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Redefined fuzzy Lie subalgebras

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Abstract

This paper introduces a new concept of a Lie subalgebra of a Lie algebra using the notion of an anti fuzzy point and its besideness to and non-quasi-coincidence with a fuzzy set, and presents some of its useful properties.

1. Introduction

The theory of Lie algebras is an area of mathematics in which we can see a harmonious between the methods of classical analysis and modern algebra. This theory, a direct outgrowth of a central problem in the calculus, has today become a synthesis of many separate disciplines, each of which has left its own mark. Theory of Lie groups were developed by the Norwegian mathematician Sophus Lie in the late nineteenth century in connection with his work on systems of differential equations. Lie algebras were also discovered by Sophus Lie when he first attempted to classify certain smooth subgroups of general linear groups. The groups he considered are called Lie groups. The importance of Lie algebras for applied mathematics and for applied physics has also become increasingly evident in recent years. In applied mathematics, Lie theory remains a powerful tool for studying differential equations, special functions and perturbation theory. Lie theory finds applications not only in elementary particle physics and nuclear physics, but also in such diverse fields as continuum mechanics, solid-state physics, cosmology and control theory. Lie algebra is also used by electrical engineers, mainly in the mobile robot control. For the basic information of Lie algebras, the readers are refereed to [7, 12, 17].

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In 1965, Zadeh [26] introduced the notion of a fuzzy subset of a set as a method for representing uncertainty. Since then it has become a vigorous area of research in different domains such as engineering, medical science, social science, physics, statistics, graph theory, artificial intelligence, signal processing, multiagent systems, pattern recognition, robotics, computer networks, expert systems, decision making, automata theory. Yehia introduced the notions of fuzzy ideals and fuzzy subalgebras of Lie algebras in [24] and studied some results. Since then, the concepts and results of Lie algebras have been broadened to the fuzzy setting frames (see, [1, 2, 3, 4, 6, 13, 14, 18, 20, 21, 24]).

This paper introduces a new concept of a subalgebra of a Lie algebra using the notion of an anti fuzzy point and its besideness to and non-quasi-coincidence with a fuzzy set, and presents some of its useful properties.

2. Preliminaries

A *Lie algebra* is a vector space L over a field F (equal to $\mathbf{R}$ or $\mathbf{C}$) on which is defined the multiplication $L \times L \to L$, denoted by $(x, y) \to [x, y]$, satisfying the following axioms:

(L_1) $[x, y]$ is bilinear,

(L_2) $[x, x] = 0$ for all $x \in L$,

(L_3) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in L$ (Jacobi identity).

In this paper by L will be denoted a Lie algebra. We note that the multiplication in a Lie algebra is not associative, but it is *anti commutative*, i.e., $[x, y] = -[y, x]$ for all $x, y \in L$. A subspace H of L closed under $[\cdot, \cdot]$ will be called a *Lie subalgebra*.

Definition 2.1. A *fuzzy set* ν on L, i.e., a real mapping $\nu : L \to R$ such that $0 \leqslant \nu(x) \leqslant 1$ for all $x \in L$, is called an *anti fuzzy Lie subalgebra* of L if

(I) $\nu(x + y) \leqslant \max\{\nu(x), \nu(y)\}$,

(II) $\nu(\alpha x) \leqslant \nu(x)$,

(III) $\nu([x, y]) \leqslant \min\{\nu(x), \nu(y)\}$

hold for all $x, y \in L$ and $\alpha \in F$.

As a consequence of the Transfer Principle proved in [22] we obtain

Theorem 2.2. *Let ν be a fuzzy set on L. Then ν is a fuzzy Lie subalgebra of L if and only if*
$$L(\nu;t) = \{x \in L : \nu(x) \leqslant t\}$$
is a Lie subalgebra of L for all $t \in (0,1]$. □

The membership degree expresses the degree of belongingness of elements to a fuzzy set. The membership degree 1 denote that an element completely belongs to its corresponding fuzzy set, and the membership degree 0 denote that an element does not belong to the fuzzy set. The membership degrees on the interval $(0,1)$ denote the partial membership to the fuzzy set. Sometimes, the membership degree means the satisfaction degree of elements to some property or constraint corresponding to a fuzzy set. The membership degrees on the interval $(0,1]$ denote that elements somewhat satisfy the property.

A fuzzy set ν on L of the form

$$\nu(y) = \begin{cases} t \in [0,1) & \text{if } y = x, \\ 1, & \text{if } y \neq x \end{cases}$$

is called an *anti fuzzy point* with support x and value t and is denoted by x_t. A fuzzy set ν in L is said to be *non-unit* if there exists $x \in L$ such that $\nu(x) < 1$. An anti fuzzy point x_t is said to "*besides to*" a fuzzy set ν, written as $x_t \prec \nu$ if $\nu(x) \leqslant t$. An anti fuzzy point x_t is said to be "*non-quasicoincident with*" a fuzzy set ν, denoted by $x_t \vdash \nu$ if $\nu(x) + t \leqslant 1$.

3. Redefined fuzzy Lie subalgebras

Let α and β denote one of the symbols $\prec$, $\vdash$, $\prec\vee\vdash$ or $\prec\wedge\vdash$ unless otherwise specified.

Definition 3.1. A fuzzy set ν in L is called an $(\alpha,\beta)^*$-*fuzzy Lie subalgebra* of L if it satisfies the following conditions:

(1) $x_s\alpha\nu, y_t\alpha\nu \Rightarrow (x+y)_{\max(s,t)}\beta\nu$,

(2) $x_s\alpha\nu \Rightarrow (mx)_s\beta\nu$,

(3) $x_s\alpha\nu, y_t\alpha\nu \Rightarrow ([x,y])_{\min(s,t)}\beta\nu$

for all x, $y \in L$, $m \in F$, s, $t \in [0,1)$.

Notations: The following notations will be used:

- "$x_t \prec \nu$" and "$x_t \vdash \nu$" will be denoted by $x_t \prec\wedge\vdash \nu$.
- "$x_t \prec \nu$" or "$x_t \vdash \nu$" will be denoted by $x_t \prec\vee\vdash \nu$.
- The symbol $\overline{\prec\wedge\vdash}$ means neither $\prec$ nor $\vdash$ hold.

Remark. If ν is a fuzzy set in L such that $\nu(x) \geqslant 0.5$ for all $x \in L$. Then $\{x_t | x_t \prec\wedge\vdash \mu\} = \emptyset$.

The proof of the following proposition is trivial.

Proposition 3.2. *For any fuzzy set ν in L, Definition 2.1 is equivalent to the following conditions:*

(4) $x_s, y_t \prec \nu \Rightarrow (x+y)_{\max(s,t)} \prec \nu$,

(5) $x_s \prec \nu \Rightarrow (mx)_s \prec \nu$,

(6) $x_s, y_t \prec \nu \Rightarrow ([x,y])_{\min(s,t)} \prec \nu$,

for all $x, y \in L$, $m \in F$, $s, t \in [0,1)$. □

For a fuzzy set ν in a Lie algebra L, we denote $L^* = \{x \in L : \nu(x) < 1\}$.

Proposition 3.3. *If ν is a non-unit $(\prec, \prec)^*$-fuzzy Lie subalgebra of L, then L^* is a Lie subalgebra of L.*

Proof. Let $x, y \in L^*$. Then $\nu(x) < 1$ and $\nu(y) < 1$.

(1) Assume $\nu(x+y) = 1$. Then we can see that $x_{\nu(x)} \prec \nu$ and $y_{\nu(y)} \prec \nu$, but $(x+y)_{\max(\nu(x),\nu(y))} \overline{\prec} \nu$ since $\nu(x+y) = 1 > \max(\nu(x), \nu(y))$. This is clearly a contradiction, and hence $\nu(x+y) < 1$, which shows that $x+y \in L^*$.

(2) Assume $\nu(mx) = 1$. Then we can see that $x_{\nu(x)} \prec \nu$, but $(mx)_{\nu(x)} \overline{\prec} \nu$ since $\nu(mx) = 1 > \nu(x)$. This is clearly a contradiction, and hence $\nu(mx) < 1$, which shows that $mx \in L^*$.

(3) Assume $\nu([x,y]) = 1$. Then we can see that $x_{\nu(x)} \prec \nu$ and $y_{\nu(y)} \prec \nu$, but $([x,y])_{\min(\nu(x),\nu(y))} \overline{\prec} \nu$ since $\nu([x,y]) = 1 > \min(\nu(x), \nu(y))$. This is clearly a contradiction, and hence $\nu([x,y]) < 1$, which shows that $[x,y] \in L^*$. Hence L^* is a Lie subalgebra of L. □

Proposition 3.4. *If ν is a non-unit $(\prec, \vdash)^*$-fuzzy Lie subalgebra of L, then the set L^* is a Lie subalgebra of L.*

Proof. Let x, $y \in L^*$. Then $\nu(x) < 1$ and $\nu(y) < 1$.

(1) Suppose that $\nu(x+y) = 1$, then

$$\nu(x+y) + \max(\nu(x), \nu(y)) \geqslant 1.$$

Hence $(x+y)_{\max(\nu(x),\nu(y))}\overline{\vdash}\nu$, which is a contradiction since $x_{\nu(x)} \prec \nu$ and $y_{\nu(y)} \prec \nu$. Thus $\nu(x+y) < 1$, so $x+y \in L^*$.

(2) Suppose that $\nu(mx) = 1$, then

$$\nu(mx) + \nu(x) \geqslant 1.$$

Hence $mx_{\nu(x)}\overline{\vdash}\nu$, a contradiction since $x_{\nu(x)} \prec \nu$. Thus $\nu(mx) < 1$, so $mx \in L^*$.

(3) Suppose that $\nu([x,y]) = 1$, then

$$\nu([x,y]) + \min(\nu(x),\nu(y)) \geqslant 1.$$

Hence $[x,y]_{\min(\nu(x),\nu(y))}\overline{\vdash}\nu$, which is a contradiction since $x_{\nu(x)} \prec \nu$ and $y_{\nu(y)} \prec \nu$. Thus $\nu([x,y]) < 1$, so $[x,y] \in L^*$. Hence L^* is a Lie subalgebra of L. □

Proposition 3.5. *If ν is a non-unit $(\vdash,\prec)^*$-fuzzy Lie subalgebra of L, then L^* is a Lie subalgebra of L.*

Proof. Let $x,y \in L^*$. Then $\nu(x) < 1$ and $\nu(y) < 1$. Thus $x_0 \vdash \nu$ and $y_0 \vdash \nu$.

(1) If $\nu(x+y) = 1$, then $\nu(x+y) = 1 > 0 = \max(0,0)$. Therefore, $(x+y)_{\max(0,0)}\overline{\prec}\nu$, which is a contradiction. It follows that $\nu(x+y) < 1$ so that $x+y \in L^*$.

(2) If $\nu(mx) = 1$, then $\nu(mx) = 1 > 0$. Therefore, $mx_0\overline{\prec}\nu$, a contradiction. It follows that $\nu(mx) < 1$ so that $mx \in L^*$.

(3) If $\nu([x,y]) = 1$, then $\nu([x,y]) = 1 > 0 = \min(0,0)$. Therefore, $[x,y]_{\min(0,0)}\overline{\prec}\nu$, which is a contradiction. It follows that $\nu([x,y]) < 1$ so that $[x,y] \in L^*$. Hence L^* is a Lie subalgebra of L. □

Proposition 3.6. *If ν is a non-unit $(\vdash,\vdash)^*$-fuzzy Lie subalgebra of L, then L^* is a Lie subalgebra of L.*

Proof. Let $x,y \in L^*$. Then $\nu(x) < 1$ and $\nu(y) < 1$.

(1) If $\nu(x+y) = 1$, then $\nu(x+y)+\max(0,0) = 1$, and so $(x+y)_{\max(0,0)}\overline{\vdash}\nu$. This is impossible, and hence $\nu(x+y) < 1$, i.e., $x+y \in L^*$.

(2) If $\nu(mx) = 1$, then $\nu(mx) + 0 = 1$, and so $(mx)_0\overline{\vdash}\nu$. This is impossible, and hence $\nu(mx) < 1$, i.e., $mx \in L^*$.

(3) If $\nu([x,y]) = 1$, then $\nu([x,y])+\min(0,0) = 1$, and so $[x,y]_{\min(0,0)}\overline{\vdash}\nu$. This is impossible, and hence $\nu([x,y]) < 1$, i.e., $[x,y] \in L^*$. Hence L^* is a Lie subalgebra of L. □

Proposition 3.7. *If ν is a non-unit $(\prec, \prec\vee\vdash)^*$-fuzzy Lie subalgebra of L, then L^* is a Lie subalgebra of L.*

Proof. Let x, $y\in L^*$. Then $\nu(x) < 1$ and $\nu(y) < 1$. Thus $\nu(x) = s_1$ and $\nu(y) = s_1$ for some s_1, $s_2\in [0, 1)$. It follow that $x_{s_1} \prec \nu$ and $y_{s_2} \prec \nu$ so that $(x+y)_{\max(s_1,s_2)} \prec\vee\vdash \nu$, i.e., $(x+y)_{\max(s_1,s_2)} \prec \nu$ or $(x+y)_{\max(s_1,s_2)} \vdash \nu$. If $(x+y)_{\max(s_1,s_2)} \prec \nu$, then $\nu(x+y) \leqslant \max(s_1, s_2) < 1$ and hence $x+y \in L^*$. On the other hand, If $(x + y)_{\max(s_1,s_2)} \vdash \nu$, then $\nu(x + y) \leqslant \nu(x + y) + \max(s_1, s_2) < 1$, and hence $x + y \in L^*$. Verification of conditions (2) and (3) in Definition 3.1 is similar, we omit the details. $\square$

By using similar argumentations we can also prove the following two propositions.

Proposition 3.8. *If ν is a non-unit $(\vdash, \prec \vee \vdash)^*$-fuzzy Lie subalgebra of L, then L^* is a Lie subalgebra of L.* $\square$

Proposition 3.9. *If ν is a non-unit $(\prec, \prec\wedge\vdash)^*$-,$(\prec\vee\vdash, \vdash)^*$-, $(\prec\vee\vdash, \prec)^*$-, $(\prec\vee\vdash, \prec\wedge\vdash)^*$-, $(\vdash, \prec\wedge\vdash)^*$-, or $(\prec\vee\vdash, \prec \vee \vdash)^*$-fuzzy Lie subalgebra of L, then L^* is a Lie subalgebra of L.* $\square$

Definition 3.10. A fuzzy set ν in L is called an $(\prec, \prec \vee \vdash)^*-$*fuzzy Lie subalgebra* of L if the following conditions are satisfied:

(a) $x_s, y_t \prec \nu \Rightarrow (x + y)_{\max(s,t)} \prec\vee\vdash \nu$,

(b) $x_s \prec \nu \Rightarrow (mx)_s \prec\vee\vdash \nu$,

(c) $x_s, y_t \prec \nu \Rightarrow ([x, y])_{\min(s,t)} \prec\vee\vdash \nu$

for all $x, y \in L$, $m \in F$, $s, t \in [0, 1)$.

Example 3.11. Let V be a vector space over a field F such that $dim(V) = 5$. Let $V = \{e_1, e_2, \ldots, e_5\}$ be a basis of a vector space over a field F with Lie brackets as follows:

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = e_5, \quad [e_1, e_4] = e_5, \quad [e_1, e_5] = 0,$$

$$[e_2, e_3] = e_5, \quad [e_2, e_4] = 0, \quad [e_2, e_5] = 0, \quad [e_3, e_4] = 0,$$

$$[e_3, e_5] = 0, \quad [e_4, e_5] = 0, \quad [e_i, e_j] = -[e_j, e_i]$$

and $[e_i, e_j] = 0$ for all $i = j$. Then V is a Lie algebra over F. We define a fuzzy set $\nu : V \to [0, 1]$ by

$$\nu(x) := \begin{cases} 0.25 & \text{if } x = 0, \\ 0.46 & \text{if } x \in \{e_3, e_5\}, \\ 0 & \text{if } x \in \{e_1, e_2, e_4\}. \end{cases}$$

By routine computations, it is easy to see that ν is an $(\prec, \prec\vee\vdash)^*$-fuzzy Lie subalgebra of L.

Theorem 3.12. *Let ν be a fuzzy set in a Lie algebra L. Then ν is an $(\prec, \prec\vee\vdash)^*-$fuzzy Lie subalgebra of L if and only if*

(d) $\nu(x+y) \leqslant \max(\nu(x), \nu(y), 0.5)$,

(e) $\nu(mx) \leqslant \max(\nu(x), 0.5)$,

(f) $\nu([x,y]) \leqslant \min(\nu(x), \nu(y), 0.5)$

hold for all $x, y \in L$, $m \in F$.

Proof. $(a) \Rightarrow (d)$: Let $x, y \in L$. We consider the following two cases:

(1) $\max(\nu(x), \nu(y)) > 0.5$,

(2) $\max(\nu(x, \nu(y)) \leqslant 0.5$.

Case (1): Assume that $\nu(x+y) > \max(\nu(x), \nu(y), 0.5)$ Then $\nu(x+y) > \max(\nu(x), \nu(y))$. Take s such that $\nu(x+y) > s > \max(\nu(x), \nu(y))$. Then $x_s \prec \nu$, $y_s \prec \nu$, but $(x+y)_s \overline{\prec\vee\vdash}\nu$, which is contradiction with (a).

Case (2): Assume that $\nu(x+y) > 0.5$. Then $x_{0.5}$, $y_{0.5} \prec \nu$ but $(x+y)_{0.5} \overline{\prec\vee\vdash}\nu$, a contradiction. Hence (d) holds.

$(d) \Rightarrow (a)$: Let x_s, $y_t \prec \nu$, then $\nu(x) \leqslant s$, $\nu(y) \leqslant t$. Now, we have

$$\nu(x+y) \leqslant \max(\nu(x), \nu(y), 0.5) \leqslant \max(s, t, 0.5).$$

If $\max(s,t) < 0.5$, then $\nu(x+y) \leqslant 0.5 \Rightarrow \nu(x+y) + \max(s,t) < 1$. On the other hand, if $\max(s,t) \geqslant 0.5$, then $\nu(x+y) \leqslant \max(s,t)$. Hence $(x+y)_{\max(s,t)} \prec\vee\vdash \nu$.

The verification of $(b) \Leftrightarrow (e)$ and $(c) \Leftrightarrow (f)$ is analogous and we omit the details. This completes the proof. □

Theorem 3.13. *Let ν be an $(\prec, \prec\vee\vdash)^*$-fuzzy Lie subalgebra of L.*

(i) *If there exists $x \in L$ such that $\nu(x) \leqslant 0.5$, then $\nu(0) \leqslant 0.5$.*

(ii) *If $\nu(0) > 0.5$, then ν is an $(\prec, \prec)^*$-fuzzy Lie subalgebra of L.*

Proof. (i) Let $x \in L$ such that $\nu(x) \leqslant 0.5$. Then $\nu(-x) = \max(\nu(x), 0.5) = 0.5$. Hence $\nu(0) = \nu(x-x) \leqslant \max(\nu(x), \nu(-x), 0.5) = 0.5$.

(ii) If $\nu(0) > 0.5$ then $\nu(x) > 0.5$ for all $x \in L$. Thus we conclude that $\nu(x+y) \leqslant \max(\nu(x), \nu(y))$, $\nu(mx) \leqslant \nu(x)$, $\nu([x,y]) \leqslant \min(\nu(x), \nu(y))$ for all $x, y \in L$, $m \in F$. Hence ν is an $(\prec, \prec)^*$-fuzzy Lie subalgebra of L. □

Theorem 3.14. *Let ν be a fuzzy set of fuzzy Lie subalgebra of L. Then ν is an $(\prec, \prec \vee \vdash)^*$-fuzzy Li subalgebra of L if and only if each nonempty $L(\nu; t)$, $t \in [0.5, 1)$ is a Lie subalgebra of L.*

Proof. Assume that ν is an $(\prec, \prec \vee \vdash)^*$ fuzzy Lie subalgebra of L and let $t \in [0.5, 1)$. If $x, y \in L(\nu; t)$ and $m \in F$, then $\nu(x) \leq t$ and $\nu(y) \leq t$. Thus,

$$\nu(x + y) \leqslant \max(\nu(x), \nu(y), 0.5) \leqslant \max(t, 0.5) = t,$$

$$\nu(mx) \leqslant \max(\nu(x), 0.5) \leqslant \max(t, 0.5) = t,$$

$$\nu([x, y]) \leqslant \max(\nu(x), \nu(y), 0.5) \leqslant \max(t, 0.5) = t,$$

and so $x+y, mx, [x, y] \in L(\nu; t)$. This shows that $L(\nu; t)$ is a Lie subalgebra of L.

Conversely, let ν be a fuzzy set such that $L(\nu; t)$ is a Lie subalgebra of L, for all $t \in [0.5, 1)$. If there exist $x, y \in L$ such that such that $\nu(x + y) > \max(\nu(x), \nu(y), 0.5)$, then we can take $t \in (0, 1)$ such that

$$\nu(u + v) > t > \max(\nu(x), \nu(y), 0.5).$$

Thus x, $y \in L(\nu; t)$ and $t > 0.5$, and so $x + y \notin L(\nu; t)$, which contradicts to the assumption that all $L(\nu; t)$ are Lie ideals. Therefore,

$$\nu(x + y) \leqslant \max(\nu(x), \nu(y), 0.5).$$

The verification is analogous for other conditions and we omit the details. Hence ν is an $(\prec, \prec\vee\vdash)^*$ fuzzy Lie subalgebra of L. □

Theorem 3.15. *Let ν be a fuzzy set in a Lie algebra L. Then $L(\nu; t)$ is a Lie subalgebra of L if and only if*

(g) $\min(\nu(x + y), 0.5) \leqslant \max(\nu(x), \nu(y))$,

(h) $\min(\nu(mx), 0.5) \leqslant \nu(x)$,

(i) $\min(\nu([x, y]), 0.5)) \leqslant \max(\nu(x), \nu(y))$

for all $x, y \in L$, $m \in F$.

Proof. Suppose that $L(\nu; t)$ is a Lie subalgebra of L. Let $\min(\nu(x+y), 0.5) > \max(\nu(x), \nu(y)) = t$ for some $x, y \in L$, then $t \in [0.5, 1)$, $\nu(x + y) > t$, $x \prec L(\nu; t)$ and $y \prec L(\nu; t)$. Since $x, y \prec L(\nu; t)$ and $L(\nu; t)$ is a Lie subalgebra of L, so $x + y \prec L(\nu; t)$ or $\nu(x + y) \leqslant t$, which is contradiction with $\nu(x + y) > t$. Hence (d) holds. For (e), (f) the verification is analogous.

Conversely, suppose that (d), (e) and (f) hold. Assume that $t \in [0.5, 1)$, $x, y \prec L(\nu; t)$. Then

$$0.5 > t \geqslant \max(\nu(x), \nu(y)) \geqslant \min(\nu(x+y), 0.5) \Rightarrow \nu(x+y) \leqslant t,$$

$$0.5 > t \geqslant \nu(x) \geqslant \min(\nu(mx), 0.5) \Rightarrow \nu(mx) \leqslant t,$$

$$0.5 > t \geqslant \max(\nu(x), \nu(y)) \geqslant \min(\nu([x, y], 0.5) \Rightarrow \nu([x, y]) \leqslant t,$$

and so $x+y \prec L(\nu; t)$, $mx \prec L(\nu; t)$,$[x, y] \prec L(\nu; t)$. This shows that $L(\nu; t)$ is a Lie subalgebra of L. □

Definition 3.16. An $(\prec, \prec \vee \vdash)^*$-fuzzy Lie subalgebra of L is called *proper* if Im ν has at least two elements. Two $(\prec, \prec \vee \vdash)^*$- fuzzy Lie subalgebras ν_1 and ν_2 are said to be *equivalent* if they have the same family of level Lie subalgebras.

Theorem 3.17. *Any proper $(\prec, \prec \vee \vdash)^*$-fuzzy Lie subalgebra of L for which the cardinality of $\{\nu(x) : \nu(x) > 0.5\} \leqslant 2$ can be expressed as the union of two proper non-equivalent $(\prec, \prec \vee \vdash)^*$-fuzzy Lie subalgebras of L.*

Proof. Let ν be a proper $(\prec, \prec \vee \vdash)^*$-fuzzy Lie subalgebra of L such that $\{\nu(x) : \nu(x) > 0.5\} = \{t_1, t_2, \ldots, t_n\}$ where $t_1 < t_2 < \ldots < t_n$ and $n \geqslant 2$. Then

$$\nu_{0.5} \subseteq \nu_{t_1} \subseteq \ldots \subseteq \nu_{t_n} = L$$

is the chain of $(\prec, \prec \vee \vdash)^*$-Lie subalgebras of ν. Define μ_1 and μ_2 by

$$\mu_1(x) = \begin{cases} t_1, & \text{if } x \in \nu_{t_1}, \\ t_2, & \text{if } x \in \nu_{t_2} \setminus \nu_{t_1}, \\ \vdots & \\ t_n, & \text{if } x \in \nu_{t_n} \setminus \nu_{t_{n-1}}, \end{cases}$$

$$\mu_2(x) = \begin{cases} \nu(x), & \text{if } x \in \nu_{0.5}, \\ n, & \text{if } x \in \nu_{t_2} \setminus \nu_{0.5}, \\ t_3, & \text{if } x \in \nu_{t_3} \setminus \nu_{t_2}, \\ \vdots & \\ t_n, & \text{if } x \in \nu_{t_n} \setminus \nu_{t_{n-1}}, \end{cases}$$

respectively, where $t_3 > n > t_2$. Then μ_1 and μ_2 are $(\prec, \prec \vee \vdash)^*$-fuzzy Lie subalgebras of L with

$$\nu_{t_1} \subseteq \nu_{t_2} \subseteq \ldots \subseteq \nu_{t_n}$$

and

$$\nu_{t_{0.5}} \subseteq \nu_{t_2} \subseteq \ldots \subseteq \nu_{t_n}$$

being respectively chains of $(\prec, \prec \vee \vdash)^*$-fuzzy Lie subalgebras of μ_1 and μ_2.

Hence ν can be expressed as the union of two proper non-equivalent $(\prec, \prec \vee \vdash)^*$-fuzzy subalgebras of L. $\square$

Theorem 3.18. *Let $\{\nu_i : i \in \Lambda\}$ be a family of $(\prec, \prec)^*$-fuzzy Lie subalgebras of L. Then $\nu = \bigcup_{i\in\Lambda} \nu_i$ is an $(\prec, \prec)^*$-fuzzy Lie subalgebra of L.*

Proof. Let $x_s \prec \nu$ and $y_t \prec \nu$, where $s, t \in [0, 1)$. Then $\nu(x) \leqslant s$ and $\nu(y) \leqslant t$. Thus we have $\nu_i(x) \leqslant s$ and $\nu_i(y) \leqslant t$ for all $i \in \Lambda$. Hence $\nu_i(x + y) \leqslant \max(s, t)$. Therefore, $\nu(x + y) \leqslant \max(s, t)$, which implies that $(x + y)_{\max\{s,t\}} \prec \nu$. For other conditions the verification is analogous. $\square$

Theorem 3.19. *Let $\{\nu_i : i \in \Lambda\}$ be a family of $(\prec, \prec \vee \vdash)^*$-fuzzy Lie subalgebra of L. Then $\nu := \bigcap_{i\in\Lambda} \nu_i$ is an $(\prec, \prec \vee \vdash)^*$-fuzzy Lie subalgebra of L.*

Proof. By Theorem 3.12, we have $\nu(x+y) \leqslant \max(\nu(x), \nu(y), 0.5)$, and hence

$$\begin{aligned}
\nu(x + y) &= \inf_{i\in\Lambda} \nu_i(x + y)\\
&\leqslant \inf_{i\in\Lambda} \max(\nu_i(x), \nu_i(y), 0.5)\\
&= \max(\inf_{i\in\Lambda} \nu_i(x), \inf_{i\in\Lambda} \nu_i(y), 0.5)\\
&= \max(\textstyle\bigcap_{i\in\Lambda} \nu_i(x), \bigcap_{i\in\Lambda} \nu_i(y), 0.5)\\
&= \max(\nu(x), \nu(y), 0.5).
\end{aligned}$$

For other conditions the verification is analogous. By Theorem 3.12, it follows that ν is an $(\prec, \prec\vee\vdash)^*$-fuzzy Lie subalgebra of L. $\square$

Remark. Let $\{\nu_i : i \in \Lambda\}$ be a family of $(\prec, \prec\vee\vdash)^*$-fuzzy Lie subalgebras of L. Is $\nu = \bigcup_{i\in\Lambda} \nu_i$ an $(\prec, \prec\vee\vdash)^*$-fuzzy Lie subalgebra of L? When? The following example shows that it is not an $(\prec, \prec\vee\vdash)^*$-fuzzy Lie subalgebra in general.

Example 3.20. Let V be a vector space over a field F such that $dim(V) = 5$. Let $V = \{e_1, e_2, e_3, e_4, e_5\}$ be its basis and let Lie brackets will be defined as in Example 3.11. If we define fuzzy sets $\mu_1, \mu_2 : V \to [0, 1]$ by putting

$$\mu_1(x) := \begin{cases} 0.6 & \text{if } x = 0,\\ 1 & \text{if } x \in \{e_3, e_5\},\\ 0 & \text{if } x \in \{e_1, e_2, e_4\},\end{cases}$$

$$\mu_2(x) := \begin{cases} 0.3 & \text{if } x = 0, \\ 1 & \text{if } x \in \{e_3, e_5\}, \\ 0 & \text{if } x \in \{e_1, e_2, e_4\}, \end{cases}$$

then both μ_1 and μ_2 will be $(\prec, \prec \vee \vdash)^*$-fuzzy Lie subalgebras of L, but $\mu_1 \cup \mu_2$ is not an $(\prec, \prec\vee\vdash)^*$-fuzzy Lie subalgebra of L since

$$\begin{aligned} 1 = \max(\mu_1(e_3), \mu_2(e_3)) &= (\mu_1 \cup \mu_2)(e_3) = (\mu_1 \cup \mu_2)([e_1, e_2]) \\ &\leqslant \min((\mu_1 \cup \mu_2)(e_1), (\mu_1 \cup \mu_2)(e_2), 0.5) = \min(0, 0, 0.5) = 0. \end{aligned}$$

Theorem 3.21. *Let $\{\nu_i : i \in \Lambda\}$ be a family of $(\prec, \prec \vee \vdash)^*$-fuzzy Lie subalgebras of L such that $\nu_i \subseteq \nu_j$ or $\nu_j \subseteq \nu_i$ for all $i, j \in \Lambda$. Then the fuzzy set $\nu := \bigcup_{i\in\Lambda} \nu_i$ is an $(\prec, \prec\vee\vdash)^*$-fuzzy Lie subalgebra of L.*

Proof. By Theorem 3.12, we have $\nu(x+y) \leqslant \max(\nu(x), \nu(y), 0.5)$, and hence

$$\begin{aligned} \nu(x+y) &= \sup\nolimits_{i\in\Lambda} \nu_i(x+y) \\ &\leqslant \sup\nolimits_{i\in\Lambda} \max(\nu_i(x), \nu_i(y), 0.5) \\ &= \max(\sup\nolimits_{i\in\Lambda} \nu_i(x), \sup\nolimits_{i\in\Lambda} \nu_i(y), 0.5) \\ &= \max(\textstyle\bigcup_{i\in\Lambda} \nu_i(x), \bigcup_{i\in\Lambda} \nu_i(y), 0.5) \\ &= \max(\nu(x), \nu(y), 0.5). \end{aligned}$$

It is easy to see that

$$\sup_{i\in\Lambda} \max(\nu_i(x), \nu_i(y), 0.5) \geqslant \bigcup_{i\in\Lambda} \max(\nu_i(x), \nu_i(y), 0.5).$$

Suppose that

$$\sup_{i\in\Lambda} \max(\nu_i(x), \nu_i(y), 0.5) \neq \bigcup_{i\in\Lambda} \max(\nu_i(x), \nu_i(y), 0.5),$$

then there exists s such that

$$\sup_{i\in\Lambda} \max(\nu_i(x), \nu_i(y), 0.5) > s > \bigcup_{i\in\Lambda} \max(\nu_i(x), \nu_i(y), 0.5).$$

Since $\nu_i \subseteq \nu_j$ or $\nu_j \subseteq \nu_i$ for all $i, j \in \Lambda$, there exists $k \in \Lambda$ such that $s > \max(\nu_k(x), \nu_k(y), 0.5)$. On the other hand, $\max(\nu_i(x), \nu_i(y), 0.5) > s$ for all $i \in \Lambda$, a contradiction. Hence

$$\begin{aligned} \sup\nolimits_{i\in\Lambda} \max\{\nu_i(x), \nu_i(y), 0.5\} &= \max(\textstyle\bigcup_{i\in\Lambda} \nu_i(x), \bigcup_{i\in\Lambda} \nu_i(y), 0.5) \\ &= \max\{\nu(x), \nu(y), 0.5\}. \end{aligned}$$

The verification of other conditions is analogous. By Theorem 3.12, it follows that ν is an $(\prec, \prec\vee\vdash)^*$-fuzzy Lie subalgebra of L. $\square$

Finally we study anti fuzzy Lie subalgebras with thresholds.

Definition 3.22. Let $m_1, m_2 \in [0,1]$ and $m_1 < m_2$. If ν is a fuzzy set of a Lie algebra L, then ν is called an *anti fuzzy Lie subalgebra with thresholds* (m_1, m_2) if

(1) $\min(\nu(x+y), m_1) \leqslant \max(\nu(x), \nu(y), m_2)$,

(2) $\min(\nu(mx), m_1) \leqslant \max(\nu(x), m_2)$,

(3) $\min(\nu([x,y]), m_1) \leqslant \max(\nu(x), \nu(y), m_2)$

for all $x, y \in L$, $m \in F$.

Theorem 3.23. *A fuzzy set ν of Lie algebra L is an anti fuzzy Lie subalgebra with thresholds (m_1, m_2) of L if and only if $L(\nu; t)(\neq \emptyset)$, for any $t \in (m_1, m_2]$, is a Lie subalgebra of L.*

Proof. Assume that ν is an anti fuzzy Lie subalgebra with thresholds (m_1, m_2) of L. Let $x, y \in L(\nu; t)$. Then $\nu(x) \leqslant t$ and $\nu(y) \leqslant t$, $t \in (m_1, m_2]$. Then it follows that

$$\min(\nu(x+y), m_1) \leqslant \max(\nu(x), \nu(y), m_2) = t \Longrightarrow \nu(x+y) \leqslant t,$$

$$\min(\nu(mx), m_1) \leqslant \max(\nu(x), m_2) = t \Longrightarrow \nu(mx) \leqslant t,$$

$$\min(\nu([x,y]), m_1) \leqslant \min(\nu(x), \nu(y), m_2) = t \Longrightarrow \nu([x,y]) \leqslant t,$$

and hence $x+y, mx, [x,y] \in L(\nu; t)$. This shows that $L(\nu; t)$ is a Lie subalgebra of L.

Conversely, assume that ν is a fuzzy set such that $L(\nu; t) \neq \emptyset$ is a Lie subalgebra of L for $m_1, m_2 \in [0,1]$ and $m_1 < m_2$. Suppose that $\min(\nu(x+y), m_1) > \max(\nu(x), \nu(y), m_2) = t$, then $\nu(x+y) > t$, $x \in L(\nu; t)$, $y \in L(\nu; t)$, $t \in (m_1, m_2]$. Since $x, y \in L(\nu; t)$ and $L(\nu; t)$ are Lie subalgebras, $x + y \in L(\nu; t)$, i.e., $\nu(x+y) \leqslant t$. This is a contradiction. Therefore condition (1) holds. The verification of (2) and (3) is analogous. $\square$

Remark. By Definition 3.22, we have the following result: If ν is an anti fuzzy subalgebra with thresholds (m_1, m_2), then we can conclude that: ν is an anti fuzzy subalgebra when $m_1 = 0$ and $m_2 = 1$; ν is an $(\prec, \prec\vee\vdash)^*$-fuzzy Lie subalgebra when $m_1 = 0.5$ and $m_2 = 1$.

By Definition 3.22, one can define other anti fuzzy subalgebra of L, such as $[0.2, 0.6)$-fuzzy subalgebra of L, $[0.3, 0.8)$- fuzzy subalgebra of L.

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Generalized fuzzy subquasigroups

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Abstract

Different types of (α,β)-fuzzy subquasigroups, for $\alpha,\beta \in \{\in, q, \in\vee q, \in\wedge q\}$, $\alpha \neq\in \wedge q$, are investigated. Various characterizations of $(\in,\in \vee q)$-fuzzy subquasigroups are obtained. Fuzzy subquasigroups with thresholds are studied also.

1. Introduction

During the last decade, there have been many applications of quasigroups in different areas, such as cryptography [13], modern physics [15], coding theory, geometry [14]. In 1965, Zadeh introduced the notion of a fuzzy subset as a method for representing uncertainty. Since than fuzzy set theory has been developed in many directions by many scholars and has evoked great interest among mathematicians working in different fields of mathematics such as topological spaces, functional analysis, loops, groups, rings, semirings, hemirings, nearrings, vector spaces, differential equations, automation. The notion of fuzzy subgroup was made by Rosenfeld [1] in 1971. Das [5] characterized fuzzy subgroups by their level subgroups. The concept of quasi-coincidence of a fuzzy point with a fuzzy subset was introduced by Pu and Liu [17]. Using the idea of quasi-coincidence of a fuzzy point with a fuzzy subset, Bhakat and Das defined in [4] different types of fuzzy subgroups called, (α,β)-fuzzy subgroups. In particular, they introduced the concept of $(\in,\in \vee q)$-fuzzy subgroups which was an important and useful generalization of Rosenfeld's fuzzy subgroups. Dudek [7] introduced the notion of fuzzy subquasigroups and studied some their properties.

In this paper we introduce the notion of (α,β)- fuzzy subquasigroups where α, $\beta \in \{\in, q, \in\vee q, \in\wedge q\}$ and $\alpha \neq\in\wedge q$, and investigate some related properties. We characterize $(\in,\in\vee q)$- fuzzy subquasigroups by their levels subquasigroups. Finally we study fuzzy subquasigroups with thresholds.

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2. Preliminaries

In this section we review some facts which are necessary for this paper.

A groupoid $(G,\cdot)$ is called a *quasigroup* if for any $a,b\in G$ each of the equations $a\cdot x=b$, $x\cdot a=b$ have a unique solution in G. A quasigroup may be also defined as an *equasigroup*, i.e., an algebra $(G,\cdot,\backslash,/)$ with three binary operations $\cdot$, $\backslash$, $/$ satisfying the following identities:

$$(x\cdot y)/y=x,\qquad x\backslash(x\cdot y)=y,$$

$$(x/y)\cdot y=x,\qquad x\cdot(x\backslash y)=y.$$

A nonempty subset S of a quasigroup $\mathcal{G}=(G,\cdot,\backslash,/)$ is called a *subquasigroup* if it is closed with respect to these three operations, that is, if $x*y\in S$ for all $x,y\in S$ and $*\in\{\cdot,\backslash,/\}$.

A homomorphic image of an equasigroup is an equasigroup. Also every subset of an equasigroup closed with respect to these three operations is an equasigroup. In theory of quasigroups an important role play *unipotent quasigroups*, i.e., quasigroups with the identity $x\cdot x=y\cdot y$. These quasigroups are connected with Latin squares which have one fixed element on the diagonal [6]. Such quasigroups may be defined as quasigroups G with the special fixed element θ satisfying the identity $x\cdot x=\theta$. Obviously, θ is uniquely determined and it is an idempotent, but, in general, it is not the (left, right) neutral element.

To avoid repetitions we use the following convection: a *quasigroup* $\mathcal{G}$ always denotes an equasigroup $(G,\cdot,\backslash,/)$, G always denotes the nonempty set.

A mapping $\mu:G\to[0,1]$ is called a *fuzzy set* on G. For any fuzzy set μ on G and any $t\in[0,1]$, we define the set

$$U(\mu;t)=\{x\in G\mid\mu(x)\geqslant t\},$$

which is called the *upper t-level cut* of μ. The set $\underline{\mu}=\{x\in G\,|\,\mu(x)>0\}$ is called the *support* of μ.

Definition 2.1. (cf. [7]) A fuzzy set μ on G is called a *fuzzy subquasigroup* of $\mathcal{G}$ if

$$\mu(x*y)\geqslant\min\{\mu(x),\mu(y)\}$$

for all $x,y\in G$ and $*\in\{\cdot,\backslash,/\}$.

The following two results are proved in [7].

Proposition 2.2. *A fuzzy set μ on a quasigroup $\mathcal{G}$ is a fuzzy subquasigroup if and only if every its nonempty upper level cut is a subquasigroup of $\mathcal{G}$.* $\square$

Proposition 2.3. *If μ is a fuzzy subquasigroup of a unipotent quasigroup $(G,\cdot,\backslash,/,\theta)$, then $\mu(\theta)\geqslant\mu(x)$ for all $x\in G$.* $\square$

Definition 2.4. A fuzzy set μ of the form

$$\mu(y)=\begin{cases} t\in(0,1] & \text{for} \quad y=x,\\ 0 & \text{for} \quad y\neq x,\end{cases}$$

is called a *fuzzy point* with the support x and the value t and is denoted by x_t.

For any fuzzy set μ the symbol $x_t\in\mu$ means that $\mu(x)\geqslant t$. In the case $\mu(x)+t>1$ we say that a fuzzy point x_t is *quasicoincident* with a fuzzy set μ and write $x_tq\mu$. The symbol $x_t\in\vee q\mu$ means that $x_t\in\mu$ or $x_tq\mu$. Similarly, $x_t\in\wedge q\mu$ denotes that $x_t\in\mu$ and $x_tq\mu$. $x_t\overline{\in}\mu$, $x_t\overline{q}\mu$ and $x_t\overline{\in\vee q}\mu$ mean that $x_t\in\mu$, $x_tq\mu$ and $x_t\in\vee q\mu$ do not hold, respectively.

3. (α,β)-fuzzy subquasigroups

Let α and β denote one of the symbols $\in$, q, $\in\vee q$ or $\in\wedge q$ unless otherwise specified.

Definition 3.1. *A fuzzy set μ in $\mathcal{G}$ is called a (α,β)-fuzzy subquasigroup of $\mathcal{G}$, if it satisfies the following condition:*

$$x_{t_1}\alpha\mu,\ y_{t_2}\alpha\mu\Longrightarrow(x*y)_{\min\{t_1,t_2\}}\beta\mu$$

for all $x,y\in G$, $t_1,t_2\in(0,1]$, $\alpha\neq\in\wedge q$ and $*\in\{\cdot,\backslash,/\}$.

Remark 3.2. (1) It is easy to construct 12 different types of fuzzy subquasigroups by the replacement of $\alpha(\neq\in\wedge q)$ and β in the Definition 3.1 by any two of $\{\in,q,\in\vee q,\in\wedge q\}$.
(2) Why $\alpha\neq\in\wedge q$? Since for a fuzzy set μ such that $\mu(x)\leqslant 0.5$ for all $x\in G$ and $x_t\in\wedge q\mu$ for some $t\in(0,1]$, we have $\mu(x)\geqslant t$ and $\mu(x)+t>1$. Thus

$$1<\mu(x)+t\leqslant\mu(x)+\mu(x)=2\mu(x),$$

so, $\mu(x)>0.5$. Hence $\{x_t\,|\,x_t\in\wedge q\mu\}=\emptyset$. This explains why $\alpha=\in\wedge q$ can be omitted in the above definition.

(3) $(\in, \in)$-fuzzy subquasigroups are in fact fuzzy subquasigroups.

(4) (α, β)-fuzzy subquasigroups are a generalization of fuzzy subquasigroups described in [7].

It is not difficult to see that the following proposition is true.

Proposition 3.3. *Every $(\in, \in)$-fuzzy subquasigroup is an $(\in, \in\vee q)$-fuzzy subquasigroup.* □

Corollary 3.4. *For any subset S of $\mathcal{G}$, the characteristic function χ_S of S is an $(\in, \in\vee q)$-fuzzy subquasigroup of $\mathcal{G}$ if and only if S is a subquasigroup of $\mathcal{G}$.*

Proof. Suppose that characteristic function χ_S is an $(\in, \in\vee q)$-fuzzy subquasigroup of $\mathcal{G}$. Let $x, y \in S$. Then $\chi_S(x) = 1 = \chi_S(y)$, and so $x_1 \in \chi_S$ and $y_1 \in \chi_S$. It follows that $(x*y)_1 = (x*y)_{\min\{1,1\}} \in\vee q\chi_S$, which implies $\chi_S(x*y) > 0$. Thus $x*y \in S$, and hence χ_S is a fuzzy subquasigroup of $\mathcal{G}$.

Conversely, if S is a fuzzy subquasigroup of $\mathcal{G}$, then χ_S is an $(\in, \in)$-fuzzy subquasigroup of $\mathcal{G}$ and, by Proposition 3.3, it is an $(\in, \in\vee q)$-fuzzy subquasigroup of $\mathcal{G}$. □

Proposition 3.5. *Every $(\in\vee q, \in\vee q)$-fuzzy subquasigroup is an $(\in, \in\vee q)$-fuzzy subquasigroup.*

Proof. Let μ be an $(\in\vee q, \in\vee q)$-fuzzy subquasigroup of $\mathcal{G}$. Let $x, y \in \mathcal{G}$ and $t_1, t_2 \in (0,1]$ be such that $x_{t_1} \in \mu$ and $y_{t_2} \in \mu$. Then $x_{t_1} \in\vee q\mu$ and $y_{t_2} \in\vee q\mu$. Thus $(x*y)_{\min\{t_1,t_2\}} \in\vee q\mu$, which proves that μ is an $(\in, \in\vee q)$-fuzzy subquasigroup of $\mathcal{G}$. □

The converse statement of Proposition 3.5 is not true as we can see in the following example.

Example 3.6. The set $G = \{0, a, b, c\}$ with the multiplication:

$\cdot$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

is a commutative quasigroup (Klein's group) in which the operations $\backslash$ and $/$ coincide with the group inverse operation.

Consider on this quasigroup the fuzzy set μ such that $\mu(0) = 0.5$, $\mu(a) = 0.6$ and $\mu(b) = \mu(c) = 0.3$. By routine computations, it is easy to verify that:

(1) μ is an $(\in, \in\vee q)$-fuzzy subquasigroup,

(2) μ is not an $(\in, \in)$-fuzzy subquasigroup because $a_{0.65} \in \mu$ and $a_{0.67} \in \mu$, but $(a * a)_{\min\{0.65,0.67\}} = 0_{0.65}\overline{\in}\mu$,

(3) μ is not an $(q, \in\vee q)$-fuzzy subquasigroup because $a_{0.51}q\mu$ and $b_{0.81}q\mu$, but $(a * b)_{\min\{0.51,0.81\}} = c_{0.51}\overline{\in\vee q}\mu$,

(4) μ is not an $(\in\vee q, \in\vee q)$-fuzzy subquasigroup because $a_{0.63} \in q\mu$ and $c_{0.77} \in q\mu$, but $(a * c)_{\min\{0.63,0.77\}} = c_{0.63}\overline{\in\vee q}\mu$. □

Now we prove some basic properties of (α, β)-fuzzy quasigroups.

Lemma 3.7. *If μ is a nonzero $(\in, \in)$-fuzzy subquasigroup of $\mathcal{G}$, then $\underline{\mu}$ is a subquasigroup of $\mathcal{G}$.*

Proof. If $\underline{\mu}$ is not a subquasigroup, then $\mu(x) > 0$, $\mu(y) > 0$ and $\mu(x*y) = 0$ for some $x, y \in \underline{\mu}$. But in this case $x_{\mu(x)}, y_{\mu(y)} \in \mu$ and $(x*y)_{\min\{\mu(x),\mu(y)\}}\overline{\in}\mu$, which is a contradiction. Hence $\mu(x * y) > 0$, i.e., $x * y \in \underline{\mu}$. So, $\underline{\mu}$ is a subquasigroup. □

Lemma 3.8. *If μ is a nonzero $(\in, q)$-fuzzy subquasigroup of $\mathcal{G}$, then $\underline{\mu}$ is a subquasigroup of $\mathcal{G}$.*

Proof. Similarly as in the previous proof suppose that $x, y \in \underline{\mu}$ and $x*y \notin \underline{\mu}$. Then $\mu(x) > 0$, $\mu(y) > 0$ and $\mu(x * y) = 0$. Consequently,

$$\mu(x * y) + \min\{\mu(x), \mu(y)\} = \min\{\mu(x), \mu(y)\} \leqslant 1.$$

Hence $(x * y)_{\min\{\mu(x),\mu(y)\}}\overline{q}\mu$, which is impossible. Thus $\mu(x * y) > 0$, so $x * y \in \underline{\mu}$. □

Lemma 3.9. *If μ is a nonzero $(q, \in)$-fuzzy subquasigroup of $\mathcal{G}$, then $\underline{\mu}$ is a subquasigroup of $\mathcal{G}$.*

Proof. Let $x, y \in \underline{\mu}$. Then $\mu(x) > 0$ and $\mu(y) > 0$. Thus $\mu(x) + 1 > 1$ and $\mu(y) + 1 > 1$, which imply that $x_1q\mu$ and $y_1q\mu$. If $\mu(x * y) = 0$, then $\mu(x * y) < 1 = \min\{1, 1\}$. Therefore, $(x * y)_{\min\{1,1\}}\overline{\in}\mu$, which is a contradiction. Therefore $\mu(x * y) > 0$, i.e., $x * y \in \underline{\mu}$. □

Lemma 3.10. *If μ is a nonzero (q, q)-fuzzy subquasigroup of $\mathcal{G}$, then $\underline{\mu}$ is a subquasigroup of $\mathcal{G}$.*

Proof. Let $x, y \in \underline{\mu}$. Then $\mu(x) > 0$ and $\mu(y) > 0$. Thus $\mu(x) + 1 > 1$ and $\mu(y) + 1 > 1$. This implies that $x_1 q\mu$ and $y_1 q\mu$. If $\mu(x * y) = 0$, then $\mu(x*y)+\min\{1, 1\} = 0+1 = 1$, and so $(x*y)_{\min\{1,1\}}\overline{q}\mu$. This is impossible, and hence $\mu(x * y) > 0$, i.e., $x * y \in \underline{\mu}$. □

By using a very similar argumentation as in the proof of the above four lemmas we can prove the following theorem.

Theorem 3.11. *If μ is a nonzero (α, β)-fuzzy subquasigroup of $\mathcal{G}$, then $\underline{\mu}$ is a subquasigroup of $\mathcal{G}$.* □

Theorem 3.12. *Let S be a subquasigroup of $\mathcal{G}$. Then any fuzzy set μ of $\mathcal{G}$ such that $\mu(x) \geqslant 0.5$ for all $x \in S$ and $\mu(x) = 0$ otherwise is a $(\alpha, \in\vee q)$-fuzzy subquasigroup.*

Proof. (i) Let $x, y \in G$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \in \mu$ and $y_{t_2} \in \mu$. Then $\mu(x) \geqslant t_1$ and $\mu(y) \geqslant t_2$. Thus $x, y \in S$, and so $x * y \in S$, i.e., $\mu(x * y) \geqslant 0.5$. If $\min\{t_1, t_2\} \leqslant 0.5$, then $\mu(x * y) \geqslant 0.5 \geqslant \min\{t_1, t_2\}$. Hence $(x*y)_{\min\{t_1,t_2\}} \in \mu$. If $\min\{t_1, t_2\} > 0.5$, then $\mu(x*y)+\min\{t_1, t_2\} > 0.5 + 0.5 = 1$ and so $(x * y)_{\min\{t_1,t_2\}}q\mu$. Therefore $(x * y)_{\min\{t_1,t_2\}} \in\vee q\mu$.

(ii) Let $x, y \in G$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1}q\mu$ and $y_{t_2}q\mu$. Then $x, y \in S$, $\mu(x) + t_1 > 1$ and $\mu(y) + t_2 > 1$. Since $x * y \in S$, we have $\mu(x * y) \geqslant 0.5$. If $\min\{t_1, t_2\} \leqslant 0.5$, then $\mu(x * y) \geqslant 0.5 \geqslant \min\{t_1, t_2\}$. Hence $(x*y)_{\min\{t_1,t_2\}} \in \mu$. If $\min\{t_1, t_2\} > 0.5$, then $\mu(x*y)+\min\{t_1, t_2\} > 0.5 + 0.5 = 1$ and so $(x * y)_{\min\{t_1,t_2\}}q\mu$. Therefore $(x * y)_{\min\{t_1,t_2\}} \in\vee q\mu$.

(iii) Let $x, y \in G$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \in \mu$ and $y_{t_2}q\mu$. Then $\mu(x) \geqslant t_1$ and $\mu(y) + t_2 > 1$. Since $x, y \in S$, also $x * y \in S$, i.e., $\mu(x * y) \geqslant 0.5$. Analogously as in (i) and (ii) we obtain $(x * y)_{\min\{t_1,t_2\}} \in \mu$ for $\min\{t_1, t_2\} \leqslant 0.5$ and $(x * y)_{\min\{t_1,t_2\}}q\mu$ for $\min\{t_1, t_2\} > 0.5$. Thus $(x * y)_{\min\{t_1,t_2\}} \in\vee q\mu$.

(iv) The case $x_{t_1}q\mu$ and $y_{t_2} \in \mu$ is analogous to (iii). □

Theorem 3.13. *A fuzzy set μ of $\mathcal{G}$ is an $(\in, \in\vee q)$-fuzzy subquasigroup if and only if it satisfies the inequality*

$$\mu(x * y) \geqslant \min\{\mu(x), \mu(y), 0.5\}. \tag{1}$$

Proof. Let μ be an $(\in, \in\vee q)$-fuzzy subquasigroup of $\mathcal{G}$. Suppose that for $x, y \in G$ we have $\min\{\mu(x), \mu(y)\} < 0.5$. If $\mu(x*y) < \min\{\mu(x), \mu(y)\}$, then $x_t \in \mu$ and $y_t \in \mu$ for any t such that $\mu(x*y) < t < \min\{\mu(x), \mu(y)\}$. but in this case $(x*y)_{\min\{t,t\}} = (x*y)_t \overline{\in\vee q}\mu$, a contradiction. This means that in the case $\min\{\mu(x), \mu(y)\} < 0.5$ must be $\mu(x*y) \geqslant \min\{\mu(x), \mu(y)\}$.

If $\min\{\mu(x), \mu(y)\} \geqslant 0.5$, then $x_{0.5} \in \mu$ and $y_{0.5} \in \mu$, which imply

$$(x*y)_{\min\{0.5,0.5\}} = (x*y)_{0.5} \in\vee q\mu.$$

Hence $\mu(x*y) \geqslant 0.5$. Otherwise, $\mu(x*y) + 0.5 < 0.5 + 0.5 = 1$, a contradiction. Consequently, $\mu(x*y) \geqslant 0.5 = \min\{\mu(x), \mu(y), 0.5\}$ for all $x, y \in G$.

Conversely, assume that the inequality mentioned in the above theorem is valid. Let $x, y \in G$ and $t_1, t_2 \in (0,1]$ be such that $x_{t_1} \in \mu$ and $y_{t_2} \in \mu$. Then $\mu(x) \geqslant t_1$ and $\mu(y) \geqslant t_2$. In the case $\mu(x*y) \geqslant \min\{t_1, t_2\}$ we obtain $(x*y)_{\min\{t_1,t_2\}} \in \mu$. In the case $\mu(x*y) < \min\{t_1, t_2\}$ we have $\min\{\mu(x), \mu(y)\} \geqslant 0.5$. If not, then

$$\mu(x*y) \geqslant \min\{\mu(x), \mu(y), 0.5\} \geqslant \min\{\mu(x), \mu(y)\} \geqslant \min\{t_1, t_2\},$$

which is a contradiction. So, in this case

$$\mu(x*y) + \min\{t_1, t_2\} > 2\mu(x*y) \geqslant 2\min\{\mu(x), \mu(y), 0.5\} = 1,$$

i.e., $(x*y)_{\min\{t_1,t_2\}} q\mu$. Hence μ is an $(\in, \in\vee q)$-fuzzy subquasigroup of $\mathcal{G}$. $\square$

Corollary 3.14. *Any $(\in, \in\vee q)$-fuzzy subquasigroup μ of $\mathcal{G}$ satisfying the inequality $\mu(x) < 0.5$ is an ordinary fuzzy subquasigroup of $\mathcal{G}$.* $\square$

Theorem 3.15. *A fuzzy set μ of $\mathcal{G}$ is its $(\in, \in\vee q)$-fuzzy subquasigroup if and only if for every $t \in (0, 0.5]$ each nonempty level $U(\mu; t)$ is a subquasigroup of $\mathcal{G}$.*

Proof. Assume that μ is an $(\in, \in\vee q)$-fuzzy subquasigroup of $\mathcal{G}$ and let $t \in (0, 0.5]$ be such that $U(\mu; t) \neq \emptyset$. If $x, y \in U(\mu; t)$, then $\mu(x) \geqslant t$ and $\mu(y) \geqslant t$. Thus

$$\mu(x*y) \geqslant \min\{\mu(x), \mu(y), 0.5\} \geqslant \min\{t, 0.5\} = t.$$

So, $x*y \in U(\mu; t)$. Hence $U(\mu; t)$ is a subquasigroup of $\mathcal{G}$.

Conversely, suppose that each nonempty level $U(\mu; t)$, $t \in (0, 0.5]$, is a subquasigroup of $\mathcal{G}$. If there are $x, y \in G$ such that

$$\mu(x*y) < \min\{\mu(x), \mu(y), 0.5\},$$

then also

$$\mu(x * y) < t_1 < \min\{\mu(x), \mu(y), 0.5\}$$

for some t_1. This means that $x, y \in U(\mu; t_1)$ and $x * y \notin U(\mu; t_1)$, which contradicts to the assumption that all $U(\mu; t)$ are subquasigroups. Therefore

$$\mu(x * y) < \min\{\mu(x), \mu(y), 0.5\}.$$

So, μ is an $(\in, \in\!\vee q)$-fuzzy subquasigroup of $\mathcal{G}$. □

Theorem 3.16. *The nonempty intersection of any family of* $(\in, \in\!\vee q)$*-fuzzy subquasigroups of* $\mathcal{G}$ *is an* $(\in, \in\!\vee q)$*-fuzzy subquasigroups of* $\mathcal{G}$.

Proof. Let $\{\lambda_i : i \in \Lambda\}$ be a fixed family of $(\in, \in\!\vee q)$-fuzzy subquasigroups of $\mathcal{G}$ and let λ be the nonempty intersection of this family. If $x_{t_1}, y_{t_2} \in \lambda$ and $(x * y)_{\min\{t_1,t_2\}} \overline{\in\!\vee q} \lambda$ for some $x, y \in G$ and $t_1, t_2 \in (0, 1]$, then

$$\lambda(x * y) < \min\{t_1, t_2\} \quad \text{and} \quad \lambda(x * y) + \min\{t_1, t_2\} \leqslant 1.$$

Thus $\lambda(x * y) < 0.5$.

Since each λ_i is an $(\in, \in\!\vee q)$-fuzzy subquasigroup, the family $\{\lambda_i : i \in \Lambda\}$ can be divided into two disjoint parts:

$$\Lambda' = \{\lambda_i \,|\, \lambda_i(x * y) \geqslant \min\{t_1, t_2\}\}$$

and

$$\Lambda'' = \{\lambda_i \,|\, \lambda_i(x * y) < \min\{t_1, t_2\} \quad \text{and} \quad \lambda_i(x * y) + \min\{t_1, t_2\} > 1\}.$$

If $\lambda_i(x * y) \geqslant \min\{t_1, t_2\}$ for all λ_i, then also $\lambda(x * y) \geqslant \min\{t_1, t_2\}$, which is a contradiction. So, for some λ_i we have $\lambda_i(x * y) < \min\{t_1, t_2\}$ and $\lambda_i(x * y) + \min\{t_1, t_2\} > 1$. Thus $\min\{t_1, t_2\} > 0.5$, whence $\lambda_i(x) \geqslant \lambda(x) \geqslant t_1 \geqslant \min\{t_1, t_2\} > 0.5$ for all $\lambda_i \in \Lambda''$. Similarly $\lambda_i(y) > 0.5$ for all $\lambda_i \in \Lambda''$. Now suppose that $t = \lambda_i(x * y) < 0.5$ for some λ_i. Let $t' \in (0, 0.5)$ be such that $t < t'$, then $\lambda_i(x) > 0.5 > t'$ and $\lambda_i(y) > 0.5 > t'$, that is $x_{t'} \in \lambda_i$ and $y_{t'} \in \lambda_i$ but $\lambda_i(x * y) = t < t'$ and $\lambda_i(x * y) + t' < 1$. So, $(x * y)_{t'} \overline{\in\!\vee q} \lambda_i$. This contradicts that λ_i is a $(\in, \in\!\vee q)$ fuzzy subquasigroup of $\mathcal{G}$. Hence $\lambda_i(x * y) \geqslant 0.5$ for all λ_i, and thus $\lambda(x * y) \geqslant 0.5$. This is impossible because for all $x, y \in G$ we have $\lambda(x * y) < 0.5$. Therefore $(x * y)_{\min\{t_1,t_2\}} \in\!\vee q \lambda$. □

For any fuzzy subset μ of $\mathcal{G}$ and any $t \in (0, 1]$ we consider two subsets:

$$Q(\mu; t) = \{x \in G \,|\, x_t q \mu\} \quad \text{and} \quad [\mu]_t = \{x \in G \,|\, x_t \in\!\vee q \mu\}.$$

It is clear that $[\mu]_t = U(\mu;t) \cup Q(\mu;t)$.

In Theorem 3.15 we have shown that a fuzzy subset μ of a quasigroup $\mathcal{G}$ is an $(\in, \in\!\vee q)$-fuzzy subquasigroup of $\mathcal{G}$ if and only if $U(\mu;t) \neq \emptyset$ is a subquasigroup of $\mathcal{G}$ for all $0 < t \leqslant 0.5$. Now we show a similar result for $[\mu]_t$.

Theorem 3.17. *A fuzzy subset μ of $\mathcal{G}$ is an $(\in, \in\!\vee q)$-fuzzy subquasigroup of $\mathcal{G}$ if and only if $[\mu]_t$ is a subquasigroup of $\mathcal{G}$ for all $t \in (0, 0.5]$.*

Proof. Let μ be an $(\in, \in\!\vee q)$-fuzzy subquasigroup of $\mathcal{G}$. Let $x, y \in [\mu]_t$ for some $t \in (0, 0.5]$. Then $\mu(x) \geqslant t$ or $\mu(x) + t > 1$ and $\mu(y) \geqslant t$ or $\mu(y)+t > 1$. Since μ is an $(\in, \in\!\vee q)$-fuzzy subquasigroup, we have $\mu(x*y) \geqslant \min\{\mu(x), \mu(y), 0.5\}$ (Theorem 3.13). This implies $\mu(x * y) \geqslant \min\{t, 0.5\} = t$. So, $x * y \in [\mu]_t$.

Conversely, let μ be a fuzzy subset of $\mathcal{G}$ and let $[\mu]_t$ be a subquasigroup of $\mathcal{G}$ for all $t \in (0, 0.5]$. If $\mu(x * y) < t < \min\{\mu(x), \mu(y), 0.5\}$ for some $t \in (0, 0.5]$, then $x, y \in [\mu]_t$ and $x*y \in [\mu]_t$. Hence $\mu(x*y) \geqslant t$ or $\mu(x*y)+t > 1$, a contradiction. Therefore $\mu(x*y) \geqslant \min\{\mu(x), \mu(y), 0.5\}$ for all $x, y \in G$. □

Lemma 3.18. *Let μ be an arbitrary fuzzy set defined on $\mathcal{G}$ and let $x \in G$. Then $\mu(x) = t$ if and only if $x \in U(\mu;t)$, $x \notin U(\mu;s)$ for all $s > t$.* □

Theorem 3.19. *Let $\{A_t\}_{t\in\Gamma}$, where $\Gamma \subseteq (0, 0.5]$ be a collection of subquasigroups of $\mathcal{G}$ such that $G = \bigcup\limits_{t\in\Gamma} A_t$, and for $s, t \in \Gamma$, $s < t$ if and only if $A_t \subset A_s$. Then a fuzzy set μ defined by*

$$\mu(x) = \sup\{t \in \Gamma \mid x \in A_t\},$$

is an $(\in, \in\!\vee q)$-fuzzy subquasigroup of $\mathcal{G}$.

Proof. According to Theorem 3.15, it is sufficient to show that for every $t \in (0, 0.5]$, each nonempty $U(\mu;t)$ is a subquasigroup of $\mathcal{G}$. We consider two cases:

$$(i) \quad t = \sup\{s \in \Gamma \mid s < t\}$$

$$(ii) \quad t \neq \sup\{s \in \Gamma \mid s < t\}.$$

In the first case

$$x \in U(\mu;t) \longleftrightarrow (x \in A_s \ \forall s < t) \longleftrightarrow x \in \bigcap_{s<t} A_s.$$

So, $U(\mu;t) = \bigcap\limits_{s<t} A_s$, which is a subquasigroup of $\mathcal{G}$. In the second case, we have $U(\mu;t) = \bigcup\limits_{s\geq t} A_s$. Indeed, if $x \in \bigcup\limits_{s\geqslant t} A_s$, then $x \in A_s$ for some $s \geqslant t$. Thus $\mu(x) \geqslant s \geqslant t$, i.e., $x \in U(\mu;t)$. This proves $\bigcup\limits_{s\geqslant t} A_s \subset U(\mu;t)$. To prove the converse inclusion consider $x \notin \bigcup\limits_{s\geqslant t} A_s$. Then $x \notin A_s$ for all $s \geqslant t$. Since $t \neq \sup\{s \in \Gamma \,|\, s < t\}$, there exists $\varepsilon > 0$ such that $(t-\varepsilon, t) \cap \Gamma = \emptyset$. Hence $x \notin A_s$ for all $s > t - \varepsilon$, which means that if $x \in A_s$, then $s \leqslant t - \varepsilon$. Thus $\mu(x) \leq t - \varepsilon < t$, and so $x \notin U(\mu;t)$. Therefore $U(\mu;t) = \bigcup\limits_{s\geqslant t} A_s$. Since, as it is not difficult to verify, $\bigcup\limits_{s\geqslant t} A_s$ is a subquasigroup of $\mathcal{G}$, we see that $U(\mu;t)$ is a subquasigroup in any case. □

Theorem 3.20. *For any chain $A_0 \subset A_1 \subset A_2 \subset \ldots \subset A_n = G$ of subquasigroups of $\mathcal{G}$ there exists an $(\in, \in\!\vee q)$-fuzzy subquasigroup of $\mathcal{G}$ for which level sets coincide with this chain.*

Proof. Let $t_0, t_1, \ldots, t_n$ be a finite decreasing sequence in $[0,1]$. Consider the fuzzy set μ on $\mathcal{G}$ defined by $\mu(A_0) = t_0$ and $\mu(A_k \setminus A_{k-1}) = t_k$ for $0 < k \leqslant n$. Let $x, y \in G$. If $x, y \in A_k \setminus A_{k-1}$, then $x * y \in A_k$ and

$$\mu(x * y) \geqslant t_k = \min\{\mu(x), \mu(y)\}.$$

Now let $x \in A_i \setminus A_{i-1}$ and $y \in A_j \setminus A_{j-1}$, where $i \neq j$. If $i > j$, then $A_j \subset A_i$, $\mu(x) = t_i < t_j = \mu(y)$, $x * y \in A_i$. Thus

$$\mu(x * y) \geqslant t_i = \min\{\mu(x), \mu(y)\}.$$

Analogously for $i < j$. So, μ is a fuzzy subquasigroup. It is not difficult to see that it is an $(\in, \in\!\vee q)$-fuzzy subquasigroup.

Such defined μ has only the values $t_0, t_1, \ldots, t_n$. Their level subsets are subquasigroups and form the chain

$$U(\mu;t_0) \subset U(\mu;t_1) \subset \ldots \subset U(\mu;t_n) = G.$$

We now prove that $U(\mu;t_k) = A_k$ for $0 \leqslant k \leqslant n$. Indeed,

$$U(\mu;t_0) = \{x \in G \,|\, \mu(x) \geqslant t_0\} = A_0.$$

Moreover, $A_k \subseteq U(\mu;t_k)$ for $0 < k \geqslant n$. If $x \in U(\mu;t_k)$, then $\mu(x) \geqslant t_k$ and so $x \notin A_i$ for $i > k$. Hence $\mu(x) \in \{t_0, t_1, \ldots, t_k\}$, which implies $x \in A_i$ for some $i \leqslant k$. Since $A_i \subseteq A_k$, it follows that $x \in A_k$. Consequently, $U(\mu;t_k) = A_k$ for every $0 < k \leqslant n$. This completes the proof. □

4. Fuzzy subquasigroups with thresholds

Definition 4.1. Let $0 \leqslant \lambda_1 < \lambda_2 \leqslant 1$ be fixed. A fuzzy set μ of a quasigroup $\mathcal{G}$ is called a *fuzzy subquasigroup with thresholds* (λ_1, λ_2), if

$$\max\{\mu(x*y), \lambda_1\} \geqslant \min\{\mu(x), \mu(y), \lambda_2\}$$

for all $x, y \in G$.

It is not difficult to see that:

- for $\lambda_1 = 0$ and $\lambda_2 = 1$ we have an ordinary fuzzy subquasigroup,
- for $\lambda_1 = 0$ and $\lambda_2 = 0.5$ we have an $(\in, \in\!\vee q)$-fuzzy subquasigroup,
- a fuzzy subquasigroup is a fuzzy subquasigroup with some thresholds,
- also any $(\in, \in\!\vee q)$-fuzzy subquasigroup is a fuzzy subquasigroup with some thresholds.

Example 4.2. Let $\mathcal{G}$ be a commutative quasigroup defined in Example 3.6 and let $\mu(0) = 0.5$, $\mu(a) = 0.7$, $\mu(b) = 0.4$, $\mu(c) = 0.3$. Then:

1. μ is a fuzzy subquasigroup with thresholds $\lambda_1 = 0.4$ and $\lambda_2 = 0.65$, but it is not a fuzzy subquasigroup with thresholds $\lambda_1 = 0.6$ and $\lambda_2 = 0.8$ since $\max\{\mu(a*a), 0.6\} = 0.6 < 0.7 = \min\{\mu(a), \mu(a), 0.8\}$,
2. μ is a fuzzy subquasigroup with thresholds $\lambda_1 = 0.77$ and $\lambda_2 = 0.88$, but it is not an ordinary fuzzy subquasigroup because $\mu(a*b) = \mu(c) = 0.3 < 0.4 = \min\{\mu(a), \mu(b)\}$. □

Theorem 4.3. *A fuzzy set μ of a quasigroup $\mathcal{G}$ is a fuzzy subquasigroup with thresholds (λ_1, λ_2) if and only if for all $t \in (\lambda_1, \lambda_2]$ each nonempty $U(\mu; t)$ is a subquasigroup of $\mathcal{G}$.*

Proof. The proof is similar to the proof of Theorem 3.15. □

Note that in the above theorem the restriction $t \in (\lambda_1, \lambda_2]$ is essential. $U(\mu; t)$ for $t \in (0, \lambda_1]$ may not be a subquasigroup of $\mathcal{G}$.

Example 4.4. The set $\mathbb{Z}$ of all integers with three operations $\circ$, $\backslash$, $/$ defined as follows: $x \circ y = x - y$, $x \backslash y = x - y$, $x/y = x + y$, is a quasigroup. Consider the following fuzzy set

$$\mu(x) = \begin{cases} 0 & \text{if} \quad x < 0 \quad \text{and} \quad x \neq 2k, \\ 0.3 & \text{if} \quad x > 0 \quad \text{and} \quad x \neq 2k, \\ 0.5 & \text{if} \quad x = 2n \quad \text{and} \quad x \neq 4k, \\ 0.8 & \text{if} \quad x = 4n \quad \text{and} \quad x \neq 8k, \\ 0.9 & \text{if} \quad x = 8n \quad \text{and} \quad x < 0, \\ 1 & \text{if} \quad x = 8n \quad \text{and} \quad x > 0, \end{cases}$$

where k and n are arbitrary integers. Then

$$U(\mu;t)=\begin{cases} \mathbb{Z} & \text{for} \quad t=0, \\ 2\mathbb{Z}\cup\mathbb{Z}^{+} & \text{for} \quad t\in(0,0.3], \\ 2\mathbb{Z} & \text{for} \quad t\in(0.3,0.5], \\ 4\mathbb{Z} & \text{for} \quad t\in(0.5,0.8], \\ 8\mathbb{Z} & \text{for} \quad t\in(0.8,0.9], \\ 8\mathbb{Z}^{+} & \text{for} \quad t\in(0.9,1], \end{cases}$$

where $p\mathbb{Z}$ denotes the set of all integers divided by p, $\mathbb{Z}^{+}$ – the set of all positive integers. It is clear that for $t\in(0.3,0.9]$ each $U(\mu;t)$ is a subguasigroup of this quasigroup. For $t\in(0,0.3]$ and $t\in(0.9,1]$ $U(\mu;t)$ are not subquasigroups. So, in view of Theorem 4.3, μ is a fuzzy subquasigroup with thresholds $\lambda_1=0.3$ and $\lambda_2=0.9$. But μ is not a fuzzy subquasigroup since

$$\mu(3\circ 8)=\mu(-5)=0\ngeqslant 0.3=\min\{\mu(3),\mu(8)\}.$$

It is not an $(\in,\in\vee q)$-fuzzy subquasigroup too because $3_{0.2}\in\mu$ and $8_{0.5}\in\mu$ but $(3\circ 8)_{\min\{0.2,0.5\}}\overline{\in\vee q}\mu$. □

Theorem 4.5. *Let $f:\mathcal{G}_1\to\mathcal{G}_2$ be an epimorphism of quasigroups and let μ and ν be fuzzy subquasigroups of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Then $f(\mu)$ defined by*

$$f(\mu)(y)=\sup\{\mu(x)\,|\,f(x)=y \quad \text{for all } y\in\mathcal{G}_2\}$$

and $f^{-1}(\nu)$ defined by

$$f^{-1}(\nu)(x)=\nu(f(x)) \quad \text{for all} \quad x\in\mathcal{G}_1$$

are fuzzy subquasigroups of $\mathcal{G}_2$ and $\mathcal{G}_1$, respectively. Moreover, if μ and ν are with thresholds (λ_1,λ_2), then also $f(\mu)$ and $f^{-1}(\nu)$ are with thresholds (λ_1,λ_2).

Proof. Let $y_1,y_2\in G_2$. Then

$$\begin{aligned} \max\{f(\mu)(y_1*y_2),\lambda_1\} &= \max\{\sup\{\mu(x_1*x_2),|f(x_1*x_2)=y_1*y_2\},\lambda_1\} \\ &= \sup\{\max\{\mu(x_1*x_2),\lambda_1\}\mid f(x_1*x_2)=y_1*y_2\} \\ &\geqslant \sup\{\min\{\mu(x_1),\mu(x_2),\lambda_1\}|f(x_1)=y_1,f(x_2)=y_2\} \\ &= \min\{\sup\{\mu(x_1)\mid f(x_1)=y_1\}, \\ &\qquad \sup\{\mu(x_2)\mid f(x_2)=y_2\},\lambda_2\} \\ &= \min\{f(\mu)(y_1),f(\mu)(y_2),\lambda_2\}. \end{aligned}$$

Similarly, for $x, y \in G_1$ we obtain

$$\begin{aligned}\max\{f^{-1}(\nu)(x*y), \lambda_1\} &= \max\{\nu(f(x*y)), \lambda_1\} = \max\{\mu(f(x)*f(y)), \lambda_1\} \\ &\geqslant \min\{\nu(f(x)), \nu(f(y)), \lambda_2\} = \min\{f^{-1}(\nu)(x), f^{-1}(\nu)(y), \lambda_2\},\end{aligned}$$

which completes the proof. □

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Secondary representation of semimodules over a commutative semiring

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Abstract

In this paper, we analyze some results on the theory secondary representation of semimodules over a commutative semiring with non-zero identity analogues to the theory secondary representation of modules over a commutative ring with non-zero identity.

1. Introduction

Semimodules constitute a fairly natural generalization of modules, with broad applications in the mathematical foundations of computer science [4]. The main part of this paper is devoted to stating and proving analogues to several well-known results in the theory of modules.

For the sake of completeness, we state some definitions and notations used throughout. By a *commutative semiring* we mean an algebraic system $R = (R, +, \cdot)$ such that $(R, +)$ and $(R, \cdot)$ are commutative semigroups, connected by $a(b + c) = ab + ac$ for all $a, b, c \in R$, and there exists $0 \in R$ such that $r + 0 = r$ and $r0 = 0r = 0$ for all $r \in R$. Throughout this paper let R be a commutative semiring. A (*left*) *semimodule M over a semiring R* is a commutative additive semigroup which has a zero element, together a mapping from $R \times M$ into M (sending (r, m) to rm) such that $(r+s)m = rm + sm$, $r(m + p) = rm + rp$, $r(sm) = (rs)m$ and $0m = r0_M = 0_M$ for all $m, p \in M$ and $r, s \in R$.

Let M be a semimodule over the semiring R, and let N be a subset of M. We say that N is a *subsemimodule* of M, or an *R-subsemimodule* of M, percisely when N is itself an R-semimodule with respect to the operations

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for M (so $0_M \in N$). It is easy to see that if $r \in R$, then

$$rM = \{rm : m \in M\}$$

is a subsemimodule of M. The semiring R is considered to be also a semimodule over itself. In this case, the subsemimodules of R are called *ideals* of R. A *subtractive subsemimodule* (= *k-subsemimodule*) N is a subsemimodule of M such that if $x, x + y \in N$, then $y \in N$ (so $\{0_M\}$ is a k-subsemimodule of M). If M is a semimodule over a semiring R, then M is *Artinian* if any non-empty set of k-subsemimodules of M has minimal member with respect to the set inclusion. This definition is equivalent to the descending chain condition on k-subsemimodules of M. A *prime ideal* of R is a proper ideal P of R in which $x \in P$ or $y \in P$ whenever $xy \in P$.

A subsemimodule N of a semimodule M over a semiring R is called a *partitioning subsemimodule* (=*Q_M-subsemimodule*) if there exists a non-empty subset Q_M of M such that

(1) $RQ_M \subseteq Q_M$;

(2) $M = \cup\{q + N : q \in Q_M\}$;

(3) If $q_1, q_2 \in Q_M$ then $(q_1 + N) \cap (q_2 + N) \neq \emptyset$ if and only if $q_1 = q_2$.

It is easy to see (cf. [5]) that if $M = Q_M$, then $\{0\}$ is a Q_M-subsemimodule of M.

Remark 1.1. Let M be a semimodule over a semiring R, and let N be a Q_M-subsemimodule of M. We put $M/N = \{q + N : q \in Q_M\}$. Then M/N forms a commutative additive semigroup which has zero element under the binary operation $\oplus$ defined as follows: $(q_1 + N) \oplus (q_2 + N) = q_3 + N$ where $q_3 \in Q_M$ is the unique element such that $q_1 + q_2 + N \subseteq q_3 + N$. Note that by the definition of Q_M-subsemimodule, there exists a unique $q_0 \in Q_M$ such that $0_M + N \subseteq q_0 + N$. Then $q_0 + N$ is a zero element of M/N.

Now let $r \in R$ and suppose that $q_1 + N, q_2 + N \in M/N$ are such that $q_1 + N = q_2 + N$ in M/N. Then $q_1 = q_2$, we must have $rq_1 + N = rq_2 + N$. Hence we can unambiguously define a mapping from $R \times M/N$ into M/N (sending $(r, q_1 + N)$ to $rq_1 + N$) and it is routine to check that this turns the commutative semigroup M/N into an R-semimodule. We call this R-semimodule the *residue class semimodule* or *factor semimodule* of M modulo N [4].

We need the following theorem proved in [5, Lemma 2.4, Proposition 2.5, Theorem 2.6, Theorem 2.7 and Theorem 2.10].

Theorem 1.2. *Assume that N is a Q_M-subsemimodule of a seminodule M over a semiring R and let T, L be k-subsemimodules of M containing N. Then the following hold:*

(*i*) *If $q_0 + N$ is a zero in M/N, then $q_0 + N = N$.*

(*ii*) *N is a k-subsemimodule of M.*

(*iii*) *$L/N = \{q + N : q \in Q_M \cap L\}$ is a k-subsemimodule of M/N.*

(*iv*) *If H is a k-subsemimodule of M/N, then $H = K/N$ for some k-subsemimodule K of M.*

(*v*) *$T/N = L/N$ if and only if $T = L$.* □

2. Secondary semimodules

We begin with the key lemma of this paper.

Lemma 2.1. *Let M be a semimodule over a semiring R, N an Q_M-subsemimodule of M and q_0 the unique element Q_M such that $q_0 + N$ is the zero in M/N. Then the following hold:*

(*i*) *$q_0 \in N$ and if $q \in N \cap Q_M$, then $q \in N$.*

(*ii*) *If $q_1, q_2 \in Q_M$ and $a, b \in N$ with $q_1 + a = q_2 + b$, then $q_1 = q_2$.*

(*iii*) *If for each $n \in N$, there exists $n' \in N$ such that $n + n' = 0$, then $N = a + N = \{a + n : n \in N\}$ for every $a \in N$.*

Proof. (*i*) Since by Theorem 1.2, $q_0 + N = N$ is a k-subsemimodule of M, we must have $q_0 \in N$. Moreover, since $q + q_0 \in (q + N) \cap (q_0 + N)$, we get $q = q_0 \in N$.

(*ii*) Since $q_1 + a \in (q_1 + N) \cap (q_2 + N)$, we must have $q_1 = q_2$.

(*iii*) It is suffices to show that $N \subseteq a + N$. Let $n \in N$. Since N is a Q_M subsemimodule, there is an element $q \in Q_M$ and $n' \in N$ such that $n = q + n'$, so $q \in N$ since every Q_M-submodule is a k-subsemimodule. By assumption, $a + a' = 0$ for some $a' \in N$. Hence $n = a + a' + q + n' \in a + N$, and the proof is complete. □

Assume that R is a semiring and let N be an R-subsemimodule of a semimodule M. Then N is a *relatively divisible subsemimodule* (or an *RD-subsemimodule*) if $rN = N \cap rM$ for all $r \in R$. Since $rN \subseteq N \cap rM$, we see that N is an *RD*-subsemimodule of M if and only if for all $x \in M$ and $r \in R$, $rx \in N$ implies $rx = ry$ for some $y \in N$. Hence, N is an

RD-subsemimodule of M if and only if $a \in N$ and the equation $rx = a$ has a solution in M, then it is solvable in N too.

Lemma 2.2. *Let R be a semiring, and let P, N be subsemimodules of the R-semimodule M such that $P \subseteq N \subseteq M$. Then:*

(i) *If P is an RD-subsemimodule of N and N is an RD-subsemimodule of M, then P is an RD-subsemimodule of M.*

(ii) *If P is an RD-subsemimodule of M, then P is an RD-subsemimodule of N.*

Proof. The proof is straightforward. □

Proposition 2.3. *Let R be a semiring, M an R-semimodule, P a Q_M-subsemimodule of M and N a k-subsemimodule of M such that $P \subseteq N \subseteq M$. Then:*

(i) *If N is an RD-subsemimodule of M, then N/P is an RD-subsemimodule of M/P.*

(ii) *If P is an RD-subsemimodule of M and N/P is an RD-subsemimodule of M/P, then N is an RD-subsemimodule of M.*

Proof. (i) Let $rx = q_1 + P$ be an equation over N/P that admits a solution in M/P, say, $r(q_2 + P) = q_1 + P$ where $q_2 \in Q_M$ and $q_1 \in Q_M \cap N$, so $rq_2 = q_1$. By the purity of N in M the equation $rx = q_1$ has a solution $x = a$ in N. Then $a = q_3 + b$ for some $q_3 \in Q_M \cap N$ and $b \in P$ (since N is a k-subsemimodul), so $rq_3 + rb = q_1$. Hence $rq_3 = q_1$ by Lemma 2.1. Thus $r(q_3 + P) = q_1 + P$. Hence $x = q_3 + P$ is a solution of our original equation.

(ii) Let $rx = a$ be an equation over N which has a solution $x = c$ in M. There are elements $q_1 \in N \cap Q_M$, $q_2 \in Q_M$ and $e, f \in P$ such that $a = q_1 + e$ and $c = q_2 + f$, so $rq_2 + rf = q_1 + e$. Hence $rq_2 = q_1$. Therefore, we must have $r(q_2 + P) = q_1 + P$. By purity of N/P in M/P there exist $q_3 + P \in N/P$ such that $r(q_3 + P) = q_1 + P$, where $q_3 \in N \cap Q_M$, so $rq_3 = q_1$. Since $r(q_3 + f) = rq_2 + rf = q_1 + e$, we get $x = q_3 + f$ is a solution of our original equation. □

Proposition 2.4. *Let M be a semimodule over a semiring R, N an Q_M-subsemimodule of M and $r \in R$. Let q_0 be the unique element of Q_M such that $q_0 + N$ is the zero in M/N. Then:*

(i) *$rM + N$ is an $(rQ)_M$-subsemimodule of M. In particular,*

$$(rM + N)/N = \{rq + N : rq \in rQ_M \cap (rM + N)\}$$

is a k-subsemimodule of M/N.

(*ii*) $r(M/N) = (rM + N)/N$. *In particular,* $N/N = \{q_0 + N\}$.

Proof. (*i*) Clearly, $R(rQ) \subseteq rQ$ and $\bigcup\{rq + N : q \in Q_M\} \subseteq rM + N$. For the reverse inclusion, assume that $rm + n \in rM + N$ where $m \in M$ and $n \in N$. There are elements $q \in Q$ and $n_1 \in N$ such that $m = q + n_1$ since N is a Q_M-subsimimodule of M, so $rm + n = rq + rn_1 + n \in rq + N$. Hence $rM + N = \cup\{rq + N : q \in Q\}$. It is easy to see that if $rq_1, rq_2 \in rQ$, then $(rq_1 + N) \cap (rq_2 + N) \neq \emptyset$ if and only if $rq_1 = rq_2$. It follows from Theorem 1.2 that $rM + N$ is a k-subsemimodule of M containing N. Then $(rM + N)/N$ is a k-subsemimodule of M/N by Theorem 1.2.

(*ii*) Since the inclusion $(rM + N)/N \subseteq r(M/N)$ is trivial, we will prove the reverse inclusion. Let $r(q + N) = rq + N \in r(M/N)$. Since $rq \in (rM + N) \cap rQ$, we must have $r(q + N) \in (rM + N)/N$ by (*i*), and we have equality. Finally, $N/N = \{q + N : q \in N \cap Q_M\} = \{q_0 + N\}$ by Lemma 2.1. □

Let R be a semiring with identity. An R-semimodule M is said to be *secondary* if $M \neq 0$ and if, for each $r \in R$, the endomorphism $\varphi_{r,M}$ (i.e., multiplication by r in M) is either surjective or nilpotent. Equivalently, M is secondary if and only if either $rM = M$ or $r^n M = 0$ for some n for every $r \in R$. It is easy to see that the nilradical of M is a prime ideal P, and M is said to be P-secondary [7].

Proposition 2.5. *Let N be a proper Q_M-subsemimodule of a P-secondary semimodule M over a semiring R. Then M/N is a P-secondary R-semimodule.*

Proof. Assume that q_0 is the unique element Q_M such that $q_0 + N$ is the zero in M/N and let $r \in R$. If $r \in P$, then $r(M/N) = (rM + N)/N = (M + N)/N = M/N$ by Proposition 2.4. If $r \notin P$, then there is a positive integer s such that $r^s(M/N) = (r^s M + N)/N = N/N = \{q_0 + N\}$, as required. □

Theorem 2.6. *Assume that R is a semiring and let N be a non-zero proper RD-subsemimodule* (*resp. pure subsemimodule*) *of an R-semimodule M. If N is a Q_M-subsemimodule of M, then M is P-secondary if and only if N and M/N are secondary.*

Proof. If M is secondary, then M/N is secondary by Proposition 2.7. To see that N is secondary, assume that $a \in R$. If $a \in P$, then $a^n N \subseteq a^n M = 0$ for some n. So suppose that $a \notin P$. Then $aN = N \cap aM = N \cap M = N$ since N is an RD-submodule. Conversely, assume that both N and M/N are secondary and let q_0 be the unique element Q_M such that $q_0 + N$ is the zero in M/N. Let $r \in R$. If $r \in P$, then $r^m(M/N) = (r^m M + N)/N = N/N = \{q_0 + N\}$ by Proposition 2.6 and $r^m N = 0$ for some m. Hence $r^m M \subseteq N$ by Proposition 2.4 and Theorem 1.2, and $0 = r^m N = r^m M \cap N = r^m M$. If $r \notin P$, then $rM + N = M$, $rN = N$ and $N = rN = N \cap rM$, so we must have $rM = M$. Thus M is secondary. □

Let R be a semiring. An element $a \in R$ is said to be *regular* if there exists $b \in R$ such that $a = a^2 b$, and R is said to be regular if each of its elements is regular.

Theorem 2.7. *Assume that R is a regular semiring and let N be a non-zero proper Q_M-subsemimodule of an R-semimodule M. Then M is secondary if and only if N and M/N are secondary.*

Proof. By Theorem 2.6, it suffices to show that every subsemimodule of M is a RD-subsemimodule of M. Let N be a subsemimodule of M. It is enough to show that if $n \in N$ and the equation $rx = n$ (where $r \in R$) has a solution in M, say m, then it is solvable in N. By assumption, there is an element $s \in R$ such that $r = r^2 s$. Hence $r(sn) = r^2 sm = rm = n$. Therefore, the equation $rx = n$ has a solution $x = sn$ in N. □

Lemma 2.8. *Let R be a semiring. Then finite sum of P-secondary semimodules is P-secondary.*

Proof. Let $M = M_1 + \ldots + M_k$, where for each i, M_i is P-secondary. Let $a \in R$. If $a \in P$, then there is a positive integer n such that $a^n M_i = 0$ for every i. Hence $a^n M = 0$. Similarly, if $a \notin P$, then $aM = M$. Thus M is P-secondary. □

Let M be a semimodule over a semiring R. A *secondary representation* of M is an expression of M as a sum of secondary submodules, say $M = N_1 + \ldots + N_k$. The representation is said to be *minimal* if (1) the prime ideals $\mathrm{nilrad}(N_i) = P_i$ are distinct and (2) none of the summand N_i is redundant. By Lemma 2.8, any secondary representation of M can be refined to a minimal one. If M has a secondary representation, we shall say that M is *representable* [7].

Definition 2.9. Let R be a semiring. An R-semimodule M is *sum-irreducible* if $M \neq 0$ and the sum of any two proper subsemimodules of M is always a proper subsemimodule. An R-semimodule M is *strongly subtractive* if every subsemimodule of M is a k-subsemimodule and for each $m \in M$ there exists $m' \in M$ such that $m + m' = 0$ [2].

Theorem 2.10 *Every strongly subtractive Artinian semimodule M over a semiring R has a secondary representation.*

Proof. First, we show that if M is sum-irreducible, then M is secondary. Suppose M is not secondary. Then there is an element $r \in R$ such that $rM \neq M$ and $r^n M \neq 0$ for all positive integers n. By assumption, there exists a positive integer k such that $r^k M = r^{k+1} M = \dots$ Set $M_1 = \mathrm{Ker}\varphi_{r^k,M}$ and $M_2 = r^k M$. Then M_1 and M_2 are proper subsemimodules of M. Let $x \in M$. Then $r^k x = r^{2k} y$ for some $y \in M$. We can write $y + y' = 0$ for some $y' \in M$. Hence $r^k y + r^k y' = 0$, $r^{2k} y + r^{2k} y' = 0$ and $x = (x + r^k y') + r^k y$, where $x + r^k y' \in M_1$ and $r^k y \in M_2$. Hence $M = M_1 + M_2$, and therefore M is not sum-irreducible.

Next, suppose that M is not representable. Then the set of non-zero subsemimodules of M which are not representable has a minimal element N. Certainly N is not secondary and $N \neq 0$. Hence N is the sum of two strictly smaller subsemimodules N_1 and N_2. By the minimality of N, each N_1, N_2 is representable, and therefore so also is N, which is a contradiction. $\square$

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Groups homeomorphisms: topological characteristics, invariant measures and classifications

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Abstract

It is a survey of main results on groups of homeomorphisms of the real line and the circle obtained in the last years.

1. Introduction

One of main problems of the theory of groups is the problem of a classification of abstract groups. Such classification can be based on the Tarski's number connected with the Day's problem. It is known, that the Tarski's number distinguishes among themselves no more than account set of subclasses of the paradoxical groups, but not distinguishes the amenable groups. Nevertheless, the classification is possible on a basis of the scale of the values given by the growth group for finitely generated amenable groups. Unfortunately, there is not any correspondence between good known canonical subclasses of groups and characteristic given by the growth group in a class of finitely generated amenable groups, even though such correspondence takes place for special subclasses of groups. Other important method of investigation of the abstract groups is their realization in the form of subgroups of some selected groups with well investigated properties. The groups of actions on locally compact space and, in particular, the groups of homeo-

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morphisms of the real line and circle belong to such groups. In addition the topological and metric invariants arise for groups of homeomorphisms. From the noted invariants, the invariant measures and topological characteristic connected with them will be considered. The presence of additional invariants generates some natural factorization of such groups. For quotient groups the classification mentioned above appears more informative, as it will be shown in the presented work.

2. The amenability and paradoxical partitions. Tarski's number and Day's problem

The major characteristics of groups are connected with the concept of the amenability and, in particular, the metric invariants. This fact is known since the early works of Krylov and Bogolyubov on invariant measures for groups acting on a compact set.

Definition 1. The discrete group G is called the *amenable group* if it admits a G-invariant probability measure, i.e., the map $\mu : P(G) \longrightarrow [\,0,1\,]$, where $P(G)$ is the collection of all subsets of G, such that

1) μ is finitely additive,
2) $\mu(gA) = \mu(A)$ for all $g \in G$ and $A \subseteq G$,
3) $\mu(G) = 1$.

For a discrete group G by $B(G)$ we will denote the space of all bounded functions on G with the sup-norm.

A linear function m on $B(G)$ is called a *left-invariant mean*, if:

1) $m(\bar{f}) = \overline{m(f)}$,
2) $m(f) \geqslant 0, \quad f \geqslant 0, \quad m(1) = 1$,
3) $m(gf) = m(f)$, where $gf(\bar{g}) = f(g^{-1}\bar{g})$ for all $g, \bar{g} \in G$.

Using this concept we can give the equivalent definition of the amenability.

Definition 1*. The discrete group G is called the *amenable group* if on G there is a left-invariant mean.

Definition 2. A group G is *paradoxical*, if it admits the paradoxical partition, i.e., there are subsets $A_1, \ldots, A_n$, $B_1, \ldots, B_m$ of G and elements

$g_1, \dots, g_n$, $h_1, \dots, h_m$ such that

$$G = \begin{cases} A_1 \bigsqcup \cdots \bigsqcup A_n \bigsqcup B_1 \bigsqcup \cdots \bigsqcup B_m \\ g_1 A_1 \bigsqcup \cdots \bigsqcup g_n A_n \\ h_1 B_1 \bigsqcup \cdots \bigsqcup h_m B_m. \end{cases}$$

Theorem 1. (Tarski's alternative [44])
The group G either is amenable or paradoxical. □

The set AG of all amenable groups is closed with respect to the following four operations:

(1) taking of subgroups,

(2) taking of quotient groups,

(3) extensions of groups by elements from AG (G is an extension of a group H by F if $H \trianglelefteq G$ and $G/H \cong F$),

(4) the directed union of amenable subgroups $\{H_\alpha\}$ $(\bigcup_\alpha H_\alpha$ where for any H_{α_1}, H_{α_2} there is $H_\gamma \supset H_{\alpha_1} \cup H_{\alpha_2})$.

Note that any group containing a free subgroup with two generators is paradoxical. Moreover, if a subgroup H of a group G or a quotient group of G/H is paradoxical, then G is paradoxical too.

Definition 3. The smallest number $\tau = n+m$ of all paradoxical partitions of a paradoxical group G is called the *Tarski number* and is denoted by $\tau(G)$.

It is easy that $\tau(G) \geqslant 4$.

Fact 1. *If a subgroup H of G or a quotient group G/H is paradoxical, then $\tau(G) \leqslant \tau(H)$.* □

Fact 2. (Johnson, Dekker [46])
For a paradoxical group G we have $\tau(G) = 4$ if and only if G contains a free subgroup with two generators. □

Fact 3. *For a torsion group $\tau(G) \geqslant 6$.* □

Fact 4. *For any paradoxical group G there exists a finitely generated subgroup H such that $\tau(G) = \tau(H)$.* □

More interesting facts about amenable groups and the Tarski number one can find in the surveys [16], [20], [23] and [27].

Problem. *Is it true that for each natural $n \geqslant 4$ there is a paradoxical group G with $\tau(G) = n$?*

The following classes of groups will be considered:

EG – the class of all elementary amenable groups,

FG – the class of groups containing a free subgroup with two generators,

$F_N G$ – the class of groups without free subgroups with two generators.

It is clear that the class EG is the smallest class of groups containing all finite and abelian groups and closed with respect to the operations $(1) - (4)$ defining the class AG. Obviously

$$EG \subseteq AG \subseteq F_N G \subseteq (F_N G \cup FG) \tag{1}$$

and $F_N G \cap FG = \emptyset$. We know that $\tau(G) = 4$ for all groups from the class FG, $\tau(G) \geqslant 5$ for groups from the class $F_N G \backslash AG$ and $\tau(G) = \infty$ for groups from AG. In connection to this, in 1957 Day (cf. [17]) posed the following problem:

Day's problem. *Is it true that*

$$EG \subset AG \subset F_N G\,? \tag{2}$$

The above sequence of is inclusions is called the *dichotomy* or the *extremal property*.

Greenleaf [27] (and others) posed in 1969 the hypothesis that *a discrete group is either amenable or contains a free subgroup with two generators.* This means that

$$AG = F_N G. \tag{3}$$

Then, in 1979, Tits proved [45] the so-called Tits's alternative: *a finitely generated linear group either contains a free subgroup with two generators, or is almost solvable.*

Olshansky [35] (1980), Adyan [1] (19820 and Gromov [25] (1988) had found examples of finitely generated groups from the class $F_NG\backslash AG$. They found *examples of non-amenable finitely generated groups without free subgroups with two generators which are not finitely defined.*

In 1984, Grigorchuk solved [19] the Day's problem by the construction of a finitely generated group from $AG\backslash EG$ (now called the *Grigorchuk's group*). Later, Grigorchuk found [22] the second example of such group but this group is not finitely defined.

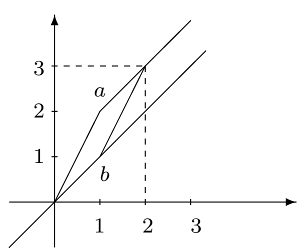
Now it is desirable to know: *is there an example of a finitely generated and finitely defined group from* $F_NG\backslash AG$? In view of Fact 1, such group will be maximally near to $\tau(G) = 5$.

The Richard Thompson's (1965) group seems to be the potential candidate of such group: *F is the set of all piecewise linear homeomorphisms* $[0,1]$, *having the breaks only in the finite number of binary rational points, and on intervals of differentiability the derivative is equal to a degree two.*

Brin and Squier have shown in 1985 (cf. [14]), that the group F is not elementary and it does not contain a free subgroup with two generators, i.e., $F \in AG\backslash EG$. Such group is isomorphic to the group with two generators and two relations [15]. Namely,

$$F = < A, B : \ [AB^{-1}, A^{-1}BA], \ [AB^{-1}, A^{-2}BA^2] > .$$

It can be realized as a group of homeomorphisms of $\mathbb{R}$ with two generators a, b having the form:



Problem 2. *Which of the statements*

1) $F \in AG\backslash EG$ *(F is amenable),*

2) $F \in F_NG\backslash AG$ *(F is not amenable)*

is true?

The answer is necessary to determine the way of further systematic investigation of groups of homeomorphisms of the real line, their metric invariants and topological characteristics.

3. The growth of a finitely generated group and a scale of correspondences

For a group $G =< g_1, \ldots, g_s >$ the important characteristic is the *growth*

$$\lambda(G) = \lim_{n\to\infty} \sqrt[n]{\gamma_G(n)},$$

where $\gamma_G(n)$ is the number of elements of the set

$$\{g : g = g_{i_1}^{\varepsilon_1}...g_{i_m}^{\varepsilon_m},\ m \leqslant n,\ i_j \in \{1, \ldots, s\},\ \varepsilon_j = \pm 1,\ j = 1, \ldots, m\}.$$

We say that the group G has the *exponential growth* if $\lambda(G) > 1$, and the *subexponential growth* if $\lambda(G) = 1$.

If for a given group G the function $\gamma_G(n)$ grows more quickly than a polynomial function, but more slowly than an exponential function, then such group is called the group with the *intermediate growth*.

For a group $G =< g_1, \ldots, g_s >$ the growth $\lambda(G)$ is always defined, and its properties do not depend on the choice of generators. For groups from the classes FG and $F_N G \setminus AG$ the growth $\lambda(G)$ is exponential. For groups from the class AG the growth $\lambda(G)$ is no more than exponential.

The growth of groups can be used to the classification of finitely generated amenable groups only.

Finitely generated groups, containing free subsemigroup with two generators, have the exponential growth. On the other hand, *if a group of homeomorphisms of the real line has two generators such that one generator is the shift on unit, the second is an affinity transformation, then this group is solvable of the step two and contains free subsemigroup with two generators, hence it has the exponential growth.*

There are examples of non-amenable finitely generated groups without free subsemigroups with two generators. Hence no connections between the property of the amenability and the existence free subsemigroups with two generators. But there are also "typical" groups with the exponential growth containing free subsemigroups with two generators.

Nevertheless, such one-to-one correspondence takes place for some classes of finitely generated groups. Namely, in 1981 Gromov proved the following theorem [24]:

Theorem 2. *A finitely generated group has the polynomial growth if and only if it is almost nilpotent.* □

Earlier, in 1974, a similar result was proved by Rosenblatt [38] for solvable groups.

Theorem 3. *A finitely generated solvable group without free subsemigroups with two generators is almost nilpotent. This means that it has the polynomial growth.* □

From Theorem 2 it follows that for Grigorchuk's group the growth is more than polynomial. Grigorchuk proved in 1984 (cf. [19]) a stronger result, which can be formulated in the following way:

Theorem 4. *The growth of the Grigorchuk's group is more than polynomial and less than exponential, i.e., this group has the intermediate growth.* □

4. About realizations of abstract groups as groups of action on the real line (circle)

We start with the theorem proved in 1996 by Ghys [26].

Theorem 5. *An account group can be realized as a group of preserving orientation homeomorphisms of the real line if and only if it is rightordered.*

This result was presented for me (independently to [26]), by Grigorchuk who observed that such realization possess one additional important property. Namely, for ordered account groups the graphs of different elements have the form of a cortege, i.e., the graph of one of them is dishosed above the graph of other, though the tangency is possible.

Starting from the construction of the Grigorchuk's group the first nontrivial example of a subgroup of $Homeo_{+}([0,1])$ (the preserving orientation homeomorphisms of an interval $[0,1]$) has been obtained as a result of embedding of $Homeo_+([0,1])$ into some group associated with the Grigorchuk's group and having intermediate growth.

Grigorchuk and Maki have proved in [21] the following theorem.

Theorem 6. *A group $Homeo_+([0,1])$ has a finitely generated subgroup with intermediate growth.* $\square$

Theorem 5 shows that groups of homeomorphisms of an interval, the real line and a circle are the universal object for the abstract theory of groups. By Theorem 6, such groups have a nontrivial structure.

5. Topological characteristics and invariant measures for groups homeomorphisms of the real line and the circle

One of the first results in this direction has been obtained in 1939 by Krylov and Bogolyubov [13], then (in 1961) by Day [18].

Theorem 7. *For discrete amenable groups G, acting continuously on a compact space, there is a G–invariant Borel measure.* $\square$

Various aspects of proofs of this theorem are analyzed in the review [3].

Note that the existence of an invariant Borel measure is equivalent to the existence of his topologically characteristic support. Therefore, it is difficult to expect the presence of a criterion of the existence of an invariant Borel measure by the terms of the amenability, or the algebraic characteristics of the initial group.

Nevertheless, Plante [36] has formulated such type criterion for some finitely generated groups. This criterion is formulated in the term of the subexponential growth of orbits of points (Theorem 8 below).

Definition 4. We say that the orbit $G(t)$ of the point $t \in \mathbb{R}$ has the *subexponential growth*, if

$$\lim_{n\to\infty} \inf \frac{1}{n} \log |G^n(t)| = 0,$$

where G^n is the set of all words of length no more than n, and $|G^n|$ is the cardinality of G^n.

The form of the condition of the subexponential growth has the asymptotic character, but, in fact, it is a topological characteristic.

Theorem 8. *Let $G \subseteq Homeo_+(\mathbb{R})$ be a finitely generated group. The existence of a Borel measure finite on compact subsets and invariant with respect to the group G is equivalent to the existence of a point $t \in \mathbb{R}$ with the orbit having the subexponential growth.* □

Unfortunately, this theorem does not admit a generalization to groups which are not finitely generated.

Nevertheless Ghys (1998) posed a hypothesis that Theorem 7 can be proved also for groups acting on the circle. This hypothesis was verified by Margulis (2000). Namely, he proved in [33]:

Theorem 9. *For any group of homeomorphisms of the circle there exists a free subgroup with two generators or a probability Borel measure invariant with respect to this group.* □

This alternative is not strong. There are groups of homeomorphisms of the circle, for which there are both a free subgroup with two generators and a probability invariant Borel measure simultaneously.

Note, that result analogous to the above theorem was obtained earlier (1984) by Solodov. His result was formulated in other terms (cf. [42]). The equivalence of these two results was proved by Beklaryan in 2002 (cf. [11]).

Denote by $Homeo(X)$ the group of homeomorphisms of $X = \mathbb{R}$, S^1 and by $Homeo_+X$ – the group of orientation-preserving homeomorphisms of $X = \mathbb{R}$, S^1.

Since for a group $G \subseteq Homeo(X)$ the set G_+ of all preserving orientation homeomorphisms defines the normal subgroup of an index no more than two, the study of such groups can be reduced to the study of groups of preserving orientation homeomorphisms of X.

For a group $G \subseteq Homeo_+(X)$ we additionally define the set

$$G^s = \{g \in G : \exists t \in \mathbb{R},\ g(t) = t\},$$

which is the union of stabilizers.

For $X = \mathbb{R}$ we also define the set

$$G^s_\infty = \{g \in G^s : \sup\{t : g(t) = t\} = +\infty,\ \inf\{t : g(t) = t\} = -\infty\}.$$

Note that G^s is not a group, in general, but

$$G^s \subseteq < G^s > \subseteq G\,. \tag{4}$$

Moreover, the following lemma is true (cf. [4]).

Lemma 1. *For $G \subseteq Homeo_+(\mathbb{R})$ we have $G^s =< G^s >$ or $< G^s >= G$.* □

This alternative is not strong. There are groups for which $G^s = G$.

Theorem 10. *For $G \subseteq Homeo_+(X)$ the quotient group $G/<G^s>$ is commutative and isomorphic to some subgroup of the additive group of X.* □

In the proof of this theorem Lemma 1 and the Hölder's theorem about archimedean groups (cf. [4]) are used.

For many special cases (for finitely generated groups, for finitely generated groups without free subsemigroups with two generators, ...) this theorem has been easier proved by Novikov [34], Imanishi [29] and Salhi [39], [41].

5.1. Topological characterizations

For any group $G \subseteq Homeo_+(X)$ we define the set:

$$Fix\, G^s = \{t \in X : \forall g \in G^s,\ g(t) = t\}.$$

Definition 5. By a *minimal set* of a group $G \subseteq Homeo(X)$ we mean such closed G-invariant subset of X which do not contains any proper closed G-invariant subsets. If there is no such set, then we say that minimal set is empty. If for a group G there exists only one minimal set, then it is denoted by $E(G)$.

A very important characterization of minimal sets was given in 1996 by Beklaryan [7]. Namely, he proved that

Theorem 11. *For a group $G \subseteq Homeo_+(\mathbb{R})$ the following four cases are possible:*

a) any minimal set is discrete and is contained in $Fix\, G^s$ (in this case $Fix\, G^s$ is the union of minimal sets), *b*) *the minimal set is a perfect anywhere dense subset of* $\mathbb{R}$ (*in this case it is a unique minimal set and it is contained in the closure of the orbit* $G(t)$ *of an arbitrary point* $t \in \mathbb{R}$),

c) *the minimal set coincides with* $\mathbb{R}$,

d) *the minimal set is empty.* □

Earlier, the minimal sets of cyclic groups of homeomorphisms of the circle were investigated in [2] and [31]. Note that in the compact case for any group $G \subseteq Homeo_{+}(S^1)$ the non-empty minimal set always exists.

Account groups of homeomorphisms and groups of diffeomorphisms of the real line were investigated by Salhi. The minimal sets of groups $G \subseteq Homeo_{+}(S^1)$ were studied by many authors.

The problem of existence of non-empty minimal sets was partially solved in [7], where the following is proved:

Proposition 1. *If a group* $G \subseteq Homeo_{+}(\mathbb{R})$

a) *is finitely generated, or*

b) $Fix\, G^s \neq \emptyset$, *or*

c) $G \neq G^s_\infty$,

then it has a non-empty minimal set. □

The proof of this proposition is based on the axiom of choice, so the minimal set cannot be described constructively. But in the case $Fix\, G^s \neq \emptyset$ we have a stronger result [6]:

Theorem 12. *Let* $G \subseteq Homeo_{+}(X)$. *If* $Fix\, G^s \neq \emptyset$, *then:*

1) *for every* $t \in Fix\, G^s$ *the set* $\mathbb{P}(G)$ *of all limit points of* $G(t)$ *does not depends on the point* t,

2) $\mathbb{P}(G) \subseteq Fix\, G^s$,

3) *either* $\mathbb{P}(G) = X$, *or* $\mathbb{P}(G)$ *is the perfect anywhere dense subset of* $\mathbb{X}$, *or* $\mathbb{P}(G) = \emptyset$,

4) *if* $\mathbb{P}(G) \neq \emptyset$, *then* G *has the unique non-discrete minimal set* $E(G)$ *and* $\mathbb{P}(G)$ *coincides with* $E(G)$,

5) *in the case* $\mathbb{P}(G) = \emptyset$, *all minimal sets are discrete, belong to* $Fix\, G^s$ *and* $Fix\, G^s$ *is the union of these minimal sets.* □

In the study of some problems, for example, in the study of tracks of groups of quasiconformal maps of the upper half plane [10], a very important role plays the possibility of replacement of the initial group of homeomorphisms by its subgroup with the same topological complexity (i.e., with the same minimal set). Therefore, the investigation of connections between subgroups of initial groups and their minimal sets represents the big interest. We present two lemmas proved in [9] as examples of such results.

Lemma 2. *If the minimal set* $E(\Gamma)$ *of a subgroup* $\Gamma \subseteq G \subseteq Homeo_+(X)$ *is non-empty and non-discrete, then the minimal set* $E(G)$ *of* G *is also non-empty and non-discrete and* $E(\Gamma) \subseteq E(G)$. □

Lemma 3. *If the minimal set* $E(\Gamma)$ *of a normal subgroup* $\Gamma \subseteq G \subseteq Homeo_+(X)$ *is non-empty and non-discrete, then it coincides with the minimal set of the initial groups* G, *i.e.,* $E(\Gamma) = E(G)$. □

The latter lemma gives the possibility to reduce the study of the initial groups of homeomorphisms and its minimal sets to the study of smallest and simplest groups and their minimal sets.

5.2. Invariant measures

Since the existence of invariant Borel measures is equivalent to the existence of their closed supports, the criterion of the existence of such measure can be formulated in terms of topological characteristics.

Theorem 13. *For* $G \subseteq Homeo_+(X)$ *the set* $Fix\, G^s$ *is either empty, or it is a Borel (probabilistic, in the case* $X = S^1$*) measure* μ, *finite on compact sets and invariant with respect to the group* G. □

The proof of this theorem [5] is based on our Theorems 10, 11 and 12.

If in the Margulis theorem (Theorem 9) the existence of an invariant measure will be guaranteed by $Fix\, G^s \neq \emptyset$, then we obtain the result proved earlier (1984) by Solodov [42]. In terms of homomorphisms (characters) this result was proved (1983) by Hector and Hirsch [28] for finitely generated groups of homeomorphisms of the circle. For arbitrary groups of homeomorphisms of the circle it has been obtained in 1996 by Beklaryan [7]. More interesting facts about various criterions of the existence of an invariant measure for groups of homeomorphisms of the real line (circle) one can find in the review [12].

Now we focus our attention on four theorems proved in [6] and their consequences.

Theorem 14. *If for $G \subseteq Homeo_+(X)$ there exists a Borel (probabilistic, in the case $X = S^1$) measure μ, finite on compact sets and invariant with respect to the group G, then $supp\,\mu \subseteq Fix\,G^s$ and $supp\,\mu = \mathbb{P}(G) = E(G)$, if $\mathbb{P}(G) \neq \emptyset$ (in this case μ is continuous). In the case $\mathbb{P}(G) = \emptyset$ the support of μ is the union of some discrete minimal sets.* □

Definition 6. A group $G \subseteq Homeo_+(X)$ is *strictly ergodic*, if there is a Borel measure, finite on compact sets and invariant with respect to the group G, and for any two invariant measures μ_1, μ_2 there is a constant $c > 0$ such that $\mu_1 = c\mu_2$.

Theorem 15. *If for the group $G \subseteq Homeo_+(X)$ there is a Borel (probabilistic, in the case $X = S^1$) measure, finite on compact sets and invariant with respect to the group G, then G is strictly ergodic if and only if*

1) $\mathbb{P}(G) \neq \emptyset$, *or*
2) $\mathbb{P}(G) = \emptyset$ *and $Fix\,G^s$ coincides with the unique non-empty minimal set.* □

Now, using the above results, especially Theorem 13, we can present the criterion of the existence of invariant measures in another form.

Theorem 16. *Let $G \subseteq Homeo_+(\mathbb{R})$. For the existence of Borel measures, finite on compact sets and invariant with respect to the group G, it is necessary and sufficient, that:*

1) *for any finitely generated subgroup $\Gamma \subseteq G$ there is a Borel measure, finite on compact sets and invariant with respect to the subgroup Γ,*

2) *there is a natural number n such that $[-n, n] \cap Fix\,\Gamma^s \neq \emptyset$ for any finitely generated subgroup Γ of G.* □

Theorem 17. *Let $G \subseteq Homeo_+(\mathbb{R})$. If the quotient group $G/< G^s >$ is non-trivial, i.e., $G/< G^s > \neq < e >$, then there is a Borel measure finite on compact sets and invariant with respect to the group G. Moreover, if the quotient group $G/< G^s >$ is non-cyclic, then the group G is strictly ergodic.* □

5.3. Combinatorial aspects

In view of Theorem 14 the support of an invariant measure is the union of minimal sets.

The natural problem is: *What are the combinatorial obstacles for a group with the non-empty minimal set to have an invariant measure*?

Various aspects of this problem were studied by many authors. Below we present some results obtained in [11] by Beklaryan.

In the formulation of these results a normal subgroup H_G of G plays an important role.

Definition 7. For a group $G \subseteq Homeo(X)$ we define the *normal subgroup* H_G in the following way:

1) if a minimal set is non-empty and non-discrete, then

$$H_G = \{h \in G_+ : E(G_+) \subseteq Fix\ < h >\},$$

2) if a minimal set is non-empty and discrete, then $H_G = G^s_+$,

   (since a minimal set is discrete, $Fix\, G^s_+$ is non-empty, consequently G^s_+ is a normal subgroup),

3) if a minimal set is empty, then we put $H_G =< e >$.

Note that $H_G = < e >$ also in the case when a minimal set coincides with the real line.

Theorem 18. *Let $G \subseteq Homeo\,(S^1)$. Then either the quotient group G/H_G contains a free subgroup with two generators, or there is a probabilistic Borel measure invariant with respect to the group G.* □

Theorem 19. *Let $G \subseteq Homeo\,(\mathbb{R})$ be a group with a non-empty minimal set. Then either the quotient group G/H_G contains a free subsemigroup with two generators, or there is a Borel measure finite on compact sets and invariant with respect to the group G.* $\square$

5.4. About analogs of the Tits's alternative

For groups $G \subseteq Homeo\,(X)$ with an invariant measure we have $H_G = G_+^s$, where G_+ is the maximal normal subgroup of all orientation-preserving homeomorphisms with the index no more than two. Thus, the quotient group G_+/H_G is commutative.

Theorem 18 about existence of an invariant measure on the circle can be reformulated to the form analogous to the Tits's alternative (cf. [11]).

Theorem 20. *For groups $G \subseteq Homeo\,(S^1)$ either the quotient group G/H_G contains a free subgroup with two generators, or contains a commutative normal subgroup G_+/H_G of index no more than two.* $\square$

For any group $G \subseteq Homeo_+\,(S^1)$ the action of an element $\tilde{g} \in G/H_G$ can be realized as action on the circle. If $\tilde{g}$ corresponds to $g \in G$, then the action of $\tilde{g}$ coincides with the action of g on the minimal set.

If we denote by KG the class of almost commutative groups, then the chain (1) can be expanded to the chain

$$KG \subseteq EG \subseteq AG \subseteq F_N G \subseteq (F_N G \cup FG)\,. \tag{5}$$

Note that for groups of homeomorphisms of the circle Theorem 18 is equivalent to the condition

$$KG = F_N G$$

for corresponding quotient groups. This means that groups of homeomorphisms of the circle satisfy the extremal property for quotient groups.

Moreover, for such finitely generated groups, the growth of the quotient group satisfies the extremal property as well: either it is polynomial (for quotient groups from $F_N G$), or it is exponential (for quotient groups from FG) and there are no quotient groups having the intermediate growth.

Theorem 19 about existence of an invariant measure on the real line can be reformulated to the form analogous to the Tits's alternative (cf. [11]).

Theorem 21. *If $G \subseteq Homeo(\mathbb{R})$ is a group with the non-empty minimal set, then it either has the quotient group G/H_G containing a free subsemigroup with two generators, or it contains the commutative normal subgroup G_+/H_G with the index no more than two.* □

Let FPG be a class of groups containing the free subsemigroup with two generators, $F_N PG$ – a class of groups without free subsemigroups with two generators. Then

$$KG \subseteq F_N PG \subseteq (F_N PG \cup FPG) . \tag{6}$$

For groups of homeomorphisms of the real line Theorem 19 is equivalent to the condition

$$KG = F_N PG$$

for corresponding quotient groups and means that groups of homeomorphisms of the real line satisfy the extremal property for quotient groups. Moreover, for such finitely generated groups, the growth of the quotient group satisfies the extremal property also: either it is polynomial (for quotient groups from $F_N PG$), or it is exponential (for quotient groups from FPG) and there are no quotient groups of intermediate growth.

Thus, an investigation of groups of homeomorphisms G of the circle (the real line) can be reduced to the study of the canonical subgroups H_G in which all algebraic properties of the initial group are concentrated. For this purpose the additional metric invariants in the form of a projectively-invariant measure [5], [7], [8], [37] and a ω-projectively-invariant measure [9] can be applied.

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Fuzzy regular congruence relations on hyper BCK-algebras

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Abstract

In this manuscript, by considering the notion of fuzzy regular congruence relation on a hyper BCK-algebra, we construct a quotient hyper BCK-algebra and then we state and prove some related theorems. Finally, we state and prove isomorphism theorems on that structure.

1. Introduction

The study of BCK-algebras was initiated by Y. Imai and K. Iséki [7] in 1966, as a generalization of the concept of set-theoretic difference and propositional calculi. Since then a great deal of literature has been produced on the theory of BCK-algebras.

The hyperstructure theory (called also multialgebras) was introduced in 1934 by F. Marty [11] at the 8th congress of Scandinavian Mathematiciens. Around the 40's, several authors worked on hypergroups, especially in France and in the United States, but also in Italy, Russia and Japan. Hyperstructures have many applications to several sectors of both pure and applied sciences.

In [10] Y. B. Jun et al. applied the hyperstructures to BCK-algebras, and introduced the notion of a hyper BCK-algebra which is a generalization of BCK-algebra, and investigated some related properties. The notion of regular congruence relation on hyper BCK-algebras have been introduced by R. A. Borzooei et al [6]. In [1], [4] and [5], the authors studied the fuzzy set theory on hyper BCK-algebras and defined the notion of a fuzzy

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congruence relation on a hyper BCK-algebra. Now, in this paper, we follow the references and we obtain some results as mentioned in the abstract.

2. Preliminaries

Definition 2.1. By a *hyper BCK-algebra* we mean a non-empty set H endowed with a hyperoperation "$\circ$" and a constant 0 satisfying the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x \circ H \ll \{x\}$,
(HK4) $x \ll y$ and $y \ll x$ imply $x = y$,

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$.

Proposition 2.2. [10] *In any hyper BCK-algebra H for all $x, y, z \in H$ the following hold:*

(i) $x \ll x$,
(ii) $0 \circ x = \{0\}$,
(iii) $x \circ y \ll x$,
(iv) $x \circ 0 = \{x\}$.

Definition 2.3. A non-empty subset I of hyper BCK-algebra H is said to be a (*weak*) *hyper BCK-ideal* if $(x \circ y \subseteq I)$ $x \circ y \ll I$ and $y \in I$ imply $x \in I$.

Definition 2.4. Let $(H_1, \circ_1)$ and $(H_2, \circ_2)$ be two hyper BCK-algebras and $f : H_1 \longrightarrow H_2$ be a function. Then, f is called

- a *homomorphism*, if $f(x \circ_1 y) = f(x) \circ_2 f(y)$, for all $x, y \in H_1$,
- an *isomorphism*, if f is a one-to-one and onto homomorphism.

Note. From now on, in this paper, H denotes a hyper BCK-algebra.

Definition 2.5. Let Θ be a binary relation on H and $A, B \subseteq H$. Then,

(i) $A\Theta B$ means that, there exist $a \in A$ and $b \in B$ such that $a\Theta b$,

(ii) $A\bar{\Theta}B$ means that, for all $a \in A$ there exists $b \in B$ and for all $b \in B$ there exists $a \in A$ such that $a\Theta b$,

(iii) Θ is called *left* (resp. *right*) *compatible* if $x\Theta y$ implies that $a \circ x\bar{\Theta} a \circ y$ $(x \circ a\bar{\Theta} y \circ a)$, for all $a, x, y \in H$,

(iv) Θ is called a *congruence* if it is left and right compatible,

(v) Θ is called *regular* if $x \circ y\Theta\{0\}$ and $y \circ x\Theta\{0\}$ imply $x\Theta y$, for all $x, y \in H$.

Theorem 2.6. [6] *Let Θ be a regular congruence relation on H, $I = [0]_\Theta$ and $H/I = \{I_x : x \in H\}$, where $I_x = [x]_\Theta$. Then H/I with hyperoperation "$\circ$" and hyperorder "$\ll$" which is defined as follows,*

$$I_x \circ I_y = \{I_z : z \in x \circ y\} \quad , \quad I_x \ll I_y \longleftrightarrow I \in I_x \circ I_y$$

is a hyper BCK-algebra which is called "quotient hyper BCK-algebra".

Definition 2.7. Let μ be a fuzzy subset of H. Then for all $t \in [0, 1]$, the *level subset* μ_t of H is defined by $\mu_t = \{x \in H : \mu(x) \geqslant t\}$. Moreover, μ satisfies the *sup property* (*inf property*), if for each non-empty subset T of X there exists $x_0 \in T$ such that $\mu(x_0) = \sup\limits_{x \in T} \mu(x)$ $\left(\mu(x_0) = \inf\limits_{x \in T} \mu(x)\right)$.

Definition 2.8. Let $f : H_1 \longrightarrow H_2$ be a homomorphism of hyper BCK-algebras and μ be a fuzzy subset of H_2. Then fuzzy subset $f^{-1}(\mu)$ of H_1 is defined by $f^{-1}(\mu)(x) = \mu(f(x))$, for all $x \in H_1$.

Definition 2.9. Let H be a hyper BCK-algebra. A function $\rho : H \times H \to [0, 1]$ is called a *fuzzy relation* on H. A fuzzy relation ρ on H is said to be a *fuzzy equivalence relation* if for all $x, y \in H$

$\rho(x, x) = \sup\limits_{(y,z) \in H^2} \rho(y, z)$, (Fuzzy reflexive)

$\rho(y, x) = \rho(x, y)$, (Fuzzy symmetric)

$\rho(x, y) \geqslant \sup_{z \in H} \min(\rho(x, z), \rho(z, y))$, (Fuzzy transitive).

Definition 2.10. Let ρ be a fuzzy equivalence relation on H. Then ρ is said to be a

- *fuzzy left compatible* if for all $u \in a \circ x$ there exists $v \in a \circ y$ and for all $v \in a \circ y$ there exists $u \in a \circ x$ such that $\rho(u, v) \geqslant \rho(x, y)$, for all $a, x, y \in H$.

- *fuzzy right compatible* if for all $z \in x \circ a$ there exists $w \in y \circ a$ and for all $w \in y \circ a$ there exists $z \in x \circ a$ such that $\rho(z, w) \geqslant \rho(x, y)$, for all $a, x, y \in H$.

- *fuzzy congruence relation* if ρ is fuzzy left and fuzzy right compatible.

Theorem 2.11. [5] *Let ρ be a fuzzy relation on H. If ρ is a fuzzy congruence relation on H then for all $t \in [0,1]$, $\rho_t = \{(x,y) \in H \times H : \rho(x,y) \geqslant t\} \neq \emptyset$, is a congruence relation on H. Conversely, if ρ satisfies the sup property and for all $t \in [0,1]$, $\rho_t \neq \emptyset$ is a congruence relation on H, then ρ is a fuzzy congruence relation on H.*

Notation: By $\mathcal{F}_R(H)$, $\mathcal{F}_E(H)$ and $\mathcal{F}_C(H)$ we mean respectively, the set of all fuzzy relations, fuzzy equivalence relations and fuzzy congruence relations on H.

3. Quotient structures

Definition 3.1. $\rho \in \mathcal{F}_R(H)$ is called *fuzzy regular* if for all $x, y \in H$,

$$\rho(x,y) \geqslant \min\left(\sup_{a \in x \circ y} \rho(a,0), \sup_{b \in y \circ x} \rho(b,0)\right).$$

Theorem 3.2. *If $\rho \in \mathcal{F}_R(H)$ is fuzzy regular, then each $\rho_t \neq \emptyset$ is a regular relation on H. Conversely, if ρ satisfies the sup property and each $\rho_t \neq \emptyset$ is a regular relation on H, then ρ is fuzzy regular on H.*

Proof. ($\Leftarrow$) Let for all $s \in [0,1]$, $\rho_s \neq \emptyset$ be a regular relation on H. We first show that ρ is a fuzzy equivalence relation. Let $t = \sup_{(y,z) \in H^2} \rho(y,z)$. Since, ρ satisfies the sup property, then $\rho_t \neq \emptyset$. Now, since ρ_t is a reflexive relation, then $(x,x) \in \rho_t$ and so $\rho(x,x) \geqslant t$ for all $x \in H$. Hence,

$$\rho(x,x) \leqslant \sup_{(y,z) \in H^2} \rho(y,z) = t \leqslant \rho(x,x)$$

and so $\rho(x,x) = \sup_{(y,z) \in H^2} \rho(y,z)$. Thus, ρ is a fuzzy reflexive relation. Moreover, it is easy to check that ρ is a fuzzy symmetric and fuzzy transitive relation. Therefore, ρ is a fuzzy equivalence relationon H. Now, let

$$t = \min\left(\sup_{a \in x \circ y} \rho(a,0), \sup_{b \in y \circ x} \rho(b,0)\right).$$

Since, ρ satisfies the sup property, then there exist $a_0 \in x \circ y$ and $b_0 \in y \circ x$ such that $\rho(a_0, 0) = \sup_{a \in x \circ y} \rho(a,0) \geqslant t$ and $\rho(b_0, 0) = \sup_{b \in y \circ x} \rho(b,0) \geqslant t$

and so $a_0\rho_t 0$ and $b_0\rho_t 0$. This implies that $x \circ y \rho_t \{0\}$ and $y \circ x \rho_t \{0\}$. Since, ρ_t is a regular relation, then $x\rho_t y$ and so

$$\rho(x, y) \geqslant t = \min\left(\sup_{a\in x\circ y} \rho(a, 0), \sup_{b\in y\circ x} \rho(b, 0)\right).$$

Therefore, ρ is a fuzzy regular relation on H.

($\Rightarrow$) Let ρ be a fuzzy regular relation on H, $t \in [0, 1]$ and $\rho_t \neq \emptyset$. We first show that ρ_t is an equivalence relation on H. Since, ρ_t is a non-empty subset of H, there exists $y, z \in H$ such that $(y, z) \in \rho_t$ and so $\rho(y, z) \geqslant t$. Since, for all $x \in H$, $\rho(x, x) = \sup_{(y,z)\in H^2} \rho(y, z) \geqslant t$, then $(x, x) \in \rho_t$ and so ρ_t is a reflexive relation. It is easy to check that ρ_t is a symmetric and transitive relation on H. Therefore, ρ_t is an equivalence relation on H. Now, let $x \circ y \rho_t \{0\}$ and $y \circ x \rho_t \{0\}$, for $x, y \in H$. Then, there exist $a \in x \circ y$ and $b \in y \circ x$ such that $a\rho_t 0$ and $b\rho_t 0$. Since, ρ is a fuzzy regular relation on H, then

$$\rho(x, y) \geqslant \min\left(\sup_{u\in x\circ y} \rho(u, 0), \sup_{v\in y\circ x} \rho(v, 0)\right) \geqslant \min(\rho(a, 0), \rho(b, 0)) \geqslant t$$

and so $x\rho_t y$. Hence, ρ_t is a regular relation on H. □

Definition 3.3. Let $\rho \in \mathcal{F}_R(H)$. Then for all $x \in H$, the fuzzy subset $\mu_x : H \longrightarrow [0, 1]$ is defined as follows: for all $y \in H$,

$$\mu_x(y) = \rho(y, x).$$

Notation. From now on, in this paper, for all $y \in H$ we let

$$\mu(y) = \mu_0(y)(= \rho(y, 0)).$$

Lemma 3.4. *Let $\rho \in \mathcal{F}_E(H)$. Then,*

(*i*) *for all $x, y \in H$, $\mu_x = \mu_y$ if and only if $\rho(x, y) = \sup_{(w,z)\in H^2} \rho(w, z)$,*

(*ii*) *if $t \in [0, 1]$ and $\rho_t \neq \emptyset$, then $[0]_{\rho_t} = \mu_t$.*

Proof. (*i*) Let $\mu_x = \mu_y$, for $x, y \in H$. Since, ρ is a fuzzy reflexive relation, then

$$\rho(x, y) = \mu_y(x) = \mu_x(x) = \rho(x, x) = \sup_{(w,z)\in H^2} \rho(w, z).$$

Conversely, let $\rho(x,y)=\sup_{(u,v)\in H^2}\rho(u,v)$, for $x,y\in H$ and $w\in H$. Since, ρ is a fuzzy symmetric and fuzzy transitive relation, then

$$\begin{aligned}\mu_x(w)&=\rho(w,x)=\rho(x,w)\geqslant\min(\rho(x,y),\rho(y,w))\\&=\min\left(\sup_{(u,v)\in H^2}\rho(u,v),\rho(y,w)\right)=\rho(y,w)=\rho(w,y)=\mu_y(w).\end{aligned}$$

Similarly, we can show that $\mu_y(w)\geqslant\mu_x(w)$. Hence, for all $w\in H$, $\mu_x(w)=\mu_y(w)$ and so $\mu_x=\mu_y$.

(ii) Let $x\in[0]_{\rho_t}$. Then, $x\rho_t 0$ and so $\mu(x)=\rho(x,0)\geqslant t$. Hence, $x\in\mu_t$. Conversely, if $x\in\mu_t$ then $\rho(x,0)=\mu(x)\geqslant t$ and so $x\rho_t 0$. Hence, $x\in[0]_{\rho_t}$. Therefore, $[0]_{\rho_t}=\mu_t$. □

Theorem 3.5. *Let ρ be a fuzzy regular congruence relation on H and*

$$H/\mu=\{\mu_x : x\in H\}.$$

If a hyperoperation "$\circ$" and a hyperorder "$\ll$" on H/μ are defined as follows:

$$\mu_x\circ\mu_y=\mu_{x\circ y}=\{\mu_z : z\in x\circ y\},$$

$$\mu_x\ll\mu_y\longleftrightarrow\ \mu\in\mu_x\circ\mu_y,$$

then $(H/\mu,\circ,\mu)$ is a hyper BCK-algebra.

Proof. First we show that a hyperoperation "$\circ$" is well-defined. Let $\mu_x=\mu_{x'}$ and $\mu_y=\mu_{y'}$. Then, by Lemma 3.4(i),

$$\rho(x,x')=\sup_{(u,z)\in H^2}\rho(u,z)=\rho(y,y').$$

Let $t=\sup_{(u,z)\in H^2}\rho(u,z)$. Hence, $x\rho_t x'$ and $y\rho_t y'$. Since, by Theorem 2.11, ρ_t is a congruence relation on H, then $x\circ y\overline{\rho_t}x'\circ y$ and $x'\circ y\overline{\rho_t}x'\circ y'$ and so $x\circ y\overline{\rho_t}x'\circ y'$. Now, let $\mu_z\in\mu_x\circ\mu_y$. Then, there exists $z'\in x\circ y$ such that $\mu_z=\mu_{z'}$. Since, $z'\in x\circ y$ and $x\circ y\overline{\rho_t}x'\circ y'$, there exists $w\in x'\circ y'$ such that $z'\rho_t w$ and so $\rho(z',w)\geqslant t=\sup_{(u,z)\in H^2}\rho(u,z)\geqslant\rho(z',w)$. Hence $\rho(z',w)=t$. Since ρ is a fuzzy equivalence relation, then for all $u\in H$,

$$\begin{aligned}\mu_z(u)&=\mu_{z'}(u)=\rho(u,z')=\rho(z',u)\geqslant\min(\rho(z',w),\rho(w,u))\\&=\min(t,\rho(w,u))=\rho(w,u)=\rho(u,w)=\mu_w(u).\end{aligned}$$

Conversely, for all $u \in H$,

$$\begin{aligned}\mu_w(u) = \rho(u,w) = \rho(w,u) \geqslant \min(\rho(w,z'),\rho(z',u)) = \min(\rho(z',w),\rho(z',u))\\ = \min(t,\rho(z',u)) = \rho(z',u) = \rho(u,z') = \mu_{z'}(u) = \mu_z(u).\end{aligned}$$

Hence, $\mu_z(u) = \mu_w(u)$, for all $u \in H$ and so $\mu_z = \mu_w$. Since, $w \in x' \circ y'$, then $\mu_z = \mu_w \in \mu'_x \circ \mu'_y$ and so $\mu_x \circ \mu_y \subseteq \mu'_x \circ \mu'_y$. Similarly, we can show that $\mu'_x \circ \mu'_y \subseteq \mu_x \circ \mu_y$. Therefore, $\mu_x \circ \mu_y = \mu_{x'} \circ \mu_{y'}$.

Now we establish the axioms of a hyper BCK-algebra.

<u>(HK1)</u> Let $\mu_v \in (\mu_x \circ \mu_z) \circ (\mu_y \circ \mu_z)$. Then, there exist $\mu_u \in \mu_x \circ \mu_z$ and $\mu_w \in \mu_y \circ \mu_z$ such that $\mu_v \in \mu_u \circ \mu_w$ and so there exists $a \in u \circ w$ such that $\mu_v = \mu_a$. Since, $a \in u \circ w \subseteq (x \circ z) \circ (y \circ z) \ll x \circ y$, then there exists $b \in x \circ y$ such that $a \ll b$ and so $0 \in a \circ b$. This implies that $\mu \in \mu_{a\circ b} = \mu_a \circ \mu_b = \mu_v \circ \mu_b \subseteq ((\mu_u \circ \mu_w) \circ (\mu_x \circ \mu_y)) \subseteq ((\mu_x \circ \mu_z) \circ (\mu_y \circ \mu_z)) \circ (\mu_x \circ \mu_y)$. Thus, $(\mu_x \circ \mu_z) \circ (\mu_y \circ \mu_z) \ll \mu_x \circ \mu_y$.

<u>(HK2)</u> Let $\mu_u \in (\mu_x \circ \mu_y) \circ \mu_z$. Then, there exists $v \in (x \circ y) \circ z$ such that $\mu_u = \mu_v$. Since, $v \in (x \circ y) \circ z = (x \circ z) \circ y$ then $\mu_u = \mu_v \in (\mu_x \circ \mu_z) \circ \mu_y$. This implies that $(\mu_x \circ \mu_y) \circ \mu_z \subseteq (\mu_x \circ \mu_z) \circ \mu_y$. Similarly, we can show that $(\mu_x \circ \mu_z) \circ \mu_y \subseteq (\mu_x \circ \mu_y) \circ \mu_z$. Thus, $(\mu_x \circ \mu_y) \circ \mu_z = (\mu_x \circ \mu_z) \circ \mu_y$.

<u>(HK3)</u> Let $\mu_z \in \mu_x \circ H/\mu$. Then, there exists $\mu_y \in H/\mu$ such that $\mu_z \in \mu_x \circ \mu_y$ and so there exists $w \in x \circ y$ such that $\mu_z = \mu_w$. Since, $x \circ y \ll x$ then $w \ll x$ and so $0 \in w \circ x$. Thus, $\mu \in \mu_{w\circ x} = \mu_w \circ \mu_x = \mu_z \circ \mu_x$. This implies that $\mu_z \ll \mu_x$ and so $\mu_x \circ H/\mu \ll \mu_x$.

<u>(HK4)</u> Let $\mu_x \ll \mu_y$ and $\mu_y \ll \mu_x$. Then, $\mu \in \mu_x \circ \mu_y$ and $\mu \in \mu_y \circ \mu_x$. Hence, there exist $z \in x \circ y$ and $w \in y \circ x$ such that $\mu_z = \mu_0 = \mu_w$ and so by Lemma 3.4(i), $\rho(z,0) = \sup_{(a,b)\in H^2} \rho(a,b) = \rho(w,0)$. Let $t = \sup_{(a,b)\in H^2} \rho(a,b)$. Then, $z\rho_t 0$ and $w\rho_t 0$. Since, $z \in x \circ y$ and $w \in y \circ x$, then $x \circ y \rho_t \{0\}$ and $y \circ x \rho_t \{0\}$. Since, by Theorem 3.2, ρ_t is a regular relation, hence $x\rho_t y$ and so $\rho(x,y) \geqslant t = \sup_{(a,b)\in H^2} \rho(a,b) \geqslant \rho(x,y)$. Thus, $\rho(x,y) = \sup_{(y,u)\in H^2} \rho(y,u)$ and so by Lemma 3.4(i), $\mu_x = \mu_y$. □

Theorem 3.6. *If ρ is a fuzzy congruence relation on H, then μ is a fuzzy hyper BCK-ideal of H.*

Proof. Let $x \ll y$, for $x, y \in H$. Then $0 \in x \circ y$. Since, $x \in x \circ 0$ and ρ is fuzzy left compatible, then $\mu(x) = \rho(x,0) \geqslant \rho(y,0) = \mu(y)$. Now, let $x, y \in H$ and $a \in x \circ y$. Since, $x \in x \circ 0$, then $\rho(x,a) \geqslant \rho(y,0)$ and since ρ is fuzzy transitive, then

$$\mu(x) = \rho(x,0) \geqslant \min(\rho(x,a),\rho(a,0)) \geqslant \min(\rho(y,0),\rho(a,0))$$

$$\geqslant \min\left(\inf_{a\in x\circ y}\rho(a,0),\rho(y,0)\right)=\min\left(\inf_{a\in x\circ y}\mu(a),\mu(y)\right).$$

This implies that μ is a fuzzy hyper BCK-ideal of H. □

4. Isomorphism theorems

Theorem 4.1. [6] *Let $f : H \longrightarrow H'$ be a homomorphism of hyper BCK-algebras. Then,*

(i) *$kerf$ is a hyper BCK-ideal of H,*

(ii) *if Θ is a regular congruence on H and $kerf = I$, then $H/I \simeq f(H)$.*

Theorem 4.2. *Let ρ be a fuzzy regular congruence relation on H and $t = \sup_{(z,w)\in H^2}\rho(z,w)$. Then there is a hyper BCK-ideal J of H/μ such that*

$$(H/\mu)/J \simeq H/\mu_t.$$

Proof. Since ρ is a fuzzy reflexive relation on H, then

$$\rho(0,0)=\sup_{(y,z)\in H^2}\rho(y,z)=t$$

and so $(0,0)\in\rho_t$ and ρ satisfies the sup property. Hence, by Theorems 2.11 and 3.2, ρ_t is a regular congruence relation on H. By Lemma 3.4(ii), $[0]_{\rho_t}=\mu_t$ and so by Theorem 2.6, H/μ_t is a hyper BCK-algebra. Let $I = \mu_t$. Then, by Theorem 2.6, $H/\mu_t = H/I = \{I_x : x\in H\}$, where $I_x=[x]_{\rho_t}$. Now, let $\psi : H/\mu \to H/I$ be defined by $\psi(\mu_x) = I_x$. Let $\mu_x = \mu_y$, for $\mu_x, \mu_y \in H/\mu$. Then, by Lemma 3.4(i), $\rho(x,y) = \sup_{(z,w)\in H^2}\rho(z,w) = t$. Thus, $x\rho_t y$ and so $I_x = I_y$; i.e. $\psi(\mu_x) = \psi(\mu_y)$. This implies that ψ is well-defined. Moreover, for all $\mu_x, \mu_y \in H/\mu$,

$$\begin{aligned}\psi(\mu_x\circ\mu_y)&=\psi(\mu_{x\circ y})=\psi(\{\mu_z : z\in x\circ y\})=\{\psi(\mu_z): z\in x\circ y\}\\&=\{I_z : z\in x\circ y\}=I_x\circ I_y=\psi(\mu_x)\circ\psi(\mu_y)\end{aligned}$$

and so ψ is a homomorphism. Now, let

$$\mu_x\Theta\mu_y \longleftrightarrow x\rho_t y$$

for all $x, y\in H$. Since, ρ_t is a regular congruence relation on H, then Θ is a regular congruence relation on H/μ, too. Now, we show that $ker\psi = [\mu]_\Theta$.

$$\begin{aligned}\mu_x\in ker\psi &\longleftrightarrow \psi(\mu_x)=I\longleftrightarrow I_x=I\longleftrightarrow x\in I=[0]_{\rho_t}\\&\longleftrightarrow x\rho_t 0\longleftrightarrow \mu_x\Theta\mu\longleftrightarrow \mu_x\in[\mu]_\Theta.\end{aligned}$$

It is clear that ψ is onto. Therefore, by Theorem 4.1(ii), $(H/\mu)/ker\psi \simeq H/\mu_t$. Now, let $J = ker\psi$. Then $(H/\mu)/J \simeq H/\mu_t$ and by Theorem 4.1(i), J is a hyper BCK-ideal of H/μ. $\square$

Theorem 4.3. *Let ρ be a fuzzy regular congruence relation on H and $\mu^* = \{x \in H : \mu(x) = \mu(0)\}$. Then $H/\mu \simeq H/\mu^*$.*

Proof. Let $t = \mu(0)$. Since, ρ is fuzzy reflexive, then $t = \mu(0) = \rho(0,0) = \sup_{(y,z)\in H^2} \rho(y,z)$. Now, we must show that $\mu^* = [0]_{\rho_t}$. Let $x \in \mu^*$. Then, $\mu(x) = \mu(0)$ and so $\rho(x,0) = \rho(0,0) = t$. Hence, $x\rho_t 0$ and so $x \in [0]_{\rho_t}$. Hence, $\mu^* \subseteq [0]_{\rho_t}$. Moreover, if $x \in [0]_{\rho_t}$, then $x\rho_t 0$; i.e, $(x,0) \in \rho_t$ and so $\rho(x,0) \geqslant t = \sup_{(y,z)\in H^2} \rho(y,z) \geqslant \rho(x,0)$. Thus, $\rho(x,0) = t = \rho(0,0)$ and so $\mu(x) = \mu(0)$; i.e, $x \in \mu^*$. Hence, $[0]_{\rho_t} \subseteq \mu^*$. Therefore, $[0]_{\rho_t} = \mu^*$ and so by Theorem 2.6, H/μ^* is well-defined.

Now, let $\psi : H/\mu \longrightarrow H/\mu^*$ be defined by $\psi(\mu_x) = I_x$, for all $x \in H$, where $I = \mu^*$. By the proof of Theorem 4.2, ψ is an epimorphism. Now, we show that ψ is one-to-one. For this, let $\mu_x \in ker\psi$. Then, $\psi(\mu_x) = I$ and so $I_x = I$. Hence, $x \in I = \mu^*$ and so $\mu(x) = \mu(0)$. Since, ρ is fuzzy reflexive, then $\rho(x,0) = \rho(0,0) = \sup_{(y,z)\in H^2} \rho(y,z)$ and so by Lemma 3.4(i), $\mu_x = \mu_0 = \mu$. Hence, $ker\psi = \{\mu\} = 0_{H/\mu^*}$ and so $H/\mu \simeq H/\mu^*$. $\square$

Theorem 4.4. (First Isomorphism Theorem)
Let ρ be a fuzzy regular congruence relation on H and $f : H \to H'$ be an epimorphism of hyper BCK-algebras such that $\mu^* = kerf$. Then $H/\mu \simeq H'$.

Proof. Let $\varphi : H/\mu \to H'$ be defined by $\varphi(\mu_x) = f(x)$, for all $x \in H$. First we show that φ is well-defined. For this, let $\mu_x = \mu_y$, for $x, y \in H$. Then by Lemma 3.4(i), $\rho(x,y) = \sup_{(z,w)\in H^2} \rho(z,w)$. Let $t = \sup_{(z,w)\in H^2} \rho(z,w)$. Hence, $x\rho_t y$. Since, by Theorem 2.11, ρ_t is a congruence relation on H, then $x \circ y\overline{\rho_t}y \circ y$ and $x \circ x\overline{\rho_t}y \circ x$. Since, $0 \in y \circ y$, then there exists $a \in x \circ y$ such that $a\rho_t 0$ and so $\rho(a,0) \geqslant t = \sup_{(z,w)\in H^2} \rho(z,w) \geqslant \rho(a,0)$. Hence, $\rho(a,0) = \sup_{(z,w)\in H^2} \rho(z,w)$. Since, ρ is fuzzy reflexive, then $\mu(0) = \rho(0,0) = \sup_{(z,w)\in H^2} \rho(z,w) = \rho(a,0) = \mu(a)$ and so $a \in \mu^* = kerf$. Hence, $0' = f(a) \in f(x \circ y) = f(x) \circ f(y)$ and so $f(x) \ll f(y)$. Similarly, since $0 \in x \circ x$, then there exists $b \in y \circ x$ such that $b \in kerf$. Hence, $0' = f(b) \in f(y \circ x) = f(y) \circ f(x)$ and so $f(y) \ll f(x)$. Thus, $f(x) = f(y)$ and so φ is well-defined. Let $\mu_x, \mu_y \in H/\mu$. Then, $\varphi(\mu_x \circ \mu_y) = \varphi(\mu_{x\circ y}) = f(x \circ y) = f(x) \circ f(y) = \varphi(\mu_x) \circ \varphi(\mu_y)$, and so φ is a homomorphism. Now, let $\mu_x \in ker\varphi$. Then $f(x) = \varphi(\mu_x) = 0'$ and so $x \in kerf = \mu^*$; i.e., $\mu(x) = \mu(0)$. Thus, $\rho(x,0) = \mu(x) = \mu(0) = \rho(0,0) = \sup_{(z,w)\in H^2} \rho(z,w)$.

Hence, by Lemma 3.4(i), $\mu_x = \mu$ and so $ker\varphi = \{\mu\}$. Hence, φ is one to one. Since, f is onto, then φ is onto, too. Therefore, φ is an isomorphism and so $H/\mu \simeq H'$. □

Theorem 4.5. *Let $f : H \to H'$ be an epimorphism of hyper BCK-algebras, ρ and σ (resp.) be fuzzy regular congruence relations on H and H', μ and μ' (resp.) be fuzzy subsets on H and H' such that $\mu_y = f^{-1}(\mu'_{f(y)})$, for all $y \in H$. Then $H/\mu \simeq H'/\mu'$.*

Proof. Let $\varphi : H/\mu \to H'/\mu'$ be defined by $\varphi(\mu_x) = \mu'_{f(x)}$, for all $x \in H$. Now, let $\mu_x = \mu_y$, for $\mu_x, \mu_y \in H/\mu$ and $z' \in H'$. Since, f is onto, then there exists $z \in H$ such that $f(z) = z'$. Hence,

$$\begin{aligned}\mu'_{f(x)}(z') &= \mu'_{f(x)}(f(z)) = f^{-1}(\mu'_{f(x)})(z) = \mu_x(z) = \mu_y(z)\\ &= f^{-1}(\mu'_{f(y)})(z) = \mu'_{f(y)}(f(z)) = \mu'_{f(y)}(z').\end{aligned}$$

Thus, $\mu'_{f(x)} = \mu'_{f(y)}$ and so φ is well-defined. Now, let $\mu_x, \mu_y \in H/\mu$. Then,

$$\varphi(\mu_x \circ \mu_y) = \varphi(\mu_{x\circ y}) = \mu'_{f(x\circ y)} = \mu'_{f(x)\circ f(y)} = \mu'_{f(x)} \circ \mu'_{f(y)} = \varphi(\mu_x) \circ \varphi(\mu_y),$$

and this implies that φ is a homomorphism. Moreover, since f is onto then φ is onto, too. Now, let $\varphi(\mu_x) = \varphi(\mu_y)$, for $\mu_x, \mu_y \in H/\mu$. Then $\mu'_{f(x)} = \mu'_{f(y)}$ and so for all $z \in H$,

$$\mu_x(z) = f^{-1}(\mu'_{f(x)})(z) = \mu'_{f(x)}(f(z)) = \mu'_{f(y)}(f(z)) = f^{-1}(\mu'_{f(y)})(z) = \mu_y(z).$$

This implies that φ is one-to-one. Therefore, φ is an isomorphism and so $H/\mu \simeq H'/\mu'$. □

Lemma 4.6. *Let ρ and σ be two fuzzy regular congruence relations on H such that $\mu_y(x) = \sigma(x, y)$, $\mu(x) = \sigma(x, 0)$, for all $x, y \in H$ and ρ satisfies the sup property. Then ρ/μ is a fuzzy regular congruence relation on H/μ, where fuzzy relation ρ/μ on H/μ is defined by $\rho/\mu(\mu_x, \mu_y) = \rho(x, y)$.*

Proof. Since, σ is a fuzzy regular congruence relation on H, then by Theorem 3.5, H/μ is well-defined. Moreover, since ρ is a fuzzy regular congruence relation on H, then by some modifications we can show that ρ/μ is a fuzzy congruence relation on H, too. Now, let

$$s = \min\left(\sup_{\mu_a\in\mu_x\circ\mu_y} \rho/\mu(\mu_a, \mu), \sup_{\mu_b\in\mu_y\circ\mu_x} \rho/\mu(\mu_b, \mu)\right).$$

Then, $\sup_{\mu_a \in \mu_x \circ \mu_y} \rho/\mu(\mu_a, \mu) \geqslant s$ and $\sup_{\mu_b \in \mu_y \circ \mu_x} \rho/\mu(\mu_b, \mu) \geqslant s$. Since, ρ satisfies the sup property, then ρ/μ so is. Thus, there exist $a_0 \in x \circ y$ and $b_0 \in y \circ x$ such that

$$\rho(a_0, 0) = \rho/\mu(\mu_{a_0}, \mu) = \sup_{\mu_a \in \mu_x \circ \mu_y} \rho/\mu(\mu_a, \mu) \geqslant s$$

and
$$\rho(b_0, 0) = \rho/\mu(\mu_{b_0}, \mu) = \sup_{\mu_b \in \mu_y \circ \mu_x} \rho/\mu(\mu_b, \mu)) \geqslant s.$$

Since, ρ is a fuzzy regular relation on H, then

$$\rho/\mu(\mu_x, \mu_y) = \rho(x, y) \geqslant \min(\sup_{a \in x \circ y} \rho(a, 0), \sup_{b \in y \circ x} \rho(b, 0))$$

$$\geqslant \min(\rho(a_0, 0), \rho(b_0, 0)) \geqslant s = \min(\sup_{\mu_a \in \mu_x \circ \mu_y} \rho/\mu(\mu_a, \mu), \sup_{\mu_b \in \mu_y \circ \mu_x} \rho/\mu(\mu_b, \mu)).$$

Hence, ρ/μ is a fuzzy regular relation on H/μ and so it is a fuzzy regular congruence relation on H/μ. □

Theorem 4.7. (Second Isomorphism Theorem)
Let ρ and σ be two fuzzy regular congruence relations on H such that $\sigma \subseteq \rho$ and there exists $a \in H$ such that $\sigma(a, a) = 1$. Let fuzzy subsets η_y and μ_y on H are defined by $\eta_y(x) = \rho(x, y)$ and $\mu_y(x) = \sigma(x, y)$, for all $x, y \in H$. Then

$$(H/\mu)/(\eta/\mu) \simeq H/\eta,$$

where $(\eta/\mu)(\mu_x) = \rho/\mu(\mu_x, \mu)$ *and* $(\rho/\mu)(\mu_x, \mu_y) = \rho(x, y)$.

Proof. Since, by Lemma 4.6, ρ/μ is a fuzzy regular congruence relation on H/μ and $(\eta/\mu)(\mu_x) = (\rho/\mu)(\mu_x, \mu)$, then $(H/\mu)/(\eta/\mu)$ is a hyper BCK-algebra. Also, it is easy to see that

$$\sup_{(z,w) \in H^2} \rho(z, w) = \sup_{(\mu_z, \mu_w) \in (H/\mu)^2} \rho/\mu(\mu_z, \mu_w).$$

Now, let $\psi : H/\mu \to H/\eta$ be defined by $\psi(\mu_x) = \eta_x$. We have to show that ψ is well-defined. Let $\mu_x = \mu_y$, for $\mu_x, \mu_y \in H/\mu$. Then, by Lemma 3.4(i), $\sigma(x, y) = \sup_{(z,w) \in H^2} \sigma(z, w) = 1$. Since, $\sigma \subseteq \rho$, then $\rho(x, y) \geqslant \sigma(x, y) = 1$ and so $\rho(x, y) = \sup_{(z,w) \in H^2} \rho(z, w)$. Hence, by Lemma 3.4($i$), $\eta_x = \eta_y$, which this shows that ψ is well-defined. It is easy to check that ψ is an epimorphism. Now,

$$\begin{aligned} ker\psi &= \{\mu_x \in H/\mu : \eta_x = \psi(\mu_x) = \eta\} \\ &= \{\mu_x \in H/\mu : \rho(x, 0) = \sup\nolimits_{(z,w) \in H^2} \rho(z, w) = \rho(0, 0)\} \\ &= \{\mu_x \in H/\mu : \rho/\mu(\mu_x, \mu) = \rho/\mu(\mu, \mu)\} \end{aligned}$$

$$= \{\mu_x \in H/\mu : (\eta/\mu)(\mu_x) = (\eta/\mu)(\mu)\} = (\eta/\mu)^*.$$

So, $(H/\mu)/(\eta/\mu) \simeq H/\eta$. □

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On prolongations of quasigroups

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Abstract

We prove that any quasigroup admissing complete or quasicomplete mapping has a prolongation to a quasigroup having one element more.

1. Introduction

By a *prolongation* of a quasigroup we mean a process which shows how, starting from a quasigroup $Q(\cdot)$ of order n, we can obtain a quasigroup $Q'(\circ)$ of order $n+1$ such that the set Q' is obtained from the set Q by the adjunction of one additional element. In other words, it is a process which shows how a given Latin square extends to a new Latin square by the adjunction of one additional row and one column. The first construction of prolongation was proposed by R. H. Bruck [7] who considered only the case of idempotent quasigroups. More general construction was given by J. Dénes and K. Pásztor [9]. Further generalizations, for special types of quasigroups, have been discussed in [2] and [3] by V. D. Belousov. In fact, the construction proposed by V. D. Belousov is more elegant form of the construction proposed by J. Dénes and K. Pásztor. G. B. Belyavskaya studied this problem together with the inverse problem, i.e., with the problem how from a given Latin square of order n one can obtain a Latin square of order $n-1$ (cf. [4, 5, 6]). Quasigroups obtained by the construction proposed by G. B. Belyavskaya are not isotopic to quasigroups obtained by the constructions proposed by R. H. Bruck and V.D. Belousov. This means that we have two different methods of construction of prolongations.

Below we present a third method. Our method can be applied to any quasigroup of order n with the property that its multiplication table possesses a partial transversal of length $n-1$, i.e., a sequence of $n-1$ distinct elements contained in distinct rows and distinct columns. All these three constructions are presented in short elegant form.

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2. Definitions and basic facts

In this paper $Q(\cdot)$ always denotes a quasigroup. The set Q' is identified with the set $Q\cup\{q\}$, where $q\notin Q$.

Any mapping σ of a quasigroup $Q(\cdot)$ defines on Q a new mapping $\overline{\sigma}$, called *conjugated to* σ, such that

$$\overline{\sigma}(x)=x\cdot\sigma(x) \tag{1}$$

for all $x\in Q$. If σ is the identity mapping ε, then $\overline{\sigma}(x)=x^2$. The set

$$\operatorname{def}(\sigma)=Q\setminus\overline{\sigma}(Q),$$

where $\overline{\sigma}(Q)=\{\overline{\sigma}(x)\,|\,x\in Q\}$, is called the *defect* of σ.

A mapping σ is *quasicomplete* on a quasigroup $Q(\cdot)$ if $\overline{\sigma}(Q)$ contains all elements of Q except one. In this case there exists an element $a\in Q$, called *special*, such that $a=\overline{\sigma}(x_1)=\overline{\sigma}(x_2)$ for some $x_1,x_2\in Q$. If $\overline{\sigma}(Q)$ contains all elements of Q, then we say that σ is *complete*. A quasigroup having at least one complete mapping is called *admissible*. V. D. Belousov proved in [3] (see also [2]) that any admissible quasigroup is isotopic to some idempotent quasigroup and has a prolongation. Since for a given admissible quasigroup the method of constructions of a prolongation proposed by V.D. Belousov gives, in fact, a quasigroup which is isotopic to a quasigroup obtained from the corresponding idempotent quasigroup (by the method proposed by R. H. Bruck) we will identify these two methods and will call it the *classical construction*.

3. Prolongations of admissible quasigroups

1. Classical constructions. The idea of the construction proposed by R. H. Bruck is presented by the following tables, where the corresponding empty cells of these tables are identical.

$$\begin{array}{c|cccccc|}
\cdot & 1 & 2 & 3 & 4 & \dots & n \\ \hline
1 & 1 & & & & & \\
2 & & 2 & & & & \\
3 & & & 3 & & & \\
4 & & & & 4 & & \\
\vdots & & & & & \ddots & \\
n & & & & & & n \\ \hline
\end{array}
\quad\longrightarrow\quad
\begin{array}{c|cccccc|c}
\circ & 1 & 2 & 3 & 4 & \dots & n & q \\ \hline
1 & q & & & & & & 1 \\
2 & & q & & & & & 2 \\
3 & & & q & & & & 3 \\
4 & & & & q & & & 4 \\
\vdots & & & & & \ddots & & \vdots \\
n & & & & & & q & n \\ \hline
q & 1 & 2 & 3 & 4 & \dots & n & q
\end{array}$$

The quasigroup $Q'(\circ)$ obtained from the quasigroup $Q(\cdot)$ is a loop with the identity q. The operation on Q' is defined according to the formula:

$$x \circ y = \begin{cases} x \cdot y & \text{for} \quad x, y \in Q, \ x \neq y, \\ x & \text{for} \quad x \in Q, \ y = q, \\ y & \text{for} \quad x = q, \ y \in Q, \\ q & \text{for} \quad x = y \in Q'. \end{cases} \tag{2}$$

In the construction for a prolongation of an admissible quasigroup $Q(\cdot)$ proposed by V. D. Belousov [3] the complete mapping σ of $Q(\cdot)$ and its conjugated mapping $\overline{\sigma}$ are used. The operation on Q' is defined by the formula:

$$x \circ y = \begin{cases} x \cdot y & \text{for} \quad x, y \in Q, \ y \neq \sigma(x), \\ \overline{\sigma}(x) & \text{for} \quad x \in Q, \ y = q, \\ \overline{\sigma}\sigma^{-1}(y) & \text{for} \quad x = q, \ y \in Q, \\ q & \text{for} \quad x \in Q, \ y = \sigma(x), \\ q & \text{for} \quad x = y = q. \end{cases} \tag{3}$$

Geometrically this means that the multiplication table (Latin square) $L' = [a'_{ij}]$ of a quasigroup $Q'(\circ)$ is obtained from the multiplication table $L = [a_{ij}]$ of a quasigroup $Q(\cdot)$ by the adjunction of one row and one column in this way that all elements from the cells $a_{i\sigma(i)}$ are moved to the last place of the i-th row and $\sigma(i)$-th column of L'. Elements of the cells $a_{i\sigma(i)}$ are replaced by $q = n + 1$. Additionally we put $a_{qq} = q$. In other words: $a'_{ij} = a_{ij}$ for $i \neq \sigma(i)$, $a'_{iq} = a_{i\sigma(i)} = \overline{\sigma}(i)$, $a'_{qj} = a_{\sigma^{-1}(j)j} = \overline{\sigma}\sigma^{-1}(j)$ and $a'_{i\sigma(i)} = a'_{qq} = q$.

Example 1. Consider the quasigroup $Q(\cdot)$ with the multiplication table

$\cdot$	1	2	3	4	5
1	1	2	3	4	5
2	4	3	1	5	2
3	2	5	4	1	3
4	5	4	2	3	1
5	3	1	5	2	4

and its two complete mappings $\sigma = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 4\,2\,1\,5\,3 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 3\,1\,2\,5\,4 \end{pmatrix}$. Then, as it is not difficult to see, $\overline{\sigma} = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 4\,3\,2\,1\,5 \end{pmatrix}$ and $\overline{\tau} = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 3\,4\,5\,1\,2 \end{pmatrix}$.

Using these two mappings we can construct two different prolongations:

$\circ_1$	1	2	3	4	5	6
1	1	2	3	6	5	4
2	4	6	1	5	2	3
3	6	5	4	1	3	2
4	5	4	2	3	6	1
5	3	1	6	2	4	5
6	2	3	5	4	1	6

$\circ_2$	1	2	3	4	5	6
1	1	2	6	4	5	3
2	6	3	1	5	2	4
3	2	6	4	1	3	5
4	5	4	2	3	6	1
5	3	1	5	6	4	2
6	4	5	3	2	1	6

The first prolongation is obtained by σ, the second by τ.

By transpositions of rows and columns, we can transform these two tables into multiplication tables of loops $Q'(\star_1)$ and $Q'(\star_2)$:

$\star_1$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	3	5	6	1	4
3	3	1	6	5	4	2
4	4	6	1	3	2	5
5	5	4	2	1	6	3
6	6	5	4	2	5	1

$\star_2$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	6	5	1	3	4
3	3	1	2	6	4	5
4	4	5	6	2	1	3
5	5	4	1	3	6	2
6	6	3	4	5	2	1

Since $\gamma(x \star_1 y) = \alpha(x) \star_2 \beta(y)$, where

$$\alpha = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 2\,4\,1\,6\,5\,3 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 4\,1\,6\,3\,5\,2 \end{pmatrix}, \qquad \gamma = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 1\,2\,4\,5\,3\,6 \end{pmatrix},$$

loops $Q'(\star_1)$ and $Q'(\star_2)$ are isotopic. This means that also prolongations $Q'(\circ_1)$ and $Q'(\circ_2)$ are isotopic. $\square$

If the diagonal of the multiplication table of a quasigroup $Q(\cdot)$ contains all elements of Q, then as σ we can select the identity mapping. In this case the formula (3) has the form:

$$x \circ y = \begin{cases} x \cdot y & \text{for} \quad x, y \in Q, \ x \neq y, \\ x^2 & \text{for} \quad x \in Q, \ y = q, \\ y^2 & \text{for} \quad x = q, \ y \in Q, \\ q & \text{for} \quad x = y \in Q'. \end{cases} \tag{4}$$

If $(Q(\cdot)$ is an idempotent quasigroup, then (4) coincides with (2) and $Q'(\circ)$ is a loop with the identity q.

Example 2. The diagonal of the multiplication table of the additive group $\mathbb{Z}_3$ contains all elements of $\mathbb{Z}_3$. So, according to (4), the prolongation $\mathbb{Z}_3'(\circ)$ has the following multiplication table:

$\circ$	0	1	2	3
0	3	1	2	0
1	1	3	0	2
2	2	0	3	1
3	0	2	1	3

Putting $\alpha = \begin{pmatrix} 0\,1\,2\,3 \\ 3\,1\,2\,0 \end{pmatrix}$ and $x \odot y = \alpha(x \circ y)$ we can see that $\mathbb{Z}_3'(\circ)$ is isotopic to the Klein's group $K_4(\odot)$.

Using $\sigma = \begin{pmatrix} 0\,1\,2 \\ 2\,0\,1 \end{pmatrix}$ and $\tau = \begin{pmatrix} 0\,1\,2 \\ 1\,2\,0 \end{pmatrix}$ we obtain two non-commutative prolongations:

$\circ$	0	1	2	3
0	0	1	3	2
1	3	2	0	1
2	2	3	1	0
3	1	0	2	3

$\circ$	0	1	2	3
0	0	3	2	1
1	1	2	3	0
2	3	0	1	2
3	2	1	0	3

These prolongations also are isotopic to the Klein's group. For the first we have $x \odot y = \alpha(x) \circ \beta(y)$, for the second $x \odot y = \beta(x) \circ \alpha(y)$, where $\alpha = \begin{pmatrix} 0\,1\,2\,3 \\ 0\,3\,2\,1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 0\,1\,2\,3 \\ 0\,1\,3\,2 \end{pmatrix}$. □

2. The construction proposed by G. B. Belyavskaya. This construction is valid for admissible quasigroups. At first we consider the case when $Q(\cdot)$ is an idempotent quasigroup. To find the prolongation $Q'(\diamond)$ of $Q(\cdot)$ we select an arbitrary element $a \in Q$. Next, in the multiplication table of $Q(\cdot)$ we replace all elements of the diagonal, except a, by q and adjunct one column and one row:

$$
\begin{array}{c|cccccc|}
\cdot & 1 & 2 & \ldots & a & \ldots & n \\
\hline
1 & 1 & & & & & \\
2 & & 2 & & & & \\
\vdots & & & \ddots & & & \\
a & & & & a & & \\
\vdots & & & & & \ddots & \\
n & & & & & & n \\
\hline
\end{array}
\quad \longrightarrow \quad
\begin{array}{c|cccccc|c}
\diamond & 1 & 2 & \ldots & a & \ldots & n & q \\
\hline
1 & q & & & & & & 1 \\
2 & & q & & & & & 2 \\
\vdots & & & \ddots & & & & \vdots \\
a & & & & a & & & q \\
\vdots & & & & & \ddots & & \vdots \\
n & & & & & & q & n \\
\hline
q & 1 & 2 & \ldots & q & \ldots & n & a
\end{array}
$$

The operation in $Q(\diamond)$ is defined in the following way:

$$
x \diamond y = \begin{cases}
x \cdot y & \text{for} \quad x, y \in Q, \ x \neq y, \\
q & \text{for} \quad x = y \in Q - \{a\}, \\
a & \text{for} \quad x = y = a, \\
x & \text{for} \quad x \in Q - \{a\}, \ y = q, \\
y & \text{for} \quad x = q, \ y \in Q - \{a\}, \\
q & \text{for} \quad x = q, \ y = a, \\
q & \text{for} \quad x = a, \ y = q, \\
a & \text{for} \quad x = y = q.
\end{cases} \tag{5}
$$

In a general case, when $Q(\cdot)$ is "only" an admissible quasigroup, we can select a complete mapping σ of Q and fix an arbitrary element $a \in Q$. Then, obviously, there exists an uniquely determined element $x_a \in Q$ such that $a = x_a \cdot \sigma(x_a)$. The prolongation $Q'(\diamond)$ of $Q(\cdot)$ can be defined by

$$
x \diamond y = \begin{cases}
x \cdot y & \text{for} \quad x, y \in Q, \ y \neq \sigma(x), \\
q & \text{for} \quad x \in Q - \{x_a\}, \ y = \sigma(x), \\
a & \text{for} \quad x = x_a, \ y = \sigma(x_a), \\
\overline{\sigma}(x) & \text{for} \quad x \in Q - \{x_a\}, \ y = q, \\
\overline{\sigma}\sigma^{-1}(y) & \text{for} \quad x = q, \ y \neq \sigma(x_a), \\
q & \text{for} \quad x = q, \ y = \sigma(x_a), \\
q & \text{for} \quad x = x_a, \ y = q, \\
a & \text{for} \quad x = y = q.
\end{cases} \tag{6}
$$

Selecting different σ and different a we obtain different prolongations.

From a formal point of view, the above construction is a generalization on the classical construction. Indeed, putting $\sigma(q) = q$ we extend σ to a complete mapping of Q'. Next, putting $a = x_a = q$ in (6) we obtain (3).

If the diagonal of the multiplication table of $Q(\cdot)$ contains all elements of Q, then as σ can be selected the identity mapping and the formula (6) can be written in the form:

$$x \diamond y = \begin{cases} x \cdot y & \text{for} \quad x, y \in Q, \ x \neq y, \\ q & \text{for} \quad x = y \in Q - \{x_a\}, \\ a & \text{for} \quad x = y = x_a, \\ x^2 & \text{for} \quad x \in Q - \{x_a\}, \ y = q, \\ y^2 & \text{for} \quad x = q, \ y \in Q - \{x_a\}, \\ q & \text{for} \quad x = q, \ y = x_a, \\ q & \text{for} \quad x = x_a, \ y = q, \\ a & \text{for} \quad x = y = q. \end{cases} \tag{7}$$

For idempotent quasigroups it coincides with (5) but, generally, prolongations obtained by the method proposed by G. B. Belyavskaya are not isotopic to prolongations obtained by the method proposed by V. D. Belousov. Below we present the corresponding example.

Example 3. The prolongation $\mathbb{Z}_3'(\diamond)$ of the additive group $\mathbb{Z}_3$ constructed according to (7), where $a = 1$, $x_a = 2$, $q = 3$, has the following multiplication table:

$\diamond$	0	1	2	3
0	3	1	2	0
1	1	3	0	2
2	2	0	1	3
3	0	2	3	1

This prolongation is isotopic to the group $\mathbb{Z}_4(+)$. The connection between $\mathbb{Z}_4(+)$ and $\mathbb{Z}_3'(\diamond)$ is given by the formula $\gamma(x+y) = \alpha(x) \diamond \alpha(y)$, where $\alpha = \begin{pmatrix} 0\,1\,2\,3 \\ 3\,1\,2\,0 \end{pmatrix}$, $\gamma = \begin{pmatrix} 0\,1\,2\,3 \\ 1\,2\,3\,0 \end{pmatrix}$. So, the prolongation of $\mathbb{Z}_3$ constructed by (7) and the prolongation of $\mathbb{Z}_3$ constructed by (4) (in Example 2) are not isotopic. $\square$

Example 4. Let $Q(\cdot)$ and σ be as in Example 1. Then, for example, for $a = 2$ we have $x_a = 3$. Whence, according to (6), we obtain the prolongation:

$\diamond$	1	2	3	4	5	6
1	1	2	3	6	5	4
2	4	6	1	5	2	3
3	2	5	4	1	3	6
4	5	4	2	3	6	1
5	3	1	6	2	4	5
6	6	3	5	4	1	2

Similarly, for $a = 3$ we have $x_a = 2$ and consequently

$\diamond$	1	2	3	4	5	6
1	1	2	3	6	5	4
2	4	3	1	5	2	6
3	6	5	4	1	3	2
4	5	4	2	3	6	1
5	3	1	6	2	4	5
6	2	6	5	4	1	3

Applying Theorem 2.5 from [10] we can verify that these prolongations are not isotopic to the prolongation obtained in Example 1. □

4. Our construction

In the previous section methods of construction of a prolongation of quasigroups that have a complete mapping were given. But, as it is proved in [12] (see also [8], p. 36) there are quasigroups which do not possess such mappings. For example, a group of order $4k+2$ has no complete mapping.

Below, we give a new method of a construction of prolongations for quasigroups that have a quasicomplete mapping. Our method can also be applied to quasigroups that have a complete mapping.

Let $Q(\cdot)$ be an arbitrary quasigroup with a quasicomplete mapping σ. Then $|\overline{\sigma}(Q)| = n-1$ and $\mathrm{def}(\sigma) = d$ for some $d \in Q$. In this case we also have $\overline{\sigma}(x_1) = \overline{\sigma}(x_2) = a$, i.e., $x_1 \cdot \sigma(x_1) = x_2 \cdot \sigma(x_2) = a$ in $Q(\cdot)$, for some $x_1, x_2, a \in Q$, $x_1 \neq x_2$.

The idea of our construction is presented by the following tables, where for simplicity it is assumed that σ is the identity mapping and all elements of Q, except x_1 and x_2, are idempotents.

$$
\begin{array}{c|cccccccc}
\cdot & 1 & 2 & \dots & x_1 & \dots & x_2 & \dots & n \\ \hline
1 & 1 &&&&&&& \\
2 && 2 &&&&&& \\
\vdots &&& \ddots &&&&& \\
x_1 &&&& a &&&& \\
\vdots &&&&& \ddots &&& \\
x_2 &&&&&& a && \\
\vdots &&&&&&& \ddots & \\
n &&&&&&&& n
\end{array}
\quad\longrightarrow\quad
\begin{array}{c|cccccccc|c}
* & 1 & 2 & \dots & x_1 & \dots & x_2 & \dots & n & q \\ \hline
1 & q &&&&&&&& 1\\
2 && q &&&&&&& 2\\
\vdots &&& \ddots &&&&&& \vdots\\
x_1 &&&& a &&&&& q\\
\vdots &&&&& \ddots &&&& \vdots\\
x_2 &&&&&& q &&& a\\
\vdots &&&&&&& \ddots && \vdots\\
n &&&&&&&& q & n\\ \hline
q & 1 & 2 & \dots & q & \dots & a & \dots & n & d
\end{array}
$$

This new table is obtained from the old one by replacing all elements of the diagonal, except $a = x_1 \cdot x_1$, by q and adding one new row and column such that $x * q = q * x = x$ for $x \in Q - \{x_1, x_2\}$, $x_1 * q = q * x_1 = q$, $x_2 * q = q * x_2 = a$, $q * q = d$.

The operation of this new quasigroup is determined by the formula:

$$
x * y = \begin{cases}
x \cdot y & \text{for} \quad x, y \in Q,\ x \neq y, \\
q & \text{for} \quad x = y \in Q - \{x_1\}, \\
a & \text{for} \quad x = y = x_1, \\
x & \text{for} \quad x \in Q - \{x_1, x_2\},\ y = q, \\
y & \text{for} \quad x = q,\ y \in Q - \{x_1, x_2\}, \\
q & \text{for} \quad x = x_1,\ y = q \ \ \textit{or} \ \ x = q,\ y = x_1, \\
a & \text{for} \quad x = x_2,\ y = q \ \ \textit{or} \ \ x = q,\ y = x_2, \\
d & \text{for} \quad x = y = q.
\end{cases} \tag{8}
$$

In the general case, when σ is an arbitrary quasicomplete mapping of Q, $\operatorname{def}(\sigma) = d$, $a = \overline{\sigma}(x_1) = \overline{\sigma}(x_2)$, $x_1 \neq x_2$ and x_1 is fixed, the operation of $Q'(*)$ has the form:

$$
x * y = \begin{cases}
x \cdot y & \text{for} \quad x, y \in Q,\ y \neq \sigma(x), \\
q & \text{for} \quad x \in Q - \{x_1\},\ y = \sigma(x), \\
a & \text{for} \quad x = x_1,\ y = \sigma(x), \\
\overline{\sigma}(x) & \text{for} \quad x \in Q - \{x_1, x_2\},\ y = q, \\
\overline{\sigma}\sigma^{-1}(y) & \text{for} \quad x = q,\ y \neq \sigma(x_1),\ y \neq \sigma(x_2), \\
q & \text{for} \quad x = x_1,\ y = q \ \ \textit{or} \ \ x = q,\ y = \sigma(x_1), \\
a & \text{for} \quad x = x_2,\ y = q \ \ \textit{or} \ \ x = q,\ y = \sigma(x_2), \\
d & \text{for} \quad x = y = q.
\end{cases} \tag{9}
$$

If in the above formula we delete x_2 and assume that σ is a complete mapping, then for $x_1 = x_a$ and $d = a$ this formula will be identical with (7). This means that our construction is a generalization of the construction proposed by G. B. Belyavskaya. Consequently, it is also a generalization of the classical construction.

Example 5. Let $Q(\cdot)$ be a quasigroup defined in Example 1. The mapping $\sigma = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 4\,5\,2\,3\,1 \end{pmatrix}$ is quasicomplete on Q, $\overline{\sigma} = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 4\,2\,5\,2\,3 \end{pmatrix}$ is its conjugated mapping, $\mathrm{def}(\sigma) = 1$, $\overline{\sigma}(2) = \overline{\sigma}(4) = 2$. Hence $d = 1$, $a = 2$, $x_1 = 2$, $x_2 = 4$. Putting $q = 6$ and using our construction we obtain the following prolongation of $Q(\cdot)$:

$*$	1	2	3	4	5	6
1	1	2	3	6	5	4
2	4	3	1	5	2	6
3	2	6	4	1	3	5
4	5	4	6	3	1	2
5	6	1	5	2	4	3
6	3	5	2	4	6	1

For $x_1 = 4$, $x_2 = 2$ our construction gives the quasigroup:

$*$	1	2	3	4	5	6
1	1	2	3	6	5	4
2	4	3	1	5	6	2
3	2	6	4	1	3	5
4	5	4	2	3	1	6
5	6	1	5	2	4	3
6	3	5	6	4	2	1

From Theorem 2.5 in [10] it follows that these two prolongations are isotopic, but they are not isotopic to the prolongation constructed in Example 1 and in Example 4. □

6. Conclusion

The Brualdi conjecture (cf. [8], p.103) says that each Latin square $n \times n$ possesses a sequence of $k \geqslant n-1$ distinct elements selected from different rows and different columns. In other words, each finite quasigroup has at least one complete or quasicomplete mapping. It is known that if a quasigroup $Q(\cdot)$ has a complete mapping, then each quasigroup isotopic to $Q(\cdot)$ has one also. Any group of odd order has a complete mapping, but, for example, groups of order $4k+2$ do not contain such mappings. More interesting facts on the Brualdi conjecture one can find in [1, 2, 8, 11] and [13].

If this conjecture is true, then from our results it follows that *each finite quasigroup has a prolongation.*

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On fuzzy relations and fuzzy quotient Γ-groups

Kostaq Hila

Abstract

The problem of the structure of fuzzy quotient Γ-groups is discussed. We introduce and define the fuzzy quotient Γ-group by using some special fuzzy relation defined in this paper, and also we prove some basic properties.

1. Introduction and preliminaries

The concept of fuzzy sets was first introduced by Zadeh in [10] and since then there has been a tremendous interest in the subject due to its various applications ranging from engineering and computer since to social behavior studies. The concept of fuzzy relations on a set was defined by Zadeh [10, 11]. The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups by Rosenfeld [5]. The notion of Γ-groups was introduced in [7] as a generalization of the notion of classical groups. In this paper we introduce and define some new special fuzzy equivalence relations. Then using these relations we define suitable fuzzy quotient Γ-subgroup of G_α/H_α and prove some basic properties.

In 1986 Sen and Saha [7] defined a Γ-semigroup as follows:

Definition 1.1. Let $M = \{a, b, c, \ldots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \ldots\}$ be two nonempty sets. If there exists a mapping $M \times \Gamma \times M \rightarrow M$ denoted by $(a, \gamma, b) \longmapsto a\gamma b$ and satisfying the identity

$$(a\alpha b)\beta c = a\alpha(b\beta c),$$

where $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, then M is called a Γ-*semigroup*.

For a Γ-semigroup M and a fixed element $\gamma \in \Gamma$ we define on M a binary operation $\circ$ by putting $a \circ b = a\gamma b$ for all $a, b \in M$. Such defined

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groupoid $(M, \circ)$ is denoted by M_γ. It is a semigroup [7]. Moreover, if it is a group for some $\gamma \in \Gamma$, then it is a group for every $\gamma \in \Gamma$ [7]. In this case we says that M is a Γ-*group*. Examples can be found in [7] and [8].

For subsets A and B of a Γ-semigroup M we define the set

$$A\Gamma B = \{a\gamma b \,|\, a \in A, b \in B \text{ and } \gamma \in \Gamma\}.$$

The interval $[0,1]$ we denoted by I, $max\{x,y\}$ by $x \vee y$, $min\{x,y\}$ by $x \wedge y$. By a fuzzy set on X we mean any mapping $\mu : X \longmapsto I$. For any fuzzy sets μ and ν on X we define

$$\begin{aligned}
\mu = \nu &\Leftrightarrow \mu(x) = \nu(x),\ \forall x \in X,\\
\mu \subseteq \nu &\Leftrightarrow \mu(x) \leqslant \nu(x)\ \forall x \in X,\\
(\mu \cup \nu)(x) &= \mu(x) \vee \nu(x),\\
(\mu \cap \nu)(x) &= \mu(x) \wedge \nu(x).
\end{aligned}$$

For a family of fuzzy sets $\{\mu_i \,|\, i \in I\}$ defined on X we put

$$(\cup \mu_i)(x) = \bigvee_{i \in I} \{\mu_i(x)\} \quad \text{and} \quad (\cap \mu_i)(x) = \bigwedge_{i \in I} \{\mu_i(x)\}.$$

Definition 1.2. A fuzzy set μ of a group G is called a *fuzzy subgroup* if

(i) $\mu(xy) \geqslant \mu(x) \wedge \mu(y)$,

(ii) $\mu(x^{-1}) \geqslant \mu(x)$

holds for all $x, y \in G$.

Obviously $\mu(e) \geqslant \mu(x)$ for every $x \in G$, where e is the identity of G.

Theorem 1.3. *A fuzzy set μ of a group G is a fuzzy subgroup G if and only if*

$$\mu(xy^{-1}) \geqslant \mu(x) \wedge \mu(y) \quad \textit{and} \quad \mu(e) \geqslant \mu(x)$$

for all $x, y \in G$.

Definition 1.4. A fuzzy subgroup μ of a group G is called a *fuzzy normal subgroup* of G if

$$\mu(xyx^{-1}) \geqslant \mu(y)$$

for all $x, y \in G$, or equivalently, if and only if

$$\mu(xy) = \mu(yx)$$

for all $x, y \in G$.

By a *fuzzy relation* on X we mean a fuzzy set $\mu : X \times X \to I$. If θ and φ are two fuzzy relations on a set X, then $\theta \leqslant \varphi$ means that $\theta(x, y) \leqslant \varphi(x, y)$ for all $x, y \in X$. Their composition $\theta \circ \varphi$ is defined by

$$(\theta \circ \varphi)(x, y) = \bigvee_{z \in X} \{\theta(x, z) \wedge \varphi(z, y)\}.$$

Definition 1.5. A fuzzy relation θ on X is a *fuzzy equivalence relation* if

(i) $\theta(x, x) = 1 \ \ \forall x \in X$,

(ii) $\theta(x, y) = \theta(y, x) \ \ \forall x, y \in X$,

(iii) $\theta \circ \theta \leqslant \theta$.

Definition 1.6. A fuzzy equivalence relation θ on a semigroup S is a *fuzzy congruence* if it is *fuzzy compatible*, that is,

$$\theta(x, y) \wedge \theta(z, t) \leqslant \theta(xz, yt)$$

for all $x, y, z, t \in S$, or equivalently, if and only if it is *fuzzy left* and *fuzzy right compatible*, i.e.,

$$\theta(x, y) \leqslant \theta(zx, zy) \quad \text{and} \quad \theta(x, y) \leqslant \theta(xz, yz)$$

for all $x, y, z, t \in S$.

2. Fuzzy relations and fuzzy congruences

We need to define a special relation β_α as follows:

Definition 2.1. Let M be a Γ-group, μ_{H_α} be a fuzzy subgroup of M_α, $\alpha \in \Gamma$ and e_α be the identity of M_α. A fuzzy relation β_α on M is defined by

$$\beta_\alpha(a, b) = \begin{cases} \mu_{H_\alpha}(a) \wedge \mu_{H_\alpha}(b), & \text{if } a \neq b, \\ \mu_{H_\alpha}(e_\alpha), & \text{if } a = b. \end{cases}$$

Proposition 2.2. *β_α is a fuzzy equivalence relation on M.*

Proof. β_α is reflexive and symmetric. It is also transitive. Indeed, for all $a, c \in M$ we have

$$\begin{aligned}(\beta_\alpha \circ \beta_\alpha)(a,c) &= \bigvee_{b\in M} \{\beta_\alpha(a,b) \wedge \beta_\alpha(b,c)\} \\ &= \bigvee_{b\in M} \{(\mu_{H_\alpha}(a) \wedge \mu_{H_\alpha}(b)) \wedge (\mu_{H_\alpha}(b) \wedge \mu_{H_\alpha}(c))\} \\ &\leqslant \bigvee_{b\in M} \{\mu_{H_\alpha}(a) \wedge \mu_{H_\alpha}(b)\} \wedge \bigvee_{b\in M} \{\mu_{H_\alpha}(b) \wedge \mu_{H_\alpha}(c)\} \\ &\leqslant \bigvee_{b\in M} \{\mu_{H_\alpha}(a)\} \wedge \bigvee_{b\in M} \{\mu_{H_\alpha}(c)\} = \mu_{H_\alpha}(a) \wedge \mu_{H_\alpha}(c) = \beta_\alpha(a,c).\end{aligned}$$

Therefore β_α is a fuzzy equivalence relation. □

Corollary 2.3. *$\beta_\alpha(x_\alpha^{-1}, y_\alpha^{-1}) = \beta_\alpha(x,y)$ for all $x, y \in M$, where $x_\alpha^{-1}, y_\alpha^{-1}$ are inverses of x and y in M_α.*

Proof. μ_{H_α} is a fuzzy subgroup of M_α. Thus

$$\beta_\alpha(x_\alpha^{-1}, y_\alpha^{-1}) = \mu_{H_\alpha}(x_\alpha^{-1}) \wedge \mu_{H_\alpha}(y_\alpha^{-1}) = \mu_{H_\alpha}(x) \wedge \mu_{H_\alpha}(y) = \beta_\alpha,$$

which completes the proof. □

Proposition 2.4. *β_α is a fuzzy congruence on M.*

Proof. Indeed,

$$\begin{aligned}\beta_\alpha(a\alpha c, b\alpha d) &= \mu_{H_\alpha}(a\alpha c) \wedge \mu_{H_\alpha}(b\alpha d) \\ &\geqslant (\mu_{H_\alpha}(a) \wedge \mu_{H_\alpha}(c)) \wedge (\mu_{H_\alpha}(b) \wedge \mu_{H_\alpha}(d)) \\ &= (\mu_{H_\alpha}(a) \wedge \mu_{H_\alpha}(b)) \wedge (\mu_{H_\alpha}(c) \wedge \mu_{H_\alpha}(d)) \\ &= \beta_\alpha(a,b) \wedge \beta_\alpha(c,d).\end{aligned}$$

This completes the proof. □

Definition 2.5. If a fuzzy set is a (normal) fuzzy subgroup of M_α/H_α, then it is called a (*normal*) *fuzzy quotient* Γ-*subgroup*. For any normal subgroup H_α of M_α we define a fuzzy set $R : M_\alpha/H_\alpha \to [0,1]$ by putting $R(x\alpha H_\alpha) = \beta_\alpha(x,h)$ for all $h \in H_\alpha$.

Proposition 2.6. *R is a normal fuzzy quotient subgroup of M_α/H_α.*

Proof. Since μ_{H_α} is a fuzzy subgroup of M_α, for all $x\alpha H, y\alpha H \in M_\alpha/H_\alpha$ we have

$$\begin{aligned} R(x\alpha H_\alpha \alpha y\alpha H_\alpha) &= \beta_\alpha(x\alpha y, h) = \mu_{H_\alpha}(x\alpha y) \wedge \mu_{H_\alpha}(h) \\ &\geqslant (\mu_{H_\alpha}(x) \wedge \mu_{H_\alpha}(y)) \wedge \mu_{H_\alpha}(h) \\ &= \big(\mu_{H_\alpha}(x) \wedge \mu_{H_\alpha}(h)\big) \wedge \big(\mu_{H_\alpha}(y) \wedge \mu_{H_\alpha}(h)\big) \\ &= \beta_\alpha(x,h) \wedge \beta_\alpha(y,h) = R(x\alpha H) \wedge R(y\alpha H) \end{aligned}$$

and

$$\begin{aligned} R(x_\alpha^{-1}\alpha H_\alpha) &= \beta_\alpha(x_\alpha^{-1}, h) = \mu_{H_\alpha}(x_\alpha^{-1}) \wedge \mu_{H_\alpha}(h) \\ &\geqslant \mu_{H_\alpha}(x) \wedge \mu_{H_\alpha}(h) = \beta_\alpha(x,h) = R(x\alpha H_\alpha). \end{aligned}$$

Thus R ia a quotient fuzzy subgroup of M_α/H_α. Since μ_{H_α} is normal

$$\begin{aligned} R(x\alpha H_\alpha \alpha y\alpha H_\alpha) &= \beta_\alpha(x\alpha y, h) = \mu_{H_\alpha}(x\alpha y) \wedge \mu_{H_\alpha}(h) \\ &= \mu_{H_\alpha}(y\alpha x) \wedge \mu_{H_\alpha}(h) = \beta_\alpha(y\alpha x, h) = R(y\alpha H_\alpha \alpha x\alpha H_\alpha). \end{aligned}$$

Hence R is a normal quotient fuzzy subgroup of M_α/H_α. □

Proposition 2.7. *If M_α/H_α is finite and R is its fuzzy quotient subgroup, then R is a fuzzy subgroup.*

Proof. Since M_α/H_α is finite, every $x\alpha H_\alpha \in M_\alpha/H_\alpha$ has finite order, say n. Then $(x\alpha H_\alpha)^n = (x\alpha)^{n-1}x\alpha H_\alpha = H_\alpha$, where H_α is the identity of M_α/H_α. Thus $(x\alpha H_\alpha)^{-1} = x_\alpha^{-1}\alpha H_\alpha = (x\alpha)^{n-2}x\alpha H_\alpha$ and

$$\begin{aligned} R(x_\alpha^{-1}\alpha H_\alpha) &= R((x\alpha)^{n-2}x\alpha H_\alpha) = \beta_\alpha((x\alpha)^{n-2}x, h) \\ &= \mu_{H_\alpha}((x\alpha)^{n-3}x\alpha x) \wedge \mu_{H_\alpha}(h) = \mu_{H_\alpha}((x\alpha)^{n-3}x\alpha x) \\ &\geqslant \mu_{H_\alpha}((x\alpha)^{n-3}x) \wedge \mu_{H_\alpha}(x) \geqslant \mu_{H_\alpha}(x) \\ &= \mu_{H_\alpha}(x) \wedge \mu_{H_\alpha}(h) = \beta_\alpha(x,h) = R(x\alpha H_\alpha). \end{aligned}$$

Hence R is a fuzzy quotient subgroup. □

Proposition 2.8. *Let R be a fuzzy quotient subgroup of a group M_α/H_α and let $x\alpha H_\alpha \in M_\alpha/H_\alpha$. Then*

$$R(x\alpha H_\alpha \alpha y\alpha H_\alpha) = R(y\alpha H_\alpha) \Longleftrightarrow R(x\alpha H_\alpha) = R(H_\alpha).$$

Proof. If $R(x\alpha H_\alpha \alpha y\alpha H_\alpha) = R(y\alpha H_\alpha)$ holds for all $y\alpha H_\alpha \in M_\alpha/H_\alpha$, then putting $y\alpha H_\alpha = H_\alpha$, we obtain $R(x\alpha H_\alpha) = R(H_\alpha)$.

Conversely, suppose that $R(x\alpha H_\alpha) = R(H_\alpha)$. Since R is a fuzzy subgroup of M_α/H_α and μ_{H_α} is a fuzzy subgroup of M_α, we have

$$\begin{aligned} R(x\alpha H_\alpha \alpha y\alpha H_\alpha) &\geqslant R(x\alpha H_\alpha) \wedge R(y\alpha H_\alpha) = R(H_\alpha) \wedge R(y\alpha H_\alpha) \\ &= \beta_\alpha(e,h) \wedge \beta(y\alpha H_\alpha) = \mu_{H_\alpha}(h) \wedge \mu_{H_\alpha}(y) \\ &= \beta_\alpha(y,h) = R(y\alpha H_\alpha). \end{aligned}$$

Interchanging $x\alpha H_\alpha \alpha y\alpha H_\alpha$ with $y\alpha H_\alpha$, we get

$$R(y\alpha H_\alpha) \geqslant R(x\alpha H_\alpha \alpha y\alpha H_\alpha).$$

Hence the proof is completed. □

Proposition 2.9. *The intersection of two normal fuzzy quotient subgroups of M_α/H_α also is a normal fuzzy quotient subgroups of M_α/H_α.*

Proof. Let R and Q be two normal fuzzy quotient subgroups of M_α/H_α. Then for ally $x\alpha H_\alpha, y\alpha H_\alpha \in M_\alpha/H_\alpha$ we have

$$\begin{aligned} (R\cap Q)(x\alpha H_\alpha \alpha y\alpha H_\alpha) &= R(x\alpha H_\alpha \alpha y\alpha H_\alpha) \wedge Q(x\alpha H_\alpha \alpha y\alpha H_\alpha) \\ &\geqslant (R(x\alpha H_\alpha) \wedge R(y\alpha H_\alpha)) \wedge (Q(x\alpha H_\alpha) \wedge Q(y\alpha H_\alpha)) \\ &= (R(x\alpha H_\alpha) \wedge Q(x\alpha H_\alpha)) \wedge (R(y\alpha H_\alpha) \wedge Q(y\alpha H_\alpha)) \\ &= (R\cap Q)(x\alpha H_\alpha) \wedge (R\cap Q)(y\alpha H_\alpha) \end{aligned}$$

and

$$\begin{aligned} (R\cap Q)(x_\alpha^{-1}\alpha H_\alpha) &= R(x_\alpha^{-1}\alpha H_\alpha) \wedge Q(x_\alpha^{-1}\alpha H_\alpha) = R(x\alpha H_\alpha) \wedge Q(x\alpha H_\alpha) \\ &\leqslant (R\cap Q)(x\alpha H_\alpha). \end{aligned}$$

Interchanging $x\alpha H_\alpha$ with $x_\alpha^{-1}\alpha H_\alpha$, we obtain $(R\cap Q)(x\alpha H_\alpha) \leqslant (R\cap Q)(x_\alpha^{-1}\alpha H_\alpha)$. Hence $R\cap Q$ is a fuzzy subgroup of M_α/H_α. It is normal because

$$\begin{aligned} (R\cap Q)(x\alpha H_\alpha \alpha y\alpha H_\alpha) &= R(x\alpha H_\alpha \alpha y\alpha H_\alpha) \wedge Q(x\alpha H_\alpha \alpha y\alpha H_\alpha) \\ &= R(y\alpha H_\alpha \alpha x\alpha H_\alpha) \wedge Q(y\alpha H_\alpha \alpha x\alpha H_\alpha) \\ &\leqslant (R\cap Q)(y\alpha H_\alpha \alpha x\alpha H_\alpha). \end{aligned}$$

This completes the proof. □

Definition 2.10. On M_α/H_α we define a fuzzy relation $\mu_{\alpha,R}$ putting

$$\mu_{\alpha,R}(x\alpha H_\alpha, y\alpha H_\alpha) = R(x\alpha H_\alpha \alpha y_\alpha^{-1}\alpha H_\alpha)$$

for all $x\alpha H_\alpha, y\alpha H_\alpha \in M_\alpha/H_\alpha$.

Proposition 2.11. *$\mu_{\alpha,R}$ is a fuzzy congruence on M_α/H_α.*

Proof. It is clear that this relation is transitive. Since

$$\begin{aligned}\mu_{\alpha,R}(x\alpha H_\alpha, y\alpha H_\alpha) &= R(x\alpha H_\alpha \alpha y\alpha H_\alpha) = R((y\alpha x_\alpha^{-1})_\alpha^{-1}\alpha H_\alpha)\\ &= R(y\alpha x_\alpha^{-1}\alpha H_\alpha) = R(y\alpha H_\alpha \alpha x_\alpha^{-1}\alpha H_\alpha)\\ &= \mu_{\alpha,R}(y\alpha H_\alpha, x\alpha H_\alpha)\end{aligned}$$

it is also symmetric. Moreover, for all $x\alpha H_\alpha, y\alpha H_\alpha \in M_\alpha/H_\alpha$ we have

$$\begin{aligned}&(\mu_{\alpha,R} \circ \mu_{\alpha,R})(x\alpha H_\alpha, y\alpha H_\alpha)\\ &\quad= \bigvee_{z\alpha H_\alpha \in M_\alpha/H_\alpha} \{\mu_{\alpha,R}(x\alpha H_\alpha, z\alpha H_\alpha) \wedge \mu_{\alpha,R}(z\alpha H_\alpha, y\alpha H_\alpha)\}\\ &\quad= \bigvee_{z\alpha H_\alpha \in M_\alpha/H_\alpha} \big\{R(x\alpha H_\alpha \alpha z_\alpha^{-1}\alpha H_\alpha) \wedge R(z\alpha H_\alpha \alpha y_\alpha^{-1}\alpha H_\alpha)\big\}\\ &\quad= \bigvee_{z\alpha H_\alpha \in M_\alpha/H_\alpha} \big\{R(x\alpha z_\alpha^{-1}\alpha H_\alpha) \wedge R(z\alpha y_\alpha^{-1}\alpha H_\alpha)\big\}\\ &\quad= \bigvee_{z\alpha H_\alpha \in M_\alpha/H_\alpha} \big\{\beta_\alpha(x\alpha z_\alpha^{-1}, h) \wedge \beta_\alpha(z\alpha y_\alpha^{-1}, h)\big\}\\ &\quad= \bigvee_{z\alpha H_\alpha \in M_\alpha/H_\alpha} \big\{\big(\mu_{H_\alpha}(x\alpha z_\alpha^{-1}) \wedge \mu_{H_\alpha}(h)\big) \wedge \big(\mu_{H_\alpha}(z\alpha y_\alpha^{-1}) \wedge \mu_{H_\alpha}(h)\big)\big\}\\ &\quad\leqslant \bigvee_{z\alpha H_\alpha \in M_\alpha/H_\alpha} \big\{\big(\mu_{H_\alpha}(x\alpha z_\alpha^{-1}) \wedge \mu_{H_\alpha}(z\alpha y_\alpha^{-1}) \wedge \mu_{H_\alpha}(h)\big\}\\ &\quad\leqslant \bigvee_{z\alpha H_\alpha \in M_\alpha/H_\alpha} \big\{\mu_{H_\alpha}(x\alpha y_\alpha^{-1}) \wedge \mu_{H_\alpha}(h)\big\} = \mu_{H_\alpha}(x\alpha y_\alpha^{-1}) \wedge \mu_{H_\alpha}(h)\\ &\quad= \beta_\alpha(x\alpha y_\alpha^{-1}, h) = R(x\alpha H_\alpha \alpha y_\alpha^{-1}\alpha H_\alpha) = \mu_{\alpha,R}(x\alpha H_\alpha, y\alpha H_\alpha).\end{aligned}$$

So, $\mu_{\alpha,R}$ is an equivalence relation.

To prove that it is a congruence observe that

$$\begin{aligned}&\mu_{\alpha,R}(x\alpha H_\alpha, y\alpha H_\alpha) \wedge \mu_{\alpha,R}(z\alpha H_\alpha, w\alpha H_\alpha)\\ &\qquad= R(x\alpha H_\alpha \alpha y_\alpha^{-1}\alpha H_\alpha) \wedge R(z\alpha H_\alpha \alpha w_\alpha^{-1}\alpha H_\alpha)\\ &\qquad= R(x\alpha y_\alpha^{-1}\alpha H_\alpha) \wedge R(z\alpha w_\alpha^{-1}\alpha H_\alpha)\\ &\qquad= \beta_\alpha(x\alpha y_\alpha^{-1}, h) \wedge \beta_\alpha(z\alpha w_\alpha^{-1}, h)\\ &\qquad= \{\big(\mu_{H_\alpha}(x\alpha y_\alpha^{-1}) \wedge \mu_{H_\alpha}(h)\big) \wedge \big(\mu_{H_\alpha}(z\alpha w_\alpha^{-1}) \wedge \mu_{H_\alpha}(h)\big)\}\\ &\qquad= \mu_{H_\alpha}(x\alpha y_\alpha^{-1}) \wedge \mu_{H_\alpha}(z\alpha w_\alpha^{-1})\\ &\qquad= \mu_{H_\alpha}(y_\alpha^{-1}\alpha x) \wedge \mu_{H_\alpha}(z\alpha w_\alpha^{-1}).\end{aligned}$$

Since μ_{H_α} is a fuzzy normal subgroup of M_α

$$\begin{aligned}\mu_{H_\alpha}(y_\alpha^{-1}\alpha x)\wedge\mu_{H_\alpha}(z\alpha w_\alpha^{-1}) \leqslant \mu_{H_\alpha}(y_\alpha^{-1}x\alpha z\alpha w_\alpha^{-1}) &= \mu_{H_\alpha}(x\alpha z\alpha w_\alpha^{-1}\alpha y_\alpha^{-1})\\ &= \mu_{\alpha,H_\alpha}(x\alpha z\alpha(y\alpha w)_\alpha^{-1})\wedge\mu_{\alpha,H_\alpha}(h)\\ &= \beta_\alpha(x\alpha z\alpha(y\alpha w)_\alpha^{-1},h)\\ &= R(x\alpha z\alpha H_\alpha\alpha(y\alpha w)_\alpha^{-1}\alpha H_\alpha)\\ &= \mu_{\alpha,R}(x\alpha z\alpha H_\alpha, y\alpha w\alpha H_\alpha),\end{aligned}$$

which completes the proof. □

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Fuzzy ideals in ordered semigroups I

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Abstract

We prove that: a regular ordered semigroup S is left simple if and only if every fuzzy left ideal of S is a constant function. We also show that an ordered semigroup S is left (resp. right) regular if and only if for every fuzzy left(resp. right) ideal f of S we have, $f(a) = f(a^2)$ for every $a \in S$. Further, we characterize some semilattices of ordered semigroups in terms of fuzzy left(resp. right) ideals. In this respect, we prove that an ordered semigroup S is a semilattice of left (resp. right) simple semigroups if and only if for every fuzzy left(resp. right) ideal f of S we have, $f(a) = f(a^2)$ and $f(ab) = f(ba)$ for all $a, b \in S$.

1. Introduction

A fuzzy subset f of a given set S is described as an arbitrary function $f : S \longrightarrow [0, 1]$, where $[0, 1]$ is the usual closed interval of real numbers. This fundamental concept of a fuzzy set, was first introduced by Zadeh in his pioneering paper [24] of 1965, provides a natural frame-work for the generalizations of some basic notions of algebra, e.g. logic, set theory, group theory, ring theory, groupoids, real analysis, measure theory, topology, and differential equations etc. Rosenfeld (see [21]) was the first who considered the case when S is a groupoid. He gave the definition of a fuzzy subgroupoid and the fuzzy left (right, two-sided) ideal of S and justified these definitions by showing that a subset A of a groupoid S is a subgroupoid or a left (right, or two-sided) ideal of S if and only if the characteristic mapping $f_A : S \to \{0, 1\}$ of A defined by

$$x \longmapsto f_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

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is, respectively, a fuzzy subgroupoid or a fuzzy left (right or two-sided) ideal of S. The concept of a fuzzy ideal in semigroups was first developed by Kuroki (see [12-17]). Fuzzy ideals and Green's relations in semigroups were studied by McLean and Kummer in [18]. Dib and Galhum in [2], introduced the definitions of a fuzzy groupoid, and a fuzzy semigroups and studied fuzzy ideals and fuzzy bi-ideals of a fuzzy semigroups. Ahsan et. al in [1] characterized semisimple semigroups in terms of fuzzy ideals. A systematic exposition of fuzzy semigroups by Mordeson, Malik and Kuroki appeared in [20], where one can find theoretical results on fuzzy semigroups and their use in fuzzy coding, fuzzy finite state machines and fuzzy languages. The monograph by Mordeson and Malik [19] deals with the applications of fuzzy approach to the concepts of automata and formal languages. Fuzzy sets in ordered semigroups/ordered groupoids were first introduced by Kehayopulu and Tsingelis in [8]. They also introduced the concepts of fuzzy bi-ideals and fuzzy quasi-ideals in ordered semigroups in (see [9] and [10]).

In [22], Shabir and Khan, introduced the concept of a fuzzy generalized bi-ideal of ordered semigroups and characterized different classes of ordered semigroups by using fuzzy generalized bi-ideals. They also gave the concept of fuzzy left (resp. bi-) filters in ordered semigroups and gave the relations of fuzzy bi-filters and fuzzy bi-ideal subsets of ordered semigroups in [23].

In this paper, which is a continuation of the work carried out by Kehayopulu-Tesingelis [11] for ordered semigroups in terms of fuzzy ideals, we characterize regular, left and right simple ordered semigroups and completely regular ordered semigroups in terms of fuzzy left (resp. right) ideals. In this respect, we prove that: A regular ordered semigroup S left simple if and only if every fuzzy left ideal f of S is a constant function. We also prove that S is left regular if and only if for every fuzzy left ideal f of S we have $f(a) = f(a^2)$ for every $a \in S$. Next we characterize semilattices of left simple ordered semigroups in terms of fuzzy left ideals of S. We prove that an ordered semigroup S is a semilattice of left simple semigroups if and only if for every fuzzy left ideal f of S we have, $f(a) = f(a^2)$ and $f(ab) = f(ba)$ for all $a, b \in S$.

2. Preliminaries

By an *ordered semigroup* (or *po-semigroup*) we mean a structure $(S, \cdot, \leqslant)$ in which

$(OS1)$ $(S, \cdot)$ is a semigroup,

$(OS2)$ $(S, \leqslant)$ is a poset,
$(OS3)$ $(\forall a, b, x \in S)(a \leqslant b \longrightarrow ax \leqslant bx$ and $xa \leqslant xb)$.

Let $(S, \cdot, \leqslant)$ be an ordered semigroup. A non-empty subset A of S is called a *left* (resp. *right*) *ideal* of S (see [7-10]) if:

(i) $SA \subseteq A$ (resp. $AS \subseteq A$) and
(ii) $(\forall a \in A)(\forall b \in S)\,(b \leqslant a \longrightarrow b \in A)$.

Let $(S, \cdot, \leqslant)$ be an ordered semigroup and $\emptyset \neq A \subseteq S$. Then A is called a *subsemigroup* of S (see [9]) if $A^2 \subseteq A$. A subsemigroup A of S is called a *bi-ideal* of S if:

(i) $ASA \subseteq A$ and
(ii) $(\forall a \in A)(\forall b \in S)\,(b \leqslant a \longrightarrow b \in A)$.

A subsemigroup A of S is called a $(1,2)$-*ideal* of S if:

(i) $ASA^2 \subseteq A$ and
(ii) $(\forall a \in A)(\forall b \in S)\,(b \leqslant a \longrightarrow b \in A)$.

By a *fuzzy subset* f of S we mean a *mapping* $f : S \longrightarrow [0, 1]$.

Definition 2.1. Let $(S, \cdot, \leqslant)$ be an ordered semigroups and f a fuzzy subset of S. Then f is called a *fuzzy left* (resp. *right*) *ideal* of S if:

(1) $(\forall x, y \in S)(x \leqslant y \longrightarrow f(x) \geqslant f(y))$.
(2) $(\forall x, y \in S)(f(xy) \geqslant f(y)(\text{resp. } f(xy) \geqslant f(x))$.

A fuzzy left and right ideal f of S is called a *fuzzy two-sided ideal* of S.

For any fuzzy subset f of S and $t \in (0, 1]$, the set

$$U(f; t) := \{x \in S \mid f(x) \geqslant t\}$$

is called the *level subset* of f.

Theorem 2.2. (cf. [8]) *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. A fuzzy subset f of S is a fuzzy left (resp. right) ideal of S if and only if for every $t \in (0, 1]$ $U(f; t) \neq \emptyset$ is a left (resp. right) ideal.* □

Example 2.3. Let $S = \{a, b, c, d, e, f\}$ be an ordered semigroup defined by the multiplication and the order below:

$\cdot$	a	b	c	d	e	f
a	a	a	a	d	a	a
b	a	b	b	d	b	b
c	a	b	c	d	e	e
d	a	a	d	d	d	d
e	a	b	c	d	e	e
f	a	b	c	d	e	f

$$\leqslant := \{(a,a),(b,b),(c,c),(d,d),(e,e),(f,e),(f,f)\}$$

Right ideals of S are: $\{a,d\},\{a,b,d\}$ and S. Left ideals of S are: $\{a\},\{d\},\{a,b\},\{a,d\},\{a,b,d\},\{a,b,c,d\},\{a,b,d,e,f\}$ and S(see [7]).

Define $f : S \longrightarrow [0,1]$ by $f(a) = 0.8$, $f(b) = 0.5$, $f(d) = 0.6$ and $f(c) = f(e) = f(f) = 0.4$. Then

$$U(f;t) := \begin{cases} S & \text{if} \quad t \in (0.2, 0.4], \\ \{a,b,d\} & \text{if} \quad t \in (0.4, 0.5], \\ \{a,d\} & \text{if} \quad t \in (0.5, 0.6], \\ \emptyset & \text{if} \quad t \in (0.8, 1]. \end{cases}$$

and $U(f;t)$ is a right ideal of S, By Theorem 2.2, f is a fuzzy right ideal of S.

Let $\emptyset \neq A \subseteq S$. The *characteristic mapping* $f_A : S \longrightarrow \{0,1\}$ of A is defined by:

$$f_A : S \longrightarrow [0,1],\ x \longmapsto f_A(x) := \begin{cases} 1 & \text{if} \quad x \in A, \\ 0 & \text{if} \quad x \notin A. \end{cases}$$

Lemma 2.4. (cf. [4, 5]) *A non-empty subset A of an ordered semigroup $(S,\cdot,\leqslant)$ is a left (resp. right and bi-) ideal of S if and only if its characteristic function f_A is a fuzzy left (resp. right and bi-) ideal of S.* □

A subset T of an ordered semigroup S is called *semiprime* (see [9]) if for every $a \in S$ from $a^2 \in T$ it follows $a \in T$, or equivalently, if for each subset A of S $A^2 \subseteq T$ implies $A \subseteq T$.

3. Characterizations of regular semigroups

An ordered semigroup S is *regular* (see [5]) if for every $a \in S$, there exists $x \in S$ such that $a \leqslant axa$ or, equivalently, if $a \in (aSa]$ for every $a \in S$, and $A \subseteq (ASA]$ for every $A \subseteq S$.

An ordered semigroup S is *left* (resp. *right*) *simple* (see [9]) if for every left (resp. right) ideal A of S, we have $A = S$. S is called *simple* if it is left simple and right simple.

Theorem 3.1. *A regular ordered semigroup S is left simple if and only if every fuzzy left ideal of S is a constant map.*

Proof. Let S be a left simple ordered semigroup, f a fuzzy left ideal of S and $a \in S$. We consider the set,

$$E_S := \{e \in S \mid e^2 \geqslant e\}.$$

Then $E_S \neq \emptyset$. In fact, since S is regular and $a \in S$, there exists $x \in S$ such that $a \leqslant axa$. It follows from $(OS3)$ that

$$(ax)^2 = (axa)x \geqslant ax,$$

and so $ax \in E_S$ and hence $E_S \neq \emptyset$.

(1) Let $t \in E_S$. Then $f(e) = f(t)$ for every $e \in E_S$. Indeed, since S is left simple and $t \in S$ we have $(St] = S$. Since $e \in S$, then $e \in (St]$ and there exists $z \in S$ such that $e \leqslant zt$. Hence $e^2 \leqslant (zt)(zt) = (ztz)t$. Since f is a fuzzy left ideal of S, we have

$$f(e^2) \geqslant f((ztz)t) \geqslant f(t).$$

Since $e \in E_S$, we have $e^2 \geqslant e$. Then $f(e) \geqslant f(e^2)$ and we have $f(e) \geqslant f(t)$. Besides, since S is left simple and $e \in S$, we have $(Se] = S$. Since $t \in E_S$, exactly on the previous case−by symmetry− we get $f(t) \geqslant f(e)$. Hence $f(t) = f(e)$, i.e., f is constant on E_S.

(2) Let $a \in S$, then $f(a) = f(t)$ for every $t \in S$. Indeed, since S is regular there exists $x \in S$ such that $a \leqslant axa$. We consider the element $xa \in S$. Then it follows by $(OS3)$ that,

$$(xa)^2 = x(axa) \geqslant xa,$$

then $xa \in E_S$ and by (1), we have $f(xa) = f(t)$. Besides, f is fuzzy left ideal of S, we have $f(xa) \geqslant f(a)$. Then $f(t) \geqslant f(a)$. On the other hand, since S is left simple and $t \in S$ then $S = (St]$. Since $a \in S$, we have $a \leqslant st$ for some $s \in S$. Since f is fuzzy left ideal of S, we have $f(a) \geqslant f(st) \geqslant f(t)$. Thus $f(t) = f(a)$, i.e., f is constant on S.

Conversely, let $a \in S$. Then the set $(Sa]$ is a left ideal of S. In fact, $S(Sa] = (S](Sa] \subseteq (SSa] \subseteq (Sa]$. If $x \in (Sa]$ and $S \ni y \leqslant x$, then $y \in ((Sa]] = (Sa]$. Since $(Sa]$ is a left ideal of S. By Lemma 2.4, the characteristic mapping

$$f_{(Sa]} : S \longrightarrow \{0,1\}, \quad x \longmapsto f_{(Sa]}(x)$$

is a fuzzy left ideal of S. By hypothesis $f_{(Sa]}$ is a constant mapping, that is, there exists $c \in \{0,1\}$ such that

$$f_{(Sa]}(x) = c \quad \text{for every } x \in S.$$

Let $(Sa] \subset S$ and let $t \in S$ such that $t \notin (Sa]$ then $f_{(Sa]}(t) = 0$. On the other hand, since $a^2 \in (Sa]$, then we have $f_{(Sa]}(a^2) = 0$, a contradiction to the fact that $f_{(Sa]}$ is a constant mapping. Hence $S = (Sa]$. $\square$

From left–right dual of Theorem 3.1, we have the following:

Theorem 3.2. *A regular ordered semigroup S is right simple if and only if every fuzzy right ideal of S is a constant mapping.* $\square$

An ordered semigroup $(S, \cdot, \leqslant)$ is *left* (resp. *right*) *regular* [4, 6], if for every $a \in S$ there exists $x \in S$ such that $a \leqslant xa^2$ (resp. $a \leqslant a^2x$) or, equivalently, if $a \in (Sa^2]$ (resp. $a \in (a^2S]$) for all $a \in S$, and $A \subseteq (SA^2]$ (resp. $A \subseteq (A^2S]$) for all $A \subseteq S$.

An ordered semigroup S is called *completely regular* (see [6]) if it is regular, left regular and right regular.

Lemma 3.3. (cf. [9]) *An ordered semigroup S is completely regular if and only if $A \subseteq (A^2SA^2]$ for every $A \subseteq S$ or, equivalently, if and only if $a \in (a^2Sa^2]$ for every $a \in S$.* $\square$

Theorem 3.4. *An ordered semigroup $(S, \cdot, \leqslant)$ is left regular if and only if for each fuzzy left ideal f of S, we have $f(a) = f(a^2)$ for all $a \in S$.*

Proof. Suppose that f is a fuzzy left ideal of S and let $a \in S$. Since S is left regular, there exists $x \in S$ such that $a \leqslant xa^2$. Since f is a fuzzy left ideal of S, we have

$$f(a) \geqslant f(xa^2) \geqslant f(a^2) \geqslant f(a).$$

Conversely, let $a \in S$. We consider the left ideal $L(a^2) = (a^2 \cup Sa^2]$ of S, generated by a^2. Then by Lemma 2.4, the characteristic mapping

$$f_{L(a^2)} : S \longrightarrow \{0, 1\}, \quad x \longmapsto f_{L(a^2)}(x)$$

is a fuzzy left ideal of S.

By hypothesis we have $f_{L(a^2)}(a) = f_{L(a^2)}(a^2)$. Since $a^2 \in L(a^2)$, we have $f_{L(a^2)}(a^2) = 1$ and $f_{L(a^2)}(a) = 1$. Then $a \in L(a^2) = (a^2 \cup Sa^2]$ and $a \leqslant y$ for some $y \in a^2 \cup Sa^2$. If $y = a^2$, then $a \leqslant y = a^2 = aa = aa^2 \in Sa^2$ and $a \in (Sa^2]$. If $y = xa^2$ for some $x \in S$, then $a \leqslant y = xa^2 \in Sa^2$, and $a \in (Sa^2]$. $\square$

From left$-$right dual of Theorem 3.4, we have the following:

Theorem 3.5. *An ordered semigroup $(S, \cdot, \leqslant)$ is right regular if and only if for each fuzzy right ideal f of S, we have $f(a) = f(a^2)$ for all $a \in S$.* $\square$

From ([9, Theorem 3]) and Theorems 3.1 and 3.4, and by Lemma 3.3, we have the following characterization theorem for completely regular ordered semigroups.

Theorem 3.6. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. Then the following are equivalent:*

(i) *S is completely regular,*

(ii) *for each fuzzy bi-ideal f of S we have $f(a) = f(a^2)$ for all $a \in S$,*

(iii) *for each fuzzy left ideal g and each fuzzy right ideal h of S we have $g(a) = g(a^2)$ and $h(a) = h(a^2)$ for all $a \in S$.* $\square$

An ordered semigroup $(S, \cdot, \leqslant)$ is called *left* (resp. *right*) *duo* if every left (resp. right) ideal of S is a two-sided ideal of S, and *duo* if every its ideal is both left and right duo.

Definition 3.7. An ordered semigroup $(S, \cdot, \leqslant)$ is called *fuzzy left* (resp. *right*) *duo* if every fuzzy left (resp. right) ideal of S is a fuzzy two-sided ideal of S. An ordered semigroup S is called *fuzzy duo* if it is both fuzzy left and fuzzy right duo.

Theorem 3.8. *A regular ordered semigroup is left (right) duo if and only if it is fuzzy left (right) duo.*

Proof. Let S be left duo and f a fuzzy left ideal of S. Let $a, b \in S$. Then the set $(Sa]$ is a left ideal of S. In fact, $S(Sa] = (S](Sa] \subseteq (SSa] \subseteq (Sa]$ and if $x \in (Sa]$ and $S \ni y \leqslant x$ then $y \in ((Sa]] = (Sa]$. Since S is left duo, then $(Sa]$ is a two-sided ideal of S. Since S is regular there exists $x \in S$ such that $a \leqslant axa$ then

$$ab \leqslant (axa)b \in (aSa)b \subseteq (Sa)S \subseteq (Sa]S \subseteq (Sa].$$

Then $ab \in ((Sa]] = (Sa]$ and $ab \leqslant xa$ for some $x \in S$. Since f is a fuzzy left ideal of S, we have

$$f(ab) \geqslant f(xa) \geqslant f(a).$$

Let $x, y \in S$ be such that $x \leqslant y$. Then $f(x) \geqslant f(y)$, because f is a fuzzy left ideal of S. Thus f is a fuzzy right deal of S and S is fuzzy left duo.

Conversely, if S is fuzzy left duo and A a left ideal of S, then the characteristic function f_A of A is a fuzzy left ideal of S. By hypothesis f_A is a fuzzy right ideal of S and by Lemma 2.4, A is a right ideal of S. Thus S is left duo. □

Theorem 3.9. *In a regular ordered semigroup every bi-ideal is a right (left) ideal if and only if every its fuzzy bi-ideal is a fuzzy right (left) ideal.*

Proof. Let $a, b \in S$ and f a fuzzy bi-ideal of S. Then $(aSa]$ is a bi-ideal of S. In fact, $(aSa]^2 \subseteq (aSa](aSa] \subseteq (aSa]$, $(aSa]S(aSa] = (aSa](S](aSa] \subseteq (aSa]$ and if $x \in (aSa]$ and $S \ni y \leqslant x \in (aSa]$ then $y \in ((aSa]] = (aSa]$. Since $(aSa]$ is a bi-ideal of S, by hypothesis $(aSa]$ is right ideal of S. Since $a \in S$ and S is regular there exists $x \in S$ such that $a \leqslant axa$ then

$$ab \leqslant (axa)b \in (aSa)S \subseteq (aSa]S \subseteq (aSa].$$

Hense $ab \leqslant aza$ for some $z \in S$. Since f is a fuzzy bi-ideal of S, we have

$$f(ab) \geqslant f(aza) \geqslant \min\{f(a), f(a)\} = f(a).$$

Let $x, y \in S$ be such that $x \leqslant y$. Then $f(x) \geqslant f(y)$ because f is a fuzzy bi-ideal of S. Thus f is a fuzzy right ideal of S.

Conversely, if A is a bi-ideal of S, then by Lemma 2.4, f_A is a fuzzy bi-ideal of S. By hypothesis f_A is a fuzzy right ideal of S. By Lemma 2.4, A is a right ideal of S. □

Definition 3.10. Let $(S, \cdot, \leqslant)$ be an ordered semigroup and f a fuzzy sub-semigoup of S. Then f is called a *fuzzy* $(1,2)$*-ideal* of S if:

(i) $x \leqslant y \longrightarrow f(x) \geqslant f(y)$,

(ii) $f(xa(yz)) \geqslant \min\{f(x), f(y), f(z)\}$

for all $x, y, z, a \in S$.

Proposition 3.11. *Every fuzzy bi-ideal of an ordered semigroup S is a fuzzy $(1,2)$-ideal of S.*

Proof. Let f be a fuzzy bi-ideal of S and let $x, y, z, a \in S$. Then

$$\begin{aligned} f(xa(yz)) &= f((xay)z) \geqslant \min\{f(xay), f(z)\} \\ &\geqslant \min\{\min\{f(x), f(y)\}, f(z)\} = \min\{f(x), f(y), f(z)\}. \end{aligned}$$

Now, let $x, y \in S$ be such that $x \leqslant y$. Then $f(x) \geqslant f(y)$, because f is a fuzzy bi-ideal of S. □

Corollary 3.12. *Every fuzzy left* (*resp. right*) *ideal f of an ordered semigroup S is a fuzzy* $(1,2)$*-ideal of S.*

The converse of the Proposition 3.11, is not true in general. However, if S is a regular ordered semigroup then we have the following Proposition:

Proposition 3.13. *A fuzzy* $(1,2)$*-ideal of a regular ordered semigroup is a fuzzy bi-ideal.*

Proof. Assume that S is regular ordered semigroup and let f be a fuzzy $(1,2)$-ideal of S. Let $x, y, a \in S$. Since S is regular and $(xSx]$ is a bi-ideal of S, so it is a right ideal of S, by Theorem 3.9. Thus

$$xa \leqslant (xSx)a \in (xSx)S \subseteq (xSx]S \subseteq (xSx],$$

whence $xa \leqslant xyx$ for some $y \in S$. Thus $xay \leqslant (xyx)y$ and we have

$$\begin{aligned} f(xay) &\geqslant f((xyx)y) \geqslant \min\{f(xyx), f(y)\} \\ &\geqslant \min\{\min\{f(x), f(x)\}, f(y)\} = \min\{f(x), f(y)\}. \end{aligned}$$

Let $x, y \in S$ be such that $x \leqslant y$. Then $f(x) \geqslant f(y)$, because f is a fuzzy $(1,2)$-ideal of S. Thus f is a fuzzy bi-ideal of S. □

4. Semilattices of left simple ordered semigroups

A subsemigroup F of an ordered semigroup $(S, \cdot, \leqslant)$ is called a *filter* of S if:

(1) $ab \in F \longrightarrow a \in F$ and $b \in F$,

(2) $c \geqslant a \in F \longrightarrow c \in F$

for all $a, b, c \in S$.

For $x \in S$, we denote by $N(x)$ the filter of S generated by x. $\mathcal{N}$ denotes the equivalence relation on S defined by

$$\mathcal{N} := \{(x, y) \in S \times S \mid N(x) = N(y)\}.$$

Definition 4.1. (cf. [7]) An equivalence relation σ on S is called *congruence* if $(a, b) \in \sigma$ implies $(ac, bc) \in \sigma$ and $(ca, cb) \in \sigma$ for every $c \in S$. A congruence σ on S is called *semilattice congruence* if $(a^2, a) \in \sigma$ and $(ab, ba) \in \sigma$ for each $a, b \in S$. If σ is a semilattice congruence on S then the σ-class $(x)_\sigma$ of S containing x is a subsemigroup of S for every $x \in S$.

Lemma 4.2. (cf. [9]) *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. Then $(x)_{\mathcal{N}}$ is a left simple subsemigroup of S, for every $x \in S$ if and only every left ideal of S is a right ideal of S and it is semiprime.* □

An ordered semigroup S is called a *semilattice of left simple semigroups* if there exists a semilattice congruence σ on S such that the σ-class $(x)_\sigma$ of S containing x is a left simple subsemigroup of S for every $x \in S$ or, equivalently, if there exists a semilattice Y and a family $\{S_\alpha\}_{\alpha \in Y}$ of left simple subsemigroups of S such that

(1) $S_\alpha \cap S_\beta = \emptyset \ \ \forall \alpha, \beta \in Y, \quad \alpha \neq \beta$,

(2) $S = \bigcup\limits_{\alpha \in Y} S_\alpha$,

(3) $S_\alpha S_\beta \subseteq S_{\alpha\beta} \ \forall \alpha, \beta \in Y$.

In ordered semigroups the semilattice congruences are defined exactly same as in the case of semigroups −without order− so the two definitions are equivalent (see [7]).

Lemma 4.3. *An ordered semigroup $(S, \cdot, \leqslant)$ is a semilattice of left simple semigroups if and only if for all left ideals A, B of S we have*

$$(A^2] = A \quad \textit{and} \quad (AB] = (BA].$$

Proof. $(\rightarrow)$ Let S be a semilattice of left simple semigroups and A, B are left ideals of S. Then there exists a semilattice Y and a family $\{S_\alpha\}_{\alpha \in Y}$ of left simple subsemigroups of S satisfying all conditions mentioned in the definition of a semilattice of left simple semigroups.

Let $a \in A$. Since $a \in S = \bigcup\limits_{\alpha \in Y} S_\alpha$, there exists $\alpha \in Y$ such that $a \in S_\alpha$. Since S_α is left simple, we have

$$S_\alpha = (S_\alpha b] = \{c \in S \mid \exists x \in S_\alpha : c \leqslant xb\}$$

for all $b \in S_\alpha$.

Since $a \in S_\alpha$, we have $S_\alpha = (S_\alpha a]$ that is $a \leqslant xa$ for some $x \in S_\alpha$. Since $x \in S_\alpha = (S_\alpha a]$, we have $x \leqslant ya$ for some $y \in S_\alpha$. Thus we have $a \leqslant xa \leqslant (ya)a \in (SA)A \subseteq AA = A^2$ and $a \in (A^2]$. Hence $A \subseteq (A^2]$. On the other hand, since A is a subsemigroup of S, hence $A^2 \subseteq A$ and we have $(A^2] \subseteq (A] = A$. Let $x \in (AB]$, then $x \leqslant ab$ for some $a \in A$ and $b \in B$. Since $a, b \in S = \bigcup\limits_{\alpha \in Y} S_\alpha$, there exist $\alpha, \beta \in Y$ such that $a \in S_\alpha$, $b \in S_\beta$. Then $ab \in S_\alpha S_\beta \subseteq S_{\alpha\beta}$ and $ba \in S_\beta S_\alpha \subseteq S_{\beta\alpha} = S_{\alpha\beta}$(since $\alpha, \beta \in Y$

and Y is a semilattice). Since $S_{\alpha\beta}$ is left simple, we have $S_{\alpha\beta} = (S_{\alpha\beta}c]$ for each $c \in S_{\alpha\beta}$. Then $ab \in (S_{\alpha\beta}ba]$ and $ab \leqslant yba$ for some $y \in S_{\alpha\beta}$. Since B is a left ideal of S, we have $yba \in (SB)A \subseteq BA$, then $x \in (BA]$. Thus $(AB] \subseteq (BA]$. By symmetry we have $(BA] \subseteq (AB]$.

($\leftarrow$) Since $\mathcal{N}$ is a semilattice congruence on S, which is equivalent to the fact that $(x)_{\mathcal{N}}$ $\forall x \in S$, is a left simple subsemigrup of S. By Lemma 4.2, it is enough to prove that every left ideal is right ideal and semiprime. Let L be a left ideal of S. Then

$$LS \subseteq (LS] = (SL] \subseteq (L] = L.$$

If $x \in L$, $S \ni y \leqslant x \in L$, then $y \in L$, since L is a left ideal of S. Thus L is a right ideal of S. Let $x \in S$ be such that $x^2 \in L$. We consider the bi-ideal $B(x)$ of S generated by x. Then

$$\begin{aligned} B(x)^2 &= (x \cup x^2 \cup xSx](x \cup x^2 \cup xSx] \subseteq ((x \cup x^2 \cup xSx)(x \cup x^2 \cup xSx)] \\ &= (x^2 \cup x^3 \cup xSx^2 \cup x^4 \cup xSx^3 \cup x^2Sx \cup x^3Sx \cup xSx^2Sx]. \end{aligned}$$

Since $x^2 \in L,\ x^3 \in SL \subseteq L,\ (xS)x^2 \subseteq SL \subseteq L,\ x^4 \in SL \subseteq L$. Then

$$B(x)^2 \subseteq (L \cup LS] = (L] = L.$$

Thus $(B(x)^2] \subseteq (L] = L$ and $x \in L$. Hence L is semiprime. □

Theorem 4.4. *An ordered semigroup $(S, \cdot, \leqslant)$ is a semilattice of left (right) simple semigroups if and only if for every fuzzy left (right) ideal f of S and all $a, b \in S$, we have*

$$f(a^2) = f(a) \quad \text{and} \quad f(ab) = f(ba).$$

Proof. Let S be a semilattice of left simple semigroups. By hypothesis, there exists a semilattice Y and a family $\{S_\alpha\}_{\alpha \in Y}$ of left simple subsemigroups of S such that:

(1) $S_\alpha \cap S_\beta = \emptyset \ \ \forall \alpha, \beta \in Y, \quad \alpha \neq \beta$,

(2) $S = \bigcup\limits_{\alpha \in Y} S_\alpha$,

(3) $S_\alpha S_\beta \subseteq S_{\alpha\beta} \ \forall \alpha, \beta \in Y$.

Let f be a fuzzy left ideal of S and $a \in S$. Then $f(a) = f(a^2)$. In fact, by Theorem 3.4, it is enough to prove that $a \in (Sa^2]$ for every $a \in S$. Let $a \in S$, then there exists $\alpha \in Y$ such that $a \in S_\alpha$. Since S_α is left simple, we have $S_\alpha = (S_\alpha a]$ and $a \leqslant xa$ for some $x \in S$.

Since $x \in S_\alpha$, we have $x \in (S_\alpha a]$ and $x \leqslant ya$ for some $y \in S_\alpha$. Thus we have

$$a \leqslant xa \leqslant (ya)a = ya^2,$$

which for $y \in S$, implies $a \in (Sa^2]$.

Let $a, b \in S$. Then, by the above, we have

$$f(ab) = f((ab)^2) = f(a(ba)b) \geqslant f(ba).$$

By symmetry we can prove that $f(ba) \geqslant f(ab)$. Hence $f(ab) = f(ba)$.

Conversely, assume that for every fuzzy left ideal f of S, we have

$$f(a^2) = f(a) \quad \text{and} \quad f(ab) = f(ba)$$

for all $a, b \in S$.

Then by condition (1) and by Theorem 3.4, we see that S is left regular. Let A be a left ideal of S and let $a \in A$. Then $a \in S$, since S is left regular, there exists $x \in S$ such that

$$a \leqslant xa^2 = (xa)a \in (SA)A \subseteq AA = A^2.$$

Hence $a \in (A^2]$ and $A \subseteq (A]$. On the other hand, since A is a left ideal of S, we have $A^2 \subseteq SA \subseteq A$, then $(A^2] \subseteq (A] = A$. Let A and B be left ideals of S and let $x \in (BA]$ then $x \leqslant ba$ for some $a \in A$ and $b \in B$. We consider the left ideal $L(ab)$ generated by ab. That is, the set $L(ab) = (ab \cup Sab]$. Then by Lemma 3.4, the characteristic function $f_{L(ab)}$ of $L(ab)$ is a fuzzy left ideal of S. By hypothesis, we have $f_{L(ab)}(ab) = f_{L(ab)}(ba)$. Since $ab \in L(ab)$, we have $f_{L(ab)}(ab) = 1$ and $f_{L(ab)}(ba) = 1$ and hence $ba \in L(ab) = (ab \cup Sab]$. Then $ba \leqslant ab$ or $ba \leqslant yab$ for some $y \in S$. If $ba \leqslant ab$ then $x \leqslant ab \in AB$ and $x \in (AB]$. If $ba \leqslant yab$ then $x \leqslant yab \in (SA)B \subseteq AB$ and $x \in (AB]$. Thus $(BA] \subseteq (AB]$. By symmetry we can prove that $(AB] \subseteq (BA]$. Therefore $(AB] = (BA]$ and by Lemma 4.3, it follows that S is a semilattice of left simple semigroups. $\square$

Proposition 4.5. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup and f a fuzzy left (resp. right) ideal of S, $a \in S$ such that $a \leqslant a^2$. Then $f(a) = f(a^2)$.*

Proof. Since $a \leqslant a^2$ and f is a fuzzy left ideal of S, we have

$$f(a) \geqslant f(a^2) = f(aa) \geqslant f(a),$$

and so $f(a) = f(a^2)$. $\square$

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The action of G_2^2 on $PL(F_p)$

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Abstract

Γ_3 is a copy of unique circuit-free connected graph all of whose vertices have degree 3, called cubic tree. The group $G_2^2 = \langle x, y, t : x^2 = t,\ y^3 = t^2 = (yt)^2 = 1\rangle$, is one of the seven finitely presented isomorphism types of subgroups of the full automorphism group $\mathrm{Aut}(\Gamma_3)$ of Γ_3. These seven groups act arc-transitively on the arcs of Γ_3 with a finite vertex stabilizer. In this paper we have found a condition on p such that the action of G_2^2 on the projective line over the finite field, $PL(F_p)$, always yields the subgroups of the alternating groups of degree $p+1$. We have shown also that the action of G_2^2 on $PL(F_p)$ is transitive.

1. Introduction

A cubic tree Γ_3 is a copy of unique circuit-free connected graph all of whose vertices have degree 3. Djoković and Miller [1] have proved that there are seven groups act arc-transitively on the arcs of Γ_3 with a finite vertex stabilizer. The group

$$G_2^2 = \langle x, y, t : x^2 = t,\ y^3 = t^2 = (yt)^2 = 1\rangle$$

is one of these seven finitely presented isomorphism types of subgroups of the full automorphism group $\mathrm{Aut}(\Gamma_3)$ of Γ_3.

Γ_3 can be constructed by the group G_2^2 as follows.
Let $\Omega = \{gH : g \in G_2^2\}$ be the collection of all distinct left cosets of the subgroup

$$H = \langle y, t : y^3 = t^2 = (yt)^2 = 1\rangle$$

of G_2^2 in G_2^2. Two cosets g_1H and g_2H can be joined by an edge if and only if $g_1^{-1}g_2 \in HxH$. Thus vertex H is joined to xH, yxH and y^2xH, whereas xH is joined to H, $xyxH$ and xy^2xH and so on as shown in the Figure 1.

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Keywords: Automorphism groups, arc-transitivity, vertex stabilizer, coset diagram, alternative group, Pythagorean prime

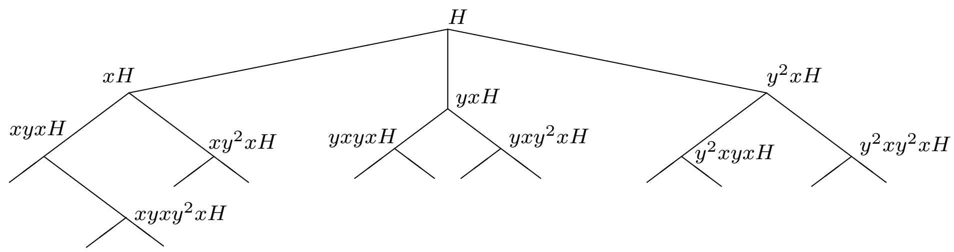

Figure 1.

In fact there is a one to one correspondence between the vertices of Γ_3 and all the reduced words in x and y (and y^2), which are different from identity, which end in x. The elements of G_2^2 induce automorphisms of Γ_3 by left multiplication. For example, the multiplication of y fixes vertex H and rotate other neighbours of vertex H, whereas multiplication of x interchanges H by xH, and the other neighbours of H with the other neighbours of xH and so on as shown in the Figure 2.

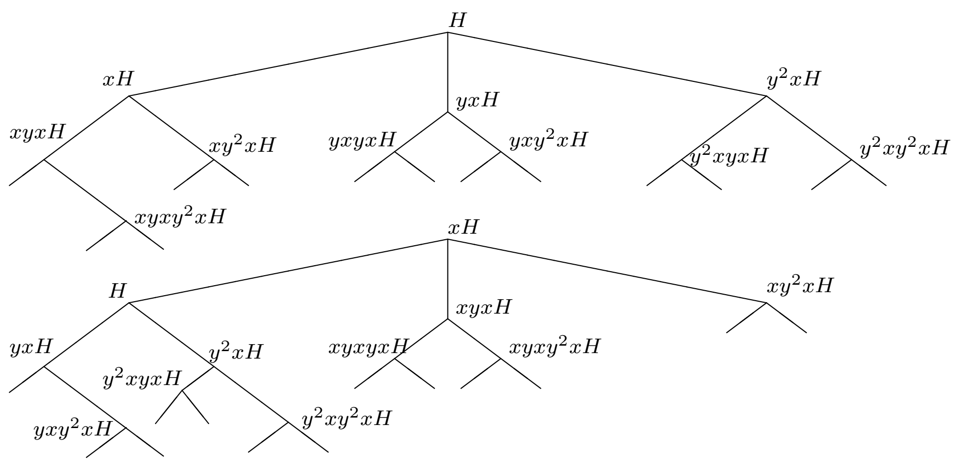

Figure 2.

In particular, action of G_2^2 is transitive on the vertices of Γ_3 and is sharply transitive on its arcs(ordered edges). In other words, the action of G_2^2 is arc-regular on Γ_3, that is, the stabilizer of each arc in G_2^2 is the identity. Of course, the cubic tree has many more automorphisms then these. Indeed, given any path $(v_0, v_1, \ldots, v_{n-1}, v_n)$ of length n in Γ_3, there are automorphisms fixing each vertex v_i on this path and interchanging the other two vertices adjacent to v_n, it follows that Γ_3 is highly arc-transitive, its full automorphism group is transitive on paths of length n, for all $n \geqslant 0$.

Now clearly the stabilizer (in full automorphism group) of any given vertex is infinite. On the other hand, there are subgroups which act transitively on the arcs of Γ_3 but which have a finite vertex stabilizer, for example, in the G_2^2 the stabilizer of the vertex H is the subgroup H itself of order 6. Up to isomorphism, there are only seven such subgroups and they are:

$$G_1 = \langle x, y : x^2 = y^3 = 1\rangle,$$
$$G_2^1 = \langle x, y, t : x^2 = y^3 = t^2 = (xt)^2 = (yt)^2 = 1\rangle,$$
$$G_2^2 = \langle x, y, t : x^2 = t,\ y^3 = t^2 = (yt)^2 = 1\rangle,$$
$$G_3 = \langle x, y, t, q : x^2 = y^3 = t^2 = q^2 = 1, tq = qt, ty = yt, qyq = y^{-1}, xt = qx\rangle,$$
$$G_4^1 = \langle x, y, t, q, r : x^2 = y^3 = t^2 = q^2 = r^2 = 1,\ tq = qt,\ tr = rt,\ rq = tqr,\ y^{-1}ty = q,\ y^{-1}qy = tq,\ ryr = y^{-1},\ xt = tx,\ xq = rx\rangle,$$
$$G_4^2 = \langle x, y, t, q, r : y^3 = t^2 = q^2 = r^2 = 1,\ x^2 = t,\ tq = qt,\ tr = rt,\ rq = tqr,\ y^{-1}ty = q,\ y^{-1}qy = tq,\ ryr = y^{-1},\ xt = tx,\ xq = rx\rangle,$$
$$G_5 = \langle x, y, t, q, r, s : x^2 = y^3 = t^2 = q^2 = r^2 = s^2 = 1,\ tq = qt,\ tr = rt,\ ts = st,\ rq = qr,\ qs = sq,\ sr = tqrs,\ ty = yt,\ y^{-1}qy = r,\ y^{-1}ry = tqr,\ xt = qx,\ xr = sx\rangle.$$

The group G_2^2 is generated by the linear fractional transformations $x(z) = \frac{z+i}{iz+1}$, $y(z) = \frac{z-1}{z}$ and $t(z) = \frac{1}{z}$, which satisfy the relations $y^3 = t^2 = (yt)^2 = 1, x^2 = t$. In [4], Q. Mushtaq and I. Ali have shown that G_2^2 is generated by x, y, t and $x^2 = t, y^3 = t^2 = (yt)^2 = 1$ are the defining relations.

The group G_2^2 acts on the projective line over the finite field, $PL(F_p)$, provided p is prime and $p-1$ is a perfect square in F_p. These primes are known as Pythagorean primes. In this short note, by p we shall mean a Pythagorean prime. The action of G_2^2 on $PL(F_p)$ results into the permutation group $G = \langle \overline{x}, \overline{y} : \overline{x}^4 = \overline{y}^3 = (\overline{xy})^k = 1\rangle$, which is homomorphic image of $\Delta(3,4,k)$. When $k = 1$, G is trivial group and when $k = 2$, the group G is isomorphic to the triangle group $\Delta(3,4,2)$, which is symmetric group S_4. If $k \geqslant 3$, G is homomorphic image of an infinite triangle group $\Delta(3,4,k)$. If $p \equiv 1 (\mathrm{mod}\, 8)$ then G is a simple subgroup of an alternating group A_{p+1}, and isomorphic to $PSL(2,p)$ because the order G is equal to $|PSL(2,p)| = \frac{p(p-1)(p+1)}{2}$. These results can be verified with the help of *GAP*. The following table gives orders of various groups corresponding to some values of the Pythagorean prime p.

$p \equiv 1(\text{mod}8)$	k	$\text{Order}(G) = \frac{p(p-1)(p+1)}{2}$
17	9	2448
41	21	34440
73	37	194472
89	15	352440
97	49	456288
113	56	721392
137	68	1285608
193	48	3594432

If p is not congruent to $1(\text{mod}\,8)$ then G is a subgroup of symmetric group S_{p+1} and the order G is $p(p-1)(p+1)$.

$p \not\equiv 1(\text{mod}\,8)$	k	$\text{Order}(G) = p(p-1)(p+1)$
5	6	120
13	14	2184
29	28	24360
37	36	50616
53	52	148824
61	62	226920
101	34	1030200
109	108	1294920
149	148	3307800
157	158	3869736

Theorem 1. *The action of G_2^2 on $PL(F_p)$, where p is the Pythagorean prime, gives a permutation group G. If $p \equiv 1(\text{mod}\,8)$ then G is a subgroup of A_{p+1}.*

Proof. Note that the group G is generated by permutations $\overline{x}$ and $\overline{y}$ where $\overline{x}$ is a product of cycles each of length 4 and $\overline{y}$ is a product of cycles each of length 3. Also since $\overline{y}$ is a product of cycles of length 3, each cycle can be decomposed into an even number of transpositions. Thus implying that $\overline{y}$ is an even permutation. In the decomposition of the permutation $\overline{x}$, each cycle can be reduced into odd number of transpositions. Let N represent number of cycles in the permutation $\overline{x}$. If N is even then $\overline{x}$ is even also. Since $\overline{x}$ has $\frac{p-1}{4}$ cycles, so $N = \frac{p-1}{4}$. Now if $p \equiv 1(\text{mod}\,8)$ then there exists an integer m such that $p = 8m+1$, and therefore $N = 2m$. Thus $\overline{x}$ is even, implies that G is generated by two even permutations $\overline{x}$ and $\overline{y}$. Hence G is always a subgroup of A_{p+1}. □

2. Higman's Coset diagrams

The idea of coset diagrams for modular group has been propounded and used by G. Higman and Q. Mushtaq in [2] and the transitivity has been discussed in [3].

An action of G_2^2 on $PL(F_p)$ can be represented by a coset diagram. The group G_2^2 is generated by the linear fractional transformations $x(z) = \frac{z+i}{iz+1}$, $y(z) = \frac{z-1}{z}$, and $t(z) = \frac{1}{z}$, which satisfy the relations $y^3 = t^2 = (yt)^2 = 1$, $x^2 = t$. A coset diagram for the group G_2^2 is defined as follows. Since the generator x has order 4, so the 4-cycles of x are represented by twisted squares, with the convention that x permutes their vertices counter-clockwise. The generator y has order 3, so the 3-cycles of y are denoted by doted edges permuting counter-clockwise. Fixed points of x and y, if they exist, are denoted by heavy dots. The generator t is an involution and therefore it is represented by symmetry along a vertical line of axis passing through the coset diagram.

For example, the action of G_2^2 on $PL(F_{17})$, is depicted by the following coset diagram.

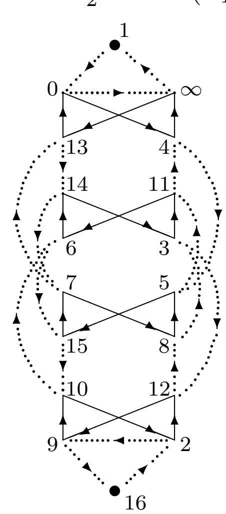

Figure 3.

According to Figure 3, in the coset diagram we begin walking along the path by starting from the vertex labelled as 1. The path $y^2x^{-1}yx^{-1}y^2x^{-1}y^2$ ends at $p-1 = 16$. Thus there exists a word $y^2x^{-1}yx^{-1}y^2x^{-1}y^2$ which connects 1 with the vertex $p-1$, that is $(1)(y^2x^{-1}yx^{-1}y^2x^{-1}y^2) = 16$. Similarly we can connect any two vertices of this coset diagram by a word. Hence the action of G_2^2 on $PL(F_{17})$ is transitive.

Theorem 2. *Let p be the Pythagorean prime. Then G_2^2 acts transitively on $PL(F_p)$.*

Proof. Since the action of G_2^2 on $PL(F_p)$ yields a permutation group G generated by $\overline{x}$ and $\overline{y}$ in whose coset diagram we can always start our walk from the vertex labelled by 1 and end at the vertex labelled by $p-1$ as shown in the Figure 4. In this coset diagram, 4-cycles of x are represented by the four sides of a twisted square, the 3-cycles of y are represented by a triangle with broken edges, whose vertices are permuted counter-clockwise. The fixed points of x and y are represented by heavy dots.

Next we wish to show that the action of G_2^2 on $PL(F_p)$ is transitive for all Pythagorean prime p. Let w be a word connecting 1 with $p-1$, that is, for:

p	$(1)w = p-1$
5	$y^2x^{-1}y^2$
13	$y^2x^{-1}y^2x^{-1}y^2$
17	$y^2x^{-1}yx^{-1}y^2x^{-1}y^2$
29	$y^2x^{-1}yx^{-1}yx^{-1}y^2$
37	$y^2x^{-1}y^2x^{-1}yx^{-1}y^2x^{-1}y^2$
41	$y^2x^{-1}yx^{-1}yx^{-1}y^2x^{-1}y^2x^{-1}y^2$
53	$y^2x^{-1}yx^{-1}yx^{-1}y^2x^{-1}yx^{-1}y^2$
61	$y^2x^{-1}yx^{-1}y^2x^{-1}yx^{-1}y^2$
73	$y^2x^{-1}yx^{-1}yx^{-1}y^2x^{-1}yx^{-1}y^2x^{-1}y^2$
89	$y^2x^{-1}yx^{-1}yx^{-1}y^2x^{-1}yx^{-1}y^2x^{-1}y^2$
97	$y^2x^{-1}yx^{-1}yx^{-1}yx^{-1}y^2x^{-1}y^2x^{-1}y^2x^{-1}yx^{-1}yx^{-1}yx^{-1}y^2$

For, we show that there exists a path between 1 and $p-1$. We begin from 1 and apply y^2 on it to reach ∞. Next we apply x^{-1} on ∞ to reach $k=\sqrt{p-1}$, which is the right top vertex of first twisted square. Similarly, we apply a suitable y^ϵ on $\sqrt{p-1}$, where $\epsilon = \pm 1$, to reach the right top vertex of another twisted square. We again apply x^{-1} and a suitable y^ϵ to reach the right top vertex of any other twisted square. We continue in this way so that after a finite number of steps eventually we reach the vertex $p-1$. That is $(1)y^2x^{-1}y^{\epsilon}x^{-1}y^{\epsilon}x^{-1}y^{\epsilon}\ldots x^{-1}y^{\epsilon} = p-1$.

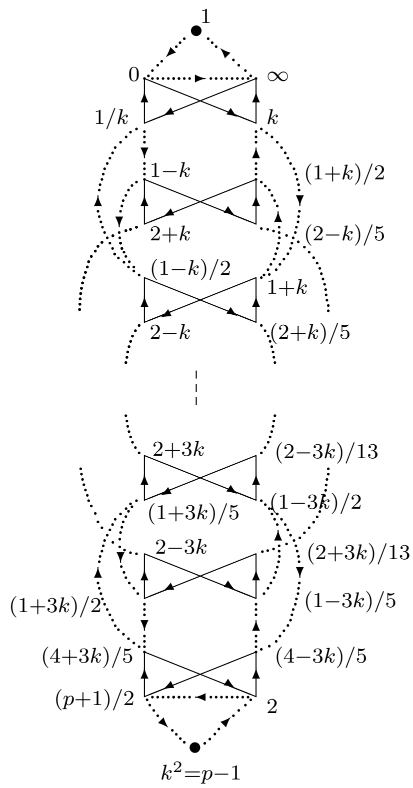

Figure 4.

This shows that the coset diagram is connected. Hence the action is transitive. □

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Simple hyper K-algebras

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Abstract

In this note we define the notion of simple hyper K-algebras and give some examples of simple hyper K-algebras. Then we investigate (weak) hyper K-ideals, normal hyper K-algebras and commutative hyper K-ideals.

1. Introduction

The study of BCK-algebra was initiated by K. Iséki [3] in 1966 as a generalization of concept of the set-theoretic difference and propositional calculus. Since the many researches worked in this area. Hyper structures (called also multialgebras) were introduced in 1934 by F. Marty [5] at the 8^{th} congress of Scandinavian Mathematicians. Around the 40 years several authors worked on hyper groups, specially in France and United States, but also in Italy, Russia, Japan and Iran.

Hyper structures have many applications to several sectors of both pure and applied sciences. Recently Y. B. Jun et al. [4] introduced and studied hyper BCK-algebras which are generalization of BCK-algebras. R. A. Borzooei and M. M. Zahedi [1, 10] constructed the hyper K-algebras , (weak) hyper K-ideals and defined simple hyper K-algebras of order 3. T. Roodbari and M. M. Zahedi [8] defined 9 different types of commutative hyper K-ideals. In this paper we define the notion of simple hyper K-algebras and give some examples of simple hyper K-algebras. Then we investigate (weak) hyper K-ideals, normal hyper K-algebras and commutative hyper K-ideals.

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Keywords: hyper K-algebra, simple hyper K-algebra, (weak) hyper K-ideal, normal hyper K-algebra, commutative hyper K-ideal.

2. Preliminaries

Definition 2.1. Let H be a nonempty set and " $\circ$ " be a *hyperoperation* on H, that is " $\circ$ " is a function from $H \times H$ to the family $\mathcal{P}^*(H)$ of all nonempty subsets of H. Then H is called a *hyper K-algebra* if it contains a constant "0" and satisfies the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) < x \circ y$,
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x < x$,
(HK4) $x < y,\ \ y < x \longrightarrow x = y$,
(HK5) $0 < x$,

where $x < y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A < B$ means that there are $a \in A$ and $b \in B$ such that $a < b$. By $A \circ B$ we denote the union of all $a \circ b$ such that $a \in A$, $b \in B$.

Theorem 2.2. *Let $(H, \circ, 0)$ be a hyper K-algebra. Then for all $x, y, z \in H$ and for all nonempty subsets A, B and C of H the following hold*:

(i)	$x \circ y < z \longleftrightarrow x \circ z < y$,	(ii)	$(x \circ z) \circ (x \circ y) < y \circ z$,
(iii)	$x \circ (x \circ y) < y$,	(iv)	$x \circ y < x$,
(v)	$A \subseteq B \longrightarrow A < B$,	(vi)	$x \in x \circ 0$,
(vii)	$(A \circ C) \circ (A \circ B) < B \circ C$,	$(viii)$	$(A \circ C) \circ (B \circ C) < A \circ B$,
(ix)	$A \circ B < C \Leftrightarrow A \circ C < B$,	(x)	$A \circ B < A$.

Definition 2.3. Let I be a nonempty subset of a hyper K-algebra $(H, \circ, 0)$ and $0 \in I$. Then I is called

(i) a *weak hyper K-ideal* of H if $x \circ y \subseteq I$ and $y \in I$ imply $x \in I$,
(ii) a *hyper K-ideal* of H if $x \circ y < I$ and $y \in I$ imply that $x \in I$.

Definition 2.4. A nonempty subset I of H such that $0 \in I$ is called a *commutative hyper K-ideal of*

- *type* 1, if $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $z \in I$ imply $(x \circ (y \circ (y \circ x))) \subseteq I$,
- *type* 2, if $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $z \in I$ imply $(x \circ (y \circ (y \circ x))) \bigcap I \neq \emptyset$,
- *type* 3, if $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $z \in I$ imply $(x \circ (y \circ (y \circ x))) < I$,
- *type* 4, if $((x \circ y) \circ z) \subseteq I$, $z \in I$ imply $(x \circ (y \circ (y \circ x))) \subseteq I$,
- *type* 5, if $((x \circ y) \circ z) \subseteq I$ and $z \in I$ imply $(x \circ (y \circ (y \circ x))) \bigcap I \neq \emptyset$,
- *type* 6, if $((x \circ y) \circ z) \subseteq I$ and $z \in I$ imply $(x \circ (y \circ (y \circ x))) < I$,
- *type* 7, if $((x \circ y) \circ z) < I$, $z \in I$ imply $(x \circ (y \circ (y \circ x))) \subseteq I$,
- *type* 8, if $((x \circ y) \circ z) < I$ and $z \in I$ imply $(x \circ (y \circ (y \circ x))) \bigcap I \neq \emptyset$,
- *type* 9, if $((x \circ y) \circ z) < I$ and $z \in I$ imply $(x \circ (y \circ (y \circ x))) < I$.

Definition 2.5. An element a of a hyper K-algebra $(H, \circ, 0)$ is called a *hyper atom* if $x < a$ implies $x = 0$ or $x = a$. By $A(H)$ we denote the set of all hyper atoms of H. If in H there exists an element e such that $x < e$ for all $x \in H$, then H is called a *bounded hyper K-algebra*.

Definition 2.6. A hyper K-algebra $(H, \circ, 0)$ in which for all $x, y \in H$, $x < y$ implies $x \in y \circ (y \circ x)$ is called *quasi-commutative*. A hyper K-algebra satisfying the identity $x \circ (x \circ y) = y \circ (y \circ x)$ for all $x, y \in H$ is called *commutative*.

Theorem 2.7. *If $(H, \circ, 0)$ is a quasi-commutative hyper K-algebra, then the hyper K-ideal $\{0\}$ is a commutative hyper K-ideal of type* 9 and 6.

Definition 2.8. Let $(H, \circ, 0)$ be a hyper K-algebra and S be a nonempty subset of H. Then the sets

$${}_{l1}S = \{x \in H \,|\, a < (a \circ x), \forall a \in S\}, \quad {}_{l2}S = \{x \in H \,|\, a \in (a \circ x), \forall a \in S\},$$
$$S_{r1} = \{x \in H \,|\, x < (x \circ a), \forall a \in S\}, \quad S_{r2} = \{x \in H \,|\, x \in (x \circ a), \forall a \in S\}$$

are called *left hyper stabilizers of type* 1 (type 2, respectively) and *right hyper stabilizer of type* 1 (type 2, respectively).

In the case $S = \{s\}$, for simplicity, we will write ${}_{li}s$ and s_{ri} instead of ${}_{li}\{s\}$ and ${}_{ri}\{s\}$.

Definition 2.9. A hyper K-algebra $(H, \circ, 0)$ is called a *left* (*right*) *hyper normal of type i* if ${}_{li}a$ (respectively a_{ri}) is a hyper K-ideal of H for any $a \in H$ and $i = 1, 2$. If H is both left and right hyper normal K-algebra of type i, then H is called a *hyper normal K-algebra of type i*.

3. Simple hyper K-algebra

Definition 3.1. A hyper K-algebra $(H, \circ, 0)$ is called *simple* if for all distinct elements $a, b \in H - \{0\}$ we have $a \not< b$ and $b \not< a$.

Theorem 3.2. *Let H be a nonempty set and $0 \in H$. Define a hyper operation "$\circ$" on H by putting*

$$x \circ y = \begin{cases} \{x\} & \text{if} \quad x \neq y, \ y = 0, \\ \{x, y\} & \text{if} \quad x \neq y, \ y \neq 0, \\ \{0, x\} & \text{if} \quad x = y, \end{cases}$$

for all $x, y \in H$. Then $(H, \circ, 0)$ is a simple hyper K-algebra.

Proof. Since axioms (HK3), (HK4) and (HK5) are obvious, we verify only (HK1) and (HK2). For this we consider the following cases:

Case (i). $x \neq y$, $x \neq z$ and $y = z = 0$. Then
$(x \circ z) \circ (y \circ z) = \{x\} < \{x\} = x \circ y$ and $(x \circ y) \circ z = (x \circ z) \circ y = \{x\}$.

Case (ii). $x \neq y$, $x \neq z$, $z \neq 0$ and $y = 0$. Then
$(x \circ z) \circ (y \circ z) = \{x, z\} < \{x\} = x \circ y$ and $(x \circ y) \circ z = (x \circ z) \circ y = \{x, z\}$.

Case (iii). $x \neq y$, $x \neq z$, $y \neq z$, $y \neq 0$ and $z \neq 0$. Then
$(x \circ z) \circ (y \circ z) = \{0, x, y, z\} < \{x, y\} = x \circ y$ and $(x \circ y) \circ z = \{x, y, z\} = (x \circ z) \circ y$.

Case (iv). $x \neq y$, $y \neq z$, $x = z$, $y = 0$ and $z \neq 0$. Then
$(x \circ z) \circ (y \circ z) = \{0, x\} < \{x\} = x \circ y$ and $(x \circ y) \circ z = (x \circ z) \circ y = \{0, x\}$.

Case (v). $x \neq y$, $x \neq z$, $y \neq z$, $z = 0$ and $y \neq 0$. Then
$(x \circ z) \circ (y \circ z) = \{x, y\} < \{x, y\} = x \circ y$ and $(x \circ y) \circ z = (x \circ z) \circ y = \{x, y\}$.

Case (vi). $x \neq y$, $x \neq z$, $y = z$, $y \neq 0$ and $z \neq 0$. Then
$(x \circ z) \circ (y \circ z) = \{0, x, y\} < \{x, y\} = x \circ y$ and $(x \circ y) \circ z = (x \circ z) \circ y = \{0, x, y\}$.

Case (vii). $x \neq y$, $y \neq z$, $y \neq 0$ and $x = z = 0$. Then
$(x \circ z) \circ (y \circ z) = \{0, y\} < \{0, y\} = x \circ y$ and $(x \circ y) \circ z = (x \circ z) \circ y = \{0, y\}$.

Case $(viii)$. $x \neq y$, $y \neq z$, $x = z$, $y \neq 0$ and $z \neq 0$. Then
$(x \circ z) \circ (y \circ z) = \{0, x, y\} < \{x, y\} = x \circ y$ and $(x \circ y) \circ z = (x \circ z) \circ y = \{0, x, y\}$.

Case (ix). $x \neq z$, $y \neq z$, $x = y$ and $z = 0$. Then
$(x \circ z) \circ (y \circ z) = \{x\} < \{0, x\} = x \circ y$ and $(x \circ y) \circ z = (x \circ z) \circ y = \{0, x\}$.

Case (x). $x \neq z$, $y \neq z$, $x = y$ and $z \neq 0$. Then
$(x \circ z) \circ (y \circ z) = \{0, x, z\} < \{0, x\} = x \circ y$ and $(x \circ y) \circ z = (x \circ z) \circ y = \{0, x, z\}$.

Case (xi). $x = z = y$. Then
$(x \circ z) \circ (y \circ z) = \{0, x\} < \{0, x\} = x \circ y$ and $(x \circ y) \circ z = (x \circ z) \circ y = \{0, x\}$.
Therefore $(H, \circ, 0)$ is a simple hyper K-algebra. □

Corollary 3.3. *A hyper K-algebra defined in Theorem 3.2 is commutative and normal of types 1 and 2.*

Proof. The commutativity is obvious. Also $a_{ri} =_{li} a = H$ for all $a \in H$ and $i = 1, 2$. □

Example 3.4. Consider the following two hyper K-algebras defined on $H = \{0, 1, 2, 3\}$:

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0,2,3\}$
1	$\{1\}$	$\{0,1\}$	$\{1,2,3\}$	$\{1,2,3\}$
2	$\{2\}$	$\{2\}$	$\{0,2,3\}$	$\{2\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\{0,3\}$

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0\}$	$\{2\}$
3	$\{3\}$	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$

The first hyper K-algebra is simple, the second is not simple, because $3 < 2$.

It is not difficult to see that the following theorem is true.

Theorem 3.5. *A hyper K-algebra is simple if and only if it contains only hyper atoms.* $\square$

Theorem 3.6. *For a simple hyper K-algebra the following statements hold.*

(i) $a \circ 0 = \{a\}$ *for all* $a \in H - \{0\}$,

(ii) $a \in a \circ b$ *for all distinct elements* $a, b \in H$,

(iii) $H - \{a\} \subseteq H \circ a$ *for all* $a \in H$,

(iv) $a \in b \circ c \longleftrightarrow c \in b \circ a$ *for distinct elements* $a, c \in H$ *and* $b \in H - \{0\}$,

(v) $x < x \circ a \longleftrightarrow x \in x \circ a$ *for all* $a, x \in H$,

(vi) $A < A \circ b \longleftrightarrow A \cap (A \circ b) \neq \emptyset$ *for all* $b \in H$ *and* $\emptyset \neq A \subseteq H$,

(vii) $(x \circ y) \circ z < x \circ (y \circ z)$ *for all* $x, y, z \in H$,

$(viii)$ *If* $0 \in I \subseteq H$, *then* $A \circ B < I \longleftrightarrow (A \circ B) \cap I \neq \emptyset$ *for all nonempty subsets* A *and* B *of* H.

Proof. (i) We have $a \in a \circ 0$. Now let $b \in a \circ 0$. Then $0 \in (a \circ 0) \circ b = (a \circ b) \circ 0$. Thus there is $t \in a \circ b$ such that $0 \in t \circ 0$ i.e., $t < 0$. Hence $t = 0$ and so $a < b$. Since H is simple and $a \in H - \{0\}$, then $a = b$. Therefore $a \circ 0 = \{a\}$.

(ii) If $a = 0$, then it is clear that $0 \in 0 \circ b$, for all $b \in H$. Now let $a, b \in H, a \neq 0$ and $a \neq b$. Since by Theorem 2.2(iv) $a \circ b < a$, then there is $t \in a \circ b$ such that $t < a$. Thus $t = 0$ or $t = a$. Hence $a \neq b$ and $a \neq 0$ imply that $t \neq 0$. Therefore $t = a$ and so $a \in a \circ b$.

(iii) Let $x \in H - \{a\}$. Then $x \neq a$ and so by (ii) we have $x \in x \circ a$. Therefore $x \in H \circ a$.

(iv) Let $a \in b \circ c$. Then $0 \in (b \circ c) \circ a = (b \circ a) \circ c$. Thus there exists $t \in b \circ a$ such that $0 \in t \circ c$ and so $t < c$. Hence $t = 0$ or $t = c$. Since $b \neq a$ and $b \neq 0$, then $t \neq 0$. So $t = c$. Therefore $c \in b \circ a$. The proof of the converse statement is similar.

(v) Let $x < x \circ a$. Then there exists $t \in x \circ a$ such that $x < t$. Thus $x = 0$ or $x = t$. If $x = 0$, then by (HK5), $0 \in 0 \circ a$. If $x = t$, then $x \in x \circ a$. Conversely, let $x \in x \circ a$. Then by Theorem 2.2(v), $x < x \circ a$.

(vi) Let $A \neq \emptyset$ and $A < A \circ b$. Then there exists $a \in A$ and $t \in A \circ b$ such that $a < t$. Thus $a = 0$ or $a = t$. If $a = 0$, then $0 \in A \cap A \circ b$. If $a = t$, then $a \in A \cap A \circ b$. Therefore $A \cap A \circ b \neq \emptyset$. The proof of the converse statement is obvious.

(vii) If $x = y$ or $x = z$, then $0 \in (x \circ y) \circ z$. So $(x \circ y) \circ z < x \circ (y \circ z)$. Now let $x \neq y$ and $x \neq z$. Then by (ii), $x \in (x \circ y) \cap (x \circ z)$. Thus $x \in x \circ z \subseteq (x \circ y) \circ z$. If $y = z$, then $0 \in y \circ z$ and so $x \in x \circ (y \circ z)$. Hence $(x \circ y) \circ z < x \circ (y \circ z)$. If $y \neq z$, then by (ii), $y \in y \circ z$, so $x \in x \circ y \subseteq x \circ (y \circ z)$. Therefore $(x \circ y) \circ z < x \circ (y \circ z)$.

$(viii)$ Let $0 \in I$ and $A \circ B < I$. Then there exists $t \in A \circ B$ and $i \in I$ such that $t < i$. So $t = 0$ or $t = i$. If $t = 0$, then $0 \in (A \circ B) \cap I$. If $t = i$, then $i \in (A \circ B) \cap I$. Therefore $(A \circ B) \cap I \neq \emptyset$. The converse statement is clear. □

Corollary 3.7. *A simple hyper K-algebra is normal of type 1 if and only if it is normal of type 2.* □

Theorem 3.8. *In simple hyper K algebras every subset containing 0 is a weak hyper K-ideal.*

Proof. Let $0 \in A \subseteq H$, $x \circ y \subseteq A$ and $y \in A$. If $x = y$, then $x \in A$. If $x \neq y$, then by Theorem 3.6(ii), $x \in x \circ y \subseteq A$ and so $x \in A$. □

Corollary 3.9. *Every hyper K-subalgebra of a simple hyper K-algebra is a weak hyper K-ideal.* □

Since by Theorem 3.6(v), we have ${}_{l1}A ={}_{l2} A$ and $A_{r1} = A_{r2}$, for all nonempty subset $A \subseteq H$, in the sequel we will write ${}_lA$ instead of ${}_{l1}A$ and A_r instead of A_{r1}.

Corollary 3.10. *In simple hyper K-algebras A_r and ${}_lA$ are weak hyper K-ideals for any nonempty subset A of H.* □

Definition 3.11. A hyper K-algebra H is called *left (right) weak normal of type i* if ${}_{il}a$ (respectively a_{ri}) is a weak hyper K-ideal of H for any $a \in H$.

Theorem 3.12. *Every simple hyper K-algebra is a left (right) weak normal K-algebra of type $i = 1, 2$.* □

Theorem 3.13. *Let H be a simple hyper K-algebra and let $a \neq 0$. Then $H - \{a\}$ is a hyper K-ideal of H if and only if $|a \circ x| = 1$ for all $x \in H - \{a\}$.*

Proof. Let $H-\{a\}$ be a hyper K-ideal and on the contrary, let there exists $x \in H-\{a\}$ such that $|a\circ x|>1$. Since $a \in a\circ x$, then there is $z \in H-\{a\}$ such that $z \in a \circ x$. Thus $a \circ x < H-\{a\}$. Since $H-\{a\}$ is a hyper K-ideal, then $a \in H-\{a\}$, which is a contradiction. Therefore $|a\circ x| = 1$, for all $x \in H-\{a\}$.

Conversely, let $|a\circ x| = 1$, for all $x \in H-\{a\}$. Since by Theorem 3.6(ii), $a \in a\circ x$, for all $x \in H-\{a\}$, then $a \circ x = \{a\}$. Thus $a\circ x \not< H-\{a\}$, for all $x \in H-\{a\}$. Therefore $H-\{a\}$ is a hyper K-ideal. □

Theorem 3.14. *Let $\emptyset \neq A \subseteq H$ and $T = \{a \in A \,|\, a \notin a\circ a\}$.*

(1) *If $T=\emptyset$, then A_r and ${}_lA$ are hyper K-ideals of H.*

(2) *If $T \neq \emptyset$ and $|a\circ x| = 1$ for all $a \in T$ and $x \in H-\{a\}$, then A_r and ${}_lA$ are hyper K-ideals of H.*

Proof. (1) By Theorem 3.6(ii) $A_r = \{x \in H \,|\, x \in x\circ a \ \forall a \in A\} = H$. Thus A_r is a hyper K-ideal.

(2) $A_r = H-T = \bigcap_{a\in T}(H-\{a\})$. So, by Theorem 3.13, A_r is a hyper K-ideal. □

The following example shows that the converse of Theorem 3.14(2) is not true in general. The condition "$|a\circ x| = 1$ for all $x \in H-\{a\}$" in Theorem 3.14(ii) is necessary.

Example 3.15. Consider the hyper K-algebra

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0,2\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1,2\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0\}$	$\{2\}$
3	$\{3\}$	$\{3\}$	$\{2,3\}$	$\{0\}$

and $A=\{1,2\}$. Then $T=\{1,2\}$ and $A_r=\{0,3\}$ is a hyper K-ideal, but $2 \in H-\{1\}$, $|1\circ 2| = 2$. For $A=\{1\}$, we see that $T=\{1\}$ and $A_r = \{0,2,3\}>$ But A_r is not a hyper K-ideal, because $|1\circ 2| = 2 \neq 1$.

As a consequence of Theorems 3.13 and 3.14. we obtain

Corollary 3.16. *Let $a \neq 0$ be an element of a simple hyper K-algebra H.*

(a) *If $a \in a\circ a$, then a_r and ${}_la$ are a hyper K-ideals of H.*

(b) *If $a \notin a\circ a$, then a_r and ${}_la$ are hyper K-ideals of H if and only if $|a\circ x| = 1$ for all $x \in H-\{a\}$.*

As a consequence of the above results we obtain

Corollary 3.17. *A simple hyper K-algebra H such that $a \in a \circ a$ for every $a \in H$ is right (left) normal of type $i = 1, 2$.* □

Corollary 3.18. *In a simple hyper K-algebra all sets of the form $\{0, a\}$ are hyper K-ideals.* □

Corollary 3.19. *A bounded simple hyper K-algebra has at most two elements.*

4. Commutative hyper K-ideals

Directly from the definition of commutative hyper K-ideals and Theorem 3.6 it follows that in simple hyper K-algebras commutative hyper K-ideals of types 1 and 7 coincides. Similarly, commutative hyper K-ideals of types 2, 3, 8 and 9. Also 5 and 6.

Theorem 4.1. *A simple hyper K-algebra is quasi-commutative.*

Proof. Let $x < y$. Then $x = 0$ or $x = y$. If $x = 0$, then $0 \in y \circ y \subseteq y \circ (y \circ 0)$. If $x = y$, then $y \in y \circ 0 \subseteq y \circ (y \circ y)$. Therefore $x \in y \circ (y \circ x)$. □

Corollary 4.2. *In any simple hyper K-algebra, $I = \{0\}$ is a commutative hyper K-ideal of type $i = 2, 3, 5, 6, 8, 9$.*

Proof. The proof follows from Theorems 4.1 and 2.7. □

Theorem 4.3. *If $a \circ a = \{0\}$ holds for all elements of a simple hyper K-algebra, then $I = \{0\}$ is its commutative hyper K-ideal of type 4.*

Proof. Let $(x \circ y) \circ z \subseteq I$ and $z \in I$. Then $x \circ y \subseteq (x \circ y) \circ 0 \subseteq I$ and so $x < y$. Thus $x = 0$ or $x = y$. If $x = 0$, then $x \circ (y \circ (y \circ x)) = 0 \circ (y \circ (y \circ 0)) = 0 \circ (y \circ y) = 0 \circ 0 = I$. If $x = y$, then $y \circ (y \circ (y \circ y)) = y \circ y = I$. Therefore I is a commutative hyper K-ideal of type 4. □

Remark 4.4. The hyper K-algebra defined by the table

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,2\}$
1	$\{1\}$	$\{0,2\}$	$\{1,2\}$
2	$\{2\}$	$\{2\}$	$\{0\}$

proves that the condition "$a \circ a = \{0\}$ for all $a \in H$" in the above theorem is necessary. Indeed, $1 \circ 1 \neq \{0\}$ and $I = \{0\}$ is not a commutative hyper K-ideal of type 4, because $(0 \circ 1) \circ 0 = I$, while $0 \circ (1 \circ (1 \circ 0)) = \{0, 2\} \nsubseteq I$.

Theorem 4.5. *In a simple hyper K-algebra $I = \{0\}$ is a commutative hyper K-ideal of type 7 (and 1) if and only if $a \circ a = \{0\}$ for all $a \in H$.*

Proof. Let $I = \{0\}$ be a commutative hyper K-ideal of type 7. Then $(y \circ y) \circ 0 < I$ and $0 \in I$ imply that $y \circ y \subseteq y \circ (y \circ 0) \subseteq y \circ (y \circ (y \circ y)) \subseteq I$. Thus $y \circ y = \{0\}$, for all $y \in H$. The proof of the converse statement is similar to the proof of Theorem 4.4. □

Theorem 4.6. *In a simple hyper K-algebra H the set $I = H - \{a\}$ is a commutative hyper K-ideal of type 6 (and 5) for any $a \neq 0$.*

Proof. Let $(x \circ y) \circ z \subseteq I$ and $z \in I$. If $x = y$, then $0 \in x \circ (y \circ (y \circ x))$ and so $x \circ (y \circ (y \circ x)) < I$. If $x \neq y$, then $x \in x \circ 0 \subseteq x \circ (y \circ y) \subseteq x \circ (y \circ (y \circ x))$. Now we show that $x \neq a$. On the contrary let $x = a$. Then $x \neq z$ and so by Theorem 3.6(ii), $x \in x \circ z \subseteq (x \circ y) \circ z \subseteq I$, which is a contradiction. Hence $x \neq a$ implies that $x \circ (y \circ (y \circ x)) < I$. □

Theorem 4.7. *Let a be a non-zero element of a simple hyper K-algebra H such that $|a \circ x| = 1$ for all $x \in H - \{a\}$. Then $I = H - \{a\}$ is a commutative hyper K-ideal of type 9 (and 2, 3, 8).*

Proof. Let $(x \circ y) \circ z < I$ and $z \in I$. If $x = y$, then $x \circ (y \circ (y \circ x)) < I$. For $x \neq y$ we consider two cases: (i) $x \neq a$, (ii) $x = a$. In the first case we have $x \in x \circ (y \circ (y \circ x))$ and so $x \circ (y \circ (y \circ x)) < I$. in the second, from $|a \circ y| = |a \circ z| = 1$ it follows $\{a\} = a \circ z = (a \circ y) \circ z < I$. Thus there exists $t \in I$ such that $a < t$. So $a = 0$ or $a = t$, which is impossible. Therefore I is a commutative hyper K-ideal of type 9. □

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Prime bi-ideals in ternary semigroups

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Abstract

We introduced the notions of prime, semiprime and strongly prime bi-ideals in ternary semigroups. The space of strongly prime bi-ideals is topologized. We characterize different classes of ternary semigroups by the properties of their quasi-ideals and bi-ideals. We also characterize those ternary semigroups for which each bi-ideal is strongly prime.

1. Introduction

Ternary algebraic operations and cubic relations were considered in the 19th century by several mathematicians such as Cayley and Sylvester. Ternary structures and their generalization, the so called n-ary structures, raise certain hopes in view of their possible applications in Physics. Some significant physical applications are given in [1, 2, 11, 10]. Ternary semigroups provide natural examples of ternary algebras.

In [8], Good and Hughes introduced the notion of bi-ideals and in [15], Steinfeld introduced the notion of quasi-ideals in semigroups. In [13] the concepts of prime bi-ideals, strongly prime bi-ideals and semiprime bi-ideals in semigroups is introduced. In [14], Sioson studied some properties of quasi-ideals of ternary semigroups. In [4], Dixit and Dewan studied about the quasi-ideals and bi-ideals of ternary semigroups. Connections of some types of ideals in ternary and n-ary semigroups with the regularity of these semigroups are described in [6]. Applications of ideals to the divisibility theory in ternary and n-ary semigroups and rings one can find in [5].

In this paper we characterized some classes of ternary semigroups by the properties of their quasi-ideals and bi-ideals.

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2. Preliminaries

A ternary semigroup is an algebraic structure $(S, [\ \ ])$ such that S is a non-empty set and $[\ \ ] : S \times S \times S \longrightarrow S$ is a ternary operation satisfying the following associative law:

$$[[abc]de] = [a[bcd]e] = [ab[cde]].$$

For simplicity we will write $[abc]$ as abc.

It is clear that any ordinary semigroup $(S, *)$ induces a ternary semigroup $(S, [\,])$ by putting $[abc] = a * b * c$. But there are ternary semigroups which are not of this form. Connections between ternary semigroups and some ordinary semigroups are described in [3]. Criterion when ternary semigroup has the above form is proved in [7].

An element e in a ternary semigroup S is called *idempotent* if $eee = e$.

If a ternary semigroup S contains an element 0 such that $0ab = a0b = ab0 = 0$ for all $a, b \in S$, then 0 is called a *zero element* of S. If S has no zero then it is easy to adjoin an extra element 0 to form a ternary semigroup with zero. In this case we define $0ab = a0b = ab0 = 0$ for all $a, b \in S$ and $000 = 0$. In this case $S \cup \{0\}$ becomes a ternary semigroup with zero. A non-empty subset T of a ternary semigroup S is called a *ternary subsemigroup* of S if and only if $TTT = T^3 \subseteq T$. A subset T satisfying the identity $TTT = T$ is called an *idempotent subset*. By a *left* (*right, middle*) *ideal* of a ternary semigroup S we mean a non-empty subset A of S such that $SSA \subseteq A$ ($ASS \subseteq A$, $SAS \subseteq A$). By a *two sided ideal*, we mean a subset of S which is both a left and a right ideal of S. If a non-empty subset of S is a left, right and middle ideal of S, then it is called an *ideal* of S. It is clear that every one-sided ideal, middle ideal and two-sided ideal is a ternary subsemigroup. Let X be a non-empty subset of a ternary semigroup S. Then intersection of all left ideals of S containing X is a left ideal of S containing X, furthermore it is the smallest left ideal of S containing X and is called the *left ideal of S generated by X*. It is denoted by $\langle X \rangle_l$. Clearly,

$$\langle X \rangle_l = X \cup SSX,$$

Similarly,

$$\begin{aligned} \langle X \rangle_r &= X \cup XSS, \\ \langle X \rangle_m &= X \cup SXS \cup SSXSS, \\ \langle X \rangle_t &= X \cup SSX \cup XSS \cup SSXSS, \\ \langle X \rangle &= X \cup XSS \cup SSX \cup SXS \cup SSXSS, \end{aligned}$$

are the right, middle, two sided ideals , and an ideal of S generated by X, respectively.

An element a in a ternary semigroup S is called *regular* if there exists an element $x \in S$ such that $axa = a$, that is $a \in aSa$. A ternary semigroup S is called *regular* if all its elements are regular.

Definition 1. (cf. [14]) A non-empty subset Q of a ternary semigroup S is called a it quasi-ideal of S if

(i) $(QSS) \cap (SQS) \cap (SSQ) \subseteq Q$,

(ii) $(QSS) \cap (SSQSS) \cap (SSQ) \subseteq Q$.

Every right, left and middle ideal in a ternary semigroup is a quasi-ideal but the converse is not true in general. Every quasi-ideal of a ternary semigroup S is a ternary subsemigroup of S.

Definition 2. (cf. [4]) By a *bi-ideal* of a ternary semigroup S we mean a ternary subsemigroup B of S such that $BSBSB \subseteq B$.

Proposition 1. (cf. [4]) *The intersection of a family of quasi-ideals (bi-ideals) in a ternary semigroup is either empty or a quasi-ideal (bi-ideal).* □

Corollary 1. (cf. [4]) *The intersection of a right ideal R and a left ideal L of a ternary semigroup S is a quasi-ideal of S.* □

Proposition 2. (cf. [4]) *Every quasi-ideal of a ternary semigroup is a bi-ideal.* □

Proposition 3. (cf. [14]) *A ternary semigroup S is regular if and only if $R \cap M \cap L = RML$ for every right ideal R, middle ideal M and left ideal L of S.* □

2. Regular ternary semigroups

Theorem 1. *A commutative ternary semigroup is regular if and only if every its ideal is idempotent.*

Proof. Straightforward. □

Theorem 2. *If every quasi-ideal Q of S is idempotent, then S is a regular ternary semigroup.*

Proof. Let R be a right ideal, M a middle ideal and L a left ideal of S, then $(R \cap M \cap L)$ is a quasi-ideal of S. Since each quasi-ideal is idempotent so,

$$(R \cap M \cap L) = (R \cap M \cap L)^3 \subseteq RML.$$

On the other hand, $RML \subseteq R \cap M \cap L$ always. Thus $RML = R \cap M \cap L$. Hence by Proposition 3, S is a regular ternary semigroup. □

Theorem 3. *For a ternary semigroup S, the following assertions are equivalent:*

(i) *S is regular,*

(ii) $R \cap L = RSL$ *for every right ideal R and every left ideal L of S,*

(iii) $\langle a\rangle_r \cap \langle b\rangle_l = \langle a\rangle_r S\langle b\rangle_l$ *for every $a, b \in S$,*

(iv) $\langle a\rangle_r \cap \langle a\rangle_l = \langle a\rangle_r S\langle a\rangle_l$ *for every $a \in S$.*

Proof. $(i) \to (ii)$ Assume that S is a regular ternary semigroup. Let R and L be right and left ideals of S, respectively. Since $RSL \subseteq RSS \subseteq R$ and $RSL \subseteq SSL \subseteq L$, therefore $RSL \subseteq R \cap L$. Let $a \in R \cap L$, then there exists $x \in S$ such that $a = axa$. As $axa \in RSL$, thus $R \cap L \subseteq RSL$. Hence $R \cap L = RSL$.

$(ii) \to (iii)$ and $(iii) \to (iv)$ are trivial.

$(iv) \to (i)$ Consider $a \in S$, then

$$\begin{aligned} a \in \langle a\rangle_r \cap \langle a\rangle_l = \langle a\rangle_r S\langle a\rangle_l &= (a \cup aSS)S(a \cup SSa) \\ &= aSa \cup aSSSa \cup aSSSa \cup aSSSSSa \subseteq aSa, \end{aligned}$$

which implies $a \in aSa$. So $a = axa$ for some $x \in S$. Hence S is regular. □

Theorem 4. *The following assertions on a ternary semigroup S are equivalent:*

(i) *S is regular,*

(ii) $B = BSB$ *for every bi-ideal of S,*

(iii) $Q = QSQ$ *for every quasi-ideal Q of S.*

Proof. $(i) \to (ii)$ Suppose S is a regular ternary semigroup and let b be any element of B. Then there exists $x \in S$ such that $b = bxb$. As $b = bxb \in BSB$, so $B \subseteq BSB$. Now let $y \in BSB$, then $y = b_1sb_2$ for some $b_1, b_2 \in B$ and $s \in S$. Since S is regular so b_1 can be written as $b_1 = b_1tb_1$ for some $t \in S$, thus $y = b_1sb_2 = b_1tb_1sb_2 \in BSBSB \subseteq B$, which implies $BSB \subseteq B$. Hence $BSB = B$.

$(ii) \to (iii)$ Since every quasi-ideal of S is a bi-ideal, so by (ii), $Q = QSQ$ for every quasi-ideal Q of S.

$(iii) \to (i)$ Suppose $Q = QSQ$ for every quasi-ideal Q of S. Let R be a right ideal, M be a middle ideal and L be a left ideal of S, then $Q = R \cap M \cap L$ is a quasi-ideal. Now $R \cap M \cap L = Q = QSQ = QSQSQ = (R \cap M \cap L)S(R \cap M \cap L)S(R \cap M \cap L) \subseteq RSMSL \subseteq RML$.

Also $RML \subseteq R \cap M \cap L$ always. Therefore $RML = R \cap M \cap L$. Hence by Proposition 3, S is regular. □

Proposition 4. *If B is a bi-ideal of a regular ternary semigroup S and T_1, T_2 are non-empty subsets of S, then BT_1T_2, T_1BT_2 and T_1T_2B are bi-ideals of S.*

Proof. Let S be a regular ternary semigroup, B a bi-ideal of S and T_1, T_2 are non-empty subsets of S. Then,

$$\begin{aligned}(BT_1T_2)(BT_1T_2)(BT_1T_2) &\subseteq B(T_1T_2B)(T_1T_2B)T_1T_2\\ &\subseteq B(SSB)(SSB)T_1T_2 = B(SSBSS)(BT_1T_2)\\ &\subseteq B(SSSSS)BT_1T_2 \subseteq B(SSS)BT_1T_2\\ &\subseteq (BSB)T_1T_2 = BT_1T_2\end{aligned}$$

because in a regular ternary semigroup $B = BSB$. Thus BT_1T_2 is a ternary subsemigroup of S. Also

$$\begin{aligned}(BT_1T_2)S(BT_1T_2)S(BT_1T_2) &= B(T_1T_2S)B(T_1T_2S)BT_1T_2\\ &\subseteq B(SSS)B(SSS)BT_1T_2\\ &\subseteq (BSBSB)T_1T_2 \subseteq BT_1T_2.\end{aligned}$$

Hence BT_1T_2 is a bi-ideal of S.

Similarly, we can show that T_1BT_2, T_1T_2B are bi-ideals of S. □

Corollary 2. *If B_1, B_2 and B_3 are bi-ideals of a regular ternary semigroup S then $B_1B_2B_3$ is a bi-ideal of S.*

Corollary 3. *If Q_1, Q_2, Q_3 are quasi-ideals of a regular ternary semigroup S then $Q_1Q_2Q_3$ is a bi-ideal.*

Theorem 5. *A ternary semigroup in which all bi-ideals are idempotent is regular.*

Proof. Let R be a right ideal, M be a middle ideal and L be a left ideal of S. Then $R \cap M \cap L$ is a bi-ideal. Therefore by the hypothesis

$$R \cap M \cap L = (R \cap M \cap L)^3 = (R \cap M \cap L)(R \cap M \cap L)(R \cap M \cap L) \subseteq RML.$$

Also, $RML \subseteq R \cap M \cap L$ always. Hence $R \cap M \cap L = RML$. Thus by Proposition 3, S is a regular ternary semigroup. □

Theorem 6. *The following assertions are equivalent for a ternary semigroup S:*

(i) *S is regular,*

(ii) *$I \cap B = BIB$ every middle ideal I and for every bi-ideal B,*

(iii) *$I \cap Q = QIQ$ for every middle ideal I and every quasi-ideal Q.*

Proof. $(i) \to (ii)$ Suppose S is a regular ternary semigroup, I a middle ideal and B a bi-ideal of S. Since $BIB \subseteq SIS \subseteq I$ and by Theorem 4, $BIB \subseteq BSB = B$. Therefore $BIB \subseteq I \cap B$. Now let $a \in I \cap B$. Since S is regular, so there exists $x \in S$ such that $a = axa$. Thus we have $a = axa = (axa)xa = a(xax)a \in BIB$ which shows that $I \cap B \subseteq BIB$. Hence $BIB = I \cap B$.

$(ii) \to (iii)$ Since every quasi-ideal of a ternary semigroup S is also a bi-ideal, so by (ii), we have $I \cap Q = QIQ$.

$(iii) \to (i)$ Let Q be a quasi-ideal of S. Then by (iii), we can write $Q = S \cap Q = QSQ$. Hence by Theorem 5, S is regular. □

Theorem 7. *For a ternary semigroup S, the following conditions are equivalent:*

(i) *S is regular,*

(ii) *$B \cap L \subseteq BSL$ for every bi-ideal B and every left ideal L,*

(iii) *$Q \cap L \subseteq QSL$ for every quasi-ideal Q and every left ideal L,*

(iv) *$B \cap R \subseteq RSB$ for every bi-ideal B and every right ideal R,*

(v) *$Q \cap R \subseteq RSQ$ for every quasi-ideal Q and every right ideal R.*

Proof. $(i) \to (ii)$ Let $a \in B \cap L$. Since S is regular, so there exists $x \in S$ such that $a = axa$. As $a = axa \in BSL$, therefore $B \cap L \subseteq BSL$.

$(ii) \to (iii)$ Since every quasi-ideal of S is a bi-ideal, so by (ii), we have $Q \cap L \subseteq QSL$.

$(iii) \to (i)$ Assume that $Q \cap L \subseteq QSL$, for every quasi-ideal Q and every left ideal L of S. We show that S is regular. Let R be any right ideal of S. Take $Q = R$ the by (iii) $R \cap L \subseteq RSL$, but $RSL \subseteq R \cap L$ always. Hence $RSL = R \cap L$. Thus by Theorem 3, S is regular.

Similarly we can show that $(i) \to (iv) \to (v) \to (i)$. □

Theorem 8. *For a ternary semigroup S, the following conditions are equivalent:*

(i) *S is regular,*

(ii) *$B_1 \cap B_2 \subseteq (B_1SB_2) \cap (B_2SB_1)$ for any bi-ideals B_1 and B_2,*

(iii) *$B \cap Q \subseteq (BSQ) \cap (QSB)$ for any bi-ideal B and quasi-ideal Q,*

(iv) *$B \cap L \subseteq (BSL) \cap (LSB)$ any bi-ideal B and for any left ideal L,*

(v) *$Q \cap L \subseteq (QSL) \cap (LSQ)$ for any left ideal L and quasi-ideal Q,*

(vi) *$R \cap L \subseteq (RSL) \cap (LSR)$ any right ideal R and for any left ideal L,*

(vii) *$B \cap R \subseteq (BSR) \cap (RSB)$ any bi-ideal B and for any right ideal R,*

$(viii)$ *$Q \cap R \subseteq (QSR) \cap (RSQ)$ for any right ideal R and any quasi-ideal Q.*

Proof. $(i) \to (ii)$ Suppose S is a regular ternary semigroup and B_1, B_2 are bi-ideals of S. Let $a \in B_1 \cap B_2$. Then there exists $x \in S$ such that $a = axa$. As $a = axa \in (B_1SB_2)$ and $a = axa \in (B_2SB_1)$, thus $B_1 \cap B_2 \subseteq (B_1SB_2) \cap (B_2SB_1)$.

$(ii) \to (iii)$ Since every quasi-ideal of S is a bi-ideal, therefore by (ii), we have $B \cap Q \subseteq (BSQ) \cap (QSB)$ for any bi-ideal B and for any quasi-ideal Q of S.

$(iii) \to (iv)$ Since every one-sided ideal of S is a quasi-ideal, therefore by (iii), we have $B \cap L \subseteq (BSL) \cap (LSB)$ for any bi-ideal B and for any left ideal L of S.

$(iv) \to (v)$ As every quasi-ideal of S is also a bi-ideal, therefore by (iv), we have $Q \cap L \subseteq (QSL) \cap (LSQ)$, for any left ideal L and for any quasi-ideal Q of S.

$(v) \to (vi)$ Since every one-sides ideal of S is a quasi-ideal, therefore by (v), we have $R \cap L \subseteq (RSL) \cap (LSR)$, for any right ideal R and for any left ideal L of S.

$(vi) \to (i)$ Suppose $R \cap L \subseteq (RSL) \cap (LSR)$, for any right ideal R and for any left ideal L of S. Now (vi) implies $R \cap L \subseteq (RSL) \cap (LSR) \subseteq RSL$. On the other hand, $RSL \subseteq R \cap L$ always. Thus $RSL = R \cap L$. Thus by Theorem 3, S is regular.

Similarly we can show that $(i) \longleftrightarrow (vii) \longleftrightarrow (viii)$. □

3. Weakly regular ternary semigroups

Definition 3. A ternary semigroup S is said to be *right* (resp. *left*) *weakly regular*, if for each $x \in S$, $x \in (xSS)^3$ (resp. $x \in (SSx)^3$).

Every regular ternary semigroup is right (left) weakly regular but the converse is not true.

Lemma 1. *A ternary semigroup S is right weakly regular if and only if $R \cap I = RII$, for every right ideal R and for every two-sided ideal I of S.*

Proof. Suppose S is right weakly regular and $x \in J \cap I$. Since S is right weakly regular, therefore $x \in (xSS)^3$, that is $x = (xs_1t_1)(xs_1t_2)(xs_3t_3)$ for some $s_1, s_2, s_3, t_1, t_2, t_3 \in S$. Thus $x = (xs_1t_1)(xs_1t_2)(xs_3t_3) \in JII$, hence $J \cap I \subseteq JII$. On the other hand, $JII \subseteq J \cap I$ always. So, $J \cap I = JII$.

Conversely, assume that $J \cap I = JII$, for all right ideals J and for all two-sided ideals I of S. We show that S is right weakly regular. Suppose $x \in S$. Let J be the right and I be the two-sided ideal of S generated by x, that is $J = x \cup xSS$, $I = x \cup SSx \cup xSS \cup SSxSS$. Then

$$
\begin{aligned}
x \in J \cap I &= JII \\
&= (x \cup xSS)(x \cup SSx \cup xSS \cup SSxSS)(x \cup SSx \cup xSS \cup SSxSS) \\
&= (xx \cup xSSx \cup xxSS \cup xSSxSS \cup xSSx \cup xSSSSx \cup xSSxSS \\
&\qquad\qquad \cup xSSSSxSS)(x \cup SSx \cup xSS \cup SSxSS) \\
&= (xx \cup xSSx \cup xxSS \cup xSSxSS)(x \cup SSx \cup xSS \cup SSxSS) \\
&\subseteq x^3 \cup xxSSx \cup x^3SS \cup xxSSxSS \cup xSSxx \cup xSSxSSx \cup xSSxxSS \\
&\qquad\qquad \cup xSSxSSxSS.
\end{aligned}
$$

Simple calculations shows that in any case $x \in (xSS)^3$. Hence S is right weakly regular. □

Theorem 9. *For a ternary semigroup S, the following conditions are equivalent:*

(*i*) *S is right weakly regular,*

(*ii*) *$B \cap I \cap R \subseteq BIR$ for every bi-ideal B, every two-sided ideal I and every right ideal R of S,*

(*iii*) *$Q \cap I \cap R \subseteq QIR$ for every quasi-ideal Q, every two-sided ideal I and every right ideal R of S.*

Proof. $(i) \to (ii)$ Let S be a right weakly regular ternary semigroup and $x \in B \cap I \cap J$. Since S is right weakly regular, therefore $x \in (xSS)^3$, that is $x = (xs_1t_1)(xs_2t_2)(xs_3t_3)$ for some $s_1, s_2, s_3, t_1, t_2, t_3 \in S$. Thus

$$x = (xs_1t_1)(xs_2t_2)(xs_3t_3) = x(s_1t_1xs_2t_2)(xs_3t_3) \in BIJ.$$

Hence $B \cap I \cap J \subseteq BIJ$.

$(ii) \to (iii)$ Since every quasi-ideal of S is a bi-ideal, (ii) implies (iii).

$(iii) \to (i)$ Let R be a right ideal and I a two sided ideal of S. Take $Q = R$, and $J = I$, then we have $Q \cap I \cap J = R \cap I \cap I = R \cap I$ and $QIJ = RII$. Thus by (iii) it follows that $R \cap I \subseteq RII$. But $RII \subseteq R \cap I$ always. Hence $R \cap I = RII$ and so by Lemma 1, S is right weakly regular. □

Theorem 10. *For a ternary semigroup S the following conditions are equivalent:*

- (i) *S is right weakly regular,*
- (ii) *$B \cap I \subseteq BII$ for every bi-ideal B and every two-sided ideal I,*
- (iii) *$Q \cap I \subseteq QII$ for every quasi-ideal Q and every two-sided ideal I.*

Proof. $(i) \to (ii)$ Let $x \in B \cap I$, where B is a bi-ideal and I is a two-sided ideal of S. Since S is right weakly regular, therefore $x \in (xSS)^3$. Consequently $x = (xs_1t_1)(xs_2t_2)(xs_3t_3)$ for some $s_1, s_2, s_3, t_1, t_2, t_3 \in S$. Thus $x = x(s_1t_1xs_2t_2)(xs_3t_3) \in BII$. Hence $B \cap I \subseteq BII$.

$(ii) \to (iii)$ Since every quasi-ideal of S is also a bi-ideal, therefore we have $Q \cap I \subseteq QII$ for every quasi-ideal Q and every two-sided ideal I of S.

$(iii) \to (i)$ Let R be a right ideal of S and I be a two sided ideal of S. Take $Q = R$, then by hypothesis $R \cap I \subseteq RII$. On the other hand $RII \subseteq R \cap I$ is always true. Thus $R \cap I = RII$, for every right ideal R and for every two-sided ideal I of S. Thus by Lemma 1, S is right weakly regular. □

4. Prime, strongly prime and semiprime bi-ideals

Throughout this section S will be considered as the ternary semigroup with zero.

Definition 4. A bi-ideal B of a ternary semigroup S is called

- *prime* if $B_1B_2B_3 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$ for any bi-ideals B_1, B_2, B_3 of S,

- *strongly prime* if $B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$ for any bi-ideals B_1, B_2, B_3 of S,
- *semiprime* if $B_1^3 \subseteq B$ implies $B_1 \subseteq B$ for any bi-ideal B_1 of S.

Remark 1. Every strongly prime bi-ideal of a ternary semigroup S is a prime bi-ideal and every prime bi-ideal is a semiprime bi-ideal. A prime bi-ideal is not necessarily a strongly prime bi-ideal and a semiprime bi-ideal is not necessarily a prime bi-ideal.

Example 1. Let $S = \{0, a, b\}$ and $abc = (a * b) * c$ for all $a, b, c \in S$, where $*$ is defined by the table:

$*$	0	a	b
0	0	0	0
a	0	a	a
b	0	b	b

Then S is a ternary semigroup.

Bi-ideals in S are: $\{0\}$, $\{0, a\}$, $\{0, b\}$ and $\{0, a, b\}$. All bi-ideals are prime and hence semiprime. The prime bi-ideal $\{0\}$ is not strongly prime, because

$$\begin{aligned}&(\{0,a\}\{0,b\}\{0,a,b\}) \cap (\{0,b\}\{0,a,b\}\{0,a\}) \cap (\{0,a,b\}\{0,a\}\{0,b\})\\&= \{0,a\} \cap \{0,b\} \cap \{0,a,b\} = \{0\},\end{aligned}$$

but neither $\{0, a\}$ nor $\{0, b\}$ nor $\{0, a, b\}$ is contained in $\{0\}$.

Example 2. Let S be a left zero ternary semigroup, that is $xyz = x$ for all $x, y, z \in S$ and let $|S| > 1$. We extend s to $S^0 = S \cup \{0\}$, where $0 \notin S$, by putting $xyz = x$ for $x, y, z \in S$ and $xyz = 0$ in all other cases. Then all subsets B_1, B_2, B_3 of S^0 containing 0 we have $B_1S^0B_1S^0B_1 = B_1$ and $B_1B_2B_3 = B_1$. Thus every subset of S^0 containing 0 is a bi-ideal of S^0 and every bi-ideal of S^0 is prime. If B is a bi-ideal of S^0 such that $|S^0 \backslash B| \geqslant 3$, then B is not strongly prime, since for any distinct elements $a, b, c \in S^0 \backslash B$,

$$\begin{aligned}&(B \cup \{a\})(B \cup \{b\})(B \cup \{c\}) \cap (B \cup \{b\})(B \cup \{c\})(B \cup \{a\})\\&\cap (B \cup \{c\})(B \cup \{a\})(B \cup \{b\}) = (B \cup \{a\}) \cap (B \cup \{b\}) \cap (B \cup \{c\}) = B\end{aligned}$$

but neither $B \cup \{a\}$ nor $B \cup \{b\}$ nor $B \cup \{c\}$ is contained in B. In particular, $\{0\}$ is a prime bi-ideal of S^0 which is not strongly prime.

Example 3. Let $0 \in S$ and $|S| > 3$. Then S with the ternary operation defined by

$$xyz = \begin{cases} x & \text{if } x = y = z, \\ 0 & \text{otherwise,} \end{cases}$$

is a ternary semigroup with zero. Since for all subsets B_1, B_2, B_3 of S containing 0 we have $B_1SB_1SB_1 = B_1$ and $B_1B_2B_3 = B_1 \cap B_2 \cap B_3$, all these subsets are semiprime bi-ideals.

Note that a semiprime bi-ideal B of S such that $|S \backslash B| \geqslant 3$ is not a prime bi-ideal because for distinct $a, b, c \in S \backslash B$, we have

$$(B \cup \{a\})\,(B \cup \{b\})\,(B \cup \{c\}) = (B \cup \{a\}) \cap (B \cup \{b\}) \cap (B \cup \{c\}) = B,$$

but neither $(B \cup \{a\})$ nor $(B \cup \{b\})$ nor $(B \cup \{c\})$ is contained in B. In particular, $\{0\}$ is a semiprime bi-ideal but it is not prime.

It is not difficult to verify that the following proposition is true.

Proposition 5. *The intersection of any family of prime bi-ideals of a ternary semigroup S is a semiprime bi-ideal.* □

5. Irreducible and strongly irreducible bi-ideals

Definition 5. A bi-ideal B of a ternary semigroup S is called *irreducible* (*strongly irreducible*) if $B_1 \cap B_2 \cap B_3 = B$ (resp. $B_1 \cap B_2 \cap B_3 \subseteq B$) implies $B_1 = B$ or $B_2 = B$ or $B_3 = B$ (resp. $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$) for all bi-ideals B_1, B_2, B_3 of S.

Every strongly irreducible bi-ideal of a ternary semigroup S is an irreducible bi-ideal but the converse is not true in general.

Example 4. Let $S = \{0, 1, 2, 3, 4, 5\}$ and $abc = (a * b) * c$ for all $a, b, c \in S$, where $*$ is defined by the table

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	2	3	1	1
3	0	1	1	1	2	3
4	0	1	4	5	1	1
5	0	1	1	1	4	5

Then S is a ternary semigroup with bi-ideals: $\{0\}$, $\{0,1\}$, $\{0,1,2\}$, $\{0,1,3\}$, $\{0,1,4\}$, $\{0,1,5\}$, $\{0,1,2,4\}$, $\{0,1,3,5\}$, $\{0,1,2,3\}$, $\{0,1,4,5\}$ and S. Bi-ideals $\{0\}$, $\{0,1,2,4\}$, $\{0,1,3,5\}$, $\{0,1,2,3\}$, $\{0,1,4,5\}$ and S are irreducible. Strongly irreducible are only $\{0\}$ and S.

Proposition 6. *Every strongly irreducible semiprime bi-ideal is strongly prime.*

Proof. Let B be a strongly irreducible semiprime bi-ideal of S. Suppose B_1, B_2 and B_3 are bi-ideals of S such that

$$B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B.$$

Since

$$(B_1 \cap B_2 \cap B_3)(B_1 \cap B_2 \cap B_3)(B_1 \cap B_2 \cap B_3) \subseteq B_1B_2B_3,$$
$$(B_1 \cap B_2 \cap B_3)(B_1 \cap B_2 \cap B_3)(B_1 \cap B_2 \cap B_3) \subseteq B_2B_3B_1,$$
$$(B_1 \cap B_2 \cap B_3)(B_1 \cap B_2 \cap B_3)(B_1 \cap B_2 \cap B_3) \subseteq B_3B_1B_2,$$

we have

$$(B_1 \cap B_2 \cap B_3)^3 \subseteq B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B.$$

But B is semiprime, so $(B_1 \cap B_2 \cap B_3) \subseteq B$.

Also since B is strongly irreducible, so we have either $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. Thus B is a strongly prime bi-ideal of S. $\square$

Proposition 7. *Let B be a bi-ideal of S. For any $a \in S \setminus B$ there exists an irreducible bi-ideal I of S such that $B \subseteq I$ and $a \notin I$.*

Proof. Suppose $\Im = \{B_i : i \in I\}$ be the collection of all bi-ideals of S which contains B and does not contain a, then $\Im \neq \emptyset$ because $B \in \Im$. Evidently $\Im$ is partially ordered under inclusion. If Ω is any totally ordered subset of $\Im$ then $\bigcup \Omega$ is a bi-ideal of S containing B and not containing a. Hence by Zorn's lemma, there exists a maximal element I in $\Im$. We show that I is an irreducible bi-ideal of S. Let C, D and E be any three bi-ideals of S such that $I = C \cap D \cap E$. If all of three bi-ideals C, D and E properly contain I then according to the hypothesis $a \in C$, $a \in D$ and $a \in E$. Hence $a \in C \cap D \cap E = I$. This contradicts the fact that $a \notin I$. Thus either $I = C$ or $I = D$ or $I = E$. Hence I is irreducible. $\square$

Theorem 11. *For a regular ternary semigroup S, the following assertions are equivalent:*

(*i*) *every bi-ideal of S is idempotent,*

(*ii*) $B_1 \cap B_2 \cap B_3 = B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2$ *for every bi-ideals of S,*

(*iii*) *every bi-ideal of S is semiprime,*

(*iv*) *each proper bi-ideal of S is the intersection of all irreducible semiprime bi-ideals of S which contain it.*

Proof. $(i) \to (ii)$ Let B_1, B_2 and B_3 be bi-ideals of S. Then by the hypothesis

$$B_1 \cap B_2 \cap B_3 = (B_1 \cap B_2 \cap B_3)^3 \subseteq B_1B_2B_3.$$

Similarly,

$$B_1 \cap B_2 \cap B_3 \subseteq B_2B_3B_1 \quad \text{and} \quad B_1 \cap B_2 \cap B_3 \subseteq B_3B_1B_2.$$

Thus

$$B_1 \cap B_2 \cap B_3 \subseteq B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2\,. \tag{1}$$

By Corollary 2, $B_1B_2B_3$, $B_2B_3B_1$ and $B_3B_1B_2$ are bi-ideals. Also by Proposition 1, $B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2$ is a bi-ideal. Thus by hypothesis

$$\begin{aligned} B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 &= (B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2)^3 \\ &\subseteq (B_1B_2B_3)\,(B_3B_1B_2)\,(B_2B_3B_1) \\ &\subseteq (B_1SS)\,(SB_1S)\,(SSB_1) \\ &= B_1(SSS)B_1(SSS)B_1 \subseteq B_1SB_1SB_1 \subseteq B_1. \end{aligned}$$

Similarly,

$$B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B_2$$

and

$$B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B_3.$$

Thus

$$B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B_1 \cap B_2 \cap B_3. \tag{2}$$

Hence from (1) and (2),

$$B_1 \cap B_2 \cap B_3 = B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2.$$

$(ii) \to (i)$ Obvious.

$(i) \to (iii)$ Let B and B_1 be any two bi-ideals of S such that $B_1^3 \subseteq B$, then by hypothesis $B_1 = B_1^3 \subseteq B$. Hence every bi-ideal of S is semiprime.

$(iii) \to (iv)$ Let B be a proper bi-ideal of S, then B is contained in the intersection of all irreducible bi-ideals of S which contain B. Proposition 7, guarantees the existence of such irreducible bi-ideals. If $a \notin B$, then there exists an irreducible bi-ideal of S which contains B but does not contain a. Thus B is the intersection of all irreducible bi-ideals of S which contain B.

By hypothesis each bi-ideal is semiprime, so each bi-ideal is the intersection of irreducible semiprime bi-ideals of S which contains it.

$(iv) \to (i)$ Let B be a bi-ideal of a ternary semigroup S. If $B^3 = S$, then clearly B is idempotent. If $B^3 \neq S$, then B^3 is a proper bi-ideal of S containing B^3, so by the hypothesis,

$$B^3 = \bigcap \left\{ B_\alpha : B_\alpha \text{ is irreducible semiprime bi-ideal of } S \text{ containing } B^3 \right\}.$$

Since each B_α is semiprime bi-ideal, $B^3 \subseteq B_\alpha$ implies $B \subseteq B_\alpha$. Therefore $B \subseteq \bigcap B_\alpha = B^3$ implies $B \subseteq B^3$, but $B^3 \subseteq B$. Hence $B^3 = B$. □

Proposition 8. *If each bi-ideal of a ternary semigroup S is idempotent, then a bi-ideal B of S is strongly irreducible if and only if B is strongly prime.*

Proof. Suppose that a bi-ideal B is strongly irreducible and let B_1, B_2, B_3 are bi-ideals of S such that

$$B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B.$$

By Theorem 11,

$$B_1 \cap B_2 \cap B_3 = B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2,$$

so we have

$$B_1 \cap B_2 \cap B_3 \subseteq B.$$

Since B is strongly irreducible so, either $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. Thus B is strongly prime.

On the other hand, if B is strongly prime and $B_1 \cap B_2 \cap B_3 \subseteq B$ for some bi-ideals B_1, B_2 and B_3 of S, then, in view of Theorem 11,

$$B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B,$$

whence we conclude either $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. Therefore B is strongly irreducible. □

Next we characterize those ternary semigroups for which each bi-ideal is strongly irreducible and also those ternary semigroups in which each bi-ideal is strongly prime.

Theorem 12. *Each bi-ideal of a regular ternary semigroup S is strongly prime if and only if every bi-ideal of S is idempotent and the set of bi-ideals of S is totally ordered by inclusion.*

Proof. Suppose that each bi-ideal of S is strongly prime, then each bi-ideal of S is semiprime. Thus by Theorem 11, every bi-ideal of S is idempotent. We show that the set of bi-ideals of S is totally ordered by inclusion. Let B_1 and B_2 be any two bi-ideals of S, then by Theorem 11,

$$B_1 \cap B_2 = B_1 \cap B_2 \cap S = B_1B_2S \cap B_2SB_1 \cap SB_1B_2.$$

Thus

$$B_1B_2S \cap B_2SB_1 \cap SB_1B_2 \subseteq B_1 \cap B_2.$$

As each bi-ideal is strongly prime, therefore $B_1 \cap B_2$ is strongly prime bi-ideal. Hence either $B_1 \subseteq B_1 \cap B_2$ or $B_2 \subseteq B_1 \cap B_2$ or $S \subseteq B_1 \cap B_2$. Now, if $B_1 \subseteq B_1 \cap B_2$, then $B_1 \subseteq B_2$; if $B_2 \subseteq B_1 \cap B_2$, then $B_2 \subseteq B_1$; if $S \subseteq B_1 \cap B_2$, then $B_1 = S = B_2$. Thus set of bi-ideals of S is totally ordered under inclusion.

Conversely, assume that every bi-ideal of S is idempotent and the set of bi-ideals of S is totally ordered under inclusion. We show that each bi-ideal of S is strongly prime. Let B, B_1, B_2 and B_3 be bi-ideals of S such that

$$B_1B_2B_3 \cap B_2B_3B_1 \cap B_3B_1B_2 \subseteq B.$$

Since every bi-ideal of S is idempotent so by Theorem 11,

$$B_1 \cap B_2 \cap B_3 \subseteq B.$$

Since the set of all bi-ideals of S is totally ordered under inclusion so for B_1, B_2, B_3 we have the following six possibilities:

(1) $B_1 \subseteq B_2,\ B_2 \subseteq B_3,\ B_1 \subseteq B_3$, (2) $B_1 \subseteq B_2,\ B_3 \subseteq B_2,\ B_1 \subseteq B_3$,
(3) $B_1 \subseteq B_2,\ B_3 \subseteq B_2,\ B_3 \subseteq B_1$, (4) $B_2 \subseteq B_1,\ B_2 \subseteq B_3,\ B_1 \subseteq B_3$,
(5) $B_2 \subseteq B_1,\ B_3 \subseteq B_2,\ B_3 \subseteq B_1$, (6) $B_2 \subseteq B_1,\ B_3 \subseteq B_1,\ B_2 \subseteq B_3$.

In these cases we have

(1) $B_1 \cap B_2 \cap B_3 = B_1$, (2) $B_1 \cap B_2 \cap B_3 = B_1$; (3) $B_1 \cap B_2 \cap B_3 = B_3$,
(4) $B_1 \cap B_2 \cap B_3 = B_2$, (5) $B_1 \cap B_2 \cap B_3 = B_3$, (6) $B_1 \cap B_2 \cap B_3 = B_2$.

Thus either $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$, which proves that B is strongly prime. $\square$

Theorem 13. *If the set of bi-ideals of a regular ternary semigroup S is totally ordered, then every bi-ideal of S is idempotent if and only if each bi-ideal of S is prime.*

Proof. Suppose every bi-ideal of S is idempotent. Let B, B_1, B_2, B_3 be bi-ideals of S such that

$$B_1B_2B_3 \subseteq B.$$

As in the proof of the previous theorem we obtain $B_1 \subseteq B_2$, $B_2 \subseteq B_3$, $B_1 \subseteq B_3$, whence we conclude $B_1B_1B_1 \subseteq B_1B_2B_3 \subseteq B$, i.e., $B_1^3 \subseteq B$. By Theorem 11, B is a semiprime bi-ideal, so $B_1 \subseteq B$. Similarly for other cases we have $B_2 \subseteq B$ or $B_3 \subseteq B$.

Conversely, assume that every bi-ideal of S is prime. Since the set of bi-ideals of S is totally ordered under inclusion, so the concepts of primeness and strongly primeness coincide. Hence by Theorem 13, every bi-ideal of S is idempotent. □

Theorem 14. *For a ternary semigroup S the following are equivalent:*

(i) *the set of bi-ideals of S is totally ordered under inclusion,*

(ii) *each bi-ideal of S is strongly irreducible,*

(iii) *each bi-ideal of S is irreducible.*

Proof. $(i) \to (ii)$ Let $B_1 \cap B_2 \cap B_3 \subseteq B$ for some bi-ideals of S. Since the set of bi-ideals of S is totally ordered under inclusion, therefore either $B_1 \cap B_2 \cap B_3 = B_1$ or B_2 or B_3. Thus either $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. Hence B is strongly irreducible.

$(ii) \to (iii)$ If $B_1 \cap B_2 \cap B_3 = B$ for some bi-ideals of S, then $B \subseteq B_1$, $B \subseteq B_2$ and $B \subseteq B_3$. On the other hand by hypothesis we have, $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. Thus $B_1 = B$ or $B_2 = B$ or $B_3 = B$. Hence B is irreducible.

$(iii) \to (i)$ Suppose each bi-ideal of S is irreducible. Let B_1, B_2 be bi-ideals of S, then $B_1 \cap B_2$ is also a bi-ideal of S. Since $B_1 \cap B_2 \cap S = B_1 \cap B_2$, the irreducibility of $B_1 \cap B_2$ implies that either $B_1 = B_1 \cap B_2$ or $B_2 = B_1 \cap B_2$ or $S = B_1 \cap B_2$, i.e., either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$ or $B_1 = B_2$. Hence the set of bi-ideals of S is totally ordered under inclusion. □

Let $\mathcal{B}$ be the family of all bi-ideals of S and $\mathcal{P}$ – the family of all proper strongly prime bi-ideals of S. For each $B \in \mathcal{B}$ we define

$$\Theta_B = \{J \in \mathcal{P} : B \nsubseteq J\},$$
$$\Im(\mathcal{P}) = \{\Theta_B : B \text{ is a bi-ideal of } S\}.$$

Theorem 15. *If S is ternary semigroup with the property that every bi-ideal of S is idempotent then $\Im(\mathcal{P})$ forms a topology on the set $\mathcal{P}$.*

Proof. As $\{0\}$ is a bi-ideal of S, so $\Theta_{\{0\}} = \{J \in \mathcal{P} : \{0\} \nsubseteq J\} = \emptyset$ because 0 belong to every bi-ideal. Since S is a bi-ideal of S, we have $\Theta_S = \{J \in \mathcal{P} : S \nsubseteq J\} = \mathcal{P}$ because $\mathcal{P}$ is the collection of all proper strongly prime bi-ideals in S. Thus $\emptyset$ and $\mathcal{P}$ belongs to $\Im(\mathcal{P})$.

Let $\{\Theta_{B_\alpha} : \alpha \in I\} \subseteq \Im(\mathcal{P})$. Then

$$\bigcup_{\alpha\in I} \Theta_{B\alpha} = \{J \in \mathcal{P} : B_\alpha \nsubseteq J \text{ for some } \alpha \in I\} = \{J \in \mathcal{P} : \widehat{\bigcup_{\alpha\in I} B_\alpha} \nsubseteq J\},$$

which is equal to $\Theta_{\widehat{\bigcup_{\alpha\in I} B_\alpha}} \in \Im(\mathcal{P})$, where $\widehat{\bigcup_{\alpha\in I} B_\alpha}$ means the bi-ideal of S generated by $\bigcup_{\alpha\in I} B_\alpha$.

Let Θ_{B_1} and Θ_{B_2} be arbitrary two elements from $\Im(\mathcal{P})$. We show that $\Theta_{B_1} \cap \Theta_{B_2} \in \Im(\mathcal{P})$. If $J \in \Theta_{B_1} \cap \Theta_{B_2}$, then $J \in \mathcal{P}$, $B_1 \nsubseteq J$ and $B_2 \nsubseteq J$. Suppose that $B_1 \cap B_2 = B_1 \cap B_2 \cap S \subseteq J$. By Theorem 11, we have $B_1B_2S \cap B_2SB_1 \cap SB_1B_2 \subseteq J$. Since J is a strongly prime bi-ideal, therefore either $B_1 \subseteq J$ or $B_2 \subseteq J$ ($S \nsubseteq J$ because J is a proper bi-ideal of S), which is a contradiction. Hence $B_1 \cap B_2 \nsubseteq J$, i.e., $J \in \Theta_{B_1\cap B_2}$. Thus $\Theta_{B_1} \cap \Theta_{B_2} \subseteq \Theta_{B_1\cap B_2}$.

On the other hand if $J \in \Theta_{B_1\cap B_2}$, then $J \in \mathcal{P}$ and $B_1 \cap B_2 \nsubseteq J$, which means that $B_1 \nsubseteq J$ and $B_2 \nsubseteq J$. Therefore, $J \in \Theta_{B_1}$ and $J \in \Theta_{B_2}$, i.e., $J \in \Theta_{B_1} \cap \Theta_{B_2}$. Hence $\Theta_{B_1\cap B_2} \subseteq \Theta_{B_1} \cap \Theta_{B_2}$. Thus $\Theta_{B_1\cap B_2} = \Theta_{B_1} \cap \Theta_{B_2}$, so $\Theta_{B_1} \cap \Theta_{B_2} \in \Im(\mathcal{P})$. This proves that $\Im(\mathcal{P})$ is a topology on $\mathcal{P}$. □

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