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On stability of multicriteria investment Boolean problem with Wald's efficiency criteria

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Abstract. Based on Markowitz's portfolio theory we construct the multicriteria Boolean problem with Wald's maximin efficiency criteria and the Pareto-optimality principle. We obtained lower and upper attainable bounds for the stability radius of the problem in the cases of linear metric l_1 in the portfolio and the market state spaces and of the Chebyshev metric l_∞ in the criteria space.

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Back in the early XX century J. Hadamard included the stability in the concept of the correct mathematical problem as a necessary condition that reflects some physical reality. Subsequently it was found that many mathematical problems are unstable to small changes in input data (parameters). In 1960 this led to the creation of the theory of ill-posed problems, basics of which were laid by A. N. Tikhonov, M. M. Lavrentiev, V. K. Ivanov and others (see e. g. [1–3]).

Usually, the stability of the optimization problem (both scalar and vector) is understood as one of the classical properties of continuity or semi-continuity optimal mapping [4–7]. In the case of the discrete problem the definition of the stability rephrases easily in terms of the existence of 'the stability ball', i. e. a surroundings of the initial data in the problem parameter space, that any 'perturbed' problem with the parameters from this surroundings has some property of invariance to the original problem.

The widespread occurrence of discrete optimization models has given a start to the interest of many experts to studying various types of stability aspects, parametric and post-optimal analysis of both scalar (single criterion) and vector (multicriteria) discrete optimization problems (e. g. monographs [7–9], surveys [10–12], and annotated bibliographies [13, 14]).

One of the well-known approaches to the stability analysis of multicriteria discrete optimization problems is focused on obtaining quantitative characteristics of the stability and consists in finding an ultimate level of perturbations of the initial data of the problem that do not result in new Pareto-optimal solutions. The majority of the results in this field is related to deriving formulas or estimates for the stability radius of multicriteria problems of Boolean and integer programming with linear criteria [12, 15–18].

In the present paper we continue the started in [19–25] research of varies types of the stability of multicriteria non-linear investment problems, formulation of which is based on Markowitz's classical portfolio theory. Here we obtained lower and upper attainable bounds for the stability radius of the multicriteria investment problem with Wald's maximin economic efficiency criteria and the Pareto-optimal principle in the case of the linear metric l_1 in the portfolio and the market state spaces, and the Chebyshev metric l_∞ in the efficiency criteria space. We notice that in [26] with such combination of metrics l_1 and l_∞ i the similar lower and upper bounds of the stability radius of the multicriteria investment problem with Wald's ordered minimax criteria were announced.

1 Problem statement and basic definitions

We consider the multicriteria discrete variant of Markowitz's investment managing problem [27]. To this end, we introduce the following notations. Let $N_n = \{1, 2, \dots, n\}$ be the set of alternative investment projects (assets); N_m be the set of possible market states (situation); $x = (x_1, x_2, \dots, x_n)^T \in X \subseteq \mathbf{E}^n$ be the investment portfolio with components $x_j = 1$ if investment project $j \in N_n$ is implemented, and $x_j = 0$ otherwise. Here $\mathbf{E} = \{0, 1\}$.

There are several approaches to evaluate the efficiency of investment projects (NPV, NFV, PI et al.), which take into account risk and uncertainty in different ways (see e. g. [28–31]). Let N_s be the set of project efficiency indicator. An investment portfolio x is evaluated by $\sum\limits_{j \in N_n} e_{ijk} x_j$, where e_{ijk} is the predicted economic efficiency of the indicator $k \in N_s$ of the investment project $j \in N_n$ in the case when the market is in the state $i \in N_m$. In this context the initial data of the problem is a 3-dimensional matrix of the project efficiency E of the size $m \times n \times s$ with elements e_{ijk} from $\mathbf{R}$.

Let the following vector objective function

$$f(x, E) = (f_1(x, E_1), f_2(x, E_2), \dots, f_s(x, E_s)),$$

be given on a set of investment portfolios X whose components are Wald's maximin criteria (extreme pessimism) [32]

$$f_k(x, E_k) = \min_{i \in N_m} E_{ik} x = \min_{i \in N_m} \sum_{j \in N_n} e_{ijk} x_j \to \max_{x \in X}, \qquad k \in N_s,$$

where $E_k \in \mathbf{R}^{m \times n}$ is the k-th cut of the 3-dimension matrix $E = [e_{ijk}] \in \mathbf{R}^{m \times n \times s}$, $E_{ik} = (e_{i1k}, e_{i2k}, ..., e_{ink})$ is the i-th row of that cut. Thus, the investor in the unstable economic state, following Wald's criteria, takes extreme caution and optimizes portfolio efficiency $E_{ik} x$ assuming that the market is in the worst state. Such caution is appropriate, because the investment is the exchange of a certain value today for an uncertain value in the future.

A multicriteria investment Boolean problem $Z^s(E)$, $s \geq 1$, means the problem of searching the Pareto set $P^s(E)$, i. e. the Pareto-optimal investment portfolios, where

$$P^s(E) = \{x \in X: \ P^s(x, E) = \emptyset\},$$
$$P^s(x, E) = \{x' \in X: \ x' \underset{E}{\succ} x\},$$
$$x' \underset{E}{\succ} x \quad \Leftrightarrow \quad g(x', x, E) \geq \mathbf{0}^{(s)} \ \ \& \ \ g(x', x, E) \neq \mathbf{0}^{(s)},$$
$$g(x', x, E) = (g_1(x', x, E_1), g_2(x', x, E_2), \dots, g_s(x', x, E_s)),$$
$$g_k(x', x, E_k) = f_k(x', E_k) - f_k(x, E_k) = \max_{i \in N_m} \min_{i' \in N_m} (E_{i'k}x' - E_{ik}x), \ k \in N_s,$$
$$\mathbf{0}^{(s)} = (0, 0, \dots, 0)^T \in \mathbf{R}^s.$$

In the portfolio space $\mathbf{R}^n$ and the market state space $\mathbf{R}^m$ we define the linear metric l_1, and in the efficiency criteria space $\mathbf{R}^s$ we define the Chebyshev metric l_∞, i. e. for any matrix $E \in \mathbf{R}^{m \times n \times s}$

$$\|E_{ik}\|_1 = \sum_{j \in N_n} |e_{ijk}|, \quad i \in N_m, \quad k \in N_s,$$
$$\|E_k\|_{11} = \sum_{i \in N_m} \|E_{ik}\|_1 = \sum_{i \in N_m} \sum_{j \in N_n} |e_{ijk}|, \quad k \in N_s,$$
$$\|E\|_{11\infty} = \max_{k \in N_s} \|E_k\|_{11} = \max_{k \in N_s} \sum_{i \in N_m} \|E_{ik}\|_1 = \max_{k \in N_s} \sum_{i \in N_m} \sum_{j \in N_n} |e_{ijk}|.$$

Thus, for any indexes $i \in N_m$ and $k \in N_s$,the following inequalities are true:

$$\|E_{ik}\|_1 \leq \|E_k\|_{11} \leq \|E\|_{11\infty}.$$

Apart from that, using the evident relation $E_{ik}x \geq -\|E_{ik}\|_1$, $x \in \mathbf{E}^n$, it is easy to see that for any portfolios x, x' the following inequalities hold:

$$E_{ik}x - E_{i'k}x' \geq -\|E_k\|_{11}, \quad i, \ i' \in N_m, \quad k \in N_s. \tag{1}$$

As usually [12,15,17], the stability radius of the problem $Z^s(E)$, $s \geq 1$, is defined as the number

$$\rho = \rho(m, n, s) = \begin{cases} \sup \ \Xi, & \text{if } \Xi \neq \emptyset, \\ 0, & \text{if } \Xi = \emptyset, \end{cases}$$

where

$$\Xi = \{\varepsilon > 0: \ \forall E' \in \Omega(\varepsilon) \quad (P^s(E + E') \subseteq P^s(E))\},$$

$\Omega(\varepsilon) = \{E' \in \mathbf{R}^{m \times n \times s}: \ 0 < \|E'\|_{11\infty} < \varepsilon\}$ is the set of perturbing matrices, $P^s(E + E')$ is the Pareto set of the perturbed problem $Z^s(E + E')$. Thus, the stability radius defines an extreme level of perturbations of the elements of the matrix E such that new Pareto-optimal portfolios do not appear. In this context the stability of the problem $Z^s(E)$ is when the set Ξ is not empty, i. e. $\rho(m, n, s) > 0$.

Thus, the problem stability $Z^s(E)$ can be considered as the discrete analogue of the upper Hausdorff semicontinuity problem [5–7] at point E of the optimal mapping

$$P^s: \quad \mathbf{R}^{m \times n \times s} \to 2^{\mathbf{E}^n},$$

i.e. the point-set mapping which puts in correspondence the set of Pareto-optimal portfolios to each point of the space of problem parameters.

Obviously, if the equality $P^s(E) = X$ holds, the stability radius of the problem $Z^s(E)$ equals infinity. Therefore, in what follows, we will not consider this case and will call the problem $Z^s(E)$ for which the set $X \setminus P^s(E)$ is nonempty nontrivial one.

2 Stability radius bounds

For a nontrivial problem $Z^s(E)$ denote

$$\varphi = \varphi(m,n,s) = \min_{x \notin P^s(E)} \max_{x' \in P^s(x,E)} \min_{k \in N_s} \max_{i \in N_m} \min_{i' \in N_m} (E_{i'k}x' - E_{ik}x).$$

Whereas for any portfolio $x \notin P^s(E)$ the set $P^s(x,E)$ is not empty, then we have the formula

$$\forall x \notin P^s(E) \quad \forall x' \in P^s(x,E) \qquad (x' \underset{E}{\succ} x).$$

Therefore, $\varphi \geq 0$.

Theorem 1. *Given $Z^s(E)$. The stability radius $\rho(m,n,s)$ of the multicriteria nontrivial investment problem $Z^s(E)$, $s \geq 1$, has the following lower and upper bounds:*

$$\varphi(m,n,s) \leq \rho(m,n,s) \leq mn\varphi(m,n,s).$$

Proof. To prove Theorem 1, we will first prove the inequality $\rho \geq \varphi$. This inequality is obvious if $\varphi = 0$. Let $\varphi > 0$. According to the definition of φ for any portfolio $x \notin P^s(E)$ there exists a Pareto-optimal portfolio $x^0 \in P^s(x,E)$ such that

$$\max_{i \in N_m} \min_{i' \in N_m} (E_{i'k}x^0 - E_{ik}x) \geq \varphi, \quad k \in N_s.$$

Hence, considering inequality (1), for any matrix $E' \in \mathbf{R}^{m \times n \times s}$ and any index $k \in N_s$ we have

$$g_k(x^0, x, E_k + E'_k) = \max_{i \in N_m} \min_{i' \in N_m} (E_{i'k}x^0 - E_{ik}x + E'_{i'k}x^0 - E'_{ik}x)$$
$$\geq \max_{i \in N_m} \min_{i' \in N_m} (E_{i'k}x^0 - E_{ik}x) - \|E'_k\|_{11} \geq \varphi - \|E'_k\|_{11}.$$

Therefore, assuming that $E' \in \Omega(\varphi)$, we obtain $g_k(x^0, x, E_k + E'_k) > 0$, $k \in N_s$. This means that $x^0 \underset{E+E'}{\succ} x$, i.e. x is not the Pareto-optimal portfolio of the perturbed problem $Z^s(E + E')$. Summarizing and taking into account $x \notin P^s(E)$, we conclude that

$$\forall E' \in \Omega(\varphi) \quad (P^s(E + E') \subseteq P^s(E)).$$

Hence, the inequality $\rho(m,n,s) \geq \varphi(m,n,s)$ is true.

Then let us prove the inequality $\rho \leq mn\varphi$. According to the definition of the number φ there exists a portfolio $x^* \notin P^s(E)$ such that for any portfolio $x \in P^s(x^*, E)$ there exists an index $l = l(x) \in N_s$ such that

$$\max_{i \in N_m} \min_{i' \in N_m} (E_{i'l}x - E_{il}x^*) \leq \varphi. \tag{2}$$

Then we assume $\varepsilon > mn\varphi$ and consider the perturbing matrix $E^0 = [e^0_{ijk}] \in \mathbf{R}^{m\times n\times s}$, elements of which we define as follows:

$$e^0_{ijk} = \begin{cases} \delta, & \text{if } i \in N_m,\ x^*_j = 1,\ k \in N_s, \\ -\delta & \text{otherwith,} \end{cases}$$

where $\varphi < \delta < \varepsilon/mn$. We note that the elements of the matrix E^0 do not depend on a portfolio x, and therefore they do not depend on an index l. Taking into account the structure of the matrix E^0, we obtain

$$\|E^0_{ik}\|_1 = n\delta, \quad i \in N_m, \quad k \in N_s,$$

$$\|E^0\|_{11\infty} = \|E^0_k\|_{11} = mn\delta, \quad k \in N_s.$$

Therefore, $E^0 \in \Omega(\varepsilon)$. Moreover, all the rows E^0_{ik}, $i \in N_m$ of any cuts E^0_k, $k \in N_s$, are the same and consist of the components δ and $-\delta$. We denote the same row by A and obtain

$$A(x - x^*) = -\delta\|x - x^*\|_1 \leq -\delta < -\varphi \leq 0. \tag{3}$$

Hence, considering (2) and the structure of the perturbing matrix E^0, we conclude that for any portfolio $x \in P^s(x^*, E)$ the following relations are true:

$$g_l(x, x^*, E_l + E^0_l) = \min_{i\in N_m}(E_{il} + A)x - \min_{i\in N_m}(E_{il} + A)x^*$$
$$= \max_{i\in N_m}\min_{i'\in N_m}(E_{i'l}x - E_{il}x^*) + A(x - x^*) < 0.$$

Therefore, we obtain

$$\forall x \in P^s(x^*, E) \quad (x \notin P^s(x^*, E + E^0)). \tag{4}$$

Let now the portfolio $x \notin P^s(x^*, E)$. Then the following two cases are possible.

Case 1. $g(x, x^*, E) = \mathbf{0}^{(s)}$. Then according to relations (3) for any index $k \in N_s$ we have

$$g_k(x, x^*, E_k + E^0_k) = \min_{i\in N_m}\ (E_{ik} + A)x - \min_{i\in N_m}\ (E_{ik} + A)x^*$$
$$= g_k(x, x^*, E_k) + A(x - x^*) < 0.$$

Case 2. There exists an index $p \in N_s$ such that $g_p(x, x^*, E_p) < 0$. Then using again (3) we obtain $g_p(x, x^*, E_p + E^0_p) < 0$.

Thus, $x \notin P^s(x^*, E+E^0)$ if $x \notin P^s(x^*, E)$. Considering (4), as a result we obtain $P^s(x^*, E + E^0) = \emptyset$, i.e. x^* is a Pareto-optimal portfolio of the perturbed problem $Z^s(E + E^0)$. Since $x^* \notin P^s(E)$ we may conclude that

$$\forall\varepsilon > mn\varphi \quad \exists E^0 \in \Omega(\varepsilon) \qquad (P^s(E + E^0) \not\subseteq P^s(E)).$$

Hence, the inequality $\rho(m, n, s) \leq mn\varphi(m, n, s)$ is true. □

Corollary 1. *The stability radius $\rho(m, n, s)$ equals zero if and only if $\varphi(m, n, s)$ equals zero.*

3 Attainability of the lower bound

Let us show that the lower bound for the problem stability radius, indicated in Theorem 1, is attainable.

Theorem 2. *There exists a class of multicriteria investment problems $Z^s(E)$, $s \geq 1$ such that for the stability radius of every problem of this class the following formula is true:*

$$\rho(m, n, s) = \varphi(m, n, s). \tag{5}$$

Proof. We will consider the class of problems $Z^s(E)$ such that the following terms are right:

$$X = \{x^0, x^*\}, \quad P^s(x^*, E) = \{x^0\},$$

i.e. $x^0 \underset{E}{\succ} x^*$, $x^* \notin P^s(E)$, $x^0 \in P^s(E)$. Then there exists an index $l \in N_s$ such that

$$g_l(x^0, x^*, E_l) = \varphi. \tag{6}$$

We also suppose that there exists an index $p \in N_n$ such that $x^0_p = 1$ and $x^*_p = 0$.

Further we introduce the notation

$$i(x^0) = \arg\min\{E_{il}x^0 : \ i \in N_m\},$$
$$i(x^*) = \arg\min\{E_{il}x^* : \ i \in N_m\}.$$

The numbers $i(x^0)$ and $i(x^*)$ can be either the same or different. The further proof does not depend on it.

For any number $\varepsilon > \varphi$ we define the elements of the perturbing matrix $E^0 = [e^0_{ijk}] \in \mathbf{R}^{m\times n\times s}$ by the rule

$$e^0_{ijk} = \begin{cases} -\delta, & \text{if} \ \ i = i(x^0), \quad j = p, \quad k = l, \\ 0 & \text{otherwise}, \end{cases} \tag{7}$$

where

$$\varphi < \delta < \varepsilon. \tag{8}$$

Then the next equalities are obvious:

$$E^0_{i(x^0)l}x^0 = -\delta, \tag{9}$$

$$E^0_{il}x^0 = 0, \quad i \in N_m \setminus \{i(x^0)\}, \tag{10}$$

$$E^0_{il}x^* = 0, \quad i \in N_m, \tag{11}$$

$$\|E^0\|_{11\infty} = \|E^0_l\|_{11} = \|E^0_{il}\|_1 = \delta, \quad i \in N_m.$$

Therefore, $E^0 \in \Omega(\varepsilon)$.

Using (9) and (10), we obtain

$$f_l(x^0, E_l + E^0_l) = \min\Big\{(E_{i(x^0)l} + E^0_{i(x^0)l})x^0, \ \min_{i\neq i(x^0)}(E_{il} + E^0_{il})x^0\Big\} =$$

$$= \min\Big\{ f_l(x^0, E_l) - \delta,\ \min_{i \neq i(x^0)} E_{il} x^0 \Big\} = f_l(x^0, E_l) - \delta. \tag{12}$$

And from (11) the following relations are true:

$$f_l(x^*, E_l + E_l^0) = \min\Big\{ (E_{i(x^*)l} + E^0_{i(x^*)l}) x^*,\ \min_{i \neq i(x^*)} (E_{il} + E^0_{il}) x^* \Big\} =$$

$$= \min\Big\{ f_l(x^*, E_l),\ \min_{i \neq i(x^*)} E_{il} x^* \Big\} = f_l(x^*, E_l).$$

Hence, consistently applying (12), (6) and (8), we have

$$g_l(x^0, x^*, E_l + E_l^0) = g_l(x^0, x^*, E_l) - \delta = \varphi - \delta < 0.$$

Therefore, $x^0 \notin P^s(x^*, E + E^0)$, i.e. $P^s(x^*, E + E^0) = \emptyset$. It proves that x^* is a Pareto-optimal investment portfolio of the perturbed problem $Z^s(E + E^0)$. Thence, because of $x^* \notin P^s(E)$ we derive

$$\forall \varepsilon > \varphi \quad \exists E^0 \in \Omega(\varepsilon) \qquad (P^s(E + E^0) \not\subseteq P^s(E)).$$

Thus, $\rho(m, n, s) \leq \varphi(m, n, s)$. Hence, by Theorem 1 the formula (5) is true. $\square$

Remark 1. If $m = 1$ then $i(x^0) = i(x^*)$. Therefore, as we noted earlier, the proof of Theorem 2 given above is true in this case. Hence, there exists a class of multicriteria linear Boolean programming problems $Z^s_B(E)$ whose stability radius equals $\varphi(1, n, s)$.

We give a numerical example that illustrates the statement of Theorem 2.

Example. Let $m = 2$, $n = 3$, $s = 2$; $X = \{x^0, x^*\}$, $x^0 = (0, 1, 1)^T$, $x^* = (1, 1, 0)^T$; $E \in \mathbf{R}^{2\times 3\times 2}$ is the matrix with cuts

$$E_1 = \begin{pmatrix} 5 & 1 & 2 \\ 2 & 0 & 4 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 6 & 2 & 3 \\ 2 & 1 & 5 \end{pmatrix}.$$

Then $p = 3$, $f(x^0, E) = (E_1 x^0, E_2 x^0) = (3, 5)$, $f(x^*, E) = (E_1 x^*, E_2 x^*) = (2, 3)$, $g(x^0, x^*, E) = (1, 2)$. Hence, $x^* \notin P^2(E)$, $\{x^0\} = P^2(x^*, E)$, $l = 1$, $i(x^0) = 1$, $i(x^*) = 2$. Therefore, $\varphi = \varphi(2, 3, 2) = \min\{1, 2\} = 1$. Further we will show that $\rho(2, 3, 2) \leq \varphi = 1$.

Since $e^0_{i(x^0)pl} = e^0_{131}$ then defining the cuts E_1^0 and E_2^0 of the perturbing matrix E^0 according to the rule (7), we obtain

$$E_1^0 = \begin{pmatrix} 0 & 0 & -\delta \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\delta > \varphi = 1$. Then it is easy to see in view of $l = 1$ that

$$g_1(x^0, x^*, E_1 + E_1^0) = g_1(x^0, x^*, E_1) - \delta = 1 - \delta < 0.$$

Hence, $x^* \in P^2(E + E^0)$. This inclusion and $\|E^0\|_{11\infty} = \delta > 1$, $x^* \notin P^2(E)$ gives $\rho(2, 3, 2) \leq 1$. Therefore, considering Theorem 1, we conclude that $\rho(2, 3, 2) = \varphi(2, 3, 2) = 1$.

4 Attainability of the upper bound

Let us show that the upper bound of the stability radius of the problem $Z^s(E)$ is attainable for $m = s = 1$. It is easy to see that in the particular case for $m = 1$ our problem $Z^s(E)$ transforms into a multicriteria linear Boolean programming problem, which we will write in the convenient form

$$Z^s_B(E): \quad f_k(x, E_k) = E_k x \to \max_{x \in X}, \quad k \in N_s,$$

where $X \subseteq \mathbf{E}^n$, E_k is the k-th row of the matrix $E = [e_{kj}] \in \mathbf{R}^{s\times n}$. Such case can be interpreted as the situation when the investor has not another alternative market state. As earlier, the metric l_∞ is in the criteria space $\mathbf{R}^s$, and the metric l_1 is in the solution space $\mathbf{R}^n$.

Theorem 3. *For $m = s = 1$ there exists a class of scalar linear Boolean programin problems $Z^1_B(E)$, $E \in \mathbf{R}^{1\times n}$ such that for the stability radius of every problem of this class the following formula is true:*

$$\rho(1, n, 1) = n\varphi(1, n, 1). \tag{13}$$

Proof. Let us show that there exists a class with $X = \{x^*, x^1, x^2, \dots, x^n\} \subset \mathbf{E}^n$, $n \geq 2$, where $x^* = \mathbf{0}^{(n)}$, $x^j = e^j$, $j \in N_n$. Here e^j is the j-th column of an identity matrix of size $n \times n$. Let $E = (a, a, \dots, a) \in \mathbf{R}^n$ in view of $m = s = 1$, where $a > 0$. Therefore, we have $f(x^*, E) = Ex^* = 0$, $f(x^j, E) = Ex^j = a$, $j \in N_n$, i.e. $x^* \notin P^1(E)$, $x^j \in P^1(E) = P^1(x^*, E)$, $j \in N_n$. Hence according to the definition of $\varphi(1, n, 1)$ the inequality $\varphi = \varphi(1, n, 1) = a$ is valid.

Let now $E' = (e'_1, e'_2, \dots, e'_n)$ be a perturbing row vector from the row set $\Omega(na)$, i.e. $\|E'\|_1 = \sum\limits_{j \in N_n} |e'_j| < na$. It is easy to prove by contrary that there exists an index p such that $|e'_p| < a$. Therefore, we derive

$$g(x^p, x^*, E + E') = (E + E')(x^p - x^*) = a + e'_p > 0.$$

Hence we see that for any perturbing row $E' \in \Omega(n\varphi)$ the portfolio x^* is not a Pareto-optimal portfolio of the perturbed problem $Z^1(E + E')$. Thus, in view of $x^* \notin P^1(E)$ we get $\rho(1, n, 1) \geq n\varphi(1, n, 1)$. Therefore, according to Theorem 1 the equality (13) is true. □

From Theorems 1–3 following Remark 1 the well-known result followws.

Corollary 2 [33]. *The stability radius $\rho(1, n, s)$, $s \geq 1$, of the multicriteria nontrivial linear Boolean programing problem $Z^s_B(E)$ has the following lower and upper bounds:*

$$\varphi(1, n, s) \leq \rho(1, n, s) \leq n\varphi(1, n, s).$$

Remark 2. We note that in [18] lower and upper bounds of the stability radius of the multicriteria linear Boolean programing problem $Z^s_B(E)$, which is searching the Pareto set, were obtained when $X = \{x \in \mathbf{E}^n : \ Ax \leq b\}$, every problem parameter is under perturbation, i.e. both the elements of the matrix $E \in \mathbf{R}^{s\times n}$ and the elements of the matrix $A \in \mathbf{R}^{q\times n}$ and the vector $b \in \mathbf{R}^q$ are perturbed, while the same Chebyshev metric l_∞ is in every space of problem parameters $\mathbf{R}^n$, $\mathbf{R}^s$ and $\mathbf{R}^q$.

5 Stability conditions

Let us introduce the Slater set [34] of the problem $Z^s(E)$:

$$Sl^s(E) = \{x \in X : \quad Sl^s(x, E) = \emptyset\},$$

where $Sl^s(x,E) = \{x' \in X : \quad \forall k \in N_s \quad (g_k(x', x, E_k) > 0)\}$. It is obvious that $P^s(E) \subseteq Sl^s(E)$ and $P^s(x,E) \supseteq Sl^s(x,E)$ for any $E \in \mathbf{R}^{m\times n\times s}$ and $x \in X$.

Theorem 4. *For a multicriteria nontrivial investment problem $Z^s(E)$, $s \geq 1$, the statements below are equivalent:*
(i) problem $Z^s(E)$ is stable,
(ii) $P^s(E) = Sl^s(E)$,
(iii) $\varphi(m,n,s) > 0$.

Proof. $(i) \Rightarrow (ii)$. Assume that problem $Z^s(E)$ is stable but $P^s(E) \neq Sl^s(E)$. Then there exists an investment portfolio $x^* \in Sl^s(E) \setminus P^s(E)$. Therefore, $Sl^s(x^*, E) = \emptyset$ and $P^s(x^*, E) \neq \emptyset$. This means that

$$\forall x \in P^s(x^*, E) \quad \exists l \in N_s \qquad (g_l(x, x^*, E_l) = 0).$$

Hence, $\varphi(m,n,s) = 0$ and according to Corollary 1 $\rho(m,n,s) = 0$, which contradicts the stability of the problem $Z^s(E)$.

$(ii) \Rightarrow (iii)$. If $P^s(E) = Sl^s(E)$, then for any portfolio $x \notin P^s(E)$ the set $Sl^s(x,E)$ is empty. Therefore, there exists a portfolio $x^0 \in X$ such that the inequalities $g_k(x^0, x, E_k) > 0$, $k \in N_s$, are true, i.e. $x^0 \in P^s(x,E)$. Thus,

$$\forall x \notin P^s(E) \quad \exists x^0 \in P^s(x,E) \quad \forall k \in N_s \qquad (g_k(x^0, x, E_k) > 0).$$

Hence, $\varphi(m,n,s) > 0$.

$(iii) \Rightarrow (i)$. According to Theorem 1, this implication is obvious. □

Since $P^1(E) = Sl^1(E)$, from Theorem 4 follows

Corollary 3. *A scalar investment problem $Z^1(E)$ is stable for any matrix $E \in \mathbf{R}^{m\times n}$.*

Remark 3. Since any two norms are equivalent in finite-dimensional linear spaces [35], the result of Theorem 4 is true for any norms in the space $\mathbf{R}^{m\times n\times s}$ of problem parameters.

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Determining the Optimal Paths in Networks with Rated Transition Time Costs

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Abstract. We formulate and study the problem of determining the optimal paths in networks with rated transition time costs on edges. Polynomial time algorithms for determining the optimal solution of this problem are proposed and grounded. The proposed algorithms generalize algorithms for determining the optimal paths in the weighted directed graphs.

Mathematics subject classification: 90B10, 90B20.
Keywords and phrases: Networks, Optimal paths, Time transition cost, Total rated cost, Polynomial time algorithm.

1 Introduction and Problem Formulation

In this paper we formulate and study an optimal path problem on networks that extends the minimum cost path problem in the weighted directed graphs.

Let $G = (X, E)$ be a finite directed graph with vertex set $X, |X| = n$ and edge set E where to each directed edge $e = (u, v) \in E$ a cost c_e is associated. Assume that for two given vertices x, y there exists a directed path $P(x, y) = \{x = x_0, e_0, x_1, e_1, x_2, e_2, \dots, x_k = y\}$ from x to y. For this directed path we define the total rated cost

$$C(x_0, x_k) = \sum_{t=0}^{k-1} \lambda^t c_{e_t},$$

where λ is a positive value. So, in this path the costs c_{e_t} of directed edges e_t are rated by $\lambda^t c_{te}$ when we pass from x to y. We consider the problem of determining a path from x to y with minimal total rated cost in the case with fixed number of transitions on the edges and in the case with free number of transitions on the edges. If $\lambda = 1$ then the formulated problem becomes the well known problem of determining the shortest path from x to y. The considered problem can be regarded as the problem of determining the optimal paths in a dynamic network determined by the graph $G = (X, E)$ with cost functions $c_e(t) = \lambda^t c_e$ on edges $e \in E$ that depend on time. Therefore if the number k of edges for the optimal path is fixed then we can apply the dynamic programming algorithm or time-expanded network method from [1, 3–5] which determines the solution of the problem using $O(|x|^3 k)$ elementary operations. In this paper we show that for the considered problem the linear programming approach can be applied which allows us to ground more efficient polynomial time algorithms for determining the optimal paths.

2 Algorithms for Solving the Problem with Free Number of Transitions on Edges

In this section we consider the optimal path problem without restrictions on the number of transitions on edges and show that it can be efficiently solved using the linear programming approach. The basic linear programming model we shall use for this problem is the following:
Minimize

$$\phi(\alpha) = \sum_{e\in E} c_e\alpha_e \tag{1}$$

subject to

$$\begin{cases} \sum\limits_{e\in E^-(u)} \alpha_e - \lambda \sum\limits_{e\in E^+(u)} \alpha_e = 1, \quad u = x; \\ \sum\limits_{e\in E^-(u)} \alpha_e - \lambda \sum\limits_{e\in E^+(u)} \alpha_e = \ 0, \ \ \forall u \in X \setminus \{x, y\}; \\ \qquad \alpha_e \geq 0, \ \ \forall e \in E, \end{cases} \tag{2}$$

where $E^-(u)$ is the set of directed edges that originate in the vertex $u \in X$ and $E^+(u)$ is the set of directed edges that enter u.

The following theorem holds.

Theorem 1. *If $\lambda \geq 1$ and in G there exists a directed path $P(x, y)$ from a given starting vertex x to a given final vertex y then for nonnegative costs c_e of edges $e \in E$ the linear programming problem (1), (2) has solutions. If α^*_e for $e \in E$ represents an optimal basic solution of this problem then the set of directed edges $E^* = \{e \in E|\alpha^*_e > 0\}$ determines an optimal directed path from x to y.*

Proof. Assume that $\lambda \geq 1$ and in G there exists at least a directed path $P(x, y) = \{x = x_0, e_0, x_1, e_1, x_2, e_2, \ldots, x_k = y\}$ from x to y. Denote by $E_P = \{e_0, e_1, e_2, \ldots, e_{k-1}\}$ the set of edges of directed path $P(x, y)$. Then it is easy to check that

$$\alpha_e = \begin{cases} \lambda^t, & if \ \ e = e_t \in E_P; \\ 0, & if \ \ e \in E \setminus E_P \end{cases} \tag{3}$$

represents a solution of system (2). Moreover we can see that if the directed path $P(x, y)$ does not contain directed cycles then the solution determined according to (3) corresponds to a basic solution of system (2). So, if in G there exists a directed path from x to y then the set of solutions of system (3) is not empty. Taking into account that the costs c_e, $e \in E$ are nonnegative we obtain that the optimal value of objective function (1) is bounded, i.e. the linear programming problem (1), (2) has solutions.

Now let us prove that an arbitrary basic solution of system (2) corresponds to a simple directed path $P(x, y)$ from x to y. Let $\alpha = (\alpha_{e_1}, \alpha_{e_2}, \ldots, \alpha_{e_m})$ be a feasible solution of problem (1), (2) and denote $E_\alpha = \{e \in E|\alpha_e > 0\}$. Then it is easy to observe that the set of directed edges $E_\alpha \subseteq E$ in G induces a subgraph

$G_\alpha = (X_\alpha, E_\alpha)$ in which vertex x is a source and y is a sink vertex. Indeed, if this is not so then we can determine a subset of vertices X'_α from X_α that can be reached in G_α from x and X'_α does not contain vertex y. In G_α we can select the subgraph $G'_\alpha = (X'_\alpha, E'_\alpha)$ induced by the subset of vertices X'_α and we can calculate

$$S = \sum_{u \in X'_\alpha} \sum_{e \in E^-(u)} \alpha_e,$$

where $E'^-(u) = \{e = (v,u) \in E' | v \in X'_\alpha\}$. It is easy to observe that the value S can be also expresses as follows

$$S = \sum_{u \in X'_\alpha} \sum_{e \in E^+(u)} \alpha_e,$$

where $E'^+(u) = \{e = (u,v) \in E' | v \in X'_\alpha\}$. If we sum the equalities from (3) that correspond to $u \in X'_\alpha$ then we obtain

$$\sum_{u \in X'_{alpha}} \sum_{e \in E^-(u)} \alpha_e - \lambda \sum_{u \in X'_{alpha}} \sum_{e \in E^+(u)} \alpha_e = 1$$

which involves $(1-\lambda)S = 1$. However this couldn't take place because $\lambda \geq 1$ and $S \geq 0$, i.e. we obtain the contradiction. So, if $\alpha \geq 1$ then in G_α there exists at least a directed path from x to y. Taking into account that an arbitrary vertex u in G_α contains at least an entering edge $e = (v,u)$ and at least an outgoing directed edge $e = (u,w)$ we may conclude that G_α has a structure of directed graph, where x is a source and y is a sink.

Thus, to prove that a basic solution $\alpha = (\alpha_{e_1}, \alpha_{e_2}, \dots, \alpha_{e_m})$ corresponds to a directed graph G_α that has a structure of a simple directed path from x to y it is sufficient to show that G_α has a structure of an acyclic directed graph and G does not contain parallel directed paths $P'(u,w), P''(u,w)$ from a vertex $u \in X_\alpha$ to $w \in X_\alpha$. We can prove the first part of the mentioned property as follows. If α is a basic solution and G_α contains a directed cycles then there exists a directed path $P(x,y) = \{x = x_0, e_0, x_1, e_1, x_2, e_2, \dots, x_r, e_r, \dots x_k = y\}$ from x to y that contains a directed cycle $\{x_r, e_r, x_{r+1}, e_{r+1}, \dots, x_{r+s-1}, e_{r+s-1}, x_r\}$ with the set of edges $E^0 = \{e_r, e_{r+1}, \dots, e_{r+s-1}\}$. If we denote the set of edges of the directed path $P^1(x, x_r) = \{x = x_0, e_0, x_1, e_1, x_2, e_2, \dots, x_r\}$ from x to x_r by $E^1 = \{e_0, e_1, e_2, \dots, e_{r-1}\}$ and we denote the set of edges of the directed path $P^2(x_r, y) = \{x_r = x_{r+s}, e_{r+s}, x_{r+s+1}, e_{r+s+1}, \dots, x_k = y\}$ from $x_r = x_{r+s}$ to $x_k = y$ by $E^2 = \{e_{r+s}, e_{r+s+1}, \dots, e_{k-1}\}$ then for a small positive θ we can construct the following feasible solution

$$\alpha'_e = \begin{cases} \alpha_e, & \forall e \in E_\alpha \setminus (E^0 \cup E^2); \\ \alpha_{e_{r+i}} - \lambda^i \theta, & i = 0, 1, \dots, s-1; \\ \alpha_{e_{r+s+i}} - \lambda^{s+i}\theta + \lambda^i \theta, & i = 0, 1, \dots, k-r-s-1. \end{cases}$$

Here θ can be chosen in such a way that $\alpha'_e = 0$ at least for an edge $e \in E^0 \cup E^2$. So, the number of nonzero components of the solution $\alpha' = (\alpha'_{e_1}, \alpha'_{e_2}, \dots, \alpha'_{e_m})$ is less than the number of nonzero components of solution α.

Now let us show that for a basic solution the graph G_α couldn't contain parallel directed paths from vertex x_r to vertex $w \in X_\alpha$. We prove this again by contradiction. We assume that in G_α we have two directed paths $P'(x_r, w) = \{x_r, e'_r, x'_{r+1}, \dots, e'_k, x'_k = w\}$ and $P''(x_r, y) = (x_r, y)\{x_r, e_{r+1}, x''_{r+1}, \dots, e''_l, x''_l = w\}$ from x to w with the corresponding edge sets $E' = \{e'_r, e'_{r+1}, \dots, e'_k\}$ and $E'' = \{e''_r, e''_{r+1}, \dots, e''_k\}$. Then for a small positive θ we can construct the following solution

$$\alpha'_e = \begin{cases} \alpha_e, & if \;\; e \in E_\alpha \setminus (E' \cup E''); \\ \alpha_{e'_{r+i}} - \lambda^i\theta, & if \; e = e'_{r+i} \in E', \; i = 0, 1, \dots, k - r; \\ \alpha_{e''_{r+i}} + \lambda^i\theta, & if \; e = e_{r+i}\text{''} \in E'', \; i = 0, 1, \dots, l - r. \end{cases}$$

Here we can chose θ in such a way that $\alpha'_{e_l} = 0$ at least for an edge $e_l \in E' \cup E''$, i.e. we obtain that the number of nonzero components of the solution α' is less then the number of nonzero components of α. Thus, if α is a basic solution then the corresponding graph G_α has a structure of a simple directed path from x to y. This means that if $\alpha^* = (\alpha^*_{e_1}, \alpha^*_{e_2}, \dots, \alpha^*_{e_m})$ is an optimal basic solution $\alpha^* = (\alpha^*_{e_1}, \alpha^*_{e_2}, \dots, \alpha^*_{e_m})$ of problem (1), (2) then the set of directed edges $E^* = \{e \in E | \alpha^*_e > 0\}$ determines an optimal directed path from x to y. □

Corollary 1. *If $\alpha \ge 1$ and vertex y is reachable in G from x then for an arbitrary basic solution α of system (2) the corresponding graph G_α has a structure of directed path from x to y.*

Corollary 2. *Assume that $0 < \lambda < 1$ and the graph G contains directed cycles. Then for a basic solution α of system (2) either the corresponding graph G_α has a structure of directed path from x to y or this graph does not contain directed paths from x to y; in the second case G_α contains a unique directed cycle that can be reached from x by using a unique directed path that connects vertex x with this cycle. Moreover, if G_α does not contain directed paths from x to y then it consists of the set of vertices and edges $\{x = x_0, e_0, x_1 e_1, x_2, e_2, \dots, x_r, e_r, x_{r+1}, e_{r+1}, \dots, x_{r+s-1}, e_{r+s-1}, x_r\}$ with a unique directed cycle $\{x_r, e_r, x_{r+1}, e_{r+1}, \dots, x_{r+s-1}, e_{r+s-1}, x_r\}$ where the nonzero components α_e of α can be expressed as follows*

$$\alpha_e = \begin{cases} \lambda^t, & if \; e = e_t, \; t = 0, 1, \dots, r - 1; \\ \lambda^{r+i}/(1 - \lambda^s), & if \; e = e_{r+i}, \; i = 0, 1, \dots, s - 1. \end{cases} \tag{4}$$

Remark 1. If $0 < \lambda < 1$ then the linear programming problem (1), (2) may have an optimal basic solution α^* for which the graph G_{α^*} does not contain a directed path from x to y. This corresponds to the case when in G the optimal path from x to y does not exist.

Now we show that the linear programming model (1), (2) can be extended for the problem of determining the optimal paths from every $x \in X\{y\}$ to y. We can see that if $\lambda \geq 1$ then there exists the tree of optimal paths from every $x \in X\{y\}$ to y and this tree of optimal paths can be found on the basis of the following theorem.

Theorem 2. *Assume that $\lambda \geq 1$ and in G for an arbitrary $u \in X \setminus \{y\}$ there exists at least a directed path $P(u, y)$ from u to y. Additionally we assume that the costs c_e of edges $e \in E$ are nonnegative. Then the linear programming problem:*
Minimize

$$\phi(\alpha) = \sum_{e \in E} c_e \alpha_e \tag{5}$$

subject to

$$\begin{cases} \sum\limits_{e \in E^-(u)} \alpha_e - \lambda \sum\limits_{e \in E^+(u)} \alpha_e = 1, \qquad \forall u \in X \setminus \{y\}, \\ \qquad \alpha_e \geq 0, \quad \forall e \in E \end{cases} \tag{6}$$

has solutions. Moreover, if $\alpha^* = (\alpha^*_{e1}, \alpha^*_{e2}, \dots, \alpha^*_{em})$ is an optimal basic solution of problem (5), (6) then the set of directed edges $E^* = \{e \in E | \alpha^*_e > 0\}$ determines a tree of optimal directed paths G_{α^*} from every $u \in X \setminus \{y\}$ to y.

Proof. Let $\alpha = (\alpha_{e_1}, \alpha_{e_2}, \dots, \alpha_{e_m})$ be a feasible solution of problem (5), (6) and consider the set of directed edges $E_\alpha = \{e \in E | \alpha_e > 0\}$ that corresponds to this solution. Then in the graph $G_\alpha = (X, E_\alpha)$ induced by the set of edges E_α the vertex y is attainable from every $x \in X$. An arbitrary basic solution α of system (6) corresponds to a graph G_α which has a structure of directed tree with sink vertex y. Moreover the optimal value of the objective function of the problem is bounded. Therefore if we find an optimal basic solution α^* of the problem (5), (6) then we determine the corresponding tree of optimal paths G_{α^*}. $\square$

If the graph $G = (X, E)$ does not contain directed cycles then Theorem 1 and Theorem 2 can be extended for an arbitrary positive λ, i.e. in this case the following theorem holds.

Theorem 3. *If $G = (X, E)$ has a structure of an acyclic directed graph with sink vertex y then for an arbitrary $\lambda \geq 0$ and arbitrary costs $c_e, e \in E$ there exists the solution of the linear programming problem (1), (2). Moreover, if α^* is an optimal basic solution of this problem then the set of directed edges $E^* = \{e \in E | \alpha^*_e > 0\}$ determines an optimal directed path from x to y.*

Proof. The proof of this theorem is similar to the proof of Theorems 2. In this case the set of edges E_α for a basic solution of problem (5), (6) induces the graph $G_\alpha = (X_\alpha, E_\alpha)$ that has a structure of directed tree with sink vertex y. Therefore the set of edges E_{α^*} for an optimal basic solution of problem (5), (6) corresponds to a directed tree $G_{\alpha^*} = (X_{\alpha^*}, E_{\alpha^*})$ of optimal paths from every $u \in X$ to sink vertex y. $\square$

As we have shown (see Corollary 2 and Remark 1) if $0 < \lambda < 1$ and the graph $G = (X, E))$ contains directed cycles then the linear programming problem (1), (2) may not find the optimal path from x to y even for the case with positive costs $c_e, \forall e \in E$ because such an optimal path in G may not exist. Below we illustrate an example of the problem with $\lambda = 1/2$ and the network represented in Figure 1. In the considered network the vertices are represented by circles and edges by arcs. Inside the circles the numbers of the vertices are written and near the arcs the values α^*_e that corresponds to the optimal solution of the problem with $x = 4,\ y = 1$ and $c_{(4,2)} = 1,\ c_{(2,1)} = 10,\ c_{(2,3)} = 1,\ c_{(3,2)} = 1$ are written. The optimal basic solution of the linear programming problem (1), (2) for the considered example is $\alpha^*_{(4,2)} = 1,\ \alpha^*_{(2,1)} = 0,\ \alpha^*_{(2,3)} = 2/3,\ \alpha^*_{(3,2)} = 1/3$ and the graph G_{α^*} is induced by the set of edges $\{(4,2),(2,3),(3,2)\}$. Here we can see that the values $\alpha^*_{(4,2)} = 1,\ \alpha^*_{(2,3)} = 2/3,\ \alpha^*_{(3,2)} = 1/3$ satisfy condition (4). The corresponding graph G_{α^*} does not contain the directed path from vertex 4 to 1, i.e the optimal path from vertex 4 to 1 does not exist.

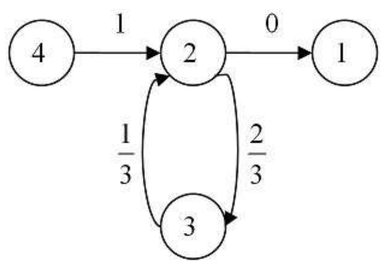

Figure 1.

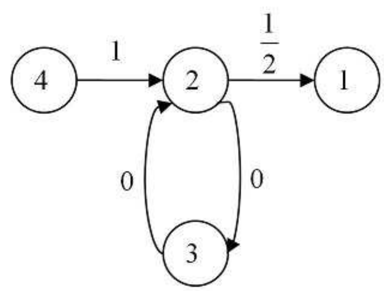

Figure 2.

In Figure 2 the optimal solution of problem (1), (2) with $x = 4,\ y = 1$ and $c_{(4,2)} = 1,\ c_{(2,1)} = 1,\ c_{(2,3)} = 2,\ c_{(3,2)} = 2$ is represented. In this case the optimal basic solution of problem (1), (2) is $\alpha^*_{(4,2)} = 1,\ \alpha^*_{(2,1)} = 1/2,\ \alpha^*_{(2,3)} = 0,\ \alpha^*_{(3,2)} = 0$. The corresponding nonzero components of this solution generate in G the subgraph $G_{\alpha^*} = (X_\alpha, E_{\alpha^*})$, where $E_{\alpha^*} = \{(4,2),(2,1)\}$. The set of edges E_{α^*} generates a unique directed path from vertex 4 to 1, i.e. in the considered case there exists the optimal path from vertex 4 to 1.

If for problem (5), (6) we consider the dual problem then on the basis of duality theorems of linear programming we can prove the following result.

Theorem 4. *Assume that $\lambda \geq 1$ and the costs $c_e, e \in E$ are strict positive. Let $\beta^*_u,\ \forall u \in X$ be a solution of the following linear programming problem:*
Maximize

$$\psi(\beta) = \sum_{x \in X \setminus \{x\}} \beta_x \tag{7}$$

subject to

$$\beta_u - \lambda\beta_v \le c_{u,v}, \forall(u,v) \in E^0, \tag{8}$$

where

$$E^0 = \{e = (u,v) \in E | u \in X \setminus \{y\}, v \in X.$$

If $\beta^*_u \ \ u \in X$ *is an optimal basic solution of problem (7), (8) then an arbitrary tree* $T = (X, E'_{\beta^*})$ *with sink vertex* y *of the graph* $G_{\beta^*} = (X, E_{\beta^*})$ *induced by the set of directed edges*

$$E_{\beta^*} = \{e = (x,y) \in E | \beta^*_u - \lambda\beta^*_v = c_{u,v}\}$$

represents the tree of optimal paths from $x \in X \setminus \{y\}$ *to* y*. An optimal basic solution of problem (7), (8), can be found starting with* $\beta^*_v = 0$ *for* $v = y$ *and* $\beta^*_u = \infty$ *for* $u \in X \setminus \{y\}$ *and then repeat* $|X| - 1$ *tames the following calculation procedure: replace* β^*_u *for* $u \in X \setminus \{y\}$ *by* $\beta^*_u = \min\limits_{v \in X(u)} \left\{\lambda\beta^*_v + c_{u,v}\right\}$*, where* $X(v) = \{u \in X | (u,v) \in E\}$*.*

Proof. Assume that $\alpha^*_e,\ e \in E$ and $\beta^*_u,\ u \in X$ represent the optimal solutions of the primal linear programming problem (5), (6) and the dual linear programming problem (7), (8), respectively. Then according to dual theorems of linear programming these solutions satisfy the following condition:

$$\alpha^*_{u,v}(\beta^*_u - \lambda\beta^*_v - c_{u,v}) = 0 \quad \forall(u,v) \in E^0. \tag{9}$$

So, if $\alpha^*_e,\ e \in E$ is an optimal basic solution then $\beta^*_u - \lambda\beta^*_v - c_{u,v} = 0$ for an arbitrary $e = (u,v) \in E_{\alpha^*}$. Taking into account that the corresponding graph G_{α^*} for an optimal basic solution α^* has a structure of the directed tree with sink vertex y then we obtain this tree coincides with the tree of optimal paths T_{β^*} that determines the solution $\beta^*_u,\ u \in X$ of the problem (7), (8).

Now let us prove that the procedure for calculating the values β^*_x determines correctly the optimal solution of the dual problem. Indeed, if in G the vertex y is attainable from each $v \in X$ then the rank of system (8) is equal to $|X| - 1$. This means that for an arbitrary optimal basic solution not more than $|X| - 1$ its components may be different from zero. Therefore we can take $\beta^*_y = 0$. After that taking into account the condition (9) we can find β^*_u for $u \in X \setminus \{y\}$ using the calculation procedure from the theorem starting with $\beta^*_v = 0$ for $v = y$ and $\beta^*_u = \infty$ for $u \in X \setminus \{y\}$. □

Thus, based on Theorem 4 we can find the tree of optimal paths in G for the problem with free number of transitions as follow.

We determine the values β^*_u for $u \in X$ using the following steps:

Preliminary step (step 0): Fix $\beta^*_y = 0$, and $\beta^*_u = \infty$ for $u \in X \setminus \{y\}$;

General step (step k ($k \ge 1$)): For every $u \in X \setminus \{y\}$ replace the value β^*_u by $\beta^*_u = \min\limits_{v \in X(u)} \left\{\lambda\beta^*_v + c_{u,v}\right\}$, where $X(v) = \{u \in X | (u,v) \in E\}$. If $k < |X| - 1$ then go to next step; otherwise stop.

If β_u^* for $u \in X$ are known then we determine the set of directed edges E_{β^*} and the corresponding directed graph $G_{\beta^*} = (X, E_{\beta^*})$. After that we find a directed tree $T_{\beta^*} = (X, E'_{\beta^*})$ in G_{β^*}. Then T_{β^*} represents the tree of optimal paths from $x \in X$ to y.

It is ease to observe that the proposed algorithm allows us to solve the considered problems in general case with the same complexity as the problem with $\lambda = 1$, i.e this algorithm in the case $\lambda \geq 1$ extends the algorithm for shortest path problems (see [2, 3]).

3 Algorithms for Solving the Problem with Fixed Number of Transitions on Edges

The optimal path problem with fixed number of transitions from starting vertex to final one can be formulated and studied using the following linear programming model:
Minimize

$$\phi_{x,y}(\alpha) = \sum_{e \in E} c_e \alpha_e \tag{10}$$

subject to

$$\begin{cases} \sum\limits_{e \in E^-(u)} \alpha_e - \lambda \sum\limits_{e \in E^+(u)} \alpha_e = \ 1, & u = x; \\ \sum\limits_{e \in E^-(u)} \alpha_e - \lambda \sum\limits_{e \in E^+(u)} \alpha_e = \ 0, & \forall u \in X \setminus \{x, y\}; \\ \sum\limits_{e \in E^-(u)} \alpha_e - \lambda \sum\limits_{e \in E^+(u)} \alpha_e = -\lambda^{k-1}, & u = y; \\ \alpha_e \geq 0, \ \forall e \in E. \end{cases} \tag{11}$$

This model is valid for an arbitrary $\lambda > 0$ $(\lambda \neq 1)$. If we solve the linear programming problem (10), (11) then find an optimal solution α^* that determines the optimal value of objective function and the corresponding graph G_{α^*}. However such an approach for solving this problem does not allow to determine the order of the edges from G_{α^*} that form the optimal path $P(x, y)$ with fixed number of transitions from x to y. The algorithms based on linear programming in this case determine in polynomial time only the optimal cost of the optimal path and the corresponding graph G_{α^*}.

In order to determine the optimal path $P(x, y)$ with a given number of transitions K from x to y it is necessary to solve the sequence of $K|X - 1|$ linear programming problem (10), (11) with fixed starting vertex for $k = 1, 2, \dots, K$ and for an arbitrary final vertex $y \in X \setminus \{x\}$.For each such a problem we determine the optimal value w $\phi_{x,y}(\alpha^{k^*})$ and the corresponding graph G_{α^k}. After that starting from final vertex y we find the optimal path $P(x, y)$ as follows: we fix a directed edge $e^{K-1} = (u^{K-1}, u^K = y)$ for which $\phi_{x,y}(\alpha^{K^*}) = \phi_{x,u^{K-1}}(\alpha^{K-1^*} + \lambda^{K-1} c_{e^{K-1}})$, then find a directed edge $e^{K-2} = (u^{K-2}, u^K = y)$ for which $\phi_{x,u^{K-1}}(\alpha^{K-1^*}) =$

$\phi_{x,u^{K-2}}(\alpha^{K-1^*} + \lambda^{K-2} c_{e^{K-2}})$ and so on. In such a way we find the vertices $x = u^o, u^1, \dots, u^k = y$ of the path $P(x, y)$.

More useful algorithms for solving the problem with fixed number of transitions on edges of the network are the dynamic programming algorithms and the time-expanded network method from [4–6]. To apply these algorithms it is sufficient to consider the network with cost functions $c_e(t) = \lambda^t c_e$ on edges $e \in E$.

4 Conclusion

The optimal paths problem on networks with rated transition time costs on edges generalizes the shortest path problem in weighted directed graphs. The proposed linear programming approach for studying this problem allows to ground polynomial time algorithms for determining the optimal paths in networks with rated costs on edges. The elaborated algorithms generalizes algorithms for determining the optimal paths in weighted directed graphs and may be useful for determining the solution for the dynamic version of minimum cost flow problem on networks with the costs on edges that depend on flow and on time (the case with separable cost functions).

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A note on six-dimensional planar Hermitian submanifolds of Cayley algebra

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Abstract. Six-dimensional planar Hermitian submanifolds of Cayley algebra are considered. It is proved that if such a submanifold of the octave algebta satisfies the U-Kenmotsu hypersurfaces axiom, then it is Kählerian. It is also proved that a symmetric non-Kählerian Hermitian six-dimensional submanifold of the Ricci type does not admit totally umbilical Kenmotsu hypersurfaces.

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1 Introduction

The almost Hermitian structures (AH-structures) belong to the most important and meaningful differential-geometrical structures. The existence of 3-vector cross products on Cayley algebra gives a lot of substantive examples of almost Hermitian manifolds. As it is well known, every 3-vector cross product on Cayley algebra induces a 1-vector cross product (or, what is the same in this case, an almost Hermitian structure) on its six-dimensional oriented submanifold (see [10–12]). Such almost Hermitian structures (in particular, Hermitian, special Hermitian, nearly-Kählerian, Kählerian etc) were studied by a number of remarkable geometers: E. Calabi, J.-T. Cho, R. Deszcs, F. Dillen, N. Ejiri, S. Funabashi, A. Gray, Guoxin Wei, Haizhong Li, H. Hashimoto, V. F. Kirichenko, J. S. Pak, K. Sekigawa, L. Verstraelen, L. Vranchen and others. For example, a complete classification of nearly-Kählerian [15], Kählerian [16] and locally symmetric Hermitian structures [17] on six-dimensional submanifolds of the octave algebra has been obtained.

The almost contact metric structures are also remarkable and very important differential-geometrical structures. These structures are studied from the point of view of differential geometry as well as of modern theoretical physics. We mark out the close connection of almost contact metric and almost Hermitian structures. For instance, an almost contact metric structure is induced on an oriented hypersurface of an almost Hermitian manifold [22].

In the present paper, we consider six-dimensional Hermitian planar submanifolds of Cayley algebra. We shall prove the following main results.

Theorem 1. *If a six-dimensional Hermitian planar submanifold of Cayley algebra satisfies the U-Kenmotsu hypersurfaces axiom, then it is Kählerian.*

Theorem 2. *A symmetric non-Kahlerian Hermitian six-dimensional submanifold of the Ricci type does not admit totally umbilical Kenmotsu hypersurfaces.*

This article is the continuation of the authors' researches in the area of planar Hermitian submanifolds of Cayley algebra (see [2, 6, 7] and others).

2 Preliminaries

Let us consider an almost Hermitian manifold, i. e. a $2n$-dimensional manifold M^{2n} with a Riemannian metric $g = \langle \cdot,\, \cdot \rangle$ and an almost complex structure J. Moreover, the following condition must hold

$$\langle JX,\ JY \rangle = \langle X,\ Y \rangle\,, \quad X, Y \in \aleph(M^{2n}),$$

where $\aleph(M^{2n})$ is the module of smooth vector fields on M^{2n}. All considered manifolds, tensor fields and similar objects are assumed to be of the class C^∞.

The specification of an almost Hermitian structure on a manifold is equivalent to the setting of a G-structure, where G is the unitary group $U(n)$ [19]. Its elements are the frames adapted to the structure (A-frames). They look as follows:

$$(p,\, \varepsilon_1,\, \ldots, \varepsilon_n,\, \varepsilon_{\hat{1}},\, \ldots,\, \varepsilon_{\hat{n}}\,),$$

where ε_a are the eigenvectors corresponding to the eigenvalue $i = \sqrt{-1}$, and $\varepsilon_{\hat{a}}$ are the eigenvectors corresponding to the eigenvalue $-i$. Here the index a ranges from 1 to n, and we state $\hat{a} = a + n$.

Therefore, the matrices of the operator of the almost complex structure and of the Riemannian metric written in an A-frame look as follows, respectively:

$$\left(J^k_j\right) = \left(\begin{array}{c|c} iI_n & 0 \\ \hline 0 & -iI_n \end{array}\right), \quad (g_{kj}) = \left(\begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array}\right),$$

where I_n is the identity matrix; $k, j = 1, \ldots, 2n$.

We recall that the fundamental form (or Kählerian form) of an almost Hermitian manifold is determined by the relation

$$F(X,\ Y) = \langle X,\ JY \rangle\,, \quad X, Y \in \aleph(M^{2n}).$$

By direct computing it is easy to obtain that in A-frame the fundamental form matrix looks as follows:

$$(F_{kj}) = \left(\begin{array}{c|c} 0 & iI_n \\ \hline -iI_n & 0 \end{array}\right).$$

An almost Hermitian manifold is called Hermitian if its structure is integrable. The following identity characterizes the Hermitian structure [13, 19]:

$$\nabla_X(F)(Y,Z) - \nabla_{JX}(F)(JY,Z) = 0,$$

where $X, Y, Z \in \aleph(M^{2n})$. The first group of the Cartan structural equations of a Hermitian manifold written in an A-frame looks as follows [19]:

$$d\omega^a = \omega^a_b \wedge \omega^b + B^{ab}_c \omega^c \wedge \omega_b,$$

$$d\omega_a = -\omega^b_\alpha \wedge \omega_b + B^c_{ab} \omega_c \wedge \omega^b,$$

where $\left\{B^{ab}_c\right\}$ and $\{B^c_{ab}\}$ are components M^{2n} of the Kirichenko tensors of $[1,5]$; $a, b, c = 1, ..., n$.

We recall also that an almost contact metric structure on an odd-dimensional manifold N is defined by the system of tensor fields $\{\Phi,\ \xi,\ \eta,\ g\}$ on this manifold, where ξ is a vector field, η is a covector field, Φ is a tensor of the type (1, 1) and $g = \langle \cdot,\ \cdot \rangle$ is the Riemannian metric [8, 9]. Moreover, the following conditions are fulfilled:

$$\eta(\xi) = 1,\ \Phi(\xi) = 0,\ \eta \circ \Phi = 0,\ \Phi^2 = -id + \xi \otimes \eta,$$

$$\langle \Phi X, \Phi Y \rangle = \langle \Phi X, \Phi Y \rangle - \eta\,(X)\,\eta\,(Y)\,,\ X, Y \in \aleph(N),$$

where $\aleph(M^{2n})$ is the module of smooth vector fields on N. As an example of an almost contact metric structure we can consider the cosymplectic structure that is characterized by the following condition:

$$\nabla\eta = 0, \quad \nabla\Phi = 0,$$

where ∇ is the Levi-Civita connection of the metric. It has been proved that the manifold which admits the cosymplectic structure is locally equivalent to the product $M \times R$, where M is a Kählerian manifold [20].

As it was mentioned, the almost contact metric structures are closely connected to the almost Hermitian structures. For instance, if $(N, \{\Phi,\ \xi,\ \eta,\ g\})$ is an almost contact metric manifold, then an almost Hermitian structure is induced on the product $N \times R$ [8]. If this almost Hermitian structure is integrable, then the input almost contact metric structure is called normal. As it is known, a normal contact metric structure is called Sasakian [8]. On the other hand, we can characterize the Sasakian structure by the following condition [19]:

$$\nabla_X(\Phi)Y = \langle X, Y \rangle\, \xi - \eta(Y)X,\ \ X, Y \in \aleph(N).$$

For example, Sasakian structures are induced on totally umbilical hypersurfaces in a Kahlerian manifold [8]. As it is well known, the Sasakian structures have many remarkable properties and play a fundamental role in contact geometry.

In 1972 Katsuei Kenmotsu introduced a new class of almost contact metric structures [14] defined by the condition:

$$\nabla_X(\Phi)Y = \langle \Phi X, Y \rangle\, \xi - \eta(Y)\Phi X,\ X, Y \in \aleph(N).$$

The Kenmotsu manifolds are normal and integrable, but they are not contact, consequently, they can not be Sasakian. In spite of the fact that the conditions for these kinds of manifolds are similar, the properties of Kenmotsu manifolds are to some extent antipodal to the Sasakian manifolds properties [18]. Note that the remarkable investigation [18] in this field contains a detailed description of Kenmotsu manifolds as well as a collection of examples of such manifolds. We mark out also the recent fundamental and profound work by G. Pitis that contains a survey of most important results on geometry of Kenmotsu manifolds [21].

3 Proof of theorems

At first, we remind that an almost Hermitian manifold M^{2n} satisfies the U-Kenmotsu hypersurfaces axiom if a totally umbilical Kenmotsu hypersurface passes through every point of this manifold.

Let $O \equiv R^8$ be the Cayley algebra. As it is well-known [12], two non-isomorphic three-fold vector cross products are defined on it by means of the relations:

$$P_1(X,\ Y,\ Z) = -X(\bar{Y}Z) + \langle X,\ Y\rangle\, Z + \langle Y,\ Z\rangle\, X - \langle Z, X\rangle\, Y,$$

$$P_2(X,\ Y,\ Z) = -(X\bar{Y})Z + \langle X,\ Y\rangle\, Z + \langle Y,\ Z\rangle\, X - \langle Z, X\rangle\, Y,$$

where $X,\ Y,\ Z \in \mathbf{O}$, $\langle\cdot,\ \cdot\rangle$ is the scalar product in $\mathbf{O}$ and $X \to \bar{X}$ is the conjugation operator. Moreover, any other three-fold vector cross product in the octave algebra is isomorphic to one of the above-mentioned two.

If $M^6 \subset \mathbf{O}$ is a six-dimensional oriented submanifold, then the induced almost Hermitian structure $\{J_y, g = \langle\cdot,\ \cdot\rangle\}$ is determined by the relation

$$J_t(X) = P_t(X,\, e_1,\, e_2), \quad t = 1,\ 2,$$

where $\{e_1,\ e_2\}$ is an arbitrary orthonormal basis of the normal space of M^6 at the point p, $X \in T_p(M^6)$ [12].

We recall that the point $p \in M^6$ is called general [16, 17], if

$$e_0 \notin T_p(M^6),$$

where e_0 is the unit of Cayley algebra. A submanifold $M^6 \subset \mathbf{O}$, consisting only of general points, is called a general-type submanifold [16]. In what follows, all submanifolds M^6 that will be considered are assumed to be of general type.

Let N be an arbitrary oriented hypersurface of a six-dimensional Hermitian submanifold $M^6 \subset \mathbf{O}$ of Cayley algebra, let σ be the second fundamental form of immersion of N into M^6. The Cartan structural equations of the almost contact metric structure on such a hypersurface look as follows [22]:

$$d\omega^\alpha = \omega^\alpha_\beta \wedge \omega^\beta + B^{\alpha\beta}_\gamma \omega^\gamma \wedge \omega_\beta +$$

$$+\left(\sqrt{2}B^{\alpha 3}_\beta + i\sigma^\alpha_\beta\right)\omega^\beta \wedge \omega + \left(-\frac{1}{\sqrt{2}}B^{\alpha\beta}_3 + i\sigma^{\alpha\beta}\right)\omega_\beta \wedge \omega,$$

$$d\omega_\alpha = -\omega_\alpha^\beta \wedge \omega_\beta + B^\gamma_{\alpha\beta}\omega_\gamma \wedge \omega^\beta +$$
$$+\left(\sqrt{2}B^\beta_{\alpha 3} - i\sigma^\beta_\alpha\right)\omega_\beta \wedge \omega + \left(-\frac{1}{\sqrt{2}}B^3_{\alpha\beta} - i\sigma_{\alpha\beta}\right)\omega^\beta \wedge \omega,$$
$$d\omega = \left(\sqrt{2}B^{3\alpha}_\beta - \sqrt{2}B^\alpha_{3\beta} - 2i\sigma^\alpha_\beta\right)\omega^\beta \wedge \omega_\alpha + \left(B^{3\beta}_3 - i\sigma^\beta_3\right)\omega \wedge \omega_\beta.$$

Here the indices α, β, γ range from 1 to 2. Taking into account that the Cartan structural equations of a Kenmotsu structure look as follows [18]:

$$d\omega^\alpha = \omega^\alpha_\beta \wedge \omega^\beta + \omega \wedge \omega^\alpha;$$
$$d\omega_\alpha = -\omega^\beta_\alpha \wedge \omega_\beta + \omega \wedge \omega_\alpha;$$
$$d\omega = 0,$$

we get the conditions whose simultaneous fulfillment is a criterion for the structure induced on N to be Kenmotsu:

$$1)\ B^{\alpha\beta}_\gamma = 0;$$
$$2)\ \sqrt{2}B^{\alpha 3}_\beta + i\sigma^\alpha_\beta = -\delta^\alpha_\beta;$$
$$3)\ -\frac{1}{\sqrt{2}}B^{\alpha\beta}_3 + i\sigma^{\alpha\beta} = 0; \tag{1}$$
$$4)\ \sqrt{2}B^{3\alpha}_\beta - \sqrt{2}B^\alpha_{3\beta} - 2i\sigma^\alpha_\beta = 0;$$
$$5)\ B^{3\beta}_3 - i\sigma^\beta_3 = 0;$$

and the formulae, obtained by complex conjugation (no need to write them explicitly).

From $(1)_3$ we obtain:

$$\sigma^{\alpha\beta} = -\frac{i}{\sqrt{2}}B^{\alpha\beta}_3.$$

By alternating this relation we get:

$$0 = \sigma^{[\alpha\beta]} = -\frac{i}{\sqrt{2}}B^{[\alpha\beta]}_3 = -\frac{i}{2\sqrt{2}}\left(B^{\alpha\beta}_3 - B^{\beta\alpha}_3\right) = -\frac{i}{\sqrt{2}}B^{\alpha\beta}_3.$$

That is why $B^{\alpha\beta}_3 = 0$, therefore

$$\sigma^{\alpha\beta} = 0.$$

Similarly, from $(1)_5$ we obtain:

$$\sigma^\beta_3 = 0.$$

So, we can rewrite the conditions (1) as follows:

$$1)\ B^{\alpha\beta}_\gamma = 0; \quad 2)\ \sigma^{\alpha\beta} = 0; \quad 3)\ \sigma^\beta_3 = 0; \quad 4)\ \sigma^\alpha_\beta = i\sqrt{2}B^{\alpha 3}_\beta + i\delta^\alpha_\beta \tag{2}$$

and the formulae, obtained by complex conjugation.

Next, let us use the expressions for Kirichenko tensors of six-dimensional Hermitian submanifolds of Cayley algebra [3, 4, 16]:

$$B_c^{ab} = \frac{1}{\sqrt{2}}\varepsilon^{abh} D_{hc}, \; B_{ab}^c = \frac{1}{\sqrt{2}}\varepsilon_{abh} D^{hc}, \tag{3}$$

where

$$\varepsilon^{abc} = \varepsilon_{123}^{abc}, \varepsilon_{abc} = \varepsilon_{abc}^{123}$$

are the components of the third-order Kronecher tensor [16] and

$$D_{hc} = \pm T_{hc}^8 + iT_{hc}^7,$$
$$D^{hc} = D_{\hat{h}\hat{c}} = \pm T_{\hat{h}\hat{c}}^8 - iT_{\hat{h}\hat{c}}^7.$$

Here $\{T_{hc}^{\varphi}\}$ are the components of the configuration tensor (in A. Gray's notation) of the Hermitian submanifold $M^6 \subset \mathbf{O}$; the index φ ranges from 7 to 8 and the indices a, b, c, h range from 1 to 3 [3, 4, 16].

Taking into account (2) and (3), we get:

$$\begin{aligned}
\sigma_{\hat{1}1} = \sigma_1^1 &= i\sqrt{2}B_1^{13} + i\delta_1^1 = i\sqrt{2}(\frac{1}{\sqrt{2}}\varepsilon^{13\gamma}D_{\gamma 1}) + i = -iD_{12} + i;\\
\sigma_{\hat{2}2} = \sigma_2^2 &= i\sqrt{2}B_2^{23} + i\delta_2^2 = i\sqrt{2}(\frac{1}{\sqrt{2}}\varepsilon^{23\gamma}D_{\gamma 2}) + i = iD_{12} + i;\\
\sigma_{\hat{1}2} = \sigma_2^1 &= i\sqrt{2}B_2^{13} + i\delta_2^1 = i\sqrt{2}(\frac{1}{\sqrt{2}}\varepsilon^{13\gamma}D_{\gamma 2}) = -iD_{22};\\
\sigma_{\hat{2}1} = \sigma_1^2 &= i\sqrt{2}B_1^{23} + i\delta_1^2 = i\sqrt{2}(\frac{1}{\sqrt{2}}\varepsilon^{23\gamma}D_{\gamma 1}) = iD_{11}.
\end{aligned} \tag{4}$$

If N is a totally umbilical submanifold of M^6, then for its second fundamental form we have:

$$\sigma_{ps} = \lambda g_{ps}, \quad \lambda - const, \quad p, s = 1, 2, 3, 4, 5. \tag{5}$$

Taking into account that the matrix of the contravariant metric tensor of the hypersurface N looks as follows [2]:

$$(g^{ps}) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

we conclude from (2), (4), and (5) that

$$B_{31}^2 = 0, \; B_{32}^1 = 0.$$

Consequently,

$$\frac{1}{\sqrt{2}}\varepsilon_{31h} D^{h2} = 0 \Leftrightarrow \varepsilon_{312} D^{22} = 0 \Leftrightarrow$$

$$\Leftrightarrow D^{22} = 0 \Leftrightarrow D_{22} = 0;$$

$$\frac{1}{\sqrt{2}}\varepsilon_{32h}\, D^{h1} = 0 \Leftrightarrow \varepsilon_{321}\, D^{11} = 0 \Leftrightarrow$$

$$\Leftrightarrow D^{11} = 0 \Leftrightarrow D_{11} = 0.$$

Knowing the identity from [4]

$$(D_{12})^2 = D_{11} D_{22}, \tag{6}$$

we conclude that $D_{\alpha\beta} = 0$. Moreover, from (2) it follows that

$$B^{\alpha\beta}_{\gamma} = 0 \Leftrightarrow \frac{1}{\sqrt{2}}\varepsilon^{\alpha\beta 3}\, D_{3\gamma} = 0 \Leftrightarrow D_{3\gamma} = 0;$$

$$B^{\alpha\beta}_{3} = 0 \Leftrightarrow \frac{1}{\sqrt{2}}\varepsilon^{\alpha\beta 3}\, D_{33} = 0 \Leftrightarrow D_{33} = 0.$$

So, the matrix (D_{ab}) vanishes:

$$D_{ab} \equiv 0. \tag{7}$$

As we can see the condition (7) is fulfilled at every point of totally umbilical Kenmotsu hypersurface of six-dimensional Hermitian submanifold of the octave algebra. But this condition is a criterion for the six-dimensional submanifold $M^6 \subset \mathbf{O}$ to be Kählerian [3, 16]. That is why if $M^6 \subset \mathbf{O}$ satisfies with the U-Kenmotsu hypersurfaces axiom, then it is a Kählerian manifold. So, the Theorem 1 is completely proved.

As it was mentioned above, the paper [17] by V. F. Kirichenko contains a complete classification of six-dimensional Kählerian submanifolds of Cayley algebra. Now, we can state that this paper contains a complete classification of six-dimensional planar Hermitian submanifolds of Cayley algebra satisfying the U-Kenmotsu hypersurfaces axiom. We remark also that the property to satisfy the U-Kenmotsu hypersurfaces axiom essentially simplify the structure of the six-dimensional planar Hermitian submanifold of the octave algebra.

Locally symmetric $M^6 \subset \mathbf{O}$ are important and substantive examples of six-dimensional Hermitian planar submanifolds of Cayley algebra [4]. As we have just mentioned, the most interesting work on this subject is the article by V. F. Kirichenko [17]. In this paper, the notion of six-dimensional Hermitian Ricci type submanifolds was introduced. We note that the point $p \in M^6$ is called special if

$$T_p(M^6) \subset L(e_0)^{\perp},$$

where $L(e_0)^{\perp}$ is the orthogonal supplement of the unit of Cayley algebra. Otherwise, the point p is called simple. It is evident that the set of all simple points in M^6 forms an open submanifold $M^6_0 \subset M^6$, on which canonically is determined the one-dimensional distribution Z induced by the orthogonal projections of e_0 on the tangent spaces $T_p(M^6)$ for all points $p \in M_0^6$. Such a distribution Z as well as the one-dimensional space $Z_p \in T_p(M^6)$, $p \in M_0^6$, are called exceptional [17].

In accordance with the definition [4,17], a Hermitian $M^6 \subset \mathbf{O}$ is called a manifold of the Ricci type if its Ricci curvature at every point $p \in M_0^6$ in the direction of the exceptional space Z_p gets the minimum value.

Now, we use the complete classification of locally symmetric Hermitian $M^6 \subset \mathbf{O}$ of the Ricci type obtained by V. F. Kirichenko: every Hermitian locally symmetric submanifold $M^6 \subset \mathbf{O}$ of the Ricci type is locally holomorphically isometric either to C^3 or to the product of Kählerian manifolds C^2 and CH^1 , "twisted" along CH^1. (Here CH^1 denotes the complex hyperbolic space.)

In [17] it is also proved that the matrices (D_{ab}), (T_{ab}^8) and T_{ab}^8 with a corresponding choice of the frame look as follows, respectively:

$$\begin{pmatrix} D_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \begin{pmatrix} T_{ab}^8 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \begin{pmatrix} T_{ab}^7 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Moreover, for the case of $C^2 \times CH^1$ the conditions

$$D_{11} \neq 0;\ T_{ab}^8 \neq 0;\ T_{ab}^7 \neq 0$$

are simultaneously fulfilled.

Applying (4) and (6), we obtain the matrix of the second fundamental form of the immersion of Kenmotsu hypersurface in such a locally symmetric submanifold $M^6 \subset \mathbf{O}$ of the Ricci type:

$$(\sigma_{ps}) = \begin{pmatrix} 0 & 0 & 0 & -i & -iD^{11} \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & \sigma_{33} & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ iD_{11} & i & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that the condition (5) can not hold. That is why we conclude that the Kenmotsu hypersurface in a non-Kählerian locally symmetric submanifold $M^6 \subset \mathbf{O}$ of the Ricci type can not be totally umbilical. So, Theorem 2 is also completely proved.

Computing the determinant of the matrix of the second fundamental form of the immersion of Kenmotsu hypersurface in a non-Kahlerian locally symmetric submanifold $M^6 \subset \mathbf{O}$ of the Ricci type we have:

$$det(\sigma_{ps}) = \sigma_{33} \begin{vmatrix} 0 & 0 & -i & -iD^{11} \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ iD_{11} & i & 0 & 0 \end{vmatrix} = i\,\sigma_{33} \begin{vmatrix} 0 & -i & -iD^{11} \\ 0 & 0 & -i \\ i & 0 & 0 \end{vmatrix} = \sigma_{33}.$$

We obtain that the matrix is degenerate if and only if $\sigma_{33} = 0$. Knowing that this equality is equivalent to the condition of minimality of a Kenmotsu hypersurface in a Hermitian manifold $\sigma(\xi, \xi) = 0$ [2], we get the following additional result.

Corollary. *The Kenmotsu hypersurface of a locally symmetric submanifold* $M^6 \subset \mathbf{O}$ *of the Ricci type is minimal if and only if its second fundamental form matrix is degenerate.*

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Estimation of the extreme survival probabilities from censored data

Ion Grama, Jean-Marie Tricot and Jean-François Petiot

Abstract. The Kaplan-Meier nonparametric estimator has become a standard tool for estimating a survival time distribution in a right censoring schema. However, if the censoring rate is high, this estimator does not provide a reliable estimation of the extreme survival probabilities. In this paper we propose to combine the nonparametric Kaplan-Meier estimator and a parametric-based model into one construction. The idea is to fit the tail of the survival function with a parametric model while for the remaining to use the Kaplan-Meier estimator. A procedure for the automatic choice of the location of the tail based on a goodness-of-fit test is proposed. This technique allows us to improve the estimation of the survival probabilities in the mid and long term. We perform numerical simulations which confirm the advantage of the proposed method.

Mathematics subject classification: 62N01, 62N02, 62G32.
Keywords and phrases: Adaptive estimation, censored data, model selection, prediction, survival analysis, survival probabilities.

1 Introduction

Let $(X_i, C_i, Z_i)'$, $i = 1, ..., n$ be i.i.d. replicates of the vector $(X, C, Z)'$, where X and C are the survival and right censoring times and Z is a categorical covariate. It is supposed that X_i and C_i are conditionally independent given Z_i, $i = 1, ..., n$. We observe the sample $(T_i, \Delta_i, Z_i)'$, $i = 1, ..., n$, where $T_i = \min\{X_i, C_i\}$ is the observation time and $\Delta_i = 1_{\{X_i \leq C_i\}}$ is the failure indicator. Let $F(x|z)$, $x \geq x_0 \geq 0$ and $F_C(x|z)$, $x \geq x_0$ be the conditional distributions of X and C, given $Z = z$, respectively. In this paper we address the problem of estimation of the survival function $S_F(x|z) = 1 - F(x|z)$ when $x \geq x_0$ is large. The function S_F is traditionally estimated using the Kaplan-Meier nonparametric estimator (Kaplan and Meier [14]). Its properties have been extensively studied by numerous authors, including Fleming and Harrington [7], Andersen, Borgan, Gill and Keiding [2], Kalbfleisch and Prentice [13], Klein and Moeschberger [16]. However, in various practical applications, when the time x is close or exceeds the largest observed data, the predictions based on the Kaplan-Meier and related estimators are rather uninformative.

For illustration purposes we consider the well known PBC (primary biliary cirrhosis) data from a clinical trial analyzed in Fleming and Harrington [7]. In this trial one observes the censored survival times of two groups of patients: the first one ($Z = 1$) was given the DPCA (D-penicillamine drug) treatment and the second

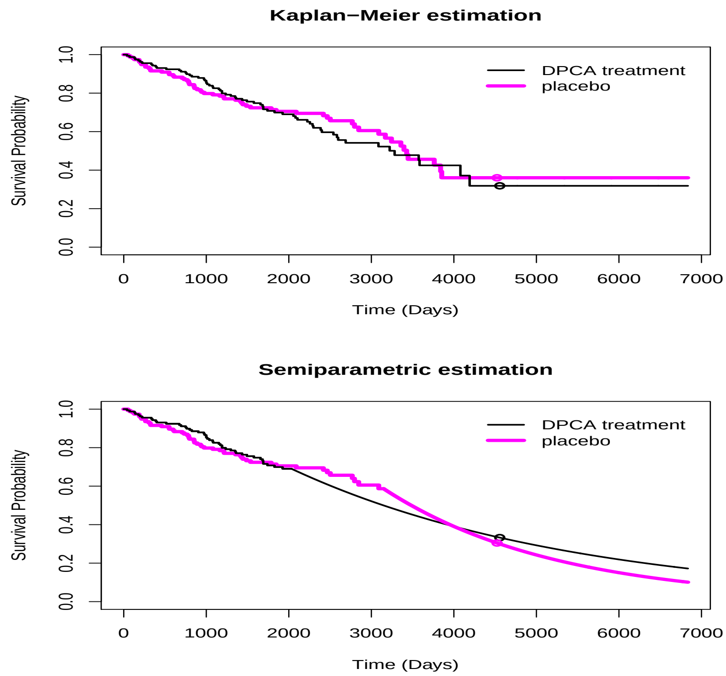

Figure 1. We compare two types of prediction of the survival probabilities in DPCA and placebo groups: on the top picture the prediction is based on the Kaplan-Meier estimation and on the bottom picture the prediction uses a semiparametric approach. The points on the curves correspond to the largest observation time in each group.

one is the control group ($Z = 0$). The overall censoring rate is about 60%. Here we consider only the group covariate and we are interested to compare the extreme survival probabilities of the patients under study in the two groups. In Figure 1 (top picture) we display the Kaplan-Meier nonparametric curves of the treatment and the control (placebo) groups. From these curves it seems difficult to infer whether the DPCA treatment has an effect on the survival probability. For instance at time $x = 4745$ (13 years) using the Kaplan-Meier nonparametric estimator (KM), one gets an estimated survival probability $\widehat{S}_{KM}(x|z = 0) = 0.3604$ for the control group and $\widehat{S}_{KM}(x|z = 1) = 0.3186$ for the DPCA treatment group. In this example and in many other applications one has to face the following two drawbacks. First, the estimated survival probabilities $\widehat{S}_{KM}(x|z)$ are constant for x beyond the largest (non-censored) survival time, which is not quite helpful for prediction purposes. Second, for this particular data set, the Kaplan-Meier estimation suggests that the DPCA treatment group has an estimated long term survival probability slightly lower than that of the control group, which can be explained by the high variability of $\widehat{S}_{KM}(x|z)$ for large x. These two points clearly rise the problem of correcting the behavior of the tail of the Kaplan-Meier estimator.

A largely accepted way to estimate the survival probabilities $S_F(x|z)$ for large x, is the parametric-based model fitting the hole data starting from the origin. Its advantages are pointed out in Miller [18], however, it is well known that the bias model can be high if it is misspecified. The more flexible nonparametric Kaplan-Meier estimator would generally be preferred for estimating certain functionals of the survival curve, as argued in Meier, Karrison, Chappell and Xie [17]. In this paper we propose to combine the nonparametric Kaplan-Meier estimator and the parametric-based model into one construction which we call semiparametric Kaplan-Meier estimator (SKM). Our new estimator incorporates a threshold t in such a way that $S_F(x|z)$ is estimated by the Kaplan-Meier estimator for $x \leq t$ and by a parametric-based estimate for $x > t$. The main theoretical contribution of the paper is to show that with an appropriate choice of the threshold t such an estimate is consistent if the tail is correctly specified. In the case when the tail is misspecified we show by simulations that the method is robust. Denote by $\widehat{S}_t$ the resulting estimator of S_F, where the parametric-based model is the exponential distribution with mean θ. By simulations we have found that $\widehat{S}_{\widehat{t}}$, endowed with a data driven threshold $\widehat{t}$, outperforms the Kaplan-Meier estimator. As it is seen from Figure 1 (bottom picture), we obtain at $x = 4745$ the estimated survival probability $\widehat{S}_{\widehat{t}_0}(x|z=0) = 0.2739$ for the control group and $\widehat{S}_{\widehat{t}_1}(x|z=1) = 0.3150$ for the DPCA treatment group, where $\widehat{t}_0$ and $\widehat{t}_1$ are the corresponding data driven thresholds. Our predictions are recorded in Table 2 and seem to be more adequate than those based on the Kaplan-Meier estimation. We refer to Section 7, where this example is described in more details.

In Figures 2 we display the root of the mean squared error of the predictions of $S_F(x|z)$ based on the Kaplan-Meier and the proposed semiparametric Kaplan-Meier estimators as functions of the observation time x. This is an example where the exponential model for survival and censoring tails are misspecified. The errors are computed within a Monte-Carlo simulation study of size $M = 2000$ with a gamma distribution modeling the survival and censoring times which do not exhibit exponential behavior in the tail (see Section 6 and Example 2 of Section 2 for details). The advantage of the proposed semiparametric estimator over the Kaplan-Meier estimator can be clearly seen by comparing the two MSE curves. The MSE of the semiparametric estimator is much smaller than that of the Kaplan-Meier estimator for large observation times $x > q_{0.99}$ but also for mid range observation time values, for example $x \in [8, q_{0.99}]$, where $q_{0.99}$ is the 0.99-quantile of the distribution F. The proposed extensions of the nonparametric curves are particularly suited for predicting the survival probabilities in the case when the proportion of the censored times is large. This is the case of the mentioned simulated data where the mean censoring rate is about 77%. Note also that we get an improvement over the Kaplan-Meier estimator even for very low sample sizes like $n = 20$.

The proposed estimator $\widehat{S}_t$ is sensible to the choice of the threshold t. The main difficulty is to choose t small enough, so that the parametric-based part contains enough observation times to ensure a reliable prediction in the tail. At the same time one should choose t large enough in order to prevent from a large bias due to an inadequate tail fitting. The very important problem of the automatic choice of the

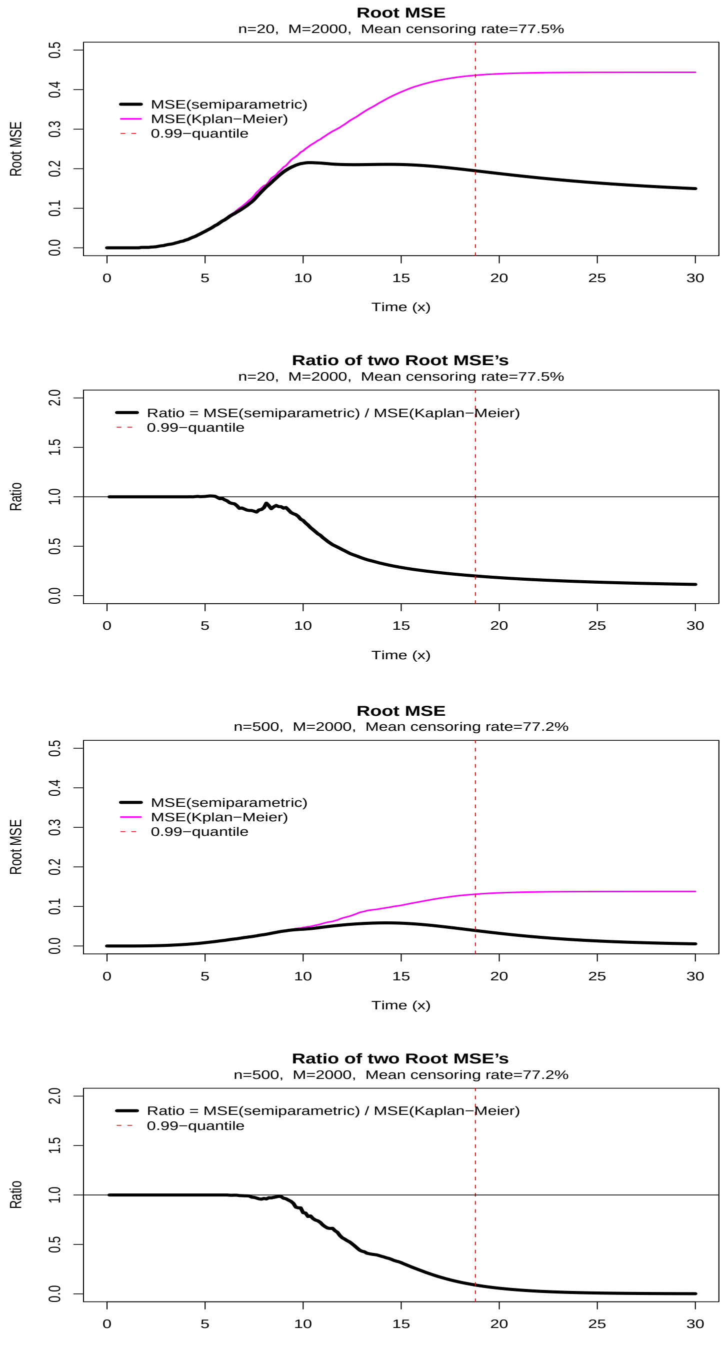

Figure 2. The lines 1, 3 (from the top) display simulated root MSE's of the Kaplan-Meier and semiparametric Kaplan-Meier estimators as functions of the time x. On the lines 2, 4 we show the ratio of the two root MSE's displayed on the lines 1, 3. The vertical dashed line is the 0.99 quantile of the true distribution of the survival time. The sample sizes are either $n = 20$ or $n = 500$.

threshold $\widehat{t}$ is treated in Section 5, where a procedure which we call *testing-pursuit-selection* is performed in two stages: First we test sequentially the null hypothesis that the proposed parametric-based model fits the data until we detect a chosen alternative. Secondly we select the best model among the accepted ones by penalized model selection. Therefore our testing-pursuit-selection procedure is actually also a goodness-of-fit test for the proposed parametric-based model. The resulting data driven estimator of the tail depends heavily on the testing procedure.

The approach developed here can be applied in conjunction with other techniques of prediction such as accelerated life testing, see Wei [22], Tseng, Hsieh and Wang [21], Escobar and Meeker [6] and extreme values estimation, see Hall [10], Hall and Welsh [11, 12], Dress [5], Grama and Spokoiny [8]. We refer also to Grama, Tricot and Petiot [9] for a related result concerning the approximation of the tail by the Cox model [4].

The case of continuous multivariate covariate Z in the context of a Cox model and the use of fitted tails other than the exponential can be treated by similar methods. The models which take into account the cure effects can be reduced to ours after removing the cure fraction. However, these problems are beyond the scope of the paper.

The paper is organized as follows. In Section 2 we introduce the main notations and give the necessary background. The main results of the paper about the consistency of the proposed estimators are stated in Sections 3 and 4. The automatic threshold selection procedure is described in Section 5. In Section 6 we give some simulation results and analyze the performance of the studied estimators. An application to real data is done in Section 7 and a conclusion in Section 8.

2 The model and background definitions

Assume that the survival and right censoring times arise from variables X and C which take their values in $[x_0, \infty)$, where $x_0 \geq 0$. Consider that X and C may depend on the categorical covariate Z with values in the set $\mathcal{Z} = \{0, ..., m\}$. The related conditional distributions $F(x|z)$ and $F_C(x|z)$, $x \geq x_0$, given $Z = z$, are supposed to belong to the set $\mathcal{F}$ of distributions with strictly positive density on $[x_0, \infty)$. Let $f_F(\cdot|z)$ and $S_F(\cdot|z) = 1 - F(\cdot|z)$ be the conditional density and survival functions of X, given $Z = z$. The corresponding conditional hazard function is $h_F(\cdot|z) = f_F(\cdot|z) / S_F(\cdot|z)$, given $Z = z$. Similarly, C has the conditional density $f_C(\cdot|z)$, survival function $S_C(\cdot|z)$ and hazard function $h_C(\cdot|z) = f_C(\cdot|z) / S_C(\cdot|z)$, given $Z = z$. We also assume the independence between X and C, conditionally with respect to Z. Let the observation time and the failure indicator be

$$T = \min\{X, C\} \quad \text{and} \quad \Delta = 1_{\{X \leq C\}},$$

where 1_B is the indicator function taking the value 1 on the event B and 0 otherwise. Let $P_{F,F_C}(dx, d\delta|z)$, $x \in [x_0, \infty)$, $\delta \in \{0, 1\}$ be the conditional distribution of the vector $\mathbf{Y} = (T, \Delta)'$, given $Z = z$. The density of P_{F,F_C} is

$$p_{F,F_C}(x, \delta|z) = f_F(x|z)^\delta S_F(x|z)^{1-\delta} f_C(x|z)^{1-\delta} S_C(x|z)^\delta, \tag{2.1}$$

where $x \in [x_0, \infty)$, $\delta \in \{0, 1\}$.

Let $z_i \in \mathcal{Z}$ be the observed value of the covariate Z_i, where Z_i, $i = 1, ..., n$ are i.i.d. copies of Z, and let $\mathbf{Y}_i = (T_i, \Delta_i)'$, $i = 1, ..., n$ be a sample of n vectors, where each vector $\mathbf{Y}_i$ has the conditional distribution $P_{F,F_C}(\cdot|z_i)$, given $Z_i = z_i$, for $i = 1, ..., n$. It is clear that, given $Z = z \in \mathcal{Z}$, the vectors $\mathbf{Y}_i$, $i \in \{j : z_j = z\}$ are i.i.d. .

In this paper the problem is to improve the nonparametric Kaplan-Meier estimators of the $m + 1$ survival probabilities $S_F(x|z) = 1 - F(x|z)$, $z \in \mathcal{Z}$, for large values of x. To this end, we fit the tail of $F(\cdot|z)$ by the exponential distribution with mean $\theta > 0$. Consider the following conditional semiparametric quasi-model

$$F_{\theta,t}(x|z) = \begin{cases} F(x|z), & x \in [x_0, t], \\ 1 - (1 - F(t|z)) \exp\left(-\frac{x-t}{\theta}\right), & x > t, \end{cases} \tag{2.2}$$

where $t \geq x_0$ is a nuisance parameter and $F(\cdot|z) \in \mathcal{F}$, $z \in \mathcal{Z}$ are functional parameters. The conditional density, survival and hazard functions of $F_{\theta,t}$ are denoted by $f_{F_{\theta,t}}$, $S_{F_{\theta,t}}$ and $h_{F_{\theta,t}}$, respectively. Note that $h_{F_{\theta,t}}(x|z) = 1/\theta$, for $x > t$.

The χ^2 entropy between two equivalent probability measures P and P_0 is defined by $\chi^2(P, P_0) = \int dP/dP_0 dP - 1$. By Jensen's inequality $\chi^2(P, P_0) \geq 0$.

Definition 2.1. *Let $F, F_C \in \mathcal{F}$ and $z \in \mathcal{Z}$. The tail of the distribution $F(\cdot|z)$ belongs to the domain of attraction of the exponential model under the right censoring schema if there exists a constant $\theta_z > 0$ such that*

$$\lim_{t\to\infty} \chi^2\left(P_{F,F_C}(\cdot|z), P_{F_{\theta_z,t},F_C}(\cdot|z)\right) = 0. \tag{2.3}$$

Below we give two examples when (2.3) is verified.

Example 1 (asymptotically constant hazards). Consider asymptotically constant survival and censoring hazard functions. This model can be related to the families of distributions in Hall [10], Hall and Welsh [11], Dress [5] and Grama and Spokoiny [8] for the extreme value models. Let $A > 0$, $\theta_{\max} > \theta_{\min} > 0$ be some constants. Consider that the survival time X has a hazard function $h_F(\cdot|z)$ such that for some $\theta_z \in (\theta_{\min}, \theta_{\max})$ and $\alpha_z > 0$,

$$|\theta_z h_F(\theta_z x|z) - 1| \leq A \exp(-\alpha_z x), \quad x \geq x_0. \tag{2.4}$$

Condition (2.4) means that $h_F(x|z)$ converges to θ_z^{-1} exponentially fast as $x \to \infty$. Substituting $\alpha_z = \alpha_z' \theta_z$, (2.4) gives $\left|h_F(x|z) - \theta_z^{-1}\right| \leq A' \exp(-\alpha_z' x)$, where $A' = A/\theta_{\min}$.

Similarly, let $M > 0$, $\gamma_{\max} > \gamma_{\min} > 0$, $\mu > 1$ be some constants. Assume that the hazard function $h_C(\cdot|z)$ of the censoring time C satisfies for some $\gamma_z \in (\gamma_{\min}, \gamma_{\max})$,

$$|\theta_z h_C(\theta_z x|z) - \gamma_z| \leq M(1 + x)^{-\mu}, \quad x \geq x_0. \tag{2.5}$$

Condition (2.5) is equivalent to saying that $h_C(x|z)$ approaches γ_z/θ_z polynomially fast as $x \to \infty$. Substituting $\gamma_z = \gamma_z' \theta_z$, (2.5) gives $|h_C(x|z) - \gamma_z'| \leq M' x^{-\mu}$, where $M' = M\theta_{\max}^{\mu}/\theta_{\min}$.

For example, conditions (2.4) and (2.5) are satisfied if F and F_C coincide with the re-scaled Cauchy distribution $K_{\mu,\theta}$ defined below. Let ξ be a variable with the positive Cauchy distribution $K(x) = 2\pi^{-1}\arctan(x)$, $x \geq 0$. We define the re-scaled Cauchy distribution by $K_{\mu,\theta}(x) = 1 - \frac{1-K(\exp((x-\mu)/\theta))}{1-K(\exp(-\mu/\theta))}$, where μ and θ are the location and scale parameters. The distribution $K_{\mu,\theta}$ can be seen as the excess distribution of the variable $\theta\log\xi + \mu$ over the threshold 0. The plots of the density $f_{K_{\mu,\theta}}$ related to $K_{\mu,\theta}$ for various values of parameters are given in Figure 4 (lines 1, 3). We leave to the reader the verification that $K_{\mu,\theta}$ fulfills (2.4) with $\theta_z = \theta$, $\alpha_z = 2$ and (2.5) with $\gamma_z = 1$. The distribution $K_{\mu,\theta}$ will be used in Section 6 to simulate survival and censoring times.

Example 2 (non-constant hazards). Now we consider the case when the hazard functions are not asymptotically constant. For instance, this is the case when the survival and censoring times have both gamma distributions. The numerical results presented in Figure 2 and Table 1 and discussed in Section 6 show that the approach of the paper works when conditions (2.4) and (2.5) are not satisfied.

The heuristic argument behind these experimental findings is as follows. Denote by $Q^{(t)}(x) = P(\xi \leq t + x|\xi \geq t)$, $x \geq 0$, the excess distribution of ξ over the threshold t, where ξ is a random variable with distribution Q. Let G_θ be the exponential distribution with mean θ. Obviously $G_\theta^{(t)} = G_\theta$. By simple re-normalization, the χ^2 entropy in (2.3) can be rewritten as follows:

$$\chi^2\left(P_{F,F_C}(\cdot|z), P_{F_{\theta_z,t},F_C}(\cdot|z)\right) = S_F(t|z)\, S_C(t|z) \times \chi^2(P_{F^{(t)},F_C^{(t)}}(\cdot|z), P_{G_{\theta_z},F_C^{(t)}}(\cdot|z)). \tag{2.6}$$

Clearly from (2.6), Definition 2.1 is fulfilled if, as $t \to \infty$,

$$\chi^2(P_{F^{(t)},F_C^{(t)}}(\cdot|z), P_{G_{\theta_z},F_C^{(t)}}(\cdot|z)) \to 0, \tag{2.7}$$

which means that beyond the threshold t, the excess distribution $F^{(t)}(\cdot|z)$ is "well" approximated by an exponential distribution with parameter θ_z, for some $t > 0$. However (2.3) can be satisfied even if (2.7) may not hold, more precisely when

$$\chi^2(P_{F^{(t)},F_C^{(t)}}(\cdot|z), P_{G_{\theta_z},F_C^{(t)}}(\cdot|z)) = o\left(\frac{1}{S_F(t|z)}\right), \tag{2.8}$$

where $S_F(t|z) \to 0$ as $t \to \infty$. This means that the tail probabilities can be estimated by our approach even if the exponential model is misspecified for the tail.

3 Consistency of the estimator with fixed threshold

Define the quasi-log-likelihood by $\mathcal{L}_t(\theta|z) = \sum_{i=1}^n \log p_{F_{\theta,t},F_C}(T_i, \Delta_i|z_i)\, 1_{\{z_i=z\}}$, where $F_{\theta,t}$ is defined by (2.2) with parameters $\theta > 0$, $t \geq x_0$ and $F(\cdot|z) \in \mathcal{F}$, $z \in \mathcal{Z}$. Taking into account (2.1) and dropping the terms related to the censoring, the partial quasi-log-likelihood is

$$\mathcal{L}_t^{\text{part}}(\theta|z) = \sum_{T_i \leq t,\ z_i = z} \Delta_i \log h_{F_{\theta,t}}(T_i|z) - \sum_{T_i > t,\ z_i = z} \Delta_i \log\theta \tag{3.1}$$

$$-\sum_{T_i \le t,\ z_i=z} \int_{x_0}^{T_i} h_{F_{\theta,t}}(v|z)\,dv - \sum_{T_i>t,\ z_i=z} \left(\int_{x_0}^{t} h_{F_{\theta,t}}(v)\,dv + \theta^{-1}(T_i - t) \right),$$

for fixed $z \in \mathcal{Z}$ and $t \ge x_0$. Maximizing $\mathcal{L}_t^{\text{part}}(\theta|z)$ in θ, obviously yields the estimator

$$\widehat{\theta}_{z,t} = \frac{\sum_{T_i>t,\ z_i=z}(T_i - t)}{\widehat{n}_{z,t}}, \tag{3.2}$$

where by convention $0/0 = \infty$ and $\widehat{n}_{z,t} = \sum_{T_i>t,\ z_i=z} \Delta_i$ is the number of observed survival times beyond the threshold t.

The estimator of $S_F(x)$, for $x_0 \le x \le t$, is easily obtained by standard nonparametric maximum likelihood approach due to Kiefer and Wolfowiz [15] (see also Bickel, Klaassen, Ritov and Wellner [3], Section 7.5). We use the product Kaplan-Meier (KM) estimator (with ties) defined by

$$\widehat{S}_{KM}(x|z) = \prod_{T_i \le x} \left(1 - d_i(z)/r_i(z)\right),\ x \ge x_0,$$

where $r_i(z) = \sum_{j=1}^n 1_{\{T_j \ge T_i,\ z_j=z\}}$ is the number of individuals at risk at T_i and $d_i(z) = \sum_{j=1}^n 1_{\{T_j=T_i,\Delta_j=1,\ z_j=z\}}$ is the number of individuals died at T_i (see Klein and Moeschberger [16], Section 4.2 and Kalbfleisch and Prentice [13]). The *semi-parametric fixed-threshold Kaplan-Meier estimator* (SFKM) of the survival function takes the form

$$\widehat{S}_t(x|z) = \begin{cases} \widehat{S}_{KM}(x|z), & x \in [x_0, t], \\ \widehat{S}_{KM}(t|z)\exp\left(-\frac{x-t}{\widehat{\theta}_{z,t}}\right), & x > t, \end{cases} \tag{3.3}$$

where $\exp\left(-(x-t)/\widehat{\theta}_{z,t}\right) = 1$ if $\widehat{\theta}_{z,t} = \infty$. Similarly, it is possible to use the Nelson-Aalen nonparametric estimator (Nelson [19, 20], Aalen [1]) instead of the Kaplan-Meier one.

Consider the Kullback-Leibler divergence $\mathcal{K}(\theta', \theta) = \int \log(dG_{\theta'}/dG_\theta)\,dG_{\theta'}$ between two exponential distributions with means θ' and θ. By convention, $\mathcal{K}(\infty, \theta) = \infty$. It is easy to see that $\mathcal{K}(\theta', \theta) = \psi(\theta'/\theta - 1)$, with $\psi(x) = x - \log(x+1)$, $x > -1$ and that there are two constants c_1 and c_2 such that $(\theta'/\theta - 1)^2 \le c_1 \mathcal{K}(\theta', \theta) \le c_2(\theta'/\theta - 1)^2$, when $|\theta'/\theta - 1|$ is small enough.

The following theorem provides a rate of convergence of the estimator $\widehat{\theta}_{z,t}$ as function of the χ^2-entropy between P_{F,F_C} and $P_{F_{\theta,t},F_C}$. Let $\mathbb{P}$ be the joint distribution of the sample $\mathbf{Y}_i$, $i = 1, ..., n$ and $\mathbb{E}$ be the expectation with respect to $\mathbb{P}$. In the sequel, the notation $\alpha_n = O_{\mathbb{P}}(\beta_n)$ means that there is a positive constant c such that $\mathbb{P}(\alpha_n > c\beta_n, \beta_n < \infty) \to 0$ as $n \to \infty$, for any two sequences of positive possibly infinite variables α_n and β_n.

Theorem 3.1. *Let $z \in \mathcal{Z}$. For any $\theta_z > 0$ (possibly depending on z) and $t \ge x_0$, it holds*

$$\mathcal{K}\left(\widehat{\theta}_{z,t}, \theta_z\right) = O_{\mathbb{P}}\left(\frac{n}{\widehat{n}_{z,t}}\chi^2\left(P_{F,F_C}(\cdot|z), P_{F_{\theta_z,t},F_C}(\cdot|z)\right) + \frac{4\log n}{\widehat{n}_{z,t}}\right). \tag{3.4}$$

For any $z \in \mathcal{Z}$ and $\theta_z > 0$ the optimal rate of convergence is obtained when the terms in the right hand side of (3.4) are balanced, i.e. when $t = t_{z,n}$ is chosen such that

$$\chi^2 \left(P_{F,F_C} (\cdot|z), P_{F_{\theta_z,t_{z,n}},F_C} (\cdot|z) \right) = O \left(\frac{\log n}{n} \right) \quad \text{as} \quad n \to \infty, \tag{3.5}$$

where $t_{z,n}$ may depend on z. It is easy to verify that, if the tail of the distribution $F(\cdot|z)$ belongs to the domain of attraction of the exponential model under the right censoring schema, a sequence $t_{z,n} \geq x_0$ satisfying (3.5) always exists.

From Theorem 3.1 we deduce the following:

Theorem 3.2. *Let $z \in \mathcal{Z}$. Assume that the distribution $F(\cdot|z)$ belongs to the domain of attraction of the exponential model under the right censoring schema and $t_{z,n}$ is a sequence satisfying (3.5). Then*

$$\mathcal{K} \left(\widehat{\theta}_{z,t_{z,n}}, \theta_z \right) = O_{\mathbb{P}} \left(\frac{\log n}{\widehat{n}_{z,t_{z,n}}} \right). \tag{3.6}$$

Using the two sided bound for the Kullback-leibler entropy between exponential laws stated before, from Theorem 3.2 we conclude that $\widehat{\theta}_{z,t_{z,n}}$ converges to θ_z at the usual $\left(\widehat{n}_{z,t_{z,n}}\right)^{-1/2}$ rate up to a $\log n$ factor: $\left(\widehat{\theta}_{z,t_{z,n}} - \theta_z \right)^2 = O_{\mathbb{P}} \left(\frac{\log n}{\widehat{n}_{z,t_{z,n}}} \right)$, provided that there are two constants $\theta_{\min}$ and $\theta_{\max}$ such that $0 < \theta_{\min} \leq \theta_z \leq \theta_{\max} < \infty$.

Furthermore, the rate of convergence of the estimator $\widehat{\theta}_{z,t_{z,n}}$ can be expressed in terms of $S_F(\cdot|z)$, $S_C(\cdot|z)$ and the sample size n, by giving a lower bound for $\widehat{n}_{z,t_{z,n}}$. To ensure such a bound we have to introduce two additional assumptions.

The first assumption involves the *conditional censoring rate function*

$$q_{F,F_C}(t|z) = \int_t^\infty S_{F,t}(x|z) f_{C,t}(x|z) \, dx \leq 1, \ t \geq x_0, \ z \in \mathcal{Z}, \tag{3.7}$$

where $S_{F,t}(x|z) = S_F(x|z) / S_F(t|z)$, $x \geq t$ is the conditional survival function related to the survival time X, given $X > t$, and $f_{C,t}(x|z) = f_C(x|z) / S_C(t|z)$, $x \geq t$ is the conditional density function related to the censoring time C, given $C > t$. The quantity $q_{F,F_C}(t|z)$ controls the proportion of the censored times among the observation times exceeding t. In particular if $t = x_0$, then $q_{F,F_C}(x_0|z) = \text{Prob}(X > C|z)$ is simply the mean censoring rate (given $Z = z$).

We assume that the conditional censoring rate function $q_{F,F_C}(\cdot|z)$ is separated from 1, i.e. that there are constants $r_0 \geq x_0$ and $q_0 < 1$, such that, for any $z \in \mathcal{Z}$ and any $t \geq r_0$,

$$q_{F,F_C}(t|z) \leq q_0. \tag{3.8}$$

Assumption (3.8) is verified, for instance, if $F(\cdot|z)$ and $F_C(\cdot|z)$ are exponential with intensities λ_X and λ_C respectively: in this case $q_{F,F_C}(t|z) = \lambda_C / (\lambda_C + \lambda_X)$, $t \geq 0$. It is also verified if distributions F and F_C meet (2.4) and (2.5). The trajectory of $q_{F,F_C}(\cdot|z)$ with F and F_C satisfying the two last conditions is plotted in Figure 4 (lines 2, 4).

The second assumption involves the number of individuals with profile $z \in \mathcal{Z}$: $n_z = \sum_{i=1}^n 1\left(z_i = z\right)$. We assume that there is a constant $\kappa \in (0,1]$ such that, for any $z \in \mathcal{Z}$,

$$n_z \geq \kappa\, n. \tag{3.9}$$

Lemma 3.3. *Assume that conditions (3.8) and (3.9) are satisfied. Then for every $t \geq r_0$, it holds $\mathbb{E}\widehat{n}_{z,t} \geq \kappa\ n\left(1-q_0\right) S_C\left(t|z\right) S_F\left(t|z\right)$ and $\mathbb{P}\left(\widehat{n}_{z,t} < \mathbb{E}\widehat{n}_{z,t}/2\right) \leq \exp\left(-\mathbb{E}\widehat{n}_{z,t}/8\right)$. Moreover, if the sequence $t_{z,n}$ is such that $\mathbb{E}\widehat{n}_{z,t_{z,n}} \to \infty$ as $n \to \infty$, then it holds $\mathbb{P}\left(\widehat{n}_{z,t_{z,n}} \geq \mathbb{E}\widehat{n}_{z,t_{z,n}}/2\right) \to 1$ as $n \to \infty$.*

As a simple consequence of Theorem 3.2 and Lemma 3.3 we have:

Theorem 3.4. *Assume conditions (3.8) and (3.9). Assume that the distribution $F\left(\cdot|z\right)$ belongs to the domain of attraction of the exponential model under the right censoring schema, $t_{z,n}$ is a sequence satisfying (3.5) and*

$$nS_C\left(t_{z,n}|z\right) S_F\left(t_{z,n}|z\right) \to \infty \text{ as } n \to \infty. \tag{3.10}$$

Then

$$\mathcal{K}\left(\widehat{\theta}_{z,t_{z,n}}, \theta_z\right) = O_{\mathbb{P}}\left(\frac{\log n}{nS_C\left(t_{z,n}|z\right) S_F\left(t_{z,n}|z\right)}\right).$$

4 Explicit computation of the rate of convergence

The results of the previous section show that the rate of convergence of the estimator $\widehat{\theta}_{z,t_{z,n}}$ depends on the survival functions $S_F\left(\cdot|z\right)$ and $S_C\left(\cdot|z\right)$ and on the sequences $t_{z,n}$. In order to derive a rate of convergence expressed only in terms of the sample size n we have to make additional assumptions on F and F_C. Moreover, we find minimal (up to one term expansion) threshold $t_{z,n}$ for which (3.5) holds true.

Our first result concerns the case when $h_C\left(\cdot|z\right)$ is separated from 0.

Theorem 4.1. *Assume conditions (3.8) and (3.9). Assume that $h_F\left(\cdot|z\right)$ satisfies (2.4), that there are positive constants $t_{\min}$ and $c_{\min}$ such that $h_C\left(x|z\right) \geq c_{\min}$ for any $x \geq t_{\min}$ and that*

$$S_C\left(t_{z,n}|z\right) n^{\frac{2\alpha_z}{1+2\alpha_z}} \log^{\frac{1}{1+2\alpha_z}} n \to \infty \text{ as } n \to \infty. \tag{4.1}$$

Then,

$$\mathcal{K}\left(\widehat{\theta}_{z,t_{z,n}}, \theta_z\right) = O_{\mathbb{P}}\left(\frac{\left(n^{-1}\log n\right)^{\frac{2\alpha_z}{1+2\alpha_z}}}{S_C\left(t_{z,n}|z\right)}\right), \tag{4.2}$$

where

$$t_{z,n} = \frac{\theta_z}{1+2\alpha_z}\log n + o\left(\log n\right).$$

Assume additionally that $S_C(t_{z,n}|z) \geq c_0 > 0$, which means that with positive probability there are large censoring times. Then the rate of convergence in (4.2) becomes $\left(n^{-1}\log n\right)^{\frac{2\alpha_z}{1+2\alpha_z}}$ for any $z \in \mathcal{Z}$.

Under the additional condition that $h_C(\cdot|z)$ satisfies (2.5) we have the following result:

Theorem 4.2. *Assume condition (3.9). Assume that $h_F(\cdot|z)$ satisfies (2.4) and $h_C(\cdot|z)$ satisfies (2.5). Then,*

$$\mathcal{K}\left(\widehat{\theta}_{z,t_{z,n}}, \theta_z\right) = O_{\mathbb{P}}\left(\left(\frac{\log n}{n}\right)^{\frac{2\alpha_z}{1+\gamma_z+2\alpha_z}}\right), \tag{4.3}$$

where

$$t_{z,n} = \frac{\theta_z}{1+\gamma_z+2\alpha_z}\log n + o(\log n).$$

We give some hints about the optimality of the rate in (4.3). Assume that the survival time X is exponential, i.e. $h_F(x|z) = \theta_z^{-1}$ for all $x \geq x_0$ and $z \in \mathcal{Z}$. This ensures that condition (2.4) is satisfied with any $\alpha > 0$. Assume conditions (2.5) and (3.9). If there are two constants $\theta_{\min}$ and $\theta_{\max}$ such that $0 < \theta_{\min} \leq \theta_z \leq \theta_{\max} < \infty$, (4.3) implies $\left|\widehat{\theta}_{z,t_{z,n}} - \theta_z\right| = O_{\mathbb{P}}\left(\left(n^{-1}\log n\right)^{\frac{\alpha}{1+\gamma_z+2\alpha}}\right)$, for any $\alpha > 0$. This rate becomes arbitrarily close to the $n^{-1/2}$ rate as $\alpha \to \infty$, since $\lim_{\alpha\to\infty} \alpha/(1+\gamma_z+2\alpha) \to 1/2$. Thus the estimator $\widehat{\theta}_{z,t_{z,n}}$ almost recovers the usual parametric rate of convergence as α becomes large whatever is $\gamma_z > 0$.

In the case when there are no censoring ($\gamma_z = 0$), after an exponential rescaling our problem can be reduced to that of the estimation of extreme index. If $\gamma_z \to 0$ our rate becomes close to $n^{-\frac{2\alpha_z}{1+2\alpha_z}}$, which is known to be optimal in the context of the extreme value estimation, see Dress [5] and Grama and Spokoiny [8]. So our result nearly recovers the best possible rate of convergence in this setting.

5 Testing and automatic selection of the threshold

In this section a procedure of selecting the adaptive estimator $\widehat{\theta}_z = \widehat{\theta}_{z,\widehat{t}_{z,n}}$ from the family of fixed threshold estimators $\widehat{\theta}_{z,t}$, $t \geq x_0$ is proposed. Here the adaptive threshold $\widehat{t}_{z,n}$ is obtained by a sequential testing procedure followed by a selection using a penalized maximum likelihood. This motivates our condensed terminology *testing-pursuit-selection* used in the sequel. The testing part is actually a multiple goodness-of-fit testing for the proposed parametric-based models, while the threshold $\widehat{t}_{z,n}$ can be seen as a data driven substitute for the theoretical threshold $t_{z,n}$ defined in Theorems 4.1 and 4.2 and in more general Theorems 3.2 and 3.4. For a discussion on the proposed approach we refer the reader to Section 3 of Grama and Spokoiny [8]. In the sequel, for simplicity of notations, we abbreviate $\widehat{t}_z = \widehat{t}_{z,n}$.

Define a semiparametric change-point distribution by

$$F_{\mu,s,\theta,t}(x|z) = \begin{cases} F(x|z), & x \in [x_0, s], \\ 1 - (1 - F(s|z)) \exp\left(-\frac{x-s}{\mu}\right), & x \in (s,t], \\ 1 - (1 - F(s|z)) \exp\left(-\frac{t-s}{\mu}\right) \exp\left(-\frac{x-t}{\theta}\right), & x > t, \end{cases}$$

for $\mu, \theta > 0$, $x_0 \leq s < t$ and $F(\cdot|z) \in \mathcal{F}$. As in Section 3 we find the maximum quasi-likelihood estimators $\widehat{\theta}_{z,t}$ of θ and $\widehat{\mu}_{z,s,t}$ of μ for fixed $z \in \mathcal{Z}$ and $x_0 \leq s < t$, which are given by (3.2) and

$$\widehat{\mu}_{z,s,t} = \frac{\widehat{n}_{z,s}\widehat{\theta}_{z,s} - \widehat{n}_{z,t}\widehat{\theta}_{z,t}}{\widehat{n}_{z,s,t}},$$

where $\widehat{n}_{z,s,t} = \sum_{s<T_i \leq t,\ z_i = z} \Delta_i$ and by convention $0 \cdot \infty = 0$ and $0/0 = \infty$.

Consider a constant $D > 0$, which will be the critical value in the testing procedure below. Let $k_0 \geq 3$ be a starting index and k_{step} be an increment for k. Let δ', δ'' be two positive constants such that $0 < \delta', \delta'' < 0.5$. The values $k_0, k_{step}, \delta', \delta''$ and D are the parameters of the procedure to be calibrated empirically. Without loss of generality, we consider that the T_i's are arranged in the decreasing order: $T_1 \geq ... \geq T_n$. The threshold t will be chosen in the set $\{T_1, ..., T_n\}$.

The *testing-pursuit-selection* procedure which we propose is performed in two stages. First we test the null hypothesis $\mathcal{H}_{T_k}(z) : F = F_{\theta,T_k}(\cdot|z)$ against the alternative $\widetilde{\mathcal{H}}_{T_k}(z) : F = F_{\mu,T_k,\theta,T_l}(\cdot|z)$ for some $\delta' k \leq l \leq (1-\delta'')k$, sequentially in $k = k_0 + ik_{step}$, $i = 0, ..., [n/k_{step}]$, until $\mathcal{H}_{T_k}(z)$ is rejected. Denote by $\widehat{k}_z$ the obtained break index and define the break time $\widehat{s}_z = T_{\widehat{k}_z}$. Second, using $\widehat{k}_z$ and $\widehat{s}_z$ define the adaptive threshold by $\widehat{t}_z = T_{\widehat{l}_z}$ with the adaptive index

$$\widehat{l}_z = \underset{\delta' \widehat{k}_z \leq l \leq (1-\delta'')\widehat{k}_z}{\operatorname{argmax}} \left\{ \mathcal{L}_{T_l}\left(\widehat{\theta}_{z,T_l}|z\right) - \mathcal{L}_{T_l}\left(\widehat{\theta}_{z,\widehat{s}_z}|z\right) \right\}, \tag{5.1}$$

where the term $\mathcal{L}_{T_l}\left(\widehat{\theta}_{z,\widehat{s}_z}|z\right)$ is a penalty for getting close to the break time $\widehat{s}_z$. The resulting adaptive estimator of θ_z is defined by $\widehat{\theta}_z = \widehat{\theta}_{z,\widehat{t}_z}$ and the *semiparametric adaptive-threshold Kaplan-Meier estimator* (SAKM) of the survival function is defined by $\widehat{S}_{\widehat{t}_z}(\cdot|z)$.

For testing $\mathcal{H}_{T_k}(z)$ against $\widetilde{\mathcal{H}}_{T_k}(z)$ we use the statistic

$$LR_{\max}(T_k|z) = \max_{\delta' k \leq l \leq (1-\delta'')k} LR(T_k, T_l|z), \tag{5.2}$$

where $LR(s,t|z)$ is the quasi-likelihood ratio test statistic for testing $\mathcal{H}_s(z) : F = F_{\theta,s}(\cdot|z)$ against the alternative $\widetilde{\mathcal{H}}_{s,t}(z) : F = F_{\mu,s,\theta,t}(\cdot|z)$. To compute (5.2), note that by simple calculations, using (3.1) and (3.2),

$$\mathcal{L}_t\left(\widehat{\theta}_{z,t}|z\right) - \mathcal{L}_t(\theta|z) = \widehat{n}_{z,t}\mathcal{K}\left(\widehat{\theta}_{z,t}, \theta\right), \tag{5.3}$$

where by convention $0 \cdot \infty = 0$. Similarly to (5.3), the quasi-likelihood ratio test statistic $LR\left(s,t|z\right)$ is given by

$$LR\left(s,t|z\right) = \widehat{n}_{z,s,t}\mathcal{K}\left(\widehat{\mu}_{z,s,t},\widehat{\theta}_{z,s}\right) + \widehat{n}_{z,t}\mathcal{K}\left(\widehat{\theta}_{z,t},\widehat{\theta}_{z,s}\right) \tag{5.4}$$

with the same convention. Note that, by (5.3), the second term in (5.4) can be viewed as the penalized quasi-log-likelihood

$$\begin{aligned} LR_{\text{pen}}\left(s,t|z\right) &= \mathcal{L}_t\left(\widehat{\theta}_{z,t}|z\right) - \mathcal{L}_t\left(\widehat{\theta}_{z,s}|z\right) \\ &= \widehat{n}_{z,t}\mathcal{K}\left(\widehat{\theta}_{z,t},\widehat{\theta}_{z,s}\right). \end{aligned}$$

Our testing-pursuit-selection procedure reads as follows:
Step 1. Set the starting index $k = k_0$.
Step 2. Compute the test statistic for testing $\mathcal{H}_{T_k}\left(z\right)$ against $\widetilde{\mathcal{H}}_{T_k}\left(z\right)$:

$$LR_{\max}\left(T_k|z\right) = \max_{\delta' k \le l \le (1-\delta'')k} LR\left(T_k,T_l|z\right)$$

Step 3. If $k \le n - k_{step}$ and $LR_{\max}\left(T_k|z\right) \le D$, increase k by k_{step} and go to Step 2. If $k > n - k_{step}$ or $LR_{\max}\left(T_k|z\right) > D$, let $\widehat{k}_z = k$,

$$\widehat{l}_z = \underset{\delta'\widehat{k}_z \le l \le (1-\delta'')\widehat{k}_z}{\operatorname{argmax}} LR_{\text{pen}}\left(T_{\widehat{k}_z},T_l|z\right),$$

take the adaptive threshold as $\widehat{t}_z = T_{\widehat{l}_z}$ and exit.

It may happen that with $k = k_0$ it holds $LR_{\max}\left(T_{k_0}|z\right) > D$, which means that the hypothesis that the tail is fitted by the exponential model, starting from T_{k_0}, is rejected. In this case we resume the procedure with a new augmented k_0, say with k_0 replaced by $[\nu_0 k_0]$, where $\nu_0 > 1$. Finally, if for each such k_0 it holds $LR_{\max}\left(T_{k_0}|z\right) > D$, we conclude that the tail of the model cannot be fitted with the proposed parametric tail and we estimate the tail by the Kaplan-Meier estimator. Therefore our testing-pursuit-procedure can be seen as well as a goodness-of-fit test for the tail.

Note that the Kullback-Leibler entropy $\mathcal{K}\left(\theta',\theta\right)$ is scale invariant, i.e. satisfies the identity $\mathcal{K}\left(\theta',\theta\right) = \mathcal{K}\left(\alpha\theta',\alpha\theta\right),$ for any $\alpha > 0$ and $\theta',\theta > 0$. Therefore the critical value D can be determined by Monte Carlo simulations from standard exponential observations. The choice of parameters of the proposed selection procedure is discussed in Section 6.

6 Simulation results

We illustrate the performance of the semiparametric estimator (3.3) with fixed and adaptive thresholds in a simulation study. The survival probabilities $S_F\left(x|z\right),$ for large values of x, are of interest.

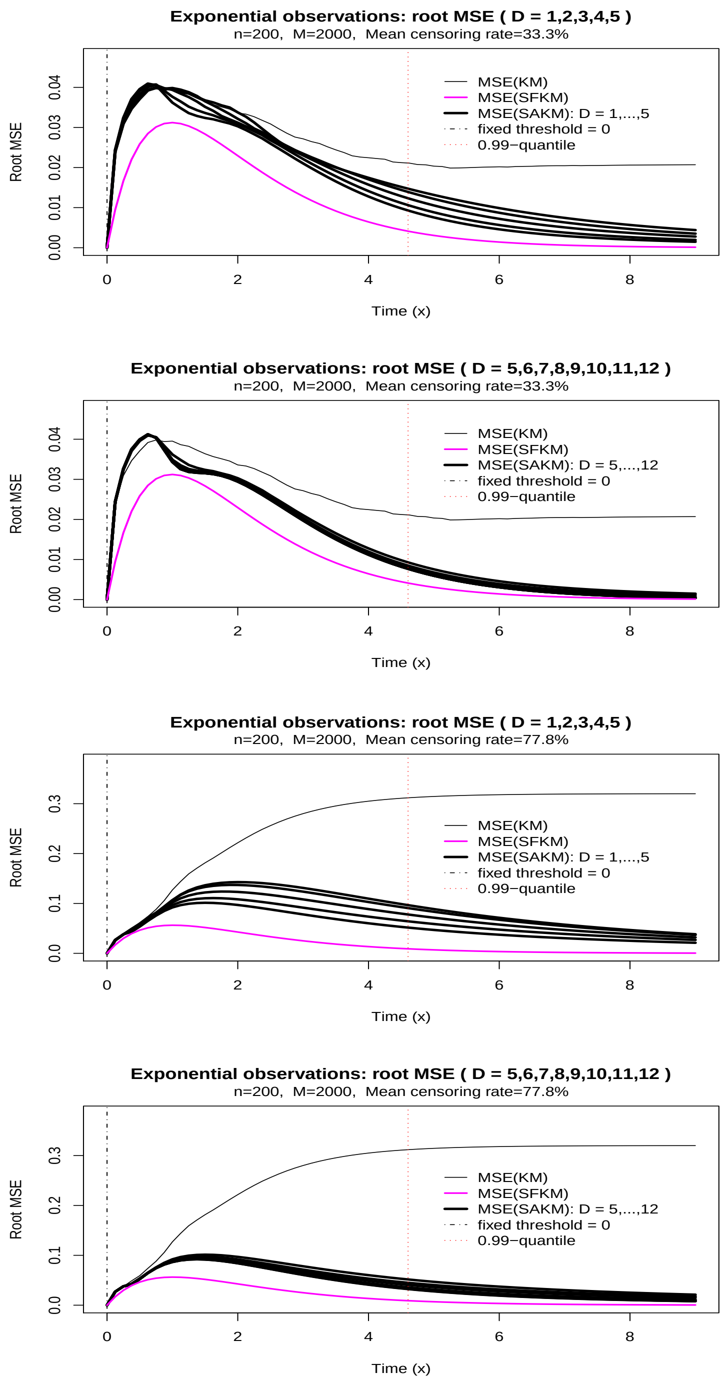

Figure 3. The lines 1, 3 (from the top) display the root type I MSE's of the Kaplan-Meier estimator (KM), of the semiparametric fixed-threshold Kaplan-Meier estimator (SFKM) with threshold fixed at 0 (which coincides with the exponential model) and of the semiparametric adaptive-threshold Kaplan-Meier estimator (SAKM) with $D = 1,2,3,4,5$. The lines 2, 4 display the same but with $D = 5,6,7,8,9,10,11,12$. The mean censoring rate is either 33.3% or 77.8%.

The mean squared error (MSE) of an estimator $\widehat{S}(\cdot|z)$ of the true survival function $S_F(\cdot|z)$ is defined by $MSE_{\widehat{S}}(x|z) = \mathbb{E}\left(\widehat{S}(x|z) - S_F(x|z)\right)^2$. The quality of the estimator $\widehat{S}(\cdot|z)$ with respect to the Kaplan-Meier estimator $\widehat{S}_{KM}(\cdot|z)$ is measured by the ratio $R_{\widehat{S}}(x|z) = MSE_{\widehat{S}}(x|z) / MSE_{\widehat{S}_{KM}}(x|z)$.

Without loss of generality, we can assume that the covariate Z takes a fixed value z. In each study developed below, we perform $M = 2000$ Monte-Carlo simulations.

We start by giving some hints on the choice of the parameters $k_0, k_{step}, \delta', \delta''$ of the testing-pursuit-selection procedure in Section 5. The initial value k_0 controls the variability of the test statistic $LR_{\max}(T_k|z)$, $k \geq k_0$. We have fixed k_0 as a proportion of the initial sample size: $k_0 = n/10$. The choice $k_{step} = 5$ is made to speed up the computations. The parameters δ' and δ'' restrict the high variability of the test statistic $LR(T_k, T_l|z)$ when the change point $T_l \in [T_k, T_{k_0}]$ is close to the ends of the interval. The values $\delta' = 0.3$ and $\delta'' = 0.1$ are retained experimentally. Our simulations show that the adaptive procedure does not depend much on the choice of the parameters $k_0, k_{step}, \delta', \delta''$.

To choose the critical value D we analyze the type I MSE of the SAKM estimator, i.e. the MSE under the null hypothesis that the survival times $X_1, ..., X_n$ are i.i.d. standard exponential. We perform two simulations using i.i.d. exponential censoring times $C_1, ..., C_n$ with rates 0.5 and 3.5. The size is fixed at $n = 200$, but the results are quite similar for other sizes. The root MSE's as functions of the time x are given in Figure 3. For comparison, in Figure 3 we also included the MSE's corresponding to the parametric-based exponential modeling which coincides with the SFKM estimator having the threshold fixed at 0. Note that the MSE's calculated when the critical values are $D = 1, 2, 3, 4, 5$, decrease as D increases (see the lines 1, 3), while for $D = 5, 6, 7, 8, 9, 10, 11, 12$ the MSE's almost do not depend on D (see the lines 2, 4). The simulations show that the type I MSE decreases as D increases and stabilizes for $D \geq 5$. From these plots we conclude that the limits for the critical value D can be set between $D_0 = 5$ and $D_1 = 7$ without important loss in the type I MSE.

It is interesting to note that the adaptive threshold $\widehat{t}_z$ is relatively stable to changes of D. A typical trajectory of the test statistic $LR_{\max}(T_k|z)$ as function of T_k is drawn in Figure 7 (top). Despite the fact that the break time $\widehat{s}_z = T_{\widehat{k}_z}$ strongly depends on the critical value D (in this picture $D = 5.8$), we found that the adaptive threshold $\widehat{t}_z = T_{\widehat{l}_z}$, which maximizes the penalized quasi-log-likelihood $LR_{\text{pen}}\left(T_{\widehat{k}_z}, T_l|z\right)$ in Figure 7 (bottom), is stable to the local changes of the break time $\widehat{s}_z = T_{\widehat{k}_z}$ and thus is also quasi-stable to relatively small changes of D.

For our simulations we fix the value $D = 6$. Below we give some evidence that the SAKM estimator with this critical value has a reasonable type II MSE, under the hypothesis that the X_i's have a distribution F alternative to the standard exponential. Our simulations show that the type II MSE's are quite similar for several families we have tested. We have chosen the following two typical cases which are representative for all these families.

Study case 1 (low tail censoring rate). We generate a sequence of $n = 200$ i.i.d. survival times $X_i,\ i = 1, ..., n$ from the re-scaled Cauchy distribution K_{μ_X,θ_X} with location parameter $\mu_X = 40$ and scale parameter $\theta_X = 5$ (see Section 2). The censoring times C_i, $i = 1, ..., n$ are i.i.d. from the re-scaled Cauchy distribution K_{μ_C,θ_C} with location parameter $\mu_C = \mu_X - 20 = 20$ and scale parameter $\theta_C = 2\theta_X = 10$. To give an overview of the variation of the censoring rate along the magnitude of X_i, we display the density functions of the survival and censoring times X_i and C_i in Figure 4. We also display the conditional censoring rate curve $q_{F,F_C}(t|z)$ as function of t. The (overall) mean censoring rate in this example corresponds to the starting point of the curve and is about 88% (horizontal dashed line in Figure 4, line 2). As $t \to \infty$ this curve decreases to the limit $\lim_{t\to\infty} q_{F,F_C}(t|z) = \theta_X/(\theta_C + \theta_X) = 1/3$, which means that the censoring rate for high observation times is about 33% (the right limit of the curve in Figure 4, line 2).

Study case 2 (high tail censoring rate). We take the same sample size $n = 200$. The X_i's, $i = 1, ..., n$ are i.i.d. from K_{μ_X,θ_X} with $\mu_X = 30$ and $\theta_X = 20$. The C_i's, $i = 1, ..., n$ are i.i.d. from K_{μ_C,θ_C} with $\mu_C = \mu_X + 10 = 40$ and $\theta_C = \theta_X/10 = 2$. In this case the (overall) mean censoring rate is about 40% (horizontal dashed line), however the conditional censoring rate in the tail is nearly equal to the limit $\lim_{t\to\infty} q_{F,F_C}(t|z) = \theta_X/(\theta_C + \theta_X) = 10/11$, i.e. is about 91% (see Figure 4, line 4).

We evaluate the performance of the SFKM and SAKM estimators $\widehat{S}_t(x|z)$ and $\widehat{S}_{\hat{t}_z}(x|z)$ with respect to the KM estimator $\widehat{S}_{KM}(x|z)$. In Figure 5 we display the root $MSE_{\widehat{S}}(x|z)$ (lines 1, 3) and the ratio $R_{\widehat{S}}(x|z)$ (lines 2, 4) for the three estimators as functions of the time x. From these plots we can see that both root $MSE_{\widehat{S}_t}(x|z)$ and root $MSE_{\widehat{S}_{\hat{t}_z}}(x|z)$ are equal to the root $MSE_{\widehat{S}_{KM}}(x|z)$ for small values of x and become smaller for large values of x, which shows that the SFKM and SAKM estimators improve the KM estimator.

In Figure 6 (lines 1, 3), for each fixed x, we show the confidence bands containing 90% of the values of $\widehat{S}_{KM}(x|z)$ and $\widehat{S}_{\hat{t}_z}(x|z)$. From these plots we see the ability of the model to fit the data and at the same time to give satisfactory predictions. Compared to those provided by the KM estimator which predicts a constant survival probability for large x, our predictions are more realistic.

In Figure 6 (lines 2, 4) we show the bias square and the variance of $\widehat{S}_{KM}(\cdot|z)$ and $\widehat{S}_{\hat{t}_z}(\cdot|z)$. From these plots we see that the variance of $\widehat{S}_{\hat{t}_z}(\cdot|z)$ is smaller than that of $\widehat{S}_{KM}(\cdot|z)$ in the two study cases. We conclude the same for their biases. However, the bias of $\widehat{S}_{KM}(\cdot|z)$ is large in the study case 2 (Figure 6, Line 4) because of a high conditional censoring rate in the tail (see Figure 4, line 4).

The case of non-constant hazards (see Example 2 of Section 2). The previous study is performed for models satisfying conditions (2.4) and (2.5). Now we consider the case when these conditions are not satisfied. Let X and C be generated from gamma distributions whose hazard rate function can be easily verified not to be asymptotically constant (in fact it is slowly varying at infinity). The survival time

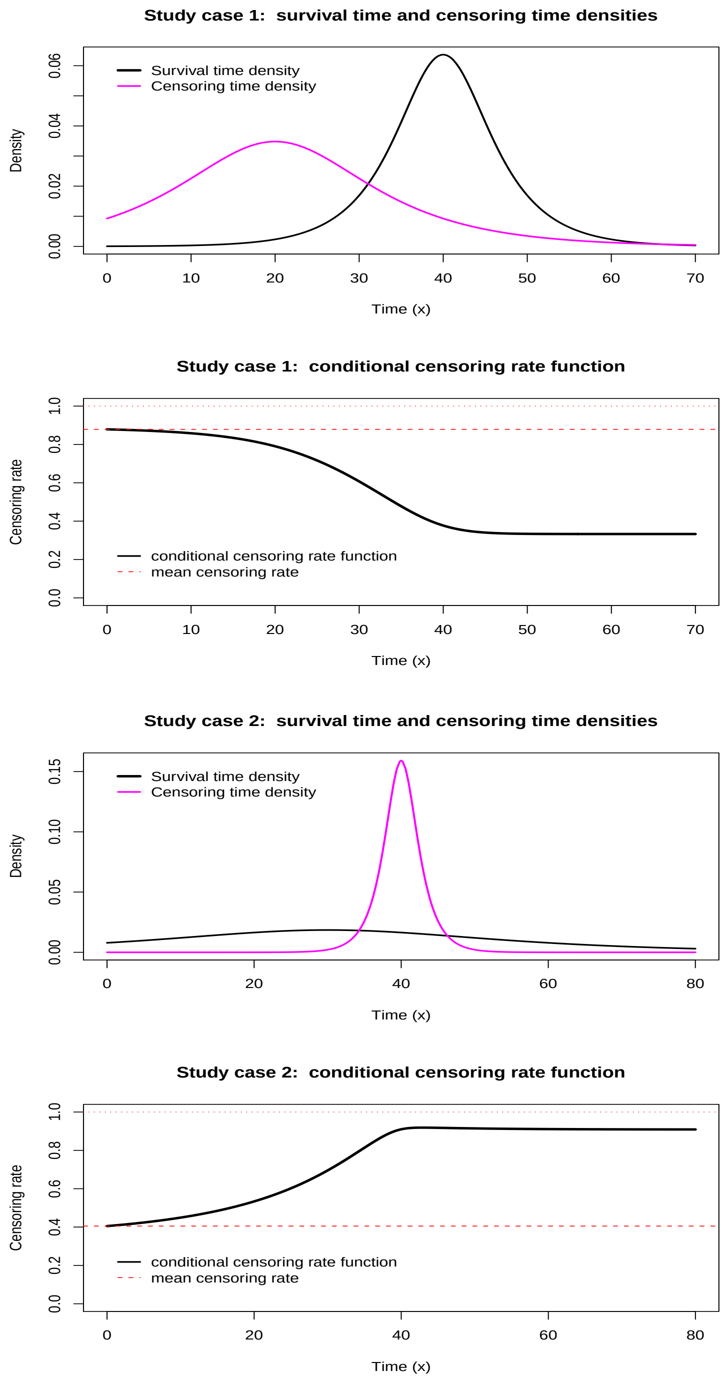

Figure 4. The lines 1, 3 (from the top) display the density functions of the survival and censoring times for study cases 1 and 2 (low and high tail censoring rates respectively). The lines 2, 4 display the conditional censoring rate $q_{F,F_C}(t|z)$ as function of the threshold t, for the two cases.

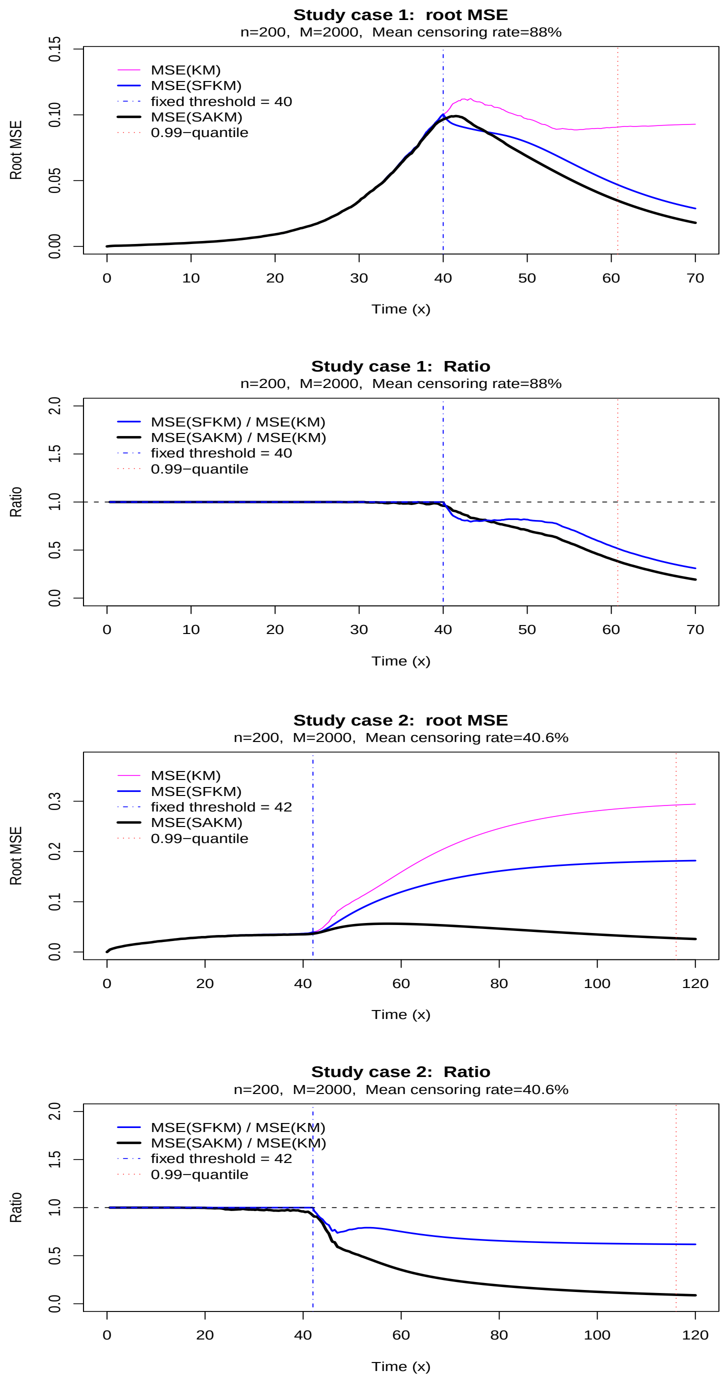

Figure 5. The lines 1, 3 (from the top) display the root type II MSE's of three estimators: $\widehat{S}_{KM}$ (KM), $\widehat{S}_t$ (SFKM) and $\widehat{S}_{\widehat{t}_z}$ (SAKM). The lines 2, 4 display the corresponding ratios of the root type II MSE's on the lines 1, 3. The critical value D in the SAKM is set to 6.

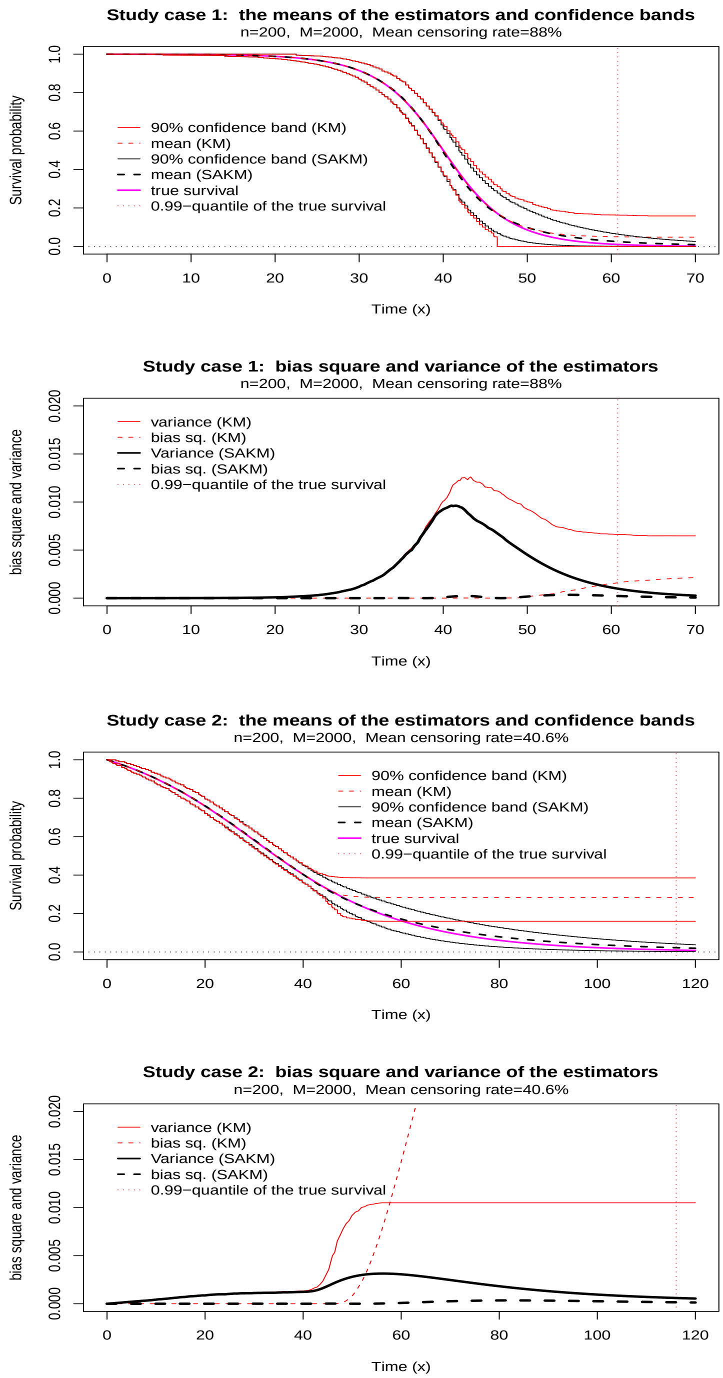

Figure 6. The lines 1, 3 (from the top) display the true survival S_F and the estimated means of $\widehat{S}_{KM}$ (KM) and $\widehat{S}_{\widehat{t}_z}$ (SAKM). We give confidence bands containing 90% of the trajectories for each fixed time x. The lines 2, 4 display the corresponding biases square and variances.

Table 1. Simulations with gamma distributions for survival and censoring times

x	5	6	7	8	9	10	11	12	13
$S_F\ (x\|z)$	0.9682	0.9161	0.8305	0.7166	0.5874	0.4579	0.3405	0.2424	0.1658
Mean of $\widehat{S}_{\widehat{t}_z}\ (x\|z)$	0.9679	0.9159	0.8318	0.7107	0.5686	0.4504	0.3575	0.2853	0.2287
Mean of $\widehat{S}_{KM}\ (x\|z)$	0.9679	0.9159	0.8306	0.7160	0.5875	0.4581	0.3399	0.2472	0.1888
Root $MSE_{\widehat{S}_{\widehat{t}_z}}\ (x\|z)$	0.0135	0.0225	0.0336	0.0461	0.0552	0.0606	0.0702	0.0831	0.0940
Root $MSE_{\widehat{S}_{KM}}\ (x\|z)$	0.0135	0.0225	0.0345	0.0466	0.0604	0.0758	0.0933	0.1144	0.1284
x	14	15	16	17	18	19	20	21	22
$S_F\ (x\|z)$	0.1094	0.0699	0.0433	0.0261	0.0154	0.0089	0.0050	0.0028	0.0015
Mean of $\widehat{S}_{\widehat{t}_z}\ (x\|z)$	0.1841	0.1487	0.1205	0.0979	0.0798	0.0652	0.0534	0.0439	0.0361
Mean of $\widehat{S}_{KM}\ (x\|z)$	0.1586	0.1453	0.1411	0.1403	0.1402	0.1402	0.1402	0.1402	0.1402
Root $MSE_{\widehat{S}_{\widehat{t}_z}}\ (x\|z)$	0.0997	0.0998	0.0952	0.0876	0.0785	0.0690	0.0599	0.0515	0.0441
Root $MSE_{\widehat{S}_{KM}}\ (x\|z)$	0.1384	0.1503	0.1627	0.1731	0.1804	0.1850	0.1877	0.1893	0.1902

X is gamma with shape parameter 10 and rate parameter 1 and the censoring time C is gamma with shape parameter 8.5 and rate parameter 1.2. The mean censoring rate in this example is about 77%. The results of the simulations are given in Figure 2 ($n = 20$ and $n = 500$) and Table 1 ($n = 500$) for $\widehat{S}_{KM}(\cdot|z)$ and $\widehat{S}_{\widehat{t}_z}(\cdot|z)$. They show that for these distributions the SAKM estimator gives a smaller root MSE than the KM estimator even when the sample size is low ($n = 20$) and x is in the range of the data.

7 Application to real data

As an illustration we deal with the well known randomized trial in primary biliary cirrhosis (PBC) from Fleming and Harrington [7] (see Appendix D.1). PBC is a rare but fatal chronic liver disease and the analyzed event is the patient's death. The trial was open for patient registration between January 1974 and May 1984. The observations lasted until July 1986, when the disease and survival status of the patients where recorded. There where $n = 312$ patients registered for the clinical trial, including 125 patients who died. The censored times where recorded either for patients which had been lost to follow up or had undergone liver transplantation or was still alive at the study analysis time (July 1986). The number of censored times is 187 and the censoring rate is about 59.9%. The last observed time is 4556 which is a censored time. Ties occur for the following three times: 264, 1191 and 1690. So there are 122 separate times for which we can observe at least one event. Two treatment groups of patients where compared: the first one ($Z = 1$) of size $n_1 = 158$ was given the DPCA (D-penicillamine drug). The second group ($Z = 0$) of size $n_0 = 154$ was the control (placebo) group. In this example we consider only the group covariate. We are interested to predict the survival probabilities of the patients under study in both groups.

The survival curves based on the KM and SAKM estimators for each group are displayed in Figure 1 (top and bottom pictures respectively). The numerical results on the predictions appear in Table 2. In this table, the time is running from 3 years ($x = 1095$ days) up to 20 years ($x = 7300$) with the step 1 year equivalent to 365 days for convenience.

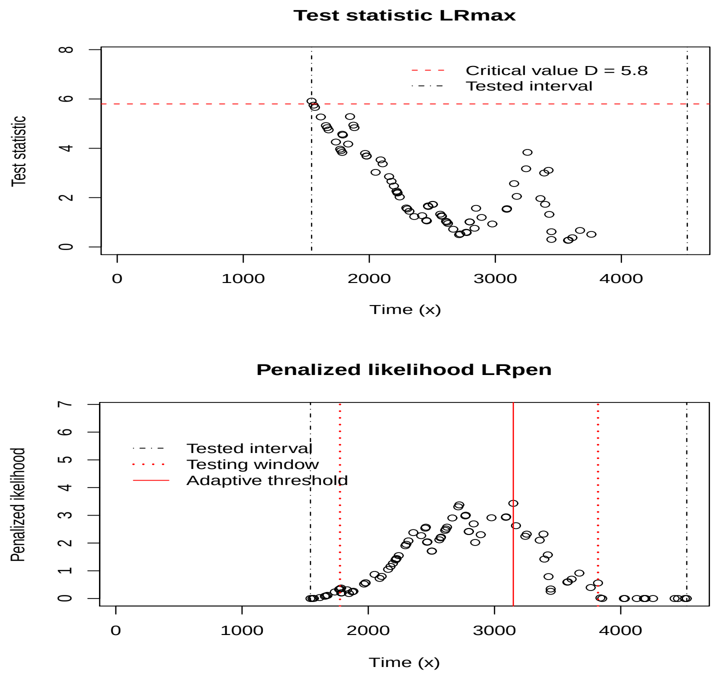

Figure 7. For the placebo group of PBC data we display the test statistics $LR_{\max}(T_k|z)$ as function of T_k (top) and $LR_{\text{pen}}(T_{\widehat{k}}, T_l|z)$ as function of T_l (bottom). The tested interval and the testing window are given by $[T_{\widehat{k}}, T_{k_0}]$ and $[T_{(1-\delta'')\widehat{k}}, T_{\delta'\widehat{k}}]$ respectively. The critical value D is fixed to 5.8.

Based on the usual KM estimator, the following two conclusions can be made: A1) The constant predictions for extreme survival probabilities in both groups appear to be too optimistic after the largest (non-censored) survival time. B1) The DPCA treatment appears to be less efficient than placebo in the long term. The statistical analysis with the SAKM estimator leads to more realistic conclusions: A2) The survival probabilities of each group extrapolate the tendency of the KM estimator as the time is increasing, and B2) the DPCA treatment is more efficient than placebo. For example, from the results in Table 2 we obtain that the survival probability in 20 years is about 2 times higher for the DPCA group than for the

Table 2. Predicted survival probabilities for PBC data

x : years	3	4	5	6	7	8	9	10	11
x : days	1095	1460	1825	2190	2555	2920	3285	3650	4015
DPCA: KM	0.8256	0.7635	0.7077	0.6613	0.5842	0.5417	0.4778	0.4247	0.4247
DPCA: SAKM	0.8256	0.7635	0.7077	0.6595	0.5934	0.5340	0.4805	0.4323	0.3890
Placebo: KM	0.7911	0.7398	0.7146	0.6950	0.6566	0.6055	0.5461	0.4563	0.3604
Placebo: SAKM	0.7911	0.7398	0.7146	0.6950	0.6566	0.6055	0.5497	0.4619	0.3881
x : years	12	13	14	15	16	17	18	19	20
x : days	4380	4745	5110	5475	5840	6205	6570	6935	7300
DPCA: KM	0.3186	0.3186	0.3186	0.3186	0.3186	0.3186	0.3186	0.3186	0.3186
DPCA: SAKM	0.3501	0.3150	0.2834	0.2550	0.2295	0.2065	0.1858	0.1672	0.1505
Placebo: KM	0.3604	0.3604	0.3604	0.3604	0.3604	0.3604	0.3604	0.3604	0.3604
Placebo: SAKM	0.3260	0.2739	0.2302	0.1934	0.1625	0.1365	0.1147	0.0964	0.0810

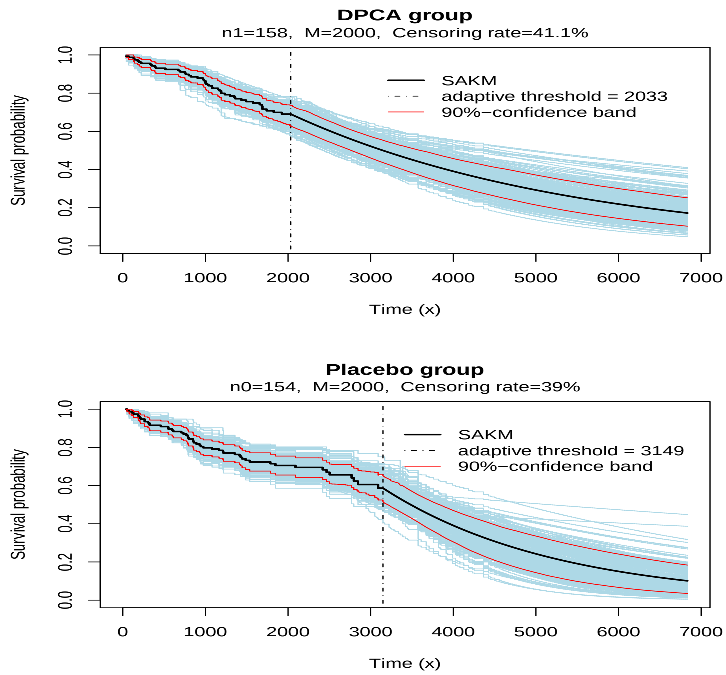

Figure 8. For PBC data, we display the pointwise bootstrap 90% confidence intervals for predicted probabilities: DPCA group (on top) and placebo (control) group (on bottom). We also display 100 bootstrap trajectories of the predicted probabilities for each group.

control group.

From the top picture of Figure 7 we see that the test statistic $LR_{\max}(T_k|0)$ for the control group $(Z=0)$ reaches the critical value $D = 5.8 \in [D_0, D_1]$ for $k = \widehat{k}_0 = 90$. Thus the hypotheses $\mathcal{H}_{s_0}(0)$ was rejected for the break time $\widehat{s}_0 = T_{\widehat{k}_0} = 1542$. The adaptive threshold $\widehat{t}_0$ is chosen via the maximization of the penalized quasi-log-likelihood (5.1). In the bottom picture of Figure 7 we see that the maximum is attained for the adaptive index $\widehat{l}_0 = 30$ and threshold $\widehat{t}_0 = T_{\widehat{l}_0} = 3149$. Thus, our testing-pursuit-selection procedure has captured the "convex bump" on the control Kaplan-Meier curve (for $Z=0$) between the times 2000 and 3500, which is easily seen in the bottom picture of Figure 1.

The pointwise (in x) 0.9-confidence bootstrap intervals for the predicted probabilities $\widehat{S}_{\widehat{t}_1}(x|1)$ and $\widehat{S}_{\widehat{t}_0}(x|0)$ are displayed in Figure 8 (top for DPCA treatment group $Z=1$ and bottom for control group $Z=0$). Here $\widehat{t}_1 = 2033$ and $\widehat{t}_0 = 3149$ are the adaptive thresholds computed from the original sample. The adaptive estimators of the mean parameters θ_1 and θ_0 are respectively $\widehat{\theta}_{1,\widehat{t}_1} = 3457.85$ and $\widehat{\theta}_{0,\widehat{t}_0} = 2096.22$. We generated $M = 2000$ bootstrap samples of size $n = 312$ taken at random from the general sample gathering the data coming from the two groups. For the m-th bootstrap sample the SAKM estimators $\widehat{S}^{(m)}_{\widehat{t}_1^{(m)}}(x|1)$ and $\widehat{S}^{(m)}_{\widehat{t}_0^{(m)}}(x|0)$ are computed as functions of x with their own adaptive thresholds $\widehat{t}_1^{(m)}$ and $\widehat{t}_0^{(m)}$.

8 Conclusion

This article deals with estimation of the survival probability in the framework of censored survival data. While the Kaplan-Meier estimator provides a flexible estimate of the survival function in the range of the data it can be improved for prediction of the extreme values, especially when the censoring rate is high. We propose a new approach based on the Kaplan-Meier estimator by adjusting a parametric correction to the tail beyond a given threshold t.

First we determine the rate of convergence of the corresponding estimators of the parameters in the adjusted model for a sequence of deterministic thresholds $t = t_{z,n}$ for each category z of the model covariate. This is done under the assumption that the hazard function is fitted by a constant in the sense that conditions (2.4) and (2.5) are satisfied. It is interesting to note that the rate of convergence depends not only on the class of survival time distributions but also on the class of censoring time distributions. By simulations we show that our approach is robust if the (survival and censoring) fitted tails are misspecified.

In applications the threshold t usually is not known. To overcome this we propose a testing-pursuit-selection procedure which yields an adaptive threshold $t = \widehat{t}_{z,n}$ in two stages: a sequential hypothesis testing and an adaptive choice of the threshold based on the maximization of a penalized quasi-log-likelihood. This testing-pursuit-selection procedure provides also a goodness-of-fit test for the parametric-based part of the model.

We perform numerical simulations with both the fixed and adaptive threshold estimators. Our simulations show that both estimators improve the Kaplan-Meier estimator not only in the long term, but also in a mid range inside the data. Comparing the fixed threshold and adaptive threshold estimators, we found that the adaptive choice of the threshold significantly improves on the quality of the predictions of the survival function.

We have seen that the quality of estimation of the extreme survival probabilities depends on the conditional censoring rate function, which describes the variations of the censoring rate as the time increases. The improvement over the Kaplan-Meier estimator is especially effective when the conditional censoring rate is high in the tail.

A Appendix: Proofs of the results

A.1 Auxiliary assertions

The following lemma plays the crucial role in the proof of our main results. Assume that $\mathbf{Y}_1, ..., \mathbf{Y}_n$ are i.i.d. with common distribution Q. Let $\mathbb{Q}$ be the joint distribution of $\mathbf{Y} = (\mathbf{Y}_1, ..., \mathbf{Y}_n)$. Let Q_1, Q_0 be two probability measures on $\mathbb{R}$ such that Q, Q_0 and Q_1 are equivalent. Define the quasi-log-likelihood ratio by

$$\mathcal{L}\left(Q_1, Q_0\right) = \sum_{i=1}^{n} \log \frac{dQ_1}{dQ_0}\left(\mathbf{Y}_i\right).$$

Lemma A.1. *For any $x \geq 0$, $n \geq 1$, we have*

$$\mathbb{Q}\left(\mathcal{L}\left(Q_1, Q_0\right) > x + n\chi^2\left(Q, Q_0\right)\right) \leq \exp\left[-\frac{x}{2}\right].$$

Proof. By exponential Chebyshev's inequality, for any $y > 0$,

$$\mathbb{Q}\left(\mathcal{L}\left(Q_1, Q_0\right) > y\right) \leq \exp\left[-y/2 + \log \mathbb{Q} \exp\left(\frac{1}{2}\mathcal{L}\left(Q_1, Q_0\right)\right)\right]. \tag{A.1}$$

Since $\mathbf{Y}_1, ..., \mathbf{Y}_n$ are i.i.d. with common distribution Q, we get

$$\log \mathbb{Q} \exp\left(\frac{1}{2}\mathcal{L}\left(Q_1, Q_0\right)\right) = n \log Q \sqrt{\frac{dQ_1}{dQ_0}}. \tag{A.2}$$

By Holder's inequality $Q\left(\sqrt{dQ_1/dQ_0}\right) \leq \sqrt{Q\left(dQ/dQ_0\right)} = \sqrt{1+\chi^2\left(Q, Q_0\right)}$. Using the last bound and (A.1), (A.2), it follows

$$\begin{aligned}\mathbb{Q}\left(\mathcal{L}\left(Q_1, Q_0\right) > y\right) &\leq \exp\left\{-\frac{y}{2} + \frac{n}{2}\log\left(1+\chi^2\left(Q, Q_0\right)\right)\right\} \\ &\leq \exp\left\{-\frac{y}{2} + \frac{n}{2}\chi^2\left(Q, Q_0\right)\right\}.\end{aligned}$$

Letting $y = x + n\chi^2\left(Q, Q_0\right)$ completes the proof. □

Now we produce an exponential bound for the quasi-log-likelihood ratio

$$\mathcal{L}_t\left(\theta'|z\right) - \mathcal{L}_t\left(\theta|z\right) = \sum_{z_i=z} \log \frac{p_{F_{\theta',t}, F_C}}{p_{F_{\theta,t}, F_C}}\left(\mathbf{Y}_i|z\right).$$

Lemma A.2. *For any $\theta, \theta' \in \mathbb{R}$, $z \in \mathcal{Z}$ and any $x \geq 0$ it holds*

$$\mathbb{P}\left(\mathcal{L}_t\left(\theta'|z\right) - \mathcal{L}_t\left(\theta|z\right) > x + n\chi^2\left(P_{F,F_C}\left(\cdot|z\right), P_{F_{\theta,t},F_C}\left(\cdot|z\right)\right)\right) \leq \exp\left(-\frac{x}{2}\right).$$

Proof. Let $I_z = \{i : z_i = z\}$. Note that $\mathbf{Y}_i = \left(T_i, \Delta_i\right)'$, $i \in I_z$, are i.i.d. variables. We apply Lemma A.1 with $\mathbf{Y} = \{\mathbf{Y}_i : i \in I_z\}$ and $Q = P_{F,F_C}\left(\cdot|z\right)$, $Q_0 = P_{F_{\theta,t},F_C}\left(\cdot|z\right)$, $Q_1 = P_{F_{\theta',t},F_C}\left(\cdot|z\right)$, which ends the proof. □

Next, we give an exponential bound for the maximum quasi-log-likelihood ratio which permits to obtain a rate of convergence of $\widehat{\theta}_{z,t}$.

Lemma A.3. *For any $\theta > 0$, $t \geq x_0$ and any $x \geq 0$ it holds*

$$\mathbb{P}\left(\widehat{n}_{z,t}\mathcal{K}\left(\widehat{\theta}_{z,t}, \theta\right) > x + n\chi^2\left(P_{F,F_C}\left(\cdot|z\right), P_{F_{\theta,t},F_C}\left(\cdot|z\right)\right) + 2\log n\right) \leq 2\exp\left(-\frac{x}{2}\right),$$

where $z \in \mathcal{Z}$ and by convention $0 \cdot \infty = 0$.

Proof. We prove that

$$\mathbb{P}\left(\widehat{n}_{z,t}\mathcal{K}\left(\widehat{\theta}_{z,t},\theta\right)>y\right)\leq 2n\exp\left(-x/2\right)=2\exp\left(-x/2+\log n\right), \tag{A.3}$$

where $y=x+n\chi^2\left(P_{F,F_C}\left(\cdot|z\right),P_{F_{\theta,t},F_C}\left(\cdot|z\right)\right)\geq 0.$

Since $\widehat{n}_{z,t}\widehat{\theta}_{z,t}=\sum_{T_i>t,\ z_i=z}\left(T_i-t\right)$, by direct calculations, we have $\mathcal{L}_t\left(\theta'|z\right)-\mathcal{L}_t\left(\theta|z\right)=\widehat{n}_{z,t}\Lambda_z\left(\theta'\right)$, where $\Lambda_z\left(u\right)=\log\left(\theta/u\right)-\left(u^{-1}-\theta^{-1}\right)\widehat{\theta}_{z,t}$. Using that $\mathcal{K}\left(\theta',\theta\right)=\theta'/\theta-1-\log\left(\theta'/\theta\right)$, we deduce $\mathcal{K}\left(\widehat{\theta}_{z,t},\theta\right)=\Lambda_z\left(\widehat{\theta}_{z,t}\right)$. Denote for brevity $g\left(u,k\right)=\left(\log\left(\theta/u\right)-y/k\right)/\left(u^{-1}-\theta^{-1}\right),\ u\neq\theta$. Note that, for $0<u<\theta$ the inequality $k\Lambda_z\left(u\right)>y$ is equivalent to $g\left(u,k\right)>\widehat{\theta}_{z,t}$ and for $u>\theta$ the inequality $k\Lambda_z\left(u\right)>y$ is equivalent to $g\left(u,k\right)<\widehat{\theta}_{z,t}$. Moreover the function $g\left(u,k\right)$ has a maximum for $0<u<\theta$ and a minimum for $u>\theta$.

Let $\theta^+\left(k\right)=\arg\max_{0\leq u<\theta}g\left(u,k\right)$ and $\theta^-\left(k\right)=\arg\min_{u>\theta}g\left(u,k\right)$. Then

$$\begin{aligned}\left\{\widehat{n}_{z,t}\Lambda_z\left(\widehat{\theta}_{z,t}\right)>y,\ \widehat{\theta}_{z,t}<\theta\right\} &= \left\{g\left(\widehat{\theta}_{z,t},\widehat{n}_{z,t}\right)>\widehat{\theta}_{z,t},\ \widehat{\theta}_{z,t}<\theta\right\}\\ &\subset \left\{g\left(\theta^+\left(\widehat{n}_{z,t}\right),\widehat{n}_{z,t}\right)>\widehat{\theta}_{z,t},\ \widehat{\theta}_{z,t}<\theta\right\}\\ &= \left\{\widehat{n}_{z,t}\Lambda_z\left(\theta^+\left(\widehat{n}_{z,t}\right)\right)>y,\ \widehat{\theta}_{z,t}<\theta\right\}\\ &\subset \left\{\widehat{n}_{z,t}\Lambda_z\left(\theta^+\left(\widehat{n}_{z,t}\right)\right)>y\right\}.\end{aligned}$$

In the same way, we get $\left\{\widehat{n}_{z,t}\Lambda_z\left(\widehat{\theta}_{z,t}\right)>y,\ \widehat{\theta}_{z,t}>\theta\right\}\subset\left\{\widehat{n}_{z,t}\Lambda_z\left(\theta^-\left(\widehat{n}_{z,t}\right)\right)>y\right\}$. Since $\Lambda_z\left(\widehat{\theta}_{z,t}\right)=\mathcal{K}\left(\widehat{\theta}_{z,t},\theta\right)$ and $\mathcal{K}\left(\widehat{\theta}_{z,t},\theta\right)=0$ if $\widehat{\theta}_{z,t}=\theta$, these inclusions imply

$$\begin{aligned}\left\{\widehat{n}_{z,t}\mathcal{K}\left(\widehat{\theta}_{z,t},\theta\right)>y\right\} &\subset \left\{\widehat{n}_{z,t}\Lambda_z\left(\theta^+\left(\widehat{n}_{z,t}\right)\right)>y\right\}\\ &\quad\cup\left\{\widehat{n}_{z,t}\Lambda_z\left(\theta^-\left(\widehat{n}_{z,t}\right)\right)>y\right\}.\end{aligned} \tag{A.4}$$

From (A.4), we get

$$\begin{aligned}&\mathbb{P}\left(\widehat{n}_{z,t}\mathcal{K}\left(\widehat{\theta}_{z,t},\theta\right)>y\right)\\ \leq\ &\mathbb{P}\left(\widehat{n}_{z,t}\Lambda_z\left(\theta^+\left(\widehat{n}_{z,t}\right)\right)>y\right)+\mathbb{P}\left(\widehat{n}_{z,t}\Lambda_z\left(\theta^-\left(\widehat{n}_{z,t}\right)\right)>y\right)\\ \leq\ &\sum_{k=1}^n\mathbb{P}\left(\widehat{n}_{z,t}\Lambda_z\left(\theta^+\left(k\right)\right)>y\right)+\sum_{k=1}^n\mathbb{P}\left(\widehat{n}_{z,t}\Lambda_z\left(\theta^-\left(k\right)\right)>y\right).\end{aligned} \tag{A.5}$$

By Lemma A.2, it follows, for $k=1,...,n$, $\mathbb{P}\left(\widehat{n}_{z,t}\Lambda_z\left(\theta^{\pm}\left(k\right)\right)>y\right)\leq\exp\left(-x/2\right)$. Then, by (A.5), we get (A.3), which ends the proof. □

A.2 Proof of Theorems 3.1 and 3.2

Theorem 3.1 follows immediately from Lemma A.3 if we set $x=2\log n$. Theorem 3.2 is a consequence of Theorem 3.1 and (3.5).

A.3 Proof of Lemma 3.3

By (2.1) it follows that $\mathbb{E}\widehat{n}_{z,t} = \sum_{z_i=z} \int_t^{\infty} f_F(x|z) S_C(x|z)\, dx$. Therefore, integrating by parts, we have $\mathbb{E}\widehat{n}_{z,t} = n_z S_F(t|z) S_C(t|z)(1 - q_{F,F_C}(t|z))$. Using (3.8) proves the first assertion.

Denote, for brevity, $\xi_i = 1_{\{T_i > t, \Delta_i = 1\}}$ and $p = \mathbb{P}(T_i > t, \Delta_i = 1) 1_{\{z_i = z\}}$. Then $\widehat{n}_{z,t} = \sum_{z_i=z} \xi_i$ and $\mathbb{E}\widehat{n}_{z,t} = n_z p$. Using exponential Chebyshev's inequality, for any $x > 0$ and any $u > 0$, we obtain

$$\mathbb{P}(\widehat{n}_{z,t} \leq \mathbb{E}\widehat{n}_{z,t} - x) \leq \exp\left(-ux + n_z p \frac{u^2}{2}\right).$$

Choosing $u = 1/2$ and $x = \mathbb{E}\widehat{n}_{z,t}/2$, we get $\mathbb{P}(\widehat{n}_{z,t} \leq \mathbb{E}\widehat{n}_{z,t}/2) \leq \exp(-n_z p/8)$, which proves the second assertion.

A.4 Proof of Theorem 4.1

Lemma A.4. *Assume that Q and Q_0 are two equivalent probability measures on a measurable space. Then*

$$\chi^2(Q, Q_0) \leq \int \left(\log \frac{dQ_0}{dQ}\right)^2 \exp\left(\left|\log \frac{dQ_0}{dQ}\right|\right) dQ.$$

Proof. Consider the convex function $g(x) = (x-1)^2/x$. Then $\chi^2(Q, Q_0) = \int g(dQ_0/dQ)\, dQ$. Since $(x-1)^2 \leq x^2 \log^2 x = \exp(2\log x)\log^2 x$ for $x \geq 1$, and $(x-1)^2 \leq \log^2 x$ for $x \in (0,1)$, we get $g(x) \leq \log^2 x \exp(|\log x|)$ for $x > 0$. □

We deduce Theorem 4.1 from Theorem 3.4. Let $z \in \mathcal{Z}$ and $t \geq x_0$. Consider the distance $\rho_t(h_1, h_2) = \sup_{x>t} |h_1(x) - h_2(x)|$, where h_1, h_2 are two non-negative functions. First we prove the following bound:

$$\chi^2\left(P_{F,F_C}(\cdot|z), P_{F_{\theta_z,t},F_C}(\cdot|z)\right) = O\left(S_C(t|z) S_F(t|z) \rho_t^2\right) \text{ as } t \to \infty. \tag{A.6}$$

By Lemma A.4,

$$\begin{aligned}\chi^2\left(P_{F,F_C}(\cdot|z), P_{F_{\theta_z,t},F_C}(\cdot|z)\right) \leq \int_{x_0}^{\infty} &\left(\log \frac{dP_{F,F_C}}{dP_{F_{\theta_z,t},F_C}}(x,\delta|z)\right)^2 \\ &\times \exp\left(\left|\log \frac{dP_{F,F_C}}{dP_{F_{\theta_z,t},F_C}}(x,\delta|z)\right|\right) P_{F,F_C}(dx, d\delta|z).\end{aligned} \tag{A.7}$$

According to (2.1), for any $x > t$,

$$\begin{aligned}\log \frac{dP_{F,F_C}}{dP_{F_{\theta_z,t},F_C}}(x,\delta|z) &= \log \frac{h_F(x|z)^{\delta} S_F(x|z)}{h_{F_{\theta_z,t}}(x|z)^{\delta} S_{F_{\theta_z,t}}(x|z)} \\ &= \delta \log \frac{h_F(x|z)}{\theta_z^{-1}} - \int_t^x \left(h_F(v|z) - \theta_z^{-1}\right) dv.\end{aligned}$$

For brevity, we denote $\rho_t = \rho_t\left(h_F(\cdot|z), \theta_z^{-1}\right)$. Since $\log(1+u) \le 2|u|$, for $u > -1/2$, it follows that

$$\left|\log \frac{dP_{F,F_C}}{dP_{F_{\theta_z,t},F_C}}(x,\delta|z)\right| \le c\rho_t(1+(x-t)), \tag{A.8}$$

whenever $\rho_t \le 1/(2\theta_{\min})$, where $c = \max\{2\theta_{\max}, 1\}$.

Denoting $g_{\rho_t}(x) = (1+x)^2 \exp(c\rho_t(1+x))$, from (A.7) and (A.8), we get

$$\begin{aligned} & \chi^2\left(P_{F,F_C}(\cdot|z), P_{F_{\theta_z,t},F_C}(\cdot|z)\right) \\ \le\ & c^2\rho_t^2 \int_{(t,\infty)\times\{0,1\}} g_{\rho_t}(x-t)\, p_{F,F_C}(x,\delta|z)\,\nu(dx,d\delta) \\ =\ & c^2\rho_t^2 \int_t^\infty \sum_{\delta\in\{0,1\}} g_{\rho_t}(x-t)\, f_F(x|z)^\delta\, S_F(x|z)^{1-\delta}\, f_C(x|z)^{1-\delta}\, S_C(x|z)^\delta\, dx. \end{aligned}$$

Since $S_C(x) \le S_C(t)$ and $S_F(x) \le S_F(t)$, for $x \ge t$, we obtain

$$\begin{aligned} & \chi^2\left(P_{F,F_C}(\cdot|z), P_{F_{\theta_z,t},F_C}(\cdot|z)\right) \\ \le\ & c^2\rho_t^2 S_F(t|z)\, S_C(t|z) \int_t^\infty g_{\rho_t}(x-t)\left(\frac{f_F(x|z)}{S_F(t|z)} + \frac{f_C(x|z)}{S_C(t|z)}\right) dx. \end{aligned}$$

From (2.4), $h_F(x|z)$ is bounded from below for x large enough:

$$\begin{aligned} h_F(x|z) &\ge \theta_z^{-1}\left(1 - |\theta_z h_F(x|z) - 1|\right) \\ &\ge \theta_{\max}^{-1}\left(1 - A\exp\left(-\alpha_{\min}\frac{x}{\theta_z}\right)\right) \\ &\ge 1/(2\theta_{\max}), \end{aligned}$$

whenever $x \ge t_{\min} = \theta_{\max}\log(2A)/\alpha_{\min}$, where $\alpha_{\min} = \min_{z\in\mathcal{Z}}\alpha_z$. This implies

$$\frac{S_F(x|z)}{S_F(t|z)} = \exp\left(-\int_t^x h_F(v|z)\,dv\right) \le \exp(-c_0(x-t)),$$

where $c_0 = 1/(2\theta_{\max})$. Integrating by parts, for any $t \ge t_{\min}$,

$$\begin{aligned} & \int_t^\infty g_{\rho_t}(x-t)\frac{f_F(x|z)}{S_F(t|z)}dx \\ =\ & \left[-g_{\rho_t}(x-t)\frac{S_F(x|z)}{S_F(t|z)}\right]_t^\infty + \int_t^\infty \frac{S_F(x|z)}{S_F(t|z)} g'_{\rho_t}(x-t)\,dx. \end{aligned}$$

If $\rho_t \le c_0/(2c)$, we have

$$\begin{aligned} & \int_t^\infty g_{\rho_t}(x-t)\frac{f_F(x|z)}{S_F(t|z)}dx \\ \le\ & \exp(c\rho_t) + \int_0^\infty (1+x)(2+c\rho_t(1+x))\exp(c\rho_t(1+x) - c_0 x)\,dx \end{aligned}$$

$$\leq \exp\left(\frac{c_2}{2}\right)\left(2+\frac{8}{c_0}+\frac{16}{c_0^2}\right) = O\left(1\right).$$

In the same way, conditions $h_C\left(x|z\right) \geq c_{\min}$, for $x \geq t_{\min}$ and $\rho_t \leq c_{\min}/\left(2c\right)$ imply, for $t \geq t_{\min}$,

$$\int_{t_{z,n}}^{\infty} g_{\rho_{t_{z,n}}}\left(x-t\right)\frac{f_C\left(x|z\right)}{S_C\left(t|z\right)}dx = O\left(1\right).$$

Putting together these bounds, yields (A.6).

Next, we find a sequence $t_{z,n}$ which verifies (3.5) and (3.10).

Since $S_C\left(t|z\right) \leq 1$, for verifying (3.5), it remains to find $t = t_{z,n}$ such that

$$S_F\left(t_{z,n}|z\right)\rho_{t_{z,n}}^2 = O\left(\frac{\log n}{n}\right). \tag{A.9}$$

Recall that $\alpha_z' = \alpha_z/\theta_z$ and $\gamma_z' = \gamma_z/\theta_z$ (see Example in Section 2). To prove (A.9), we note that, by (2.4),

$$\begin{aligned} S_F\left(t_{z,n}\right) &= \exp\left(-\int_{x_0}^{t_{z,n}} h_F\left(v|z\right)dv\right) \\ &\leq \exp\left(-\int_{x_0}^{t_{z,n}}\left(\theta_z^{-1}-\theta_z^{-1}Ae^{-\alpha_z' v}\right)dv\right) \\ &= O\left(\exp\left(-\theta_z^{-1}\left(t_{z,n}-x_0\right)\right)\right) \end{aligned} \tag{A.10}$$

and, again by condition (2.4),

$$\rho_{t_{z,n}}^2 = O\left(\exp\left(-2\alpha_z' t_{z,n}\right)\right). \tag{A.11}$$

Using (A.9), (A.10) and (A.11) we find $t_{z,n}$ from the following equation

$$\exp\left(-\left(\theta_z^{-1}+2\alpha_z'\right)t_{z,n}\right) = O\left(\frac{\log n}{n}\right).$$

The solution has the following expansion:

$$t_{z,n} = \frac{1}{\theta_z^{-1}+2\alpha_z'}\log n + o\left(\log n\right). \tag{A.12}$$

Thus (A.9) and consequently (3.5) are verified.

Now we prove (3.10). In the same way as in (A.10), we get

$$S_F\left(t_{z,n}\right) \geq \exp\left(-\theta_z^{-1}\left(t_{z,n}-x_0\right)-\frac{A}{\alpha_z}\exp\left(-\alpha_z' x_0\right)\right). \tag{A.13}$$

From (A.13) and (A.12), we get the following lower bound

$$nS_F\left(t_{z,n}|z\right) \geq n\exp\left(-\theta_z^{-1}t_{z,n}-c_1\right)$$

$$
\begin{aligned}
&\geq \quad n \exp\left(-\frac{\log n - \log\log n}{1+2\alpha_z} - c_1\right) \\
&\geq \quad c_2 n^{1-\frac{1}{1+2\alpha_z}} \log^{\frac{1}{1+2\alpha_z}} n \\
&= \quad c_2 n^{\frac{2\alpha_z}{1+2\alpha_z}} \log^{\frac{1}{1+2\alpha_z}} n, \qquad\qquad (A.14)
\end{aligned}
$$

where c_1, c_2 are some positive constants and n is large enough. Now condition (3.10) follows from (A.14) and from (4.1).

Assertion (4.2) follows from Theorem 3.4 using (A.14).

A.5 Proof of Theorem 4.2

As in the proof of Theorem 4.1 we verify (3.5) and (3.10). From (2.5) it follows

$$
S_C\left(t_{z,n}|z\right) \leq \exp\left(-\gamma_z'\left(t_{z,n}-x_0\right) + \frac{M}{\mu-1}\left(1+x_0\right)^{-\mu+1}\right). \qquad (A.15)
$$

From (A.6), (A.10), (A.11) and (A.15), we have

$$
\begin{aligned}
\chi^2\left(P_{F,F_C}\left(\cdot|z\right), P_{F_{\theta_z,t_{z,n}},F_C}\left(\cdot|z\right)\right) &= O\left(S_C\left(t_{z,n}|z\right) S_F\left(t_{z,n}|z\right) \rho^2_{t_{z,n}}\right) \\
&= O\left(\exp\left(-\left(\gamma_z' + \theta_z^{-1} + 2\alpha_z'\right) t_{z,n}\right)\right).
\end{aligned}
$$

We find $t_{z,n}$ as the solution of the equation

$$
\exp\left(-\left(\gamma_z' + \theta_z^{-1} + 2\alpha_z'\right) t_{z,n}\right) = O\left(\frac{\log n}{n}\right),
$$

which gives $t_{z,n} = \left(\theta_z^{-1} + \gamma_z' + 2\alpha_z'\right)^{-1} \log n + o\left(\log n\right)$. Thus (3.5) is verified. Condition (3.10) follows from

$$
\begin{aligned}
nS_C\left(t_{z,n}|z\right) S_F\left(t_{z,n}|z\right) &\geq n \exp\left(-\gamma_z' - \theta_z^{-1} t_{z,n} - c_1\right) \\
&\geq n \exp\left(-\left(\theta_z^{-1} + \gamma_z'\right) \frac{\log n - \log\log n}{\theta_z^{-1} + \gamma_z' + 2\alpha_z'} - c_1\right) \\
&\geq c_2 n^{1-\frac{1+\gamma_z}{1+\gamma_z+2\alpha_z}} \log^{\frac{1+\gamma_z}{1+\gamma_z+2\alpha_z}} n \\
&= c_2 n^{\frac{2\alpha_z}{1+\gamma_z+2\alpha_z}} \log^{\frac{1+\gamma_z}{1+\gamma_z+2\alpha_z}} n, \qquad\qquad (A.16)
\end{aligned}
$$

where c_1, c_2 are some positive constants and n is large enough.

The proof of (3.8) is based on similar arguments as in Section A.4.

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Chebyshev-Grüss-type inequalities via discrete oscillations

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Abstract. The classical form of Grüss' inequality, first published by G. Grüss in 1935, gives an estimate of the difference between the integral of the product and the product of the integrals of two functions. In the subsequent years, many variants of this inequality appeared in the literature. The aim of this paper is to introduce a new approach, presenting a new Chebyshev-Grüss-type inequality and applying to different well-known linear, not necessarily positive, operators. Some conjectures are presented. We also compare the new inequalities with some older results. In some cases this new approach gives better estimates than the ones already known.

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1 Introduction

Here we list some classical results which we will need in the sequel.

The functional given by

$$T(f,g) := \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \cdot \frac{1}{b-a}\int_a^b g(x)dx,$$

where $f, g : [a, b] \to \mathbb{R}$ are integrable functions, is well known in the literature as the classical Chebyshev functional (see [7]).

We first recall the following result.

Theorem 1 (see [20]). *Let $f, g : [a, b] \to \mathbb{R}$ be bounded integrable functions, both increasing or both decreasing. Furthermore, let $p : [a, b] \to \mathbb{R}_0^+$ be a bounded and integrable function. Then*

$$\int_a^b p(x)dx \int_a^b p(x)\cdot f(x)\cdot g(x)dx \geq \int_a^b p(x)\cdot f(x)dx \int_a^b p(x)\cdot g(x)dx. \qquad (1)$$

If one of the functions f or g is nonincreasing and the other nondecreasing, then inequality (1) is reversed.

Remark 1. Inequality (1) is known as Chebyshev's inequality. It was first introduced by P. L. Chebyshev in 1882 in [6]. If $p(x) = 1$ for $a \leq x \leq b$, then inequality (1) is equivalent to

$$\frac{1}{b-a}\int_a^b f(x)\cdot g(x)dx \geq \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\cdot\left(\frac{1}{b-a}\int_a^b g(x)dx\right).$$

The next result is the Grüss-type inequality for the Chebyshev functional.

Theorem 2. *(Grüss, 1935, see [14]) Let f, g be integrable functions from $[a, b]$ into $\mathbb{R}$, such that $m \leq f(x) \leq M$, $p \leq g(x) \leq P$, for all $x \in [a, b]$, where $m, M, p, P \in \mathbb{R}$. Then*

$$|T(f, g)| \leq \frac{1}{4}(M-m)(P-p).$$

The functional L, given by $L(f) := \frac{1}{b-a}\int_a^b f(x)dx$, is linear and positive and satisfies $L(e_0) = 1$; here we denote $e_i(x) = x^i$, for $i \geq 0$. In the sequel, we recall some bounds for what we call the generalized Chebyshev functional

$$T_L(f, g) := L(f \cdot g) - L(f) \cdot L(g) \tag{2}$$

and give some new results.

Remark 2. We will use the terminology "Chebyshev-Grüss-type inequalities", referring to Grüss-type inequalities for (special cases of) generalized Chebyshev functionals. These inequalities have the general form

$$|T_L(f, g)| \leq E(L, f, g),$$

where E is an expression in terms of certain properties of L and some kind of oscillations of f and g.

Another result we recall is a special form of a theorem given by D. Andrica and C. Badea (see [3]):

Theorem 3. *Let $I = [a, b]$ be a compact interval of the real axis, $B(I)$ be the space of real-valued and bounded functions defined on I and L be a linear positive functional satisfying $L(e_0) = 1$ where $e_0 : I \ni x \mapsto 1$. Assuming that for $f, g \in B(I)$ one has $m \leq f(x) \leq M$, $p \leq g(x) \leq P$ for all $x \in I$, the following holds:*

$$|T_L(f, g)| \leq \frac{1}{4}(M-m)(P-p).$$

Remark 3. Note that the positive linear functional is not present on the right hand side of the estimate.

The following pre-Chebyshev-Grüss inequality was given by A. Mc. D. Mercer and P. R. Mercer (see [18]) in 2004.

Theorem 4. *For a positive linear functional $L : B(I) \to \mathbb{R}$, with $L(e_0) = 1$, one has:*

$$|T_L(f,g)| \leq \frac{1}{2} \min\{(M-m)L\left(|g-G|\right), (P-p)L\left(|f-F|\right)\}$$

where $m \leq f(x) \leq M$, $p \leq g(x) \leq P$ for all $x \in I$, $F := Lf$ and $G := Lg$.

Remark 4. This is a more adequate result, considering that the positive linear functional appears on both the left and the right hand side of the inequality.

Let $C(X) = C_{\mathbb{R}}((X,d))$ be the Banach lattice of real-valued continuous functions defined on the compact metric space (X,d) and consider positive linear operators $H : C(X) \to C(X)$ reproducing constant functions. For $x \in X$ we take $L = \epsilon_x \circ H$, so $L(f) = H(f;x)$. We are interested in the degree of non-multiplicativity of such operators. Consider two functions $f, g \in C(X)$ and define the positive bilinear functional

$$T(f,g;x) := H(f \cdot g; x) - H(f;x) \cdot H(g;x).$$

Definition 1. Let $f \in C(X)$. If, for $t \in [0,\infty)$, the quantity

$$\omega_d(f;t) := \sup\left\{|f(x) - f(y)|\,,\ d(x,y) \leq t\right\}$$

is the usual modulus of continuity, then its least concave majorant is given by

$$\widetilde{\omega_d}(f,t) = \begin{cases} \sup_{0\leq x\leq t\leq y\leq d(X), x\neq y} \frac{(t-x)\omega_d(f,y)+(y-t)\omega_d(f,x)}{y-x} & \text{for } 0 \leq t \leq d(X)\ , \\ \omega_d(f, d(X)) & \text{if } t > d(X)\ , \end{cases}$$

and $d(X) < \infty$ is the diameter of the compact space X.

In [24] (see Theorem 3.1.) the following was shown.

Theorem 5. *If $f, g \in C(X)$, where (X,d) is a compact metric space, and $x \in X$ is fixed, then the inequality*

$$|T(f,g;x)| \leq \frac{1}{4}\widetilde{\omega_d}\left(f; 4\sqrt{H(d^2(\cdot,x);x)}\right) \cdot \widetilde{\omega_d}\left(g; 4\sqrt{H(d^2(\cdot,x);x)}\right)$$

holds, where $\widetilde{\omega_d}$ is the least concave majorant of the usual modulus of continuity and $H(d^2(\cdot,x);x)$ is the second moment of the operator H.

For $X = [a,b]$, we have a slightly better result (see Theorem 4.1. in [24]); a slightly weaker inequality had been obtained earlier in [1].

Theorem 6. *If $f, g \in C[a,b]$ and $x \in [a,b]$ is fixed, then the inequality*

$$|T(f,g;x)| \leq \frac{1}{4}\widetilde{\omega}\left(f; 2\sqrt{H((e_1-x)^2;x)}\right) \cdot \widetilde{\omega}\left(g; 2\sqrt{H((e_1-x)^2;x)}\right) \tag{3}$$

holds.

Remark 5. Here the moduli of continuity are oscillations defined with respect to functions f on the whole domain $X = [a, b]$. In order to improve some results, we propose a new approach, in which the oscillations are related to the support of the involved functional.

Remark 6. The inequality (3) is sharp in the sense that a positive linear operator reproducing constant and linear functions and functions $f, g \in C[a, b]$ exist such that equality occurs.

Example 1. Consider $f = g := e_1$. Then we have

$$\omega(f, t) = \omega(e_1, t) = \sup\{|x - y| : |x - y| \leq t\} = t.$$

Since $\omega(f, \cdot)$ is linear, we get $\widetilde{\omega}(f, \cdot) = \omega(f, \cdot)$. The left-hand side in Theorem 6 is

$$|T(f, g; x)| = H(e_2; x) - (H(e_1; x))^2$$

and the right-hand side is

$$\begin{aligned}\frac{1}{4}\widetilde{\omega}\left(f; 2\sqrt{H((e_1 - x)^2; x)}\right) \cdot \widetilde{\omega}\left(g; 2\sqrt{H((e_1 - x)^2; x)}\right) &= \frac{1}{4} \cdot (2\sqrt{H((e_1 - x)^2; x)})^2 \\ &= H((e_1 - x)^2; x).\end{aligned}$$

By choosing a positive linear operator $H : C[a, b] \to [a, b]$ such that $He_0 = e_0$ and $He_1 = e_1$, we get

$$\begin{aligned}H((e_1 - x)^2; x) &= H(e_2 - 2xe_1 + x^2; x) \\ &= H(e_2; x) - 2xH(e_1; x) + x^2 = H(e_2; x) - x^2 \\ &= H(e_2; x) - (H(e_1; x))^2,\end{aligned}$$

so we obtain equality between the two sides.

2 A Chebyshev-Grüss-type inequality: new approach

2.1 The compact topological space case

Let μ be a (not necessarily positive) Borel measure on the compact topological space X.

Let $\int_X d\mu(x) = 1$, and set $L(f) = \int_X f(x)d\mu(x)$, for $f \in C(X)$. Then, for $f, g \in C(X)$, we have

$$\begin{aligned}L(fg) - L(f)L(g) &= \int_X f(x)g(x)d\mu(x) - \int_X f(x)d\mu(x) \cdot \int_X g(y)d\mu(y) \\ &= \iint_{X^2} f(x)g(x)d(\mu \otimes \mu)(x, y) - \iint_{X^2} f(x)g(y)d(\mu \otimes \mu)(x, y)\end{aligned}$$

$$= \iint_{X^2} f(x)(g(x) - g(y))d(\mu \otimes \mu)(x, y).$$

Similarly,

$$L(fg) - L(f)L(g) = \iint_{X^2} f(y)(g(y) - g(x))d(\mu \otimes \mu)(x, y).$$

By addition,

$$2(L(fg) - L(f)L(g)) = \iint_{X^2} (f(x) - f(y))(g(x) - g(y))d(\mu \otimes \mu)(x, y). \tag{4}$$

Let

$$osc_L(f) := \max\{|f(x) - f(y)| : (x, y) \in supp(\mu \otimes \mu)\},$$

where $supp(\mu \otimes \mu)$ is the support of the tensor product of the Borel measure μ with itself (see [2]) and let $\Delta := \{(x, x) : x \in X\}$. From (4) we get

$$L(fg) - L(f)L(g) = \frac{1}{2} \iint_{X^2 \setminus \Delta} (f(x) - f(y))(g(x) - g(y))d(\mu \otimes \mu)(x, y).$$

Then we have the following result.

Theorem 7. *The Chebyshev-Grüss-type inequality in this case is given by*

$$|L(fg) - L(f)L(g)| \leq \frac{1}{2} \cdot osc_L(f) \cdot osc_L(g) \, |\mu \otimes \mu| \, (X^2 \setminus \Delta),$$

for $f, g \in C(X)$ and $|\mu \otimes \mu|$ is the absolute value of the tensor product of the Borel measure μ with itself (see Chapter 1 in [2]).

Example 2. Let $X = [0, 1]$ and consider the functional

$$L(f) = a \int_0^1 f(t)dt + (1 - a)f\left(\frac{1}{2}\right), \text{ for } 0 \leq a \leq 1.$$

Then $L(f) = \int_0^1 f(t)d\mu$, where the Borel measure μ is given by

$$\mu = a\lambda + (1 - a)\varepsilon_{\frac{1}{2}}$$

on X, with λ the Lebesgue measure on $[0, 1]$ and $\varepsilon_{\frac{1}{2}}$ the measure concentrated at $\frac{1}{2}$. Then the tensor product of μ with itself is

$$\begin{aligned} \mu \otimes \mu &= \left(a\lambda + (1 - a)\varepsilon_{\frac{1}{2}}\right) \otimes \left(a\lambda + (1 - a)\varepsilon_{\frac{1}{2}}\right) \\ &= a^2(\lambda \otimes \lambda) + a(1 - a)(\lambda \otimes \varepsilon_{\frac{1}{2}}) + (1 - a)a(\varepsilon_{\frac{1}{2}} \otimes \lambda) + (1 - a)^2(\varepsilon_{\frac{1}{2}} \otimes \varepsilon_{\frac{1}{2}}). \end{aligned}$$

$\mu \otimes \mu$ is a positive measure, so $|\mu \otimes \mu| = \mu \otimes \mu$, and

$$\begin{aligned}&\mu \otimes \mu\left([0,1]^2 \setminus \Delta\right) = [a^2(\lambda \otimes \lambda) + a(1-a)(\lambda \otimes \varepsilon_{\frac{1}{2}})\\ &+ a(1-a)(\varepsilon_{\frac{1}{2}} \otimes \lambda) + (1-a)^2(\varepsilon_{\frac{1}{2}} \otimes \varepsilon_{\frac{1}{2}})]\left([0,1]^2 \setminus \Delta\right)\\ &= a^2 + 2a(1-a) = a(2-a).\end{aligned}$$

The inequality becomes :

$$|L(fg) - L(f)L(g)| \le \frac{1}{2} \cdot a(2-a) \cdot osc_L(f) \cdot osc_L(g),$$

for two functions $f, g \in C[0,1]$.

2.2 The discrete linear functional case

Let X be an arbitrary set and $B(X)$ the set of all real-valued, bounded functions on X. Take $a_n \in \mathbb{R},\ n \ge 0$, such that $\sum_{n=0}^{\infty} |a_n| < \infty$ and $\sum_{n=0}^{\infty} a_n = 1$. Furthermore, let $x_n \in X,\ n \ge 0$ be arbitrary mutually distinct points of X. For $f \in B(X)$ set $f_n := f(x_n)$. Now consider the functional $L : B(X) \to \mathbb{R}$, $Lf = \sum_{n=0}^{\infty} a_n f_n$. L is linear and $Le_0 = 1$.

Then the relations

$$\begin{aligned}L(f \cdot g) - L(f) \cdot L(g) &= \sum_{n=0}^{\infty} a_n f_n g_n - \sum_{n=0}^{\infty} a_n f_n \cdot \sum_{m=0}^{\infty} a_m g_m\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} a_m\right) a_n f_n g_n - \sum_{n,m=0}^{\infty} a_n a_m f_n g_m\\ &= \sum_{n=0}^{\infty} a_n^2 f_n g_n + \sum_{n,m=0;m\neq n}^{\infty} a_m a_n f_n g_n\\ &- \sum_{n=0}^{\infty} a_n^2 f_n g_n - \sum_{n,m=0;m\neq n}^{\infty} a_n a_m f_n g_m\\ &= \sum_{n,m=0;m\neq n}^{\infty} a_n a_m f_n (g_n - g_m)\\ &= \sum_{0\le n<m<\infty} a_n a_m f_n (g_n - g_m) + \sum_{0\le n>m<\infty} a_n a_m f_n (g_n - g_m)\\ &= \sum_{0\le n<m<\infty} a_n a_m f_n (g_n - g_m) - \sum_{0\le n<m<\infty} a_n a_m f_m (g_n - g_m)\\ &= \sum_{0\le n<m<\infty} a_n a_m (f_n - f_m)(g_n - g_m)\end{aligned}$$

hold.

Theorem 8. *The Chebyshev-Grüss-type inequality for the above linear, not necessarily positive, functional L is given by:*

$$|L(fg) - L(f) \cdot L(g)| \leq osc_L(f) \cdot osc_L(g) \cdot \sum_{0 \leq n < m < \infty} |a_n a_m|,$$

where $f, g \in B(X)$ and we define the oscillations to be:

$$\begin{aligned} osc_L(f) &:= \sup\{|f_n - f_m| : 0 \leq n < m < \infty\}, \\ osc_L(g) &:= \sup\{|g_n - g_m| : 0 \leq n < m < \infty\}. \end{aligned}$$

Theorem 9. *In particular, if $a_n \geq 0$, $n \geq 0$, then L is a positive linear functional and we have:*

$$|L(fg) - Lf \cdot Lg| \leq \frac{1}{2} \cdot \left(1 - \sum_{n=0}^{\infty} a_n^2\right) \cdot osc_L(f) \cdot osc_L(g),$$

for $f, g \in B(X)$ and the oscillations given as above.

Remark 7. The above inequality is sharp in the sense that we can find a functional L such that equality holds.

Example 3. Let us consider the following functional

$$Lf := (1-a)f(0) + af(1), \text{ for } 0 \leq a \leq 1.$$

For this functional we have

$$L(fg) - Lf \cdot Lg = (1-a)f(0)g(0) + af(1)g(1) - [(1-a)f(0) + af(1)] \cdot [(1-a)g(0) + ag(1)],$$

so after some calculations we get that the left-hand side is

$$\begin{aligned} |L(fg) - Lf \cdot Lg| &= \left| \underbrace{a(1-a)}_{\geq 0} \cdot [f(0) - f(1)] \cdot [g(0) - g(1)] \right| \\ &= a(1-a) \, |f(0) - f(1)| \cdot |g(0) - g(1)| \end{aligned}$$

and the right-hand side is

$$\begin{aligned} \frac{1}{2}\left(1 - \sum_{n=0}^{\infty} a_n^2\right) \cdot osc_L(f) \cdot osc_L(g) &= \frac{1}{2} \cdot [1 - a^2 - (1-a)^2] \cdot |f(0) - f(1)| \cdot |g(0) - g(1)| \\ &= a(1-a) \, |f(0) - f(1)| \cdot |g(0) - g(1)| \, . \end{aligned}$$

3 A new Chebyshev-Grüss-type inequality for the Bernstein operator

Consider the classical Bernstein operators

$$B_n f(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{nk}(x), \ f \in \mathbb{R}^{[0,1]}, \ x \in [0,1],$$

where $b_{nk}(x) := \binom{n}{k} x^k (1-x)^{n-k}$. According to Theorem 9, for each $x \in [0,1]$, $f, g \in B[0,1]$ we have

$$|B_n(f \cdot g)(x) - B_n f(x) \cdot B_n g(x)| \leq \frac{1}{2}\left(1 - \sum_{k=0}^{n} b_{nk}^2(x)\right) \cdot osc_{B_n}(f) \cdot osc_{B_n}(g), \quad (5)$$

where

$$osc_{B_n}(f) := \max\{|f_k - f_l| : 0 \leq k < l \leq n\}$$

and $f_k := f\left(\frac{k}{n}\right)$; similar definitions apply to g.

Example 4. If we consider $f, g \in B[0,1]$ to be Dirichlet functions defined by

$$f(x) := \begin{cases} 1 & \text{for } x \in \mathbb{Q}, \\ 0 & \text{for } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

and analogously for g, with $f_k := f\left(\frac{k}{n}\right)$ (the same for g), then we observe that the oscillations in the above inequality vanish, so the right hand-side is zero.

Let $\varphi_n(x) := \sum_{k=0}^{n} b_{nk}^2(x)$, $x \in [0,1]$. Since

$$\left(\frac{1}{n+1}\sum_{k=0}^{n} b_{nk}^2(x)\right)^{\frac{1}{2}} \geq \frac{1}{n+1}\sum_{k=0}^{n} b_{nk}(x) = \frac{1}{n+1},$$

we get

$$\varphi_n(x) \geq \frac{1}{n+1}, \ x \in [0,1], \quad (6)$$

and therefore

$$|B_n(f \cdot g)(x) - B_n f(x) \cdot B_n g(x)| \leq \frac{n}{2(n+1)} \cdot osc_{B_n}(f) \cdot osc_{B_n}(g), \ x \in [0,1]. \quad (7)$$

Let us remark that equality is attained in (6) iff $n = 1$ and $x = \frac{1}{2}$. In fact, inspired also by some computations with Maple, we make the following conjectures:

Conjecture 1. φ_n is convex on $[0,1]$.

Conjecture 2. φ_n is decreasing on $\left[0, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, 1\right]$.

Conjecture 3. $\varphi_n(x) \geq \varphi_n\left(\frac{1}{2}\right)$, $x \in [0,1]$.

Since $\varphi_n\left(\frac{1}{2} - t\right) = \varphi_n\left(\frac{1}{2} + t\right)$, $t \in \left[0, \frac{1}{2}\right]$, we see that Conjecture 1 $\Rightarrow$ Conjecture 2 $\Rightarrow$ Conjecture 3.

On the other hand, it can be proved that

$$\varphi_n\left(\frac{1}{2}\right) = 4^{-n}\binom{2n}{n},\ \varphi_n'\left(\frac{1}{2}\right) = 0,\ \varphi_n''\left(\frac{1}{2}\right) = 4^{2-n}\binom{2n-2}{n-1},$$

and so $\frac{1}{2}$ is a minimum point for φ_n. Conjecture 3 claims that it is an absolute minimum point; in other words,

$$\varphi_n(x) \geq \frac{1}{4^n}\binom{2n}{n},\ x \in [0,1]. \tag{8}$$

The following confirmation of Conjecture 3 is due to Dr. Th. Neuschel (Katholieke Universiteit Leuven).

Lemma 1. *For $n \in \mathbb{N}$ and $x \in [0,1]$, we have*

$$\sum_{k=0}^{n} \binom{n}{k}^2 x^{2k}(1-x)^{2(n-k)} \geq \frac{1}{4^n}\binom{2n}{n}.$$

Proof. For symmetry reasons, it suffices to prove the statement only for $0 \leq x \leq \frac{1}{2}$. In the sequel we denote P_n to be the $n-$th Legendre polynomial, given by

$$P_n(x) := \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}^2 (x+1)^k (x-1)^{n-k}.$$

We make a change of variable, namely set $y := \frac{1-2x+2x^2}{1-2x} \geq 1$ and we get

$$(y - \sqrt{y^2-1})^n \cdot P_n(y) = \sum_{k=0}^{n}\binom{n}{k}^2 x^{2k}(1-x)^{2(n-k)} := \varphi_n(x),$$

so we have to show that

$$(y - \sqrt{y^2-1})^n \cdot P_n(y) \geq \frac{1}{4^n}\binom{2n}{n}$$

holds, for $y \geq 1$. The inequality holds for $y = 1$ and $y = \infty$. In the last case, the inequality is even sharp. Now it is enough to show :

$$\frac{d}{dy}\{(y - \sqrt{y^2-1})^n P_n(y)\} \leq 0 \text{ for } y > 1.$$

This is equivalent to the following statement

$$P_n'(y) \leq \frac{n}{\sqrt{y^2-1}} P_n(y) \text{ for } y > 1.$$

Using the formula

$$\frac{y^2-1}{n}P_n'(y) = yP_n(y) - P_{n-1}(y),$$

we now have to prove the following:

$$(y-\sqrt{y^2-1})P_n(y) \le P_{n-1}(y) \text{ for } y>1,$$

which is equivalent to

$$P_n(y) \le (y+\sqrt{y^2-1})P_{n-1}(y) \text{ for } y>1. \tag{9}$$

The inequality (9) can be proved by induction. For $n=1$ the inequality holds. We assume that the inequality holds also for n and we want to show:

$$P_{n+1}(y) \le (y+\sqrt{y^2-1})P_n(y) \text{ for } y>1.$$

Using Bonnet's recursion formula

$$P_{n+1}(y) = \frac{2n+1}{n+1}yP_n(y) - \frac{n}{n+1}P_{n-1}(y),$$

we now have to show that the following holds:

$$\left(\frac{2n+1}{n+1}y - (y+\sqrt{y^2-1})\right)P_n(y) \le \frac{n}{n+1}P_{n-1}(y).$$

After evaluation

$$\begin{aligned}\left(\frac{2n+1}{n+1}y - (y+\sqrt{y^2-1})\right)P_n(y) &\le \frac{n}{n+1}(y-\sqrt{y^2-1})P_n(y)\\ &\le \frac{n}{n+1}(y-\sqrt{y^2-1})(y+\sqrt{y^2-1})P_{n-1}(y)\\ &= \frac{n}{n+1}P_{n-1}(y),\end{aligned}$$

we obtain the result. □

In order to compare (6) and (8), it is not difficult to prove the inequalities

$$\frac{1}{n+1} < \frac{1}{2\sqrt{n}} < \frac{1}{4^n}\binom{2n}{n} < \frac{1}{\sqrt{2n+1}},\ n\ge 2.$$

More precise inequalities can be found in [9]:

$$\frac{1}{\sqrt{\pi(n+3)}} < \frac{1}{4^n}\binom{2n}{n} < \frac{1}{\sqrt{\pi(n-1)}},\ n\ge 2.$$

Because we have proved that Conjecture 3 is true, we have the following result.

Theorem 10. *The new Chebyshev-Grüss-type inequality for the Bernstein operator is:*

$$|B_n(f\cdot g)(x)-B_nf(x)\cdot B_ng(x)|\leq \frac{1}{2}\left(1-\frac{1}{4^n}\binom{2n}{n}\right)\cdot osc_{B_n}(f)\cdot osc_{B_n}(g),\ x\in[0,1]. \tag{10}$$

In comparison, using the second moment of the Bernstein polynomial

$$B_n((e_1-x)^2;x)=\frac{x(1-x)}{n},$$

and letting $H=B_n$ in Theorem 6, the classical Chebyshev-Grüss-type inequality looks as follows:

$$|B_n(f\cdot g)(x)-B_nf(x)\cdot B_ng(x)|\leq \frac{1}{4}\widetilde{\omega}\left(f;2\sqrt{\frac{x(1-x)}{n}}\right)\cdot\widetilde{\omega}\left(g;2\sqrt{\frac{x(1-x)}{n}}\right), \tag{11}$$

which implies

$$|B_n(f\cdot g)(x)-B_nf(x)\cdot B_ng(x)|\leq \frac{1}{4}\widetilde{\omega}\left(f;\frac{1}{\sqrt{n}}\right)\cdot\widetilde{\omega}\left(g;\frac{1}{\sqrt{n}}\right), \tag{12}$$

for two functions $f,g\in C[0,1]$ and $x\in[0,1]$ fixed.

Remark 8. In (5) and (11), the right-hand side depends on x and vanishes when $x\to 0$ or $x\to 1$. The maximum value of it, as a function of x, is attained for $x=\frac{1}{2}$, and (7), (10), (12) illustrate this fact. On the other hand, in (5) the oscillations of f and g are relative only to the points $0,\frac{1}{n},\ldots,\frac{n-1}{n},1$, while in (11) the oscillations, expressed in terms of $\widetilde{\omega}$, are relative to the whole interval $[0,1]$.

4 Grüss-type inequalities for the Lagrange operator

Consider $f\in C[-1,1]$ and the infinite matrix $X=\{x_{k,n}\}_{k=1}^{n}{}_{n=1}^{\infty}$ with

$$-1\leq x_{1,n}<x_{2,n}<\ldots<x_{n,n}\leq 1,\ \text{for } n=1,2,\ldots.$$

The Lagrange fundamental functions are given by

$$l_{k,n}(x)=\frac{\omega_n(x)}{\omega_n'(x_{k,n})(x-x_{k,n})},\ 1\leq k\leq n,$$

where $\omega_n(x)=\prod_{k=1}^{n}(x-x_{k,n})$ and the Lagrange operator (see [26]) $L_n:C[-1,1]\to\Pi_{n-1}$ is

$$L_n(f;x):=\sum_{k=1}^{n}f(x_{k,n})l_{k,n}(x).$$

The Lebesgue function of the interpolation is:

$$\Lambda_n(x) := \sum_{k=1}^{n} |l_{k,n}(x)|.$$

It is also known (see [8], p. 13) that $\|L_n\| < \infty$ and

$$\|L_n\| = \|\Lambda_n\|_\infty$$

hold.

Proposition 1 (Properties of the Lagrange operator)**.**

i) The Lagrange operator is linear but only in exceptional cases positive.

ii) $L_n(f; x_{k,n}) = f(x_{k,n}),\ 1 \le k \le n.$

iii) The Lagrange operator is idempotent: $L_n^2 = L_n$.

iv) L_n *satisfies* $\sum_{k=1}^{n} l_{k,n}(x) = 1$.

Remark 9. The Lebesgue function has been studied for different node systems. In the sequel, we will use some known results for Chebyshev nodes and give classical and new Chebyshev-Grüss-type inequalities.

4.1 A Chebyshev-Grüss-type inequality for the Lagrange operator at Chebyshev nodes

The Lagrange operator with Chebyshev nodes (see [5, 8]) is given as follows. Let $T_n(x) = \cos(n \cos^{-1} x)$ and $X = \{\cos[\pi(2k-1)/2n]\}$, i. e., when

$$x_{k,n} = \cos t_{k,n} = \cos \frac{2k-1}{2n} \cdot \pi \ (k = 1, 2, \ldots, n; n = 1, 2, \ldots)$$

are the Chebyshev roots.

Remark 10. It can be shown that the Lebesgue constant for Chebyshev nodes is a lot smaller than for equidistant nodes. That's why we concentrate on this case in our paper.

A Chebyshev-Grüss-type inequality for the Lagrange operator with this node system, similar to the one in Theorem 5, is given by:

Theorem 11. *For* $f, g \in C[-1, 1]$ *and all* $x \in [-1, 1]$, *the inequality*

$$\begin{aligned} |T(f, g; x)| &\le \frac{1}{4} \|L_n\| (1 + \|L_n\|) \widetilde{\omega}(f; 2) \cdot \widetilde{\omega}(g; 2) \\ &\le \frac{1}{2} \left(1 + \frac{3}{\pi} \log n + \frac{2}{\pi} \log^2 n\right) \omega(f; 2) \cdot \omega(g; 2) \end{aligned}$$

holds; here ω *denotes the first order modulus.*

Proof. The idea of this proof is similar to the one of Theorem 2 in [1] and that of Theorem 3.1. in [24]. Recall, however, that we have to work without the assumption of positivity. We consider the bilinear functional

$$T(f,g;x) := L_n(f \cdot g;x) - L_n(f;x) \cdot L_n(g;x).$$

Let $f, g \in C[-1,1]$ and $r, s \in Lip_1$, where $Lip_1 = \{f \in C[-1,1] : \sup_{x \neq x_0} \frac{|f(x)-f(x_0)|}{|x-x_0|} < \infty\}$ and the seminorm on Lip_1 is defined by $|f|_{Lip_1} := \sup_{x \neq x_0} \frac{|f(x)-f(x_0)|}{|x-x_0|}$. We are interested in estimating

$$\begin{aligned} |T(f,g;x)| &= |T(f-r+r, g-s+s;x)| \\ &\leq |T(f-r,g-s;x)| + |T(f-r,s;x)| + |T(r,g-s;x)| + |T(r,s;x)| \,. \end{aligned} \tag{13}$$

First note that for $f, g \in C[-1,1]$ one has

$$|T(f,g;x)| \leq \|L_n\| \left(1 + \|L_n\|\right) \|f\| \cdot \|g\| \,.$$

For $r, s \in Lip_1$ we have the estimate

$$\begin{aligned} |T(r,s;x)| &= |T((r-r(0)),(s-s(0);x)| \\ &= |L_n((r-r(0)) \cdot (s-s(0));x) - L_n(r-r(0);x) \cdot L_n(s-s(0);x)| \\ &\leq \|L_n\| \cdot \|r-r(0)\| \cdot \|s-s(0)\| + \|L_n\|^2 \cdot \|r-r(0)\| \cdot \|s-s(0)\| \\ &\leq \|L_n\| \left(1+\|L_n\|\right) \cdot |r|_{Lip_1} \cdot |s|_{Lip_1} \,. \end{aligned}$$

Moreover, for $r \in Lip_1$ and $g \in C[-1,1]$ the inequality

$$\begin{aligned} |T(r,g;x)| &= |T(r-r(0),g;x)| \\ &= |L_n((r-r(0)) \cdot g;x) - L_n(r-r(0);x) \cdot L_n(g;x)| \\ &\leq \|L_n\| \cdot \|(r-r(0)) \cdot g\| + \|L_n\|^2 \cdot \|r-r(0)\| \cdot \|g\| \\ &\leq \|L_n\| \left(1+\|L_n\|\right) \cdot \|g\| \cdot \|r-r(0)\| \\ &\leq \|L_n\| \left(1+\|L_n\|\right) \cdot \|g\| \cdot |r|_{Lip_1} \end{aligned}$$

holds. Note that in both cases considered so far we used

$$\begin{aligned} |r(x)-r(0)| &= \frac{|r(x)-r(0)|}{|x-0|} \cdot |x-0| \\ &\leq |r|_{Lip_1} \cdot |x| \,, \end{aligned}$$

for $x \in [-1,1]$, i.e.,

$$\|r(x)-r(0)\| \leq |r|_{Lip_1} \,.$$

Similarly, if $f \in C[-1,1]$ and $s \in Lip_1$ we have

$$|T(f,s;x)| \leq \|L_n\| \left(1+\|L_n\|\right) \cdot \|f\| \cdot |s|_{Lip_1} \,.$$

Then inequality (13) becomes

$$\begin{aligned}|T(f,g;x)| &\le |T(f-r,g-s;x)| + |T(f-r,s;x)| + |T(r,g-s;x)| + |T(r,s;x)|\\ &\le \|L_n\|\,(1+\|L_n\|)\cdot\left\{\|f-r\|+|r|_{Lip_1}\right\}\cdot\left\{\|g-s\|+|s|_{Lip_1}\right\}.\end{aligned}$$

The latter expression involves terms figuring in the K – functional

$$\begin{aligned}&K(f,t;C[-1,1],Lip_1)\\ &=\inf\{\|f-g\|+t\cdot|g|_{Lip_1} : g\in Lip_1\},\end{aligned}$$

for $f\in C[-1,1]$, $t\ge 0$. It is known that (see, e. g., [22])

$$K\left(f,\frac{t}{2}\right)=\frac{1}{2}\cdot\widetilde{\omega}(f;t),$$

an equality to be used in the next step.

We now pass to the infimum over r and s, respectively, which leads to

$$\begin{aligned}|T(f,g;x)| &\le \|L_n\|\,(1+\|L_n\|)\cdot K(f,1;C,Lip_1)\cdot K(g,1;C,Lip_1)\\ &=\|L_n\|\,(1+\|L_n\|)\cdot\frac{1}{2}\cdot\widetilde{\omega}(f;2)\cdot\frac{1}{2}\cdot\widetilde{\omega}(g;2)\\ &=\frac{1}{4}\,\|L_n\|\,(1+\|L_n\|)\omega(f;2)\cdot\omega(g;2).\end{aligned}$$

T. Rivlin (see [23]) proved the following inequality in the case of Lagrange interpolation at Chebyshev nodes:

$$0.9625<\|L_n\|-\frac{2}{\pi}\log n<1,$$

so using this result we get

$$\begin{aligned}&\|L_n\|<\frac{2}{\pi}\log n+1\\ &\Rightarrow 1+\|L_n\|<2\left(\frac{1}{\pi}\log n+1\right)\\ &\Rightarrow \|L_n\|\,(1+\|L_n\|)<2\left(1+\frac{3}{\pi}\log n+\frac{2}{\pi^2}\log^2 n\right)\end{aligned}$$

which implies the result. □

4.2 A new Chebyshev-Grüss-type inequality for the Lagrange operator with Chebyshev nodes

Theorem 12. *For $f,g\in C[-1,1]$ and $x\in[-1,1]$ fixed, the following inequality*

$$|T(f,g;x)|\le osc_{L_n}(f)\cdot osc_{L_n}(g)\cdot\sum_{1\le k<m\le n}|l_{k,n}(x)\cdot l_{m,n}(x)|$$

$$\leq osc_{L_n}(f) \cdot osc_{L_n}(g) \cdot \left\{ \frac{\Lambda_n^2(x) - c\left[1 + (\cos^2 nt) \cdot \frac{\pi^2}{6}\right]}{2} \right\}$$

holds, for a suitable constant c and $x = \cos t$.

Proof. The first inequality follows from Theorem 8 (with an obvious modification). The sum on the right-hand side of the first inequality can be expressed as follows:

$$\sum_{1 \leq k < m \leq n} |l_{k,n}(x) \cdot l_{m,n}(x)| = \left[\left(\sum_{i=1}^{n} |l_{i,n}(x)| \right)^2 - \left(\sum_{i=1}^{n} l_{i,n}^2(x) \right) \right] /2$$
$$= \left[\Lambda_n^2(x) - \left(\sum_{i=1}^{n} l_{i,n}^2(x) \right) \right] /2.$$

In order to estimate the sum $\sum_{i=1}^{n} l_{i,n}^2(x)$, we use the proof of Theorem 2.3. from [15] to get (case $\alpha = 2$):

$$\sum_{i=1}^{n} l_{i,n}^2(x) \geq c \left(1 + |\cos nt|^2 \sum_{i=1}^{n} i^{-2} \right),$$

where $x = \cos t$ and c is a suitable constant. After some calculation, the sum becomes

$$\sum_{1 \leq k < m \leq n} |l_{k,n}(x) \cdot l_{m,n}(x)| = \frac{\Lambda_n^2(x)}{2} - \frac{c\left(1 + (\cos nt)^2 \cdot \frac{\pi^2}{6}\right)}{2},$$

so we obtain our desired inequality. $\square$

5 Chebyshev-Grüss-type inequalities for piecewise linear interpolation at equidistant knots

We consider the operator $S_{\Delta_n} : C[0,1] \to C[0,1]$ (see [12]) at the points $0, \frac{1}{n}, \ldots, \frac{k}{n}, \ldots, \frac{n-1}{n}, 1$, which can be explicitely described as

$$S_{\Delta_n}(f;x) = \frac{1}{n} \sum_{k=0}^{n} \left[\frac{k-1}{n}, \frac{k}{n}, \frac{k+1}{n}; |\alpha - x| \right]_\alpha f\left(\frac{k}{n}\right),$$

where $[a,b,c;f] = [a,b,c;f(\alpha)]_\alpha$ denotes the divided difference of a function $f : D \to \mathbb{R}$ on the (distinct knots) $\{a,b,c\} \subset D$, w. r. t. α.

Proposition 2 (Properties of S_{Δ_n})**.**

i) S_{Δ_n} *is a positive, linear operator preserving linear functions.*

ii) S_{Δ_n} *preserves monotonicity and convexity/concavity.*

iii) $S_{\Delta_n}(f;0) = 0,\ S_{\Delta_n}(f;1) = f(1)$.

iv) If $f \in C[0,1]$ *is convex, then* $S_{\Delta_n} f$ *is also convex and we have:* $f \le S_{\Delta_n} f$.

The operator S_{Δ_n} can also be defined as follows:

$$S_{\Delta_n} f(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) u_{n,k}(x),$$

for $f \in C[0,1]$ and $x \in [0,1]$, where $u_{n,k} \in C[0,1]$ are piecewise linear continuous functions, such that

$$u_{n,k}\left(\frac{l}{n}\right) = \delta_{kl},\ k,l = 0,\dots,n.$$

5.1 A Chebyshev-Grüss-type inequality for S_{Δ_n}

In order to obtain a classical Chebyshev-Grüss-type inequality using S_{Δ_n}, we need the second moment of the operator. For $x \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$, this is given by

$$\begin{aligned} S_{\Delta_n}((e_1 - x)^2; x) &= n\left(x - \frac{k-1}{n}\right)\left(\frac{k}{n} - x\right)\left[\left(\frac{k}{n} - x\right) - \left(\frac{k-1}{n} - x\right)\right] \\ &= \left(x - \frac{k-1}{n}\right)\left(\frac{k}{n} - x\right), \end{aligned}$$

which is maximal when $x = \frac{2k-1}{2n}$. This implies

$$S_{\Delta_n}((e_1 - x)^2; x) \le \frac{1}{4n^2}.$$

By taking $H = S_{\Delta_n}$ in Theorem 6, the Chebyshev-Grüss-type inequality for S_{Δ_n} is given in the following.

Theorem 13. *If* $f, g \in C[0,1]$ *and* $x \in [0,1]$ *is fixed, then the inequality*

$$\begin{aligned} |T(f,g;x)| &\le \frac{1}{4}\tilde{\omega}\left(f; 2 \cdot \sqrt{S_{\Delta_n}((e_1 - x)^2; x)}\right) \cdot \tilde{\omega}\left(g; 2 \cdot \sqrt{S_{\Delta_n}((e_1 - x)^2; x)}\right) \\ &\le \frac{1}{4}\tilde{\omega}\left(f; \frac{1}{n}\right) \cdot \tilde{\omega}\left(g; \frac{1}{n}\right) \end{aligned}$$

holds.

5.2 A new Chebyshev-Grüss-type inequality for S_{Δ_n}

In this case, we need to find the minimum of the sum $\tau_n(x) := \sum_{k=0}^{n} u_{n,k}^2(x)$. For a particular interval $\left[\frac{k-1}{n}, \frac{k}{n}\right]$, we get that

$$\tau_n(x) := \sum_{k=0}^{n} u_{n,k}^2(x) = (nx - k + 1)^2 + (k - nx)^2,\ \text{for } k = 1,\dots,n.$$

For $k = 1$, we have $x \in \left[0, \frac{1}{n}\right]$ and $\tau_n(x) = (nx - 1)^2$, while for $k = n$, we get $x \in \left[\frac{n-1}{n}, 1\right]$ and $\tau_n(x) = (nx - n + 1)^2$. So $\tau_n(x) = (nx - k + 1)^2 + (k - nx)^2$ is minimal if and only if $x = \frac{2k-1}{2n}$ and the minimum value of $\tau_n(x)$ is $\frac{1}{2}$.

Theorem 14. *The new Chebyshev-Grüss-type inequality for S_{Δ_n} is*

$$\begin{aligned} |T(f, g; x)| &\le \frac{1}{2}\left(1 - \sum_{k=0}^{n} u_{n,k}^2(x)\right) \cdot osc_{S_{\Delta_n}}(f) \cdot osc_{S_{\Delta_n}}(g) \\ &\le \frac{1}{2}\left(1 - \frac{1}{2}\right) \cdot osc_{S_{\Delta_n}}(f) \cdot osc_{S_{\Delta_n}}(g) \\ &\le \frac{1}{4} \cdot osc_{S_{\Delta_n}}(f) \cdot osc_{S_{\Delta_n}}(g), \end{aligned}$$

with

$$\begin{aligned} osc_{S_{\Delta_n}}(f) &:= \max\left\{|f_k - f_l| : 0 \le k < l \le n\right\}, \\ osc_{S_{\Delta_n}}(g) &:= \max\left\{|g_k - g_l| : 0 \le k < l \le n\right\}, \end{aligned}$$

where $f_k := f\left(\frac{k}{n}\right)$.

Remark 11. This inequality implies the classical Chebyshev-Grüss-type inequality because $|f_k - f_l| \le M - m$ and $|g_k - g_l| \le P - p$, respectively. It is easy to give examples in which our approach gives strictly better inequalities.

6 Chebyshev-Grüss-type inequalities for Mirakjan-Favard-Szász operators

The Mirakjan-Favard-Szász operators (see [2]) were introduced by G. M. Mirakjan (see [19]) and studied by different authors, e. g., J. Favard and O. Szász (see [10] and [27]). The classical $n-$th Mirakjan-Favard-Szász operator M_n is defined by

$$M_n(f; x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \tag{14}$$

for $f \in E_2$, $x \in [0, \infty) \subset \mathbb{R}$ and $n \in \mathbb{N}$. E_2 is the Banach lattice

$$E_2 := \{f \in C([0, \infty)) : \frac{f(x)}{1 + x^2} \text{ is convergent as } x \to \infty\},$$

endowed with the norm

$$\|f\|_* := \sup_{x \ge 0} \frac{|f(x)|}{1 + x^2}.$$

The series on the right-hand side of (14) is absolutely convergent and E_2 is isomorphic to $C[0, 1]$; (see [2], Sect. 5.3.9).

6.1 A new Chebyshev-Grüss-type inequality for Mirakjan-Favard-Szász operators

This is our first application of Theorem 9 for operators defined for functions given on an infinite interval. We set

$$\sigma_n(x) := e^{-2nx} \sum_{k=0}^{\infty} \frac{(nx)^{2k}}{(k!)^2}$$

and we want to find the infimum:

$$\inf_{x \geq 0} \sigma_n(x) := \iota \geq 0.$$

Because $\sigma_n(x) \geq \iota$, we obtain the following result.

Theorem 15. *For the Mirakjan-Favard-Szász operator we have*

$$\begin{aligned} |T(f,g;x)| &\leq \frac{1}{2}\left(1 - \sigma_n(x)\right) \cdot osc_{M_n}(f) \cdot osc_{M_n}(g) \\ &\leq \frac{1}{2}(1-\iota) \cdot osc_{M_n}(f) \cdot osc_{M_n}(g), \end{aligned}$$

where $f, g \in C_b[0,\infty)$, $osc_{M_n}(f) = \sup\{|f_k - f_l| : 0 \leq k < l < \infty\}$, *with* $f_k := f\left(\frac{k}{n}\right)$ *and a similar definition applying to* g. $C_b[0,\infty)$ *is the set of all continuous, real-valued, bounded functions on* $[0,\infty)$.

Lemma 2. *The relation* $\inf_{x\geq 0} \sigma_n(x) = \iota = 0$ *holds.*

Proof. We first need to prove that

$$\lim_{x\to\infty} e^{-2nx} I_0(2nx) = 0$$

holds, for a fixed n and I_0 being the modified Bessel function of the first kind of order 0. The power series expansion for modified Bessel functions of the first kind of order 0 is

$$I_0(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^{2k}(k!)^2},$$

so for a fixed n we have

$$I_0(2nx) = \sum_{k=0}^{\infty} \frac{(nx)^{2k}}{(k!)^2}$$

and

$$e^{-2nx} \cdot I_0(2nx) = e^{-2nx} \cdot \sum_{k=0}^{\infty} \frac{(nx)^{2k}}{(k!)^2} = \varphi_n(x).$$

We now use Lebesgue's dominated convergence theorem and the integral expression

$$I_0(2nx) = \frac{1}{\pi} \int_{-1}^{1} e^{-2ntx} \cdot \frac{1}{\sqrt{1-t^2}} dt,$$

$$e^{-2nx} \cdot I_0(2nx) = \frac{1}{\pi} \int_{-1}^{1} e^{-2nx(1+t)} \cdot \frac{1}{\sqrt{1-t^2}} dt,$$

for n fixed and we conclude that $\sigma_n(x) \to 0$, as $x \to \infty$, because we see from above that $e^{-2nx} \cdot I_0(2nx) \to 0$, for $x \to \infty$. $\square$

Corollary 1. *The new Chebyshev-Grüss-type inequality for the Mirakjan-Favard-Szász operator is:*

$$|T(f,g;x)| \le \frac{1}{2} \cdot osc_{M_n}(f) \cdot osc_{M_n}(g),$$

where $f, g \in C_b[0,\infty)$, $osc_{M_n}(f) = \sup\{|f_k - f_l| : 0 \le k < l < \infty\}$ *and a similar definition applying to* g.

7 Chebyshev-Grüss-type inequalities for Baskakov operators

In the book of F. Altomare and M. Campiti [2] (Sect. 5.3.10), the classical positive, linear Baskakov operators $(A_n)_{n\in\mathbb{N}}$ are defined as follows:

$$A_n(f;x) := \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}},$$

for every $f \in E_2$, $x \in [0,\infty)$ and $n \ge 1$.

7.1 A new Chebyshev-Grüss-type inequality for Baskakov operators

The procedure in this subsection completely parallels that of Section 6.1. We set

$$\vartheta_n(x) := \frac{1}{(1+x)^{2n}} \sum_{k=0}^{\infty} \binom{n+k-1}{k}^2 \left(\frac{x}{1+x}\right)^{2k}, \text{ for } x \ge 0.$$

We need to find the infimum:

$$\inf_{x\ge 0} \vartheta_n(x) := \epsilon \ge 0.$$

Because $\vartheta_n(x) \ge \epsilon$, we obtain the following result.

Theorem 16. *For the Baskakov operator one has*

$$\begin{aligned} |T(f,g;x)| &\le \frac{1}{2}\left(1 - \vartheta_n(x)\right) \cdot osc_{A_n}(f) \cdot osc_{A_n}(g) \\ &\le \frac{1}{2}(1-\epsilon) \cdot osc_{A_n}(f) \cdot osc_{A_n}(g), \end{aligned}$$

where $f, g \in C_b[0,\infty)$, $osc_{A_n}(f) = \sup\{|f_k - f_l| : 0 \le k < l < \infty\}$, $f_k := f\left(\frac{k}{n}\right)$ *and a similar definition applying to* g.

Lemma 3. *The relation* $\inf_{x\geq 0}\vartheta_n(x)=\epsilon=0$ *holds for all* $n\geq 1$.

Proof. In [4] the following functions were defined. For $I_c=[0,\infty)(c\in\mathbb{R},c\geq 0)$, $n>0$, $k\in\mathbb{N}_0$ and $x\in I_c$, we have

$$p_{n,k}^{[c]}(x):=(-1)^k\binom{-\frac{n}{c}}{k}(cx)^k(1+cx)^{-\frac{n}{c}-k}, c\neq 0.$$

For $c=1$, we get

$$p_{n,k}^{[1]}(x)=p_{n,k}(x)=(-1)^k\binom{-n}{k}x^k(1+x)^{-n-k}=\binom{n+k-1}{k}x^k(1+x)^{-n-k}=:a_{n,k}(x),$$

so we obtain the fundamental functions of the Baskakov operator. The following kernel function was defined in [4]:

$$T_{n,c}(x,y)=\sum_{k=0}^{\infty}p_{n,k}^{[c]}(x)\cdot p_{n,k}^{[c]}(y),\ \text{for } x,y\in I_c.$$

We are interested in the case $c=1$ and $x=y$, so the above kernel becomes

$$T_{n,1}(x,x)=\sum_{k=0}^{\infty}p_{n,k}^2(x)=\sum_{k=0}^{\infty}a_{n,k}^2(x)=:\vartheta_n(x). \tag{15}$$

For $n=1$, we get

$$\vartheta_1(x)=T_{1,1}(x,x)=\frac{1}{(1+x)^2}\sum_{k=0}^{\infty}\left(\frac{x}{1+x}\right)^{2k}=\frac{1}{1+2x}\longrightarrow 0,\ \text{for } x\to\infty.$$

For $n>1$,

$$T_{n,1}(x,x)=\frac{1}{\pi}\int_0^1(\phi(x,x,t))^n\frac{dt}{\sqrt{t(1-t)}},$$

where, for $\phi(x,x,t)=[1+4x(1-t)+4x^2(1-t)]^{-1}$, it holds:

$$0<\phi(x,x,t)\leq 1,\forall t\in[0,1],\forall x\geq 0.$$

Therefore

$$T_{2,1}(x,x)\geq T_{3,1}(x,x)\geq T_{4,1}(x,x)\geq\ldots\geq 0,\forall x\geq 0. \tag{16}$$

Now for $n=2$, we have

$$T_{2,1}(x,x)=\sum_{k=0}^{\infty}p_{2,k}^2(x)=\frac{1}{(1+x)^4}\sum_{k=0}^{\infty}(k+1)^2\left(\frac{x}{1+x}\right)^{2k}.$$

Let $\left(\frac{x}{1+x}\right)^2=y$. Then

$$\sum_{k=0}^{\infty}(k+1)^2y^k=\frac{1+y}{(1-y)^3}.$$

Thus

$$T_{2,1}(x,x) = \frac{2x^2+2x+1}{(2x+1)^3} \to 0, \text{ for } x \to \infty. \tag{17}$$

For $n \geq 3$ it holds from (16) that $0 \leq T_{n,1}(x,x) \leq T_{2,1}(x,x)$. Combining this with (17), we get

$$\lim_{x\to 0} T_{n,1}(x,x) = 0, \forall n \geq 1,$$

and so the proof is finished. □

An inequality analogous to the one in Corollary 1 is now immediate.

8 Chebyshev-Grüss-type inequalities for Bleimann-Butzer-Hahn operators

In the same book [2] (Sect. 5.2.8), the Bleimann-Butzer-Hahn operators are also presented. For every $n \in \mathbb{N}$ the positive linear operator $H_n : C_b([0,\infty)) \to C_b([0,\infty))$ is defined by

$$H_n(f;x) := \frac{1}{(1+x)^n} \sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right)\binom{n}{k} x^k,$$

for every $f \in C_b([0,\infty))$, $x \geq 0$, $n \in \mathbb{N}$, $n \geq 1$.

8.1 A new Chebyshev-Grüss-type inequality for Bleimann-Butzer-Hahn operators

We set

$$\psi_n(t) = \frac{1}{(1+t)^{2n}} \sum_{k=0}^{n} \binom{n}{k}^2 t^{2k},$$

for $t \geq 0$. We make a change of variable, namely set $x = \frac{t}{t+1} \in [0,1)$. Then we get

$$\begin{aligned}\psi_n(t) &= \sum_{k=0}^{n} \binom{n}{k}^2 \left(\frac{t}{t+1}\right)^{2k} \left(\frac{1}{t+1}\right)^{2n-2k} \\ &= \sum_{k=0}^{n} \binom{n}{k}^2 x^{2k}(1-x)^{2n-2k}.\end{aligned}$$

So $\psi_n(t) = \varphi_n(x)$, i.e., $\inf_{t\geq 0} \psi_n(t) = \inf_{x\in[0,1]} \varphi_n(x) = \frac{1}{4^n}\binom{2n}{n}$, as shown in Lemma 1.

This leads to

Theorem 17. *The new Chebyshev-Grüss-type inequality in this case is:*

$$|T(f,g;x)| \leq \frac{1}{2}\left(1 - \frac{1}{4^n}\binom{2n}{n}\right) \cdot osc_{H_n}(f) \cdot osc_{H_n}(g), \tag{18}$$

with $f, g \in C_b[0,\infty)$, $x \in [0,\infty)$ *and*

$$osc_{H_n}(f) := \sup\left\{|f_k - f_l| : 0 \leq k < l \leq n\right\},$$

for $f_k := f\left(\frac{k}{n-k+1}\right)$ *and a similar definition applying to* g.

9 Chebyshev-Grüss-type inequalities for King-type operators

P. P. Korovkin [17] introduced in 1960 a theorem saying that if $\{L_n\}$ is a sequence of positive linear operators on $C[a,b]$, then

$$\lim_{n\to\infty} L_n(f)(x) = f(x)$$

for each $f \in C[a,b]$ if and only if

$$\lim_{n\to\infty} L_n(e_i(x)) = e_i(x)$$

for the three functions $e_i(x) = x^i$, $i = 0,1,2$. There are a lot of well-known operators, like the Bernstein polynomials, the Mirakjan-Favard-Szász and the Baskakov operators, that preserve e_0 and e_1 (see [16]). However, these operators do not reproduce e_2. We are now interested in a non-trivial sequence of positive linear operators $\{L_n\}$ defined on $C[0,1]$, that preserve e_0 and e_2:

$$L_n(e_0)(x) = e_0(x) \text{ and } L_n(e_2)(x) = e_2(x),\ n = 0,1,2,\ldots.$$

In [16] J. P. King defined the King-type operator as follows.

Definition 2. (see [16]) Let $\{r_n(x)\}$ be a sequence of continuous functions with $0 \leq r_n(x) \leq 1$. Let $V_n : C[0,1] \to C[0,1]$ be defined by

$$\begin{aligned} V_n(f;x) &= \sum_{k=0}^{n} \binom{n}{k} (r_n(x))^k (1-r_n(x))^{n-k} f\left(\frac{k}{n}\right) \\ &= \sum_{k=0}^{n} v_{n,k}(x) \cdot f\left(\frac{k}{n}\right), \end{aligned}$$

for $f \in C[0,1]$, $0 \leq x \leq 1$. $v_{n,k}$ are the fundamental functions of the V_n operator.

Remark 12. For $r_n(x) = x$, $n = 1,2,\ldots$, the positive linear operators V_n given above reduce to the Bernstein operators.

Proposition 3 (Properties of V_n)**.**

i) $V_n(e_0) = 1$ *and* $V_n(e_1;x) = r_n(x)$*;*

ii) $V_n(e_2;x) = \frac{r_n(x)}{n} + \frac{n-1}{n}(r_n(x))^2$*;*

iii) $\lim_{n\to\infty} V_n(f;x) = f(x)$ *for each* $f \in C[0,1]$, $x \in [0,1]$, *if and only if*

$$\lim_{n\to\infty} r_n(x) = x.$$

For special ("right") choices of $r_n(x) = r_n^*(x)$, J. P. King showed in [16] that the following theorem holds.

Theorem 18. *(see Theorem 1.3. in [13]) Let* $\{V_n^*\}_{n\in\mathbb{N}}$ *be the sequence of operators defined before with*

$$r_n^*(x) := \begin{cases} r_1^*(x) = x^2 & \text{, for } n = 1, \\ r_n^*(x) = -\frac{1}{2(n-1)} + \sqrt{\left(\frac{n}{n-1}\right)x^2 + \frac{1}{4(n-1)^2}} & \text{, for } n = 2, 3, \ldots. \end{cases}$$

Then we get $V_n^*(e_2;x) = x^2$, *for* $n \in \mathbb{N}$, $x \in [0,1]$ *and* $V_n^*(e_1;x) \neq e_1(x)$. V_n^* *is not a polynomial operator.*

The fundamental functions of this operator, namely

$$v_{n,k}^*(x) = \binom{n}{k}(r_n^*(x))^k(1 - r_n^*(x))^{n-k}$$

satisfy $\sum_{k=0}^{n} v_{n,k}^*(x) = 1$, for $n = 1, 2, \ldots$.

Proposition 4 (Properties of r_n^*)**.**

i) $0 \leq r_n^*(x) \leq 1$, *for* $n = 1, 2, \ldots$, *and* $0 \leq x \leq 1$.

ii) $\lim_{n\to\infty} r_n^*(x) = x$ *for* $0 \leq x \leq 1$.

9.1 The classical Chebyshev-Grüss-type inequality for King-type operators

The second moments of the special King-type operators V_n^* are given by

$$V_n^*((e_1 - x)^2; x) = 2x(x - r_n^*(x)),$$

so we discriminate between two cases.

The first case is $n = 1$, so $r_n^*(x) = x^2$ and the second moment is

$$V_1^*((e_1 - x)^2; x) = 2x^2(1 - x),$$

so the classical Chebyshev-Grüss-type inequality is given as follows.

Theorem 19. *For* $L = V_1^*$, *we have the inequality:*

$$|T(f,g;x)| \leq \frac{1}{4}\widetilde{\omega}\left(f; 2x\sqrt{2(1-x)}\right) \cdot \widetilde{\omega}\left(g; 2x\sqrt{2(1-x)}\right).$$

For the second case, $n = 2, 3, \ldots$, we have

$$r_n^*(x) = -\frac{1}{2(n-1)} + \sqrt{\left(\frac{n}{n-1}\right)x^2 + \frac{1}{4(n-1)^2}},$$

so the second moments look more complicated:

$$V_n^*((e_1 - x)^2; x) = 2x(x - r_n^*(x)).$$

In this case we get the following:

Theorem 20. *For $L = V_n^*(x)$ and $n = 2, 3, \ldots$, the inequality*

$$|T(f, g; x)| \leq \frac{1}{4}\widetilde{\omega}\left(f; 2\sqrt{2x(x - r_n^*(x))}\right) \cdot \widetilde{\omega}\left(g; 2\sqrt{2x(x - r_n^*(x)}\right)$$

holds.

9.2 A new Chebyshev-Grüss-type inequality for King-type operators

We need $\sum_{k=0}^{n}(v_{n,k}^*(x))^2$ to be minimal. Let $\varphi_n(x) := \sum_{k=0}^{n}(v_{n,k}^*(x))^2$.
For $n = 1$, we have that

$$\varphi_1(x) = \sum_{k=0}^{1}(v_{1,k}^*(x))^2 = (v_{1,0}^*(x))^2 + (v_{1,1}^*(x))^2 = 2x^4 - 2x^2 + 1$$

and this attains its minimum for $x = \frac{\sqrt{2}}{2}$. This minimum is

$$\varphi_1\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{2}.$$

Theorem 21. *The new Chebyshev-Grüss-type inequality for $n = 1$ then looks as follows:*

$$\begin{aligned} |T(f, g; x)| &\leq \frac{1}{4} \cdot osc_{V_1^*}(f) \cdot osc_{V_1^*}(g) \\ &= \frac{1}{4} \cdot |f_0 - f_1| \cdot |g_0 - g_1|. \end{aligned}$$

For $n = 2, 3, \ldots$, the problem of finding the minimum is more difficult, since

$$\begin{aligned} \varphi_n(x) &= \sum_{k=0}^{n}(v_{n,k}^*(x))^2 \\ &= \sum_{k=0}^{n}\binom{n}{k}^2 (r_n^*(x))^{2k}(1 - r_n^*(x))^{2(n-k)}. \end{aligned}$$

In any case, the estimate

$$\varphi_n(x) = \sum_{k=0}^{n} (v^*_{n,k}(x))^2 \geq \frac{1}{n+1}$$

holds, for $x \in [0,1]$ and $n = 2, 3, \ldots$. As a proof for this,

$$\sqrt{\frac{\sum_{k=0}^{n} v^*_{n,k}(x))^2}{n+1}} \geq \frac{\sum_{k=0}^{n} v^*_{n,k}(x)}{n+1} = \frac{1}{n+1}.$$

Then we get

$$1 - \sum_{k=0}^{n} (v^*_{n,k}(x))^2 \leq 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

Theorem 22. *For $n = 2, 3, \ldots$ there holds*

$$|V^*_n(fg)(x) - V^*_n(f;x) \cdot V^*_n(g;x)| \leq \frac{n}{2(n+1)} \cdot osc_{V^*_n}(f) \cdot osc_{V^*_n}(g).$$

Note added in proof: Regarding the conjectures from Section 3, Dr. Th. Neuschel (Katholieke Universiteit Leuven) also validated Conjecture 2 (see paper of G. Nikolov [21] for more details). Conjecture 1 was discussed and proved in recent papers by I. Gavrea and M. Ivan in [11], and by G. Nikolov in [21], independently. Conjecture 3 is the weakest of the three, but sufficient for our purposes.

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Closure operators in the categories of modules. Part III (Operations in $\mathbb{CO}$ and their properties)

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Abstract. This article is a continuation of the works [1] and [2] (Part I and Part II) and contains some results on the family $\mathbb{CO}$ of all closure operators of a module category R-Mod. The principal operations in $\mathbb{CO}$ (meet, join, product, coproduct) are studied and their properties are elucidated. Also the question on the preservation of types of closure operators with respect to these operations is investigated.

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1 Introduction. Preliminary notions

Continuing the investigation of closure operators of a module category [1, 2], in this part of the work the principal operations are analyzed, which are defined in the family of all closure operators $\mathbb{CO}$ of a module category R-Mod: the meet, join, product and coproduct [1–5]. The properties of these operations will be studied, as well as the relations between them. Moreover, the types of operators are indicated (weakly hereditary, idempotent, hereditary, maximal, minimal, cohereditary) which are preserved by the application of these operations.

The main definitions and some preliminary results can be found in [1–6]. For convenience we would remind some necessary definitions and facts.

Let R be a ring with unit and R-Mod be the category of unitary left R-modules. For the module $M \in R$-Mod we denote by $\mathbb{L}(M)$ the lattice of all submodules of M. *A closure operator* in R-Mod is a function C which associates to every pair $N \subseteq M$, where $N \in \mathbb{L}(M)$, a submodule of M denoted by $C_M(N)$ with the conditions: (c_1) $N \subseteq C_M(N)$; (c_2) if $N, P \in \mathbb{L}(M)$ and $N \subseteq P$, then $C_M(N) \subseteq C_M(P)$ (monotony); (c_3) if $f : M \to M'$ is an R-morphism and $N \subseteq M$, then $f\big(C_M(N)\big) \subseteq C_{M'}\big(f(N)\big)$ (continuity). We denote by $\mathbb{CO}$ the family of all closure operators of a category R-Mod.

The *principal operations* in $\mathbb{CO}$ are defined as follows, where $N \in \mathbb{L}(M)$:

1. *The meet* $\bigwedge\limits_{\alpha \in \mathfrak{A}} C_\alpha$ of a family $\{C_\alpha \in \mathbb{CO} \mid \alpha \in \mathfrak{A}\}$:

$$\big(\bigwedge_{\alpha \in \mathfrak{A}} C_\alpha\big)_M(N) = \bigcap_{\alpha \in \mathfrak{A}} \big[(C_\alpha)_M(N)\big]; \tag{1.1}$$

2. *The join* $\bigvee\limits_{\alpha\in\mathfrak{A}} C_\alpha$ of a family $\{C_\alpha \in \mathbb{CO} \mid \alpha \in \mathfrak{A}\}$:

$$\Big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\Big)_M(N) = \sum_{\alpha\in\mathfrak{A}} \big[(C_\alpha)_M(N)\big]; \tag{1.2}$$

3. *The product* $C \cdot D$ of two closure operators $C, D \in \mathbb{CO}$:

$$\big(C \cdot D\big)_M(N) = C_M\big(D_M(N)\big); \tag{1.3}$$

4. *The coproduct* $C \# D$ of two closure operators $C, D \in \mathbb{CO}$:

$$\big(C \# D\big)_M(N) = C_{D_M(N)}(N). \tag{1.4}$$

It is easy to observe that by the rules (1.1)–(1.4) we obtain the closure operators and the family $\mathbb{CO}$ of all closure operators of R-Mod is a complete "big lattice" with respect to the meet and join (which will be named the lattice operations). As to the other two operations we can remark that they are associative and for every $C, D \in \mathbb{CO}$ we have: $C \cdot D \geq C \vee D$, $C \# D \leq C \wedge D$ [3].

We remind in continuation the most important *types of closure operators* [1–3]. The operator $C \in \mathbb{CO}$ is called:

1) *weakly hereditary* if for every $N \subseteq M$ is true the relation:

$$C_{C_M(N)}(N) = C_M(N); \tag{1.5}$$

2) *idempotent* if for every $N \subseteq M$ we have:

$$C_M\big(C_M(N)\big) = C_M(N); \tag{1.6}$$

3) *hereditary* if for every submodules $L \subseteq N \subseteq M$ the relation holds:

$$C_N(L) = C_M(L) \cap N; \tag{1.7}$$

4) *cohereditary* if for every submodules $K, N \in \mathbb{L}(M)$ we have:

$$\big(C_M(N) + K\big)/K = C_{M/K}\big((N + K)/K\big); \tag{1.8}$$

5) *maximal* if for every $N \subseteq M$ is true the relation:

$$C_M(N)/N = C_{M/N}(\bar{0}); \tag{1.9}$$

or: for every submodules $K \subseteq N \subseteq M$ we have:

$$C_M(N)/K = C_{M/K}(N/K); \tag{1.9$'$}$$

6) *minimal* if for every $N \subseteq M$ is true the relation:

$$C_M(N) = C_M(0) + N; \tag{1.10}$$

or: for every submodules $L \subseteq N \subseteq M$ we have:

$$C_M(N) = C_M(L) + N. \tag{1.10$'$}$$

We remark the following known facts:

a) every hereditary closure operator is weakly hereditary;

b) every cohereditary closure operator is idempotent;

c) the operator $C \in \mathbb{CO}$ is cohereditary if and only if it is maximal and minimal ([2], Lemma 6.2).

2 Operations in $\mathbb{CO}$: distributivity

In this section we will study the interaction between the lattice operations $(\wedge)$, $(\vee)$ of $\mathbb{CO}$ and the operations $(\cdot)$, $(\#)$ of product and coproduct. We begin with the following *relations of distributivity.*

Proposition 2.1. *For every family of closure operators* $\{C_\alpha \in \mathbb{CO} \mid \alpha \in \mathfrak{A}\}$ *and for every operator* $D \in \mathbb{CO}$ *the following relations hold:*

$$\Big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\Big) \cdot D = \bigwedge_{\alpha\in\mathfrak{A}} (C_\alpha \cdot D); \tag{2.1}$$

$$\Big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\Big) \cdot D = \bigvee_{\alpha\in\mathfrak{A}} (C_\alpha \cdot D); \tag{2.2}$$

$$\Big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\Big) \# D = \bigwedge_{\alpha\in\mathfrak{A}} (C_\alpha \# D); \tag{2.3}$$

$$\Big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\Big) \# D = \bigvee_{\alpha\in\mathfrak{A}} (C_\alpha \# D). \tag{2.4}$$

Proof. (2.1). For every $N \subseteq M$ from the definitions of operations it follows:

$$\Big[\Big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\Big) \cdot D\Big]_M(N) = \Big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\Big)_M\big(D_M(N)\big) = \bigcap_{\alpha\in\mathfrak{A}} \big[(C_\alpha)_M\big(D_M(N)\big)\big] =$$

$$= \bigcap_{\alpha\in\mathfrak{A}} \big[(C_\alpha \cdot D)_M(N)\big] = \Big[\bigwedge_{\alpha\in\mathfrak{A}} (C_\alpha \cdot D)\Big]_M(N),$$

therefore the relation (2.1) is true.

(2.2). By definition for every $N \subseteq M$ we have:

$$\Big[\Big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\Big) \cdot D\Big]_M(N) = \Big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\Big)_M\big(D_M(N)\big) = \sum_{\alpha\in\mathfrak{A}} \big[(C_\alpha)_M\big(D_M(N)\big)\big] =$$

$$= \sum_{\alpha\in\mathfrak{A}} \big[(C_\alpha \cdot D)_M(N)\big] = \Big[\bigvee_{\alpha\in\mathfrak{A}} (C_\alpha \cdot D)\Big]_M(N),$$

hence the relation (2.2) holds.

(2.3). For $N \subseteq M$ by definition we obtain:

$$\Big[\Big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\Big) \# D\Big]_M(N) = \Big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\Big)_{D_M(N)}(N) = \bigcap_{\alpha\in\mathfrak{A}} \big[(C_\alpha)_{D_M(N)}(N)\big] =$$

$$= \bigcap_{\alpha\in\mathfrak{A}} \big[(C_\alpha \# D)_M(N)\big] = \Big[\bigwedge_{\alpha\in\mathfrak{A}} (C_\alpha \# D)\Big]_M(N),$$

which proves (2.3).

(2.4). Similarly, for every $N \subseteq M$ we have:

$$\Big[\big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\big) \# D\Big]_M(N) = \big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\big)_{D_M(N)}(N) = \sum_{\alpha\in\mathfrak{A}} \big[(C_\alpha)_{D_M(N)}(N)\big] =$$

$$= \sum_{\alpha\in\mathfrak{A}} \big[(C_\alpha \# D)_M(N)\big] = \Big[\bigvee_{\alpha\in\mathfrak{A}} (C_\alpha \# D)\Big]_M(N),$$

hence (2.4) is true. □

The other relations of distributivity of the indicated types can be obtained by some *supplementary conditions* on the operators. To concretize this idea we need the following two auxiliary statements.

Lemma 2.2. *If* $C \in \mathbb{CO}$ *is a* ***hereditary*** *closure operator, then it preserves the intersection in the superior term, i.e. for every family of submodules* $\{N_\alpha \in \mathbb{L}(M) \,|\, \alpha \in \mathfrak{A}\}$ *and every submodule* $K \subseteq N_\alpha$ $(\alpha \in \mathfrak{A})$ *the following relation holds:*

$$C_{\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha}(K) = \bigcap_{\alpha\in\mathfrak{A}} \big[C_{N_\alpha}(K)\big]. \tag{2.5}$$

Proof. From the heredity of $C \in \mathbb{CO}$ (see (1.7)) in the situation $K \subseteq N_\alpha \subseteq M$ we obtain $C_{N_\alpha}(K) = C_M(K) \cap N_\alpha$ for every $\alpha \in \mathfrak{A}$, therefore $\bigcap\limits_{\alpha\in\mathfrak{A}} \big[C_{N_\alpha}(K)\big] = C_M(K) \cap \big(\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha\big)$.

On the other hand, by the hereditary of C in the situation $K \subseteq \bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha \subseteq M$ we have $C_{\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha}(K) = C_M(K) \cap \big(\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha\big)$ and comparing with the previous relation we obtain (2.5). □

Lemma 2.3. *If* $C \in \mathbb{CO}$ *is a* ***minimal*** *closure operator, then it preserves the sum in the inferior term, i.e. for every family of submodules* $\{N_\alpha \in \mathbb{L}(M) \mid \alpha \in \mathfrak{A}\}$ *the relation is true:*

$$C_M\Big(\sum_{\alpha\in\mathfrak{A}} N_\alpha\Big) = \sum_{\alpha\in\mathfrak{A}} \big[C_M(N_\alpha)\big]. \tag{2.6}$$

Proof. Let $L \subseteq N_\alpha \subseteq M$. From the minimality of C (see (1.10′)) it follows that

$$\sum_{\alpha\in\mathfrak{A}} \big[C_M(N_\alpha)\big] = \sum_{\alpha\in\mathfrak{A}} \big[C_M(L) + N_\alpha\big] = C_M(L) + \big(\sum_{\alpha\in\mathfrak{A}} N_\alpha\big).$$

By the minimality of C in the situation $L \subseteq \sum\limits_{\alpha\in\mathfrak{A}} N_\alpha \subseteq M$ we have

$$C_M\big(\sum_{\alpha\in\mathfrak{A}} N_\alpha\big) = C_M(L) + \big(\sum_{\alpha\in\mathfrak{A}} N_\alpha\big),$$

hence (2.6) is true. □

Using the Lemmas 2.2 and 2.3 we obtain the following relations of distributivity.

Proposition 2.4. a) *If the closure operator $C \in \mathbb{CO}$ is **hereditary**, then for every family of closure operators $\{D_\alpha \in \mathbb{CO} \mid \alpha \in \mathfrak{A}\}$ the following relation holds:*

$$C \# \big(\bigwedge_{\alpha\in\mathfrak{A}} D_\alpha\big) = \bigwedge_{\alpha\in\mathfrak{A}} (C \# D_\alpha). \tag{2.7}$$

b) *If the closure operator $C \in \mathbb{CO}$ is **minimal**, then for every family of closure operators $\{D_\alpha \in \mathbb{CO} \mid \alpha \in \mathfrak{A}\}$ the relation is true:*

$$C \cdot \big(\bigvee_{\alpha\in\mathfrak{A}} D_\alpha\big) = \bigvee_{\alpha\in\mathfrak{A}} (C \cdot D_\alpha). \tag{2.8}$$

Proof. a) For every $N \subseteq M$ from the definitions it follows that:

$$\big[C \# \big(\bigwedge_{\alpha\in\mathfrak{A}} D_\alpha\big)\big]_M(N) = C_{(\bigwedge_{\alpha\in\mathfrak{A}} D_\alpha)_M(N)}(N) = C_{\bigcap_{\alpha\in\mathfrak{A}}[(D_\alpha)_M(N)]}(N);$$

$$\big[\bigwedge_{\alpha\in\mathfrak{A}} \big(C \# D_\alpha\big)\big]_M(N) = \bigcap_{\alpha\in\mathfrak{A}} [(C \# D_\alpha)_M(N)] = \bigcap_{\alpha\in\mathfrak{A}} [C_{(D_\alpha)_M(N)}(N)].$$

By assumption the operator C is hereditary, therefore it preserves the intersection in superior term (Lemma 2.2). The application of (2.5) in our case shows that the right sides of the previous relations coincide, therefore (2.7) is true.

b) For every $N \subseteq M$ we have:

$$\big[C \cdot \big(\bigvee_{\alpha\in\mathfrak{A}} D_\alpha\big)\big]_M(N) = C_M\big[\big(\bigvee_{\alpha\in\mathfrak{A}} D_\alpha\big)_M(N)\big] = C_M\big[\sum_{\alpha\in\mathfrak{A}} \big((D_\alpha)_M(N)\big)\big];$$

$$\big[\bigvee_{\alpha\in\mathfrak{A}} (C \cdot D_\alpha)\big]_M(N) = \sum_{\alpha\in\mathfrak{A}} \big[\big(C \cdot D_\alpha\big)_M(N)\big] = \sum_{\alpha\in\mathfrak{A}} \big[C_M\big((D_\alpha)_M(N)\big)\big].$$

The operator C is minimal, hence it preserves the sum in the inferior term (Lemma 2.3). By the relation (2.6) we obtain that the right sides of the previous equalities coincide. This proves (2.8). $\square$

To give a complete picture we can mention also the last two possible cases of distributivity of considered operations, which are obtained by some supplementary assumptions on the closure operators.

Proposition 2.5. a) *If the closure operator $C \in \mathbb{CO}$ preserves the intersection in the inferior term, i.e.*

$$C_M\big(\bigcap_{\alpha\in\mathfrak{A}} N_\alpha\big) = \bigcap_{\alpha\in\mathfrak{A}} \big[C_M(N_\alpha)\big], \tag{2.9}$$

where $\{N_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq \mathbb{L}(M)$, then for every family of closure operators $\{D_\alpha \in \mathbb{CO} \mid \alpha \in \mathfrak{A}\}$ the relation holds:

$$C \cdot \big(\bigwedge_{\alpha\in\mathfrak{A}} D_\alpha\big) = \bigwedge_{\alpha\in\mathfrak{A}} \big(C \cdot D_\alpha\big). \tag{2.10}$$

b) *If the closure operator* $C \in \mathbb{CO}$ *preserves the sum in the superior term, i.e.*

$$C_{\sum\limits_{\alpha\in\mathfrak{A}}(N_\alpha)}(N) = \sum_{\alpha\in\mathfrak{A}} \big[C_{N_\alpha}(N)\big], \tag{2.11}$$

where $\{N_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq \mathbb{L}(M)$, $N \subseteq N_\alpha$ $(\alpha \in \mathfrak{A})$, *then for every family of closure operators* $\{D_\alpha \in \mathbb{CO} \mid \alpha \in \mathfrak{A}\}$ *the relation is true:*

$$C \;\#\; \big(\bigvee_{\alpha\in\mathfrak{A}} D_\alpha\big) = \bigvee_{\alpha\in\mathfrak{A}} \big(C \;\#\; D_\alpha\big). \tag{2.12}$$

Proof. a) For every $N \subseteq M$ we have:

$$\big[C \cdot \big(\bigwedge_{\alpha\in\mathfrak{A}} D_\alpha\big)\big]_M(N) = C_M\big[\big(\bigwedge_{\alpha\in\mathfrak{A}} D_\alpha\big)_M(N)\big] = C_M\big[\bigcap_{\alpha\in\mathfrak{A}} \big((D_\alpha)_M(N)\big)\big];$$

$$\big[\bigwedge_{\alpha\in\mathfrak{A}} (C \cdot D_\alpha)\big]_M(N) = \bigcap_{\alpha\in\mathfrak{A}} [(C \cdot D_\alpha)_M(N)] = \bigcap_{\alpha\in\mathfrak{A}} \big[C_M\big((D_\alpha)_M(N)\big)\big].$$

By assumption the operator C preserves the intersection in the inferior term, and so applying the relation (2.9) we see that the right sides of the previous equalities coincide, therefore (2.10) is true.

b) Similarly, for every $N \subseteq M$ we have:

$$\big[C \,\#\, \big(\bigvee_{\alpha\in\mathfrak{A}} D_\alpha\big)\big]_M(N) = C_{(\bigvee\limits_{\alpha\in\mathfrak{A}} D_\alpha)_M(N)}(N) = C_{\sum\limits_{\alpha\in\mathfrak{A}}[(D_\alpha)_M(N)]}(N);$$

$$\big[\bigvee_{\alpha\in\mathfrak{A}} (C \;\#\; D_\alpha)\big]_M(N) = \sum_{\alpha\in\mathfrak{A}} [(C \;\#\; D_\alpha)_M(N)] = \sum_{\alpha\in\mathfrak{A}} \big[C_{(D_\alpha)_M(N)}(N)\big].$$

By hypothesis C preserves the sum in the superior term, and applying (2.11) now we obtain (2.12). □

3 Principal operations and preservation of types of operators

Now we will study the question on the behaviour of closure operators when the principal operations are applied. For that we consider consecutively all principal operations of $\mathbb{CO}$ and show the types of closure operators which are preserved by the application of given operation. Some similar facts are mentioned in [3].

a) *The join in* $\mathbb{CO}$

Proposition 3.1. *If the closure operators* C_α $(\alpha \in \mathfrak{A})$ *of* $\mathbb{CO}$ *are* ***weakly hereditary****, then the operator* $\bigvee\limits_{\alpha\in\mathfrak{A}} C_\alpha$ *also is weakly hereditary.*

Proof. By the monotony and weak heredity of C_α $\big($see (1.5)$\big)$, for every $N \subseteq M$ and $\alpha \in \mathfrak{A}$ we have:

$$(C_\alpha)_{\sum\limits_{\alpha\in\mathfrak{A}}[(C_\alpha)_M(N)]}(N) \supseteq (C_\alpha)_{(C_\alpha)_M(N)}(N) = (C_\alpha)_M(N),$$

and from the relation $\sum\limits_{\alpha\in\mathfrak{A}}[(C_\alpha)_M(N)] \subseteq M$ the inverse inclusion follows. Therefore

$$(C_\alpha)_{\sum\limits_{\alpha\in\mathfrak{A}}[(C_\alpha)_M(N)]}(N) = (C_\alpha)_M(N)$$

for every $\alpha \in \mathfrak{A}$, consequently

$$\sum_{\alpha\in\mathfrak{A}}[(C_\alpha)_{\sum\limits_{\alpha\in\mathfrak{A}}[(C_\alpha)_M(N)]}(N)] = \sum_{\alpha\in\mathfrak{A}}[(C_\alpha)_M(N)].$$

From the definition of join in $\mathbb{CO}$ now we have:

$$\big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\big)_{(\bigvee\limits_{\alpha\in\mathfrak{A}} C_\alpha)_M(N)}(N) = \big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\big)_M(N),$$

i.e. the operator $\bigvee\limits_{\alpha\in\mathfrak{A}} C_\alpha$ is weakly hereditary. $\square$

Proposition 3.2. *If the closure operators C_α $(\alpha \in \mathfrak{A})$ of $\mathbb{CO}$ are **maximal**, then the operator $\bigvee\limits_{\alpha\in\mathfrak{A}} C_\alpha$ is also maximal.*

Proof. By definition $\big($see (1.9$'$)$\big)$ for every submodules $K \subseteq N \subseteq M$ and every $\alpha \in \mathfrak{A}$ we have $[(C_\alpha)_M(N)]\,/\,K = (C_\alpha)_{M/K}(N/\,K)$. Using this relation we obtain:

$$\big[\big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\big)_M(N)\big]/\,K = \big[\sum_{\alpha\in\mathfrak{A}}\big((C_\alpha)_M(N)\big)\big]/\,K = \sum_{\alpha\in\mathfrak{A}}\big[\big((C_\alpha)_M(N)\big)/\,K\big] =$$

$$= \sum_{\alpha\in\mathfrak{A}}\big[(C_\alpha)_{M/K}(N/\,K)\big] = \big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\big)_{M/K}(N/\,K),$$

which means that the operator $\bigvee\limits_{\alpha\in\mathfrak{A}} C_\alpha$ is maximal. $\square$

Proposition 3.3. *If the closure operators C_α $(\alpha \in \mathfrak{A})$ of $\mathbb{CO}$ are **minimal**, then the operator $\bigvee\limits_{\alpha\in\mathfrak{A}} C_\alpha$ is also minimal.*

Proof. We consider the situation: $L \subseteq N \subseteq M$. The minimality of C_α $\big($see (1.10$'$)$\big)$ implies $(C_\alpha)_M(N) = (C_\alpha)_M(L) + N$. Using this relation we obtain:

$$\big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\big)_M(N) = \sum_{\alpha\in\mathfrak{A}}\big[(C_\alpha)_M(N)\big] = \sum_{\alpha\in\mathfrak{A}}\big[(C_\alpha)_M(L) + N\big] =$$

$$= \big[\sum_{\alpha\in\mathfrak{A}}\big((C_\alpha)_M(L)\big)\big] + N = \big[\big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\big)_M(L)\big] + N,$$

therefore the operator $\bigvee\limits_{\alpha\in\mathfrak{A}} C_\alpha$ is minimal. $\square$

Taking into account that $C \in \mathbb{CO}$ is cohereditary if and only if it is maximal and minimal (see Section 1), from Propositions 3.2 and 3.3 follows

Corollary 3.4. *If the operators C_α $(\alpha \in \mathfrak{A})$ are **cohereditary**, then the operator $\bigvee_{\alpha \in \mathfrak{A}} C_\alpha$ is also cohereditary.* □

b) *The meet in $\mathbb{CO}$*

Proposition 3.5. *If the operators C_α $(\alpha \in \mathfrak{A})$ of $\mathbb{CO}$ are **hereditary**, then the operator $\bigwedge_{\alpha \in \mathfrak{A}} C_\alpha$ is also hereditary.*

Proof. By definition $\big($see (1.7)$\big)$ the heredity of C_α means that for every submodules $L \subseteq N \subseteq M$ we have $(C_\alpha)_N(L) = (C_\alpha)_M(L) \cap N$. Therefore:

$$\Big[\big(\bigwedge_{\alpha \in \mathfrak{A}} C_\alpha\big)_M(L)\Big] \cap N = \Big[\bigcap_{\alpha \in \mathfrak{A}} \big((C_\alpha)_M(N)\big)\Big] \cap N = \bigcap_{\alpha \in \mathfrak{A}} \Big[\big((C_\alpha)_M(L)\big) \cap N\Big] =$$

$$= \bigcap_{\alpha \in \mathfrak{A}} \big[(C_\alpha)_N(L)\big] = \big(\bigwedge_{\alpha \in \mathfrak{A}} C_\alpha\big)_N(L),$$

so $\bigwedge_{\alpha \in \mathfrak{A}} C_\alpha$ is hereditary. □

Proposition 3.6. *If the operators C_α $(\alpha \in \mathfrak{A})$ of $\mathbb{CO}$ are **maximal**, then the operator $\bigwedge_{\alpha \in \mathfrak{A}} C_\alpha$ is also maximal.*

Proof. In the situation $K \subseteq N \subseteq M$ the maximality of C_α $\big($see (1.9$'$)$\big)$ implies the relation $[(C_\alpha)_M(N)]\,/\,K = (C_\alpha)_{M/K}(N/K)$. Therefore $\bigcap_{\alpha \in \mathfrak{A}} \big[\big((C_\alpha)_M(N)\big)/K\big] = \bigcap_{\alpha \in \mathfrak{A}} \big[(C_\alpha)_{M/K}(N/K)\big]$, and so $\big[\bigcap_{\alpha \in \mathfrak{A}} \big((C_\alpha)_M(N)\big)\big]/K = \bigcap_{\alpha \in \mathfrak{A}} \big[(C_\alpha)_{M/K}(N/K)\big]$. Now by the definition of the meet in $\mathbb{CO}$ it is clear that $\big[\big(\bigwedge_{\alpha \in \mathfrak{A}} C_\alpha\big)_M(N)\big]/K = \big(\bigwedge_{\alpha \in \mathfrak{A}} C_\alpha\big)_{M/K}(N/K)$, i.e. the operator $\bigwedge_{\alpha \in \mathfrak{A}} C_\alpha$ is maximal. □

c) *The product in $\mathbb{CO}$*

Proposition 3.7. *If the closure operators $C, D \in \mathbb{CO}$ are **maximal**, then the operator $C \cdot D$ is also maximal.*

Proof. Let $K \subseteq N \subseteq M$. The maximality of C and D implies the relations $C_M(N)/K = C_{M/K}(N/K)$ and $D_M(N)/K = D_{M/K}(N/K)$, which permit to obtain:

$$\big[(C \cdot D)_M(N)\big]/K = \big[C_M\big(D_M(N)\big)\big]/K = C_{M/K}\big[D_M(N)/K\big] =$$

$$= C_{M/K}\big[D_{M/K}(N/K)\big] = (C \cdot D)_{M/K}(N/K).$$

This shows that the operator $C \cdot D$ is maximal. □

Proposition 3.8. *If the closure operators $C, D \in \mathbb{CO}$ are **minimal**, then the operator $C \cdot D$ is also minimal.*

Proof. Let $L \subseteq N \subseteq M$. By the minimality of C and D we have $C_M(N) = C_M(L) + N$ and $D_M(N) = D_M(L) + N$. From the second relation we obtain $(C \cdot D)_M(N) = C_M\big(D_M(N)\big) = C_M\big(D_M(L) + N\big)$, and from the first relation in the situation $D_M(L) \subseteq D_M(L) + N \subseteq N$ we have:

$$C_M\big(D_M(L) + N\big) = C_M\big(D_M(L) + (D_M(L) + N)\big) = \big[C_M\big(D_M(L)\big)\big] + N.$$

Since $\big[\big(C \cdot D\big)_M(L)\big] + N = \big[C_M\big(D_M(L)\big)\big] + N$, now it is clear that $\big(C \cdot D\big)_M(N) = \big(C \cdot D\big)_M(L) + N$, i.e. the operator $C \cdot D$ is minimal. $\square$

From Propositions 3.7 and 3.8 follows

Corollary 3.9. *If the closure operators $C, D \in \mathbb{CO}$ are **cohereditary**, then the operator $C \cdot D$ is also cohereditary.* $\square$

The preservation of some properties of closure operators under the application of the operation of product can be obtained by some *addititional conditions* on the operators. We show in continuation two examples of such situations.

Example 1. Let $C, D \in \mathbb{CO}$ and $C \cdot D = D \cdot C$. If the operators C and D are *idempotent*, then the operator $C \cdot D$ is also idempotent.

Example 2. Let $C \in \mathbb{CO}$ preserves the intersection in the inferior term: $C_M(N_1 \cap N_2) = C_M(N_1) \cap C_M(N_2)$, where $N_1, N_2 \in \mathbb{L}(M)$. If the operators $C, D \in \mathbb{CO}$ are *hereditary*, then the operator $C \cdot D$ is also hereditary. Indeed, if $L \subseteq N \subseteq M$, then by hypotheses $C_N(L) = C_M(L) \cap N$ and $D_N(L) = D_M(L) \cap N$. Since C preserves the intersections, we have $C_M\big[D_M(L) \cap N\big] = \big[C_M\big(D_M(L)\big)\big] \cap C_M(N)$. This relation together with the heredity of C and D implies:

$$\big(C \cdot D\big)_N(L) = C_N\big(D_N(L)\big) = C_N\big[D_M(L) \cap N\big] =$$

$$= \big[C_M\big(D_M(L) \cap N\big)\big] \cap N = \big[\big(C_M(D_M(L))\big) \cap C_M(N)\big] \cap N =$$

$$= \big[C_M\big(D_M(L)\big)\big] \cap N = \big[\big(C \cdot D\big)_M(L)\big] \cap N,$$

i.e. $C \cdot D$ is hereditary.

d) *The coproduct in* $\mathbb{CO}$

Proposition 3.10. *If the closure operators $C, D \in \mathbb{CO}$ are **hereditary**, then the operator $C \# D$ is also hereditary.*

Proof. Let $L \subseteq N \subseteq M$. By the definition of coproduct and heredity of C in the situation $L \subseteq D_N(L) \subseteq M$ we obtain:

$$(C \# D)_N(L) = C_{D_N(L)}(L) = C_M(L) \cap D_N(L).$$

On the other hand, by definition we have:

$$\big[(C \# D)_M(L)\big] \cap N = \big[C_{D_M(L)}(L)\big] \cap N,$$

and applying the heredity of C in the situation $L \subseteq D_M(L) \subseteq M$, we obtain $C_{D_M(L)}(L) = C_M(L) \cap D_M(L)$. These facts together with the heredity of D $\big($i.e. $D_M(L) \cap N = D_N(L)\big)$ show that

$$\big[(C \# D)_M(L)\big] \cap N = \big[C_{D_M(L)}(L)\big] \cap N =$$

$$= \big[C_M(L) \cap D_M(L)\big] \cap N = C_M(L) \cap D_N(L).$$

Comparing with the foregoing, we conclude that $(C \# D)_N(L) = \big[(C \# D)_M(L)\big] \cap N$, i.e. $C \# D$ is hereditary. $\square$

Proposition 3.11. *If the operators $C, D \in \mathbb{CO}$ are* ***maximal****, then the operator $C \# D$ is also maximal.*

Proof. Let $K \subseteq N \subseteq M$. By the maximality of C and D we have $C_M(N)/K = C_{M/K}(N/K)$ and $D_M(N)/K = D_{M/K}(N/K)$. These relations and the definition of coproduct imply:

$$\big[(C \# D)_M(N)\big]/K = \big[C_{D_M(N)}(N)\big]/K = C_{D_M(N)/K}(N/K) =$$

$$= C_{D_{M/K}(N/K)}(N/K) = (C \# D)_{M/K}(N/K),$$

therefore $C \# D$ is maximal. $\square$

Proposition 3.12. *If the closure operators $C, D \in \mathbb{CO}$ are* ***coherediary****, then the operator $C \# D$ is also cohereditary.*

Proof. Let $K, N \in \mathbb{L}(M)$. Since C and D are cohereditary we have:

$$\big[C_M(N) + K\big]/K = C_{M/K}\big[(N+K)/K\big];$$

$$\big[D_M(N) + K\big]/K = D_{M/K}\big[(N+K)/K\big].$$

From these relations and the definition of coproduct we obtain:

$$\big[\big((C \# D)_M(N)\big) + K\big]/K = \big[\big(C_{D_M(N)}(N)\big) + K\big]/K =$$

$$= C_{(D_M(N)+K)/K}\big((N+K)/K\big) = C_{D_{M/K}((N+K)/K)}\big((N+K)/K\big) =$$

$$= (C \# D)_{M/K}\big((N+K)/K\big),$$

hence the operator $C \# D$ is cohereditary. $\square$

Similarly to the case of product (see Example 1) the commutativity $C \# D = D \# C$ implies the preservation of weak heredity, i.e. if C, D are *weakly hereditary*, then the operator $C \# D$ is also weakly hereditary, which can be proved by the direct verification.

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On the number of group topologies on countable groups

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Abstract. If a countable group G admits a non-discrete Hausdorff group topology, then the lattice of all group topologies of the group G admits:
– continuum c of non-discrete metrizable group topologies such that $\sup\{\tau_1, \tau_2\}$ is the discrete topology for any two of these topologies;
– two to the power of continuum of coatoms in the lattice of all group topologies.

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1 Introduction

This article is a continuation of article [1]. Main results of this article are Theorems 3.1 and 3.2.

Statements 3.1.2 and 3.1.3 of Theorem 3.1 are stronger than Theorem 14 from [1], and Statement 3.1.4 affirms that if a countable group G admits a non-discrete Hausdorff group topology, then the lattice of all group topologies of the group G admits two to the power of continuum of coatoms.

Moreover (see Theorem 3.2), for a countable group the requirement of the existence of a non-discrete Hausdorff group topology in Theorem 14 and Theorem 13 from [1] can be weakened to the requirement of the existence of a group topology in which the topological group does not have a finite basis of the filter of all neighborhoods of the unity element.

2 Notations and preliminaries

For proof of the main results we need the following notations and results:

Notations 2.1.

– $|A|$ is the cardinality of the set A;

– $\mathbb{N}$ is the set of all natural numbers;

– c is the continuum cardinality and $\omega(c)$ is the minimal transfinite number of the cardinality c;

– $\widetilde{\mathbb{N}}$ is a set of cardinality c of infinite subsets of the set $\mathbb{N}$ such that $A \bigcap B = \emptyset$ for any $A, B \in \widetilde{\mathbb{N}}$ and $A \neq B$ (the existence of such a set $\widetilde{\mathbb{N}}$ is proved in [3, Example 3.6.18]);

Notation 2.2. If $G(\cdot)$ is a group and x is some variable, then the free product of the group $G(\cdot)$ and of the free cyclic group generated by the element x is denoted by $G(x)$, i.e. $G(x)$ consists of elements of the form $g_1 \cdot x^{k_1} \cdot g_2 \cdot x^{k_2} \cdot \ldots \cdot g_s \cdot x^{k_s} \cdot g_{s+1}$, where $g_i \in G$ for $1 \leq i \leq s+1$ and k_j is an integer for $1 \leq j \leq s$.

Definitions 2.3.

– Elements of the group $G(x)$ are called *words in the variable x over the group $G(\cdot)$*;

– If $g \in G(\cdot)$ and $f(x) \in G(x)$, then the expression $f(x) = g$ will be called *an equation over the group $G(\cdot)$*;

– An element $a \in G(\cdot)$ is called *a root of the equation* $f(x) = g$ if $f(a) = g$.

Definitions 2.4.

– A partially ordered set $(X, \leq)$ is called *a lattice* if for any elements $a, b \in X$ there exist $\inf\{a, b\}$ and $\sup\{a, b\}$;

– A lattice $(X, \leq)$ is called *complete* if for any non-empty subset $S \subseteq X$ there exist $\inf S$ and $\sup S$;

– Lattices $(X, \leq)$ and $(Y, \leq)$ are called *anti-isomorphic* if there exists a bijective mapping $\Psi : (X, \leq) \to (Y, \leq)$ such that $\Psi(\inf\{a, b\}) = \sup\{\Psi(a), \Psi(b)\}$ and $\Psi(\sup\{a, b\}) = \inf\{\Psi(a), \Psi(b)\}$ for all elements $a, b \in X$.

The map $\Psi : (X, \leq) \to (Y, \leq)$ will be called *a lattice anti-isomorphism*;

– If a lattice $(X, \leq)$ has the greatest element 1, then an element $a \neq 1$ of the lattice $(X, \leq)$ is called *a coatom* if $b = 1$ for any element $b \in X$ such that $a < b$.

Notation 2.5. If $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$ are sequences of symmetrical subsets of a group $G(\cdot)$ (i.e. $(A_i)^{-1} = A_i$ and $(B_i)^{-1} = B_i$) such that $e \in \bigcap\limits_{i=1}^{\infty} B_i$, then for any natural number n by induction we define the set $F_n(B_1, \ldots, B_n; A_1, \ldots A_n)$: we take $F_1(B_1; A_1) = \{g \cdot h \cdot g^{-1} | g \in A_1, h \in B_1\} \bigcup B \cdot B$ and

$$F_{n+1}(B_1, \ldots, B_{n+1}; A_1, \ldots A_{n+1}) = F_1\big((B_1 \bigcup F_n(B_2, \ldots, B_{n+1}; A_2, \ldots A_{n+1}); A_1\big).$$

Theorem 2.6 (see for example [2, page 203 and page 205]). *A set Ω of subsets of a group $G(\cdot)$ is a basis of the filter of all neighborhoods of the unity element e for a Hausdorff group topology on the group $G(\cdot)$ if and only if the following conditions are satisfied:*

1) $\bigcap\limits_{V \in \Omega} V \supseteq \{e\}$;

2) *For every V_1 and $V_2 \in \Omega$ there exists $V_3 \in \Omega$ such that $V_3 \subseteq V_1 \cap V_2$;*

3) *For every $V_1 \in \Omega$ there exists $V_2 \in \Omega$ such that $V_2 \cdot V_2 \subseteq V_1$;*

4) *For every $V_1 \in \Omega$ there exists $V_2 \in \Omega$ such that $V_2^{-1} \subseteq V_1$;*

5) *For every $V_1 \in \Omega$ and any element $g \in G$ there exists $V_2 \in \Omega$ such that $g \cdot V_2 \cdot g^{-1} \subseteq V_1$.*

Moreover, this group topology is Hausdorff if and only if $\bigcap\limits_{V \in \Omega} V = \{e\}$

Proposition 2.7 (see [1]). *If $V_1, V_2, \ldots$ and $S_1, S_2, \ldots$ are some sequences of subsets of a group $(G, (\cdot))$, then (see I.5) for subsets $F_k(V_1, \ldots, V_k; S_1, \ldots, S_k)$ the following statements are true:*

2.7.1. *If $e \in V_1$, then $V_1 \subseteq V_1 \cdot V_1 \subseteq F_1(V_1; S_1)$ and $g \cdot V_1 \cdot g^{-1} \subseteq F_1(V_1; S_1)$ for any $g \in S_1$;*

2.7.2. *If $k \in \mathbb{N}$ and S_i and V_i are symmetric and finite sets for $1 \leq i \leq k$, then $F_k\big(V_1, \ldots, V_k; S_1, \ldots, S_k\big)$ is a symmetric and finite set;*

2.7.3. $F_k\big(\{e\}, \ldots, \{e\}; S_1, \ldots, S_k\big) = \{e\}$ *for any* $k \in \mathbb{N}$;

2.7.4. *If $U_i \subseteq V_i$ and $T_i \subseteq S_i$ for each $1 \leq i \leq k$, then*

$$F_k\big(U_1, \ldots, U_k; T_1, \ldots, T_k\big) \subseteq F_k\big(V_1, \ldots, V_k; S_1, \ldots, S_k\big);$$

2.7.5. *If $k, p \in \mathbb{N}$, and $e \in V_i$ for $i \leq k$ and $V_{k+j} = \{e\}$ for $1 \leq j \leq p$, then*

$$F_k\left(V_1, \ldots, V_k; S_1, \ldots, S_k\right) = F_{k+p}\left(V_1, \ldots, V_{k+p}; S_1, \ldots, S_{k+p}\right);$$

2.7.6. *If an integer $k \geq 2$, then the equality*

$$F_k\big(V_1, \ldots, V_k; S_1, \ldots, S_k\big) =$$

$$F_k\Big(V_1 \cup F_{k-1}\big(V_2, \ldots, V_k; S_2, \ldots, S_k\big), \ldots, V_{k-1} \cup F_1(V_k; S_k), V_k; S_1, \ldots, S_k\Big)$$

is true;

2.7.7. *If $e \in V_i$ for each $1 \leq i \leq k$, then $V_t \subseteq F_k\Big(V_1, \ldots, V_k; S_1, \ldots, S_k\Big)$ for each $1 \leq t \leq k$;*

2.7.8. *If $k, s \in \mathbb{N}$ and $e \in V_i$ for each $1 \leq i \leq k + s$, then*

$$F_{k+1}\Big(V_s, \ldots, V_{k+s}; S_s, \ldots, S_{k+s}\Big) \subseteq F_{k+s-t+1}\big(V_t, \ldots, V_{k+s}; S_1, \ldots, S_{k+s}\big)$$

for any $k, s, t \in \mathbb{N}$ and $t \leq s$.

Notation 2.8. Let $G(\cdot) = \{e, g_1^{\pm 1}, \ldots\}$ be a countable group, and for each positive integer n let be $S_n = \{g_1^{\pm 1}, g_2^{\pm 1}, \ldots, g_n^{\pm 1}\}$.

For each pair of natural numbers (i, j) we define subsets $V_{(i,j)}$ and $S_{(i,j)}$ of the group $G(\cdot)$ and for each three natural numbers (i, j, k) such that $1 \leq k \leq j$ define a set $\Phi_{(i,j,k)}(x)$ of equations in the variable x over the group $G(\cdot)$ as follows:

$V_{(1,j)} = \{e\}$, $S_{(1,j)} = S_j$ and $\Phi_{(1,j,k)}(x) = \big\{x = c \big| c \in S_k\big\}$ for all $j, k \in \mathbb{N}$ and $k \leq j$.

Suppose that for a natural number p the sets $V_{(i,j)}$, $S_{(i,j)}$ and $\Phi_{(i,j,k)}(x)$ are defined for $i \leq p$ and all $j, k \in \mathbb{N}$ such that $k \leq j$.

If $p + 1$ is an even natural number, then we take:

$V_{(p+1,j)} = \{e\}$ for $j \geq p + 1$;

$V_{(p+1,j)} = V_{(p,j)} \cup \{g, g^{-1}\}$, where $g \in G \setminus \bigcup\limits_{s=1}^{j} S_{(p,j)}$ (if $G \setminus \bigcup\limits_{s=1}^{j} S_{(p,j)} = \emptyset$, then we take $V_{(p+1,j)} = V_{(p,j)}$) for all $j < p+1$;

$\Phi_{(p+1,j,k)}(x) = \Phi_{(p,j,k)}(x)$ for all $k < j \in \mathbb{N}$;

$S_{(p+1,j)} = \{g \in G | g \in \bigcup\limits_{k=1}^{j} \Phi_{(p+1,j,k)}\}$ for all $j \in \mathbb{N}$.

If $p+1$ is an odd natural number, then we take:

$V_{(p+1,j)} = \{e\}$ for $j \geq p+1$;

$V_{(p+1,j)} = F_{p+1-j}\big(V_{(p,j+1)}, \dots, V_{(p,p+1)}; S_{j+1}, \dots, S_{p+1}\big) \bigcup V_{(p,j)}$ for $j < p+1$;

$\Phi_{(p+1,j,j)}(x) = \{x = g \big| g \in S_j\}$ for all $j \in N$ and $\Phi_{(p+1,j,k)}(x) = \big\{f(x) = g \big| \ f(x) \in F_{j,k}\big(V_{(p+1,k+1)}, \dots, V_{(p+1,j-1)}, V_{(p,j)} \cup \{x, x^{-1}\}; S_{k+1}, \dots, S_j\big)$ and $g \in S_k\big\}$ for any $k, j \in \mathbb{N}$ and $k < j$;

$S_{(p+1,j)} = S_{(p,j)}$ for every $j \in \mathbb{N}$.

So, we identified subsets of $V_{(i,j)}$ and $S_{(i,j)}$ of the group $G(\cdot)$ for each pair of positive integers (i, j) and the set $\Phi_{(i,j,k)}(x)$ of equations on the group $G(\cdot)$ for each triples of positive integers (i, j, k) such that $1 \leq k \leq j$.

Theorem 2.9 (see [1, Theorem 11]). *If a countable group $G(\cdot)$ admits a non-discrete Hausdorff group topology τ and $M = \big\{f_1(x) = a_1, \dots, f_m(x) = a_m\big\}$ is a finite set of equations over the group $G(\cdot)$ for which the unity element e is not a root of any of these equations, then in the topological group (G, τ) there exists a neighborhood W of the unity element e such that each its element is not a root of any of these equations.*

From Theorem 2.6 follows

Theorem 2.10. *If Ω is a set of group topologies on a group $G(\cdot)$ and for each topology $\tau \in \Omega$ in a topological group $(G(\cdot), \tau)$ a basis $\mathbf{B}_\tau$ of the filter of all neighborhoods of the unity element e is given, then the set*

$$\big\{ \bigcap_{\tau \in M} V_\tau | M \text{ is a finite subset in } \Omega \text{ and } V_\tau \in \mathbf{B}_\tau \big\}$$

is a basis of the filter of all neighborhoods of the unity element in the topological group $(G(\cdot), \sup \Omega)$.

From the definition of the prototype of any topology follows

Theorem 2.11. *Let $f : G(\cdot) \to \overline{G}(\cdot)$ be some group homomorphism from the group $G(\cdot)$ in the group $\overline{G}(\cdot)$. If $\overline{\tau}$ is a group topology on the group $\overline{G}(\cdot)$ and τ is the prototype of the topology $\overline{\tau}$ relative to the homomorphism f (i. e. $\tau = \{f^{-1}(\overline{U}) | \overline{U} \in \overline{\tau}\}$), then τ is a group topology on the group $G(\cdot)$ and for any basis $\overline{\mathbf{B}}$ of the filter of neighborhoods of the unity element in the topological group $(\overline{G}(\cdot), \overline{\tau})$, the set $\mathbf{B} = \{f^{-1}(\overline{V}) | \overline{V}) \in \overline{B}\}$ is a basis of the filter of neighborhoods of the unity element in the topological group $(G(\cdot), \tau)$.*

Similarly to the proof of step II of Theorem 13 in [1], is proved:

Theorem 2.12. *Let $G(\cdot) = \{e, g_i^{\pm 1} | \in \mathbb{N}\}$ be a countable group and let $\{h_k = g_{i_k} | k \in \mathbb{N}\}$ be a sequence of elements of the group $G(\cdot)$ such that*

$$h_i \notin F_n\big(\{e, g_1^{\pm 1}\}, \dots, \{e, g_{i-1}^{\pm 1}\}, \{e\}\{e, g_{i+1}^{\pm 1}, \dots, \{e, g_n^{\pm 1}\}; S_1, \dots, S_n\}\big)$$

for every $i, n \in \mathbb{N}$. *Then the following statements are true:*

2.12.1. *If C is an infinite subset of the set of all natural numbers $\mathbb{N}$ and*

$$U_{i,C} = \begin{cases} \{h_{k_i}, e, h_{k_i}^{-1}\} & \text{if} \quad i \in C, \\ \{e\} & \text{if} \quad i \notin C \end{cases}$$

for every $i \in \mathbb{N}$, then the set

$$\Big\{\widehat{U}_i(C) \mid \widehat{U}_i(C) = \bigcup_{j=1}^{\infty} F_{j+1}\big(U_{i,C}, \dots U_{i+j,C}; S_i, \dots, S_{i+j}\big), i \in \mathbb{N}\Big\}$$

is a basis of the filter of neighborhoods of the unity element for some group topology $\tau(C)$ in the group $G(\cdot)$;

2.12.2. *If A, B are subsets of the set $\mathbb{N}$ such that $A \setminus B$ and $B \setminus A$ are infinite subsets, then the topologies $\tau(A)$ and $\tau(B)$ are incomparable.*

Definition 2.13. An element $d \in X$ is called a maximal element in a partially ordered set $(X, \leq)$ if $d = z$ for any element z in X such that $d \leq z$.

Theorem 2.14 (see [3, page 28, the Kuratowski-Zorn's lemma]). *If $(X, \leq)$ is a partially ordered set such that for any linearly ordered subset $(A, \leq) \subseteq (X, \leq)$ there exists an element $a \in X$ such that $x \leq a$ for every $x \in A$, then for any $y \in X$ there exists a maximal element $d \in X$ in the partially ordered set $(X, \leq)$ such that $y \leq d$.*

Proposition 2.15(see [3, Corollary 3.6.12]). *If $(\beta\mathbb{N}, \tau)$ is Stone-Čech compacification, then the following statements are true:*

2.15.1. The set $\mathbb{N}$ is a dense subset of the topological space $(\beta\mathbb{N}, \tau)$;

2.15.2. The topological space $(\beta\mathbb{N}, \tau)$ is Hausdorff;

2.15.3. The cardinality of the set $\beta\mathbb{N}$ is equal to 2^c.

Proposition 2.16. *For any element $a \in \beta\mathbb{N} \setminus \mathbb{N}$ and any neighborhood U of the element a in the topological space $(\beta\mathbb{N}, \tau)$, the set $U \bigcap \mathbb{N}$ is infinite.*

Proof. Assume the contrary, i. e. that some element $a \in \beta\mathbb{N} \setminus \mathbb{N}$ has a neighborhood U such that $U \bigcap \mathbb{N}$ is a finite set.

Since every finite set is closed in any Hausdorff space and $a \notin \mathbb{N}$, then $V = U \setminus (\mathbb{N} \bigcap U)$ is a neighborhood of the element a in the topological space $(\beta\mathbb{N}, \tau)$, and $V \bigcap \mathbb{N} = \emptyset$.

This contradicts the Statement 2.15.1. □

3 Basic results

Theorem 3.1. *Let a countable group $G(\cdot)$ admit some Hausdorff non-discrete group topology τ_0 such that the topological group (G, τ_0) has a countable basis of the filter of all neighborhoods of the unity element. Then:*

3.1.1. *The group $G(\cdot)$ admits a continuum of non-discrete group topologies stronger than τ_0 and such that the following conditions are true:*

– the space of the topological group is Hausdorff;

– the unity element has a countable basis of the filter of all neighborhoods;

– any two of these topologies are comparable.

3.1.2. *The group $G(\cdot)$ admits a continuum of non-discrete group topologies stronger than τ_0 and such that for each of these topologies the following conditions are true:*

– the space of the topological group is Hausdorff;

– the unity element has a countable basis of the filter of all neighborhoods;

– sup$\{\tau_1, \tau_2\}$ is the discrete topology for any two of these topologies $\tau_1 \neq \tau_2$;

3.1.3. *There exist 2^c (two to the power of continuum) non-discrete group topologies stronger than τ_0 and such that sup$\{\tau_1, \tau_2\}$ is the discrete topology for any two of these topologies $\tau_1 \neq \tau_2$;*

3.1.4. *There exist 2^c coatoms in the lattice of all group topologies on the group $G(\cdot)$.*

Proof. Proof of Statement 3.1.1 see in the proof of Theorem 14 in [1].

Proof of Statement 3.1.2. Let $G = \{e, g_1^{\pm 1}, \dots\}$ be a numbering of elements of the group $G(\cdot)$ and let $S_n = \{g_1^{\pm 1}, \dots, g_n^{\pm 1}\}$ for any $n \in \mathbb{N}$. Then there exists a countable basis $\{V_1, V_2, \dots\}$ of the filter of neighborhoods of the unity element in the topological group (G, τ_0) which consists of symmetric subsets such that $V_k \cap S_k = \emptyset$ and $g \cdot V_{k+1} \cdot g^{-1} \subseteq V_k$ for any $k \in \mathbb{N}$ and any $g \in S_k$.

It is easily proved by induction on k that $F_k\big(V_{i+1}, \dots, V_{i+k}; S_{i+1}, \dots, S_{i+k}\big) \subseteq V_i$ for any $i, k \in \mathbb{N}$.

The proof of the theorem will be realized in several steps.

Step I. Construction of an auxiliary sequence $h_1, h_2, \dots$ of elements of $G(\cdot)$ and of an increasing sequence $k_1, k_2, \dots$ of natural numbers.

By induction on n we construct a sequence $k_1, k_2, \dots$ of natural numbers such that $k_i \geq i$ for every $i \in \mathbb{N}$ and a sequence $h_1, h_2, \dots$ of elements of the set $G \backslash \{e\}$ such that $\{e, h_i, h_i^{-1}\} \subseteq V_{k_i}$ for any positive integer i and for any subsets of $A \subseteq \{k_1, \dots, k_n\}$ and $B \subseteq \{k_1, \dots, k_n\}$ such that $A \bigcap B = \emptyset$ and the condition holds:

$$F_n(U_{1,A}, \dots, U_{n,A}; S_1, \dots, S_n) \bigcap F_n(U_{1,B}, \dots, U_{n,B}; S_1, \dots, S_n) = \{e\},$$

where

$$U_{i,C} = \begin{cases} \{h_i, e, h_i^{-1}\} & \text{if} \quad i \in C, \\ \{e\} & \text{if} \quad i \notin C \end{cases}$$

for any subset $C \subseteq \{1, \ldots, n\}$.

If $n = 1$, then we take $k_1 = 2$ and h_1 an arbitrary element of the set $V_2 \setminus \{e\}$.

If A and B are subsets of $\{1\}$ such that $A \bigcap B = \emptyset$, then either $A = \emptyset$ or $B = \emptyset$, and hence (see Statement 2.7.3), either $F_1(U_{1,A}; S_1) = F_1(\{e\}; S_1) = \{e\}$, or $F_1(U_{1,B}; S_1) = F_1(\{e\}; S_1) = \{e\}$. So $F_1(U_{1,A}; S_1) \bigcap F_1(U_{1,B}; S_1) = \{e\}$, and hence, for the natural number $k_1 = 2$ and the element h_1 all conditions specified above are true.

Suppose that we have already defined positive integers $k_1 < k_2, \ldots < k_n$ such that $k_i > i$ and elements $h_1, h_2, \ldots, h_n$ from the set $G \setminus \{e\}$ such that all conditions specified above are true.

For any sets $A \subseteq \{1, \ldots, n\}$ and $B \subseteq \{1, \ldots, n\}$ such that $A \bigcap B = \emptyset$ we consider the set $\Psi_{(A,B)}(x)$ of equations over the group $G(\cdot)$ of the form $f(x) = g$, where

$$f(x) \in F_{n+1}(U_{1,A}, \ldots, U_{n,A}, \{x, e, x^{-1}\}; S_1, \ldots, S_{n+1})$$

and $g \in F_n(U_{1,B}, \ldots, U_{n,B}\}; S_1, \ldots, S_n) \setminus \{e\}$.

We prove that the unity element e is not a root for any equation of the set $\Psi_{(A,B)}(x)$.

Assume the contrary, i.e. that $f(e) = g$ for some equation $f(x) = g$ from the set $\Psi_{(A,B)}(x)$. Then $g \in F_{k_n}(U_{1,B}, \ldots, U_{n,B}\}; S_1, \ldots, S_n) \setminus \{e\}$, and (see Statement 2.7.5)

$$g = f(e) \in F_{n+1}(U_{1,A}, \ldots, U_{n,A}, \{e\}; S_1, \ldots, S_{n+1}) = F_n(U_{1,A}, \ldots, U_{n,A}, ; S_1, \ldots, S_n),$$

and hence,

$$g \in F_n(U_{1,A}, \ldots, U_{n,A}, ; S_1, \ldots, S_n) \bigcap \big(F_n(U_{1,B}, \ldots, U_{n,B}\}; S_1, \ldots, S_n) \setminus \{e\}\big).$$

We have the contradiction with the inductive assumption that

$$F_n(U_{1,A}, \ldots, U_{n,A}; S_1, \ldots, S_n) \bigcap F_n(U_{1,B}, \ldots, U_{n,B}; S_1, \ldots, S_n) = \{e\},$$

and hence the unity element e is not a root of any equation of the set $\Psi_{(A,B)}(x)$.

Then, by Theorem 2.9 there exists a neighborhood $W_{(A,B)}$ of the unity element in the topological group $(G(\cdot), \tau_0)$ such that any element $h \in W_{\{A,B\}}$ is not a root of any equation of the set $\Psi_{(A,B)}(x)$.

If now $\widetilde{M} = \big\{(A, B) | A, B \subseteq \{1, \ldots, n + 1\} \text{ and } A \bigcap B = \emptyset\big\}$, then from the finiteness of the set $\widetilde{M}$ it follows that $\bigcap\limits_{(A,B) \in \widetilde{M}} W_{(A,B)}$ is a neighborhood of the unity element in a topological group $(G(\cdot), \tau_0)$, and any element $h \in \bigcap\limits_{(A,B) \in \widetilde{M}} W_{(A,B)} \setminus \{e\}$ is not a root of any equation of the set $\bigcup\limits_{(A,B) \in \widetilde{M}} \Psi_{(A,B)}(x)$.

Since the set $\{V_1, V_2, \dots\}$ is a basis of the filter of neighborhoods of the unity element in the topological group $(G(\cdot), \tau_0)$, then there exists a natural number $k_{n+1} > n$ such that $V_{k_{n+1}} \subseteq \bigcap\limits_{(A,B)\in\widetilde{M}} W_{(A,B)}$.

We take any element $h_{n+1} \in V_{k_{n+1}} \subseteq V_{n+1}$, and prove that the conditions specified above are true also for the number $n+1$, i.e. these conditions are satisfied for the sequence of natural numbers $k_1, k_2, \dots k_{n+1}$ and the sequence of elements $h_1, \dots, h_{n+1}$.

Since $k_{n+1} > n+1$ and $h_{n+1} \in V_{k_{n+1}}$, then it remains only to prove that

$$F_{n+1}(U_{1,A}, \dots, U_{n+1,A}; S_1, \dots, S_{n+1} \bigcap F_{n+1}(U_{1,B}, \dots, U_{n+1,B}; S_1, \dots, S_{n+1} = \{e\},$$

for any of subsets A, B of the set $\{1, \dots, n+1\}$ for which $A \bigcap B = \emptyset$.

Assume the contrary, i.e. that

$$F_{n+1}(U_{1,A}, \dots, U_{n+1,A}; S_1, \dots, S_{n+1}) \bigcap F_{n+1}(U_{1,B}, \dots, U_{n+1,B}; S_1, \dots, S_{n+1}) \neq \{e\}$$

for some subsets $A, B \subseteq \{1, \dots, n+1\}$ such that $A \bigcap B = \emptyset$.

Then either $A \not\subseteq \{1, 2, \dots, n\}$ or $B \not\subseteq \{1, \dots, n\}$.

Suppose, for definiteness, that $A \not\subseteq \{1, \dots, n\}$.

Since $A \bigcap B = \emptyset$ then $B \subseteq \{1, \dots, n\}$ and since $n+1 \in A$ then $U_{n+1,A} = \{h_{n+1}, e, h_{n+1}\}$.

Then since the element $h_{n+1} \neq e$ and it is not a root of any equation of the set $\Psi_{(A,B)}(x)$ then from the definition of the set $\Psi_{(A,B)}(x)$ and Statement I.7.5 it follows that $\{e\} =$

$$F_{n+1}(U_{1,A}, \dots, U_{n,A}, \{h_{n+1}, e, h_{n1}^{-1}\}; S_1, \dots, S_{n+1}) \bigcap F_n(U_{1,B}, \dots, U_{n,B}; S_1, \dots, S_n) =$$

$$F_{n+1}(U_{1,A}, \dots, U_{n,A}, \{h_{n+1}, e, h_{n+1}^{-1}\}; S_1, \dots, S_{n+1}) \bigcap$$

$$F_{n+1}(U_{1,B}, \dots, U_{n,B}, \{e\}; S_1, \dots, S_{n+1}) =$$

$$F_{n+1}(U_{1,A}, \dots, U_{n+1,A}; S_1, \dots, S_{n+1}) \bigcap F_{n+1}(U_{1,B}, \dots, U_{n+1,B}; S_1, \dots, S_{n+1}) \neq \{e\}.$$

We have a contradiction, and hence the conditions specified above are true for the sequence of natural numbers $k_1, k_2, \dots k_{n+1}$ and the sequence of elements $h_1, h_2, \dots, h_{n+1}$.

So, we have constructed the sequence $k_1, k_2, \dots$ of natural numbers $k_i \geq i$ and the sequence $h_1, h_2, \dots$ of elements of the set $G \setminus \{e\}$ such that $\{e, h_i, h_i^{-1}\} \subseteq V_{k_i}$ and which satisfy the following condition:

$$F_m(U_{1,A}, \dots, U_{m,A}; S_1, \dots, S_m) \bigcap F_m(U_{1,B}, \dots, U_{m,B}; S_1, \dots, S_m) = \{e\}$$

for any positive integer m and any subsets $A, B \subseteq \{1, \dots, m\}$ such that $A \bigcap B = \emptyset$.

Step II. Construction of a set T of group topologies of cardinality of continuum and such that $sup\{\tau_1, \tau_2\}$ is the discrete topology for all different topologies $\tau_1, \tau_2 \in$ T.

If $j \in \mathbb{N}$, $A = \{k_j\}$ and $B = \mathbb{N} \setminus \{k_j\}$, then $A \bigcap B) = \emptyset$, and hence,

$$F_n(U_{1,A}, \ldots, U_{n,A}; S_1, \ldots, S_n) \bigcap F_n(U_{1,B}, \ldots, U_{n,B}; S_{k_1}, \ldots, S_{k_n}) = \{e\}$$

for any positive integer n. Then

$$h_j \notin F_n\Big(\{e, h_1, h_1^{-1}\}, \ldots, \{e, h_{j-1}, h_{j-1}^{-1}\}, \{e\}, \{e, h_{j+1}, h_{j+1}^{-1}\}, \ldots, \{e, h_n, h_n^{-1}\};$$

$S_1, S_2, \ldots, S_n\Big)$ and hence, the sequence $k_1, k_2, \ldots$ of natural numbers and the sequence of elements $h_1, h_2, \ldots$ satisfy the conditions specified in the proof of Theorem 13 from [1].

Now we consider:

– the set $U_{i,A} = \{e\}$ if $k_i \notin A$ and $U_{i,A} = \{h_i, e, h_i^{-1}\}$ if $k_i \in A$ for any positive integer i and any set $A \in \widetilde{\mathbb{N}}$ (for the definition of the set $\widetilde{\mathbb{N}}$, see I.1);

– the set $U_{(i+1,j),A} = F_j(U_{i+1,A}, \ldots, U_{i+j,A}; S_{i+1}, \ldots, S_{i+j})$ for every pair (i, j) of natural numbers.

Then (see [1, Step II of the proof of Theorem 13]) the set $\{\hat{U}_i(A) = \bigcup\limits_{j=1}^{\infty} U_{(ij),A} | i \in N\}$ is a basis of the filter of neighborhoods of the unity element for some group topology $\tau(A)$ on the group $G(\cdot)$ such that the following conditions are true:

– the space of the topological group $(G(\cdot), \tau(A))$ is Hausdorff;

– the unity element has a countable basis of the filter of neighborhoods;

– the topology $\tau(A)$ is stronger than the topology τ_0 for any set $A \in \widetilde{\mathbb{N}}$.

We show that if $A, B \in \widetilde{\mathbb{N}}$ and $A \neq B$, then $sup\{\tau(A), \tau(B)\}$ is the discrete topology.

In fact, since $A \bigcap B = \emptyset$, then $\{e\} \subseteq U_{(1,m),A} \bigcap U_{(1,s),B} \subseteq$

$$F_{n+m}(U_{1,A}, \ldots, U_{n+m,A}; S_1, \ldots, S_{n+m}) \bigcap F_{n+s}(U_{1,B}, \ldots, U_{n+s,B}; S_1, \ldots, S_{n+s}) \subseteq$$

$$F_{n+m+s}(U_{1,A}, \ldots, U_{n+m+s,A}; S_1, \ldots, S_{n+m+s}) \bigcap$$
$$F_{n+m+s}(U_{1,B}, \ldots, U_{n+m+s,B}; S_1, \ldots, S_{n+m+s}) = \{e\}$$

for any positive integers m and s, and hence,

$$\hat{U}_1(A) \bigcap \hat{U}_1(B) = \Big(\bigcup_{j=1}^{\infty} U_{(1,j),A}\Big) \bigcap \Big(\bigcup_{j=1}^{\infty} U_{(1,j),B}\Big) = \{e\}.$$

Since $\hat{U}_1(A)$ and $\hat{U}_1(B)$ are neighborhoods of the unity element in the topological group $(G(\cdot) sup\{\tau(A), \tau(B)\})$, then $\{e\} = \hat{U}_1(A) \bigcap \hat{U}_1(B)$ is a neighborhood of the identity in the topological group $(G(\cdot), \sup\{\tau(A), \tau(B)\})$, and hence, $\sup\{\tau(A), \tau(B)\}$ is the discrete topology for any different sets $A, B \in \widetilde{\mathbb{N}}$.

From the fact that the topology $\tau(A)$ is non-discrete for any set $A \in \widetilde{\mathbb{N}}$ it follows that $\tau(A) \neq \tau(B)$ for any different sets $A, B \in \widetilde{\mathbb{N}}$.

Statement 3.1.2 is proved.

Proof of Statement 3.1.3. Since the set $A = U \bigcap \mathbb{N}$ is infinite (see Proposition 2.16) for each element $a \in \beta\mathbb{N} \setminus \mathbb{N}$ and any neighborhood U of the element a of the topological space $(\beta\mathbb{N}, \tau)$, then we consider the topology $\tau_{a,U} = \tau(U \bigcap \mathbb{N})$, which was defined in the proof of Statement 3.1.2.

If $a \in \beta\mathbb{N} \setminus \mathbb{N}$ and Ω_a is the set of all neighborhoods U of the element a in the topological space $(\beta\mathbb{N}, \tau)$, we consider the topology $\tau_a = sup\{\tau_{a,U} | U \in \Omega_a\}$, and show that $sup\{\tau_a, \tau_b\}$ is the discrete topology for any distinct elements $a, b \in \beta\mathbb{N}\setminus\beta\mathbb{N}$.

From the fact that the space $(\beta\mathbb{N}, \tau)$ is Hausdorff it follows that there exist neighborhoods U and V of points a and b, respectively, such that $U \bigcap V = \emptyset$, and hence, $(U \bigcap \mathbb{N}) \bigcap (V \bigcap \mathbb{N}) = \emptyset$. Then (see the end of the proof of Statement 3.1.2), $sup\{\tau_{a,U}, \tau_{a,V}\}$ is the discrete topology, and as $sup\{\tau_a, \tau_b\} \geq sup\{\tau_{a,U}, \tau_{a,V}\}$, then $sup\{\tau_a, \tau_b\}$ is the discrete topology.

Since for each $c \in \beta\mathbb{N} \setminus \mathbb{N}$ the topology τ_c is a non-discrete topology, then the set $\{\tau_c | c \in \beta\mathbb{N} \setminus \mathbb{N}\}$ has cardinality 2^c, and since $sup\{\tau_a, \tau_b\}$ is the discrete topology for any distinct elements $a, b \in \beta\mathbb{N} \setminus \mathbb{N}$ then Statement 3.1.3 is proved.

Proof of Statement 3.1.4. If $\mathcal{T}_\prime$ is the set of all group topologies on the group $G(\cdot)$ and τ_d^* is the discrete topology, then $(\mathcal{T}_\prime, \subseteq)$ is a complete lattice. From Theorem 2.10 it follows that for any linearly ordered subset $(\mathcal{T}, \subseteq)$ of non-discrete topologies the set $\{e\}$ is not a neighborhood of the unity element in the topological group $(G(\cdot), \sup \mathcal{T})$, and hence, $\sup \mathcal{T} \in \mathcal{T}_\prime \setminus \{\tau_d^*\}$. Then, by Theorem 2.14, for any topology τ_a where $a \in \beta\mathbb{N}\setminus\mathbb{N}$, which is defined in the proof of Statement 3.1.3, there exists a maximum element τ'_a in partially ordered set $\in \mathcal{T}_\prime \setminus \{\tau_d^*\}$ such that $\tau_a \leq \tau'_a$. Then for each $a \in \beta\mathbb{N} \setminus \mathbb{N}$ the topology τ'_a is a coatom in the lattice $(\mathcal{T}_\prime, \subseteq)$.

Since $\tau_d^* = \sup\{\tau_a, \tau_b\} \leq \sup\{\tau'_a, \tau'_b\} \leq \tau_d^*$ for different $a, b \in \beta\mathbb{N}\setminus\mathbb{N}$, then $\tau'_a \neq \tau'_b$ for different $a, b \in \beta\mathbb{N}\setminus\mathbb{N}$, and hence, the set $\{\tau'_a \mid a \in \beta\mathbb{N}\setminus\mathbb{N}\}$ has the cardinality 2^c (two to the power of continuum).

Statement 3.1.4 is proved and, hence, the theorem is completely proved. □

Theorem 3.2. *Let $G(\cdot)$ be a countable group and let $\mathcal{T}_0$ be the set of all group topologies on the group $G(\cdot)$ and $\mathcal{T}_1$ be the set of all group topologies on the group $G(\cdot)$ such that for any of these topologies the topological group $(G(\cdot), \tau)$ has a finite basis of the filter of all neighborhoods of unity element. Then the following statements are true:*

3.2.1. *The partially ordered set $(\mathcal{T}_1, \subseteq)$ is a lattice which is anti-isomorphic to the lattice $(\mathcal{N}, \subseteq)$ of all normal subgroups of the group $G(\cdot)$;*

3.2.2. *If $\mathcal{T}_0 \neq \mathcal{T}_1$, then in the group $G(\cdot)$ there exist continuum of group topologies in each of which the topological group has a countable basis of the filter of all neighborhoods of the unity element such that any two topologies are comparable;*

3.2.3. *If $\mathcal{T}_0 \neq \mathcal{T}_1$, then in the group $G(\cdot)$ there are 2^c (two to the power of continuum) of group topologies any two of which are incomparable.*

Proof. **Proof of Statement 3.2.1.** As for any normal subgroup N of the group $G(\cdot)$ the set $\{N\}$ satisfies all conditions of Theorem 2.6, then it is a basis if the filter of neighborhoods of the group $G(\cdot)$ for some group topology $\tau(N)$, and in this topology the topological group has a finite basis of the filter of all neighborhoods of the unity element, i. e. $\tau(N) \in \mathcal{T}_1$.

Now, if $\tau_0 \in \mathcal{T}_1$ and $\mathcal{B}$ is some finite basis of the filter of all neighborhoods of unity element in the topological group $(G(\cdot), \tau_0)$, then $N(\tau_0) = \bigcap\limits_{V \in \mathcal{B}} V$ is an open normal subgroup of $G(\cdot)$, and hence, $N(\tau_0)$, is a neighborhood of the unity element in the topological group $(G(\cdot), \tau_0)$. Then $\tau(N(\tau_0)) = \tau_0$.

So, we have proved that $\mathcal{T}_1 = \{\tau(N) | N \in \mathcal{N}\}$. As $\tau(N_1) \leq \tau(N_2)$ if and only if $N_1 \supseteq N_2$, then $(\mathcal{T}_1, \leq)$ is a lattice, which is anti-isomorphic (see I.4) to the lattice $(\mathcal{N}, \subseteq)$.

Statement 3.2.1 is proved.

Proof of Statement 3.2.2. Let $G(\cdot)$ be a group such that $\mathcal{T}_0 \neq \mathcal{T}_1$ and $\tau_0 \in \mathcal{T}_0 \setminus \mathcal{T}_1$. If $\mathbf{B}$ is some basis of the filter of all neighborhoods of the unity element in the topological group $(G(\cdot), \tau_0)$, and $\mathbf{N} = \bigcap\limits_{V \in \mathbf{B}} V$, then N is a closed normal subgroup of the topological group $(G(\cdot), \tau_0)$.

Since $\tau_0 \notin \mathcal{T}_1$, then $\mathbf{N}$ is not a neighborhood of the unity element in the topological group $(G(\cdot), \tau_0)$ (otherwise the set $\{\mathbf{N}\}$ would be a basis of the filter of neighborhoods of unity element in the topological group $(G(\cdot), \tau_0)$).

Then the factor-group $(\overline{G}(\cdot), \overline{\tau}_0) = (G(\cdot), \tau_0)/\mathbf{N}$ is a non-discrete, Hausdorff topological group, and by Statement 3.1.1, a set $\overline{\mathbf{T}}$ of cardinality of continuum of group topologies exists on the group $G(\cdot)/\mathbf{N}$, in each of which a topological group has a countable basis of the filter of neighborhoods of the unity element and any two of them are comparable.

Since the canonical homomorphism $f : G(\cdot) \to G(\cdot)/\mathbf{N}$ is a surjective map, then $f\big(f^{-1}(\overline{V})\big) = \overline{V}$ for any subset $\overline{V} \subseteq G(\cdot)/\mathbf{N}$.

Let now $\overline{\tau}_1, \overline{\tau}_2 \in \overline{\mathbf{T}}$, and let τ_1 and τ_2 be the prototype topologies $\overline{\tau_1}$ and $\overline{\tau_2}$ with respect to the homomorphism f, respectively.

If $\overline{\tau}_1 \leq \overline{\tau}_2$, then $\tau_1 = \{f^{-1}(\overline{U}) | \overline{U} \in \overline{\tau}_1\} \subseteq \{f^{-1}(\overline{V}) | \overline{V} \in \overline{\tau}_2\} = \tau_2$, and if $\tau_1 \leq \tau_2$, then $\overline{\tau}_1 = \{f\big(f^{-1}(\overline{U})\big) | \overline{U} \in \overline{\tau}_1\} = \{f(U) | U \in \tau_1\} \subseteq \{f^{-1}(V) | V \in \tau_2\} = \{f\big(f^{-1}(\overline{V})\big) | \overline{U} \in \overline{\tau}_2\} = \overline{\tau}_2$, and hence, $\tau_1 \leq \tau_2$ if and only if $\overline{\tau}_1 \leq \overline{\tau}_2$.

Then a set of cardinality of continuum of group topologies exists on the group $G(\cdot)$ in each of which the respective topological group has a countable basis of the filter of neighborhoods of the unity element and any two of them are comparable.

Statement 3.2.2 is proved.

Statement 3.2.3 can be proved by analogy with the proof of Statement 3.2.2 if you use Statement 3.1.3 and the fact that if $sup\{\overline{\tau}_1, \overline{\tau}_2\} = \tau_d^*$ for non-discrete topologies $\overline{\tau}_1$ and $\overline{\tau}_2$, then the topologies τ_1 and τ_2 are incomparable.

This theorem is proved. □

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Invariant transformations of loop transversals. 1a. The case of automorphism

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Abstract. One special class of invariant transformations of loop transversals in groups is investigated. Transformations from this class correspond to arbitrary automorphisms of transversal operations of loop transversals mentioned above.

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Keywords and phrases: Quasigroup, loop, transversal, automorphism.

1 Introduction

The notion of a transversal in a group to its own subgroup is well known and has been studied during the last 70 years (since R. Baer's work [1]). Loop transversals (transversals whose transversal operations are loops) in some fixed groups to their own subgroups present special interest.

This investigation is a continuation and important part of [6]. In the present work we will investigate such transformations of loop transversals which correspond to the most symmetric transformation of transversal operations – to an automorphism. We will use the statements from [6] and obtain the basic results of this work as corollaries.

Let us remember some necessary definitions and preliminary statements.

2 Necessary definitions and statements

Definition 1. A system $< E, \cdot >$ is called a **left (right) quasigroup** if the equation $(a \cdot x = b)$ (the equation $(y \cdot a = b)$) has exactly one solution in the set E for any fixed $a, b \in E$. If for some element $e \in E$ we have

$$e \cdot x = x \cdot e = x \quad \forall x \in E,$$

then a left (right) quasigroup $< E, \cdot, e >$ is called a **left (right) loop** (the element $e \in E$ is called a **unit**). A left quasigroup $< E, \cdot >$ which is simultaneously a right quasigroup is called simply a **quasigroup**. Similarly, a left loop which is simultaneously a right loop is called a **loop**.

Definition 2. Let G be a group and H be its subgroup. Let $\{H_i\}_{i\in E}$ be the set of all left (right) cosets in G to H, and we assume $H_1 = H$. A set $T = \{t_i\}_{i\in E}$ of representativities of the left (right) cosets (by one from each coset H_i and $t_1 = e \in H$) is called a **left (right) transversal** in G to H. If a left transversal T is simultaneously a right one, it is called a **two-sided transversal**.

On any left transversal T in a group G to its subgroup H it is possible to define the following operation (*transversal operation*) :

$$x \overset{(T)}{\cdot} y = z \quad \overset{def}{\Longleftrightarrow} \quad t_x t_y = t_z h,\ h \in H.$$

Definition 3. If a system $< E, \overset{(T)}{\cdot}, 1 >$ is a loop, then such left transversal $T = \{t_x\}_{x\in E}$ is called a **loop transversal**.

At last remind the definitions of a left multiplicative group and of a left inner permutation group of a loop.

Definition 4. Let $< E, \cdot, e >$ be a loop. Then a group

$$LM(< E, \cdot, e >) \overset{def}{=} < L_a \mid a \in E >,$$

generated by all left translations L_a of loop $< E, \cdot, e >$, is called a **left multiplicative group** of the loop $< E, \cdot, e >$. Its subgroup

$$LI(< E, \cdot, e >) \overset{def}{=} < l_{a,b} \mid l_{a,b} = L^{-1}_{a\cdot b} L_a L_b, : a, b \in E >$$

generated by all permutations $l_{a,b}$, is called a **left inner permutation group** of the loop $< E, \cdot, e >$.

Definition 5 (see [2])**.** A mapping $\Phi = (\alpha, \beta, \gamma)$ (α, β, γ are permutations on a set E) of the operation $< E, \cdot >$ on the operation $< E, \circ >$ is called an **isotopy** if

$$\gamma(x \cdot y) = \alpha(x) \circ \beta(y) \quad \forall x, y \in E.$$

If $\Phi = (\gamma, \gamma, \gamma)$, then such an isotopy is called an **isomorphism**. If $\Phi = (\gamma, \gamma, \gamma)$, and $< E, \cdot > = < E, \circ >$ then such an isomorphism is called an **automorphism**.

3 The transformations which correspond to automorphisms of the transversal operations of loop transversals

Let $T = \{t_x\}_{x\in E}$ be a loop transversal in a group G to its subgroup H, and $< E, \overset{(T)}{\cdot}, 1 >$ is its transversal operation. Consider the following group:

$$M_G(T) \overset{def}{=} < \alpha \mid \alpha \in St_1(S_E),\ LM(< E, \overset{(T)}{\cdot}, 1 >) \subseteq \alpha \widehat{G} \alpha^{-1} >,$$

it is generated by all permutations $\alpha \in St_1(S_E)$ which satisfy the condition

$$LM(< E, \overset{(T)}{\cdot}, 1 >) \subseteq \alpha \widehat{G} \alpha^{-1}.$$

Lemma 1. *The following propositions are true:*

1. $N_{St_1(S_E)}(\widehat{G}) \subseteq M_G(T) \subseteq St_1(S_E)$,
2. $M_G(T)$ *is maximal among subgroups* $M \subseteq St_1(S_E)$ *which satisfy the following property:*
$$LM(< E, \overset{(T)}{\cdot}, 1 >) = \bigcap_{\alpha \in M} (\alpha \widehat{G} \alpha^{-1}).$$

Proof. See Lemma 6 from [6]. □

Lemma 2. *Let* $\varphi : E \to E$ *be an automorphism of the loop* $< E, \overset{(T)}{\cdot}, 1 >$ *(note that* $\varphi(1) = 1$*). Then*

1. $\widehat{T} = h_0^{-1} \widehat{T} h_0$ *for some* $h_0 \in H^* = M_G(T)$*;*
2. $\varphi \equiv h_0$ *and* $LI(< E, \overset{(T)}{\cdot}, 1 >) \subseteq h_0 \widehat{H} h_0^{-1}$.

Proof. It is an evident corollary of the Lemma 7 from [6]. □

Lemma 3. *Let* $T = \{t_x\}_{x \in E}$ *be a fixed loop transversal in* G *to* H*. Let* $h_0 \in N_{St_1(S_E)}(H)$ *be an element such that:*

$$t_{x'} \overset{def}{=} h_0^{-1} t_x h_0 \quad \forall x \in E.$$

Then $\varphi \equiv h_0 \in Aut(< E, \overset{(T)}{\cdot}, 1 >)$.

Proof. It is an evident corollary of the Lemma 8 from [6]. □

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