i)}(u) + w_n^{(i)}(u)\mathbf{p}_2^{(m-i)}(u)), \; \forall m \in \mathbb{N}.$$

Substitution of equations (2) into the last equation yields that

$$\mathbf{p}^{(m)}(0) = (1 - w_n(0))\mathbf{p}_1^{(m)}(0) + w_n(0)\mathbf{p}_2^{(m)}(0) = \mathbf{p}_1^{(m)}(0), \; \forall m \in \{1, 2, ..., n-1\}$$

and

$$\mathbf{p}^{(m)}(1) = (1 - w_n(1))\mathbf{p}_1^{(m)}(1) + w_n(1)\mathbf{p}_2^{(m)}(1) = \mathbf{p}_2^{(m)}(1), \; \forall m \in \{1, 2, ..., n-1\}.$$

Besides taking into account Equations (4) it is obtained that

$$\begin{gathered}\mathbf{p}^{(n)}(0) = -w_n^{(n)}(0)\mathbf{p}_1(0) + (1 - w_n(0))\mathbf{p}_1^{(n)}(0) + \\ + w_n^{(n)}(0)\mathbf{p}_2(0) + w_n(0)\mathbf{p}_2^{(n)}(0) = \mathbf{p}_1^{(n)}(0)\end{gathered}$$

and

$$\begin{gathered}\mathbf{p}^{(n)}(1) = -w_n^{(n)}(1)\mathbf{p}_1(1) + (1 - w_n(1))\mathbf{p}_1^{(n)}(1) + \\ + w_n^{(n)}(1)\mathbf{p}_2(1) + w_n(1)\mathbf{p}_2^{(n)}(1) = \mathbf{p}_2^{(n)}(1).\end{gathered}$$

Therefore the boundary conditions described by Equations (6) are also fulfilled.

The polynomials $w_n(u)$ can be considered as a generalization of the polynomial $w_1(u)$, which is widely used in geometric applications for blending. Blending by means of the polynomials $w_1(u)$ and $(1 - w_1(u))$ is called linear. It can be seen that the proposed approach for blending parametric curves ensures C^n parametric continuity of a blending curve with the blended curves at its boundaries. Polynomial blending which ensures G^n geometric continuity is considered in other articles by Hartmann [2], Meek and Walton [5].

4 Construction of spline curves by curve blending

The considered approach to blending of parametric curves can be used for the construction of spline curves in a linear space. Suppose that it is necessary to construct a spline curve $\mathbf{p}(u) \in C^n$, $n \in \mathbb{N}$, interpolating a sequence of knot points $\mathbf{p}_i$, $i \in \{1, 2, ..., l\}$, which belong to a linear space. In this case segments $\mathbf{p}_i(u)$, $0 < i < l$, of the spline curve are constructed by blending two predefined parametric curves $\mathbf{p}_{i,1}(u)$ and $\mathbf{p}_{i,2}(u)$ using Equation (7) as follows:

$$\mathbf{p}_i(u) = (1 - w_n(u))\mathbf{p}_{i,1}(u) + w_n(u)\mathbf{p}_{i,2}(u) \tag{10}$$

where the parametric curves $\mathbf{p}_{i,1}(u)$ and $\mathbf{p}_{i,2}(u)$ must satisfy the following conditions:

$$\mathbf{p}_{i,1}(0) = \mathbf{p}_{i,2}(0) = \mathbf{p}_i, \ \mathbf{p}_{i,1}(1) = \mathbf{p}_{i,2}(1) = \mathbf{p}_{i+1}. \tag{11}$$

Besides in order to ensure C^n continuity of the spline curve $\mathbf{p}(u)$ the parametric curves $\mathbf{p}_{i-1,2}(u)$ and $\mathbf{p}_{i,1}(u)$ must be smoothly joined at the point $\mathbf{p}_i$ that is

$$\mathbf{p}_{i-1,2}(1) = \mathbf{p}_{i,1}(0) = \mathbf{p}_i \tag{12}$$

and

$$\mathbf{p}_{i-1,2}^{(m)}(1) = \mathbf{p}_{i,1}^{(m)}(0), \ \forall m \in \{1, 2, ..., n\}. \tag{13}$$

Therefore in order to apply the proposed technique to the construction of spline curves the following problem must be solved: how to choose the parametric curves $\mathbf{p}_{i-1,2}(u)$ and $\mathbf{p}_{i,1}(u)$ which satisfy Equations (12) and (13). A solution of this problem depends on the application which uses the technique or more precisely on the modeled physical process. For example, circular arcs have been using for blending curves in the paper [7] because the application was intended for robot trajectory planning. At that time most robots supported only techniques for the interpolation of circular arcs and therefore it was not difficult to use deformation of circular arcs for the construction of spline trajectories. Generally, since spline curves constructed by the proposed technique have local shape control it is reasonable to suppose that the proposed technique will be very suitable to solve problems for on-line point interpolation.

5 Interpolating Bézier spline curves with local control

In geometric design a problem of choosing the model curve is motivated by two reasons: shape and smoothness control of modeled curves. Nowadays it is accepted that Bézier and B-spline curves are most suitable for this purpose. Taking into account these considerations and since the polynomials $w_n(u)$ are represented by means of Bernstein polynomials, Bézier curves are chosen for representation of the parametric curves $\mathbf{p}_{i,1}(u)$ and $\mathbf{p}_{i,2}(u)$ used for blending.

In order to reduce degree of the designed Bézier spline curve it is reasonable to use Bézier curves of the most low degree for blending. Therefore quadric Bézier curves are chosen for parametric curves $\mathbf{p}_{i,1}(u)$ and $\mathbf{p}_{i,2}(u)$. In order to ensure unique choice of the parametric curves $\mathbf{p}_{i,1}(u)$ and $\mathbf{p}_{i,2}(u)$ it is supposed that the parametric curves $\mathbf{p}_{i-1,2}(u)$ and $\mathbf{p}_{i,1}(u)$ are smoothly joined at the knot point $\mathbf{p}_i$. Analytical representation of such quadric Bézier curves $\mathbf{p}_{i-1,1}(u)$ and $\mathbf{p}_{i,2}(u)$ can be obtained from the following conditions:

$$\mathbf{p}_{i-1,2}(1) = \mathbf{p}_{i,1}(0),\ \mathbf{p}'_{i-1,2}(1) = \mathbf{p}'_{i,1}(0),\ \mathbf{p}''_{i-1,2}(1) = \mathbf{p}''_{i,1}(0). \tag{14}$$

In order to simplify index notations consider two quadric Bézier curves

$$\mathbf{p}_j(u) = (1-u)^2\mathbf{p}_{j,0} + 2(1-u)u\mathbf{p}_{j,1} + u^2\mathbf{p}_{j,2},\ j \in \{1,2\},$$

which have the following boundary points:

$$\mathbf{p}_1(0) = \mathbf{p}_0,\ \mathbf{p}_1(1) = \mathbf{p}_2(0) = \mathbf{p}_1,\ \mathbf{p}_2(1) = \mathbf{p}_2 \tag{15}$$

and are smoothly joined at the point $\mathbf{p}_1$, that is

$$\mathbf{p}'_1(1) = \mathbf{p}'_2(0),\ \mathbf{p}''_1(1) = \mathbf{p}''_2(0) \tag{16}$$

Resolution of these equations yields the following values of unknown knot and control points of the quadric Bézier curves $\mathbf{p}_1(u)$ and $\mathbf{p}_2(u)$:

$$\mathbf{p}_{1,0} = \mathbf{p}_0,\ \mathbf{p}_{1,2} = \mathbf{p}_{2,0} = \mathbf{p}_1,\ \mathbf{p}_{2,2} = \mathbf{p}_2, \tag{17}$$

$$\mathbf{p}_{1,1} = \mathbf{p}_1 - \frac{\mathbf{p}_2 - \mathbf{p}_0}{4},\ \mathbf{p}_{2,1} = \mathbf{p}_1 + \frac{\mathbf{p}_2 - \mathbf{p}_0}{4}. \tag{18}$$

It follows from these constructions that the quadric Bézier curves $\mathbf{p}_1(u)$ and $\mathbf{p}_2(u)$ are smoothly joined at the knot point $\mathbf{p}_1$ and therefore can be used for the construction of spline curves with any degree of continuity. Actually the parametric curves $\mathbf{p}_1(u)$ and $\mathbf{p}_2(u)$ are two segments of the same parabola. Besides the segments are joined at such the point $\mathbf{p}_1$ that the distance from the point to the line connecting the points $\mathbf{p}_0$ and $\mathbf{p}_2$ is maximal for all points belonging to the parabola.

Using Equations (17) and (18) Bézier curves $\mathbf{p}_{i,1}(u)$ and $\mathbf{p}_{i,2}(u)$ which are used for the construction of a Bézier spline curve can be determined as follows:

$$\mathbf{p}_{i,k}(u) = (1-u)^2\mathbf{p}_i + 2(1-u)u\mathbf{p}_{i,k} + u^2\mathbf{p}_{i+1},\ k \in \{1,2\}, \tag{19}$$

where

$$\mathbf{p}_{i,1} = \mathbf{p}_i + \frac{\mathbf{p}_{i+1} - \mathbf{p}_{i-1}}{4},\ \mathbf{p}_{i,2} = \mathbf{p}_{i+1} - \frac{\mathbf{p}_{i+2} - \mathbf{p}_i}{4}. \tag{20}$$

That is the parametric curves $\mathbf{p}_{i,1}(u)$ and $\mathbf{p}_{i,2}(u)$ are blended in Equation (10) for the construction of the spline curve segment $\mathbf{p}_i(u)$.

Find another representation of the spline curve segment $\mathbf{p}_i(u)$ which clarifies its geometric construction. For this purpose substitute the obtained expressions for parametric curves $\mathbf{p}_{i,1}(u)$ and $\mathbf{p}_{i,2}(u)$ into Equation (10). It is obtained that

$$\mathbf{p}_i(u) = (1-u)^2\mathbf{p}_i + 2(1-u)u((1-w_n(u))\mathbf{p}_{i,1} + w_n(u)\mathbf{p}_{i,2}) + u^2\mathbf{p}_{i+1}$$

where the control points $\mathbf{p}_{i,1}$ and $\mathbf{p}_{i,2}$ are defined by Equations (20). This representation shows that the spline curve segment $\mathbf{p}_i(u)$ can be considered as a quadric Bézier curve with a smoothly modified control point.

A spline curve segment $\mathbf{p}_i(u)$ can be also represented as a Bézier curve. To obtain this representation modify Equation (10) using Equations (19) and taking into account Property 1 of the polynomials $w_n(u)$ as follows:

$$\begin{aligned}
\mathbf{p}_i(u) &= (1 - w_n(u))\mathbf{p}_{i,1}(u) + w_n(u)\mathbf{p}_{i,2}(u) = \\
&= w_n(1-u)\mathbf{p}_{i,1}(u) + w_n(u)\mathbf{p}_{i,2}(u) = \\
&= \sum_{k=0}^{n-1} b_{2n-1,k}(u)\mathbf{p}_{i,1}(u) + \sum_{k=n}^{2n-1} b_{2n-1,k}(u)\mathbf{p}_{i,2}(u) = \\
&= \sum_{k=0}^{n-1} b_{2n-1,k}(u)(b_{2,0}(u)\mathbf{p}_i + b_{2,1}(u)\mathbf{p}_{i,1} + b_{2,2}(u)\mathbf{p}_{i+1}) + \\
&+ \sum_{k=n}^{2n-1} b_{2n-1,k}(u)(b_{2,0}(u)\mathbf{p}_i + b_{2,1}(u)\mathbf{p}_{i,2} + b_{2,2}(u)\mathbf{p}_{i+1}) = \\
&= \sum_{k=0}^{2n-1} b_{2n-1,k}(u)b_{2,0}(u)\mathbf{p}_i + \sum_{k=0}^{n-1} b_{2n-1,k}(u)b_{2,1}(u)\mathbf{p}_{i,1} + \\
&+ \sum_{k=n}^{2n-1} b_{2n-1,k}(u)b_{2,1}(u)\mathbf{p}_{i,2} + \sum_{k=0}^{2n-1} b_{2n-1,k}(u)b_{2,2}(u)\mathbf{p}_{i+1} = \\
&= \sum_{k=0}^{2n-1} b_{2n+1,k}(u)c_{0,k}\mathbf{p}_i + \sum_{k=1}^{n} b_{2n+1,k}(u)c_{1,k}\mathbf{p}_{i,1} + \\
&+ \sum_{k=n+1}^{2n} b_{2n+1,k}(u)c_{1,k}\mathbf{p}_{i,2} + \sum_{k=2}^{2n+1} b_{2n+1,k}(u)c_{2,k}\mathbf{p}_{i+1}
\end{aligned}$$

where

$$\begin{aligned}
c_{0,k} &= \frac{(2n-k)(2n-k+1)}{2n(2n+1)},\ 0 \le k \le 2n-1, \\
c_{1,k} &= \frac{k(2n-k+1)}{n(2n+1)},\ 1 \le k \le 2n, \\
c_{2,k} &= \frac{(k-1)k}{2n(2n+1)},\ 2 \le k \le 2n+1.
\end{aligned}$$

It follows from the obtained equations that the segment $\mathbf{p}_i(u)$ has the following Bézier representation:

$$\begin{aligned}
\mathbf{p}_i(u) &= b_{2n+1,0}(u)\mathbf{p}_i + b_{2n+1,1}(u)(c_{0,1}\mathbf{p}_i + c_{1,1}\mathbf{p}_{i,1}) + \\
&+ \sum_{k=2}^{n} b_{2n+1,k}(u)(c_{0,k}\mathbf{p}_i + c_{1,k}\mathbf{p}_{i,1} + c_{2,k}\mathbf{p}_{i+1}) +
\end{aligned}$$

$$+\sum_{k=n+1}^{2n-1} b_{2n+1,k}(u)(c_{0,k}\mathbf{p}_i + c_{1,k}\mathbf{p}_{i,2} + c_{2,k}\mathbf{p}_{i+1})+$$
$$+b_{2n+1,2n}(u)(c_{1,2n}\mathbf{p}_{i,2} + c_{2,2n}\mathbf{p}_{i+1}) + b_{2n+1,2n+1}(u)\mathbf{p}_{i+1}.$$

For example, segments of C^1 and C^2 continuous spline curves have the following Bézier representations:

$$\mathbf{p}_i(u) = b_{3,0}(u)\mathbf{p}_i + b_{3,1}(u)\frac{1}{3}(\mathbf{p}_i + 2\mathbf{p}_{i,2}) + +b_{3,2}(u)\frac{1}{3}(2\mathbf{p}_{i,1} + \mathbf{p}_{i+1}) + b_{3,3}(u)\mathbf{p}_{i+1},$$
$$\mathbf{p}_i(u) = b_{5,0}(u)\mathbf{p}_i + b_{5,1}(u)\frac{1}{5}(3\mathbf{p}_i + 2\mathbf{p}_{i,2}) + b_{5,2}(u)\frac{1}{10}(3\mathbf{p}_i + 6\mathbf{p}_{i,2} + \mathbf{p}_{i+1})+$$
$$+b_{5,3}(u)\frac{1}{10}(\mathbf{p}_i + 6\mathbf{p}_{i,1} + 3\mathbf{p}_{i+1}) + b_{5,4}(u)\frac{1}{5}(2\mathbf{p}_{i,1} + 3\mathbf{p}_{i+1}) + b_{5,5}(u)\mathbf{p}_{i+1},$$

respectively.

It can be seen that a segment $\mathbf{p}_i(u)$ of a C^n continuous spline curve is a Bézier curve of degree $2n+1$. Let $\mathbf{p}_{i,k},\ k \in \{1, 2, \ldots, 2n\}$, denote control points of the spline curve segment $\mathbf{p}_i(u)$. It is known that Bézier curves have convex hull property, that is a Bézier curve lies completely in the convex hull of its control points. It follows from this property that deviation of a spline curve segment $\mathbf{p}_i(u)$ from the line segment $\overline{P_iP_{i+1}}$ can be estimated as follows:

$$\varepsilon < max(dist(\overline{P_iP_{i+1}}, \mathbf{p}_{i,k})),\ \forall k \in \{1, 2, \ldots, 2n\}.$$

6 Conclusions

The approach to the construction of C^n continuous interpolating spline curves by means of blending quadric Bézier curves is introduced. Properties of the polynomials which are used for blending are considered. The considered spline curves are constructed locally, that ensures local shape control of the constructed spline curves. Bézier representations of the introduced spline curves is presented. The considered interpolating spline curves can be used in on-line geometric applications or for fast sketching and prototyping of spline curves in geometric design. It also can be noted that the proposed approach enables drawing of Bézier spline curves of C^1 continuity by means of any software packages which support drawing of cubic Bézier curves.

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Received May 29, 2013

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(75), 2014, Pages 29–35
ISSN 1024–7696

Composition followed by differentiation between weighted Bergman spaces and weighted Banach spaces of holomorphic functions

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Abstract. Let ϕ be an analytic self-map of the open unit disk $\mathbb{D}$ in the complex plane. Such a map induces through composition a linear composition operator $C_\phi : f \mapsto f \circ \phi$. We are interested in the combination of C_ϕ weith the differentiation operator D, that is in the operator $DC_\phi : f \mapsto \phi' \cdot (f \circ \phi)$ acting between weighted Bergman spaces and weighted Banach spaces of holomorphic functions.

Mathematics subject classification: 47B33, 47B38.
Keywords and phrases: Composition operator, differentiation operator, weighted Bergman spaces, weighted Banach spaces of holomorphic functions.

1 Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane. For an analytic self-map ϕ of $\mathbb{D}$ the classical *composition operator* C_ϕ is given by

$$C_\phi : H(\mathbb{D}) \to H(\mathbb{D}),\ f \mapsto f \circ \phi,$$

where $H(\mathbb{D})$ denotes the set of all analytic functions on $\mathbb{D}$. Combining this with differentiation we obtain the operator

$$DC_\phi : H(\mathbb{D}) \to H(\mathbb{D}),\ f \mapsto \phi' \cdot (f' \circ \phi).$$

Composition operators occur naturally in various problems and therefore have been widely investigated. An overview of results in the classical setting of the Hardy spaces as well as an introduction to composition operators is given in the excellent monographs by Cowen and MacCluer [5] and Shapiro [8].

Next, let us explain the setting in which we are interested. Bounded and continuous functions $v : \mathbb{D} \to]0, \infty[$ are called *weights*. For such a weight v we define

$$H_v^\infty := \{f \in H(\mathbb{D});\ \|f\|_v := \sup_{z \in \mathbb{D}} v(z)|f(z)| < \infty\}.$$

Since, endowed with the weighted sup-norm $\|.\|_v$, this is a Banach space, we say that H_v^∞ is a *weighted Banach space of holomorphic functions*. These spaces arise naturally in several problems related to e. g. complex analysis, spectral theory, Fourier analysis, partial differential and convolution equations. Concrete examples may be found in [3]. Weighted Banach spaces of holomorphic functions have been studied deeply in [2], but also in [4] and [1].

The *weighted Bergman space* is defined to be the collection of all analytic functions $f \in H(\mathbb{D})$ such that

$$A_{v,p} := \{f \in H(\mathbb{D});\ \|f\|_{v,p} := \left(\int_{\mathbb{D}} |f(z)|^p v(z)\, dA(z)\right)^{\frac{1}{p}} < \infty\}, 1 \le p < \infty$$

where $dA(z)$ denotes the normalized area measure. The investigation of Bergman spaces has quite a long and rich history. An excellent introduction to Bergman spaces is given in [6].

In this article we characterize boundedness and compactness of operators $DC_\phi : A_{v,p} \to H_w^\infty$ in terms of the involved self-map ϕ and the weights v and w.

2 Basics

We study weighted spaces generated by the following class of weights. Let ν be a holomorphic function on $\mathbb{D}$ that does not vanish and is strictly positive on $[0, 1[$. Moreover, we assume that $\lim_{r\to 1} \nu(r) = 0$. Then we define the weight v in the following way

$$v(z) := \nu(|z|^2) \text{ for every } z \in \mathbb{D}. \tag{1}$$

Examples include all the famous and popular weights, such as

1. the *standard weights* $v(z) = (1 - |z|^2)^\alpha$, $\alpha \ge 1$,
2. the *logarithmic weights* $v(z) = (1 - \log(1 - |z|^2))^\beta$, $\beta > 0$.
3. the *exponential weights* $v(z) = e^{-\frac{1}{(1-|z|^2)^\alpha}}$, $\alpha \ge 1$.

For a fixed point $a \in \mathbb{D}$, we introduce a function

$$v_a(z) := \nu(\overline{a}z) \text{ for every } z \in \mathbb{D}.$$

Since ν is holomorphic on $\mathbb{D}$, so is the function v_a. Moreover, in particular, we will often assume that there is a constant $C > 0$ such that

$$\sup_{a\in\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{v(z)}{|v_a(z)|} \le C. \tag{2}$$

In the sequel we analyze which role condition (2) plays in the zoo of conditions on weights. Lusky [7] studied weights satisfying the following conditions $(L1)$ and $(L2)$ (renamed after the author) which are defined as follows

$$(L1)\ \inf_{n\in\mathbb{N}} \frac{v(1-2^{-n-1})}{v(1-2^{-n})} > 0 \text{ and } (L2)\ \limsup_{n\to\infty} \frac{v(1-2^{-n-j})}{v(1-2^{-n})} < 1 \text{ for some } j \in \mathbb{N}.$$

Actually, weights which enjoy both conitions $(L1)$ and $(L2)$ are normal weights in the sense of Shields and Williams (see [9]). Obviously condition (2) is connected with contiion $(L2)$ in the following way. If we change (2) as follows

$$\sup_{a\in\mathbb{D}}\sup_{z\in\mathbb{D}}\frac{v(z)}{|v_a(z)|}<1, \tag{3}$$

then $(L2)$ is equivalent with (3) if we assume that $|\nu(z)|\geq\nu(|z|)$ for every $z\in\mathbb{D}$. To show this, let us first assume that $(L2)$ holds. Hence we can find $j\in\mathbb{N}$ such that

$$\frac{v(1-2^{-n-j})}{v(1-2^{-n})}<1 \text{ for every } n\in\mathbb{N}.$$

Next, we fix $z\in\mathbb{D}$ and $a\in\mathbb{D}$. Then we can find $n\in\mathbb{N}$ such that

$$|z|\geq 1-2^{-n-j} \text{ and } |az|<1-2^{-n}.$$

Now,

$$\frac{v(z)}{|\nu(az)|}\leq\frac{v(1-2^{-n-j})}{v(1-2^{-n})}<1 \text{ for every } n\in\mathbb{N}.$$

Conversely, we suppose that (3) is satisfied. We take $j=1$, fix $n\in\mathbb{N}$ and select

$$a_n:=\frac{(1-2^{-n})^2}{(1-2^{-n-1})}.$$

We obtain

$$\frac{v(1-2^{-n-1})}{v(1-2^{-n})}\leq\frac{v(z)}{|\nu(az)|}\leq\sup_{a\in\mathbb{D}}\sup_{z\in\mathbb{D}}\frac{v(z)}{|v_a(z)|}<1.$$

Thus, under some additional assumptions (2) is a weaker verson of $(L2)$. Calculations show that the standard weights as well as the logarithmic weights satisfy condition (2), while the exponential weights do not fulfill condition (2).

Finally, we need some geometric data of the unit disk. A very important tool when dealling with operators such as defined above is the so-called *pseudohyberbolic metric* given by

$$\rho(z,a):=|\sigma_a(z)|,$$

where $\sigma_a(z)=\frac{a-z}{1-\bar{a}z}$, $z,a\in\mathbb{D}$, is the Möbius transformation which interchanges a and 0.

3 Results

Lemma 1. *Let $v(z)=f(|z|)$ for every $z\in\mathbb{D}$, where $f\in H(\mathbb{D})$ is a function whose Taylor series (at 0) has nonnegative coefficients. We assume additionally that v satisfies condition (2). Then there is a constant $C>0$ such that*

$$|f(z)|\leq C^{\frac{1}{p}}\frac{\|f\|_{v,p}}{v(0)^{\frac{1}{p}}(1-|z|^2)^{\frac{2}{p}}v(z)^{\frac{1}{p}}}.$$

Proof. Recall that a weight v as defined above may be written as

$$v(z) := \max\{|g(\lambda z)|; |\lambda| = 1\} \text{ for every } z \in \mathbb{D}.$$

We will write $g_\lambda(z) := g(\lambda z)$ for every $z \in \mathbb{D}$. Next, fix $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Moreover, let $\alpha \in \mathbb{D}$ be an arbitrary point. We consider the map

$$T_{\alpha,\lambda} : A_v^p \to A_v^p,\ T_{\alpha,\lambda} f(z) = f(\sigma_\alpha(z))\sigma_\alpha'(z)^{\frac{2}{p}} g_\lambda(\sigma_\alpha(z))^{\frac{1}{p}}.$$

Then a change of variables yields

$$\begin{aligned} \|T_{\alpha,\lambda} f\|_{v,p}^p &= \int_{\mathbb{D}} v(z)|f(\sigma_\alpha(z))|^p|\sigma_\alpha'(z)|^2|g_\lambda(\sigma_\alpha(z))|\, dA(z) \\ &\leq \int_{\mathbb{D}} |f(\sigma_\alpha(z))| \frac{v(z)}{v(\sigma_\alpha(z))} |\sigma_\alpha'(z)|\, dA(z) \\ &\leq C \int_{\mathbb{D}} |f(\sigma_\alpha(z))| v(\sigma_\alpha(z))|\sigma_\alpha'(z)|^2\, dA(z) \\ &\leq C \int_{\mathbb{D}} v(t)|f(t)|^p\, dA(t) = C\|f\|_{v,p}^p. \end{aligned}$$

Now put $h_\lambda(z) := T_{\alpha,\lambda}(z)$ for every $z \in \mathbb{D}$. By the mean value property we obtain

$$v(0)|h_\lambda(0)|^p \leq \int_{\mathbb{D}} v(z)|h_\lambda(z)|^p\, dA(z) = \|h_\lambda\|_{v,p}^p \leq C\|f\|_{v,p}^p.$$

Hence

$$v(0)|h_\lambda(0)|^p = v(0)|f(\alpha)|^p(1-|\alpha|^2)^2|g_\lambda(\alpha)| \leq C\|f\|_{v,p}^p.$$

Since λ was arbitrary we obtain that

$$v(0)|f(\alpha)|^p(1-|\alpha|^2)^2 v(\alpha) \leq C\|f\|_{v,p}^p$$

Thus,

$$|f(\alpha)| \leq C^{\frac{1}{p}} \frac{\|f\|_{v,p}}{v(0)^{\frac{1}{p}}(1-|\alpha|^2)^{\frac{2}{p}} v(\alpha)^{\frac{1}{p}}}.$$

Since α was arbitrary, the claim follows. □

Lemma 2. *Let $v(z) = f(|z|)$ for every $z \in \mathbb{D}$, where $f \in H(\mathbb{D})$ is a function whose Taylor series (at 0) has nonnegative coefficients. We assume additionally that v satisfies condition (2). Then for every $f \in A_v^p$ there is $C_v > 0$ such that*

$$|f(z) - f(w)| \leq C_v\|f\|_{v,p} \max\left\{\frac{1}{(1-|z|^2)^{\frac{2}{p}} v(z)^{\frac{1}{p}}}, \frac{1}{(1-|w|^2)^{\frac{2}{p}} v(w)^{\frac{1}{p}}}\right\} \rho(z,w)$$

for every $z, w \in \mathbb{D}$.

Proof. The proof is completely analogous to the proof given in [10]. Hence we omit it here. □

Lemma 3. *Let $v(z) = f(|z|)$ for every $z \in \mathbb{D}$, where $f \in H(\mathbb{D})$ is a function whose Taylor series (at 0) has nonnegative coefficients. We assume additionally that v satisfies condition (2). Then for $f \in H_v^\infty$ and $z \in \mathbb{D}$:*

$$|f'(z)| \leq \frac{M}{v(0)^{\frac{1}{p}}(1-|z|^2)^{1+\frac{2}{p}}v(z)^{\frac{1}{p}}}\|f\|_{v,p}.$$

Proof. By Lemma 2 we have that

$$|f(z)-f(w)| \leq \frac{M}{v(0)^{\frac{1}{p}}} \max\left\{\frac{1}{(1-|z|^2)^{\frac{2}{p}}v(z)^{\frac{1}{p}}}, \frac{1}{(1-|w|^2)^{\frac{2}{p}}v(w)^{\frac{1}{p}}}\right\}\rho(z,w)\|f\|_{v,p}.$$

Now $\left|\dfrac{f(z+h)-f(z)}{|h|}\right| \leq$

$$\begin{aligned}
&\leq \frac{M}{v(0)^{\frac{1}{p}}|h|} \max\left\{\frac{1}{(1-|z+h|^2)^{\frac{2}{p}}v(z+h)^{\frac{1}{p}}}, \frac{1}{(1-|z|^2)^{\frac{2}{p}}v(z)^{\frac{1}{p}}}\right\}\rho(z+h,z)\|f\|_{v,p} \\
&= \frac{M}{v(0)^{\frac{1}{p}}|h|} \max\left\{\frac{1}{(1-|z+h|^2)^{\frac{2}{p}}v(z+h)^{\frac{1}{p}}}, \frac{1}{(1-|z|^2)^{\frac{2}{p}}v(z)^{\frac{1}{p}}}\right\}\left|\frac{z+h-z}{1-\overline{z+h}z}\right|\|f\|_{v,p} \\
&= \frac{M}{v(0)^{\frac{1}{p}}} \max\left\{\frac{1}{(1-|z+h|^2)^{\frac{2}{p}}v(z+h)^{\frac{1}{p}}}, \frac{1}{(1-|z|^2)^{\frac{2}{p}}v(z)^{\frac{1}{p}}}\right\}\left|\frac{1}{1-\overline{z+h}z}\right|\|f\|_{v,p}.
\end{aligned}$$

Finally, let h tend to zero and obtain

$$|f'(z)| \leq \frac{M}{v(0)^{\frac{1}{p}}(1-|z|^2)^{1+\frac{2}{p}}v(z)^{\frac{1}{p}}}\|f\|_{v,p}.$$

□

Proposition 1. *Let $v(z) = f(|z|)$ for every $z \in \mathbb{D}$, where $f \in H(\mathbb{D})$ is a function whose Taylor series (at 0) has nonnegative coefficients. We assume additionally that v satisfies condition (2). Then $DC_\phi : A_{v,p} \to H_w^\infty$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{w(z)|\phi'(z)|}{(1-|\phi(z)|^2)^{1+\frac{2}{p}}v(\phi(z))^{\frac{1}{p}}} < \infty. \tag{4}$$

Proof. First, we assume that (4) is satisfied. Applying Lemma 1 we obtain

$$\|DC_\phi f\|_w = \sup_{z\in\mathbb{D}} w(z)|\phi'(z)||f'(\phi(z))| \leq C \sup_{z\in\mathbb{D}} \frac{w(z)|\phi'(z)|}{(1-|\phi(z)|^2)^{1+\frac{2}{p}}v(\phi(z))^{\frac{1}{p}}}.$$

Hence $DC_\phi : A_{v,p} \to H_w^\infty$ must be bounded.

Conversely, let $a \in \mathbb{D}$ be arbitrary. Then there exists f_a^p in the unit ball of H_v^∞ such that $|f_a(a)|^p = \frac{1}{\tilde{v}(a)}$. Now put

$$g_a(z) := f_a(z)\sigma_a(z) \text{ for every } z \in \mathbb{D}.$$

Then $\|g_a\|_{v,p}^p = \int_{\mathbb{D}} |g_a(z)|^p v(z)\ dA(z) \le \sup_{z\in\mathbb{D}} v(z)|f_a(z)|^p \int_{\mathbb{D}} |\sigma(z)|^p\ dA(z) \le K$. Moreover,

$$g_a'(z) = f_a'(z)\sigma_a(z) + f_a(z)\sigma_a'(z) \text{ for every } z \in \mathbb{D}.$$

Next, we assume that there is a sequence $(z_n)_n \subset \mathbb{D}$ such that $|\phi(z_n)| \to 1$ and

$$\frac{w(z_n)|\phi'(z_n)|}{(1-|\phi(z_n)|^2)^{1+\frac{2}{p}} v(\phi(z_n))^{\frac{1}{p}}} \ge n \text{ for every } n \in \mathbb{N}.$$

Thus consider now $g_n(z) := g_{\phi(z_n)}(z)$ for every $n \in \mathbb{N}$ as defined above. Obviously $(g_n)_n$ is contained in the closed unit ball of $A_{v,p}$ and

$$c \ge w(z_n)|\phi'(z_n)||g_n'(\phi(z_n))| = \frac{w(z_n)|\phi'(z_n)|}{v(\phi(z_n))^{\frac{1}{p}}(1-|\phi(z_n)|^2)^{1+\frac{2}{p}}} \ge n$$

for every $n \in \mathbb{N}$ which is a contradiction. □

Proposition 2. *Let $v(z) = f(|z|)$, $z \in \mathbb{D}$, where $f \in H(\mathbb{D})$ is a function whose Taylor sereis (at 0) has nonnegative coefficients. Moreover, we assume that v satisfies (2). Then the operator $DC_\phi : A_{v,p} \to H_w^\infty$ is compact if and only if*

$$\limsup_{|\phi(z)|\to 1} \frac{w(z)|\phi'(z)|}{(1-|\phi(z)|^2)^{1+\frac{2}{p}} v(\phi(z))^{\frac{1}{p}}}.$$

Proof. Let $(f_n)_n$ be a bounded sequence in $A_{v,p}$ that converges to zero uniformly on the compact subsets of $\mathbb{D}$. Let $M := \sup_n \|f_n\|_{v,p} < \infty$. Given $\varepsilon > 0$ there is $r > 0$ such that if $|\phi(z)| > 0$, then

$$\frac{w(z)|\phi'(z)|}{(1-|\phi(z)|^2)^{1+\frac{2}{p}} v(\phi(z))^{\frac{1}{p}}} \le \frac{\varepsilon}{2C_v}.$$

On the other hand, since $f_n \to 0$ uniformly on $\{u;\ |u| \le r\}$, there is an $n_0 \in \mathbb{N}$ such that if $|\phi(z)| \le r$ and $n \ge n_0$, then $w(z)|f_n'(\phi(z))||\phi'(z)| < \frac{\varepsilon}{2}$. Now, an application of Lemma 3 yields

$$\sup_{z\in\mathbb{D}} w(z)|f_n'(\phi(z))||\phi'(z))| \le \sup_{|\phi(z)|\le r} w(z)|f_n'(\phi(z))||\phi'(z)| +$$
$$+ \sup_{|\phi(z)|>r} w(z)|f_n'(\phi(z))||\phi'(z)| \le \frac{\varepsilon}{2} + \sup_{|\phi(z)|>r} \frac{C_v w(z)|\phi'(z)|}{(1-|\phi(z)|^2)^{\frac{2}{p}+1} v(\phi(z))^{\frac{1}{p}}} < \varepsilon.$$

Thus, the claim follows.

Conversely, we suppose that $DC_\phi : A_{v,p} \to H_w^\infty$ is compact and that there are $\delta > 0$ and $(z_n)_n \subset \mathbb{D}$ with $|\phi(z_n)| \to 1$ such that

$$\frac{w(z_n)|\phi'(z_n)|}{(1-|\phi(z_n)|^2)^{1+\frac{2}{p}} v(\phi(z_n))^{\frac{1}{p}}} \ge \delta.$$

Since $|\phi(z_n)| \to 1$ there exist natural numbers $\alpha(n)$ with $\lim_{n\to\infty} \alpha(n) = \infty$ such that $|\phi(z_n)|^{\alpha(n)} \geq \frac{1}{2}$ for every $n \in \mathbb{N}$.

Next, for every $n \in \mathbb{N}$ we consider the function

$$g_n(z) := f_n(z)\sigma_{\phi(z_n)}^{1+\frac{2}{p}} z^{\alpha(n)},$$

where $f_n^p \in H_v^\infty$ such that $\|f_n^p\|_v \leq 1$ and $|f_n(\phi(z_n))|^p = \frac{1}{\tilde{v}(\phi(z_n))}$. Then we obtain

$$\begin{aligned} \|DC_\phi f_n\|_w &\geq w(z_n)|\phi'(z_n)||f_n'(\phi(z_n))| \\ &\geq \frac{w(z_n)|\phi'(z_n)||\phi(z_n)|^{\alpha(n)}}{\tilde{v}(\phi(z_n))^{\frac{1}{p}}(1-|\phi(z_n)|^2)^{1+\frac{2}{p}}} \\ &\geq \frac{1}{2}\frac{w(z_n)|\phi'(z_n)|}{\tilde{v}(\phi(z_n))^{\frac{1}{p}}(1-|\phi(z_n)|^2)^{1+\frac{2}{p}}} \geq \frac{1}{2}\delta. \end{aligned}$$

This is a contradiction. □

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Received September 30, 2013

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(75), 2014, Pages 36–43
ISSN 1024–7696

On π-quasigroups of type T_1

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Abstract. Quasigroups satisfying the identity $x(x \cdot xy) = y$ are called π-quasigroups of type T_1. The spectrum of the defining identity is precisely $q = 0$ or $1(\text{mod } 3)$, except for $q = 6$. Necessary conditions when a finite π-quasigroup of type T_1 has the order $q = 0$ (mod 3), are given. In particular, it is proved that a finite π-quasigroup of type T_1 such that the order of its inner mapping group is not divisible by three has a left unit. Necessary and sufficient conditions when the identity $x(x \cdot xy) = y$ is invariant under the isotopy of quasigroups (loops) are found.

Mathematics subject classification: 20N05.
Keywords and phrases: Minimal identity, π-quasigroup of type T_1, spectrum, inner mapping group, invariants under isotopy.

Let $\Sigma(Q)$ be the set of all binary quasigroup operations defined on a nonempty set Q. V. Belousov proved in [1] that the minimal length of nontrivial identities $w_1 = w_2$ in $\Sigma(Q)$, of two free elements, is five and that any such minimal identity can be represented in the form: $A(x, B(x, C(x, y))) = y$. Moreover, every identity of the given above form implies the orthogonality of some pairs of parastrophes of the quasigroup operations A, B and C.

A binary quasigroup (Q, A) is called a π-quasigroup of type $[\alpha, \beta, \gamma]$, where $\alpha, \beta, \gamma \in S_3$, if it satisfies the identity:

$$ {}^{\alpha}A(x, {}^{\beta}A(x, {}^{\gamma}A(x, y))) = y \tag{1}$$

(where ${}^{\sigma}A$ denotes the σ-parastrophe of A).

V. Belousov (1983) and, independently, F. Bennett (1989) gave a classification of all identities (1), consisting of seven classes. Denoting A by " $\cdot$ ", the representatives of these classes are the following (their types are given according to [1]): $x(x \cdot xy) = y$ (of type $T_1 = [\varepsilon, \varepsilon, \varepsilon]$); $x(y \cdot yx) = y$ (of type $T_2 = [\varepsilon, \varepsilon, l]$); $x \cdot xy = yx$ (of type $T_4 = [\varepsilon, \varepsilon, lr]$); $xy \cdot x = y \cdot xy$ (of type $T_6 = [\varepsilon, l, lr]$); $xy \cdot y = x \cdot xy$ (of type $T_8 = [\varepsilon, rl, lr]$); $xy \cdot yx = y$ (of type $T_{10} = [\varepsilon, lr, l]$); $yx \cdot xy = y$ (of type $T_{11} = [\varepsilon, lr, rl]$), where $l = (13)$, $r = (23)$. Quasigroups satisfying identities from this classification have been studied by many authors (see, for example, [1, 4–6, 10]). An open problem is to describe groups isotopic to π-quasigroups of different types.

π-Quasigroups of type T_1, i.e. binary quasigroups $(Q, \cdot)$ satisfying the identity:

$$ x \cdot (x \cdot xy) = y, \tag{2}$$

are studied in the present work. It is known that the spectrum of the defining identity is precisely $q = 0$ or $1(\text{mod } 3)$, except for $q = 6$ ([5]). Necessary conditions when a finite π-quasigroup of type T_1 has the order $q = 0 \pmod 3$ are given. In particular, we prove that a π-quasigroup of type T_1 for which the order of inner mapping group is not divisible by three always has a left unit. Necessary and sufficient conditions when the identity (2) is invariant under the isotopy of quasigroups (loops) are proved. Also, π-quasigroups of type T_1 isotopic to groups, in particular π-T-quasigroups of type T_1, are considered.

Proposition 1. *A quasigroup $(Q,\cdot)$ is a π-quasigroup of type T_1 if and only if its parastrophe $(Q,\backslash)$, where "$\backslash$" is the right division in $(Q,\cdot)$, is a π-quasigroup of type T_1.*

Proof. If $(Q,\cdot)$ is a quasigroup, then the identity (2) is equivalent to the identity $x\backslash[x\backslash(x\backslash y)] = y$. $\square$

Remark that the identity (2) in a quasigroup $(Q,\cdot)$ is equivalent to the condition $L_x^3 = \varepsilon$, $\forall x \in Q$, where $L_x : Q \mapsto Q, L_x(a) = x \cdot a, \forall a \in Q$, is the left translation in $(Q,\cdot)$. We will denote by $M(Q,\cdot)$ (respectively, $LM(Q,\cdot)$, $RM(Q,\cdot)$) the multiplication group (respectively, left multiplication group, right multiplication group) of a quasigroup $(Q,\cdot)$.

A mapping $\alpha \in M(Q,\cdot)$ is called an inner mapping of a quasigroup $(Q,\cdot)$ with respect to an element $h \in Q$ if $\alpha(h) = h$. The group of inner mappings of the quasigroup $(Q,\cdot)$ with respect to h will be denoted by I_h [2, 9].

Proposition 2. *If $(Q,\cdot)$ is a finite π-quasigroup of type T_1 without a left unit, then $\mid I_h \mid \equiv\ 0 \pmod 3$, for every $h \in Q$.*

Proof. Let $(Q,\cdot)$ be a π-quasigroup of type T_1, $h \in Q$ and let f_h be the local left unit of the element h: $f_h \cdot h = h$. Then $L_{f_h}(h) = h$, so $L_{f_h} \in I_h$. If $(Q,\cdot)$ has not a left unit, then $L_{f_h} \neq \varepsilon$ and, using the equality $L_{f_h}^3 = \varepsilon$, we get $\mid I_h \mid \equiv 0 \pmod 3$. $\square$

Proposition 3. *Let $(Q,\cdot)$ be a π-quasigroup of type T_1 and $h \in Q$. If $\mid I_h \mid \equiv 1$ or $2 \pmod 3$, then $(Q,\cdot)$ has a left unit.*

Proof. As it was remarked above, $L_{f_h} \in I_h$ and $L_{f_h}^3 = \varepsilon$, where f_h is the local left unit of the element h. If $\mid I_h \mid \equiv 1$ or $2 \pmod 3$, then every element of I_h has the order not divisible by three, so the order of the mapping L_{f_h} has the form $2k+1$ or $2k+2$. In both cases we get $L_{f_h} = \varepsilon$, which means that f_h is a left unit in $(Q,\cdot)$. $\square$

Proposition 4. *If a π-quasigroup $(Q,\cdot)$ of type T_1 has a left unit and is isotopic to an abelian group, then its left multiplication group $LM(Q,\cdot)$ is abelian.*

Proof. Let $(Q,\cdot)$ be a π-quasigroup of type T_1, with the left unit f. If $(Q,\cdot)$ is isotopic to an abelian group then, according to [3], the corresponding e-quasigroup $(Q,\cdot,\backslash,/)$ satisfies the identity

$$x\backslash(y(u\backslash v)) = u\backslash(y(x\backslash v)).$$

On the other hand, the equality $x\backslash y = x\cdot xy$ follows from (2), so the previous identity is equivalent to:

$$x \cdot x(y(u \cdot uv)) = u \cdot u(y(x \cdot xv)),$$

which, for $x = f$, implies

$$y(u \cdot uv) = u \cdot (u \cdot yv),$$

i.e. $L_y L_u^2 = L_u^2 L_y$. So as $L_u^2 = L_u^{-1}$, $\forall u \in Q$, we get $L_y L_u = L_u L_y$, for every $y, u \in Q$. □

Let $(Q, \cdot)$ be a quasigroup. Following [2,9], the left (resp. middle) nucleus of $(Q, \cdot)$ is the set $N_l = \{a \in Q \mid a \cdot xy = ax \cdot y, \forall x, y \in Q\}$ (resp. $N_m = \{a \in Q \mid xa \cdot y = x \cdot ay, \forall x, y \in Q\}$). A mapping $\lambda : Q \mapsto Q$ is called a left regular mapping of the quasigroup $(Q, \cdot)$ if $\lambda(x \cdot y) = \lambda(x) \cdot y$, $\forall x, y \in Q$.

Proposition 5. *Let $(Q, \cdot)$ be a π-quasigroup of type T_1. The following statements hold:*

1) if $(Q, \cdot)$ has a left unit, then $\lambda^3 = \varepsilon$, for every left regular mapping λ of $(Q, \cdot)$;

2) if $(Q, \cdot)$ is finite and its left nucleus N_l contains at least two elements, then $|Q| \equiv 0(mod\ 3)$;

3) if $(Q, \cdot)$ is a finite π-loop of type T_1 and its middle nucleus contains at least two elements, then $|Q| \equiv 0(mod\ 3)$.

Proof. 1. Let $(Q, \cdot)$ be a π-quasigroup of type T_1 with the left unit f and let λ be a left regular mapping of $(Q, \cdot)$. Taking $x \mapsto \lambda(x)$ in (2), get: $y = \lambda(x) \cdot (\lambda(x) \cdot (\lambda(x) \cdot y)) = \lambda(x \cdot \lambda(x \cdot \lambda(x \cdot y)))$, $\forall x, y \in Q$. Now, for $x = f$, from the last equalities follows: $\lambda^3(y) = y, \forall y \in Q$, i.e. $\lambda^3 = \varepsilon$.

2. Let $|N_l| \geq 2$ and $a \in N_l^{(\cdot)}$. Then $a \cdot xy = ax \cdot y, \forall x, y \in Q$. Using the identity (2), have: $a \cdot (a \cdot ay) = y \Rightarrow a \cdot (a^2 \cdot y) = y \Rightarrow (a \cdot a^2) \cdot y = y$, so $a \cdot a^2 = f$, where f is the left unit of $(Q, \cdot)$. So as $(N_l, \cdot)$ is a group, we get that $a^3 = e, \forall a \in N_l$. From $|N_l| \equiv 0(mod\ 3)$ and the fact that $|N_l|$ divides $|Q|$ follows $|Q| \equiv 0(mod\ 3)$.

3. Let $(Q, \cdot)$ be a finite π-loop of type T_1 with the unit f. If the middle nucleus N_m contains at least two elements, then there exists $a \in N_m \backslash \{f\}$ which satisfies the equality $x \cdot ay = xa \cdot y$, $\forall x, y \in Q$, hence $y = a(a \cdot ay) = a^2 \cdot ay = (a^2 \cdot a)y = a^3 \cdot y, \forall y \in Q \Rightarrow a^3 = f, \forall a \in N_m$ (N_m is a group), which implies $|N_m| \equiv 0(mod\ 3)$ and $|Q| \equiv 0(mod\ 3)$. □

Corollary. *If the group of left regular mappings of a finite π-quasigroup $(Q, \cdot)$ of type T_1 with a left unit has at least two elements, then $|Q| \equiv 0(mod\ 3)$.*

Proof. In this case the group of left regular mappings will contain at least one element of order three, so its order will be a multiple of three and then $|Q| \equiv 0(mod\ 3)$. □

A loop $(Q, \cdot)$ is called an LPA-loop (or a left power alternative loop) if, for $\forall m, n \in Z$ and $\forall x, y \in Q$, the following equality holds:

$$x^m \cdot x^n y = x^{m+n} y.$$

It is known that LPA-loops are power associative, i.e. each element of an LPA-loop generates an associative subloop [8]. For example, left Bol loops are LPA-loops.

Proposition 6. *An LPA-loop $(Q,\cdot)$ is a π-loop of type T_1 if and only if $x^3 = e, \forall x \in Q$, where e is the unit of $(Q,\cdot)$.*

Proof. Let $(Q,\cdot)$ be an LPA-loop with the unit e. If $(Q,\cdot)$ is a π-loop of type T_1, then $y = x \cdot (x \cdot xy) = x^3 \cdot y, \forall x, y \in Q$, so $x^3 = e$. Conversely, if $x^3 = e, \forall x \in Q$, then $x \cdot (x \cdot xy) = x^3 \cdot y = e \cdot y = y, \forall x, y \in Q$, i.e. $(Q,\cdot)$ is a π-loop of type T_1. □

Corollary 1. *A left Bol loop $(Q,\cdot)$ is a π-loop of type T_1 if and only if $x^3 = e, \forall x \in Q$, where e is the unit of $(Q,\cdot)$.*

Corollary 2. A group $(Q,\cdot)$ is a π-group of type T_1 if and only if $x^3 = e$, $\forall x \in Q$, where e is the unit of the group $(Q,\cdot)$.

Proposition 7. *Let $(Q,\cdot)$ be a π-quasigroup of type T_1 and let $(Q,\circ)$ be its isotope with the isotopy (α,β,γ). Then $(Q,\circ)$ is a π-quasigroup of type T_1 if and only if, for every $x, y \in Q$, the following equality holds:*

$$\gamma\beta^{-1}[x \cdot (x \cdot y)] = x \cdot \beta\gamma^{-1}(x \cdot \beta\gamma^{-1}y). \tag{3}$$

Proof. The isotope $(Q,\circ)$ is a π-quasigroup of type T_1 if and only if it satisfies the identity

$$x \circ (x \circ (x \circ y)) = y. \tag{4}$$

Using the isotopy $x \circ y = \gamma^{-1}(\alpha(x) \cdot \beta(y))$, the identity (4) gets the form:

$$\gamma^{-1}(\alpha(x) \cdot \beta\gamma^{-1}(\alpha(x) \cdot \beta\gamma^{-1}(\alpha(x) \cdot \beta(y))) = y.$$

Taking $x \mapsto \alpha^{-1}x$ and $y \mapsto \beta^{-1}y$ in the last identity, we have the equality:

$$\gamma^{-1}(x \cdot \beta\gamma^{-1}(x \cdot \beta\gamma^{-1}(x \cdot y))) = \beta^{-1}(y),$$

which is equivalent to

$$x \cdot \beta\gamma^{-1}(x \cdot \beta\gamma^{-1}(x \cdot y)) = \gamma\beta^{-1}(y).$$

So as $(Q,\cdot)$ satisfies (2), the last equality implies

$$\beta\gamma^{-1}(x \cdot \beta\gamma^{-1}(x \cdot y)) = x \cdot (x \cdot \gamma\beta^{-1}(y)),$$

hence

$$x \cdot \beta\gamma^{-1}(x \cdot y) = \gamma\beta^{-1}(x \cdot (x \cdot \gamma\beta^{-1}(y))), \tag{5}$$

for $\forall x, y \in Q$. Denoting $x \cdot y = z$ and using (2), have $y = x \cdot xz$, so (5) takes the form:

$$x \cdot \beta\gamma^{-1}(z) = \gamma\beta^{-1}(x \cdot (x \cdot \gamma\beta^{-1}(x \cdot xz))),$$

which implies $\beta\gamma^{-1}(x\cdot\beta\gamma^{-1}(z)) = x\cdot(x\cdot\gamma\beta^{-1}(x\cdot xz))$, hence $x\cdot(\beta\gamma^{-1}(x\cdot\beta\gamma^{-1}(z))) = \gamma\beta^{-1}(x\cdot xz)$, for $\forall x, z \in Q$.

Conversely, if the π-quasigroup $(Q,\cdot)$ of type T_1 satisfies (3) then, taking $y \mapsto \gamma\beta^{-1}(y)$, we have:

$$x\cdot\beta\gamma^{-1}(x\cdot y) = \gamma\beta^{-1}(x\cdot(x\cdot\gamma\beta^{-1}(y))),$$

which, for $y = \beta\gamma^{-1}(x\cdot y)$, implies

$$x\cdot\beta\gamma^{-1}(x\cdot\beta\gamma^{-1}(x\cdot y)) = \gamma\beta^{-1}(x\cdot(x\cdot xy)) = \gamma\beta^{-1}(y).$$

Using the isotopy $x\cdot y = \gamma(\alpha^{-1}(x)\circ\beta^{-1}(y))$, from the last equalities we get:

$$\gamma(\alpha^{-1}(x)\circ(\alpha^{-1}(x)\circ(\alpha^{-1}(x)\circ\beta^{-1}(y)))) = \gamma\beta^{-1}(y),$$

or, replacing $x \mapsto \alpha(x)$, $y \mapsto \beta(y)$ and using the fact that γ is a bijection, we obtain: $x\circ(x\circ(x\circ y)) = y$, i.e. $(Q,\circ)$ is a π-quasigroup of type T_1. $\square$

Corollary 1. *Let $(Q,\cdot)$ be a π-loop of type T_1. If the isotope $(Q,\circ)$, where $(\circ) = (\cdot)^{(\alpha,\beta,\varepsilon)}$, is a π-quasigroup of type T_1, then $\beta^3 = \varepsilon$.*

Proof. If $(Q,\circ)$ is a π-loop of type T_1, then $(Q,\cdot)$ satisfies the equality (3). Taking $x = e$ in (3), where e is the unit of the loop $(Q,\cdot)$, we get $\beta(y) = \beta^{-2}(y), \forall y \in Q$, so $\beta^3 = \varepsilon$. $\square$

Corollary 2. *The identity (2) is invariant under quasigroup isotopies with equal second and third components.*

Proof. Let $(Q,\cdot)$ be a quasigroup satisfying the identity (2) and let consider the isotope $(\circ) = (\cdot)^{(\alpha,\beta,\beta)}$. Using the equality $x\cdot y = \beta(\alpha^{-1}(x)\cdot\beta^{-1}(y))$, from (2) follows

$$\beta(\alpha^{-1}x\circ(\alpha^{-1}x\circ(\alpha^{-1}x\circ\beta^{-1}y))) = y,$$

$\forall x, y \in Q$, which, for $x \mapsto \alpha(x)$ and $y \mapsto \beta(y)$, implies:

$$x\circ(x\circ(x\circ y)) = y,$$

$\forall x, y \in Q$, so $(Q,\circ)$ is a π-quasigroup of type T_1. $\square$

Proposition 8. *The identity (2) is universal in a loop $(Q,\cdot)$ if and only if $(Q,\cdot)$ satisfies the identity:*

$$x\cdot b(b\cdot x(b(b\cdot xy))) = by. \tag{6}$$

Proof. Let $(Q,\cdot)$ be a loop with universal identity (2). Then $(Q,\cdot)$ and every its loop isotope satisfy (2). Let $a, b \in Q$ and $(\circ) = (\cdot)^{(R_a^{-1}, L_b^{-1}, \varepsilon)}$. According to Proposition 5, the loop $(Q,\cdot)$ satisfies (3):

$$L_b(x\cdot xy) = x\cdot L_b^{-1}(x\cdot L_b^{-1}(y)),$$

$\forall x, y \in Q$. Taking $y \mapsto x \cdot y$ and using (2), from the last identity follows

$$L_b(y) = x \cdot L_b^{-1}(x \cdot L_b^{-1}(x \cdot y)), \tag{7}$$

$\forall x, y \in Q$. So as the loop $(Q, \cdot)$ satisfies the equality $L_b^3 = \varepsilon$, for $\forall b \in Q$, (7) is equivalent to

$$L_b(y) = x \cdot L_b^2(x \cdot L_b^2(x \cdot y)),$$

$\forall x, y \in Q$. Conversely, if a loop $(Q, \cdot)$ satisfies the identity (6) then, taking $b = e$, where e is the unit of $(Q, \cdot)$, we get the identity (2), i.e. $(Q, \cdot)$ is a π-loop of type T_1. So as every loop isotope of a loop is isomorphic to an LP-isotope, we may consider only LP-isotopes of $(Q, \cdot)$. Let $a, b \in Q$ and $(\circ) = (\cdot)^{(R_a^{-1}, L_b^{-1}, \varepsilon)}$. Using (6) and the equality $L_b^3 = \varepsilon$, have: $x \cdot L_b^{-1}(x \cdot L_b^{-1}(x \cdot y)) = b \cdot y \Rightarrow R_a^{-1}x \cdot L_b^{-1}(R_a^{-1}x \cdot L_b^{-1}(R_a^{-1}x \cdot L_b^{-1}y)) = b \cdot L_b^{-1}y = y \Rightarrow x \circ (x \circ (x \circ y)) = y$, i. e. $(Q, \circ)$ is a π-loop of type T_1. $\square$

Example 1. The couple $(Z_3^3, \circ)$, where Z_3 is the field of residues modulo 3 and the operation $(\circ)$ is defined as follows:

$$(i, j, k) \circ (p, q, r) = (i + p, j + q, k + r + ijp),$$

$\forall (i, j, k), (p, q, r) \in Z_3^3$, is a non-associative loop for which the identity (2) is universal. Remark that the left nucleus of $(Z_3^3, \circ)$ is $N_l = \{(0, 0, 0), (0, 0, 1), (0, 0, 2)\}$.

Example 2. The loop $(Q, *)$, where $Q = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ and the operation "$*$" is given by the table below, is a π-loop of type T_1, for which the identity (2) is not universal.

$*$	0	1	2	3	4	5	6	7	8
0	2	0	1	4	6	8	3	5	7
1	0	1	2	3	4	5	6	7	8
2	3	2	4	5	1	0	7	8	6
3	4	3	5	6	8	7	1	2	0
4	8	4	6	0	7	2	5	1	3
5	7	5	8	1	2	3	0	6	4
6	5	6	7	2	0	4	8	3	1
7	1	7	3	8	5	6	4	0	2
8	6	8	0	7	3	1	2	4	5

T-Quasigroups are defined and partially studied in [7]. A quasigroup $(Q, \cdot)$ is called a T-quasigroup if there exists an abelian group $(Q, +)$, its automorphisms $\varphi, \psi \in Aut(Q, +)$, and an element $g \in Q$ such that, for every $x, y \in Q$, the following equality holds:

$$x \cdot y = \varphi(x) + \psi(y) + g.$$

The tuple $((Q, +), \varphi, \psi, g)$ is called a T-form and the group $(Q, +)$ is called a T-group of the T-quasigroup $(Q, \cdot)$.

Proposition 9. *A T-quasigroup $(Q,\cdot)$, with a T-form $T=((Q,+),\varphi,\psi,g)$, is a π-quasigroup of type T_1 if and only if $\psi^2+\psi+\varepsilon=\omega$, where $\omega: Q\to Q$, $\omega(x)=0$, $\forall x\in Q$, 0 is the neutral element of the group $(Q,+)$.*

Proof. Let $(Q,\cdot)$ be a T-quasigroup with a T-form $T=((Q,+),\varphi,\psi,g)$. Then

$$x\cdot y=\varphi(x)+\psi(y)+g, \tag{8}$$

$\forall x,y\in Q$. If $(Q,\cdot)$ is a π-quasigroup of type T_1, then if satisfies the identity (2). Using (8), the identity (2) takes the form:

$$\varphi(x)+\psi\varphi(x)+\psi^2\varphi(x)+\psi^3(y)+\psi^2(g)+\psi(g)+g=y, \tag{9}$$

$\forall x,y\in Q$. Taking $x=y=0$ in (9), where 0 is the neutral element of the group $(Q,+)$, we get:

$$\psi^2(g)+\psi(g)+g=0. \tag{10}$$

Also, taking $x=0$ in (9), we have $\psi^3(y)=y,\forall y\in Q$, i. e.

$$\psi^3=\varepsilon, \tag{11}$$

where $\varepsilon: Q\mapsto Q,\varepsilon(x)=x,\forall x\in Q$. Now, using (10) and (11), the equality (9) implies: $\varphi(x)+\psi\varphi(x)+\psi^2\varphi(x)=0$, hence $(\varepsilon+\psi+\psi^2)\varphi(x)=0$, $\forall x\in Q$. So as φ is a bijection, the last equality implies

$$\varepsilon+\psi+\psi^2=\omega. \tag{12}$$

Conversely, if the equality (12) holds, then

$$\psi^3-\varepsilon=(\psi-\varepsilon)(\varepsilon+\psi+\psi^2)=\omega,$$

hence (11) holds. Using (11) and (12), we get: $y=\omega(x)+\psi^3(y)+\omega(g)=(\varepsilon+\psi+\psi^2)\varphi(x)+\psi^3(y)+\psi^2(g)+\psi(g)+g=x\cdot(x\cdot xy)=y$, $\forall x,y\in Q$, so $(Q,\cdot)$ is a π-quasigroup of type T_1. □

The following example shows that the class of π-T-quasigroups of type T_1 is not empty.

Example 3. The quasigroup $(Z_7,\cdot)$, where

$$x\cdot y=\bar{5}x+\bar{2}y+\bar{3},$$

$\forall x,y\in Q$, is a π-T-quasigroup of type T_1 with the T-form $((Z_7,+),\varphi,\psi,\bar{3})$, where $\varphi(x)=\bar{5}x$, $\psi(x)=\bar{2}x$, $\forall x\in Z_7$.

Proposition 10. *If $(Q,\cdot)$ is a finite π -T-quasigroup of type T_1 with a left unit, then $|Q|\equiv 0(mod\ 3)$.*

Proof. Let $(Q,\cdot)$ be a finite π-T-quasigroup of type T_1 with a T-form $T=((Q,+),\varphi,\psi,g)$ and with the left unit f. Then,

$$x = f \cdot x = \varphi(f) + \psi(x) + g, \tag{13}$$

so, taking $x = 0$ where 0 is the neutral element of the T-group $(Q,+)$, we get $\varphi(f) = -g$. From the last equality and (13), we obtain $\psi = \varepsilon$, where ε is the identical mapping on Q. According to Proposition 9, if $(Q,\cdot)$ is a π-quasigroup of type T_1, then $\psi^2 + \psi + \varepsilon = \omega$, hence $3\varepsilon = \omega$, i.e. $x + x + x = 0$, $\forall x \in Q$, which implies $|Q| \equiv 0 (\mathrm{mod}\ 3)$. □

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Received November 25, 2013

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(75), 2014, Pages 44–50
ISSN 1024–7696

Irreducible triangulations of the Möbius band *

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Abstract. A complete list of irreducible triangulations is identified on the Möbius band.

Mathematics subject classification: 05C10 (Primary), 57M20, 57N05 (Secondary).
Keywords and phrases: Triangulation of surface, irreducible triangulation, Möbius band.

1 Introduction

Let $S \in \{S_g, N_k\}$ be the closed orientable surface S_g of genus g or the closed non-orientable surface N_k of non-orientable genus k. In particular, S_0 is the sphere and N_1 is the projective plane. Let D be an open disk in S and let $S - D$ denote S with D removed; therefore, the boundary $\partial(S - D)$ $(=\partial D)$ is homeomorphic to a circle. In particular, $S_0 - D$ is the disk and $N_1 - D$ is the Möbius band. We use the notation Σ whenever we assume the general case: $\Sigma \in \{S, S - D\}$.

If a graph G is 2-cell embedded in Σ, the components of $\Sigma - G$ are called *faces*. A *triangulation* of Σ with a simple graph G (without loops or multiple edges) is a 2-cell embedding $T : G \to \Sigma$ in which each face is bounded by a 3-cycle (that is, a cycle of length 3) of G and any two faces are either disjoint, share a single vertex, or share a single edge. We denote by $V = V(T)$, $E = E(T)$, and $F = F(T)$ the sets of vertices, edges, and faces of T, respectively. The cardinality $|V(T)|$ is called the *order* of T. By $G(T)$ we denote the graph $(V(T), E(T))$ of triangulation T. Two triangulations T_1 and T_2 are called *isomorphic* if there is a bijection, called an *isomorphism*, $\varphi : V(T_1) \to V(T_2)$ such that $uvw \in F(T_1)$ if and only if $\varphi(u)\varphi(v)\varphi(w) \in F(T_2)$. Throughout this paper we distinguish triangulations only up to isomorphism. For $\Sigma = S - D$, let ∂T $(= \partial D)$ denote the boundary cycle of T. The vertices and edges of ∂T are called *boundary vertices* and *boundary edges* of T.

A triangulation is called *irreducible* if no edge can be shrunk without producing multiple edges or changing the topological type of the underlying surface. The term "irreducible triangulation" is more accurately introduced in Section 2. The irreducible triangulations of Σ form a basis for the family of all triangulations of Σ, in the sense that any triangulation of Σ can be obtained from a member of

*This work has been partially supported by PAI FQM-164; PAI FQM-189; MTM 2010-20445. 2099

the basis by repeatedly applying the *splitting* operation (introduced in Section 2) a finite number of times. Barnette and Edelson [2] and independently Negami [9] have proved that for every closed surface S the basis of irreducible triangulations is finite. At present such bases are known for seven closed surfaces: the sphere (Steinitz and Rademacher [10]), projective plane (Barnette [1]), torus (Lawrencenko [6]), Klein bottle (25 Lawrencenko and Negami's [8] triangulations plus 4 more irreducible triangulations found later by Sulanke [12]) as well as S_2, N_3, and N_4 (Sulanke [13, 14]). Boulch, Colin de Verdière, and Nakamoto [3] have established upper bounds on the order of an irreducible triangulation of $S - D$. In this paper we obtain a complete list of irreducible triangulations of $N_1 - D$.

2 Preliminaries

Let T be a triangulation of Σ. An unordered pair of distinct adjacent edges vu and vw of T is called a *corner* of T at vertex v, denoted by $\langle u, v, w\rangle$. The *splitting* of a corner $\langle u, v, w\rangle$, denoted by $\mathrm{sp}\langle u, v, w\rangle$, is the operation which consists in cutting T open along the edges vu and vw and then closing the resulting hole with two new triangular faces, $v'v''u$ and $v'v''w$, where v' and v'' denote the two images of v appearing as a result of cutting. Under this operation, vertex v is extended to the edge $v'v''$ and the two faces having this edge in common are inserted into the triangulation. Especially in the case $\{\Sigma = S - D \wedge uv \in E(T) \wedge v \in V(\partial T)\}$, the operation $\mathrm{sp}\langle u, v]$ of *splitting a truncated corner* $\langle u, v]$ produces a single triangular face $uv'v''$, where $v'v'' \in E(\partial(\mathrm{sp}\langle u, v](T)))$.

Under the inverse operation, *shrinking* the edge $v'v''$, denoted by $\mathrm{sh}\rangle v'v''\langle$, this edge collapses to a single vertex v, the faces $v'v''u$ and $v'v''w$ collapse to the edges vu and vw, respectively. Therefore $\mathrm{sh}\rangle v'v''\langle \circ\, \mathrm{sp}\langle u, v, w\rangle(T) = T$. It should be noticed that in the case $\{\Sigma = S - D \wedge v'v'' \in E(\partial T)\}$, there is only one face incident with $v'v''$, and only that single face collapses to an edge under $\mathrm{sh}\rangle v'v''\langle$. Clearly, the operation of splitting doesn't change the topological type of Σ. We demand that the shrinking operation must preserve the topological type of Σ as well; moreover, multiple edges must not be created in a triangulation. A 3-cycle of T is called *nonfacial* if it doesn't bound a face of T. In the case in which an edge $e \in E(T)$ occurs in some nonfacial 3-cycle, if we still insist on shrinking e, multiple edges would be produced, which would expel $\mathrm{sh}\rangle e\langle(T)$ from the class of triangulations. An edge e is called *shrinkable*, or a *cable* if $\mathrm{sh}\rangle e\langle(T)$ is still a triangulation of Σ; otherwise the edge is called *unshrinkable*, or a *rod*. The subgraph of $G(T)$ made up of all cables is called the *cable-subgraph* of $G(T)$.

The only impediment to edge shrinkability in a triangulation T of a closed surface S is identified in [1, 2, 6]: an edge $e \in E(T)$ is a rod if and only if e satisfies the following condition:

(2.1) e is in a nonfacial 3-cycle of $G(T)$.

The impediments to edge shrinkability in a triangulation T of a punctured surface $S - D$ are identified in [3]: an edge $e \in E(T)$ is a rod if and only if e satisfies either condition (2.1) or the following condition:

(2.2) e is a *chord* of D — that is, the end vertices of e are in $V(\partial D)$ but $e \notin E(\partial D)$.

A triangulation is said to be *irreducible* if it is free of cables or in other words, each edge is a rod. For instance, a single triangle is the only irreducible triangulation of the disk $S_0 - D$ although its edges don't meet either of conditions (2.1) and (2.2). Thus, we have yet one more impediment to edge shrinkability:

(2.3) e is a boundary edge in the case the boundary cycle is a 3-cycle.

Although condition (2.3) is a specific case of condition (2.1) (unless $S = S_0$) and is not explicitly stated in [3], it deserves especial mention.

3 The structure of irreducible punctured triangulations

In the remainder of this paper we assume that $S \neq S_0$. Let T be an irreducible triangulation of $S - D$. Let us restore the disk D in T, add a vertex p in D and join p to the vertices in ∂D. We thus obtain a triangulation, T^*, of the closed surface S. In this setting we call D the *patch*, call p the *central vertex of the patch*, and say that T is obtained from the corresponding triangulation T^* of S by the *patch removal*. Notice that T^* may turn out to be an irreducible triangulation of S, but not necessarily.

A vertex of a triangulation R of S is called a *pylonic vertex* if that vertex is incident with all cables of R. A triangulation that has at least one cable and at least one pylonic vertex is called a *pylonic triangulation*. It should be noticed that there exist triangulations of the torus with exactly one cable, and thereby with two different pylonic vertices; however, if a pylonic triangulation R has at least two cables, R has a unique pylonic vertex.

Lemma 1. *If T^* has at least two cables, then the central vertex p of the patch is the only pylonic vertex of T^*.*

Proof. Using the assumption that T is irreducible and the fact that each cable of T^* fails to satisfy condition (2.1), it can be easily seen that in the case T^* is not irreducible, all cables of T^* have to be entirely in $D \cup \partial D$ and, moreover, there is no cable that is entirely in ∂D. In particular, we observe that any chord of D is a rod in T because it meets condition (2.2), and is also a rod in T^* because it meets condition (2.1). □

4 Irreducible triangulations of the Möbius band

Barnette's theorem [1] states that there exist two irreducible triangulations of N_1; those are presented in Figure 1: P_1 and P_2. (For each hexagon identify each antipodal pair of points in the boundary to obtain an actual triangulation of N_1.) By repeatedly applying the splitting operation to P_1 and P_2, we can generate all triangulations of N_1. Sulanke [11] has generated by computer all triangulations of N_1 with up to 19 vertices; in particular, among them there are 20 triangulations with up to 8 vertices. Independently, the authors of the present paper have identified the same list of 20 triangulations by hand (Figure 1), using the automorphisms of P_1 and P_2. An *automorphism* of a triangulation P is an isomorphism of P with itself. The set of all automorphisms of P forms a group, called the *automorphism group* of P (denoted $\mathrm{Aut}(P)$).

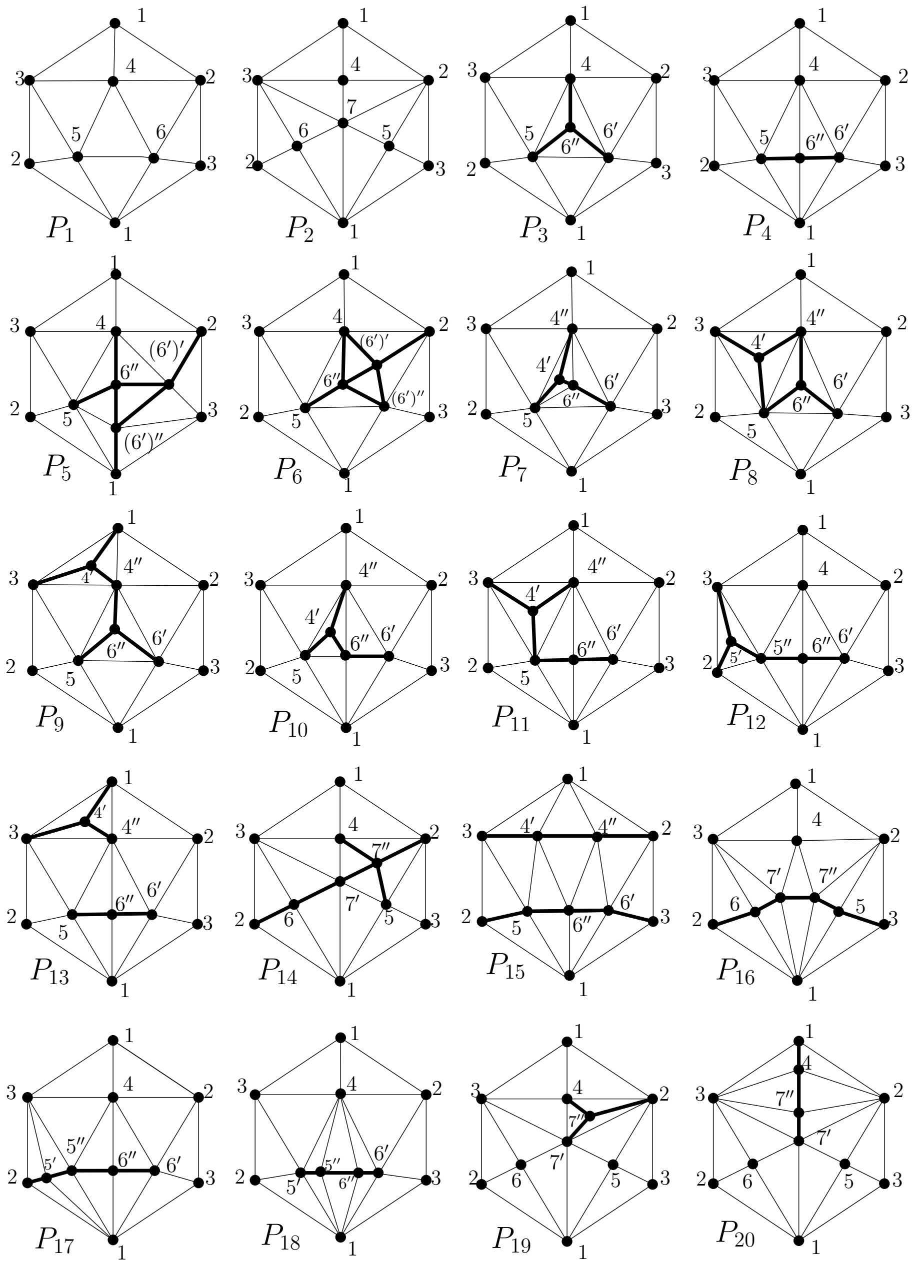

Figure 1. All projective plane triangulations with up to 8 vertices

Lemma 2 (see [11]). *There are precisely one (up to isomorphism) triangulation of N_1 with 6 vertices, three with 7 vertices, and sixteen with 8 vertices. They are shown in Figure 1, in which the bold edges indicate the cable-subgraphs of the triangulations.*

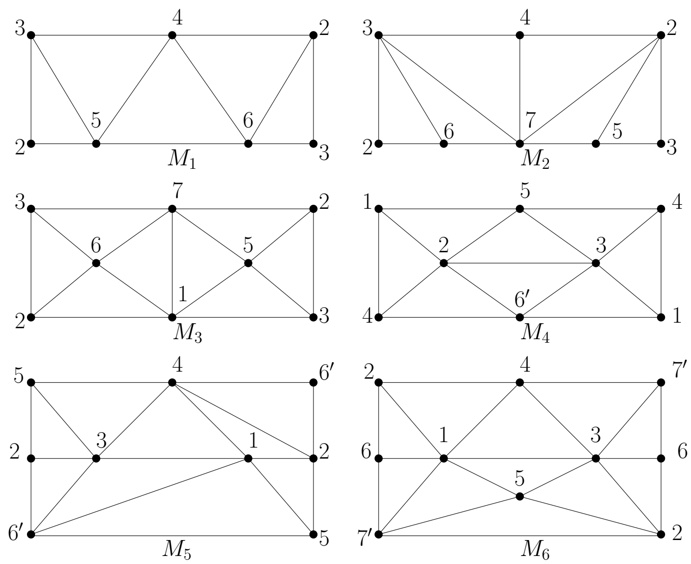

Figure 2. Irreducible triangulations of the Möbius band

Theorem 1. *There are precisely six non-isomorphic irreducible triangulations of the Möbius band, namely M_1 to M_6, shown in Figure 2 in which the left and right sides of each rectangle are identified with opposite orientation to obtain an actual triangulation of the Möbius band.*

Proof. Observe that in Figure 1 only the following three non-irreducible members have a pylonic vertex: P_3 and P_4 with pylonic vertex $6''$, and P_{19} with pylonic vertex $7''$. It can be easily proved that if a triangulation of N_1 has at least two cables but has no pylonic vertex, then no pylonic vertex can be created under further splitting of the triangulation. On the other hand, it can be easily seen that any one splitting applied to the pylonic triangulations P_3, P_4, or P_{19} destroys their pylonicity. Therefore, by Lemma 1, each irreducible triangulation of $N_1 - D$ is obtainable either by removing a vertex from an irreducible triangulation in $\{P_1, P_2\}$, or by removing the pylonic vertex from a pylonic triangulation in $\{P_3, P_4, P_{19}\}$. It is known [4,5,7] that $\mathrm{Aut}(P_1)$ acts transitively on the vertex set $V(P_1)$, while under the action of $\mathrm{Aut}(P_2)$ the set $V(P_2)$ breaks into two orbits as follows: $\mathrm{orbit}_1 = \{1,2,3,7\}$, $\mathrm{orbit}_2 = \{4,5,6\}$. Therefore, all irreducible triangulations of $N_1 - D$ are covered by the followings: $M_1 = P_1$ minus vertex 1 (subtracted with the incident edges and faces), $M_2 = P_2$ minus vertex 1, $M_3 = P_2$ minus vertex 4, $M_4 = P_4$ minus vertex $6''$, $M_5 = P_3$ minus vertex $6''$, $M_6 = P_{19}$ minus vertex $7''$. To see that these triangulations are pairwise non-isomorphic, observe that they have different vertex degree sequences except for the pair $\{M_3, M_4\}$; however, all boundary vertices have degree 5 in M_3 but not all in M_4. $\square$

Acknowledgements. The authors gratefully acknowledge support from PAI FQM-164, PAI FQM-189, and MTM 2010-20445. The authors also want to thank Esperanza María Suárez Sánchez for her feedback and valuable comments. The second author wishes to thank the University of Seville for its hospitality and inspiration while this paper was being written.

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Received November 25, 2013

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BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(75), 2014, Pages 51–59
ISSN 1024–7696

On 2-primal Ore extensions over Noetherian Weak σ-rigid rings

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Abstract. Let R be a ring, σ an endomorphism of R and δ a σ-derivation of R. In this article, we discuss skew polynomial rings over 2-primal weak σ-rigid rings. We show that if R is a 2-primal Noetherian weak σ-rigid ring, then $R[x;\sigma,\delta]$ is a 2-primal Noetherian weak $\overline{\sigma}$-rigid ring.

Mathematics subject classification: 16S36, 16N40, 16P40, 16S32, 16W20, 16W25.
Keywords and phrases: Minimal prime, 2-primal, prime radical, automorphism, derivation, weak σ-rigid rings.

1 Introduction

A ring R always means an associative ring with identity $1 \neq 0$. The fields of complex numbers and rational numbers are denoted by $\mathbb{C}$ and $\mathbb{Q}$ respectively. The set of prime ideals of R is denoted by $Spec(R)$. The set of minimal prime ideals of R is denoted by $Min.Spec(R)$. The prime radical and the set of nilpotent elements of R are denoted by $P(R)$ and $N(R)$, respectively.

Let R be a ring, σ an endomorphism of R and δ a σ-derivation of R, i. e. $\delta : R \to R$ is an additive mapping satisfying $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$. Recall that the skew polynomial ring $R[x;\sigma,\delta]$ is the set of polynomials

$$\{\sum_{i=0}^{n} x^i a_i \colon a_i \in R,\ n \in \mathbb{N}\}$$

with usual addition of polynomials and multiplication subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We denote $R[x;\sigma,\delta]$ by $O(R)$. If I is an ideal of R such that I is σ-stable (i. e. $\sigma(I) = I$) and is also δ-invariant (i. e. $\delta(I) \subseteq I$), then clearly $I[x;\sigma,\delta]$ is an ideal of $O(R)$, and we denote it as usual by $O(I)$. We note that $O(I) = I(O(R))$. This article concerns the study of skew polynomial rings (Ore extensions) in terms of 2-primal rings.

2-Primal Rings

Recall that a ring R is 2-primal if and only if $N(R) = P(R)$, i. e. if the prime radical is a completely semiprime. An ideal I of a ring R is called completely semiprime if $a^2 \in I$ implies $a \in I$. We note that a reduced ring (a ring with no non zero nilpotent elements) is 2-primal and so is a commutative ring. Also let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field. Then R is 2-primal.

2-Primal rings have been studied in recent years and are being treated by authors for different structures. In [10], Greg Marks discusses the 2-primal property of $R[x;\sigma,\delta]$, where R is a local ring, σ an automorphism of R and δ a σ-derivation of R. In Greg Marks [10], it has been shown that for a local ring R with a nilpotent maximal ideal, the Ore extension $R[x;\sigma,\delta]$ will or will not be 2-primal depending on the δ-stability of the maximal ideal of R. In the case where $R[x;\sigma,\delta]$ is 2-primal, it will satisfy an even stronger condition; in the case where $R[x;\sigma,\delta]$ is not 2-primal, it will fail to satisfy an even weaker condition. Minimal prime ideals of 2-primal rings have been discussed by Kim and Kwak in [7].

$\sigma(*)$-rings

Let R be a ring and σ an endomorphism of R. Then σ is said to be a rigid endomorphism if $a\sigma(a) = 0$ implies that $a = 0$, for $a \in R$, and R is said to be a σ-rigid ring (Krempa [8]).

For example let $R = \mathbb{C}$, and $\sigma : \mathbb{C} \to \mathbb{C}$ be the map defined by $\sigma(a+ib) = a-ib$, a, $b \in \mathbb{R}$. Then it can be seen that σ is a rigid endomorphism of R.

In Theorem 3.3 of [8], Krempa has proved the following:

Let R be a ring, σ *an endomorphism of* R and δ a σ-derivation of R. If σ is a monomorphism, then the skew polynomial ring $R[x;\sigma,\delta]$ is reduced if and only if R is reduced and σ is rigid. Under these conditions any minimal prime ideal (annihilator) of $R[x;\sigma;\delta]$ is of the form $P[x;\sigma;\delta]$ where P is a minimal prime ideal (annihilator) in R.

Definition 1 (see [9], Kwak)**.** Let R be a ring and σ an endomorphism of R. Then R is said to be a $\sigma(*)$-ring if $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$.

Example 1. Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field. Then $P(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Let $\sigma : R \to R$ be defined by $\sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. Then it can be seen that σ is an endomorphism of R and R is a $\sigma(*)$-ring.

Remark 1. A $\sigma(*)$-ring need not be a σ-rigid. For let $0 \neq a \in F$ in above example (Example 1). Then

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \sigma \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ but } \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Kwak in [9] establishes a relation between a 2-primal ring and a $\sigma(*)$-ring. The property is also extended to the skew polynomial ring $R[x;\sigma]$. It has been proved in Theorem 5 of [9] that if R is a 2-primal ring and σ is an automorphism of R, then

R is a $\sigma(*)$-ring if and only if $\sigma(P) = P$ for all $P \in Min.Spec(R)$. In Theorem 12 of [9] it has been proved that if R is a $\sigma(*)$-ring with $\sigma(P(R)) = P(R)$, then $R[x;\sigma]$ is 2-primal if and only if $P(R)[x;\sigma] = P(R[x;\sigma])$.

2 Preliminaries

We have the following:

Proposition 1. *Let R be a Noetherian ring and σ an automorphism of R. If R is a $\sigma(*)$-ring, then R is 2-primal.*

Proof. Let $a \in R$ be such that $a^2 \in P(R)$. Then $a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)\sigma^2(a) = a\sigma(a^2)\sigma^2(a) \in \sigma(P(R))$. Now R is Noetherian, so $\sigma(P(R)) = P(R)$. Therefore $a\sigma(a)\sigma(a\sigma(a)) \in P(R)$ which implies that $a\sigma(a) \in P(R)$ and so $a \in P(R)$. Hence R is 2-primal. □

The following example shows that a 2-primal ring need not be a $\sigma(*)$-ring:

Let $R = F[x]$ be the polynomial ring over a field F. Then R is an integral domain and so is 2-primal with $P(R) = 0$. Let $\sigma : R \to R$ be an endomorphism defined by $\sigma(f(x)) = f(0)$ for $f(x) \in F[x]$. Let $f(x) = xa$, $a \in F$. Then $f(x)\sigma(f(x)) = 0 \in P(R)$, but $f(x) \notin P(R)$.

Weak σ-rigid rings:

Definition 2 (see Ouyang [12])**.** Let R be a ring and σ an endomorphism of R. Then R is said to be a weak σ-rigid ring if $a\sigma(a) \in N(R)$ if and only if $a \in N(R)$ for $a \in R$.

Example 2 (see Example 2.1 of Ouyang [12])**.** Let σ be an endomorphism of a ring R such that R is a σ-rigid ring. Let

$$A = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \,\middle|\, \text{a, b, c, d} \in R \right\}$$

be a subring of $T_3(R)$, the ring of upper triangular matrices over R. Now σ can be extended to an endomorphism $\overline{\sigma}$ of A by $\overline{\sigma}((a_{ij})) = (\sigma(a_{ij}))$. Then it can be seen that A is a weak $\overline{\sigma}$-rigid ring.

Ouyang has proved in [12] that if σ is an endomorphism of a ring R, then R is σ-rigid if and only if R is weak σ-rigid and reduced.

Let R be a Noetherian ring and σ an automorphism of R. We now give a characterization for R to be a weak σ-rigid ring.

Theorem 1. *Let R be a commutative Noetherian ring. Let σ be an automorphism of R. Then R is a weak σ-rigid ring if and only if $N(R)$ is a completely semiprime ideal of R.*

Proof. R is commutative implies that $N(R)$ is an ideal of R. We show that $\sigma(N(R)) = N(R)$. We have $\sigma(N(R)) \subseteq N(R)$ as $\sigma(N(R))$ is a nilpotent ideal of R. Now for any $n \in N(R)$, there exists $a \in R$ such that $n = \sigma(a)$. So

$$I = \sigma^{-1}(N(R)) = \{a \in R \text{ such that } \sigma(a) = n \in N(R)\}$$

is an ideal of R. Now I is nilpotent, so $I \subseteq N(R)$, which implies that $N(R) \subseteq \sigma(N(R))$. Hence $\sigma(N(R)) = N(R)$.

Now let R be a weak σ-rigid ring. Let $a \in R$ be such that $a^2 \in N(R)$. Then

$$a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)\sigma^2(a) \in \sigma(N(R)) = N(R).$$

Therefore, $a\sigma(a) \in N(R)$ and hence $a \in N(R)$. So $N(R)$ is completely semiprime.

Conversely let $N(R)$ be completely semiprime. Let $a \in R$ be such that $a\sigma(a) \in N(R)$. Now $a\sigma(a)\sigma^{-1}(a\sigma(a)) \in N(R)$ implies that $a^2 \in N(R)$, and so $a \in N(R)$. Hence R is a weak σ-rigid ring. □

Completely prime ideals

Let R be a ring. Recall that an ideal $P \neq R$ is completely prime if R/P is a domain or equivalently if $ab \in P$ implies $a \in P$ or $b \in P$ for a, $b \in R$ (McCoy [11]). In commutative rings completely prime and prime have the same meaning. We also note that every completely prime ideal of a ring R is a prime ideal, but the converse need not be true.

We note that in a 2-primal ring R, for example a reduced ring, all minimal prime ideals are completely prime.

Regarding the relation between the completely prime ideals of a ring R and those of $O(R)$, the following result has been proved in Bhat [1]:

Theorem 2.4 of [1]. Let R be a ring, σ an automorphism of R and δ a σ-derivation of R. Then:

1. For any completely prime ideal P of R with $\delta(P) \subseteq P$ and $\sigma(P) = P$, $O(P)$ is a completely prime ideal of $O(R)$.

2. For any completely prime ideal U of $O(R)$, $U \cap R$ is a completely prime ideal of R.

The following result gives a characterization of a Notherian $\sigma(*)$-ring R, where σ is an automorphism of R.

Theorem 2 (see [2]). *Let R be a Noetherian ring and σ an automorphism of R. Then R is a $\sigma(*)$-ring if and only if for each minimal prime U of R, $\sigma(U) = U$ and U is a completely prime ideal of R.*

Proof. To make the paper self contained, we give a sketch of the proof.

Let R be a Noetherian ring such that for each minimal prime U of R, $\sigma(U) = U$ and U is a completely prime ideal of R. Let $a \in R$ be such that $a\sigma(a) \in P(R) = \cap_{i=1}^{n} U_i$, where U_i are the minimal primes of R. For each i, $a \in U_i$ or $\sigma(a) \in U_i$ and U_i is completely prime. Now $\sigma(a) \in U_i = \sigma(U_i)$ implies that $a \in U_i$. Therefore $a \in P(R)$. Hence R is a $\sigma(*)$-ring.

Conversely, suppose that R is a $\sigma(*)$-ring and let $U = U_1$ be a minimal prime ideal of R. Let $U_2, U_3, ..., U_n$ be the other minimal primes of R. Suppose that $\sigma(U) \neq U$. Then $\sigma(U)$ is also a minimal prime ideal of R. Renumber so that $\sigma(U) = U_n$. Let $a \in \cap_{i=1}^{n-1} U_i$. Then $\sigma(a) \in U_n$, and so $a\sigma(a) \in \cap_{i=1}^{n} U_i = P(R)$. Therefore $a \in P(R)$, and thus $\cap_{i=1}^{n-1} U_i \subseteq U_n$, which implies that $U_i \subseteq U_n$ for some $i \neq n$, which is impossible. Hence $\sigma(U) = U$.

Now suppose that $U = U_1$ is not completely prime. Then there exist $a, b \in R \setminus U$ with $ab \in U$. Let c be any element of $b(U_2 \cap U_3 \cap ... \cap U_n)a$. Then $c^2 \in \cap_{i=1}^{n} U_i = P(R)$. Now $c \in P(R)$ by Proposition 1 and, thus $b(U_2 \cap U_3 \cap ... \cap U_n)a \subseteq U$. Therefore $bR(U_2 \cap U_3 \cap ... \cap U_n)Ra \subseteq U$ and, as U is prime, $a \in U$, $U_i \subseteq U$ for some $i \neq 1$ or $b \in U$. None of these can occur, so U is completely prime. $\square$

From now onwards, we deal with σ-derivation δ and its higher orders, therefore, the ring R is also taken as an algebra over $\mathbb{Q}$.

Proposition 2. *Let R be a Noetherian $\sigma(*)$-ring which is also an algebra over $\mathbb{Q}$ and δ a σ-derivation of R such that $\delta(\sigma(a)) = \sigma(\delta(a))$, for all $a \in R$. Then $\delta(U) \subseteq U$ for all $U \in MinSpec(R)$.*

Proof. Let $U \in MinSpec(R)$. Then $\sigma(U) = U$ by Theorem 2. Consider the set

$$T = \{a \in U \mid \delta^k(a) \in U \text{ for all integers } k \geq 1\}.$$

First of all, we will show that T is an ideal of R. Let a, $b \in T$. Then $\delta^k(a) \in U$ and $\delta^k(b) \in U$ for all integers $k \geq 1$. Now $\delta^k(a - b) = \delta^k(a) - \delta^k(b) \in U$ for all $k \geq 1$. Therefore $a - b \in T$. Now let $a \in T$ and $r \in R$. We see that $\delta^k(ar) \in U$ and $\delta^k(ra) \in U$ for some $k \geq 1$ as both are sums of terms involving $\delta^j(a)$ for some $j \geq 1$. So T is a δ-invariant ideal of R.

We will now show that $T \in Spec(R)$. Suppose the contrary. Let $a \notin T$, $b \notin T$ be such that $aRb \subseteq T$. Let t, s be least positive integers such that $\delta^t(a) \notin U$ and $\delta^s(b) \notin U$. Now there exists $c \in R$ such that

$$\delta^t(a)c\sigma^t(\delta^s(b)) \notin U \tag{1}$$

as otherwise $\delta^t(a) \in U$ or $\delta^s(b) \in U$. Let $d = \sigma^{-t}(c)$. Now $aRb \subseteq T$ implies that $acb \subseteq T$. Therefore $\delta^{t+s}(adb) \in U$. This implies on simplification that

$$\delta^t(a)\sigma^t(d)\sigma^t(\delta^s(b)) + u \in U \tag{2}$$

where u is a sum of terms involving $\delta^l(a)$ or $\delta^m(b)$, where $l < t$ and $m < s$. Therefore by assumption $u \in U$ which implies that $\delta^t(a)\sigma^t(d)\sigma^t(\delta^s(b)) \in U$, i.e.

$\delta^t(a)c\sigma^t(\delta^s(b)) \in U$. This is a contradiction to 1. Therefore $T \in Spec(R)$. Now $T \subseteq U$, so $T = U$ as $U \in Min.Spec(R)$. Hence $\delta(U) \subseteq U$. $\square$

Remark 2. In above proposition the condition that $\delta(\sigma(a)) = \sigma(\delta(a))$, for all $a \in R$ is necessary. For example if $s = t = 1$, then $a \in U$, $b \in U$ and therefore, $\sigma^i(a) \in U$, $\sigma^i(b) \in U$ for all integers $i \geq 1$ as $\sigma(U) = U$. Now $\delta^2(adb) \in U$ implies that

$$\delta(a)\sigma(d)\delta(\sigma(b)) + \delta(a)\sigma(d)\sigma(\delta(b)) + u \in U.$$

where u is a sum of terms involving a or b, or $\sigma^i(b)$. Therefore by assumption $u \in U$. This implies that

$$\delta(a)\sigma(d)\delta(\sigma(b)) + \delta(a)\sigma(d)\sigma(\delta(b)) \in U.$$

If $\delta(\sigma(a)) \neq \sigma(\delta(a))$, for all $a \in R$, then we get nothing out of it and if $\delta(\sigma(a)) = \sigma(\delta(a))$, for all $a \in R$, we get $\delta(a)\sigma(d)\sigma(\delta(b)) \in U$ which gives a contradiction.

We now give a relation between a $\sigma(*)$-ring and a weak σ-rigid ring:

Proposition 3. *Let R be a Noetherian ring and σ an automorphism of R. Then*

1. *R is a $\sigma(*)$-ring implies that R is a weak σ-rigid ring.*
2. *R is a 2-primal weak σ-rigid ring implies that R is a $\sigma(*)$-ring.*

Proof. **1**. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring. Now Proposition 1 implies that R is 2-primal, i.e. $N(R) = P(R)$. Thus $a\sigma(a) \in N(R) = P(R)$ implies that $a \in P(R) = N(R)$. Hence R is a weak σ-rigid ring.

2. Let R be 2-primal weak σ-rigid ring. Then $N(R) = P(R)$ and $a\sigma(a) \in N(R)$ implies that $a \in N(R)$. Therefore, $a\sigma(a) \in P(R)$ implies that $a \in P(R)$. Hence R is a $\sigma(*)$-ring. $\square$

Corollary 1. *Let R be a Noetherian ring. Let σ be an automorphism of R. Then R is a 2-primal weak σ-rigid ring if and only if for each minimal prime U of R, $\sigma(U) = U$ and U is a completely prime ideal of R.*

Proof. Combine Theorem 2 and Proposition 3. $\square$

3 Skew polynomial rings over 2-primal weak σ-rigid rings

Proposition 4. *Let R be a Noetherian ring which is also an algebra over $\mathbb{Q}$ and σ an automorphism of R such that R is a $\sigma(*)$-ring. Let δ be a σ-derivation of R such that $\delta(\sigma(a)) = \sigma(\delta(a))$ for all $a \in R$. If $U \in Min.Spec(R)$, then $U(O(R)) = U[x;\sigma,\delta]$ is a completely prime ideal of $O(R) = R[x;\sigma,\delta]$.*

Proof. Let $U \in Min.Spec(R)$. Then $\sigma(U) = U$ by Theorem 2 and $\delta(U) \subseteq U$ by Proposition 2. Now R is 2-primal by Proposition 1 and furthermore U is completely prime by Theorem 2. Now consider canonical maps $\overline{\sigma}$ and $\overline{\delta}$ between R/U associated to σ and δ. It is well known that $O(R)/U(O(R)) \simeq (R/U)[x;\overline{\sigma},\overline{\delta}]$ and hence $U(O(R))$ is a completely prime ideal of $O(R)$. $\square$

Theorem 3. *Let R be a Noetherian ring which is also an algebra over $\mathbb{Q}$ and σ an automorphism of R such that R is a $\sigma(*)$-ring. Let δ be a σ-derivation of R such that $\delta(\sigma(a)) = \sigma(\delta(a))$ for all $a \in R$. If $P_1 \in Min.Spec(R)$, then $O(P_1) \in Min.Spec(O(R))$.*

Proof. Let $P_1 \in Min.Spec(R)$. Now by Theorem 2 $\sigma(P_1) = P_1$, and by Proposition 2 $\delta(P_1) \subseteq P_1$. Now Proposition 3.3 of [5] implies that $O(P_1) \in Spec(O(R))$. Suppose $O(P_1) \notin Min.Spec(O(R))$ and $P_2 \subset O(P_1)$ be a minimal prime ideal of $O(R)$. Then

$$P_2 = O(P_2 \cap R) \subset O(P_1) \in Min.Spec(O(R)).$$

Therefore $P_2 \cap R \subset P_1$ which is a contradiction, as $P_2 \cap R \in Spec(R)$. Hence $O(P_1) \in Min.Spec(O(R))$. □

Theorem 4 (see [3])**.** *Let R be a Noetherian ring which is also an algebra over $\mathbb{Q}$ and σ an automorphism of R such that R is a $\sigma(*)$-ring. Let δ be a σ-derivation of R such that $\delta(\sigma(a)) = \sigma(\delta(a))$ for all $a \in R$. Then $R[x;\sigma,\delta]$ is 2-primal if and only if $P(R)[x;\sigma,\delta] = P(R[x;\sigma,\delta])$.*

Proof. Let $R[x;\sigma,\delta]$ be 2-primal. Now Theorem 3 implies that $P(R[x;\sigma,\delta]) \subseteq P(R)[x;\sigma,\delta]$. Let

$$f(x) = \textstyle\sum_{j=0}^{n} x^j a_j \in P(R)[x;\sigma,\delta].$$

Now R is a 2-primal subring of $R[x;\sigma,\delta]$ by Proposition 1, which implies that a_j is nilpotent and thus

$$a_j \in N(R[x;\sigma,\delta]) = P(R[x;\sigma,\delta]).$$

So we have $x^j a_j \in P(R[x;\sigma,\delta])$ for each j, $0 \le j \le n$, which implies that $f(x) \in P(R[x;\sigma,\delta])$. Hence $P(R)[x;\sigma,\delta] = P(R[x;\sigma,\delta])$.

Conversely suppose that $P(R)[x;\sigma,\delta] = P(R[x;\sigma,\delta])$. We will show that $R[x;\sigma,\delta]$ is 2-primal. Let

$$g(x) = \textstyle\sum_{i=0}^{n} x^i b_i \in R[x;\sigma,\delta],\ b_n \neq 0$$

be such that

$$(g(x))^2 \in P(R[x;\sigma,\delta]) = P(R)[x;\sigma,\delta].$$

We will show that $g(x) \in P(R[x;\sigma,\delta])$. Now the leading coefficient $\sigma^{2n-1}(b_n)b_n \in P(R) \subseteq P$, for all $P \in Min.Spec(R)$. Also $\sigma(P) = P$ and P is completely prime by Theorem 3. Therefore we have $b_n \in P$, for all $P \in Min.Spec(R)$, i. e. $b_n \in P(R)$. Since $\delta(P) \subseteq P$ for all $P \in Min.Spec(R)$ by Proposition 2, we get

$$(\textstyle\sum_{i=0}^{n-1} x^i b_i)^2 \in P(R[x;\sigma,\delta]) = P(R)[x;\sigma,\delta]$$

and as above we get $b_{n-1} \in P(R)$. With the same process in a finite number of steps we get $b_i \in P(R)$ for all i, $0 \le i \le n$. Thus we have $g(x) \in P(R)[x;\sigma,\delta]$, i.e. $g(x) \in P(R[x;\sigma,\delta])$. Therefore, $P(R[x;\sigma,\delta])$ is completely semiprime. Hence $R[x;\sigma,\delta]$ is 2-primal. □

Proposition 5. *Let R be a 2-primal Noetherian ring which is also an algebra over $\mathbb{Q}$ and σ an automorphism of R such that R be a $\sigma(*)$-ring. Let δ a σ-derivation of R such that $\delta(\sigma(a)) = \sigma(\delta(a))$ for all $a \in R$. Then $O(N(R)) = N(O(R))$.*

Proof. The proof is on the same lines as in Proposition 5 of [2]. We take R to be 2-primal in place of commutative.

It is easy to see that $O(N(R)) \subseteq N(O(R))$. We will show that $N(O(R)) \subseteq O(N(R))$. Let

$$f = \textstyle\sum_{i=0}^{m} x^i a_i \in N(O(R)).$$

Then $(f)(O(R)) \subseteq N(O(R))$, and $(f)(R) \subseteq N(O(R))$. Let $((f)(R))^k = 0$, $k > 0$. Then equating the leading term to zero, we get

$$(x^m a_m R)^k = 0.$$

After simplification and equating the leading term to zero, we get

$$x^{km}\sigma^{(k-1)m}(a_m R).\sigma^{(k-2)m}(a_m R).\sigma^{(k-3)m}(a_m R)...a_m R = 0.$$

Therefore,

$$\sigma^{(k-1)m}(a_m R).\sigma^{(k-2)m}(a_m R).\sigma^{(k-3)m}(a_m R)...a_m R = 0 \subseteq P,$$

for all $P \in Min.Spec(R)$. This implies that $\sigma^{(k-j)m}(a_m R) \subseteq P$, for some j, $1 \leq j \leq k$. Therefore, $a_m R \subseteq \sigma^{-(k-j)m}(P)$. But $\sigma^{-(k-j)m}(P) = P$ by Theorem 2, so we have $a_m R \subseteq P$, for all $P \in Min.Spec(R)$. Therefore, $a_m \in P(R)$, and R being 2-primal implies that $a_m \in N(R)$. Now $x^m a_m \in O(N(R)) \subseteq N(O(R))$ implies that $\sum_{i=0}^{m-1} x^i a_i \in N(O(R))$, and with the same process, in a finite number of steps, it can be seen that $a_i \in P(R) = N(R)$, $0 \leq i \leq m-1$. Therefore, $f \in O(N(R))$. Hence $N(O(R)) \subseteq O(N(R))$ and the result follows. □

Let σ be an endomorphism of a ring R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then σ can be extended to an endomorphism (say $\overline{\sigma}$) of $R[x;\sigma,\delta]$ by $\overline{\sigma}(\sum_{i=0}^{m} x^i a_i) = \sum_{i=0}^{m} x^i \sigma(a_i)$. Also δ can be extended to a $\overline{\sigma}$-derivation (say $\overline{\delta}$) of $R[x;\sigma,\delta]$ by $\overline{\delta}(\sum_{i=0}^{m} x^i a_i) = \sum_{i=0}^{m} x^i \delta(a_i)$.

We note that if $\sigma(\delta(a)) \neq \delta(\sigma(a))$ for all $a \in R$, then the above does not hold. For example let $f(x) = xa$ and $g(x) = xb$, $a, b \in R$. Then

$$\overline{\delta}(f(x)g(x)) = x^2\{\delta(\sigma(a))\sigma(b) + \sigma(a)\delta(b)\} + x\{\delta^2(a)\sigma(b) + \delta(a)\sigma(b)\},$$

but

$$\overline{\delta}(f(x))\overline{\sigma}(g(x)) + f(x)\overline{\delta}(g(x)) = x^2\{\sigma(\delta(a))\sigma(b) + \sigma(a)\delta(b)\} + x\{\delta^2(a)\sigma(b) + \delta(a)\sigma(b)\}.$$

Theorem 5. *Let R be a 2-primal Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a weak σ-rigid ring and δ a σ-derivation of R such that $\delta(\sigma(a)) = \sigma(\delta(a))$ for all $a \in R$. Then $O(R) = R[x;\sigma,\delta]$ is a 2-primal Noetherian weak $\overline{\sigma}$-rigid ring.*

Proof. $O(R)$ is Noetherian by the Hilbert Basis Theorem (see for example, Theorem 1.12 of Goodearl and Warfield [6]). Now R being 2-primal weak σ-rigid ring implies that R is a $\sigma(*)$-ring by Proposition 3. Now by Theorem 1.3 of [4] $P \in Min.Spec(O(R))$ implies that $P \cap R \in Min.Spec(R)$. Now use Theorem 3 to get that $P(R)[x;\sigma,\delta] = P(R[x;\sigma,\delta])$. Therefore, Theorem 4 implies that $O(R)$ is 2-primal. Also Theorem 7 of [2] implies that $O(R)$ is a weak $\overline{\sigma}$-rigid ring. Hence $O(R)$ is a 2-primal Noetherian weak $\overline{\sigma}$-rigid ring. $\square$

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Received November 25, 2013

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(75), 2014, Pages 60–73
ISSN 1024–7696

Max-Erlang and Min-Erlang power series distributions as two new families of lifetime distribution

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Abstract. The distribution of the minimum and maximum of a random number of independent, identically Erlang distributed random variables are studied. Some particular cases of such kind of lifetime distributions are discussed too.

Mathematics subject classification: 62K10, 60N05.
Keywords and phrases: Power series distribution, distribution of the maximum, distribution of the minimum, Erlang distribution.

1 Preliminary results

In this paper we present two new families of distribution, namely the Max-Erlang power series (MaxErlPS) distribution, respectively the Min-Erlang power series (MinErlPS) distribution. These are obtained by mixing the distribution of the maximum or the minimum of a fixed number of independent Erlang distributed random variables, where the combination is obtained by the techniques that have been treated by Adamidis and Loukas (1998, [1]) and more generally by Chahkandi and Ganjali (2009, [6]) or Baretto-Souza and Cribari (2009, [3]). Recently, the new distributions that model the reliability systems were obtained by exponential distribution with several discrete distributions (the families of the power series distributions). For example, the distribution of the minimum of a sample of random size with the exponential distribution was obtained. In this connection, the geometric distribution, the Poisson distribution and the logarithmic distribution were considered by Adamidis and Loukas (1998, [1]), Kus (2007, [10]), Tahmasbi and Rezaei (2008, [19]).

Then, the previous results have been generalized by Chahkandi and Ganjali (2009, [6]) using the compounding exponential distribution with the power series distribution, thus obtaining the exponential power series distribution (EPS) type. Later, Morais and Baretto-Souza (2011, [16]) replaced the exponential distribution with the Weibull power series distribution (WPS) of the minimum of a sequence of the independent and identically distributed random variables (i. i. d. r. v.) in a random number, studying the distribution of the strength of 1.5 cm glass fibers. The case of the maximum has been discussed and analyzed by Munteanu (2013, [17]), introducing the Max Weibull power series (MaxWPS) distribution that particularizes the complementary exponential geometric (CEG) distribution introduced by

Louzada, Roman and Cancho (2011, [14]), the complementary exponential Poisson (CEP) distribution introduced by Cancho, Louzada and Barriga (2011, [5]) and the complementary exponential logaritmic (CEL) distribution introduced by Flores, Borges, Cancho and Louzada (2013, [8]).

The geometric distribution which contains a power parameter was considered by Adamidis, Dimitrakopoulou and Loukas (2005, [2]) and later generalized by Silva, Baretto-Souza and Cordeiro (2010, [18]). The Poisson distribution being treated by Cancho, Louzada and Barriga (2011, [5]) and which then was generalized by Cordeiro, Rodriques and Castro (2011, [7]) as the COMPoisson distribution, this contains a parameter power, the power series distribution type is not considered because the maximum number will being one deterministic.

The methodology and techniques used in this article are shown in the paper of Leahu, Munteanu and Cataranciuc (2013, [12]), general framework illustrating the particular cases treated in the works of Adamidis and Loukas (1998, [1]), Kus (2007, [10]), Tahmasbi and Rezaei (2008, [19]), Leahu and Lupu (2010, [11]), Baretto-Souza, Morais and Cordeiro (2011, [4]), Morais and Baretto-Souza (2011, [16]), Cancho, Louzada and Barriga (2011, [5]), Louzada, Roman and Cancho (2011, [14]), Flores, Borges, Cancho and Louzada (2013, [8]).

Let ′ s consider r.v. Z such that $\mathbb{P}\left(Z \in \{1, 2, \ldots\}\right) = 1$.

Definition 1 (see [9]). We say that r. v. Z has a power series distribution if:

$$\mathbb{P}\left(Z = z\right) = \frac{a_z \Theta^z}{A(\Theta)},\ z = 1, 2, \ldots;\ \Theta \in (0, \tau), \tag{1}$$

where $a_1, a_2, \ldots$ are nonnegative real numbers, τ is a positive number bounded by the convergence radius of power series (*series function*) $A\left(\Theta\right) = \sum\limits_{z \geq 1} a_z \Theta^z$, $\forall\ \Theta \in (0, \tau)$, and Θ is *power parameter* of the distribution (Table 1).

PSD denotes the power series distribution functions families. If the r. v. Z has the distribution from relationship (1), then we write that $Z \in PSD$.

Table 1. The representative elements of the PSD families for various truncated distributions

Distribution	a_z	Θ	$A(\Theta)$	τ
$Binom^*(n, p)$	$\binom{n}{z}$	$\frac{p}{1-p}$	$(1+\Theta)^n - 1$	∞
$Poisson^*(\alpha)$	$\frac{1}{z!}$	α	$e^{\Theta} - 1$	∞
$Log(p)$	$\frac{1}{z}$	p	$-ln(1-\Theta)$	1
$Geom^*(p)$	1	$1-p$	$\frac{\Theta}{1-\Theta}$	1
$Pascal(k, p)$	$\binom{z-1}{k-1}$	$1-p$	$\left(\frac{\Theta}{1-\Theta}\right)^k$	1
$Bineg^*(k, p)$	$\binom{z+k-1}{z}$	p	$(1-\Theta)^{-k} - 1$	1

2 On the properties of the Max-Erlang and the Min-Erlang power series distributions

We consider that $X_i \sim Erlang(k, \lambda)$, $k \in \mathbb{N}$, $k \geq 1$, $\lambda > 0$, where $(X_i)_{i\geq 1}$ i. i. d. r. v. with the distribution function $F_{X_i}(x) \equiv F_{Erl}(x) = 1 - \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!} e^{-\lambda x}$, $x > 0$ and the pdf $f_{X_i}(x) \equiv f_{Erl}(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!}$, $x > 0$. We note that $U_{Erl} = \max\{X_1, X_2, \ldots, X_Z\}$ and $V_{Erl} = \min\{X_1, X_2, \ldots, X_Z\}$.

The results in this section are obtained using the general framework of the work [12], for which reason some proofs are not presented.

Proposition 1. *If r. v. $U_{Erl} = \max\{X_1, X_2, \ldots, X_Z\}$ and $V_{Erl} = \min\{X_1, X_2, \ldots, X_Z\}$, where $(X_i)_{i\geq 1}$ are nonnegative i. i. d. r. v., $X_i \sim Erlang(k, \lambda)$, $k \in \mathbb{N}$, $k \geq 1$, $\lambda > 0$ and $Z \in PSD$ with $\mathbb{P}(Z = z) = \frac{a_z \Theta^z}{A(\Theta)}$, $z = 1, 2, \ldots$; $\Theta \in (0, \tau)$, $\tau > 0$, r. v. $(X_i)_{i\geq 1}$ and Z being independent, then the distribution functions of the r. v. U_{Erl}, respectively V_{Erl} are the following:*

$$U_{Erl}(x) = \frac{A\left[\Theta\left(1 - e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)\right]}{A(\Theta)}, \ x > 0, \tag{2}$$

$$V_{Erl}(x) = 1 - \frac{A\left[\Theta e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right]}{A(\Theta)}, \ x > 0. \tag{3}$$

We denote a r. v. U_{Erl} following Max-Erlang power series (MaxErlPS) distribution with parameters k, λ and Θ by $U_{Erl} \sim \text{MaxErlPS}(k, \lambda, \Theta)$, respectively a r. v. V_{Erl} following Min-Erlang power series (MinErlPS) distribution with parameters k, λ and Θ by $V_{Erl} \sim \text{MinErlPS}(k, \lambda, \Theta)$.

The following results characterize the survival functions and the probability density functions (pdf) for the maximum, respectively minimum of a sequence of independent Erlang distributed random variables in a random number.

Consequence 1. *If r. v. $U_{Erl} \sim \text{MaxErlPS}(k, \lambda, \Theta)$ and $V_{Erl} \sim \text{MinErlPS}(k, \lambda, \Theta)$, then the survival functions of the r. v. U_{Erl}, respectively V_{Erl} are the following:*

$$S_{U_{Erl}}(x) = 1 - \frac{A\left[\Theta\left(1 - e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)\right]}{A(\Theta)}, \ x > 0, \tag{4}$$

$$S_{V_{Erl}}(x) = \frac{A\left[\Theta e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right]}{A(\Theta)}, \ x > 0. \tag{5}$$

Consequence 2. *The pdf′ s of the r. v. U_{Erl}, respectively V_{Erl} are the following:*

$$u_{Erl}(x) = \frac{\Theta\lambda^k x^{k-1} e^{-\lambda x} A'\left[\Theta\left(1 - e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)\right]}{A(\Theta)},\ x > 0 \tag{6}$$

and

$$v_{Erl}(x) = \frac{\Theta\lambda^k x^{k-1} e^{-\lambda x} A'\left[\Theta e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right]}{A(\Theta)},\ x > 0. \tag{7}$$

Proposition 2. *The hazard rates for the r. v. U_{Erl}, respectively V_{Erl} are characterized by the following relations:*

$$h_{U_{Erl}}(x) = \frac{u_{Erl}(x)}{1 - U_{Erl}(x)} = \frac{\Theta\lambda^k x^{k-1} e^{-\lambda x} A'\left[\Theta\left(1 - e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)\right]}{A(\Theta) - A\left[\Theta\left(1 - e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)\right]}$$

and

$$h_{V_{Erl}}(x) = \frac{v_{Erl}(x)}{1 - V_{Erl}(x)} = \frac{\Theta\lambda^k x^{k-1} e^{-\lambda x} A'\left[\Theta e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right]}{A\left[\Theta e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right]}.$$

The next result shows a characteristic of the MaxErlPS and MinErlPS distributions.

Proposition 3. *If $(X_i)_{i\geq 1}$ is a sequence of independent, identically Erlang distributed r. v., with parameters $\lambda > 0$, $k \in \{1, 2, \ldots\}$ and $Z \in PSD$ with $\mathbb{P}(Z = z) = \frac{a_z\Theta^z}{A(\Theta)}$, where $(a_z)_{z\geq 1}$ is a sequence of nonnegative real numbers, $A(\Theta) = \sum_{z\geq 1} a_z\Theta^z$, $\forall\ \Theta \in (0, \tau)$, then*

$$\lim_{\Theta\to 0^+} U_{Erl}(x) = \left[1 - \sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}e^{-\lambda x}\right]^m,\ x > 0,$$

considering $m = \min\{n \in \mathbb{N}^*,\ a_n > 0\}$.

Proof. By applying the l′ Hospital rule m-time, we have:

$$\begin{aligned}\lim_{\Theta\to 0^+} U_{Erl}(x) &= \lim_{\Theta\to 0^+}\frac{A^{(m)}\left[\Theta\left(1 - e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)\right]\cdot\left(1 - e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)^m}{A^{(m)}(\Theta)} \\ &= \frac{m!a_m\left(1 - e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)^m}{m!a_m} = \left(1 - e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)^m,\ x > 0\end{aligned}$$

and $m = \min\{n \in \mathbb{N}^*,\ a_n > 0\}$. ☐

Applying the same method of proof of Proposition 3, we obtain:

Proposition 4. *Under the conditions of the Proposition 3, when $\Theta \to 0^+$, we have that*

$$\lim_{\Theta \to 0^+} V_{Erl}(x) = 1 - \left[\sum_{i=0}^{k-1} \frac{(\lambda x)^i}{i!} e^{-\lambda x} \right]^l, \; x > 0,$$

where $l = \min\{n \in \mathbb{N}^*,\ a_n > 0\}$.

Consequence 3. *The r^{th} moments, $r \in \mathbb{N}$, $r \geq 1$ of the r.v. $U_{Erl} \sim \mathrm{MaxErlPS}(k, \lambda, \Theta)$ and $V_{Erl} \sim \mathrm{MinErlPS}(k, \lambda, \Theta)$ are given by*

$$\mathbb{E}U_{Erl}^r = \sum_{z \geq 1} \frac{a_z \Theta^z}{A(\Theta)} \mathbb{E}\left[\max\{X_1,\ X_2, \ldots, X_z\}\right]^r \tag{8}$$

and

$$\mathbb{E}V_{Erl}^r = \sum_{z \geq 1} \frac{a_z \Theta^z}{A(\Theta)} \mathbb{E}\left[\min\{X_1,\ X_2, \ldots, X_z\}\right]^r, \tag{9}$$

where pdf′ s $f_{\max\{X_1,\ X_2, \ldots, X_z\}}(x) = z f_{Erl}(x) \left[F_{Erl}(x)\right]^{z-1}$ and $f_{\min\{X_1,\ X_2, \ldots, X_z\}}(x) = z f_{Erl}(x) \left[1 - F_{Erl}(x)\right]^{z-1}$.

The distribution functions and pdf′ s of the r.v. $U_{Erl} \sim \mathrm{MaxErlPS}(k, \lambda, \Theta)$ for different combinations of the r.v. $Z \in PSD$ (Table 1), are the following:

- $Z \sim Binom^*(n, p)$; $U_{ErlB} \sim \mathrm{MaxErlB}(k, \lambda, n, p)$, $\lambda > 0$; $k, n \in \{1, 2 \ldots\}$; $p \in (0, 1)$:

$$U_{ErlB}(x) = \frac{\left(1 - p e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)^n - 1}{1 - (1-p)^n}, \; x > 0,$$

$$u_{ErlB}(x) = \frac{n p \lambda^k x^{k-1} e^{-\lambda x} \left(1 - p e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)^{n-1}}{1 - (1-p)^n}, \; x > 0.$$

- $Z \sim Poisson^*(\alpha)$; $U_{ErlP} \sim \mathrm{MaxErlP}(k, \lambda, \alpha)$, $\lambda, \alpha > 0$; $k \in \{1, 2 \ldots\}$:

$$U_{ErlP}(x) = \frac{e^{-\alpha e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}} - e^{-\alpha}}{1 - e^{-\alpha}}, \; x > 0,$$

$$u_{ErlP}(x) = \frac{\alpha \lambda^k x^{k-1} e^{-\lambda x - \alpha e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}}}{1 - e^{-\alpha}}, \; x > 0.$$

- $Z \sim Log(p)$; $U_{ErlLog} \sim \text{MaxErlLog}(k, \lambda, p),\ \lambda > 0;\ k \in \{1, 2 \ldots\};\ p \in (0,1)$:

$$U_{ErlLog}(x) = \ln\left[1 - p + p\, e^{-\lambda x} \sum_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right]^{-a},\ x > 0,$$

$$u_{ErlLog}(x) = \frac{ap\lambda^k x^{k-1} e^{-\lambda x}}{\left[1 - p + p\, e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right]},\ x > 0,$$

where $a = -1/\ln(1-p)$.

- $Z \sim Geom^*(p)$; $U_{ErlG} \sim \text{MaxErlG}(k, \lambda, p),\ \lambda > 0;\ k \in \{1, 2 \ldots\};\ p \in (0,1)$:

$$U_{ErlG}(x) = \frac{p\left(1 - e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)}{p + (1-p)e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}},\ x > 0,$$

$$u_{ErlG}(x) = \frac{p\lambda^k x^{k-1} e^{-\lambda x}}{\left[p + (1-p)e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right]^2},\ x > 0.$$

- $Z \sim Pascal(k^\star, p)$; $U_{ErlPas} \sim \text{MaxErlPas}(k, \lambda, k^\star, p),\ \lambda > 0;\ k,\ k^\star \in \{1, 2 \ldots\};$ $p \in (0,1)$:

$$U_{ErlPas}(x) = \left[\frac{p\left(1 - e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)}{p + (1-p)e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}}\right]^{k^\star},\ x > 0,$$

$$u_{ErlPas}(x) = \frac{k^\star p^{k^\star} \lambda^k x^{k-1} e^{-\lambda x} \left(1 - e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)^{k^\star - 1}}{\left[p + (1-p)e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right]^{k^\star + 1}},\ x > 0.$$

- $Z \sim Bineg^*(k^\star, p)$; $U_{ErlBineg} \sim \text{MaxErlBineg}(k, \lambda, k^\star, p),\ \lambda > 0;\ k,\ k^\star \in \{1, 2 \ldots\};\ p \in (0,1)$:

$$U_{ErlBineg}(x) = \frac{\left(1 - p + pe^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)^{-k^\star} - 1}{(1-p)^{-k^\star} - 1},\ x > 0,$$

$$u_{ErlBineg}(x) = \frac{k^\star p \lambda^k x^{k-1} e^{-\lambda x} \left(1 - p + pe^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)^{-k^\star - 1}}{(1-p)^{-k^\star} - 1},\ x > 0.$$

The above results shows that the following result is valid:

Proposition 5. *If* $(X_i)_{i\geq 1}$, $X_i \sim Erlang(k,\lambda)$, $k \in \{1,2,\ldots\}$, $\lambda > 0$ *and* $(Y_j)_{j\geq 1}$ *are i. i. d. r. v.,* $Y_j \sim MaxErlG(k,\lambda,p)$, $p \in (0,1)$, *then the r. v.* $\max\{Y_1, Y_2, \ldots, Y_{k^\star}\}$ *has the same distribution as the r.v.* $\max\{X_1, X_2, \ldots, X_Z\}$, *where* $Z \sim Pascal(k^\star, p)$, $k^\star \in \{1,2,\ldots\}$, $p \in (0,1)$.

Proof. Indeed, it is known that if $(Y_j)_{j\geq 1}$ are independent r.v. Max-Erlang-Geometric (MaxErlG) distributed, with the distribution function $U_{ErlG}(x)$, $x > 0$, we have:

$$F_{\max\{Y_1,Y_2,\ldots,Y_{k^\star}\}}(x) = \left(U_{ErlG}(x)\right)^{k^\star} = U_{ErlPas}(x),\ \forall\, x > 0,$$

where $U_{ErlPas}(x)$ represents the distribution function of the Max-Erlang-Pascal (MaxErlPas) distribution. ☐

The distribution functions and pdf $'$ s of the r. v. $V_{Erl} \sim$ MinErlPS(k,λ,Θ) for different combinations of the r. v. $Z \in PSD$ (Table 1), are the following:

- $Z \sim Binom^*(n,p)$; $V_{ErlB} \sim$ MinErlB(k,λ,n,p) $\lambda > 0$; $k,n \in \{1,2\ldots\}$; $p \in (0,1)$:

$$V_{ErlB}(x) = \frac{1 - \left(1 - p + pe^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)^n}{1-(1-p)^n},\ x > 0,$$

$$v_{ErlB}(x) = \frac{np\lambda^k x^{k-1}e^{-\lambda x}\left(1 - p + pe^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)^{n-1}}{1-(1-p)^n},\ x > 0.$$

- $Z \sim Poisson^*(\alpha)$; $V_{ErlP} \sim$ MinErlP(k,λ,α), $\lambda,\alpha > 0$; $k \in \{1,2\ldots\}$:

$$V_{ErlP}(x) = \frac{1 - e^{-\alpha\left(1 - e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)}}{1-e^{-\alpha}},\ x > 0,$$

$$v_{ErlP}(x) = \frac{\alpha\lambda^k x^{k-1}e^{-\lambda x - \alpha + \alpha e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}}}{1-e^{-\alpha}},\ x > 0.$$

- $Z \sim Log(p)$; $V_{ErlLog} \sim$ MinErlLog(k,λ,p), $\lambda > 0$; $k \in \{1,2\ldots\}$; $p \in (0,1)$:

$$V_{ErlLog}(x) = 1 + \ln\left[1 - p\, e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right]^a,\ x > 0,$$

$$v_{ErlLog}(x) = \frac{ap\lambda^k x^{k-1}e^{-\lambda x}}{\left[1 - p\, e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right]},\ x > 0,$$

  where $a = -1/\ln(1-p)$.

- $Z \sim Geom^*(p)$; $V_{ErlG} \sim \mathrm{MinErlG}(k,\lambda,p),\ \lambda>0;\ k \in \{1,2\ldots\};\ p \in (0,1)$:

$$V_{ErlG}(x) = \frac{1 - e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}}{1-(1-p)e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}},\ x>0,$$

$$v_{ErlG}(x) = \frac{p\lambda^k x^{k-1}e^{-\lambda x}}{\left[1-(1-p)e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right]^2},\ x>0.$$

- $Z \sim Pascal(k^\star,p)$; $V_{ErlPas} \sim \mathrm{MinErlPas}(k,\lambda,k^\star,p)\ \lambda>0;\ k,\ k^\star \in \{1,2\ldots\};$ $p \in (0,1)$:

$$V_{ErlPas}(x) = \left[\frac{1 - e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}}{1-(1-p)e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}}\right]^{k^\star},\ x>0,$$

$$v_{ErlPas}(x) = \frac{k^\star p^{k^\star}\lambda^k x^{k-1}e^{-k^\star\lambda x}\left(\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)^{k^\star-1}}{\left[1-(1-p)e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right]^{k^\star+1}},\ x>0.$$

- $Z \sim Bineg^*(k^\star,p)$; $V_{ErlBineg} \sim \mathrm{MinErlBineg}(k,\lambda,k^\star,p),\ \lambda>0;\ k,\ k^\star \in \{1,2\ldots\};\ p\in(0,1)$:

$$V_{ErlBineg}(x) = \frac{(1-p)^{-k^\star} - \left(1-pe^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)^{-k^\star}}{(1-p)^{-k^\star}-1},\ x>0,$$

$$v_{ErlBineg}(x) = \frac{k^\star p\lambda^k x^{k-1}e^{-\lambda x}\left(1-pe^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)^{-k^\star-1}}{(1-p)^{-k^\star}-1},\ x>0.$$

Figure 1 shows several representations of pdf ′ s of some particular MaxErlPS distributions ($\mathrm{MaxErlB}(k,\lambda,n,p)$, $\mathrm{MaxErlP}(k,\lambda,\alpha)$), for different values of their parameters: $k=2,\lambda=3.5,\alpha=7,n=21,p=1/4$.

Figure 2 shows the behavior of the pdf ′ s of the $\mathrm{MinErlB}(k,\lambda,n,p)$, $\mathrm{MinErlP}(k,\lambda,\alpha)$ for some values of the parameters: $k=2,\lambda=0.5,\alpha=5,n=25,p=1/6$.

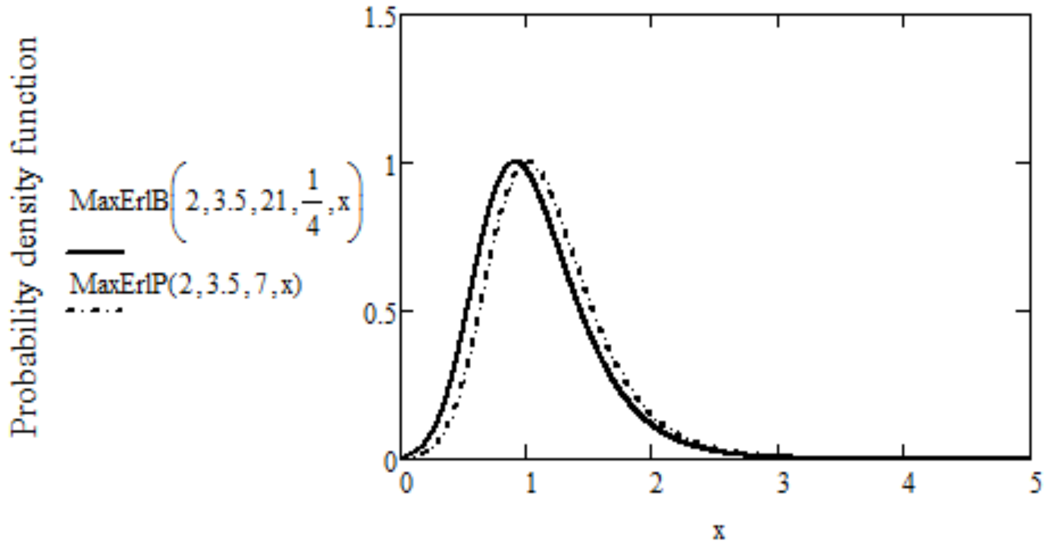

Figure 1. Pdf ′ s for Max-Erlang-Binomial and Max-Erlang-Poisson distributions

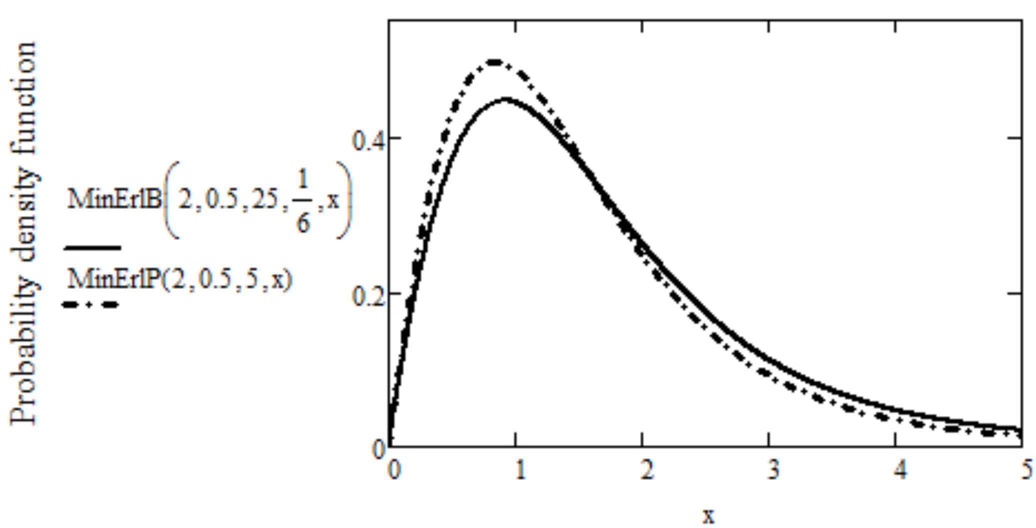

Figure 2. Pdf ′ s for Min-Erlang-Binomial and Min-Erlang-Poisson distributions

3 Special cases

In this section, we shall illustrate the characteristics of four distributions: the Max-Erlang-Binomial (MaxErB) distribution, the Min-Erlang-Binomial (MinErlB) distribution, the Max-Erlang-Poisson (MaxErlP) distribution, respectively the Min-Erlang-Poisson (MinErlP) distribution, so that later we can formulate a Poisson limit theorem.

3.1 The MaxErlB and MinErlB distributions

The MaxErlB and MinErlB distributions is defined by the distribution functions (2) and (3), with $A(\Theta) = (\Theta + 1)^n - 1$, namely:

$$U_{ErlB}(x) = \frac{\left(1 + \Theta - \Theta e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)^n - 1}{(1+\Theta)^n - 1}, \ x > 0 \tag{10}$$

and

$$V_{ErlB}(x) = \frac{(1+\Theta)^n - \left(1 + \Theta e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)^n}{(1+\Theta)^n - 1}, \ x > 0, \tag{11}$$

where n is integer pozitive.

The survival functions, defined by the relationships (4), (5), for the r.v. U_{ErlB}, respectively V_{ErlB} are the following:

$$S_{U_{ErlB}}(x) = \frac{(1+\Theta)^n - \left(1 + \Theta - \Theta e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)^n}{(1+\Theta)^n - 1}, \ x > 0$$

and

$$S_{V_{ErlB}}(x) = \frac{\left(1 + \Theta e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)^n - 1}{(1+\Theta)^n - 1}, \ x > 0.$$

By using the relationships (6), (7) and Proposition 2, the pdf $'$ s and hazard rates are given by:

$$u_{ErlB}(x) = \frac{n\Theta\lambda^k x^{k-1} e^{-\lambda x} \left(1 + \Theta - \Theta e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)^{n-1}}{(1+\Theta)^n - 1}, \ x > 0,$$

$$v_{ErlB}(x) = \frac{n\Theta\lambda^k x^{k-1} e^{-\lambda x} \left(1 + \Theta e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}\right)^{n-1}}{(1+\Theta)^n - 1}, \ x > 0$$

and

$$h_{U_{ErlB}}(x)=\frac{n\Theta\lambda^k x^{k-1}e^{-\lambda x}\left(1+\Theta-\Theta e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)^{n-1}}{(1+\Theta)^n-\left(1+\Theta-\Theta e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)^{n}},\ x>0,$$

respectively

$$h_{V_{ErlB}}(x)=\frac{n\Theta\lambda^k x^{k-1}e^{-\lambda x}\left(1+\Theta e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)^{n-1}}{\left(1+\Theta e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)^{n}},\ x>0.$$

3.2 The MaxErlP and MinErlP distributions

The MaxErlP and MinErlP distributions is defined by the distribution functions (2) and (3) with $A(\Theta)=e^{\Theta}-1$, and $\Theta=\alpha>0$, namely:

$$U_{ErlP}(x)=\frac{e^{-\Theta e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}}-e^{-\Theta}}{1-e^{-\Theta}},\ x>0 \tag{12}$$

and

$$V_{ErlP}(x)=\frac{1-e^{-\Theta\left(1-e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)}}{1-e^{-\Theta}},\ x>0. \tag{13}$$

By using Consequences 1, the survival functions for the r. v. U_{ErlP}, respectively V_{ErlP} are the following:

$$S_{U_{ErlP}}(x)=\frac{1-e^{-\Theta e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}}}{1-e^{-\Theta}},\ x>0$$

and

$$S_{V_{ErlP}}(x)=\frac{e^{-\Theta\left(1-e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)}-e^{-\Theta}}{1-e^{-\Theta}},\ x>0.$$

With the formulas (6), (7) and Proposition 2, the pdf ′ s and the hazard rates are given by:

$$u_{ErlP}(x)=\frac{\Theta\lambda^k x^{k-1}e^{-\lambda x-\Theta e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}}}{1-e^{-\Theta}},\ x>0,$$

$$v_{ErlP}(x) = \frac{\Theta \lambda^k x^{k-1} e^{-\lambda x - \Theta + \Theta e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}}}{1 - e^{-\Theta}}, \ x > 0$$

and

$$h_{U_{ErlP}}(x) = \frac{\Theta \lambda^k x^{k-1} e^{-\lambda x - \Theta e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}}}{1 - e^{-\Theta e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}}}, \ x > 0,$$

respectively

$$h_{V_{ErlP}}(x) = \frac{\Theta \lambda^k x^{k-1} e^{\Theta e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}}}{e^{\Theta e^{-\lambda x} \sum\limits_{i=0}^{k-1} \frac{(\lambda x)^i}{i!}}}, \ x > 0.$$

3.3 On the Poisson limit theorem

The next proposition shows that the MaxErlP and MinErlP distributions approximate the MaxErlB, respectively MinErlB distributions under certain conditions.

Proposition 6. ***(Poisson limit theorem)***. *The* MaxErlP *and* MinErlP *distributions can be obtained as limiting of the* MaxErlB*, respectively* MinErlB *distributions with ditribution functions given by (10) and (11) if* $n\Theta \to \alpha > 0$ *when* $n \to \infty$ *and* $\Theta \to 0^+$.

Proof. We shall study the convergence in terms of the distributions $U_{ErlB}(x)$, $V_{ErlB}(x)$, $U_{ErlP}(x)$ and $V_{ErlP}(x), \ x > 0$ of the two types of distributions.

By calculating separately three elementary limits:

$$\lim_{\substack{n\to\infty \\ \Theta\to 0^+}} (1+\Theta)^n = \lim_{\substack{n\to\infty \\ \Theta\to 0^+}} \left[(1+\Theta)^{1/\Theta}\right]^{n\Theta} = e^{\alpha},$$

$$\begin{aligned} & \lim_{\substack{n\to\infty \\ \Theta\to 0^+}} \left[1 + \Theta A(k,\lambda,x)\right]^n = \\ = \ & \lim_{\substack{n\to\infty \\ \Theta\to 0^+}} \left\{\left[1 + \Theta A(k,\lambda,x)\right]^{\frac{1}{\Theta A(k,\lambda,x)}}\right\}^{n\Theta A(k,\lambda,x)} \\ = \ & e^{\alpha A(k,\lambda,x)} \end{aligned}$$

and

$$\lim_{\substack{n\to\infty \\ \Theta\to 0^+}} \left[1 + \Theta\left(1 - A(k,\lambda,x)\right)\right]^n =$$

$$= \lim_{\substack{n\to\infty \\ \Theta\to 0^+}} \left\{ \left[1+\Theta\left(1-A(k,\lambda,x)\right)\right]^{\frac{1}{\Theta(1-A(k,\lambda,x))}} \right\}^{n\Theta(1-A(k,\lambda,x))}$$

$$= e^{\alpha(1-A(k,\lambda,x))}, \text{where } A(k,\lambda,x) = e^{-\lambda x}\sum_{i=0}^{k-1}\frac{(\lambda x)^i}{i!},$$

we obtain:

$$\lim_{\substack{n\to\infty \\ \Theta\to 0^+}} U_{ErlB}(x) = \lim_{\substack{n\to\infty \\ \Theta\to 0^+}} \frac{\left(1+\Theta-\Theta e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)^n - 1}{(1+\Theta)^n - 1}$$

$$= \frac{e^{\alpha\left(1-e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)} - 1}{e^{\alpha}-1} = U_{ErlP}(x)$$

and

$$\lim_{\substack{n\to\infty \\ \Theta\to 0^+}} V_{ErlB}(x) = \lim_{\substack{n\to\infty \\ \Theta\to 0^+}} \frac{(1+\Theta)^n - \left(1+\Theta e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}\right)^n}{(1+\Theta)^n - 1}$$

$$= \frac{e^{\alpha} - e^{\alpha e^{-\lambda x}\sum\limits_{i=0}^{k-1}\frac{(\lambda x)^i}{i!}}}{e^{\alpha}-1} = V_{ErlP}(x).$$

$\square$

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Received December 28, 2013

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BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(75), 2014, Pages 74–84
ISSN 1024–7696

A semi-isometric isomorphism on a ring of matrices

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Abstract. Let (R, ξ) be a pseudonormed ring and R_n be a ring of matrices over the ring R. We prove that if $1 \leq \gamma, \sigma \leq \infty$ and $\frac{1}{\gamma} + \frac{1}{\sigma} \geq 1$, then the function $\eta_{\xi,\gamma,\sigma}$ is a pseudonorm on the ring R_n. Let now $\varphi : (R, \xi) \to (\bar{R}, \overline{\xi})$ be a semi-isometric isomorphism of pseudonormed rings. We prove that $\Phi : (R_n, \eta_{\xi,\gamma,\sigma}) \to (\bar{R}_n, \eta_{\bar{\xi},\gamma,\sigma})$ is a semi-isometric isomorphism too for all $1 \leq \gamma, \sigma \leq \infty$ such that $\frac{1}{\gamma} + \frac{1}{\sigma} \geq 1$.

Mathematics subject classification: 16W60, 13A18.
Keywords and phrases: Pseudonormed rings, quotient rings, ring of matrices, isometric homomorphism, semi-isometric isomorphism, canonical homomorphism.

The following theorem on isomorphism is often applied in algebra and, in particular, in the ring theory:

Theorem 1. *If A is a subring of a ring R and I is an ideal of the ring R, then the quotient rings $A/(A \bigcap I)$ and $(A+I)/I$ are isomorphic rings.*

In particular, if $A \bigcap I = 0$, then the ring A is isomorphic to the ring $(A+I)/I$, i.e. the rings A and $(A+I)/I$ possess identical algebraic properties.

Since it is necessary to take into account properties of pseudonorms when studying the pseudonormed rings then one needs to consider isomorphisms which keep pseudonorms. Such isomorphisms are called isometric isomorphisms.

Theorem 1 does not always take place for pseudonormed rings. As is shown in Theorem 2.1 from [1] it is impossible to tell anything more than the validity of the inequality $\bar{\xi}(\varphi(r)) \leq \xi(r)$ in case $A \bigcap I = 0$.

The case when A is an ideal of a pseudonormed ring (R, ξ) was studied in [1], the case when A is a one-sided ideal of a pseudonormed ring (R, ξ) was studied in [2].

The following definition was introduced in [1]:

Definition 1. Let (R, ξ) and $\left(\bar{R}, \bar{\xi}\right)$ be pseudonormed rings. An isomorphism $\varphi : R \to \bar{R}$ is called a semi-isometric isomorphism if there exists a pseudonormed ring $\left(\hat{R}, \hat{\xi}\right)$ such that the following conditions are valid:

1) the ring R is an ideal in the ring $\hat{R}$;

2) $\hat{\xi}(r) = \xi(r)$ for any $r \in R$;

3) the isomorphism φ can be extended up to an isometric homomorphism $\hat{\varphi} : \left(\hat{R}, \hat{\xi}\right) \to \left(\bar{R}, \bar{\xi}\right)$ of the pseudonormed rings, i.e. $\bar{\xi}(\hat{\varphi}(\hat{r})) = \inf\left\{\hat{\xi}(\hat{r} + a) \,|\, a \in \ker \hat{\varphi}\right\}$ for all $\hat{r} \in \hat{R}$.

The following theorem was proved in [1]:

Theorem 2. *Let* (R, ξ) *and* $\left(\bar{R}, \bar{\xi}\right)$ *be pseudonormed rings and* $\varphi : R \to \bar{R}$ *be a ring isomorphism. Then the isomorphism* $\varphi : (R, \xi) \to \left(\bar{R}, \bar{\xi}\right)$ *is a semi-isometric isomorphism of the pseudonormed rings iff the inequalities* $\xi(a \cdot b) \leq \bar{\xi}(\varphi(a)) \cdot \xi(b)$, $\xi(b \cdot a) \leq \bar{\xi}(\varphi(a)) \cdot \xi(b)$ *and* $\bar{\xi}(\varphi(a)) \leq \xi(a)$ *are true for any* $a, b \in R$.

This paper is a continuation of [1] and [2] and it is devoted to the study of pseudonorms on a ring of matrices which keep a semi-isometric isomorphism.

We will use the following propositions. The proof of Propositions 1 – 3 can be found in [3]; the proof of Propositions 4, 5 can be found in [4].

Proposition 1. *Let* λ *and* λ^* *be positive real numbers such that* $\lambda > 1$, $\lambda^* > 1$ *and* $\frac{1}{\lambda} + \frac{1}{\lambda^*} = 1$. *Then the inequality*

$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k^{\lambda}\right)^{\frac{1}{\lambda}} \cdot \left(\sum_{k=1}^{n} b_k^{\lambda^*}\right)^{\frac{1}{\lambda^*}}$$

is true for all $a_k \geq 0$ *and* $b_k \geq 0$.

Proposition 2. *Let* $a_{ik} \geq 0$ *for* $1 \leq i \leq m$, $1 \leq k \leq n$. *Then the inequality*

$$\left(\sum_{k=1}^{n}\left(\sum_{i=1}^{m} a_{ik}\right)^{\lambda}\right)^{\frac{1}{\lambda}} \leq \sum_{i=1}^{m}\left(\sum_{k=1}^{n} a_{ik}^{\lambda}\right)^{\frac{1}{\lambda}}$$

is true for any $\lambda > 1$.

Proposition 3. *Let* $a_k \geq 0$ *for all* $1 \leq k \leq n$ *and* $G_\lambda(a_1, a_2, \ldots, a_n)$ *be a real function such that*

$$G_\lambda(a_1, a_2, \ldots, a_n) = \left(\sum_{k=1}^{n} a_k^{\lambda}\right)^{\frac{1}{\lambda}} \text{ for } 1 \leq \lambda < \infty,$$

$$G_\infty(a_1, a_2, \ldots, a_n) = \max_{1 \leq k \leq n} a_k \text{ for } \lambda = \infty.$$

Then the family of functions $\{G_\lambda | 1 \leq \lambda \leq \infty\}$ *has the following properties:*

1) if $\lambda_1 \leq \lambda_2$, *then* $G_{\lambda_1}(a_1, a_2, \ldots, a_n) \geq G_{\lambda_2}(a_1, a_2, \ldots, a_n)$ *for all* $a_k \geq 0$;

2) $\lim\limits_{\lambda \to +\infty} G_\lambda(a_1, a_2, \ldots, a_n) = G_\infty(a_1, a_2, \ldots, a_n)$ *for all* $a_k \geq 0$;

3) $\sup\limits_{\lambda > 1} G_\lambda(a_1, a_2, \ldots, a_n) = G_1(a_1, a_2, \ldots, a_n)$ *for all* $a_k \geq 0$.

Definition 2. A direction is a partially ordered set $(\Gamma, \leq)$ that satisfies the following condition: for any two elements $\gamma_1, \gamma_2 \in \Gamma$ there exists the third element $\gamma_3 \in \Gamma$ such that $\gamma_1 \leq \gamma_3$ and $\gamma_2 \leq \gamma_3$.

Proposition 4. *Let Γ be some set and R be a ring. If $\{\xi_\gamma \,|\gamma \in \Gamma\}$ is a family of pseudonorms on the ring R, then the following statements are valid:*

1. If Γ is a direction and for every $r \in R$ there exists $\lim\limits_{\gamma\in\Gamma} \xi_\gamma(r)$ such that $\lim\limits_{\gamma\in\Gamma} \xi_\gamma(r) \neq 0$ for every $r \neq 0$, then the function $\xi(r) = \lim\limits_{\gamma\in\Gamma} \xi_\gamma(r)$ is a pseudonorm on the ring R;

2. If the set $\{\xi_\gamma(r)\,|\gamma \in \Gamma\}$ is bounded from above for every $r \in R$, then the function $\xi(r) = \sup\limits_{\gamma\in\Gamma} \xi_\gamma(r)$ is a pseudonorm on the ring R.

Proposition 5. *Let R and $\bar{R}$ be rings and let $\varphi : R \to \bar{R}$ be a ring isomorphism. If $\{\xi_\gamma \,|\gamma \in \Gamma\}$ and $\left\{\bar{\xi}_\gamma \,|\gamma \in \Gamma\right\}$ are families of pseudonorms such that $\varphi : (R, \xi_\gamma) \to \left(\bar{R}, \bar{\xi}_\gamma\right)$ is a semi-isometric isomorphism for any $\gamma \in \Gamma$, then the following statements are true:*

1. If Γ is a direction and there exist $\lim\limits_{\gamma\in\Gamma} \xi_\gamma(r)$, $\lim\limits_{\gamma\in\Gamma} \bar{\xi}_\gamma(\bar{r})$ for every $r \in R$, $\bar{r} \in \bar{R}$ such that $\lim\limits_{\gamma\in\Gamma} \bar{\xi}_\gamma(\bar{r}) \neq 0$ for every $\bar{r} \neq 0$ and $\xi(r) = \lim\limits_{\gamma\in\Gamma} \xi_\gamma(r)$, $\bar{\xi}(\bar{r}) = \lim\limits_{\gamma\in\Gamma} \bar{\xi}_\gamma(\bar{r})$ for every $r \in R$, $\bar{r} \in \bar{R}$, then $\varphi : (R, \xi) \to \left(\bar{R}, \bar{\xi}\right)$ is a semi-isometric isomorphism;

2. If the set $\{\xi_\gamma(r)\,|\gamma \in \Gamma\}$ is bounded from above for every $r \in R$ and $\xi(r) = \sup\limits_{\gamma\in\Gamma} \xi_\gamma(r)$, $\bar{\xi}(\bar{r}) = \sup\limits_{\gamma\in\Gamma} \bar{\xi}_\gamma(\bar{r})$ for every $r \in R$, $\bar{r} \in \bar{R}$, then $\varphi : (R, \xi) \to \left(\bar{R}, \bar{\xi}\right)$ is a semi-isometric isomorphism.

We will consider a ring of matrices R_n over a pseudonormed ring (R, ξ) and a family of functions $\{\eta_{\xi,\gamma,\sigma} | 1 \leq \gamma, \sigma \leq \infty\}$ on R_n such that

$$\eta_{\xi,\gamma,\sigma}(A) = \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} \left(\xi\left(a_{ij}\right)\right)^\sigma \right)^{\frac{\gamma}{\sigma}} \right)^{\frac{1}{\gamma}},$$

$$\eta_{\xi,\gamma,\infty}(A) = \left(\sum_{i=1}^{n} \left(\max_{1\leq j\leq n} \xi\left(a_{ij}\right) \right)^{\gamma} \right)^{\frac{1}{\gamma}},$$

$$\eta_{\xi,\infty,\sigma}(A) = \max_{1\leq i\leq n} \left(\sum_{j=1}^{n} \left(\xi\left(a_{ij}\right)\right)^\sigma \right)^{\frac{1}{\sigma}},$$

$$\eta_{\xi,\infty,\infty}(A) = \max_{1\leq i,j\leq n} \xi\left(a_{ij}\right)$$

for any $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \in R_n$.

The following theorem gives conditions for γ and σ such that the functions $\eta_{\xi,\gamma,\sigma}$ define pseudonorms on the ring R_n.

Theorem 3. *Let (R, ξ) be a pseudonormed ring and let R_n be a ring of matrices over the ring R with the natural operations of addition and multiplication. Then the function $\eta_{\xi,\gamma,\sigma}$ is a pseudonorm on the ring R_n for all γ and σ such that $1 \leq \gamma, \sigma \leq \infty$ and $\frac{1}{\gamma} + \frac{1}{\sigma} \geq 1$.*

Proof. Let $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \dots & \dots & \dots \\ b_{n1} & \dots & b_{nn} \end{pmatrix}$ and $\eta = \eta_{\xi,\gamma,\sigma}$, $1 < \gamma, \sigma < \infty$ and $\frac{1}{\gamma} + \frac{1}{\sigma} \geq 1$.

It is obvious that:

1) $\eta(A) = 0 \Leftrightarrow A = 0$;

2) $\eta(-A) = \eta(A)$ for any $A \in R_n$.

Since ξ is a pseudonorm on the ring R then $\xi(a_{ij} + b_{ij}) \leq \xi(a_{ij}) + \xi(b_{ij})$ for all $1 \leq i, j \leq n$, and hence

$$\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(a_{ij}+b_{ij}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}} \leq \left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(a_{ij}\right)+\xi\left(b_{ij}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}.$$

It follows from Proposition 2 that

$$\left(\sum_{j=1}^{n}\left(\xi\left(a_{ij}\right)+\xi\left(b_{ij}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}} \leq \left(\sum_{j=1}^{n}\left(\xi\left(a_{ij}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}} + \left(\sum_{j=1}^{n}\left(\xi\left(b_{ij}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}.$$

Then

$$\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(a_{ij}\right)+\xi\left(b_{ij}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}} \leq$$
$$\left(\sum_{i=1}^{n}\left(\left(\sum_{j=1}^{n}\left(\xi\left(a_{ij}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}} + \left(\sum_{j=1}^{n}\left(\xi\left(b_{ij}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}.$$

Using Proposition 2 we obtain

$$\left(\sum_{i=1}^{n}\left(\left(\sum_{j=1}^{n}\left(\xi\left(a_{ij}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}} + \left(\sum_{j=1}^{n}\left(\xi\left(b_{ij}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}} \leq$$
$$\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(a_{ij}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}} + \left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(b_{ij}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}.$$

Therefore, $\eta(A + B) \leq \eta(A) + \eta(B)$ for any $A, B \in R_n$.

Verify the inequality $\eta(A \cdot B) \leq \eta(A) \cdot \eta(B)$ for any $A, B \in R_n$. We consider

$$\eta(A\cdot B)=\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(\sum_{k=1}^{n}a_{ik}\cdot b_{kj}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}.$$

Since ξ is a pseudonorm then

$$\xi\left(\sum_{k=1}^{n}a_{ik}\cdot b_{kj}\right)\leq\sum_{k=1}^{n}\xi\left(a_{ik}\cdot b_{kj}\right)\leq\sum_{k=1}^{n}\xi\left(a_{ik}\right)\cdot\xi\left(b_{kj}\right).$$

Hence

$$\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(\sum_{k=1}^{n}a_{ik}\cdot b_{kj}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}\leq\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\xi\left(a_{ik}\right)\cdot\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}.$$

Let σ^* be a positive real number such that $\frac{1}{\sigma}+\frac{1}{\sigma^*}=1$. It follows from Proposition 2 that

$$\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\xi\left(a_{ik}\right)\cdot\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\leq\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(a_{ik}\right)\cdot\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}=$$

$$\sum_{k=1}^{n}\left(\xi\left(a_{ik}\right)\cdot\left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\right).$$

Using Proposition 1 we obtain

$$\sum_{k=1}^{n}\left(\xi\left(a_{ik}\right)\cdot\left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\right)\leq$$

$$\left(\sum_{k=1}^{n}\left(\xi\left(a_{ik}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\cdot\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\sigma^*}{\sigma}}\right)^{\frac{1}{\sigma^*}}.$$

Then

$$\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\xi\left(a_{ik}\right)\cdot\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\leq\left(\sum_{k=1}^{n}\left(\xi\left(a_{ik}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\cdot\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\sigma^*}{\sigma}}\right)^{\frac{1}{\sigma^*}}$$

and

$$\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\xi\left(a_{ik}\right)\cdot\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}\leq$$

$$\left(\sum_{i=1}^{n}\left(\sum_{k=1}^{n}\left(\xi\left(a_{ik}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}\cdot\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\sigma^*}{\sigma}}\right)^{\frac{1}{\sigma^*}}.$$

We have the inequality

$$\eta(A\cdot B)=\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(\sum_{k=1}^{n}a_{ik}\cdot b_{kj}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}\leq$$

$$\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\xi\left(a_{ik}\right)\cdot\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}\leq\left(\sum_{i=1}^{n}\left(\sum_{k=1}^{n}\left(\xi\left(a_{ik}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}\cdot$$

$$\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\sigma^*}{\sigma}}\right)^{\frac{1}{\sigma^*}}=\eta(A)\cdot\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\sigma^*}{\sigma}}\right)^{\frac{1}{\sigma^*}}.$$

Since $\frac{1}{\gamma}+\frac{1}{\sigma}\geq 1$ and $\frac{1}{\sigma}+\frac{1}{\sigma^*}=1$ then $\frac{1}{\gamma}\geq\frac{1}{\sigma^*}$ and so $\gamma\leq\sigma^*$. Hence it follows from Proposition 3 that

$$\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\sigma^*}{\sigma}}\right)^{\frac{1}{\sigma^*}}\leq\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}.$$

Therefore $\eta(A\cdot B)\leq\eta(A)\cdot\eta(B)$ for any $A,B\in R_n$.

Thus, the function $\eta=\eta_{\xi,\gamma,\sigma}$ is a pseudonorm on the ring R_n for all γ and σ such that $1<\gamma,\sigma<\infty$ and $\frac{1}{\gamma}+\frac{1}{\sigma}\geq 1$.

It follows from Proposition 3 that

$$\eta_{\xi,1,\sigma}=\sup_{1<\gamma\leq\frac{\sigma}{\sigma-1}}\eta_{\xi,\gamma,\sigma}=\sup_{\gamma>1}\eta_{\xi,\gamma,\sigma},$$

$$\eta_{\xi,\gamma,1}=\sup_{1<\sigma\leq\frac{\gamma}{\gamma-1}}\eta_{\xi,\gamma,\sigma}=\sup_{\sigma>1}\eta_{\xi,\gamma,\sigma},$$

$$\eta_{\xi,1,1}=\sup_{\gamma>1}\eta_{\xi,\gamma,1}=\sup_{\sigma>1}\eta_{\xi,1,\sigma},$$

$$\eta_{\xi,1,\infty}=\lim_{\sigma\to+\infty}\eta_{\xi,1,\sigma},$$

$$\eta_{\xi,\infty,1}=\lim_{\gamma\to+\infty}\eta_{\xi,\gamma,1}.$$

Therefore by Proposition 4 the functions $\eta_{\xi,1,\sigma}$, $\eta_{\xi,\gamma,1}$, $\eta_{\xi,1,1}$, $\eta_{\xi,1,\infty}$, $\eta_{\xi,\infty,1}$ are pseudonorms on the ring R_n too for all $1<\gamma,\sigma<\infty$.

Thus, the function $\eta=\eta_{\xi,\gamma,\sigma}$ is a pseudonorm on the ring R_n for any γ and σ such that $1\leq\gamma,\sigma\leq\infty$ and $\frac{1}{\gamma}+\frac{1}{\sigma}\geq 1$. □

Remark 1. The conditions $\gamma,\sigma\geq 1$ and $\frac{1}{\gamma}+\frac{1}{\sigma}\geq 1$ are essential. Consider examples which show that if these conditions are not satisfied, then the function $\eta_{\xi,p,q}$ is not a pseudonorm on the ring R_n .

Let R be the ring of real numbers and $\xi(r)=|r|$ be a norm on the ring R ; let R_2 be the ring of real matrices $A=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}$, $\gamma,\sigma>0$ and $\eta=\eta_{\xi,\gamma,\sigma}$ be a pseudonorm on the ring R_2.

1. If $A = B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\frac{1}{\gamma} + \frac{1}{\sigma} < 1$, then

$$\eta(A \cdot B) = 2^{\frac{1}{\gamma}+\frac{1}{\sigma}+1} > 2^{\frac{1}{\gamma}+\frac{1}{\sigma}} \cdot 2^{\frac{1}{\gamma}+\frac{1}{\sigma}} = \eta(A) \cdot \eta(B).$$

2. If $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\gamma = \infty$, $\sigma > 1$, then

$$\eta(A \cdot B) = 2^{\frac{1}{\sigma}+1} > 2^{\frac{1}{\sigma}} \cdot 2^{\frac{1}{\sigma}} = \eta(A) \cdot \eta(B).$$

3. If $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\sigma = \infty$, $\gamma > 1$, then

$$\eta(A \cdot B) = 2^{\frac{1}{\gamma}+1} > 2^{\frac{1}{\gamma}} \cdot 2^{\frac{1}{\gamma}} = \eta(A) \cdot \eta(B).$$

4. If $A = B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\gamma = \sigma = \infty$, then

$$\eta(B \cdot A) = 2 > 1 \cdot 1 = \eta(B) \cdot \eta(A).$$

5. If $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $\sigma < 1$, then

$$\eta(A + B) = 2^{\frac{1}{\gamma}+\frac{1}{\sigma}} > 2^{\frac{1}{\gamma}} + 2^{\frac{1}{\gamma}} = \eta(A) + \eta(B).$$

6. If $A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $\gamma < 1$, then

$$\eta(A + B) = 2^{\frac{1}{\gamma}+\frac{1}{\sigma}} > 2^{\frac{1}{\sigma}} + 2^{\frac{1}{\sigma}} = \eta(A) + \eta(B).$$

7. If $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $\gamma = \infty$, $\sigma < 1$, then

$$\eta(A + B) = 2^{\frac{1}{\sigma}} > 1 + 1 = \eta(A) + \eta(B).$$

8. If $A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $\sigma = \infty$, $\gamma < 1$, then

$$\eta(A + B) = 2^{\frac{1}{\gamma}} > 1 + 1 = \eta(A) + \eta(B).$$

So $\eta_{\xi,\gamma,\sigma}$ is not a pseudonorm on the ring R_2 if the conditions $\gamma, \sigma \geq 1$ and $\frac{1}{\gamma} + \frac{1}{\sigma} \geq 1$ are violated.

Theorem 4. *Let (R, ξ), $(\bar{R}, \bar{\xi})$ be pseudonormed rings, $\varphi : (R, \xi) \to (\bar{R}, \overline{\xi})$ be a semi-isometric isomorphism, R_n and $\bar{R}_n$ be rings of matrices over the rings R and $\bar{R}$ with the pseudonorms $\eta_{\xi,\gamma,\sigma}$ and $\eta_{\bar{\xi},\gamma,\sigma}$, respectively, where $1 \leq \gamma, \sigma \leq \infty$ and $\frac{1}{\gamma} + \frac{1}{\sigma} \geq 1$. Then the mapping $\Phi : (R_n, \eta_{\xi,\gamma,\sigma}) \to (\bar{R}_n, \eta_{\bar{\xi},\gamma,\sigma})$ given by*

$$\Phi\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} = \begin{pmatrix} \varphi(a_{11}) & \dots & \varphi(a_{1n}) \\ \dots & \dots & \dots \\ \varphi(a_{n1}) & \dots & \varphi(a_{nn}) \end{pmatrix}$$

is a semi-isometric isomorphism too.

Proof. Let $1 < \gamma, \sigma < \infty$, $\frac{1}{\gamma}+\frac{1}{\sigma} \geq 1$ and let $\eta = \eta_{\xi,\gamma,\sigma}$ and $\overline{\eta} = \eta_{\overline{\xi},\gamma,\sigma}$ be pseudonorms on the rings R_n and $\bar{R}_n$. We verify the conditions of Theorem 2 for the mapping $\Phi : (R_n, \eta) \to (\bar{R}_n, \overline{\eta})$.

Let us show that the inequality $\eta(A \cdot B) \leq \overline{\eta}(\Phi(A)) \cdot \eta(B)$ is valid for any

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \dots & \dots & \dots \\ b_{n1} & \dots & b_{nn} \end{pmatrix} \in R_n.$$

Since $\xi(a+b) \leq \xi(a)+\xi(b)$ and $\xi(a \cdot b) \leq \overline{\xi}(\varphi(a)) \cdot \xi(b)$ by Theorem 2 then

$$\xi\left(\sum_{k=1}^{n} a_{ik} \cdot b_{kj}\right) \leq \sum_{k=1}^{n} \xi\left(a_{ik} \cdot b_{kj}\right) \leq \sum_{k=1}^{n} \overline{\xi}\left(\varphi\left(a_{ik}\right)\right) \cdot \xi\left(b_{kj}\right),$$

and hence

$$\eta(A \cdot B) = \left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(\sum_{k=1}^{n} a_{ik} \cdot b_{kj}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}} \leq$$
$$\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n} \overline{\xi}\left(\varphi\left(a_{ik}\right)\right) \cdot \xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}.$$

It follows from Proposition 2 that

$$\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n} \overline{\xi}\left(\varphi\left(a_{ik}\right)\right) \cdot \xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}} \leq \sum_{k=1}^{n}\left(\overline{\xi}\left(\varphi\left(a_{ik}\right)\right) \cdot \left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\right).$$

Let σ^* be a positive real number such that $\frac{1}{\sigma}+\frac{1}{\sigma^*}=1$. Using Proposition 1 we obtain the ineguality

$$\sum_{k=1}^{n}\left(\overline{\xi}\left(\varphi\left(a_{ik}\right)\right) \cdot \left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\right) \leq$$
$$\left(\sum_{k=1}^{n}\left(\overline{\xi}\left(\varphi\left(a_{ik}\right)\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}} \cdot \left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\sigma^*}{\sigma}}\right)^{\frac{1}{\sigma^*}}.$$

Then

$$\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\overline{\xi}\left(\varphi\left(a_{ik}\right)\right)\cdot\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}\leq$$

$$\left(\sum_{i=1}^{n}\left(\sum_{k=1}^{n}\left(\overline{\xi}\left(\varphi\left(a_{ik}\right)\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}\cdot\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\sigma^*}{\sigma}}\right)^{\frac{1}{\sigma^*}}.$$

Since $\frac{1}{\gamma}+\frac{1}{\sigma}\geq 1$ and $\frac{1}{\sigma}+\frac{1}{\sigma^*}=1$ then $\gamma\leq\sigma^*$. Hence it follows from Proposition 3 that

$$\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\sigma^*}{\sigma}}\right)^{\frac{1}{\sigma^*}}\leq\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}=\eta(B).$$

We have the inequality

$$\eta(A\cdot B)\leq\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\overline{\xi}\left(\varphi\left(a_{ik}\right)\right)\cdot\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}\leq$$

$$\left(\sum_{i=1}^{n}\left(\sum_{k=1}^{n}\left(\overline{\xi}\left(\varphi\left(a_{ik}\right)\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}\cdot\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(b_{kj}\right)\right)^{\sigma}\right)^{\frac{\sigma^*}{\sigma}}\right)^{\frac{1}{\sigma^*}}\leq\overline{\eta}(\Phi(A))\cdot\eta(B).$$

Let us show that the inequality $\eta(B\cdot A)\leq\overline{\eta}(\Phi(A))\cdot\eta(B)$ is true for any $A,B\in R_n$. Since $\xi\left(\sum_{k=1}^{n}b_{ik}\cdot a_{kj}\right)\leq\sum_{k=1}^{n}\xi\left(b_{ik}\cdot a_{kj}\right)\leq\sum_{k=1}^{n}\xi\left(b_{ik}\right)\cdot\overline{\xi}\left(\varphi\left(a_{kj}\right)\right)$, then

$$\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\xi\left(\sum_{k=1}^{n}b_{ik}\cdot a_{kj}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}\leq\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\xi\left(b_{ik}\right)\cdot\overline{\xi}\left(\varphi\left(a_{kj}\right)\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}$$

. It follows from Proposition 2 that

$$\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\xi\left(b_{ik}\right)\cdot\overline{\xi}\left(\varphi\left(a_{kj}\right)\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\leq\sum_{k=1}^{n}\left(\xi\left(b_{ik}\right)\cdot\left(\sum_{j=1}^{n}\left(\overline{\xi}\left(\varphi\left(a_{kj}\right)\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\right).$$

Using Proposition 1 we have

$$\sum_{k=1}^{n}\left(\xi\left(b_{ik}\right)\cdot\left(\sum_{j=1}^{n}\left(\overline{\xi}\left(\varphi\left(a_{kj}\right)\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\right)\leq$$

$$\left(\sum_{k=1}^{n}\left(\xi\left(b_{ik}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\cdot\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\overline{\xi}\left(\varphi\left(a_{kj}\right)\right)\right)^{\sigma}\right)^{\frac{\sigma^*}{\sigma}}\right)^{\frac{1}{\sigma^*}}.$$

Then

$$\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\xi\left(b_{ik}\right)\cdot\overline{\xi}\left(\varphi\left(a_{kj}\right)\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\leq$$

$$\left(\sum_{k=1}^{n}\left(\xi\left(b_{ik}\right)\right)^{\sigma}\right)^{\frac{1}{\sigma}}\cdot\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\overline{\xi}\left(\varphi\left(a_{kj}\right)\right)\right)^{\sigma}\right)^{\frac{\sigma^*}{\sigma}}\right)^{\frac{1}{\sigma^*}},$$

and hence

$$\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\xi\left(b_{ik}\right)\cdot\overline{\xi}\left(\varphi\left(a_{kj}\right)\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}\leq$$

$$\left(\sum_{i=1}^{n}\left(\sum_{k=1}^{n}\left(\xi\left(b_{ik}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}\cdot\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\overline{\xi}\left(\varphi\left(a_{kj}\right)\right)\right)^{\sigma}\right)^{\frac{\sigma^*}{\sigma}}\right)^{\frac{1}{\sigma^*}}.$$

Since $\gamma\leq\sigma^*$ then it follows from Proposition 3 that

$$\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\overline{\xi}\left(\varphi\left(a_{kj}\right)\right)\right)^{\sigma}\right)^{\frac{\sigma^*}{\sigma}}\right)^{\frac{1}{\sigma^*}}\leq\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\overline{\xi}\left(\varphi\left(a_{kj}\right)\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}=\overline{\eta}(\Phi(A)).$$

We obtain the inequality

$$\eta(B\cdot A)\leq\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\xi\left(b_{ik}\right)\cdot\overline{\xi}\left(\varphi\left(a_{kj}\right)\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}\leq$$

$$\left(\sum_{i=1}^{n}\left(\sum_{k=1}^{n}\left(\xi\left(b_{ik}\right)\right)^{\sigma}\right)^{\frac{\gamma}{\sigma}}\right)^{\frac{1}{\gamma}}\cdot\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left(\overline{\xi}\left(\varphi\left(a_{kj}\right)\right)\right)^{\sigma}\right)^{\frac{\sigma^*}{\sigma}}\right)^{\frac{1}{\sigma^*}}\leq\eta\left(B\right)\cdot\overline{\eta}(\Phi(A)).$$

The inequality $\overline{\eta}(\Phi(A))\leq\eta(A)$ follows from the inequality $\overline{\xi}(\varphi(a_{ij}))\leq\xi(a_{ij})$.

All conditions of Theorem 2 are valid. Therefore the mapping $\Phi:(R_n,\eta_{\xi,\gamma,\sigma})\to(\bar{R}_n,\eta_{\bar{\xi},\gamma,\sigma})$ is a semi-isometric isomorphism when $1<\gamma,\sigma<\infty$ and $\frac{1}{\gamma}+\frac{1}{\sigma}\geq 1$.

Since $\eta_{\xi,1,\sigma}=\sup\limits_{1<\gamma\leq\frac{\sigma}{\sigma-1}}\eta_{\xi,\gamma,\sigma}$, $\eta_{\xi,\gamma,1}=\sup\limits_{1<\sigma\leq\frac{\gamma}{\gamma-1}}\eta_{\xi,\gamma,\sigma}$, $\eta_{\xi,1,1}=\sup\limits_{\gamma>1}\eta_{\xi,\gamma,1}=\sup\limits_{\sigma>1}\eta_{\xi,1,\sigma}$, $\eta_{\xi,1,\infty}=\lim\limits_{\sigma\to+\infty}\eta_{\xi,1,\sigma}$ and $\eta_{\xi,\infty,1}=\lim\limits_{\gamma\to+\infty}\eta_{\xi,\gamma,1}$ then it follows from Proposition 5 that $\Phi:(R_n,\eta_{\xi,1,\sigma})\to(\bar{R}_n,\eta_{\bar{\xi},1,\sigma})$ for any $1<\sigma<\infty$, $\Phi:(R_n,\eta_{\xi,\gamma,1})\to(\bar{R}_n,\eta_{\bar{\xi},\gamma,1})$ for any $1<\gamma<\infty$, $\Phi:(R_n,\eta_{\xi,1,\infty})\to(\bar{R}_n,\eta_{\bar{\xi},1,\infty})$, $\Phi:(R_n,\eta_{\xi,\infty,1})\to(\bar{R}_n,\eta_{\bar{\xi},\infty,1})$ and $\Phi:(R_n,\eta_{\xi,1,1})\to(\bar{R}_n,\eta_{\bar{\xi},1,1})$ are semi-isometric isomorphisms too.

Thus the mapping $\Phi:(R_n,\eta_{\xi,\gamma,\sigma})\to(\bar{R}_n,\eta_{\bar{\xi},\gamma,\sigma})$ is a semi-isometric isomorphism for any γ and σ such that $1\leq\gamma,\sigma\leq\infty$ and $\frac{1}{\gamma}+\frac{1}{\sigma}\geq 1$. $\square$

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Received April 05, 2014

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(75), 2014, Pages 85–101
ISSN 1024–7696

Compact Global Attractors of Non-Autonomous Gradient-Like Dynamical Systems

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Abstract. In this paper we study the asymptotic behavior of gradient-like non-autonomous dynamical systems. We give a description of the structure of the Levinson center (maximal compact invariant set) for this class of systems.

Mathematics subject classification: 37B25, 37B35 37B55, 37L15, 37L30, 37L45.
Keywords and phrases: Gradient-like dynamical systems; global attractor; fixed-points; chain recurrent motions.

1 Introduction

Denote by $\mathbb{S}$ the set of all real ($\mathbb{R}$) or integer ($\mathbb{Z}$) numbers and $\mathbb{S}_+ := \{s \in \mathbb{S} : \ s \geq 0\}$. Let $\mathbb{T} := \mathbb{R}_+$ or $\mathbb{Z}_+$, X be a complete metric space and $(X, \mathbb{T}, \pi)$ be a dynamical system.

A continuous function $V : \ X \mapsto \mathbb{R}$ is said to be a (global) Lyapunov function for $(X, \mathbb{T}, \pi)$ if $V(\pi(t, x)) \leq V(x)$ for all $x \in X$ and $t \in \mathbb{T}$.

If $\pi(t, x) = x$ for all $t \geq 0$, then $x \in X$ is called a fixed (stationary) point of the dynamical system $(X, \mathbb{T}, \pi)$, and by $Fix(\pi)$ we will denote the set of all fixed points of $(X, \mathbb{T}, \pi)$.

A dynamical system $(X, \mathbb{T}, \pi)$ with the Lyapunov function V is called a gradient system if the equality $V(\pi(t, x)) = V(x)$ (for all $t \geq 0$) implies $x \in Fix(\pi)$.

The simplest example of gradient dynamical system is defined by the differential equation

$$x' = -\nabla V(x) \quad (x \in \mathbb{R}^n), \tag{1}$$

where $V : \mathbb{R}^n \mapsto \mathbb{R}$ is a continuously differentiable function and $\nabla := \{\partial_{x_1}, \dots, \partial_{x_n}\}$. Indeed, if we suppose that equation (1) admits a unique solution $\pi(t, x)$ passing through the point $x \in \mathbb{R}^n$ at the initial moment $t = 0$ and defined on $\mathbb{R}_+$, then

$$\frac{d}{dt} V(\pi(t, x)) = -|\nabla V(\pi(t, x)|^2 \leq 0 \tag{2}$$

for all $x \in \mathbb{R}^n$ and $t > 0$. From (2) we obtain $V(\pi(t, x)) \leq V(x)$ for all $t \geq 0$ and if $V(\pi(t, x)) = V(x)$ for all $t \geq 0$, then from (1) we have $x \in Fix(\pi)$, i. e., $(\mathbb{R}^n, \mathbb{R}_+, \pi)$ (the dynamical system generates by equation (1).

The asymptotic behavior of gradient dynamical systems is well studied (see, for example, [2], [3, ChIII], [9, ChV], [16, ChIII], [17], [24, ChIX] and the bibliography therein).

The aim of this paper is to study the asymptotic behavior of a class of abstract non-autonomous (gradient-like) dynamical systems. The paper is organized as follows. In the second section we give with the proof some (more or less) known results for general autonomous gradient-like semi-group dynamical systems (Lemma 1).

The third section is dedicated to the gradient systems. The main result (Theorem 2) of this section contains the sufficient conditions when a gradient dynamical system admits a compact global attractor.

In the fourth section we study the gradient dynamical systems with finite number of fixed points. For the compact dissipative gradient dynamical system we give a description of the structure of its Levinson center (Lemma 4).

The fifth section is dedicated to the study of the relation between the set of all fixed points and chain recurrent points (Theorem 4) for the gradient dynamical systems admitting a compact global attractor.

In the sixth section we introduce the notion of gradient-like non-autonomous dynamical systems. The main result of this section is Theorem 7 which contains a description of the structure of compact global attractor for a gradient-like non-autonomous dynamical system with minimal base.

2 Gradient-like systems

Denote by $\omega_x := \{y \in X :$ there exists a sequence $\{t_n\} \subset \mathbb{T}$ such that $t_n \to +\infty$ and $\pi(t_n, x) \to y$ as $n \to +\infty$ $\}$ and $J^+_x := \{y \in X :$ there exist sequences $\{t_n\} \subset \mathbb{T}$ and $\{x_n\} \subset X$ such that $t_n \to +\infty,\ x_n \to x$ and $\pi(t_n, x_n) \to y$ as $n \to +\infty$ $\}$.

Definition 1. A continuous function $\gamma : \mathbb{S} \mapsto X$ is called a full trajectory of dynamical system $(X, \mathbb{T}, \pi)$ passing through the point $x \in X$ at the initial moment $t = 0$ if $\gamma(0) = x$ and $\pi(t, \gamma(s)) = \gamma(t + s)$ for all $t \in \mathbb{T}$ and $s \in \mathbb{S}$.

By Φ_x we denote the set of all full trajectories of $(X, \mathbb{T}, \pi)$ passing through the point x at the initial moment and $\alpha_\gamma := \{y \in X :$ there exists a sequence $t_n \to -\infty$ such that $\gamma(t_n) \to y\}$.

Definition 2. A dynamical system $(X, \mathbb{T}, \pi)$ is said to be a gradient-like dynamical system if it has a global Lyapunov function.

Lemma 1. *Let $(X, \mathbb{T}, \pi)$ be a gradient-like dynamical system and V be its Lyapunov function, then the following statements hold:*

1) *for all $x \in X$ there exists a constant $C_x \in \mathbb{R}$ such that $V(p) = C_x$ for any $p \in \omega_x$ if the positive semi–trajectory $\pi(\mathbb{T}_+, x)$ is relatively compact;*

2) *if $\gamma \in \Phi_x$ and the negative semi–trajectory $\gamma(\mathbb{T}_-)$ is relatively compact, then there exists $c_\gamma \in \mathbb{R}$ such that $V(q) = c_\gamma$ for all $q \in \alpha_\gamma$;*

3) *if $x \in X$ is a non-wandering point (i.e., $x \in J^+_x$), then $V(\pi(t, x)) = V(x)$ for all $t \in \mathbb{T}_+$.*

Proof. Consider the continuous function $\psi : \mathbb{T}_+ \mapsto \mathbb{R}$ defined by the equality $\psi(t) := V(\pi(t,x))$ for all $t \geq 0$. Since $\psi(t_2) \leq \psi(t_1)$ for all $t_2 \geq t_1$ and V is upper-bounded along trajectories of $(X, \mathbb{T}, \pi)$, then there exists $\lim\limits_{t\to+\infty} V(\pi(t,x)) = C_x$. Let now $p \in \omega_x$, then there exist $t_n \to +\infty$ such that $p = \lim\limits_{t\to+\infty} \pi(t_n, x)$ and, consequently, $V(p) = \lim\limits_{n\to\infty} V(\pi(t_n, x)) = C_x$.

Consider the function $\psi : \mathbb{T}_- \mapsto \mathbb{R}$ defined by the equality $\psi(s) := V(\gamma(s))$ for all $s \in \mathbb{T}_-$, where $\gamma \in \Phi_x$. Since $\psi(s_1) \geq \psi(s_2)$ for all $s_1 \leq s_2$ $(s_1, s_2 \in \mathbb{T}_-)$ and ψ is upper-bounded on $\mathbb{T}_-$, then there exists $\lim\limits_{t\to+\infty} V(\gamma(t)) = c_\gamma$. If $q \in \alpha_\gamma$ then there exists a sequence $\{s_n\} \subseteq \mathbb{T}_-$ with $s_n \to -\infty$ such that $q = \lim\limits_{n\to\infty} \gamma(s_n)$ and $V(q) = \lim\limits_{n\to\infty} V(\gamma(s_n)) = c_\gamma$.

Let $p \in J_x^+$. Since $J_x^+ \subseteq J^+_{\pi(t,x)}$ for all $t \in \mathbb{T}$, then we have $p = \lim\limits_{n\to\infty} \pi(t_n, \tilde{x}_n)$, where $t_n \to +\infty$ and $\tilde{x}_n \to \pi(t,x)$. Thus we obtain

$$V(p) = \lim_{n\to\infty} V(\pi(t_n, \tilde{x}_n)) \leq \lim_{n\to\infty} V(\tilde{x}_n) = V(\pi(t,x)).$$

In particular, $V(x) \leq V(\pi(t,x))$ since $x \in J_x^+$. On the other hand $V(\pi(t,x)) \leq V(x)$ for all $x \in X$ and $t \geq 0$ and, consequently, we have $V(\pi(t,x)) = V(x)$ for all $t \geq 0$. $\square$

Let M be a subset of X. Denote by $W^u(M) := \{x \in X :$ there exists $\gamma \in \Phi_x$ such that $\lim\limits_{t\to-\infty} \rho(\gamma(t), M) = 0\}$.

Remark 1. The first and second statements of Lemma 1 are well known (LaSalle's invariance principle).

2. The third statement is a slight modification of a result from [4, p.131].

3 Gradient systems

Definition 3. $x \in X$ is called a stationary point (fixed point, singular point) if $\pi(t,x) = x$ for all $t \in \mathbb{T}$.

Denote by $Fix(\pi)$ the set of all fixed points of dynamical system $(X, \mathbb{T}, \pi)$ and $J_x^+ := \{p \in X :$ there are $x_n \to x$ and $t_n \to +\infty$ such that $\pi(t_n, x_n) \to p\}$.

Definition 4. $(X, \mathbb{T}, \pi)$ is called a gradient dynamical system if there exists a Lyapunov function $V : X \mapsto \mathbb{R}$ with the following property: if $V(\pi(t,x)) = V(x)$ for all $t \geq 0$ then $x \in Fix(\pi)$.

Lemma 2. *Let $(X, \mathbb{T}, \pi)$ be a gradient dynamical system, then $\Omega_X = \Omega(\pi) = Fix(\pi)$, where $\Omega_X := \overline{\bigcup\{\omega_x : \ x \in X\}}$ and $\Omega(\pi) = \{x \in X : \ x \in J_x^+\}$.*

Proof. The inclusions $Fix(\pi) \subseteq \Omega_X \subseteq \Omega(\pi)$ are evident and take place for arbitrary dynamical systems (including gradient systems too). To finish the proof of Lemma it is sufficient to establish the inclusion $\Omega(\pi) \subseteq Fix(\pi)$ for gradient systems. Let $x \in \Omega(\pi)$, then $x \in J_x^+$ and by Lemma 1 (item 3) we have $V(\pi(t,x)) = V(x)$ for all $t \geq 0$ and, consequently, $x \in Fix(\pi)$. $\square$

Definition 5. A subset $M \subseteq X$ is called bounded if its diameter $d(M) := \sup\{d(p,q) :\ p,q \in M\}$ is finite.

Remark 2. 1. A subset $M \subseteq X$ is bounded if and only if for every $x_0 \in X$ there exists a number $C_{x_0} \geq 0$ such that $\rho(x_0, x) \leq C_{x_0}$ for all $x \in M$.

2. A subset $M \subseteq X$ is unbounded if and only if there exist a point $x_0 \in X$ and a sequence $\{x_n\} \subseteq M$ such that $\rho(x_n, x_0) \to +\infty$ as $n \to \infty$.

Let A and B be two bounded subsets from X. Denote by $\beta(A,B) := \sup\{\rho(a,B) :\ a \in A\}$, where $\rho(a,B) := \inf\{\rho(a,b) :\ b \in B\}$.

Definition 6. A dynamical system $(X, \mathbb{T}, \pi)$ is said to be:

– point dissipative if there exists a nonempty compact subset $K \subseteq X$ such that

$$\lim_{t\to\infty} \rho(\pi(t,x), K) = 0 \tag{3}$$

for all $x \in X$;

– compact dissipative if it is point dissipative and equality (3) takes pace uniformly with respect to $x \in X$ on every compact subset from X.

Remark 3. If the dynamical system $(X, \mathbb{T}, \pi)$ is compact dissipative, then in X there exists a maximal compact invariant set J (Levinson center [10, ChI]) which attracts every compact subset from X.

Theorem 1 (see [10, ChI])**.** *Suppose that $(X, \mathbb{T}, \pi)$ is a point dissipative dynamical system, then it will be compact dissipative if and only if for any compact subset $K \subseteq X$ the set $\Sigma_K^+ := \{\pi(t,x) :\ t \geq 0,\ x \in K\}$ is relatively compact.*

Theorem 2. *Suppose that the following conditions hold:*

1) $(X, \mathbb{T}, \pi)$ is asymptotically compact;

2) $(X, \mathbb{T}, \pi)$ is a gradient dynamical system and $V : X \mapsto \mathbb{R}$ is its Lyapunov function;

3) the set $Fix(\pi)$ is bounded;

4) for any sequence $\{x_n\}$ with the property $\rho(x_n, x_0) \to +\infty$ as $n \to \infty$ we have $V(x_n) \to +\infty$, where x_0 is some point from X.

Then the following statements hold:

1) the dynamical system $(X, \mathbb{T}, \pi)$ is compact dissipative;

2) if the Lyapunov function V is bounded on every bounded subset from X, then the Levinson center J of $(X, \mathbb{T}, \pi)$ attracts every bounded subset M from X, i.e.,

$$\lim_{t\to+\infty} \beta(\pi^t M, J) = 0.$$

Proof. Let $x \in X$ be an arbitrary point. Note that the positive semi-trajectory Σ_x^+ of point x is a bounded set. In fact, if we suppose that it is not so, then there exist a point $x_0 \in X$ and a sequence $t_n \to +\infty$ such that $\rho(\pi(t_n, x), x_0) \to +\infty$ as $n \to \infty$. Under the conditions of Theorem we have $V(\pi(t_n, x)) \to +\infty$. On the other hand we have $V(\pi(t_n, x)) \le V(x)$ for all $n \in \mathbb{N}$. The obtained contradiction proves our statement. Since the dynamical system $(X, \mathbb{T}, \pi)$ is asymptotically compact, then the semi-trajectory Σ_x^+ is relatively compact, and consequently, ω_x is a nonempty compact set. According to Lemma 2 we have $\omega_x \subseteq Fix(\pi)$. Note that the set $Fix(\pi)$ is closed and invariant. Since the dynamical system $(X, \mathbb{T}, \pi)$ is asymptotically compact and $Fix(\pi)$ is bounded, then it is compact. Thus every semi-trajectory Σ_x^+ of $(X, \mathbb{T}, \pi)$ is relatively compact and there exists a nonempty compact subset $K := Fix(\pi) \subset X$ such that $\Omega_X \subseteq K$. This means that the dynamical system $(X, \mathbb{T}, \pi)$ is point-wise dissipative.

Let now M be an arbitrary nonempty compact subset from X. We will prove that the positive semi-trajectory Σ_M^+ of the set M is relatively compact. To this end under the conditions of Theorem it is sufficient to establish that it is bounded. Denote by $c := \max\{V(x) : \ x \in M\}$ and $M_c := \{x \in X : \ V(x) \le c\}$. Note that M_c is a bounded subset of X. Indeed, if we suppose that it is not true, then there exists a point $x_0 \in X$ and sequence $\{x_n\} \subseteq M_c$ such that $\rho(x_n, x_0) \to +\infty$ and, consequently, $V(x_n) \to +\infty$ as $n \to +\infty$. On the other hand $x_n \in M_c$ and, consequently, $V(x_n) \le c$ for all $n \in \mathbb{N}$. The obtained contradiction proves our statement. Thus the dynamical system $(X, \mathbb{T}, \pi)$ is point dissipative and semi-trajectory Σ_M^+ is relatively compact for any compact subset $M \subseteq X$. By Theorem 1 $(X, \mathbb{T}, \pi)$ is compact dissipative.

Denote by J its Levinson center and consider an arbitrary bounded subset M from X then, under the conditions of Theorem 2, the set $V(M) \subset \mathbb{R}$ is bounded. Now we will prove that the semi-trajectory Σ_M^+ is a bounded subset of X for every bounded set $M \subseteq X$. Indeed, denote by $c := \sup\{V(x) : \ x \in M\}$, then we have $V(\pi(t, x)) \le V(x) \le c$ for all $x \in M$ and $t \ge 0$ and, consequently $\Sigma_M^+ \subseteq M_c$. Thus the set Σ_M^+ is bounded and positive invariant. Since the dynamical system $(X, \mathbb{T}, \pi)$ is asymptotically compact, then the set $\Omega(M)$ is nonempty, compact and it attracts the set M, i.e.,

$$\lim_{t \to +\infty} \beta(\pi^t M, \Omega(M)) = 0,$$

where

$$\Omega(M) := \bigcap_{t \ge 0} \overline{\bigcup_{\tau \ge t} \pi(\tau, M)}.$$

Since J is a maximal compact invariant of the dynamical system $(X, \mathbb{T}, \pi)$, then $\Omega(M) \subseteq J$ and, consequently, J attracts M. The theorem is proved. $\square$

Remark 4. Note that the second statement of Theorem 2 remains true (see Theorem 3.8.5 [16, ChIII] and [21]) if we replace the condition of boundedness of the function V on every bounded subset from X by the following: for every bounded subset M

from X there exists a number $\tau \geq 0$ such that the set $\{\pi(t,x): \ x \in M, \ t \geq \tau\}$ is bounded.

4 Gradient systems with finite number of fixed points

In this section we will study the gradient dynamical systems $(X,\mathbb{T},\pi)$ with finite set $Fix(\pi)$ of fixed points.

Lemma 3. *Let $(X,\mathbb{T},\pi)$ be a dynamical system and the following conditions hold:*

1) *every positive semi-trajectory Σ^+_x is relatively compact;*
2) *$\Omega_X \subseteq Fix(\pi)$;*
3) *the set $Fix(\pi)$ is finite, i.e., there are a finite number of stationary points $p_1, p_2, \dots, p_m$ of dynamical system $(X,\mathbb{T},\pi)$ such that $\Omega_X = \{p_1, p_2, \dots, p_m\}$.*

Then every point $x \in X$ is asymptotically stationary, i.e., there exists a point $p_i \in Fix(\pi)$ $(1 \leq i \leq m)$ such that

$$\lim_{t\to+\infty} \rho(\pi(t,x), p_i) = 0. \tag{4}$$

Proof. Let $x \in X$. Since Σ^+_x is relatively compact, then the set ω_x is nonempty, compact, invariant, and it attracts the point x. On the other hand $\omega_x \subseteq \Omega_X = \{p_1, p_2, \dots, p_m\}$. Thus there are $1 \leq i_1 < i_2 < \dots < i_k \leq m$ such that $\omega_x = \{p_{i_1}, p_{i_2}, \dots, p_{i_k}\}$. Taking into account that the set ω_x is dynamically undecomposable we conclude that the set ω contains a single stationary point p_i and, consequently, we have the equality (4). □

We put $A_x := \bigcup\{\alpha_\gamma : \ \gamma \in \Phi_x\}$, $A_X := \overline{\bigcup\{\alpha_x : \ x \in X\}}$ and $\Delta_X := \Omega_X \bigcup A_X$. If $p \in Fix(\pi)$, then by $W^u(p) := \{y \in X : \ \lim\limits_{t\to-\infty} \rho(\gamma(t), p) = 0$ for certain $\gamma \in \Phi_y\}$ we denote the unstable "manifold" of p.

Lemma 4. *Suppose that the following conditions are fulfilled:*

1) *the dynamical system $(X,\mathbb{T},\pi)$ is compact dissipative and J is its Levinson center;*
2) *$\Delta_X \subseteq Fix(\pi)$;*
3) *the set $Fix(\pi)$ is finite, i.e., $Fix(\pi) = \{p_1, p_2, \dots, p_m\}$.*

Then the following equality

$$J = \bigcup\{W^u(p) : \ p \in Fix(\pi)\}$$

takes place.

Proof. Since J is a maximal compact invariant set of $(X, \mathbb{T}, \pi)$, then $Fix(\pi) \subseteq J$ and $W^u(p) \subseteq J$ for all $p \in Fix(\pi)$. Thus to finish the proof it is sufficient to establish that $J \subseteq \bigcup\{W^u(p) : \ p \in Fix(\pi)\}$. Since $J \subseteq X$ is a compact invariant set, then every motion $\gamma \in \Phi := \bigcup\{\Phi_x : \ x \in J\}$ is defined on $\mathbb{S}$ and Φ is compact with respect to compact-open topology. Consider a two-sided shift dynamical system (with uniqueness) $(\Phi, \mathbb{S}, \sigma)$. We note that $Fix(\sigma) = \{p_1, p_2, \dots, p_m\}$. The inclusion $\{p_1, p_2, \dots, p_m\} \subseteq Fix(\sigma)$ is evident. Thus to prove our statement it is sufficient to show the inclusion $Fix(\sigma) \subseteq \{p_1, p_2, \dots, p_m\}$. Let $\psi \in Fix(\sigma)$, then $\psi(t) = \psi(0)$ for all $t \in \mathbb{S}$ and, consequently, $\psi \in \Phi_{\psi(0)}$ and $\psi(0) \in \alpha_\psi \subseteq \Delta_X \subseteq Fix(\pi) = \{p_1, p_2, \dots, p_m\}$. Thus there exists a number $1 \le i \le m$ such that $\psi(0) = p_i$ and, consequently, $\psi(t) = p_i$ for all $t \in \mathbb{S}$, i.e., $\psi = p_i \in \{p_1, p_2, \dots, p_m\}$. Let $x \in J$ and $\gamma \in \Phi_x$. Denote by $\tilde{\alpha}_\gamma$ the α–limit set of the point $\gamma \in \Phi$ with respect to shift dynamical system $(\Phi, \mathbb{S}, \sigma)$. If $\psi \in \tilde{\alpha}_\gamma$, then there exists a sequence $t_n \to -\infty$ such that $\psi(t) = \lim\limits_{n \to \infty} \gamma(t + t_n)$ and the last equality takes place uniformly on every compact from $\mathbb{S}$. Thus $\psi(0) \in \alpha_\gamma \subseteq \{p_1, p_2, \dots, p_m\}$ and $\psi \in \Phi_{\psi(0)}$. Thus there exists a number i such that $\psi(0) = p_i$ and, consequently, $\psi = p_i$, i.e., $\tilde{\alpha}_\gamma \subseteq \{p_1, p_2, \dots, p_m\}$. Taking into account that the set $\tilde{\alpha}_\gamma$ is dynamically undecomposable we conclude that the set $\tilde{\alpha}_\gamma$ consists of a single stationary point p_j and, consequently, $\lim\limits_{t \to -\infty} \rho(\gamma(t), p_j) = 0$. Thus we have $x \in W^u(p_j)$ and, consequently, $J \subseteq \bigcup\{W^u(p) : p \in Fix(\pi)\}$. □

Remark 5. Lemma 4 remains true if we replace the condition $\Delta_X \subseteq Per(\pi)$ by $A_X \subseteq Per(\pi)$, where $A_X := \bigcup\{\alpha_\gamma : \gamma \in \Phi(\pi)\}$.

Remark 6. The statements close to Lemma 4 (see also Remark 5) were published in the works [3, ChIII] and [23].

Lemma 5 (see [1])**.** *Let $x \in X$ and $y \in \omega_x$, then $J^+_x \subseteq J^+_y = D^+_y$.*

Lemma 6. *Suppose that $(X, \mathbb{T}, \pi)$ is a compact dissipative dynamical system and J is its Levinson center, then the following statements hold:*

1) $\omega_x \subseteq \Omega(\pi)$ for all $x \in X$;

2) $\alpha_\gamma = \emptyset$ for all $\gamma \in \Phi_x$ and $x \notin J$;

3) $\alpha_\gamma \subseteq \Omega(\pi)$ for all $\gamma \in \Phi_x$ and $x \in J$.

Proof. Let $x \in X$. Since the dynamical system $(X, \mathbb{T}, \pi)$ is compact dissipative, then ω_x is a nonempty, compact and invariant set. If $y \in \omega_x$, then according to Lemma 5 we have $J^+_y = D^+_y$. Since $y \in D^+_y$, then $y \in J^+_y$, i.e., $\omega_x \subseteq \Omega(\pi)$.

Let now $x \notin J$, $\gamma \in \Phi_x$ and $p \in \alpha_\gamma$, then there exists a sequence $t_n \to -\infty$ such that $p = \lim\limits_{n \to \infty} \gamma(t_n)$. Denote by $K := \{\gamma(t_n)\} \bigcup \{p\}$, then the set K is compact. Since the dynamical system $(X, \mathbb{T}, \pi)$ is compact dissipative and J its Levinson center, then $\Omega(K)$ is a nonempty compact set and $\Omega(K) \subseteq J$. Note that $x = \gamma(0) = \pi(-t_n, \gamma(t_n))$ and, consequently, $x \in \Omega(K) \subseteq J$. The obtained contradiction proves our statement.

If $x \in J$, $\gamma \in \Phi_x$ and $p \in \alpha_\gamma$, then there exists a sequence $t_n \to -\infty$ such that $p = \lim_{n\to\infty} \gamma(t_n)$. Consider the sequence $\{\sigma(t_n, \gamma)\} \subseteq \Phi$. Since the space Φ is compact (with respect to compact-open topology) without loss of generality we may suppose that the sequence $\{\sigma(t_n, \gamma)\}$ is convergent. Denote by ψ its limit, then $\psi \in \tilde{\alpha}_\gamma$ and, consequently, $\psi \in J^+_\psi$. This means that there are sequence $\{\psi_n\} \to \psi$ and $t_n \to +\infty$ such that $\psi = \lim_{n\to\infty} \sigma(t_n, \psi_n)$. In particular, we have $p = \psi(0) = \lim_{n\to\infty} \sigma(t_n, \psi_n)(0) = \lim_{n\to\infty} \psi_n(t_n) = \lim_{n\to\infty} \pi(t_n, \psi_n(0))$. Since $\psi_n(0) \to \psi(0) = p$ as $n \to \infty$, then we have $p \in J^+_p$. $\square$

Corollary 1. *Suppose that $(X, \mathbb{T}, \pi)$ is a compact dissipative dynamical system and J is its Levinson center, then we have $\Delta_X \subseteq \Omega(\pi)$.*

Definition 7. Let $p, q \in Fix(\pi)$. The point p is said to be chained to q, written $q \mapsto p$, if there exists a full trajectory $\gamma \in \Phi_x$ for some $x \notin \{p, q\}$ such that $\alpha_\gamma = q$ and $\omega_x = \{p\}$. A finite sequence $\{p_1, p_2, \dots, p_k\} \subseteq Fix(\pi)$ is called a k-chain if $p_1 \mapsto p_2 \mapsto \dots \mapsto p_k$. The k-chain is called a k-cycle if $p_k = p_1$.

Definition 8. Recall that:

- a nonempty compact invariant subset M of X is said to be locally maximal for $(X, \mathbb{T}, \pi)$ if it is the maximal compact invariant set in some neighborhood of itself;
- a dynamical system $(X, \mathbb{T}, \pi)$ is called asymptotically compact if for every bounded positive invariant subset $B \subseteq X$ its ω-limit set $\Omega(B)$ is nonempty, compact and
$$\lim_{t\to\infty} \beta(\pi(t, B), \Omega(B)) = 0.$$

Lemma 7. *Suppose that the following conditions hold:*

1) *$(X, \mathbb{T}, \pi)$ is a compact dissipative dynamical system and J its Levinson center;*
2) *$(X, \mathbb{T}, \pi)$ is a gradient system and $Fix(\pi) = \{p_1, p_2, \dots, p_m\}$;*
3) *the set $Fix(\pi)$ does not contain any 1-cycles.*

Then every set $M_i := \{p_i\}$ ($i = 1, 2, \dots, m$) is locally maximal.

Proof. We will show that every subset M_i ($i = 1, 2, \dots, m$) is locally maximal. Denote by $d := \min\{d_{ij} : \ i, j = 1, 2, \dots, m \text{ and } i \neq j\}$, where $d_{ij} = \rho(p_i, p_j)$ is the distance between p_i and p_j. We will show that M_i is the maximal invariant set in $B(p_i, \delta) := \{x \in X : \ \rho(x, p_i) < \delta\}$, where $0 < \delta < d/3$. Indeed, suppose that it is not true, then there exists a compact invariant set $M \subset B(p_i, \delta)$ such that $M \neq M_i$. Let $x \in M \setminus M_i$, then there exists a full trajectory $\gamma \in \Phi_x$ such that $\gamma(\mathbb{S}) \subseteq M$. By Lemma 3 and Lemma 4 there exist points $p, q \in Fix(\pi)$ such that $\alpha_\gamma = \{p\}$ and $\omega_x = \{q\}$. Since $p, q \in M \subset B(p_i, \delta)$, then according to the choice of δ we have $p = q = p_i$, i.e., we obtain a 1-cycle $p_i \mapsto p_i$. Thus the obtained contradiction completes the proof of Lemma. $\square$

5 Chain-recurrent motions

Let $\Sigma \subseteq X$ be a compact positive invariant set, $\varepsilon > 0$ and $t > 0$.

Definition 9. The collection $\{x = x_0, x_1, x_2, \dots, x_k = y; t_0, t_1, \dots, t_k\}$ of the points $x_i \in \Sigma$ and the numbers $t_i \in \mathbb{T}$ such that $t_i \geq t$ and $\rho(x_i t_i, x_{i+1}) < \varepsilon$ $(i = 0, 1, \dots, k-1)$ is called (see, for example, [7,8], [14,15] and [24]) a (ε, t, π)–chain joining the points x and y.

We denote by $P(\Sigma)$ the set $\{(x, y) : x, y \in \Sigma, \forall\ \varepsilon > 0\ \forall\ t > 0\ \exists\ (\varepsilon, t, \pi)$-chain joining x and $y\}$. The relation $P(\Sigma)$ is closed, invariant and transitive [7,14,19,22,24].

Definition 10. The point $x \in \Sigma$ is called chain-recurrent (in Σ) if $(x, x) \in P(\Sigma)$.

We denote by $\mathfrak{R}(\Sigma)$ the set of all chain-recurrent (in Σ) points from Σ.

Definition 11. Let $A \subseteq X$ be a nonempty positive invariant set. The set A is called (see, for example, [18]) internally chain recurrent if $\mathfrak{R}(A) = A$, and internally chain transitive if the following stronger condition holds: for any $a, b \in A$ and any $\varepsilon > 0$ and $t > 0$, there is an (ε, t, π)-chain in A connecting a and b.

The set of all chain recurrent points for $(X, \mathbb{T}, \pi)$ is denoted by $\mathfrak{R}(\Sigma)$, i.e., $\mathfrak{R}(\Sigma) := \{x \in \Sigma : (x, x) \in P(\Sigma)\}$. On $\mathfrak{R}(\Sigma)$ we will introduce a relation $\sim$ as follows: $x \sim y$ if and only if $(x, y) \in P(\Sigma)$ and $(y, x) \in P(\Sigma)$. It is easy to check that the introduced relation $\sim$ on $\mathfrak{R}(\Sigma)$ is a relation of equivalence and, consequently, it is easy to decompose it into the classes of equivalence $\{\mathfrak{R}_\lambda : \lambda \in \mathcal{L}\}$ (internally chain transitive subsets), i.e., $\mathfrak{R}(\Sigma) = \sqcup\{\mathfrak{R}_\lambda :\ \lambda \in \mathcal{L}\}$. By Proposition 2.6 from [7] (see also [14] and [19,22,24] for the semi-group dynamical systems) the defined above components of the decomposition of the set $\mathfrak{R}(\Sigma)$ are closed and positive invariant.

Lemma 8 (see [18]). *Let $x \in X$, $\gamma \in \Phi_x$ and the positive (respectively, negative) semi-trajectory of the point $x \in X$ is relatively compact. Then the ω (respectively, α)-limit set of the point x is internally chain-transitive, i. e., $\mathfrak{R}(\omega_x) = \omega_x$ (respectively, $\mathfrak{R}(\alpha_\gamma) = \alpha_\gamma$).*

Theorem 3 (see [18]). *Assume that each fixed point of $(X, \mathbb{T}, \pi)$ is a locally maximal invariant set, that there is no k-cycle $(k \geq 1)$ of fixed points, and that every pre-compact orbit converges to some fixed point of $(X, \mathbb{T}, \pi)$. Then any compact internally chain-transitive set is a fixed point of $(X, \mathbb{T}, \pi)$.*

Remark 7. 1. Theorem 3 was established in [18] for the dynamical systems with discrete time, i. e., $\mathbb{T} \subseteq \mathbb{Z}$.

2. Theorem 3 for the dynamical systems with continuous time (i.e., $\mathbb{R}_+ \subseteq \mathbb{T}$) may be established with slight modifications of the proof of Theorem 3.2 [18] and using some results from [19].

Theorem 4. *Suppose that the following conditions hold:*

1) $(X, \mathbb{T}, \pi)$ *is a compact dissipative dynamical system and* J *its Levinson center;*

2) $(X, \mathbb{T}, \pi)$ *is a gradient system and* $Fix(\pi) = \{p_1, p_2, \dots, p_m\}$*;*

2) the set $Fix(\pi)$ *does not contain any* k*-cycles* $(k \geq 1)$*.*

Then $\mathfrak{R}(J) = \{p_1, p_2, \dots, p_m\}$*.*

Proof. By Lemma 7 the set $M_i := \{p_i\}$ $(i = \overline{1, m})$ is locally maximal.

Since $Fix(\pi) \subseteq \mathfrak{R}(\pi)$, then to prove this statement it is sufficient to show that $\mathfrak{R}(\pi) \subseteq Fix(\pi)$. Indeed, let $\{\mathfrak{R}_\lambda(\pi) : \lambda \in \Lambda\}$ be the family of all chain transitive components of $\mathfrak{R}(\pi)$, then $\mathfrak{R}(\pi) = \coprod\{\mathfrak{R}_\lambda(\pi) : \lambda \in \Lambda\}$. By Lemma 3 and Theorem 3 for any $\lambda \in \Lambda$ there exists a number $i \in \{1, \dots, m\}$ such that $\mathfrak{R}_\lambda(\pi) = \{p_i\}$ and consequently $\mathfrak{R}(\pi) \subseteq Fix(\pi)$. Theorem is proved. □

Remark 8. Note that using the same arguments as in the proof of Theorem 4 and some properties of the chain current set we can establish a more general statement. Namely, the following statement holds.

Suppose that the following conditions hold:

1) $(X, \mathbb{T}, \pi)$ is a compact dissipative dynamical system and J its Levinson center;

2) M is a closed and invariant subset of $\mathfrak{R}(\pi)$;

3) $M = \bigcup_{k=1}^{m} M_k$, $M_i \bigcap M_j = \emptyset$ (for all $i \neq j$) and M_k $(k = 1, \dots, m)$ is closed and invariant;

3) $\Delta_X \subseteq M$;

4) the family $M_1, \dots, M_m$ of subsets does not contain any l-cycles $(l \geq 1)$.

Then $\mathfrak{R}(J) = M$.

6 Non-autonomous gradient-like dynamical systems

Definition 12. Let $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ be two dynamical systems and $\mathbb{T}_1 \subseteq \mathbb{T}_2$. A triplet $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ is called a non-autonomous dynamical system, where $h : X \mapsto Y$ is a homomorphism from $(X, \mathbb{T}_1, \pi)$ onto $(Y, \mathbb{T}_2, \sigma)$, i.e., $h(\pi(t, x)) = \sigma(h(x), t)$ for all $x \in X$ and $t \in \mathbb{T}_1$.

Definition 13. A mapping $\phi : Y \mapsto X$ is called:

1) a section of bundle space (X, h, Y) if $h \circ \phi = Id_Y$, i.e., $h(\phi(y)) = y$ for all $y \in Y$;

2) an invariant section of non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ if $h \circ \phi = Id_Y$ and $\phi(\sigma(t, y)) = \pi(t, \phi(y))$ for all $y \in Y$ and $t \in \mathbb{T}_1$.

Definition 14. A positive invariant (respectively, negative invariant or invariant) subset M of dynamical system $(X, \mathbb{T}, \pi)$ is called dynamically decomposable if there are two positive invariant (respectively, negative invariant or invariant) subsets M_i $(i = 1, 2)$ of M such that:

1) $M_1 \bigcap M_2 = \emptyset$;

2) $M = M_1 \bigcup M_2$.

Otherwise M is called dynamically indecomposable.

Definition 15. A dynamical system $(X, \mathbb{T}, \pi)$ is called minimal if $H(x) = X$ for all $x \in X$, where $H(x) := \overline{\{\pi(t, x) : \ t \in \mathbb{T}\}}$.

Definition 16. The non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is called gradient-like if the following conditions hold:

1) the space Y is compact and $(Y, \mathbb{T}_2, \sigma)$ is minimal;

2) the dynamical system $(X, \mathbb{T}_1, \pi)$ is compact dissipative and J is its Levinson center;

3) there are a finite number of invariant sections $\phi_1, \phi_2, \dots, \phi_m$ of non-autonomous dynamical system $(X, \mathbb{T}_1, \pi)$ such that $\Delta_X = \coprod_{i=1}^{m} \phi_i(Y)$.

Let $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ be a non-autonomous dynamical system, $X_y := h^{-1}(y) = \{x \in X : \ h(x) = y\}$ $(y \in Y)$ and $p \in X$. Denote by $W^s_y(p) := \{x \in X_y : \lim\limits_{t\to+\infty} \rho(\pi(t, x), \pi(t, p)) = 0\}$ and $W^u_y(p) := \{x \in X_y : \ \lim\limits_{t\to-\infty} \rho(\gamma(t), \pi(t, p)) = 0$ for certain $\gamma \in \Phi_x$ $\}$.

Theorem 5. *Let $\langle (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ be a gradient-like non-autonomous dynamical system, then the following statements take place:*

1) for all $y \in Y$ and $x \in X_y$ there exists a unique invariant section ϕ_i such that $x \in W^s_y(\phi_i(y))$, i.e.,

$$\lim_{t\to+\infty} \rho(\pi(t, x), \phi_i(\sigma(t, y))) = 0; \tag{5}$$

2) $J_y = \bigcup_{i=1}^{m} W^u_y(\phi_i(y))$ for all $y \in Y$.

Proof. Let $y \in Y$ and $x \in X_y$. Since $(X, \mathbb{T}, \pi)$ is compact dissipative, then the positive semi-trajectory $\Sigma^+_x := \{\pi(t, x) : \ t \geq 0\}$ of point x is relatively compact and, consequently, its ω–limit set ω_x is a nonempty, compact, invariant and dynamically indecomposable set. Under the conditions of Theorem 5 we have $\omega_x \subseteq \Delta_X = \bigcup_{i=1}^{m} \phi_i(Y)$. Note that

$$\phi_i(Y) \bigcap \phi_j(Y) = \emptyset$$

for all $i \neq j$ $(1 \leq i, j \leq m)$. In fact, if we suppose the contrary, then there are $i_0 \neq j_0$ $(1 \leq i_0, j_0 \leq m)$ and y_0 such that $\phi_{i_0}(y_0) = \phi_{j_0}(y_0)$. Since Y is minimal, then for any $y \in Y$ there exists a sequence $\{t_n\} \subseteq \mathbb{T}$ such that $y = \lim_{t \to +\infty} \sigma(t_n, y_0)$ and, consequently, $\phi_{i_0}(y) = \lim_{t \to +\infty} \phi_{i_0}(\sigma(t_n, y_0)) = \lim_{t \to +\infty} \phi_{j_0}(\sigma(t_n, y_0)) = \phi_{j_0}(y)$. The obtained contradiction proves our statement. Thus there exists a unique natural number $1 \leq i \leq m$ such that $\omega_x = \phi_i(Y)$ because $\phi_i(Y)$ is a minimal set of the dynamical system $(X, \mathbb{T}, \pi)$ and, consequently,

$$\omega_x \bigcap X_y = \phi_i(y)$$

for all $y \in Y$. Now we will establish the equality (5). If we suppose that it is not true, then there are $y_0 \in Y$, $x_0 \in X_{y_0}$, $\varepsilon_0 > 0$, and $t_n \to +\infty$ such that

$$\rho(\pi(t_n, x_0), \phi_i(\sigma(t_n, y_0))) \geq \varepsilon_0. \tag{6}$$

Under the conditions of Theorem 5 we may suppose that the sequences $\{\sigma(t_n, y_0)\}$ and $\{\pi(t_n, x_0)\}$ are convergent. Denote by $\bar{y}$ and $\bar{x}$ their limits respectively then from (6) we obtain $\bar{x} \neq \phi_i(\bar{y})$. On the other hand $\bar{x} \in X_{\bar{y}} \bigcap \omega_{x_0} = \{\phi_i(\bar{y})\}$, i.e., $\bar{x} = \phi_i(\bar{y})$. The obtained contradiction completes the proof of the first statement of Theorem 5.

Now we will prove the second statement of Theorem 5. Let $y \in Y$, $x \in W^u_y(\phi_i(y))$ and $\gamma \in \Phi_x$, then $x \in J$. In fact, $\gamma(\mathbb{S})$ is relatively compact. Since $\alpha_\gamma \subseteq \Delta_X \subseteq J$, then there exists a sequence $\tau_n \to -\infty$ such that $\gamma(\tau_n) \to p \in J$. Since the Levinson center J of dynamical system $(X, \mathbb{T}, \pi)$ attracts every compact subset from X, then in particular it attracts also the compact subset $\overline{\gamma(\mathbb{S})}$ and, consequently, we have $\rho(x, J) = \lim_{n \to \infty} \rho(\pi(-\tau_n, \gamma(\tau_n)), J) = 0$. This means that $x \in J$. Thus to finish the proof it is sufficient to show that $J_y \subseteq \bigcup_{i=1}^m W^u_y(\phi_i(y))$ for all $y \in Y$. Let $y \in Y$, $x \in J_y$ and $\gamma \in \Phi_x$. Note that $\alpha_\gamma \bigcap X_y \neq \emptyset$. In fact, let $\tau_n \to -\infty$ such that $\sigma(\tau_n, y) \to y$. Since $\gamma(\mathbb{S}) \subseteq J$ we may suppose that the sequence $\{\gamma(\tau_n)\}$ is convergent, then its limit p belongs to $\alpha_\gamma \bigcap X_y$. Evidently, under the conditions of Theorem 5 we have $\alpha_\gamma \bigcap X_y \subseteq \{\gamma_1(y), \gamma_2(y), \ldots, \gamma_m(y)\}$. Thus there are natural numbers $1 \leq i_1 < i_2 < \ldots < i_k \leq m$ such that $\alpha_\gamma \bigcap X_y = \{\gamma_{i_1}(y), \gamma_{i_2}(y), \ldots, \gamma_{i_k}(y)\}$. Note that the following equality

$$\lim_{t \to -\infty} \inf_{1 \leq l \leq k} \rho(\gamma(t), \phi_{i_l}(\sigma(t, y))) = 0$$

takes place. Suppose that it is not true, then there are $\varepsilon_0 > 0$ and $\tau_n \to -\infty$ such that

$$\rho(\gamma(\tau_n), \phi_{i_l}(\sigma(\tau_n, y))) \geq \varepsilon_0 \tag{7}$$

for all $l = 1, 2, \ldots, k$. Under the conditions of Theorem 5 we may suppose that the sequences $\{\sigma(\tau_n, y)\}$ and $\{\gamma(\tau_n)\}$ converge. Denote by $\bar{y}$ (respectively, $\bar{x}$) the limit of the sequence $\{\sigma(\tau_n, y)\}$ (respectively, $\{\gamma(\tau_n)\}$), then $\bar{x} \in X_{\bar{y}}$ and $\bar{x} \in \alpha_\gamma \bigcap X_{\bar{y}} = \{\gamma_{i_1}(\bar{y}), \gamma_{i_2}(\bar{y}), \ldots, \gamma_{i_k}(\bar{y})\}$. On the other hand, passing to the limit in (7) as $n \to \infty$ we obtain $\bar{x} \notin \{\gamma_{i_1}(\bar{y}), \gamma_{i_2}(\bar{y}), \ldots, \gamma_{i_k}(\bar{y})\}$. The obtained contradiction proves our statement. Let us show that there exists a number $1 \leq i_0 \leq k$ such that

$$\lim_{t \to -\infty} \rho(\gamma(t), \phi_{i_0}(\sigma(t, y)) = 0.$$

Denote by $r := \inf\{\rho(\phi_i(y), \phi_j(y)) : \ y \in Y, 1 \le i, j \le m; \ i \ne j\}$. From (6) it follows that the number r is positive. For a number ε, $0 < \varepsilon < r/3$, we will find $L(\varepsilon) > 0$ such that

$$\inf\{\rho(\gamma(t), \phi_{i_l}(\sigma(t, y)) : 1 \le l \le k\} < \varepsilon$$

for all $t < -L(\varepsilon)$. Let $t_0 < -L(\varepsilon)$, then there exists $1 \le i_{l_0} \le k$ such that

$$\rho(\gamma(t_0), \phi_{i_{l_0}}(\sigma(t_0, y))) < \varepsilon.$$

Assume $\delta(t_0) := \sup\{\tilde{\delta} : \ \rho(\gamma(t_0), \phi_{i_{l_0}}(\sigma(t_0, y))) < \varepsilon$ for all $t \in [t_0 - \tilde{\delta}, t_0]\}$. Let us show that $\delta(t_0) = +\infty$. Suppose the contrary, then

$$\rho(\gamma(t_0'), \phi_{i_{l_0}}(\sigma(t_0', y))) \ge \varepsilon$$

where $t_0' = t_0 - \delta(t_0)$, and there exists $k_0 \ne i_0$ $(1 \le k_0 \le k)$ such that

$$\rho(\gamma(t_0'), \phi_{i_{l_0}}(\sigma(t_0', y))) < \varepsilon.$$

On the other hand,

$$\begin{aligned} \rho(\gamma(t_0'), \phi_{i_{k_0}}(\sigma(t_0', y))) \ge \rho(\phi_{i_{l_0}}(\sigma(t_0', y)), \phi_{i_{k_0}}(\sigma(t_0', y))) - \\ \rho(\phi_{i_0}(\sigma(t_0', y), \gamma(t_0')) > r - \varepsilon > 2\varepsilon. \end{aligned} \tag{8}$$

Inequality (8) contradicts the assumption. So, we found $L(\varepsilon) > 0$ and $1 \le i_0 \le k$ such that

$$\rho(\gamma(t), \phi_{i_0}(\sigma(t, y))) < \varepsilon$$

for all $t \le -L(\varepsilon)$. Let us show that the number i_0 does not depend on the choice of ε. In fact, if we suppose the contrary, then we can find numbers ε_1 and ε_2, natural numbers i_1 and i_2 $(1 \le i_1 \ne i_2 \le k)$, and $L(\varepsilon_1) > 0$ and $L(\varepsilon_2) > 0$ satisfying the conditions mentioned above. Assume $L := \max(L(\varepsilon_1), L(\varepsilon_2))$, then

$$\begin{aligned} \rho(\phi_{i_1}(\sigma(t, y)), \phi_{i_2}(\sigma(t, y))) \le \rho(\phi_{i_1}(\sigma(t, y)), \gamma(t)) + \\ \rho(\gamma(t), \phi_{i_2}(\sigma(t, y))) \le \varepsilon_1 + \varepsilon_2 < 2r/3 < r. \end{aligned} \tag{9}$$

Inequality (9) contradicts the choice of r. Theorem is completely proved. $\square$

Definition 17. Let $\langle (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h \rangle$ be a non-autonomous dynamical system and Y be a compact minimal set. A compact minimal set $M \subseteq X$ is called [20, 25] an m–fold covering of Y if *card* $h^{-1}(y) = m$ for all $y \in Y$.

Theorem 6 (see [6, 20, 25, 26]). *Let $\langle (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h \rangle$ be a non-autonomous dynamical system and Y be a compact minimal set, then the following statements are equivalent:*

1) *a compact minimal set $M \subseteq X$ is an m–fold covering of Y;*

2) (a) *there exists a $y_0 \in Y$ such that card $h^{-1}(y_0) = m$;*

(b) *the minimal set M is distal with respect to $\langle (X,\mathbb{T},\pi),(Y,\mathbb{T},\sigma),h\rangle$, i.e., $\inf_{t\in\mathbb{T}}\rho(\pi(t,x_1),\pi(t,x_2)) > 0$ for all $(x_1,x_2) \in X \bigotimes X := \{(x_1,x_2) : x_1,x_2 \in X$ with condition $h(x_1)=h(x_2)\}$.*

Theorem 7. *Let $\langle (X,\mathbb{T},\pi),(Y,\mathbb{T},\sigma),h\rangle$ be a non-autonomous dynamical system satisfying the following conditions:*

1) *Y is a compact minimal set;*
2) *the dynamical system $(X,\mathbb{T},\pi)$ is compact dissipative and J is its Levinson center;*
3) *there are a finite number of minimal compact subsets $M^1,M^2,\dots,M^k \subseteq X$ such that $\Delta_X \subseteq \bigcup_{i=1}^k M^i$;*
4) *for every $i=1,2,\dots,k$ there exists a natural number m_i such that $card M^i_y = m_i$ for all $y\in Y$, where $M_y := h^{-1}(y)$.*

Then the following statements take place:

1) *for all $y\in Y$ and $x\in X_y$ there exist a unique natural number $1\le i\le k$ and a point $p_i\in M^i_y$ such that $x\in W^s_y(p_i)$, i.e.,*

$$\lim_{t\to+\infty}\rho(\pi(t,x),\pi(t,p_i))=0; \tag{10}$$

2) *$J_y=\bigcup_{i=1}^m \bigcup_{p\in M^i_y} W^u_y(p)$ for all $y\in Y$.*

Proof. This statement can be proved with slight modification of the proof of Theorem 5. $\square$

Definition 18. The point x of dynamical system $(X,\mathbb{T},\pi)$ is called τ ($\tau\in\mathbb{T}$) periodic if Φ_x contains a motion γ such that $\gamma(t+\tau)=\gamma(t)$ for all $t\in\mathbb{S}$.

Denote by $Per(\pi)$ the set of all periodic points of $(X,\mathbb{T},\pi)$, $W^s(p) := \{x\in X : \lim_{t\to+\infty}\rho(\pi(t,x),\pi(t,p))=0\}$ and $W^u(p) := \{x\in X : \lim_{t\to-\infty}\rho(\gamma(t),\pi(t,p)) = 0$ for certain $\gamma\in\Phi_x\}$.

Corollary 2. *Suppose that $(X,\mathbb{T},\pi)$ is an autonomous dynamical system with discrete time (i.e., $\mathbb{T}\subseteq\mathbb{Z}$) and the following conditions are fulfilled:*

1) *the dynamical system $(X,\mathbb{T},\pi)$ is compact dissipative and J is its Levinson center;*
2) *the set $Per(\pi)$ contains a finite number of points, i. e., $Per(\pi)=\{p_1,p_2,\dots,p_m\}$;*
3) *$\Delta_X\subseteq Per(\pi)$.*

Then the following statements hold:

1) *for all* $x \in X$ *there exists a unique natural number* $1 \leq i \leq m$ *such that* $x \in W_y^s(p_i)$, *i. e.,*
$$\lim_{t \to +\infty} \rho(\pi(t,x), \pi(t,p_i)) = 0;$$

2) $J = \bigcup_{i=1}^m W^u(p_i)$.

Proof. Formulated statements directly follow from Theorem 7. □

Remark 9. If the dynamical system $(X, \mathbb{T}, \pi)$ with continuous time (i. e., $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{R}_+$) admits at least one nontrivial periodic point, then:

1. $Per(\pi)$ contains a continuum subset. In fact, if $p \in Per(\pi)$ is a τ–periodic point, then $\pi(t, t_1) \neq \pi(t_2, p)$ for all $t_1, t_2 \in (0, \tau)$ $(t_1 \neq t_2)$ and, consequently, $\{\pi(t,p) : \ t \in [0,\tau]\} \subseteq Per(\pi)$ and $\{\pi(t,p) : \ t \in [0,\tau]\}$ is homeomorphic to $[0,\tau]$.

2. Corollary 2 does not take place.

The last statement (second item of Remark 9) can be confirmed by the following example.

Example 1. Let us consider the dynamical system defined on the plane $\mathbb{R}^2$ by the following rule. Let the origin $O(0,0)$ be a stationary point, the unit circumference S^1 be the trajectory of the periodic motion with the period $\tau = 1$. The rest of motions will be not singular. And besides we assume that every semi-trajectory Σ_x^+ is not un. st. $\mathcal{L}^+\Sigma_x^+$ for every point $x \in \mathbb{R}^2 \setminus (S^1 \cup O)$. The described dynamical system is given by the system of differential equations which in polar coordinates looks as the following:
$$\begin{cases} \dot{r} = (r-1)^2 \\ \dot{\varphi} = r. \end{cases}$$
It is easy to see that ω–limit set of the point x is a trajectory of 1–periodic point for all $x \in \mathbb{R}^2 \setminus O$, but the point x itself is not asymptotically 1–periodic, since Σ_x^+ is not un.st. $\mathcal{L}^+\Sigma_x^+$ (see Theorem 1.3.1 from [11]). In this example we have $\Delta_X = S^1 \cup \{O\}$.

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Received May 2, 2014

BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(75), 2014, Pages 102–105
ISSN 1024–7696

One new class of cubic systems with maximum number of invariant lines omitted in the classification of J. Llibre and N. Vulpe

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Abstract. We present a new class of cubic systems with invariant lines of total multiplicity 9, including the line at infinity endowed with its own multiplicity. This class is different from the 23 classes included in the classification given in [4] by J. Llibre and N. Vulpe.

Mathematics subject classification: 34G20, 34A26, 14L30, 34C14.
Keywords and phrases: Cubic differential system, configuration of invariant straight lines, multiplicity of an invariant straight line, group action, affine invariant polynomial.

Consider real cubic systems, i. e. systems of the form:

$$\begin{aligned} \dot{x} &= p_0 + p_1(x,y) + p_2(x,y) + p_3(x,y) \equiv P(a,x,y), \\ \dot{y} &= q_0 + q_1(x,y) + q_2(x,y) + q_3(x,y) \equiv Q(a,x,y) \end{aligned} \tag{1}$$

with real coefficients and variables x and y. The polynomials p_i and q_i $(i = 0, 1, 2, 3)$ are homogeneous polynomials of degree i in x and y.

Let

$$\mathbf{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$$

be the polynomial vector field corresponding to systems (1).

A straight line $f(x,y) = ux + vy + w = 0$, $(u,v) \neq (0,0)$ satisfies

$$\mathbf{X}(f) = uP(x,y) + vQ(x,y) = (ux + vy + w)R(x,y)$$

for some polynomial $R(x,y)$ if and only if it is *invariant* under the flow of the systems. If some of the coefficients u, v, w of an invariant straight line belongs to $\mathbb{C}\setminus\mathbb{R}$, then we say that *the straight line is complex*; otherwise *the straight line is real.* Note that, since systems (1) are real, if a system has a complex invariant straight line $ux + vy + w = 0$, then it also has its conjugate complex invariant straight line $\bar{u}x + \bar{v}y + \bar{w} = 0$.

Definition 1. We say that an invariant affine straight line $f(x,y) = ux+vy+w = 0$ (respectively the line at infinity $Z = 0$) for a cubic vector field $\mathbf{X}$ has multiplicity m if there exists a sequence of real cubic vector fields $\mathbf{X}_k$ converging to $\mathbf{X}$, such

*This work is partially supported by the grant FP7-PEOPLE-2012-IRSES-316338

that each $\mathbf{X}_k$ has m (respectively $m-1$) distinct invariant affine straight lines $f_i^j = u_i^j x + v_i^j y + w_i^j = 0$, $(u_i^j, v_i^j) \neq (0,0)$, $(u_i^j, v_i^j, w_i^j) \in \mathbb{C}^3$, converging to $f = 0$ as $k \to \infty$ (with the topology of their coefficients), and this does not occur for $m+1$ (respectively m).

Definition 2 (see [5]). Consider a planar cubic system (1). We call *configuration of invariant straight lines* of this system, the set of (complex) invariant straight lines (which may have real coefficients) of the system, each endowed with its own multiplicity and together with all the real singular points of this system located on these invariant straight lines, each one endowed with its own multiplicity.

According to [1] the maximum number of invariant straight lines taking into account their multiplicities (including the line at infinity) for cubic systems is 9. A classification of all cubic systems possessing the maximum number of invariant straight lines taking into account their multiplicities have been made in [4]. In this paper the authors have detected 23 classes of such cubic systems and have constructed the corresponding canonical forms. Moreover they proved that modulo affine group and time rescaling each such class is represented by a specific cubic system without parameters.

We are interested in the classification of cubic systems which possess invariant lines of total multiplicity 8. For this purpose we split the whole family of cubic systems in four subfamilies, depending on the number of distinct infinite singularities.

It is well known that the infinite singularities (real or complex) of systems (1) are determined by the linear factors of the polynomial

$$C_3 = yp_3(x,y) - xq_3(x,y).$$

In the paper [2] the cubic systems with 4 distinct infinite singularities are examined and it was proved that there exist 17 classes of such cubic systems with invariant lines of total multiplicity 8. All possible distinct configurations of invariant straight lines are determined and the corresponding necessary and sufficient conditions are constructed in terms of affine invariant polynomials (see for instance [6, 7]).

The second family of cubic systems which possess 3 distinct infinite singularities has been examined in [3]. We proved that this family of systems can only have 5 distinct configurations of invariant lines of total multiplicity 8 and determined the corresponding affine invariant criteria.

Now the family of cubic systems with two distinct infinite singularities is under examination. And in the case when infinite singularities of cubic systems are determined by one simple and one triple factors of C_3, we have detected a new class of cubic systems with maximum number (nine) of invariant straight lines. In what follows we show that this class is omitted in the classification given by J. Llibre and N. Vulpe in [4].

Indeed, considering the family of cubic systems with 8 invariant lines (including the line at infinity and including multiplicities) which in addition possesses two infinite singularities, we found out that a subfamily of these systems could be brought via affine transformation and time rescaling to the canonical form

$$\dot{x} = x(r + 2x + x^2), \quad \dot{y} = y(r + 2x), \quad 0 \neq r \in \mathbb{R}, \tag{2}$$

depending on one parameter. We observe that these systems possess invariant lines of total multiplicity 8. More precisely, we have the invariant affine lines: $x = 0$ (triple), $y = 0$, $x^2 + 2x + r = 0$ (simple real or complex or real double) and the line at infinity ($Z = 0$), which is double.

We detected that in the case $r = 8/9$ the obtained system

$$\dot{x} = x(2 + 3x)(4 + 3x)/9, \quad \dot{y} = 2(4 + 9x)y/9 \tag{3}$$

possesses invariant lines of total multiplicity 9, and namely: $x = 0$ (triple), $x = -2/3$ (double), $x = -4/3$ and $y = 0$ (both simple) and the line at infinity ($Z = 0$) (double).

To prove this it is sufficient to present the following corresponding perturbed systems

$$\dot{x} = x(2 + 3x)(4 + 3x)/9, \quad \dot{y} = 2y(1 + \varepsilon y)(4 + 9x - 4\varepsilon y)/9,$$

which possess the following 8 invariant affine lines:

$$x = 0, \quad y = 0, \quad x = -2/3, \quad x = -4/3, \quad 3x - 4\varepsilon y = 0,$$
$$3x - 2\varepsilon y = 0, \ 1 + \varepsilon y = 0, \ 3x - 2\varepsilon y + 2 = 0$$

Thus system (3) indeed possesses invariant lines of total multiplicity 9 (including the infinite one).

On the other hand in [4] nine classes of cubic systems with two infinite singularities determined by one simple and one triple factors of C_3 are given and their corresponding configurations are presented in Figures 14–22.

Considering the configuration of invariant lines of system (3) given in Fig. 1 we observe that this configuration is different from configurations given in Figures 14–22 [4].

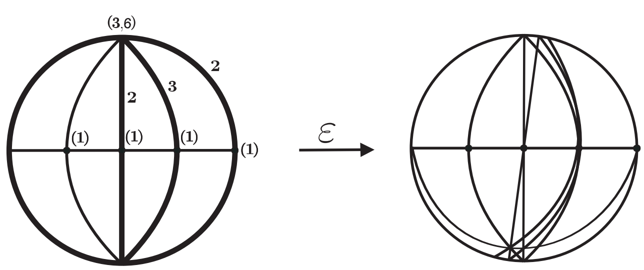

Figure 1. The configuration of invariant lines corresponding to system (3)

Remark. In the above configuration if an invariant straight line has multiplicity $k > 1$, then the number k appears near the corresponding straight line and this line is in bold face. We indicate next to the real singular points of the system, located on the invariant straight lines, their corresponding multiplicities. In the case of infinite singularities we denote by '(a, b)' the maximum number a (respectively b) of infinte (respectively finite) singularities which can be obtained by perturbation of the multiple point.

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Received July 22, 2014