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Instability of solutions for nonlinear functional differential equations of fifth order with n-deviating arguments

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Abstract. In this paper, we study the instability properties of solutions of a class of nonlinear functional differential equations of the fifth order with n-constant deviating arguments. By using the Lyapunov-Krasovskii functional approach, we obtain some interesting sufficient conditions ensuring that the zero solution of the equations is unstable.

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1 Introduction

In 1990, Li and Duan [6] proved some instability theorems for the nonlinear differential equation of the fifth order without delay,

$$x^{(5)} + f_5(x''')x^{(4)} + f_4(x'')x''' + f_3(x'') + f_2(x') + f_1(x) = 0. \tag{1}$$

Later, in a recent paper, Tunç [15] improved the results obtained for Eq. (1) to the nonlinear differential equation of the fifth order with a constant delay r,

$$x^{(5)} + f_5(x''')x^{(4)} + f_4(x'')x''' + f_3(x, x(t-r), ..., x^{(4)}, x^{(4)}(t-r))x''$$

$$+f_2(x'(t-r)) + f_1(x(t-r)) = 0.$$

In this paper, instead of these equations, we consider the nonlinear differential equations of the fifth order n-constant deviating arguments τ_i,

$$x^{(5)} + f_5(x''')x^{(4)} + f_4(x'')x''' + f_3(x, x(t-\tau_1), ..., x(t-\tau_n), ..., x^{(4)}, ..., x^{(4)}(t-\tau_n))x''$$

$$+\sum_{i=1}^{n} g_i(x'(t-\tau_i)) + \sum_{i=1}^{n} h_i(x(t-\tau_i)) = 0. \tag{2}$$

We write Eq. (2) in the system form as follows

$$x' = y, y' = z, z' = w, w' = u,$$

$$
\begin{aligned}
u' &= -f_5(w)u - f_4(z)w - f_3(x,...,x(t-\tau_n),...,u,...,u(t-\tau_n))z \\
&\quad -\sum_{i=1}^{n} g_i(y) - \sum_{i=1}^{n} h_i(x) + \sum_{i=1}^{n} \int_{t-\tau_i}^{t} g_i'(y(s))z(s)ds \\
&\quad +\sum_{i=1}^{n} \int_{t-\tau_i}^{t} h_i'(x(s))y(s)ds, \qquad (3)
\end{aligned}
$$

where τ_i are positive constants, n – fixed deviating arguments, the primes in Eq. (2) denote differentiation with respect to $t, t \in \Re_+,\ \Re_+ = [0,\infty); f_5, \quad f_4, f_3, g_i$ and h_i are continuous functions on $\Re, \Re, \Re^{2n+2}, \Re$and $\Re$, respectively, with $h_i(0) = g_i(0) = 0$, and satisfy a Lipschitz condition in their respective arguments. Hence, the existence and uniqueness of the solutions of Eq. (2) are guaranteed (see Èl'sgol'ts [1], pp. 14, 15). We assume in what follows that the functions f_i and g_i are also differentiable, and $x(t),\ y(t),\ z(t),\ w(t)$ and $u(t)$are abbreviated as $x,\ y,\ z,\ w$ and u, respectively.

To the best of our knowledge from the literature, so far, the instability of solutions for nonlinear differential equations of the fifth order with multiple deviating arguments has not been investigated. However, since 1978 up to now, the instability of solutions of various nonlinear scalar and vector differential equations of the fifth order without or with a delay has been investigated and is still being studied by researchers. In particular, for some results proceeded on this topic related to these type equations, the reader can refer to the papers of Ezeilo [2]-[4], Li and Duan [6], Li and Yu [7], Sadek [8], Sun and Hou [9], Tiryaki [10], Tunç [11]-[17], Tunç and Erdogan [18], Tunç and Karta [19], Tunç and Sevli [20]. In all these papers, the authors used some suitable Lyapunov functions or functionals as basic tool to achieve their proposed goal in the works. They also based on the Krasovskii's properties (see Krasovskii [5]) to study the instability of solutions of the equations considered therein. In this paper, we employ the Lyapunov-Krasovskii functional approach to investigate the subject for Eq. (2) by defining two new appropriate Lyapunov functionals. In fact, when we take into consideration the differential equations of the fifth order discussed in the above mentioned papers and the literature, it can be seen that all the equations studied there do not include or include only a deviating argument. However, this paper includes n-deviating arguments and is a continuation of the instability results related to the scalar nonlinear differential equations of the fifth order mentioned above (see Ezeilo [2]-[4], Li and Duan [6], Li and Yu [7], Sun and Hou [9], Tiryaki [10], Tunç [14]-[17]). The researches related to the instability of solutions are also very important in the theory and applications of differential equations, and the investigation of this topic for nonlinear differential equations of the fifth order with multiple deviating arguments takes an important place for the researchers work in this area. This work makes a contribution to the existing studies made in the literature.

Let $r \geqslant 0$ be given, and let $C = C([-r, 0],\ \Re^n)$ with

$$\|\phi\| = \max_{-r \leqslant s \leqslant 0} |\phi(s)|\,,\ \phi \in C.$$

For $H > 0$ define $C_H \subset C$ by

$$C_H = \{\phi \in C : \|\phi\| < H\}.$$

If $x : [-r,\ A) \to \Re^n$ is continuous, $0 < A \leqslant \infty$,then, for each t in $[0, A)$, x_t in C is defined by

$$x_t(s) = x(t+s), -r \leqslant s \leqslant 0, \quad t \geqslant 0.$$

Let Gbe an open subset of $C$and consider the general autonomous delay differential system with finite delay

$$\dot{x} = F(x_t), \quad x_t = x(t+\theta), \quad -r \leqslant \theta \leqslant 0, \quad t \geqslant 0,$$

where $F(0) = 0, F : G \to \Re^n$ is continuous and maps closed and bounded sets into bounded sets. It follows from the conditions on Fthat each initial value problem

$$\dot{x} = F(x_t), \quad x_0 = \phi \in G,$$

has a unique solution defined on some interval $[0, A), 0 < A \leqslant \infty$. This solution will be denoted by $x(\phi)(.)$so that $x_0(\phi) = \phi$.

Definition 1. The zero solution, $x = 0$, of $\dot{x} = F(x_t)$is stable if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|\phi\| < \delta$ implies that $|x(\phi)(t)| < \varepsilon$ for all $t \geqslant 0$. The zero solution is said to be unstable if it is not stable.

2 Main results

The first result of this paper is the following theorem.
Let $\tau = \max \tau_i,\ (i = 1, 2, ..., n)$.

Theorem 1. *In addition to all the assumptions imposed on the functions f_5, f_4, f_3, g_i and h_i appearing in Eq. (2), we assume that there exist positive constants $a_3, \bar{b}_i$, b_i and c_i such that the following conditions hold:*

$$h_i(0) = g_i(0) = 0, \quad h_i(x) \neq 0, \quad (x \neq 0), g_i(y) \neq 0, \quad (y \neq 0),$$

$$-\bar{b}_i \leqslant h_i'(x) \leqslant -b_i, \quad 0 \leqslant \left|g_i'(y)\right| \leqslant c_i, f_5(w) \leqslant 0$$

for all $x,\ y,\ w$ and

$$f_3(x, ..., x(t-\tau_n), ..., u, ..., u(t-\tau_n)) \geqslant a_3$$

for all $x, ...,\ u,\ ...,u(t-\tau_n)$.

If

$$\tau < 2\min\left\{\frac{b_i}{\bar{b}_i},\ \frac{a_3}{\sum\limits_{i=1}^{n}(\bar{b}_i+2c_i)}\right\},$$

then the zero solution of Eq. (2) is unstable.

Proof. Define the Lyapunov functional $V = V(x_t, y_t, z_t, w_t, u_t)$:

$$V = \frac{1}{2}w^2 - zu - z\int_0^w f_5(s)ds - \int_0^z f_4(s)sds - \int_0^y \sum_{i=1}^n g_i(s)ds$$

$$-y\sum_{i=1}^n h_i(x) - \sum_{i=1}^n \lambda_i \int_{-\tau_i}^0 \int_{t+s}^t y^2(\theta)d\theta ds - \sum_{i=1}^n \mu_i \int_{-\tau_i}^0 \int_{t+s}^t z^2(\theta)d\theta ds, \tag{4}$$

where s is a real variable such that the integrals $\sum\limits_{i=1}^n \lambda_i \int\limits_{-\tau_i}^0 \int\limits_{t+s}^t y^2(\theta)d\theta ds$ and $\sum\limits_{i=1}^n \mu_i \int\limits_{-\tau_i}^0 \int\limits_{t+s}^t z^2(\theta)d\theta ds$are non-negative, and $\lambda_i$and $\mu_i$are some positive constants which will be determined later in the proof.

It is clear that

$$V(0,0,0,\varepsilon,0) = \frac{1}{2}\varepsilon^2 > 0$$

for all sufficiently small ε. Hence, in every neighborhood of the origin,$(0,0,0,0,0)$, there exists a point $(0,0,0,\varepsilon,0)$ such that $V(0,0,0,\varepsilon,0) > 0$,which shows that V has the property (K_1), (see [5]).

By a direct computation from (3) and (4), we obtain

$$\frac{d}{dt}V = -\sum_{i=1}^n h_i'(x)y^2 + f_3(x,...,x(t-\tau_n),u,...,u(t-\tau_n))z^2$$

$$-w\int_0^w f_5(s)ds - z\sum_{i=1}^n \int_{t-\tau_i}^t g_i'(y(s))z(s)ds$$

$$-z\sum_{i=1}^n \int_{t-\tau_i}^t h_i'(x(s))y(s)ds - \sum_{i=1}^n (\lambda_i\tau_i)y^2 - \sum_{i=1}^n (\mu_i\tau_i)z^2$$

$$+\sum_{i=1}^n \lambda_i \int_{t-\tau_i}^t y^2(s)ds + \sum_{i=1}^n \mu_i \int_{t-\tau_i}^t z^2(s)ds.$$

The assumptions of Theorem 1 and the estimate $2|mn| \leqslant m^2 + n^2$ imply

$$-\sum_{i=1}^{n} h_i'(x)y^2 \geqslant \sum_{i=1}^{n} b_i y^2,$$

$$f_3(x, ..., x(t-\tau_n), ..., u, ..., u(t-\tau_n))z^2 \geqslant a_3 z^2,$$

$$-z\sum_{i=1}^{n}\int_{t-\tau_i}^{t} h_i'(x(s))y(s)ds \geqslant -|z|\sum_{i=1}^{n}\int_{t-\tau_i}^{t} \left|h_i'(x(s))\right| \ |y(s)|\, ds$$

$$\geqslant -|z|\sum_{i=1}^{n}\int_{t-\tau_i}^{t} \bar{b}_i \ |y(s)|\, ds$$

$$\geqslant -\frac{1}{2}\sum_{i=1}^{n}(\bar{b}_i\tau_i)z^2 - \frac{1}{2}\sum_{i=1}^{n}\bar{b}_i\int_{t-\tau_i}^{t} y^2(s)ds$$

and

$$-z\sum_{i=1}^{n}\int_{t-\tau_i}^{t} g_i'(y(s))z(s)ds \geqslant -|z|\sum_{i=1}^{n}\int_{t-\tau_i}^{t} \left|g_i'(y(s))\right| \ |z(s)|\, ds$$

$$\geqslant -|z|\sum_{i=1}^{n}\int_{t-\tau_i}^{t} c_i \ |z(s)|\, ds$$

$$\geqslant -\frac{1}{2}\sum_{i=1}^{n}(c_i\tau_i)z^2 - \frac{1}{2}\sum_{i=1}^{n}c_i\int_{t-\tau_i}^{t} z^2(s)ds$$

so that

$$\frac{d}{dt}V \geqslant \sum_{i=1}^{n}(b_i - \lambda_i\tau_i)y^2 + \{a_3 - 2^{-1}\sum_{i=1}^{n}(c_i + \bar{b}_i + 2\mu_i)\tau_i\}z^2$$

$$+\sum_{i=1}^{n}(\lambda_i - 2^{-1}\bar{b}_i)\int_{t-\tau_i}^{t} y^2(s)ds + \sum_{i=1}^{n}(\mu_i - 2^{-1}c_i)\int_{t-\tau_i}^{t} z^2(s)ds.$$

Let $\lambda_i = \frac{1}{2}\bar{b}_i, \quad \mu_i = \frac{1}{2}c_i$ and $\tau = \max\tau_i, \ (i = 1, 2, ..., n)$. Hence, we have

$$\frac{d}{dt}V \geqslant \sum_{i=1}^{n}(b_i - 2^{-1}\bar{b}_i\tau)y^2 + \{a_3 - 2^{-1}\sum_{i=1}^{n}(\bar{b}_i + 2c_i)\tau\}z^2.$$

If

$$\tau < 2\min\left\{\frac{b_i}{\bar{b}_i}, \ \frac{a_3}{\sum\limits_{i=1}^{n}(\bar{b}_i + 2c_i)}\right\},$$

then

$$\frac{d}{dt}V \geqslant \sum_{i=1}^{n}(b_i - 2^{-1}\bar{b}_i\tau)y^2 + \{a_3 - 2^{-1}\sum_{i=1}^{n}(\bar{b}_i + 2c_i)\tau\}z^2 > 0,$$

which verifies that V has the property (K_2), (see [5]).

On the other hand, $\frac{d}{dt}V = 0$ if and only if $y = z = 0$, which implies that

$$y = z = w = u = 0.$$

Besides, by $h_i(0) = g_i(0) = 0, h_i(x) \neq 0$ for all $x \neq 0, g_i(y) \neq 0$for all $y \neq 0$and the system (3), we can conclude that $\frac{d}{dt}V = 0$ if and only if $x = y = z = w = u = 0$. Thus, the property (K_3), (see [5]), holds. By the above discussion, we conclude that the zero solution of Eq. (2) is unstable. The proof of Theorem 1 is completed. □

Example 1. We consider the nonlinear differential equation of the fifth order with two deviating arguments,

$$x^{(5)} - \frac{1}{1+(x'''')^4}x^{(4)} + 9x''' + \{2 + \exp(-x^2 - x^2(t-\tau_1) - x^2(t-\tau_2))\}x''$$

$$+ \sin x'(t-\tau_1) + \sin x'(t-\tau_2) - x(t-\tau_1) - x(t-\tau_2)$$

$$-4arctgx(t-\tau_1) - arctgx(t-\tau_2) = 0. \tag{5}$$

We write Eq. (5) in system form as follows

$$x' = y, \quad y' = z, z' = w, w' = u,$$

$$u' = \frac{u}{1+w^4} - 9w - \{2 + \exp(-x^2 - x^2(t-\tau_1) - x^2(t-\tau_2)\}z$$

$$-2\sin y + 2x + 5arctgx - \int_{t-\tau_1}^{t} y(s)ds - \int_{t-\tau_2}^{t} y(s)ds$$

$$+ \int_{t-\tau_1}^{t} \cos y(s)z(s)ds + \int_{t-\tau_2}^{t} \cos y(s)z(s)ds$$

$$-4\int_{t-\tau_1}^{t} \frac{1}{1+x^2(s)}y(s)ds - \int_{t-\tau_2}^{t} \frac{1}{1+x^2(s)}y(s)ds.$$

It follows that Eq. (5) is a special case of Eq. (2) and

$$f_5(w) = -\frac{1}{1+w^4} \leqslant 0,$$

$$f_4(z) = 9,$$

$$f_3(.) = 2 + \exp\{-x^2 - x^2(t-\tau_1) - x^2(t-\tau_2)\} \geqslant 2 = a_3,$$

$$f_2(y) = \sin y, -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2},$$

$$f_2(0) = 0, g_1'(y) = \cos y, |\cos y| \leqslant 1 = c_1,$$

$$g_2(y) = \sin y, -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2},$$

$$g_2(0) = 0, g_2'(y) = \cos y, |\cos y| \leqslant 1 = c_2,$$

$$h_1(x) = x + 4arctgx, -\frac{\pi}{2} < x < \frac{\pi}{2},$$

$$h_1(0) = 0, h_1'(x) = 1 + \frac{4}{1+x^2},$$

$$\bar{b}_1 = 5 \geqslant 1 + \frac{4}{1+x^2} \geqslant 1 = b_1,$$

$$h_2(x) = x + arctgx, -\frac{\pi}{2} < x < \frac{\pi}{2},$$

$$h_2(0) = 0, h_2'(x) = 1 + \frac{1}{1+x^2},$$

$$\bar{b}_2 = 2 \geqslant 1 + \frac{1}{1+x^2} \geqslant 1 = b_2,$$

$$\tau < 2\min\{\frac{b_i}{\bar{b}_i}, \frac{a_3}{\sum\limits_{i=1}^{n}(\bar{b}_i + 2c_i)}\} = \frac{4}{11}.$$

In view of the above estimates, we conclude that all the assumptions of Theorem 1 hold. Hence, if $\tau < \frac{4}{11}$, then the zero solution of Eq. (5) is unstable.

Second, we consider the special case of Eq. (2) with $g_i(x'(t-\tau_i)) = f_2(x')$, namely, the differential equation of the fifth order n-constant deviating arguments τ_i,

$$x^{(5)} + f_5(x''')x^{(4)} + f_4(x'')x''' + f_3(x, x(t-\tau_1), ..., x(t-\tau_n), ..., x^{(4)}, ..., x^{(4)}(t-\tau_n))x''$$

$$+f_2(x') + \sum_{i=1}^{n} h_i(x(t-\tau_i)) = 0. \tag{6}$$

We write Eq. (6) in the system form as follows

$$x' = y, \quad y' = z, \quad z' = w, w' = u,$$

$$u' \quad = \quad -f_5(w)u - f_4(z)w - f_3(x, ..., x(t-\tau_n), ..., u, ..., u(t-\tau_n))z$$

$$-f_2(y)-\sum_{i=1}^{n}h_i(x)+\sum_{i=1}^{n}\int_{t-\tau_i}^{t}h_i'(x(s))y(s)ds. \tag{7}$$

The second result of this paper is the following theorem.

Let $\tau=\max\tau_i,\ (i=1,2,...,n)$.

Theorem 2. *In addition to all the assumptions imposed to the functions $f_5,\quad f_4,f_3,f_2$ and h_i that appearing in Eq. (6), we assume that there exist positive constants a_3,b_i and $\bar{b}_i$ such that the following conditions hold:*

$$h_i(0)=f_2(0)=0,\quad h_i(x)\neq 0,\quad (x\neq 0),f_2(y)\neq 0,\quad (y\neq 0),$$

$$\bar{b}_i\geqslant h_i'(x)\geqslant b_i,f_5(w)\geqslant 0$$

for arbitrary $x,\ y,\ w$ and

$$f_3(x,...,x(t-\tau_n),...,u,...,u(t-\tau_n))\leqslant -a_3$$

for all $x,...,\ u,\ ...,u(t-\tau_n)$.

If

$$\tau<2\min\{\frac{b_i}{\bar{b}_i},\ \frac{a_3}{\sum\limits_{i=1}^{n}\bar{b}_i}\},$$

then the zero solution of Eq. (6) is unstable.

Proof. Define the Lyapunov functional $V_1=V_1(x_t,y_t,z_t,w_t,u_t):$

$$V_1=\quad -\frac{1}{2}w^2+y\sum_{i=1}^{n}h_i(x)+zu+z\int_0^w f_5(s)ds$$

$$+\int_0^z f_4(s)sds+\int_0^y f_2(s)ds-\sum_{i=1}^{n}\gamma_i\int_{-\tau_i}^{0}\int_{t+s}^{t}y^2(\theta)d\theta ds, \tag{8}$$

where s is a real variable such that the integrals $\sum\limits_{i=1}^{n}\gamma_i\int\limits_{-\tau_i}^{0}\int\limits_{t+s}^{t}y^2(\theta)d\theta ds$ are non-negative, and γ_i are positive constants which will be determined later in the proof.

Let $M=\max\limits_{|z|\leqslant 1}|f_4(z)|$, there exists a positive constant $e$such that $Me<1$ and $0<e<1$.

Then, it follows that

$$V_1(0,0,e^2,0,e)=e^3+\int_0^{e^2}f_4(s)sds\geqslant e^3-\frac{1}{2}Me^4>0$$

for all sufficiently small e. Hence, in every neighborhood of the origin,$(0,0,0,0,0)$, there exists a point $(0,0,e^2,0,e)$ such that $V_1(0,0,e^2,0,e)>0$.

By an elementary differentiation, time derivative of the functional V_1 in (8) along the solutions of (7) yields

$$\frac{d}{dt}V_1=\sum_{i=1}^{n}h_i'(x)y^2-f_3(x,...,x(t-\tau_n),...,u,...,u(t-\tau_n))z^2$$

$$+w\int_0^w f_5(s)ds+z\sum_{i=1}^{n}\int_{t-\tau_i}^{t}h_i'(x(s))y(s)ds$$

$$-\sum_{i=1}^{n}(\gamma_i\tau_i)y^2+\sum_{i=1}^{n}\gamma_i\int_{t-\tau_i}^{t}y^2(s)ds.$$

The assumptions $\bar{b}_i\geqslant h_i'(x)\geqslant b_i$, $f_3(.)\leqslant -a_3$and the estimate $2\left|mn\right|\leqslant m^2+n^2$ imply that

$$\sum_{i=1}^{n}h_i'(x)y^2\geqslant\sum_{i=1}^{n}b_iy^2,$$

$$-f_3(x,...,u(t-\tau_n))z^2\geqslant a_3z^2,$$

$$z\sum_{i=1}^{n}\int_{t-\tau_i}^{t}h_i'(x(s))y(s)ds\geqslant-\left|z\right|\sum_{i=1}^{n}\int_{t-\tau_i}^{t}\left|h_i'(x(s))\right|\ \left|y(s)\right|ds$$

$$\geqslant-\left|z\right|\sum_{i=1}^{n}\int_{t-\tau_i}^{t}\bar{b}_i\ \left|y(s)\right|ds$$

$$\geqslant-\frac{1}{2}\sum_{i=1}^{n}(\bar{b}_i\tau_i)z^2-\frac{1}{2}\sum_{i=1}^{n}\bar{b}_i\int_{t-\tau_i}^{t}y^2(s)ds$$

so that

$$\frac{d}{dt}V\geqslant\sum_{i=1}^{n}(b_i-\gamma_i\tau_i)y^2+(a_3-2^{-1}\sum_{i=1}^{n}\bar{b}_i\tau_i)z^2$$

$$+\sum_{i=1}^{n}(\gamma_i-2^{-1}\bar{b}_i)\int_{t-\tau_i}^{t}y^2(s)ds.$$

Let $\gamma_i=\frac{1}{2}\bar{b}_i$ and $\tau=\max\tau_i,\ (i=1,2,...,n)$.Hence

$$\frac{d}{dt}V\geqslant\sum_{i=1}^{n}(b_i-2^{-1}\bar{b}_i\tau)y^2+(a_3-2^{-1}\sum_{i=1}^{n}\bar{b}_i\tau)z^2.$$

If

$$\tau < 2\min\left\{\frac{b_i}{\bar{b}_i},\ \frac{a_3}{\sum\limits_{i=1}^{n}\bar{b}_i}\right\},$$

then

$$\frac{d}{dt}V \geqslant \sum_{i=1}^{n}(b_i - 2^{-1}\bar{b}_i\tau)y^2 + (a_3 - 2^{-1}\sum_{i=1}^{n}\bar{b}_i\tau)z^2 > 0.$$

The remainder of the proof follows as before, Theorem 1. □

Example 2. We consider nonlinear differential equation of the fifth order with two deviating arguments,

$$x^{(5)} + \frac{1}{1+(x''')^4}x^{(4)} + x''' - \{3+\exp(-x^2 - x^2(t-\tau_1) - x^2(t-\tau_2)\}x''$$

$$+x'(t) - x(t-\tau_1) - 4arctgx(t-\tau_1)$$

$$-x(t-\tau_2) - arctgx(t-\tau_2) = 0. \tag{9}$$

We write Eq. (9) in system form as follows

$$x' = \quad y, \quad y' = \quad z, z' = \quad w, w' = \quad u,$$

$$u' = \quad -\frac{u}{1+w^4} - w + \{3+\exp(-x^2 - x^2(t-\tau_1) - u^2(t-\tau_2)\}z$$

$$-2y + x + 5arctgx$$

$$-\int_{t-\tau_1}^{t} y(s)ds - 4\int_{t-\tau_1}^{t}\frac{1}{1+x^2(s)}y(s)ds$$

$$-\int_{t-\tau_2}^{t} y(s)ds - \int_{t-\tau_2}^{t}\frac{1}{1+x^2(s)}y(s)ds.$$

It follows that Eq. (9) is a special case of Eq. (2) and

$$f_5(w) = \frac{1}{1+w^4} \geqslant 0,$$

$$f_4(z) = 1,$$

$$f_3(.) = -3 - \exp\{-x^2 - x^2(t-\tau_1) - x^2(t-\tau_2)\} \leqslant -3 = -a_3,$$

$$f_2(y) = y, f_2(0) = 0,$$

$$h_1(x) = x + 4arctgx, -\frac{\pi}{2} < x < \frac{\pi}{2},$$

$$h_1'(x) = 1 + \frac{4}{1+x^2},$$

$$\bar{b}_1 = 5 \geqslant 1 + \frac{4}{1+x^2} \geqslant 1 = b_1,$$

$$h_2(x) = x + arctgx, -\frac{\pi}{2} < x < \frac{\pi}{2},$$

$$h_2'(x) = 1 + \frac{1}{1+x^2},$$

$$\bar{b}_2 = 2 \geqslant 1 + \frac{1}{1+x^2} \geqslant 1 = b_2.$$

In view of the above estimates, we conclude that all the assumptions of Theorem 2 hold. Hence, we conclude that if $\tau < \frac{2}{5}$,then the zero solution of Eq. (9) is unstable.

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The stochastic optimal growth problem

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Abstract. This paper concerns the formulation of economic optimal control problem in a stochastic form. Equilibrium growth rate for this problem was obtained on the base of the stochastic maximum principle following the new approach [1] to the solution of optimal control stochastic problem, in which the stochastic dynamic programming formulation is transformed into formulation of the maximum principle. This approach was applied to the solution of the stochastic optimal growth problem.

Mathematics subject classification: C61, D91.
Keywords and phrases: Optimal stochastic control, Ito's processes, Ito's Lemma, Hamilton-Jacobi-Bellman equation.

1 Problem formulation

Let us consider optimal control problem [2,3] in stochastic formulation

$$\max_{(C,L_Y,L_R)} L \;=\; E\left[\int_0^\infty e^{(\beta_k-\rho)t}\frac{C^{1-\vartheta}}{1-\vartheta}dt\right],$$

subject to

$$\dot{K} = Y - C = K^\alpha A L_Y^{1-\alpha} N^{1-\alpha} - C, \quad K(0) = 0, \tag{1}$$

$$dN = b_1 L_R N dt + g dz, \quad N(0) = 0, \tag{2}$$

$$L_Y + L_R - L = 0, \tag{3}$$

here E is an expectation operator $x = (K, N)$ and $F = e^{(\beta_1-\rho)t}\dfrac{C^{1-\vartheta}}{1-\vartheta}$, the utility function with constant elasticity of substitution ϑ, ρ is the subjective rate of discount, β_k is the subsidy for capital accumulation stimulation. K is the capital stock observed in economic activity, N is the stock of innovation elaborated by $R\&D$ sector, A is the productivity parameter in the final goods production sector, L_Y is the labor force enrolled in the final goods production sector, L_R is the number of employers in $R\&D$ sector, C is the final consumption.

$$dN \;=\; b_1 L_R N dt \;+\; g dz \tag{4}$$

here dz is the stochastic Wiener process, $f = (Y - C, b_1 L_R N)$, $g = \sigma$ is a constant while gdz is normally distributed with mean zero E[gdz]=o, Var(gdz)= $\sigma_z^2 dt$, $dz = \sqrt{dt}$.

2 Problem solution

The respective optimality conditions now are:

$$0 \;=\; \max_{C,L_Y,L_R}\left[F+\frac{E(dL)}{dt}\right] \tag{5}$$

and the corresponding HJB (Hamilton – Jacoby – Bellman) equation becomes:

$$0 \;=\; \max_{C,L_Y,L_R}\left[F+\frac{\partial L}{\partial t}+\frac{\partial L}{\partial x}f+\frac{1}{2}\frac{g^2\partial^2 L}{(\partial x)^2}\right]=$$

$$=\max_{C,L_Y,L_R}\left[F\;+\;L_t+L_x f+\frac{1}{2}g^2L_{xx}\right]. \tag{6}$$

The Hamiltonian function H for the stochastic case is presented below:

$$H \;=\; F\;+\;L_x f+\tfrac{1}{2}g^2L_{xx}=e^{(\beta_1-\rho)t}\frac{C^{1-\vartheta}}{1-\vartheta}+$$

$$+L_{x_1}\left(K^{\alpha}AL_Y^{1-\alpha}N^{1-\alpha}-C\right)+L_{x_2}b_1L_RN+\nu\left(L_Y+L_R-L\right)+\tfrac{1}{2}\sigma^2L_{xx}.$$

Let's mention that the second order term in (6) is explained by the fact that state variable N being an Ito process (Lemma *Ito's*). Taking derivative of the equation (6) with respect to x gives:

$$L_{xt}+F_x+L_{xx}f\;+\;f_xL_x+\frac{1}{2}g^2L_{xxx}+\;\frac{1}{2}\left(g^2\right)_xL_{xx}=\;0 \tag{7}$$

and, therefore,

$$L_{xt}+L_{xx}f+\frac{1}{2}g^2L_{xxx}=\;-F_x-f_xL_x-\frac{1}{2}\left(g^2\right)_xL_{xx}. \tag{8}$$

Applying chain rule and considering second order contribution of the derivatives with respect to x (Lemma *Ito's*), result in:

$$dLx\;=\frac{\partial L_x}{\partial t}dt+\frac{\partial Lx}{\partial x}\frac{dx}{dt}dt+\frac{1}{2}\frac{\partial^2 L_x}{(\partial x)^2}\,dx^2\,.$$

Since, from *Ito's* Lemma $E[d(x^2)]\;=\;g^2\,dt$, the previous equation is reduced to:

$$\frac{dL_x}{dt}=\frac{\partial L_x}{\partial t}+\frac{\partial L_x}{\partial x}\frac{dx}{dt}+\frac{1}{2}\frac{\partial^2 L_x}{(\partial x)^2}\,g^2$$

$$\frac{dL_x}{dt}=L_{xt}+L_{xx}f+\frac{1}{2}L_{xxx}g^2\,. \tag{9}$$

Substituting (8) in (9) we obtain:

$$\frac{dL_x}{dt} = -F_x - f_x L_x - \frac{1}{2}\left(g^2\right)_x L_{xx}\,.$$

Equating adjoint variable μ to the first derivatives of the objective function L with respect to x $\mu_1 = L_{x_1}$, $\mu_2 = L_{x_2}$ and ω to the second derivatives of the objective function with respect to state variables $\omega_1 = L_{x_1x_1}$, the system of ordinary differential equation with respect to state variable is as follows:

$$\frac{dL_{x_1}}{dt} = -F_{x_1} - f_{x_1} L_{x_1} - \frac{1}{2}\left(\sigma^2\right)_{x_1} L_{x_1x_1} \Rightarrow \tag{10}$$

$$\Rightarrow \frac{d\mu_1}{dt} = -\mu_1 \alpha \frac{Y}{K},$$

$$\frac{dL_{x_2}}{dt} = -F_{x_2} - f_{x_2} L_{x_2} - \frac{1}{2}\left(\sigma^2\right)_{x_2} L_{x_2x_2} \Rightarrow \tag{11}$$

$$\Rightarrow \frac{d\mu_2}{dt} = -\mu_1(1-\alpha)\frac{Y}{N} - \mu_2 b_1 L_R\ ,$$

$$\frac{dL_{x_2x_2}}{dt} = -F_{x_2x_2} - 2f_{x_2} L_{x_2x_2} - \frac{1}{2}\left(\sigma^2\right)_{x_2x_2} L_{x_2x_2} \Rightarrow \tag{12}$$

$$\Rightarrow \frac{d\omega}{dt} = -2\omega b_1 L_R - \omega \frac{\sigma^2_{x_2x_2}}{2}\,.$$

From the functional maximization with respect to C, L_Y, L_R we obtain:

$$\mu_1 = C^{-\vartheta} e^{-(\beta_1+\rho)t}\,, \tag{13}$$

$$\upsilon = -\mu_1(1-\alpha)\frac{Y}{N}\,, \tag{14}$$

$$\nu = -\mu_2 b_1 N\,. \tag{15}$$

and the resulting system of conjugate equations becomes:

$$\frac{d\mu_1}{dt} = -\mu_1 \alpha \frac{Y}{K}\,, \tag{16}$$

$$\frac{d\mu_2}{dt} = -\mu_1(1-\alpha)\frac{Y}{N} - \mu_2 b_1 L\,, \tag{17}$$

$$\mu_1 = C^{-\vartheta} e^{-(\beta_1+\rho)t}\,, \tag{18}$$

$$\nu = -\mu_1(1-\alpha)\frac{Y}{N}\,, \tag{19}$$

$$\nu = -\mu_2 b_1 N\,, \tag{20}$$

$$\frac{d\omega}{dt} = -2\omega b_1 L_R - \omega \frac{{\sigma_{x_2x_2}}^2}{2}\,. \tag{21}$$

From the conjugate equations for variables μ we obtain:

$$\mu_1(1-\alpha)\frac{Y}{N} = \mu_2 b_1 N \Rightarrow \frac{\mu_1}{\mu_2} = \frac{b_1 N}{(1-\alpha)\frac{Y}{N}}\,,$$

while from (17) it results

$$\frac{\dot{\mu}_2}{\mu_2} = -\frac{\mu_1}{\mu_2}(1-\alpha)Y/N - b_1 L_R.$$

If in the previous equation to introduce the ratio between variables μ_1 and μ_2 then

$$-\frac{\dot{\mu}_2}{\mu_2} = b_1 L_Y + b_1 L_R = b_1 L.$$

From (16) it becomes $\frac{\dot{\mu}_1}{\mu_1} = -\alpha\frac{Y}{K}$, while in equilibrium the growth rates of the conjugate variables are the same, then $\frac{\dot{\mu}_1}{\mu_1} = \frac{\dot{\mu}_2}{\mu_2}$, and considering [2] that

$$g_{opt} = g_C = \frac{\dot{C}}{C} = \frac{1}{\vartheta}\left(\frac{\alpha K}{Y} + \beta_k - \rho\right),$$

and taking advantage of the last equality, we obtain

$$g_{opt} = \frac{1}{\vartheta}\left(b_1 L + \beta_k - \rho\right).$$

In conclusion, there are the same balanced growth rates of the conjugate variables μ for stochastic problem formulation as for the deterministic problem formulation [2].

3 Mayer form presentation

If the problem is represented in the Mayer linear form, $F = 0$, then:

$$\frac{dL_x}{dt} = -f_x L_x - \frac{1}{2}(g^2)_x L_{xx}\,. \tag{22}$$

Equation (22) describes dynamics of the conjugate variables in the stochastic case. The presence of the second order term in the equation follows from the fact that the state variable is the stochastic variable which is an *Ito* process.

From the equation (22) one concludes that the calculation of L_{xx} is necessary. In order to obtain some expression for the L_{xx} dynamics, the same derivation as earlier will be utilized. Resulting equation will be called a conjugate equation. Let's differentiate again equation (7) with respect to x:

$$L_{xxt} + F_{xx} + L_{xx}f_x + L_{xxx}f + L_{xx}f_x + L_x f_{xx} + \frac{1}{2}g^2 L_{xxxx} +$$

$$+\frac{1}{2}(g^2)_x L_{xxx} + \frac{1}{2}(g^2)_x L_{xxx} + \frac{1}{2}(g^2)_{xx} L_{xx} = 0 \tag{23}$$

and, therefore,

$$L_{xxt} + L_{xxx}f + \frac{1}{2}g^2 L_{xxxx} =$$

$$= -F_{xx} - 2L_{xx}f_x - L_x f_{xx} - (g^2)_x L_{xxx} - \frac{1}{2}\,(g^2)_{xx} L_{xx}\ . \tag{24}$$

Using chain rule and considering the second order contribution with respect to x we obtain:

$$dLxx\ = \frac{\partial L_{xx}}{\partial t}dt + \frac{\partial L_{xx}}{\partial x}\frac{dx}{dt}dt + \frac{1}{2}\frac{\partial^2 L_{xx}}{(\partial x)^2}\ dx^2\ . \tag{25}$$

Using *Ito's* Lemma $E[d(x^2)]\ =\ g^2\ dt$ we obtain:

$$\frac{dL_{xx}}{dt} = \frac{\partial L_{xx}}{\partial t} + \frac{\partial L_{xx}}{\partial x}\ \frac{dx}{dt} + \frac{1}{2}\frac{\partial^2 L_{xx}}{(\partial x)^2}\ g^2,$$

$$\frac{dL_{xx}}{dt} = L_{xxt} + L_{xxx}f + \frac{1}{2}\ L_{xxxx}g^2\ . \tag{26}$$

Inserting equation (24) in equation (26) we obtain:

$$\frac{dL_{xx}}{dt} = -F_{xx} - 2L_{xx}f_x - L_x f_{xx} - \left(g^2\right)_x L_{xxx} - \frac{1}{2}\,(g^2)_{xx} L_{xx}$$

and, finally, if $F = 0$ and the third order contribution is not considered (hypotheses accepted by *Ito*Lemma), it follows:

$$\frac{dL_{xx}}{dt} = -\,2L_{xx}f_x - L_x f_{xx} - \frac{1}{2}(g^2)_{xx} L_{xx}\,. \tag{27}$$

Equating conjugate variable, μ, to the prime derivative from the objective function L with respect to the state variable x and equating conjugate variable ω to the second derivative, we rewrite equations (9) and (27) in the following way:

$$d\mu/dt = -f_x\mu - 1/2(g^2)_x, \tag{28}$$

$$d\omega/dt = -2\omega f_x - \mu f_{xx} - 1/2(g^2)_{xx}\omega\,. \tag{29}$$

Summarizing the results for the stochastic case ($F\ =\ 0$), Hamiltonian function and conjugate equations shall be solved in the stochastic principle maximum formulation:

$$H\ = \mu f\ +\ 1/2g^2\omega\,,$$

$$\frac{d\mu}{dt} = -f_x\mu - \frac{1}{2}(g^2)_x\omega \quad \mu(T\ )\ =\ c\,,$$

$$\frac{d\omega}{dt} = -2\omega f_x - \mu f_{xx} - \frac{1}{2}(g^2)_{xx}\omega \quad \omega(T\ )\ =\ 0\,.$$

It must be mentioned that the resulting problem is two-dimensional.

4 Conclusions

In the present article the solution of the economic optimal control problem in the stochastic formulation is obtained. In order to obtain the solution of this problem the derivation of the respective Hamilton – Jacobi – Bellman equation was applied. This method contributed to obtaining solution in the stochastic maximum principle form containing the first order system of the conjugate differential equations. Note that the growth rate reached in stable condition for the examined problem is the same as for deterministic, with one difference – there is an additional ordinary equation characterizing additional conjugate variable (shadow price of the stochastic restriction). More complete stochastic optimal control problem with the shock above all economy and with the shocks under productivity in intermediate good sectors will be studied in the future.

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Conjugate sets of loops and quasigroups. DC-quasigroups

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Abstract. It is known that the set of conjugates (the conjugate set) of a binary quasigroup can contain 1, 2, 3 or 6 elements. We investigate loops, IP-quasigroups and T-quasigroups with distinct conjugate sets described earlier. We study in more detail the quasigroups all conjugates of which are pairwise distinct (shortly, DC-quasigroups). The criterion of a DC-quasigroup (a DC-IP-quasigroup, a DC-T-quasigroup) is given, the existence of DC-T-quasigroups for any order $n \geq 5$, $n \neq 6$, is proved and some examples of DC-quasigroups are given.

Mathematics subject classification: 20N05, 05B15.
Keywords and phrases: Quasigroup, loop, IP-quasigroup, T-quasigroup, conjugate, parastrophe, identity.

1 Introduction

A quasigroup is an ordered pair (Q, A) where Q is a nonempty set and A is a binary operation defined on Q such that each of the equations $A(a, y) = b$ and $A(x, a) = b$ is uniquely solvable for any pair of elements a, b in Q was established. It is known that the multiplication table of a finite quasigroup defines a Latin square and six (not necessarily distinct) conjugates (or parastrophes) are associated with each quasigroup (Latin square) [1, 6].

In [9] a connection between five identities of two variables and the equality of a quasigroup to some of the rest five its conjugates was established. It was also proved that the number of distinct conjugates of a finite quasigroup can be 1, 2, 3 or 6 and for any $m = 1, 2, 3, 6$ and any $n \geq 4$ there exists a quasigroup of order n with m distinct conjugates (see Theorem 6 of [9]).

In [12] a connection between different pairs of conjugates of a quasigroup was established, four identities that correspond to the equality of a quasigroup to its conjugates were given. It was also proved that any two of these four identities imply the rest two identities. All six possible sets of conjugates taking into account all possible cases of the equality ("assembling") of conjugates were described. The connection between four identities and possible conjugate sets was shown.

In this article we continue the investigation of conjugates of quasigroups started in [12], in particular, we study loops, IP-quasigroups and T-quasigroups with distinct conjugate sets described in [12].

We study in more detail quasigroups and loops all conjugates of which are pairwise distinct (these quasigrops we call distinct conjugate quasigroups or, shortly,

DC-quasigroups). Such quasigroups form an important class and arise by the research of various questions of the quasigroup theory and the Latin square theory, in particular, in the research of totally conjugate-orthogonal [5] and near totally conjugate-orthogonal quasigroups [11]. They can be also used by coding and encryption of information. The criterion of a DC-quasigroup (of a DC-IP-quasigroup, a DC-T-quasigroup) is established, some examples of DC-quasigroups are given and the existence of DC-T-quasigroups of any order $n \geq 5$, $n \neq 6$, is proved.

2 Preliminaries

Remind some necessary notions and results. To any quasigroup (Q, A) the system $\Sigma(A)$ of six (not necessarily distinct) *conjugates (parastrophes)* corresponds:

$$\Sigma(A) = (A, A^{-1}, {}^{-1}A, {}^{-1}\left(A^{-1}\right), ({}^{-1}A)^{-1}, A^*),$$

where $A(x, y) = z \Leftrightarrow A^{-1}(x, z) = y \Leftrightarrow {}^{-1}A(z, y) = x \Leftrightarrow A^*(y, x) = z$.

Using the Belousov's designation of conjugates of a quasigroup (Q, A) from [2] we have the following conjugate system $\Sigma(A)$:

$$\Sigma(A) = (A,\ {}^{r}A,\ {}^{l}A,\ {}^{lr}A,\ {}^{rl}A,\ {}^{s}A),$$

where ${}^{1}A = A$, ${}^{r}A = A^{-1}$, ${}^{l}A = {}^{-1}A$, ${}^{lr}A = {}^{-1}(A^{-1})$, ${}^{rl}A = ({}^{-1}A)^{-1}$, ${}^{s}A = A^*$.

Note that $\left({}^{-1}(A^{-1})\right)^{-1} = {}^{rlr}A = {}^{-1}\left(({}^{-1}A)^{-1}\right) = {}^{lrl}A = {}^{s}A$ and ${}^{rr}A = {}^{ll}A = A$, ${}^{\sigma\tau}A = {}^{\sigma}({}^{\tau}A)$.

Let $\overline{\Sigma}(A)$ be the set of conjugates (*the conjugate set)* of a quasigroup (Q, A). It is known [9] that $|\ \overline{\Sigma}(A)\ | = 1$, 2, 3 or 6.

A quasigroup is a totally-symmetric quasigroup (a TS-quasigroup) if it satisfies the identities $x \cdot xy = y$ and $xy = yx$. For TS-quasigroups $|\ \overline{\Sigma}(A)\ | = 1$.

The following Theorem 1 of [12] describes all possible conjugate sets for quasigroups and points out the only possible variants of equality ("assembling") of conjugates in every case.

Theorem 1 [12]. *The following conjugate sets of a quasigroup (Q, A) are only possible:* $\overline{\Sigma}_1(A) = \{A\}$; $\overline{\Sigma}_2(A) = \{A, {}^{s}A\} = \{A = {}^{lr}A = {}^{rl}A,\ {}^{l}A = {}^{r}A = {}^{s}A\}$; $\overline{\Sigma}_6(A) = \{A,\ {}^{r}A,\ {}^{l}A,\ {}^{lr}A,\ {}^{rl}A,\ {}^{s}A\}$; $\overline{\Sigma}_3(A) = \{A, {}^{lr}A, {}^{rl}A\}$ *and three cases are only possible:*

$\overline{\Sigma}_3^1(A) = \{A = {}^{r}A,\ {}^{l}A = {}^{lr}A,\ {}^{rl}A = {}^{s}A\}$;
$\overline{\Sigma}_3^2(A) = \{A = {}^{l}A,\ {}^{r}A = {}^{rl}A,\ {}^{lr}A = {}^{s}A\}$;
$\overline{\Sigma}_3^3(A) = \{A = {}^{s}A,\ {}^{r}A = {}^{lr}A,\ {}^{l}A = {}^{rl}A\}$.

For convenience we denote the classes of quasigroups (Q, A) with $\overline{\Sigma}(A) = \overline{\Sigma}_1(A), \overline{\Sigma}_2(A), \overline{\Sigma}_3^1(A), \overline{\Sigma}_3^2(A), \overline{\Sigma}_3^3(A), \overline{\Sigma}_6(A)$ by $V_1, V_2, V_3^1, V_3^2, V_3^3, V_6$, respectively.

We say that a quasigroup (Q, A) satisfies exactly one identity of the set of identities $\overline{T} = \{A(x, A(x, y)) = y, A(A(y, x), x) = y, A(x, y) = A(y, x), A(A(x, y), x) = y\}$ if it satisfies one identity and does not satisfy the rest identities of this set.

Remark 1. According to Corollary 4 [12], establishing a connection between conjugate sets described in Theorem 1 and the identities of the set $\overline{T}$ we have that V_1 is the class of quasigroups satisfying all identities of $\overline{T}$; V_2 (V_3^1, V_3^2, V_3^3) is the class of quasigroups satisfying exactly the identity $A(A(x,y),x) = y$ $(A(x, A(x,y)) = y, A(A(y,x),x) = y, A(x,y) = A(y,x)$ respectively) of $\overline{T}$ and V_6 is the class of quasigroups which satisfies none of four identities of $\overline{T}$. For a quasigroup (Q,A) of the class V_1 (of the variety of TS-quasigroups) $\mid \overline{\Sigma}(A) \mid = 1$; for a quasigroup of the class V_2 (every of the classes V_3^1, V_3^2, V_3^3) we have $\mid \overline{\Sigma}(A) \mid = 2$ ($\mid \overline{\Sigma}(A) \mid = 3$ respectively) and $\mid \overline{\Sigma}(A) \mid = 6$ for the class V_6.

Below we study loops, IP-quasigroups and T-quasigroups from the point of view of their conjugate sets.

3 Conjugate sets of loops

Let (Q,A) be a loop with the identity e, $A(I_l x, x) = A(x, I_r x) = e$, that is $I_l x = {}^{-1}x$, $I_r x = x^{-1}$. It is easy to see that if the loop (Q,A) satisfies at least one of the three identities $A(x, A(x,y)) = y$, $A(A(y,x),x) = y$, $A(A(x,y),x) = y$ of the set $\overline{T}$, then it is a loop of exponent two: $A(x,x) = e$ for any $x \in Q$. In this case $I_l = I_r = \varepsilon$.

Proposition 1. *In any of the classes $V_1, V_2, V_3^1, V_3^2, V_3^3, V_6$ of quasigroups there exists a loop of exponent two.*

Proof. Note that if a loop (Q,A) has exponent two , then all its congugates also are loops of exponent two since $L_x^r y = L_x^{-1} y$ and $R_y^l x = R_y^{-1} x$, where $L_x^r y = {}^r A(x,y)$, $R_y^l x = {}^l A(x,y)$, $L_x y = A(x,y)$, $R_y x = A(x,y)$. Any TS-loop is in V_1. The loops of exponent two given by Tables 1–5 are, respectively, in V_2, V_3^1, V_3^2, V_3^3 and V_6:

1	2	3	4	5
2	1	4	5	3
3	5	1	2	4
4	3	5	1	2
5	4	2	3	1

Tab. 1

1	2	3	4	5	6
2	1	5	6	3	4
3	6	1	5	4	2
4	3	2	1	6	5
5	4	6	2	1	3
6	5	4	3	2	1

Tab. 2

1	2	3	4	5	6
2	1	5	6	4	3
3	4	1	5	6	2
4	3	6	1	2	5
5	6	2	3	1	4
6	5	4	2	3	1

Tab. 3

1	2	3	4	5	6
2	1	6	5	3	4
3	6	1	2	4	5
4	5	2	1	6	3
5	3	4	6	1	2
6	4	5	3	2	1

Tab. 4

1	2	3	4	5	6
2	1	4	3	6	5
3	5	1	6	4	2
4	6	5	1	2	3
5	3	6	2	1	4
6	4	2	5	3	1

Tab. 5

Now consider the loops which are not loops of exponent two.

Proposition 2. *Let a loop (Q, A) be not of exponent two, then $(Q, A) \in V_3^3$ if (Q, A) is a commutative loop and $(Q, A) \in V_6$ if (Q, A) is a noncommutative loop.*

Proof. Indeed, in this case $(Q, A) \notin V_1, V_2, V_3^1, V_3^2$ since this loop satisfies none of identities of the set $\overline{T}$ corresponding to these classes. If the loop is commutative, then by Theorem 1 and Remark 1 it is in the class V_3^3. Otherwise it is in V_6. $\square$

4 Conjugate sets of IP-quasigroups

At first we recall that a quasigroup (Q, A) is called a quasigroup with the property of invertibility (an IP-quasigroup) if there exist two mappings I_l and I_r of the set Q into Q such that $A(I_l x, A(x, y)) = y$ and $A(A(y, x), I_r x)) = y$ for all $x, y \in Q$.

It is known that the mappings I_l and I_r are permutations, $I_l^2 = I_r^2 = \varepsilon$ (the identity permutation) and $I_l A(x, y) = A(I_r y, I_r x)$, $I_r A(x, y) = A(I_l y, I_l x)$ [1].

The conjugates of an IP-quasigroup have the following form:

${}^{l}A(x, y) = A(x, I_r y)$, ${}^{r}A(x, y) = A(I_l x, y)$, ${}^{lr}A(x, y) = I_l A(x, I_r y)$,

${}^{rl}A(x, y) = I_r A(I_l x, y)$, ${}^{s}A(x, y) = I_r A(I_l x, I_l y) = I_l A(I_r x, I_r y)$.

By Theorem 1 of [3] all conjugates of an IP-quasigroup are isotopic. Note that in a commutative IP-quasigroup and in an IP-loop $I_r = I_l = I$.

Proposition 3. *Let a quasigroup (Q, A) be a noncommutative IP-quasigroup. Then ${}^{r}A(x, y) = {}^{l}A(x, y)$ if and only if $I_l = I_r = I$ and $IA(x, y) = A(y, x)$.*

Proof. Let ${}^{r}A =^{l} A$, then $I_l \neq \varepsilon$ ($I_r \neq \varepsilon$): by $I_l = \varepsilon$ we have $A(I_l x, y) = A(x, y) = A(x, I_r y)$, then $I_r = \varepsilon$ and (Q, A) is commutative. But in this case from ${}^{r}A(x, y) =^{l} A(x, y)$ it follows $A(I_l x, y) = A(x, I_r y)$, $A(x, y) = A(I_l x, I_r y)$, $I_l A(x, y) = I_l A(I_l x, I_r y) = A(y, I_r I_l x)$, $I_l A(I_l x, y) = A(y, I_r x)$, $I_l I_r A(I_l y, x) = A(y, I_r x)$, $I_l I_r A(I_l y, I_r x) = A(y, x) = A(I_l y, I_r x)$, since $A(x, y) = A(I_l x, I_r y)$, whence it follows that $I_l I_r = \varepsilon$ or $I_l = I_r = I$. Taking into account that $A(y, x) = A(I_l y, I_r x)$ we obtain $IA(x, y) = A(y, x)$.

Conversely, let $I_l = I_r = I$ in a noncommutative IP-quasigroup (Q, A) and $IA(x, y) = A(y, x)$, then $A(x, y) = A(Ix, Iy)$, $A(Ix, y) = A(x, Iy)$, that is ${}^{r}A(x, y) = {}^{l}A(x, y)$. $\square$

Now we consider IP-quasigroups from the point of view of their affiliation to the classes of quasigroups $V_1, V_2, V_3^1, V_3^2, V_3^3$ and V_6.

Theorem 2. *Let a quasigroup (Q, A) be an IP-quasigroup with $I_l = I_r = I$. Then $(Q, A) \in V_1$ if and only if $I = \varepsilon$; $(Q, A) \in V_3^3$ if and only if (Q, A) is commutative and $I \neq \varepsilon$; $(Q, A) \in V_2$ if and only if (Q, A) is noncommutative and $IA(x, y) = A(y, x)$; $(Q, A) \in V_6$ if and only if (Q, A) is noncommutative and $IA(x, y) \neq A(y, x)$.*

Proof. If $I_l = I_r = I = \varepsilon$, then all conjugates coincide and $(Q, A) \in V_1$. The converse is also true. If $I \neq \varepsilon$ and (Q, A) is commutative, then $A =^{s} A$, $A \neq^{l} A$, $A \neq^{r} A$, so by Theorem 1 $(Q, A) \in V_3^3$. The converse follows from Theorem 1.

Let (Q, A) be a noncommutative IP-quasigroup. If $IA(x, y) = A(y, x)$ (in this case $I \neq \varepsilon$), then $A \neq^{s} A$, $A \neq^{l} A$, $A \neq^{r} A$ and by Proposition 3 ${}^{r}A =^{l} A$, so by

Theorem 1 $(Q,A) \in V_2$. If $(Q,A) \in V_2$, then by Theorem 1 the quasigroup (Q,A) is noncommutative and ${}^rA = {}^lA$, so by Proposition 3 $IA(x,y) = A(y,x)$.

If $IA(x,y) \neq A(y,x)$ and (Q,A) is a noncommutative quasigroup, then $A \neq^s A$, $A \neq^l A$, $A \neq^r A$ and by Proposition 3, ${}^rA \neq^l A$. It means that by Theorem 1 the quasigroup (Q,A) is contained in V_6.

If a quasigroup (Q,A) is contained in V_6, then it is noncommutative and ${}^rA \neq^l A$, so by Proposition 3 $IA(x,y) \neq A(y,x)$ (since in this case $I_l = I_r = I$). $\square$

Note that by Theorem 2 of [3] all conjugates of an IP-quasigroup (Q,A) are also IP-quasigroups if and only if there exists a permutation α such that $\alpha A(x,y) = A(y,x)$, so in the cases $(Q,A) \in V_1$, $(Q,A) \in V_2$ and $(Q,A) \in V_3^3$ conjugates of (Q,A) are IP-quasigroups.

Recall that a Moufang loop is defined by the identity $x(y \cdot xz) = (xy \cdot x)z$ and is a special case of IP-loops. From Theorem 2 and Proposition 2 the following corollaries easy follow.

Corollary 1. *Let (Q,A) be an IP-loop (a Moufang loop), then*
$(Q,A) \in V_1$ *if* $I = \varepsilon$;
$(Q,A) \in V_3^3$ *if* (Q,A) *is commutative and* $I \neq \varepsilon$;
$(Q,A) \in V_6$, *if* (Q,A) *is noncommutative.*

Note that the case $(Q,A) \in V_2$ of Theorem 2 for an IP-loop is impossible.

Corollary 2. *All abelian groups of exponent 2 are contained in the class V_1, the rest abelian group are contained in the class V_3^3. Non-abelian groups are in V_6.*

Theorem 3. *Let a quasigroup (Q,A) be an IP-quasigroup with $I_l \neq I_r$. Then*
$(Q,A) \in V_3^1$ *if and only if* $I_l = \varepsilon$.
$(Q,A) \in V_3^2$ *if and only if* $I_r = \varepsilon$.
$(Q,A) \in V_6$ *if and only if* $I_l, I_r \neq \varepsilon$.

Proof. In this case a quasigroup (Q,A) is noncommutative. If $I_l = \varepsilon$ $(I_r = \varepsilon)$ and $I_l \neq I_r$, then $A \neq {}^sA$, $A \neq {}^{lr}A$, $A \neq {}^lA$, and $A = {}^rA$ $(A \neq {}^sA$, $A \neq {}^{rl}A$, $A \neq {}^rA$ and $A = {}^lA$), so $(Q,A) \in V_3^1$ ($(Q,A) \in V_3^2$, respectively). The converse follows from Theorem 1 since then $A = {}^rA$ $(A = {}^lA)$, that is $I_l = \varepsilon$ $(I_r = \varepsilon)$. If $I_l, I_r \neq \varepsilon$ and $I_l \neq I_r$ we have $A \neq^s A$, $A \neq^l A$, $A \neq^r A$ and by Proposition 3 ${}^rA \neq^l A$, so $(Q,A) \in V_6$ according to Theorem 1. If $(Q,A) \in V_6$, then $A \neq^l A$ and $A \neq^r A$, so $I_l, I_r \neq \varepsilon$. $\square$

Example 1. In [1], p. 74, the following example of IP-quasigroup with $I_l \neq I_r$ is given. Let $(Q,\cdot)$ be a group with the identity e, θ be its automorphism of order two, (Q,A) be the quasigroup where $A(x,y) = \theta x \cdot y$. Then $(M,\circ) = (Q,\cdot) \times (Q,A)$ is an IP-quasigroup with $I_l(a,b) = (a^{-1}, b^{-1})$, $I_r(a,b) = (a^{-1}, \theta b^{-1})$, where $a \cdot a^{-1} = e$. In this quasigroup $I_l \neq I_r$ and $I_l, I_r \neq \varepsilon$ if $(Q,\cdot)$ has not exponent two, so by Theorem 3 $(M,\circ)$ is in V_6. If $(Q,\cdot)$ is a group of exponent two, then $I_l = \varepsilon$ and by Theorem 3 $M(\circ) \in V_3^1$.

Let in this example $A(x,y) = x \cdot \theta y$, $(M,\circ) = (Q,A) \times (Q,\cdot)$, $I_r(a,b) = (a^{-1}, b^{-1})$, $I_l(a,b) = (\theta a^{-1}, b^{-1})$, then

$((a,b)\circ(c,d))\circ I_r(c,d) = (a\cdot\theta c, bd)\circ(c^{-1},d^{-1}) = (a\cdot\theta c\cdot\theta c^{-1}, bd\cdot d^{-1}) = (a,b)$,
$I_l(a,b)\circ((a,b)\circ(c,d)) = (\theta a^{-1}, b^{-1})\circ(a\cdot\theta c, bd) = (\theta a^{-1}\cdot\theta a\cdot\theta^2 c, b^{-1}\cdot bd)) = (c,d)$.

Thus, $(M,\circ)$ is also an IP-quasigroup with $I_l \neq I_r$.

If the group $(Q,\cdot)$ has not exponent two, then the IP-quasigroup $(M,\circ)$ is in V_6, since $I_l, I_r \neq \varepsilon$. If the group $(Q,\cdot)$ is a group of exponent two, then $I_r = \varepsilon$, so by Theorem 3 $(M,\circ) \in V_3^2$.

5 Conjugate sets of T-quasigroups

A quasigroup (Q,A) is *a T-quasigroup* if there exist an abelian group $(Q,+)$, its automorphisms φ,ψ and an element $c\in Q$ such that $A(x,y) = \varphi x + \psi y + c$ for any $x,y\in Q$ [8].

The conjugates of a T-quasigroup $A(x,y) = \varphi x + \psi y + c$ (which are also T-quasigroups) have the following form:

${}^{s}A(x,y) = \psi x + \varphi y + c$, ${}^{r}A(x,y) = \psi^{-1}(y - \varphi x - c)$,
${}^{l}A(x,y) = \varphi^{-1}(x - \psi y - c)$, ${}^{rl}A(x,y) = \psi^{-1}(x - \varphi y - c)$,
${}^{lr}A(x,y) = \varphi^{-1}(y - \psi x - c)$ (see, for example, [10]).

Let $Ix = -x$, then $I^2 = \varepsilon$ where ε is the identity transformation, and $I\varphi = \varphi I$ for any automorphism φ of a group $(Q,+)$.

By Proposition 1 of [12] all pairs of conjugates of the conjugate system $\Sigma(A)$ of a quasigroup (Q,A) can be divided into four disjoint classes:

I. $(A,{}^{r}A)$, $({}^{l}A,{}^{lr}A)$, $({}^{rl}A,{}^{s}A)$;
II. $(A,{}^{l}A)$, $({}^{r}A,{}^{rl}A)$, $({}^{s}A,{}^{lr}A)$;
III. $(A,{}^{s}A)$, $({}^{r}A,{}^{lr}A)$, $({}^{l}A,{}^{rl}A)$;
IV. $({}^{l}A,{}^{r}A)$, $(A,{}^{lr}A)$, $({}^{r}A,{}^{s}A)$, $({}^{lr}A,{}^{rl}A)$, $(A,{}^{rl}A)$, $({}^{l}A,{}^{s}A)$

such that the equality (inequality) of components of one pair in a class implies the equality (inequality) of components of any pair in this class.

For T-quasigroups the following (Theorem 2 of [12]) was proved:

Theorem 4 [12]. *The components of any pair of a class I, II, III or IV for a T-quasigroup (Q,A): $A(x,y) = \varphi x + \psi y$ coincide if and only if $\psi = I$ for the pairs of class I; $\varphi = I$ for the pairs of class II; $\varphi = \psi$ for the pairs of class III; $\varphi^2 = I\psi$ and $\psi^2 = I\varphi$ (or $\varphi = \psi^{-1}$ and $\varphi^3 = I$) for the pairs of class IV.*

Note that in [12] the equivalence of the pair of equalities $\varphi^2 = I\psi$ and $\psi^2 = I\varphi$ to the pair of equalities $\varphi = \psi^{-1}$ and $\varphi^3 = I$ was proved.

Now we shall describe T-quasigroups with distinct conjugate sets.

Theorem 5. *Let (Q,A) be a T-quasigroup: $A(x,y) = \varphi x + \psi y$. Then*
$(Q,A) \in V_1$ if and only if $\varphi = \psi = I$;
$(Q,A) \in V_2$ if and only if $\varphi^3 = I, \varphi = \psi^{-1}$, $\varphi \neq I, \psi$;
$(Q,A) \in V_3^1$ if and only if $\psi = I, \varphi \neq I$;
$(Q,A) \in V_3^2$ if and only if $\varphi = I, \psi \neq I$;

$(Q,A) \in V_3^3$ if and only if $\varphi = \psi, \varphi \neq I$, and at least one of two inequalities $\varphi \neq \psi^{-1}$, $\varphi^3 \neq I$ is fulfilled;

$(Q,A) \in V_6$ if and only if $\varphi, \psi \neq I$, $\varphi \neq \psi$ and at least one of two inequalities $\varphi \neq \psi^{-1}$, $\varphi^3 \neq I$ and at least one of two inequalities $\varphi^2 \neq I\psi$ or $\psi^2 \neq I\varphi$ is fulfilled.

Proof. The first statement is easy checked if to take into account the definition of a TS-quasigroup.

Let $\varphi^3 = I, \varphi = \psi^{-1}$ and $\varphi \neq I, \psi$. In this case we have $\psi \neq I$ and $\varphi \neq \psi$, so by Proposition 1 of [12] and Theorem 4 $A = {}^{lr}A = {}^{rl}A$, ${}^{l}A = {}^{r}A = {}^{s}A$ (these equalities correspond to the pairs of class IV), $A \neq {}^{r}A$, $A \neq {}^{l}A$ and $A \neq {}^{s}A$. Thus, in the set $\overline{\Sigma}(A)$ there are exactly two conjugates and $(Q,A) \in V_2$. The converse follows from the form of $\overline{\Sigma}_2(A)$ for V_2 in Theorem 1 and from Theorem 4 since in this case $A = {}^{lr}A = {}^{rl}A$, ${}^{l}A = {}^{r}A = {}^{s}A$, moreover, $A \neq {}^{r}A$, $A \neq {}^{l}A$, $A \neq {}^{s}A$, since $\overline{\Sigma}_2(A)$ contains two elements.

Let $\psi = I, \varphi \neq I$, then $\varphi \neq \psi, \psi^{-1}$, so by Theorem 4 and Theorem 1 we have the set $\overline{\Sigma}_3^1(A)$, as $A = {}^{r}A$, $A \neq {}^{l}A$, $A \neq {}^{s}A$ and ${}^{l}A \neq {}^{r}A$. The converse follows from the form of $\overline{\Sigma}_3^1(A)$ in Theorem 1 as in this case $A = {}^{r}A$, $A \neq {}^{l}A$, $A \neq {}^{s}A$ and so by Theorem 4 $\psi = I, \varphi \neq I$ and $\varphi \neq \psi$ whence $\varphi \neq \psi^{-1}$.

The case of $\overline{\Sigma}_3^2(A)$ is proved analogously. Let $\varphi = \psi, \varphi \neq I$ and at least one of two inequalities $\varphi \neq \psi^{-1}$, $\varphi^3 \neq I$ be fulfilled, then $\psi \neq I$, so by Theorem 4 and Theorem 1 we have the set $\overline{\Sigma}_3^3(A)$, as $A = {}^{s}A$, $A \neq {}^{l}A$, $A \neq {}^{r}A$ and ${}^{l}A \neq {}^{r}A$. The converse follows from the form of $\overline{\Sigma}_3^3(A)$ in Theorem 1 and from Theorem 4.

Let $\varphi, \psi \neq I$, $\varphi \neq \psi$ and at least one of two equalities $\varphi = \psi^{-1}$, $\varphi^3 = I$ be not fulfilled. Then the quasigroup (Q,A) satisfies none of conditions of Theorem 4, so all conjugates of this quasigroup are distinct and $\overline{\Sigma}(A) = \overline{\Sigma}_6(A)$. Conversely, if all conjugates of a quasigroup (Q,A) are different, then by Theorem 4 in (Q,A) $\varphi, \psi \neq I$, $\varphi \neq \psi$ and at least one of two equalities of $\varphi = \psi^{-1}$, $\varphi^3 = I$ is not fulfilled. □

6 *DC*-quasigroups

Consider in more detail the class of quasigroups all six conjugates of which are distinct.

Definition 1. *A quasigroup is called a distinct conjugate quasigroup or, shortly, a DC-quasigroup if all its conjugates are distinct, that is $\mid \overline{\Sigma} \mid = 6$.*

All DC-quasigroups form the class V_6.

Theorem 6. *A quasigroup (Q,A) is a DC-quasigroup if and only if $A \neq {}^{r}A, {}^{l}A, {}^{s}A, {}^{lr}A$. A quasigroup (Q,A) is a DC-quasigroup if and only if it satisfies none of four identities of the set $\overline{T}$.*

Proof. Indeed, by Proposition 1 of [12]

if $A \neq^r A$, then ${}^lA \neq^{lr} A$ and ${}^{rl}A \neq^s A$;

if $A \neq^l A$, then ${}^rA \neq^{rl} A$ and ${}^sA \neq^{lr} A$;

if $A \neq^s A$, then ${}^rA \neq^{lr} A$ and ${}^lA \neq^{rl} A$;

if $A \neq^{lr} A$, then ${}^lA \neq^r A$, ${}^rA \neq^s A$, ${}^{lr}A \neq^{rl} A$, $A \neq^{rl} A$ and ${}^lA \neq^s A$

since the corresponding pairs of conjugates coincide simultaneously. □

Let (Q, A) be a DC-quasigroup, $A = {}^{\varepsilon}A$ where ε is the identity transformation, $C = \{\varepsilon, r, l, rl, lr, s\}$ be the set of six conjugations, as transformations of a quasigroup (Q, A). On the set C we shall define the operation $(\cdot)$, corresponding to the passage from one conjugate of a quasigroup to another one, taking into account that the multiplication is realized from the right to the left.

We obtain the group $C(\cdot)$ which is isomorphic to the symmetric group S_3 (see [1]). The multiplication table of the group $C(\cdot)$ is the following:

$\cdot$	ε	r	l	rl	lr	s
ε	ε	r	l	rl	lr	s
r	r	ε	rl	l	s	lr
l	l	lr	ε	s	r	rl
rl	rl	s	r	lr	ε	l
lr	lr	l	s	ε	rl	r
s	s	rl	lr	r	l	ε

Tab. 6

In this table rs means that at first s then r are applied, so $rs = rrlr = lr$, and $sr = rlrr = rl$.

The following statement gives some properties of DC-quasigroups.

Proposition 4. *For a DC-quasigroup the group $C(\cdot)$ is isomorphic to the symmetric group S_3.*

Any DC-quasigroup is noncommutative and nontrivial.

Any conjugate of a DC-quasigroup is a DC-quasigroup.

Any quasigroup containing a DC-subquasigroup is a DC-quasigroup.

The direct product of DC-quasigroups is a DC-quasigroup.

The direct product of a TS-quasigroup and a DC-quasigroup is a DC-quasigroup.

The direct product of two quasigroups from distinct classes of $V_2, V_3^1, V_3^2, V_3^3, V_6$ is a DC-quasigroup.

A nontrivial quasigroup which is a homomorphic image of a DC-quasigroup is not necessarily a DC-quasigroup.

Proof. The results follow from the definitions, Theorem 6, the characterization of the classes $V_1, V_2, V_3^1, V_3^2, V_3^3, V_6$ using the identities of the set $\overline{T}$ (see Remark 1) and taking into account that if a quasigroup satisfies an identity, then this identity holds in any its subquasigroup. The last statement is true since, for example, the non-abelian group S_3 which by Corollary 2 is a DC-group has a homomorphic group of order two, which is contained in the class V_1. □

By Theorem 6 of [9] for any $m = 1, 2, 3, 6$ and any $n \geq 4$ there exists a quasigroup of order n with m distinct conjugates. The proof of this theorem for a quasigroup (Q, A) with $\mid \overline{\Sigma}(A) \mid = 6$ is based on the existence of a quasigroup of order 3 satisfying none of the identities in the set T. But it is easy to check that such quasigroups do not exist, since six of 12 quasigroups of order 3 are commutative and every of the remaining six quasigroups coincide with the left or the right inverse quasigroup. So below we shall bring in small correction in the proof for the case of quasigroups with $\mid \overline{\Sigma}(A) \mid = 6$ using the idea of embedding used in the proof of Theorem 6 [9].

Theorem 7. *For every $n \geq 4$ there exists a DC-quasigroup of order n.*

Proof. It is easy to check that, for example, the quasigroup (Q, A) of order 4 with the following multiplication table:

A	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	4	3	2	1
4	3	1	4	2

Tab. 7

is a DC-quasigroup. In [7] Trevor Evans has shown that a quasigroup of order n can be embedded in a quasigroup of order t for every $t \geq 2n$. Using the quasigroup of order 4, given above, for embedding we obtain a DC-quasigroup of any order $n \geq 8$ by Proposition 4. The existence of DC-quasigroups of order $n = 5, 7$ follows, for example, from Theorem 8 below (for $n = 5$ see also Example 2 in the end). By Corollary 2 the noncommutative group S_3 of order $n = 6$ is a DC-group. □

Summarizing the above results we have the following DC-loops and DC-quasigroups.

Proposition 5. *A noncommutative loop (Q, A) which is not of exponent two is a DC-loop.*

A noncommutative IP-quasigroup (Q, A) with $I_l = I_r = I$ and $IA(x, y) \neq A(y, x)$ is a DC-quasigroup.

A noncommutative IP-loop (a noncommutative Moufang loop, a non-abelian group) is a DC-loop.

A noncommutative IP-quasigroup with $I_l \neq I_r$ and $I_l, I_r \neq \varepsilon$ is a DC-quasigroup.

A T-quasigroup (Q, A): $A(x, y) = \varphi x + \psi y$ such that $\varphi \neq I, \psi$; $\psi \neq I$ and $\varphi^2 \neq I\psi$ or $\psi^2 \neq I\varphi$ (and $\varphi \neq \psi^{-1}$ or $\varphi^3 \neq I$) is a DC-quasigroup.

Denote by s_n the number of DC-groups of order n, then using Fig. 4.3.4 of [6] with the number of all non-abelian groups of order $n < 32$ we get that $s_6 = s_{10} = s_{14} = s_{21} = s_{22} = s_{26} = s_{27} = 1$; $s_8 = s_{20} = s_{24} = s_{28} = 2$; $s_{12} = s_{18} = s_{30} = 3$; $s_{16} = 9$.

The criterion of Theorem 5 for a DC-T-quasigroup can be reformulated in the following way.

Corollary 3. *A T-quasigroup (Q, A): $A(x, y) = \varphi x + \psi y$ is a DC-quasigroup if and only if $\varphi + \varepsilon \neq \overline{0}$, $\psi + \varepsilon \neq \overline{0}$, $\varphi - \psi \neq \overline{0}$ and $\varphi^2 + \psi \neq \overline{0}$ or $\psi^2 + \varphi \neq \overline{0}$, where $\overline{0}$ is the endomorphism zero of the abelian group $(Q, +)$.*

Indeed, for example, the inequality $\varphi \neq I$ means that $\varphi x_0 \neq I x_0$ for some $x_0 \in Q$, $x_0 \neq 0$, that is $(\varphi x_0 + x_0) \neq 0$, $(\varphi + \varepsilon)x_0 \neq 0$ and $\varphi + \varepsilon \neq \overline{0}$.

An operation A of the form $A(x, y) = ax + by \pmod n$, $n \geq 3$, $a, b \neq 0$, is a T-quasigroup if and only if the numbers a, b modulo n are relatively prime to n. In this case $\varphi = L_a$, $\psi = L_b$, where $L_a x = ax \pmod n$, $x \in Q = \{0, 1, 2, ..., n-1\}$, are permutations (automorphisms of the additive group modulo n). Note that since the elements a, b modulo n are relatively prime to n, then they are invertible and belong to the multiplicative group of the residue-class ring $\pmod n$. This multiplicative group consists of all numbers from 1 to $n-1$ relatively prime to n. In this case $L_a^{-1} x = L_{a^{-1}} x \pmod n$. Taking into account that $I = L_{n-1}$ for such quasigroups we have

Corollary 4. *A T-quasigroup (Q, A): $A(x, y) = ax + by \pmod n$ is a DC-quasigroup if and only if $a, b \neq n-1$, $a \neq b$ and $a \neq b^{-1}$ or $a^3 \neq n-1$ (mod n).*

The following theorem gives some information about the spectrum of DC-T-quasigroups.

Theorem 8. *For any $n \geq 5$, $n \neq 6$, there exists a DC-T-quasigroup of order n.*

Proof. Consider a T-quasigroup (Q, A) with $A(x, y) = x + ky \pmod n$ of order n, $n \geq 5$, $n \neq 6$, such that the greatest common divisor (n, k) is equal to 1 (that is $(n, k) = 1$), $k \neq 1, n-1$, where $1 \cdot x = x \pmod n$. It is easy to see that for any finite $n \geq 5$, $n \neq 6$ the required number k exists. For this quasigroup $a = 1$, $b = k \pmod n$. Check the conditions of Corollary 4: $1, k \neq n-1 \pmod n$, $k \neq 1$ and $1 \neq k^{-1} \pmod n$. Thus, by Corollary 4 all conjugates of the quasigroup (Q, A) are different and it is a DC-T-quasigroup. □

Note that among T-quasigroups (Q, A): $A(x, y) = ax + by \pmod 4$ or $A(x, y) = ax + by \pmod 6$ there are not DC-quasigroups. That follows if we take into account Corollary 4 and that the numbers a, b modulo n are relatively prime to n.

Example 2. Find the conjugates of the DC-T-quasigroup (Q, A) with $A(x, y) = x + 2y \pmod 5$ of order 5, taking into account the form of conjugates of a T-quasigroup:

${}^{s}A(x, y) = \psi x + \varphi y = 2x + y \pmod 5$,

${}^{r}A(x, y) = \psi^{-1}(y - \varphi x) = L_{2^{-1}}(y - x) = 3y - 3x \pmod 5 = 2x + 3y \pmod 5$,

${}^{l}A(x, y) = \varphi^{-1}(x - \psi y) = x - 2y \pmod 5 = x + 3y \pmod 5$,

${}^{rl}A(x, y) = \psi^{-1}(x - \varphi y) = L_{2^{-1}} x - L_{2^{-1}} y = 3x - 3y \pmod 5 = 3x + 2y \pmod 5$,

${}^{lr}A(x, y) = \varphi^{-1}(y - \psi x) = -2x + y \pmod 5 = 3x + y \pmod 5$.

Recall that a quasigroup (Q, A) is called *totally conjugate orthogonal (near totally conjugate orthogonal), shortly, a totCO-quasigroup* [5] *(near totCO-quasigroup,*

respectively [11]) if all six its conjugates (five of its conjugates) are pairwise orthogonal. It is evident that these quasigroups are DC-quasigroups if to take into account that in an orthogonal system all quasigroups are different. In [5] it was proved that for any number n which is relatively prime to 2, 3, 5 and 7 there exists a $totCO$-quasigroup (moreover, a T-quasigroup) of order n.

Note that loops (moreover, quasigroups with right or left identity) and IP-quasigroups can not be $totCO$-quasigroups. That follows, for example, from Proposition 3 of [4] where impossibility of orthogonality of some conjugates for these quasigroups is proved.

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Center problem for cubic systems with a bundle of two invariant straight lines and one invariant conic

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Abstract. For cubic differential systems with a bundle of two invariant straight lines and one invariant conic it is proved that a weak focus is a center if and only if the first four Liapunov quantities $L_j,\ j=\overline{1,4}$, vanish.

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1 Introduction

In this paper we consider the cubic system of differential equations

$$\begin{aligned}\dot{x} &= y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x,y),\\ \dot{y} &= -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x,y),\end{aligned} \tag{1}$$

in which all variables and coefficients are assumed to be real. The origin $O(0,0)$ is a singular point of a center or a focus type for (1), i.e. a weak focus. The purpose of this paper is to find verifiable conditions for $O(0,0)$ to be a center.

It is known that the origin is a center for system (1) if and only if it has in some neighborhood of $O(0,0)$ a holomorphic integrating factor of the form

$$\mu = 1 + \sum \mu_j(x,y).$$

There exists a formal power series $F(x,y) = \sum F_j(x,y)$ such that the rate of change of $F(x,y)$ along trajectories of (1) is a linear combination of polynomials $\{(x^2+y^2)^j\}_{j=2}^{\infty}$:

$$\frac{dF}{dt} = \sum_{j=2}^{\infty} L_{j-1}(x^2+y^2)^j.$$

The quantities $L_j,\ j=\overline{1,\infty}$, are polynomials in the coefficients of system (1) and are called the Liapunov quantities. The order of the weak focus $O(0,0)$ is r if $L_1 = L_2 = \ldots = L_{r-1} = 0$ but $L_r \neq 0$.

The origin is a center for (1) if and only if $L_j = 0,\ j=\overline{1,\infty}$. By the Hilbert's basis theorem there exists a natural number N such that the infinite system $L_j = 0,\ j=\overline{1,\infty}$, is equivalent with a finite system $L_j = 0,\ j=\overline{1,N}$. The number N is known only for quadratic systems $N=3$ [13] and for cubic systems with only homogeneous cubic nonlinearities $N=5$ [18, 22]. If the cubic system (1) contains

both quadratic and cubic nonlinearities, the problem of the center was solved only in some particular cases (see for instance [1, 2, 4, 6–12, 15, 16, 19, 20]).

In this paper we solve the problem of the center for cubic differential system (1) assuming that (1) has two invariant straight lines and one invariant conic passing through one singular point, i.e. forming a bundle. The paper is organized as follows. The results concerning relation between integrability, invariant algebraic curves and Liapunov quantities are presented in Section 2. In Section 3 we find seventeen sufficient sets of conditions for the existence of a bundle of two invariant straight lines and one invariant conic. In Section 4 we obtain sufficient conditions for the existence of a center and finally we give the proof of the main result: a weak focus $O(0,0)$ is a center for cubic system (1) with a bundle of two invariant straight lines and one invariant conic if and only if the first four Liapunov quantities vanish.

2 Invariant algebraic curves, Liapunov quantities, center

An algebraic curve $\Phi(x,y)=0$ (real or complex) is said to be an invariant curve of system (1) if there exists a polynomial $K(x,y)$ such that

$$P\frac{\partial\Phi}{\partial x}+Q\frac{\partial\Phi}{\partial y}=\Phi K.$$

The polynomial K is called the cofactor of the invariant algebraic curve $\Phi=0$. We shall consider only algebraic curves $\Phi=0$ with Φ irreducible.

If the cubic system (1) has sufficiently many invariant algebraic curves $\Phi_j(x,y)=0,\ j=1,\dots,q$, then in most cases an integrating factor can be constructed in the Darboux form

$$\mu=\Phi_1^{\alpha_1}\Phi_2^{\alpha_2}\cdots\Phi_q^{\alpha_q}. \tag{2}$$

A function (2), with $\alpha_j\in\mathbb{C}$ not all zero, is an integrating factor for (1) if and only if

$$\sum_{j=1}^{q}\alpha_jK_j\equiv-\frac{\partial P}{\partial x}-\frac{\partial Q}{\partial y}.$$

System (1) is called Darboux integrable if the system has a first integral or an integrating factor of the form (2).

The method of Darboux turns out to be very useful and elegant one to prove integrability for some classes of systems depending on parameters. These last years, interesting results which relate algebraic solutions, Liapunov quantities and Darboux integrability have been published (see, for example, [3, 5, 6, 9–12, 17, 21]). The cubic systems (1) which are Darboux integrable have a center at $O(0,0)$.

Definition 1. We shall say that $(\Phi_j,\ j=\overline{1,M};\ L=N)$ is *ILC* (I – invariant algebraic curves, L – Liapunov quantities, C – center) for (1) if the existence of M algebraic curves $\Phi_j(x,y)=0$ and the vanishing of the focal values $L_\nu,\ \nu=\overline{1,N}$, implies the origin $O(0,0)$ to be a center for (1).

The works [6–9, 19, 20] are dedicated to the investigation of the problem of the center for cubic differential systems with invariant straight lines. In these papers, the problem of the center was completely solved for cubic systems with at least three invariant straight lines. The principal results of these works are gathered in the following two theorems:

Theorem 1. *$(\Phi_j(x,y),\ \Phi_j(0,0)\neq 0,\ j=\overline{1,4};\ \ L=1)$ is ILC for system (1).*

Theorem 2. *$(a_jx+b_jy+c_j,\ j=\overline{1,4};\ \ L=2)$ and $(a_jx+b_jy+c_j,\ j=\overline{1,3};\ \ L=7)$ are ILC for cubic system (1).*

The problem of the center was solved for cubic systems (1) with two parallel invariant straight lines and one invariant conic [10]; for cubic systems (1) with two homogeneous invariant straight lines and one invariant conic [11] and for a class of cubic systems (1) with a bundle of two invariant straight lines and one invariant conic [12]. The following theorem was proved:

Theorem 3. *$(x\pm iy,\ \Phi;\ \ L=2)$ and $(l_j=1+a_jx+b_jy,\ j=1,2,\ l_1||l_2,\ \Phi;\ \ L=3)$, where $\Phi=0$ is an irreducible invariant conic, are ILC for system (1).*

In this paper we shall prove that $(l_j=1+a_jx-y,\ j=1,2,\ l_1\cap l_2\cap\Phi=(0,1);\ \ L=4)$, where $\Phi=0$ is an irreducible invariant conic, is ILC for system (1).

3 Conditions for the existence of a bundle of two invariant straight lines and one invariant conic

Let the cubic system (1) have two invariant straight lines l_1, l_2 intersecting at a point (x_0,y_0). The intersection point (x_0,y_0) is a singular point for (1) and has real coordinates. By rotating the system of coordinates $(x\to x\cos\varphi-y\sin\varphi, y\to x\sin\varphi+y\cos\varphi)$ and rescaling the axes of coordinates $(x\to\alpha x,\ y\to\alpha y)$, we obtain $l_1\cap l_2=(0,1)$. In this case the invariant straight lines can be written as

$$l_j=1+a_jx-y,\ a_j\in\mathbb{C},\ j=1,2;\ \Delta_{12}=a_2-a_1\neq 0. \tag{3}$$

The straight lines (3) are invariant for (1) if and only if the following coefficient conditions are satisfied:

$$\begin{aligned}
&k=(a-1)(a_1+a_2)+g,\ \ l=-b,\ \ s=(1-a)a_1a_2,\\
&m=-a_1^2-a_1a_2-a_2^2+c(a_1+a_2)-a+d+2,\ \ r=-f-1,\\
&n=a_1a_2(-f-2)-(d+1),\ \ p=(f+2)(a_1+a_2)+b-c,\\
&q=(a_1+a_2-c)a_1a_2-g,\ \ (a-1)^2+(f+2)^2\neq 0.
\end{aligned} \tag{4}$$

Let the conditions (4) be satisfied and assume that $f=-2$ (the case $f+2\neq 0$ was considered in [12]), then $a\neq 1$ and the cubic system (1) looks:

$$\begin{aligned}
\dot x&=y+ax^2+cxy-y^2+[d+2-a-a_1^2-(a_1+a_2)(a_2-c)]x^2y+\\
&\quad[(a-1)(a_1+a_2)+g]x^3+(b-c)xy^2+y^3\equiv P(x,y),\\
\dot y&=-x-gx^2-dxy-by^2+[g+a_1a_2(c-a_1-a_2)]x^2y+\\
&\quad(a-1)a_1a_2x^3+(d+1)xy^2+by^3\equiv Q(x,y).
\end{aligned} \tag{5}$$

Next for cubic system (5) we find conditions for the existence of one invariant conic passing through the same singular point $(0,1)$, i.e. forming a bundle. Let the conic curve be given by the equation

$$\Phi(x,y) \equiv a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + 1 = 0 \tag{6}$$

with $(a_{20}, a_{11}, a_{02}) \neq 0$ and $a_{20}, a_{11}, a_{02}, a_{10}, a_{01} \in \mathbb{R}$.

For every conic curve (6) the following quantities [14]:

$$I_1 = a_{02} + a_{20}, \quad I_2 = (4a_{02}a_{20} - a_{11}^2)/4,$$
$$I_3 = (4a_{02}a_{20} - a_{01}^2 a_{20} + a_{01}a_{10}a_{11} - a_{02}a_{10}^2 - a_{11}^2)/4$$

are invariants with respect to the translation and rotation of axes. These invariants will be taken into account classifying conics. A conic (6) is reducible into two straight lines if and only if $I_3 = 0$. If $I_2 > 0$, then (6) is an ellipse, if $I_2 < 0$ – a hyperbola and if $I_2 = 0$ – a parabola.

In order the conic (6) pass through a singular point $(0,1)$ and form a bundle with the invariant straight lines (3), we shall assume $a_{01} = -a_{02} - 1$. In this case

$$\Phi(x,y) \equiv a_{20}x^2 + a_{11}xy + a_{10}x + (a_{02}y - 1)(y - 1) = 0. \tag{7}$$

The conic (7) is an invariant conic for (5) if and only if there exist numbers c_{20}, c_{11}, c_{02}, c_{10}, $c_{01} \in \mathbb{R}$, where $c_{10} = -a_{01}$, $c_{01} = a_{10}$ such that

$$P(x,y)\frac{\partial \Phi}{\partial x} + Q(x,y)\frac{\partial \Phi}{\partial y} \equiv \Phi(x,y)(c_{20}x^2 + c_{11}xy + c_{02}y^2 + (a_{02}+1)x + a_{10}y). \tag{8}$$

Identifying the coefficients of $x^i y^j$ in (8), we reduce this identity to three systems of equations $\{F_{ij} = 0\}$ for the unknowns a_{20}, a_{11}, a_{02}, a_{10}, c_{20}, c_{11}, c_{02} :

$$\begin{aligned}
F_{40} &\equiv (a-1)(a_1a_2a_{11} + 2a_1a_{20} + 2a_2a_{20}) + a_{20}(2g - c_{20}) = 0,\\
F_{31} &\equiv (a-1)(2a_1a_2a_{02} + a_1a_{11} + a_2a_{11}) - (a_2a_{11} + 2a_{20})a_1^2 -\\
&\quad - (a_1a_{11} + 2a_{20})a_2^2 + (ca_{11} - 2a_{20})a_1a_2 + (2ca_1 + 2ca_2 - 2a -\\
&\quad - c_{11} + 2d + 4)a_{20} + (2g - c_{20})a_{11} = 0,\\
F_{22} &\equiv 2(c - a_1 - a_2)a_1a_2a_{02} + (2g - c_{20})a_{02} + [c(a_1 + a_2) - a_1^2 -\\
&\quad - a_2^2 - a_1a_2 - a - c_{11} + 2d + 3]a_{11} + (2b - 2c - c_{02})a_{20} = 0,\\
F_{13} &\equiv (2 + 2d - c_{11})a_{02} + (2b - c - c_{02})a_{11} + 2a_{20} = 0,\\
F_{04} &\equiv (2b - c_{02})a_{02} + a_{11} = 0,
\end{aligned} \tag{9}$$

$$\begin{aligned}
F_{30} &\equiv (a-1)[(a_1 + a_2)a_{10} - a_1a_2(a_{02} + 1)] - ga_{11} +\\
&\quad + (2a - 1 - a_{02})a_{20} + (g - c_{20})a_{10} = 0,\\
F_{21} &\equiv [g - c_{20} + ca_1a_2 - (a_1 + a_2)a_1a_2](-a_{02} - 1) +\\
&\quad + [c(a_1 + a_2) - a_1^2 - a_1a_2 - a_2^2 - a + d + 2 - c_{11}]a_{10} +\\
&\quad + (2c - a_{10})a_{20} + (a - d + 1 + a_{02})a_{11} - 2ga_{02} = 0,\\
F_{12} &\equiv -(d + 1 - c_{11})(a_{02} + 1) - (a_{02} + 2d + 1)a_{02} +\\
&\quad + (b - c - c_{02})a_{10} + (c - b - a_{10})a_{11} - 2a_{20} = 0,\\
F_{03} &\equiv (b - c_{02})(a_{02} + 1) + (a_{10} + 2b)a_{02} - a_{10} + 2a_{11} = 0,
\end{aligned} \tag{10}$$

$$\begin{aligned} F_{20} &\equiv (a-a_{02}-1)a_{10}+g(a_{02}+1)-a_{11}-c_{20}=0,\\ F_{11} &\equiv (a_{02}+d+1)a_{01}+(a_{10}-c)a_{10}+2a_{02}-2a_{20}+c_{11}=0,\\ F_{02} &\equiv c_{02}-(a_{10}+b)(a_{02}+1)+2a_{10}-a_{11}=0. \end{aligned} \tag{11}$$

Let us denote

$$j_1=(a_1+a_2-c)a_{02}-a_{11},\ \ j_2=a_{02}a_1^2+a_{11}a_1+a_{20},$$
$$j_3=a_{02}a_2^2+a_{11}a_2+a_{20},\ \ j_4=4a_{02}a_{20}-a_{11}^2.$$

We shall study the compatibility of $\{(9),\ (10),\ (11)\}$ when $I_3\neq 0,\ \Delta_{12}\neq 0,\ a\neq 1$ and divide the investigation into five subcases: $\{j_1=0\},\ \{j_1\neq 0, j_2=0\},\ \{j_1j_2\neq 0,\ j_3=0\},\ \{j_1j_2j_3\neq 0, j_4=0\},\ \{j_1j_2j_3j_4\neq 0\}$.

3.1 Case $j_1=0$

In this case $a_{11}=a_{02}(a_1+a_2-c)$. We express c_{02}, c_{11} and c_{20} from (9), then we obtain

$$\begin{aligned} F_{40} &\equiv h_1[a_1a_2(a_1+a_2-c)a_{02}+(3a_1+3a_2-c)a_{20}]=0,\\ F_{31} &\equiv h_1[((a_2-c+3a_1)(2a_2-c)+2a_1^2)a_{02}-2a_{20}]=0, \end{aligned} \tag{12}$$

where $h_1=(a-1)a_{02}+a_{20}$.

3.1.1. $h_1=0$. In this case $a_{20}=(1-a)a_{02}$ and

$$F_{02}\equiv F_{03}=(a_{02}-1)(a_{10}+a_1+a_2+b-c)=0.$$

Let $a_{02}=1$, then express a_{10} from $F_{11}=0$, a_1 from $F_{12}=0$, a_2^2 from $F_{20}+F_{21}=0$ and g from $F_{30}=0$. We obtain the following conditions

1)
$$\begin{aligned} g=&[b^2(4a-d-6)+2bc(1-a)-2a(8a+c^2-4d-24)+\\ &+(c^2-8)(d+4)]/(4b),\ \ 2a(2b+c)-(b+c)(d+4)=0,\\ &2a_2^2+a_2(b-c)-6a+2d+10=0,\ \ a_1=(c-b-2a_2)/2 \end{aligned}$$

for the existence of a conic $2(a-1)x^2+(b+c)xy+(b-c)x-2(y-1)^2=0$.

Assume $a_{02}\neq 1$, then $F_{02}=0$ yields $c=a_1+a_2+a_{10}+b$. Express d and g from (11). If $b=0$, then we obtain the following conditions

2)

$$\begin{aligned} &a=[a_{02}(a_{02}-a_1a_2+1)-a_1a_2+a_{10}(a_1+a_2-a_{10})]/(2a_{02}),\ \ b=0,\\ &c=a_1+a_{10}+a_2,\ \ d=[a_{10}(a_1+a_2-a_{10})-a_1a_2-a_{02}(a_1a_2+2)]/a_{02},\\ &g=[a_{10}^3-3(a_1+a_2)a_{10}^2+a_{10}((2a_1+a_2)(a_1+2a_2)-a_{02}^2+(3a_1a_2+1)a_{02})\\ &\quad+2(a_{02}^2-a_1a_2-(a_1a_2+1)a_{02})(a_1+a_2)]/[2a_{02}(a_{02}-1)] \end{aligned}$$

for the existence of a conic $[a_{02}(a_{02}-a_1a_2-1)-a_1a_2+a_{10}(a_1+a_2-a_{10})]x^2+2(a_{02}y-1)a_{10}x-2(y-1)(a_{02}y-1)=0$.

If $b\neq 0$, then express a from $F_{21}=0$ and we get the following conditions

3)

$$\begin{aligned} &a=[a_{10}(c-3b-2a_{10})+b(c-b)]/(a_{02}-1),\ \ a_1=c-a_{10}-a_2-b,\\ &d=[2a_{10}(c-4b-2a_{10})-a_{02}(a_{02}+2)+3bc-3b^2+3]/(a_{02}-1),\\ &g=[(a_{10}+b)a_1a_2+2(a-1)(a_1+a_2)+(1-a)a_{10}-ba_{02}]/(a_{02}-1),\\ &F_{30}\equiv a_{02}(a_{02}-a_1a_2-1)+a_1a_2+a_{10}(a_{10}-a_1-a_2)=0 \end{aligned}$$

for the existence of a conic

$$[(1-a)x^2-(a_{10}+b)xy+y^2]a_{02}+a_{10}x-(a_{02}+1)y+1=0.$$

3.1.2. $h_1 \neq 0$. In this case we express a_{20} from $F_{31}=0$ of (12) and obtain

$$F_{40} \equiv g_1g_2g_3=0,$$

where $g_1=a_1+3a_2-c,\ g_2=a_2+3a_1-c$ and $g_3=2a_1+2a_2-c$.

3.1.2.1. $g_1=0$. If $a_{02}=1$, then we obtain the following conditions

4)

$$a=2,\ d=(-3b^2-4bc-c^2-16)/8,\ g=-b,\ a_1=(c-3b)/4,\ a_2=(b+c)/4.$$

The invariant conic is $(b+c)^2x^2-8(b+c)xy-8(b-c)x+16(y-1)^2=0$.

If $a_{02}\neq 1$, then from $F_{02}=F_{03}=0$ we have $a_{10}=2a_2-b$. We express g from $F_{20}=0$, c from $F_{30}=0$ and reduce the equations $\{F_{21}=0, F_{12}=0\}$ by d from $F_{11}=0$, then $F_{12}\equiv 0$ and $F_{21}\equiv b(a-2)I_3=0$.

If $b=0$, then $F_{11}=0$ yields $a_{02}=-(1+d+2a_2^2)$ and we get the following conditions

5)

$$b=g=0,\quad c=[a_2(2a_2^2+5a+d-4)]/(a-1),\quad a_1=c-3a_2$$

for the existence of a conic

$$a_2(1+d+2a_2^2)(a_2x-2y)x-2a_2x+(1+dy+y+2a_2^2y)(y-1)=0.$$

If $b\neq 0$ and $a=2$, then $F_{11}=0$ yields $a_{02}=-(1+d+ba_2+2a_2^2)$ and we obtain the following conditions

6)

$$a=2,\ c=2a_2^3+ba_2^2+(d+6)a_2-b,\ g=-b,\ a_1=c-3a_2$$

for the existence of a conic

$$(2a_2^2+ba_2+d+1)(a_2x-y)^2+(b-2a_2)x-(2a_2^2+ba_2+d)y-1=0.$$

3.1.2.2. The case $g_2=0$ can be reduced to $g_1=0$ if we replace a_2 by a_1.

3.1.2.3. $g_1g_2\neq 0, g_3=0$. If $a_{02}=1$, then $F_{11}=I_3\neq 0$. Let $a_{02}\neq 1$. In this case we express a_{10} from $F_{02}=F_{03}=0$, g from $F_{20}=0$, a_{02} from $F_{30}=0$, d from $F_{12}=0$ and b from $F_{21}=0$. We obtain

7)

$$b=g=0,\quad c=2(a_1+a_2),\quad d=-(a+2a_1a_2).$$

The invariant conic is

$$a_1a_2(a-1)x^2-(ay-y-1)(a_1x+a_2x-y+1)=0.$$

3.2 Case $j_1\neq 0, j_2=0$

In this case $a_{20}=-a_1(a_{02}a_1+a_{11})$. If $a_{02}=0$, then $F_{04}=j_1\neq 0$. Assume $a_{02}\neq 0$ and express c_{02}, c_{11} and c_{20} from (9), then we obtain

$$F_{40}\equiv F_{31}=u_1u_2u_3=0,$$

where $u_1=2a_{02}a_1+a_{11},\ u_2=a_{02}(a_1+a_2)+a_{11},\ u_3=(a_1a_2-ca_1+a-1)a_{02}^2+(a_2-a_1-c)a_{02}a_{11}-a_{11}^2$.

3.2.1. $u_1=0$, i.e. $a_{11}=-2a_{02}a_1$. If $a_{02}=1$, then $F_{02}\equiv F_{03}\equiv 0$. We express a and a_{10} from (11); a_1, a_2 and d from (10). In this case we get the following conditions

8)
$$a = 2,\ d = [2cg - (b+c)^2 - 2(4g^2 + 3bg + 8)]/8,$$
$$a_1 = (b+c)/4,\ a_2 = (c + 4g + b)/4$$

for the existence of a hyperbola
$$(b+c)^2x^2 - 8(b+c)xy + 8(c-b)x + 16(y-1)^2 = 0.$$

If $a_{02} \neq 1$, then $F_{02} \equiv F_{03} = 0$ yields $a_{10} = 2a_1 - b$. Reduce the equations of (10) by g from $F_{20} = 0$ and express d from $F_{11} \equiv F_{12} = 0$, a_2 from $F_{30} = 0$ and c from $F_{21} = 0$. Then we obtain the following conditions

9)
$$c = [a_1(b - bh + 5hv - v) + v(v - b - hv)]/(hv),\quad h = a - 1,$$
$$d = [a_1v(v-b) - 2a_1^2hv - bh^2 + bh + bv^2 - 2hv - v^3]/(hv),$$
$$a_2 = [(ab + 2ag - b - 3g)a_1 + gv]/(hv),\ v = b + g$$

for the existence of a hyperbola
$$((2a_1 - b)x + 1)v - (ab + 2g)y - (b - g - ab)(a_1x - y)^2 = 0.$$

3.2.2. $u_1 \neq 0,\ u_2 = 0$. In this case $a_{11} = -a_{02}(a_1 + a_2)$. If $a_{02} = 1$, then express a and a_1 from (11); g and a_{10} from $\{F_{30} = 0, F_{12} = 0\}$. We get the following conditions

$$a = 2,\quad g = 0,\quad a_1 = (b + c - 2a_2)/2,\quad 2a_2^2 - (b+c)a_2 - d - 2 = 0 \tag{13}$$

for the existence of a hyperbola $a_2(b+c-2a_2)x^2 - (b+c)xy + (c-b)x + 2(y-1)^2 = 0$.

If $a_{02} \neq 1$, then $F_{02} \equiv F_{03} = 0$ yields $a_1 = a_{10} - a_2 + b$. Reduce the equations of (10) by d from $F_{11} = 0$ and express g, a_{02}, a_{10} from $\{F_{20} = 0, F_{30} = 0, F_{21} = 0\}$, respectively. Then we obtain the following conditions

10)
$$f = -2,\quad g = 0,\quad a_1 = (b + c - 2a_2)/2,\quad 2a_2^2 - (b+c)a_2 - a - d = 0$$

for the existence of a hyperbola
$$(a-1)[(a+d)x^2 + (b+c)xy] + (b-c)x - 2(ay - y - 1)(y-1) = 0.$$

It is easy to check that (13) are contained in **11)**.

3.2.3. $u_1u_2 \neq 0,\ u_3 = 0$. If $a_{02} = 1$, then express a from $u_3 = 0$ and c, a_{10}, d, g from $\{F_{12} = 0, F_{11} = 0, F_{21} = 0, F_{30} = 0\}$, respectively. Then we obtain the following conditions

11)
$$a = 1 - a_{11}^2 - (a_1 + a_2 + b)a_{11} - a_1a_2 - ba_1,\quad c = -2a_{11} - b,$$
$$d = (a_1 - 2a_2 - b)a_{11} - a_{11}^2 + 2a_1^2 - a_1a_2 - a_2^2 - b(a_1 + a_2) - 2,$$
$$g = a_1[2a_{11}^2 + (2a_1 + 3a_2 + 2b)a_{11} + 2a_1a_2 + 2ba_1 + a_2^2 + ba_2 + 1],$$
$$F_{20} \equiv a_{11}^3 + (b + a_2)(2a_{11} + b + a_2)a_{11} + (2a_1 + 3a_{11} + 2a_2 + b)\times$$
$$(b + a_{11} + a_2)a_1 + b = 0$$

for the existence a hyperbola $a_1(a_1 + a_{11})x^2 + bx - a_{11}x(y-1) - (y-1)^2 = 0$.

Assume $a_{02} \neq 1$, then express c from $u_3 = 0$ and b, d, g from (11). If $a_{11} + a_{10} + (a_{02} - 1)a_2 = 0$, then $F_{21} = 0$ yields $a_{11} = -a_{02}a_{10}$ and we get the following conditions

12)
$$b = 0,\quad c = (2a_2^2 - 2a_1a_2 - a + 1)/(a_2 - a_1),\quad g = a_1,$$
$$d = (2a_1^3 - 4a_1^2a_2 + 2a_1a_2^2 + a_2 - 2aa_1 + aa_2)/(a_1 - a_2)$$

for the existence of a conic for system (5):

$$[(2a-1)a_1-aa_2](a_1x-a_2x+y)(a_1x-y)+(2aa_1-aa_2-a_2)y-(a_1-a_2)(a_2x+1)=0.$$

If $a_{11}+a_{10}+(a_{02}-1)a_2\neq 0$, then express a from $F_{21}=0$ and denote $v=a_{10}-a_2,\ s_1=a_{02}^2+2a_{02}a_1v-v^2,\ s_2=a_{02}^2a_2+2a_{02}a_1^2v+a_{02}a_1a_2v+a_{02}v-a_1v^2$, then $F_{30}\equiv s_1a_{11}+s_2a_{02}=0$.

If $s_1=0$, then we obtain the following conditions

13)

$$\begin{aligned}
&a=[v^3a_2(1-a_2^4)+v^2(1-a_2^4-2a_2^3z-2a_2z)+4vz(a_2^2-1)+4z^2]/(4va_2^2z),\\
&b=(z-v^2a_2-v)/(va_2),\ \ d=[v^3a_2(a_2^2+1)+v^2(2a_2^3z+a_2^2-2a_2z+1)\\
&\quad -vz(a_2^4+a_2^2+4)+2z^2(2-a_2^2)]/(2va_2^2z),\ \ a_1=(a_2^2-1)/(2a_2),\\
&c=[v^3a_2(a_2^2+1)+v^2(a_2^2+1)+4vz(a_2^2-1)+2z^2]/(2va_2z),\\
&g=[v^3a_2(1-a_2^4)+v^2(1-a_2^4+2a_2^3z-2a_2z)+2vz(a_2^4+a_2^2-2)\\
&\quad +4z^2(1-a_2^2)]/(4va_2^3z),\ \ z=a_{11}+v-a_2^2v,\ \ v=a_{10}-a_2
\end{aligned}$$

for the existence a hyperbola $x^2(2a_2^2v-a_2^4v-2a_2^2z-v+2z)+4xya_2(a_2^2v-v+z)+4xa_2(a_2+v)-4(a_2vy+1)(y-1)a_2=0$.

If $s_1\neq 0$, then express a_{11} from $F_{30}=0$ and obtain

14)

$$\begin{aligned}
&a=[(a_1-a_2)a_{02}^4+(2a_1^2v-3a_1a_2v+a_1-v)a_{02}^3+v(4a_1^2-2a_1^2a_2v-2a_1a_2-\\
&\quad -3a_1v+a_2^2+a_2v)a_{02}^2+v^2(a_1a_2^2+a_1a_2v-4a_1+2a_2+v)a_{02}+v^3]/(s_1s_3),\\
&b=[v^2(v-a_1+a_2)-va_{02}^2+va_{02}(2a_1^2-a_1a_2-2va_1+1)]/s_1,\\
&c=[a_{02}^4+(4a_1a_2+4va_1-2a_2^2-1)a_{02}^3+(4a_1^2(a_1+a_2+v)-3a_1a_2^2-4a_2-\\
&\quad -2v)va_{02}^2+v^2(a_2^2-4a_1^2-6a_1a_2-4va_1)a_{02}+v^3(a_1+2a_2+v)]/(s_1s_3),\\
&d=[a_{02}^4(a_2-2a_1)+a_{02}^3(4a_1^3-6a_1^2a_2-4va_1^2+2a_1a_2^2+2va_1a_2-2a_1+a_2)+\\
&\quad +v(2a_1^2a_2^2-4a_1^3a_2-8a_1^2+5a_1a_2+2va_1-2va_2+2)a_{02}^2+v^2(4a_1^2a_2-\\
&\quad -a_1a_2^2-2va_1a_2+5a_1+a_2)a_{02}+v^3(a_2^2-a_1a_2+va_2-1)]/(s_1s_3),\\
&g=[(a_{02}^2(2a_1-a_2+v)+2va_{02}(a_1v-1)-a_2v^2-v^3)(a_{02}+a_2v)a_1]/(s_1s_3),\\
&v=a_{10}-a_2,\ \ s_1=a_{02}^2+2a_{02}a_1v-v^2,\ \ s_3=2a_{02}a_1-a_{02}a_2-v.
\end{aligned}$$

The invariant hyperbola is of the form (7) with $a_{10}=a_2+v,\ \ a_{01}=-a_{02}-1$, $a_{20}=[a_{02}^2a_1(a_{02}(a_2-a_1)+va_1a_2+v)]/s_1,\ \ a_{11}=(-s_2a_{02})/s_1$.

3.3 Case $j_1\cdot j_2\neq 0,\ \ j_3=0$

In this case we also obtain the sets of conditions **8)–14)**.

3.4 Case $j_1\cdot j_2\cdot j_3\neq 0,\ \ j_4=0$

In this case $a_{20}=a_{11}^2/(4a_{02}),\ I_2=0$ and the conic is a parabola. We express c_{02} from $F_{04}=0$, c_{11} from $F_{13}=0$, c_{20} from $F_{22}=0$ and a_1 from $F_{31}\equiv F_{40}=0$.

3.4.1. $a_{02}=1$. In this subcase from the equations $F_{11}=0$ and $F_{12}=0$, we get respectively that $a_{10}=a_{11}+c,\ c=-(2a_{11}+b)$. We find g from $F_{30}=0$, d from $F_{21}=0$ and a_2^2 from $F_{20}=0$, then obtain the following conditions

15)

$$c=-2a_{11}-b,\ \ g=(-a_{11}^2-2ba_{11}-4h^2)/(2a_{11}),$$
$$d=(-2ha_{11}^4+b(1-h)a_{11}^3-4h(h+2)a_{11}^2-8bh^2a_{11}-16h^3)/(4ha_{11}^2),$$
$$a_1=(-3a_{11}^2-2a_{11}(a_2+b)-4h)/(2a_{11}),\ \ h=a-1,$$
$$2ha_{11}^3+a_{11}^2(6a_2h+bh+b)+4a_{11}h(a_2^2+ba_2+h)+8a_2h^2=0$$

for the existence of a parabola $a_{11}^2x^2+4a_{11}xy-4a_{11}x-4bx+4(y-1)^2=0$.

3.4.2. $a_{02}\neq 1$. In this case the equations $F_{02}\equiv F_{03}=0$ yields $a_{11}=-(a_{10}+b)a_{02}$. We reduce the equations of (10) by d from $F_{11}=0$ and a_2^2 from $F_{20}=0$. Next we find c from $F_{30}=0$ and denote $u=a_{10}+b,\ h=a-1,\ v=2(b+g)-u$, then the equation $F_{21}=0$ becomes $F_{21}\equiv r_1a_{02}+r_2=0$, where

$$r_1=(u^2+4h)v,\ \ r_2=2buv-8bh^2-4hv-u^2v.$$

Let $r_1=0$. If $u^2+h=0$, then $j_1=0$. Suppose $u^2+h\neq 0$ and $v=0$, then $b=0$ and $F_{20}\equiv f_1f_2=0$, where $f_1=a_2-u,\ f_2=a_{02}u+4ha_2-2hu-u$.

If $f_1=0$ or $f_2=0$, then we get

16)

$$a=h+1,\ \ b=0,\ \ c=(dg^2+2g^4+8g^2h+2g^2-2h^2)/(2gh),$$
$$a_1=[g(d+2g^2+2h+2)]/(2h),\ \ a_2=2g.$$

The invariant parabola is $(1+d+2g^2)(gx-y)^2-2gx-(d+2g^2)y-1=0$.

Assume $r_1\neq 0$ and express a_{02} from $F_{21}=0$, then we obtain the following conditions

17)

$$c=[u^4v(16h^2-bv)+4u^3hv(b+v-bh)+8u^2h(6h^2v-2bh^2-bv^2)$$
$$+16uh^2v^2-64h^4v]/[8h^2uv(4h+u^2)],$$
$$d=[u^5v(bv-8h^2)-4u^4hv(b+v)+4u^3hv(3bv-8h^2-8h)$$
$$+16u^2h^2v(b-2v)+32uh^2(bv^2-4bh^2-4hv)-64h^3v^2]/[16h^2uv(4h+u^2)],$$
$$a_1=[u^3v(12h^2-bv)+4u^2hv(v-2a_2h-bh+b)+8uh(6h^2v-2bh^2-bv^2)$$
$$+16h^2v(v-2a_2h)]/[8h^2v(4h+u^2)],$$
$$u^4v(bv-8h^2)+2u^3v(12a_2h^2-a_2bv-2bh-2hv)$$
$$+4u^2h(bv^2-4a_2^2hv-2a_2bhv+2a_2bv+2a_2v^2+8bh^2-8h^2v)$$
$$+16uh(bhv-2a_2bh^2-a_2bv^2+6a_2h^2v-hv^2)+32a_2h^2v(v-2a_2h)=0$$

for the existence of a parabola $2[((ux-2y+2)(ux-2y)u+8hx)v-4((ux-4y)ux+4(y-1)y)h^2]b-(4h+u^2)(ux-2y+2)^2v=0$

3.5 Case $j_1\cdot j_2\cdot j_3\cdot j_4\neq 0$

In this case we express c_{02} from $F_{04}=0$, c_{11} from $F_{13}=0$, c_{20} from $F_{22}=0$ and substitute into the equations $\{F_{40}=0,\ F_{31}=0\}$ of (7). Calculating the resultant of the equations $\{F_{40}=0,\ F_{31}=0\}$ by a we obtain

$$Res(F_{40},F_{31},a)=j_1j_2j_3j_4\neq 0.$$

In this case the system of equations $\{(9),(10),(11)\}$ is not compatible.

Remark 1. For cubic differential system (1) seventeen sets of conditions for the existence of at least two invariant straight lines and one invariant conic passing through the same singular point (0, 1) were obtained.

4 Sufficient conditions for the existence of a center

Lemma 1. *The following eighteen sets of conditions are sufficient conditions for the origin to be a center for system (5):*

i)

$$a = [a_{02}(a_{02} - a_1a_2 + 1) - a_1a_2 + a_{10}(a_1 + a_2 - a_{10})]/(2a_{02}), \quad b = 0,$$
$$c = a_1 + a_{10} + a_2, \quad d = [a_{10}(a_1 + a_2 - a_{10}) - a_1a_2 - a_{02}(a_1a_2 + 2)]/a_{02},$$
$$g = [a_{10}^3 - 3(a_1 + a_2)a_{10}^2 + a_{10}((2a_1 + a_2)(a_1 + 2a_2) - a_{02}^2 + (3a_1a_2 + 1)a_{02})$$
$$+ 2(a_{02}^2 - a_1a_2 - (a_1a_2 + 1)a_{02})(a_1 + a_2)]/[2a_{02}(a_{02} - 1)];$$

ii)

$$a = 2, \quad b = [(c(9 - 2c^2)]/[3(c^2 + 9)], \quad d = [(2(-4c^2 - 9)]/(c^2 + 9),$$
$$g = [c(2c^2 - 9)]/[3(c^2 + 9)], \quad a_1 = c/3, \quad a_2 = 0;$$

iii)

$$b = c = g = 0, \quad d = 2a_1^2 - a, \quad a_2 = -a_1;$$

iv)

$$b = g = 0, \quad d = a - 2, \quad a_2 = (1 - a)/a_1, \quad 2a_1^2 - ca_1 - 2a + 2 = 0;$$

v)

$$a = 2, \quad d = -[b^3g + 7b^2g^2 + 2b^2 + 14bg^3 - 8bg + 8g^4 + 8g^2]/(b - 2g)^2,$$
$$c = [(b + 2g)a_2]/g, \quad a_1 = [2(b + g)g]/(2g - b), \quad a_2 = [(b + 4g)g]/(2g - b)$$
$$(b + 4g)(b + 2g)(b + g)^2bg + (b^2 + bg + 6g^2)(b - 2g)^2 = 0;$$

vi)

$$c = [((b + 4g)(b + g) - a(b + 2g)^2)(g - b)]/[bg(b + g)^2],$$
$$d = [b^2(1 - a) + bg(2a - 5) + 2g^2(a - 1)]/[b(b + g)], \quad a_1 = [(b + g)g]/(g - b),$$
$$bg(b + g)^2 + (a - 1)(b - g)^2 = 0, \quad a_2 = g[a(b + 2g) - 2(b + g)]/[(a - 1)(g - b)];$$

vii)

$$a = 4g^2 + 2, \quad b = -3g, \quad c = -5g, \quad d = -2(7g^2 + 1), \quad a_1 = -2g, \; a_2 = -g;$$

viii)

$$a = [2(b^2 + bg - 6g^2)a_1^2 - (b^3 - 19bg^2 - 18g^3)a_1 - 6g^2(b + g)^2]/z,$$
$$c = [a_1(b - bh + 5hv - v) + v(v - b - hv)]/(hv),$$
$$d = [a_1v(v - b) - 2a_1^2hv - bh^2 + bh + bv^2 - 2hv - v^3]/(hv),$$
$$a_2 = [a_1(b - bh + 2hv - v) + v(v - b)]/(hv), \quad h = a - 1, \quad v = b + g,$$
$$z = 2a_1^2(8bv - 4v^2 - 3b^2) + v(5b^2 - 16bv + 10v^2)a_1 - (b^2 - 3bv + 2v^2)v^2,$$
$$(2a_1 - v)(2a_1 - v - g)z + b(2a_1 - b + g)((2b - 4g)a_1 + 4g^2 + 3bg - b^2) = 0;$$

ix)

$$c = [2(680a - 877)]/[b(245a - 316)], \quad d = (1178 - 913a)/(55a - 71),$$
$$g = 5b(1 - a), \quad a_1 = [b(35a - 44)]/(5a - 8), \quad a_2 = [b(100a - 129)]/(10a - 13),$$
$$5a^2 - 8a + 2 = 0, \quad b^2(245a - 316) - 65a + 84 = 0;$$

x)

$$a = 4/3, \quad c = 5b, \quad g = 0, \quad 36b^2d + 12b^2 + 9d^2 + 12d + 4 = 0,$$
$$a_1 = 3b - a_2, \quad 6a_2^2 - 18ba_2 - 3d - 4 = 0;$$

xi)
$$c = b,\ d = 2(1-a),\ g = 0,\ a_1 = b - a_2,\ 2a_2^2 - 2ba_2 + a - 2 = 0;$$

xii)
$$\begin{aligned}&d = (6b + 2c - 7ab - ac)/(5b - c),\ \ a_1 = (c + b - 2a_2)/2,\\ &g = 0,\ \ 36a^2 - 10ab^2 - 8abc + 2ac^2 - 96a + 5b^2 + 14bc - 3c^2 + 64 = 0,\\ &(5b - c)(2a_2^2 - (b + c)a_2) + 2(ab + ac - 3b - c) = 0;\end{aligned}$$

xiii)
$$\begin{aligned}&a = (3a_2^2 + 2ba_2 - 1)/(2a_2^2),\ \ d = (4ba_2 - a_2^4 - 2ba_2^3 - 3a_2^2 - 2)/(2a_2^2),\\ &c = 2a_2 + b - 2a_2^{-1},\ \ g = (a_2^4 - 2ba_2^3 + 2ba_2 - 1)/(2a_2^3),\ \ a_1 = (a_2^2 - 1)/(2a_2);\end{aligned}$$

xiv)
$$\begin{aligned}&a = 1 - 2a_1^2 + 3a_1a_2 - a_2^2,\ \ b = 0,\ \ c = 3a_2 - 2a_1,\\ &d = 6a_1^2 - 6a_1a_2 + a_2^2 - 2,\ \ g = a_1;\end{aligned}$$

xv)
$$\begin{aligned}&a = [v^3a_2(1 - a_2^4) + v^2(1 - a_2^4 - 2a_2^3z - 2a_2z) + 4vz(a_2^2 - 1) + 4z^2]/(4va_2^2z),\\ &b = (z - v^2a_2 - v)/(va_2),\ \ d = [v^3a_2(a_2^2 + 1) + v^2(2a_2^3z + a_2^2 - 2a_2z + 1)\\ &\quad - vz(a_2^4 + a_2^2 + 4) + 2z^2(2 - a_2^2)]/(2va_2^2z),\ \ a_1 = (a_2^2 - 1)/(2a_2),\\ &c = [v^3a_2(a_2^2 + 1) + v^2(a_2^2 + 1) + 4vz(a_2^2 - 1) + 2z^2]/(2va_2z),\\ &g = [v^3a_2(1 - a_2^4) + v^2(1 - a_2^4 + 2a_2^3z - 2a_2z) + 2vz(a_2^4 + a_2^2 - 2)\\ &\quad + 4z^2(1 - a_2^2)]/(4va_2^3z),\ \ z = a_{11} + v - a_2^2v,\ v = a_{10} - a_2;\end{aligned}$$

xvi)
$$\begin{aligned}&a = 2,\ \ b = [2(a_{11}^2 + 4)]/[a_{11}(a_{11}^2 - 4)],\ \ g = [a_{11}(a_{11}^2 + 4)]/[2(4 - a_{11}^2)],\\ &c = [2(a_{11}^4 - 3a_{11}^2 + 4)]/[a_{11}(4 - a_{11}^2)],\ \ d = [(a_{11}^2 + 4)(a_{11}^2 - 2)]/[2(4 - a_{11}^2)],\\ &a_1 = a_{11}^3/(4 - a_{11}^2),\ \ a_2 = (-a_{11})/2;\end{aligned}$$

xvii)
$$\begin{aligned}&a = h + 1,\ \ b = 0,\ \ c = (8g^2 - 3h)/(2g),\ \ a_2 = 2g,\\ &d = (-2g^4 - 2g^2 - h^2)/g^2,\ \ a_1 = (2g^2 - h)/(2g);\end{aligned}$$

xviii)
$$\begin{aligned}&a = (uv + 4)/4,\ b = [(u + v)v]/[2(v - u)],\ g = [(u + v)u]/[2(u - v)],\\ &c = ((2u - v)^2 + 4)/[2(u - v)],\ d = [(2u^2 - 2uv - v^2 + 12)u]/[4(v - u)],\\ &a_1 = u/2,\ a_2 = (2u^2 - uv + 4)/[2(u - v)].\end{aligned}$$

Proof. In each of the cases **i)** − **xviii)** the system (5) has two invariant straight lines of the form (3) and one invariant conic $\Phi = 0$. The system (5) has a Darboux integrating factor of the form
$$\mu = l_1^{\alpha_1} l_2^{\alpha_2} \Phi^{\alpha_3}.$$

In the case i): $\Phi = [a_{02}(a_{02} - a_1a_2 - 1) - a_1a_2 + a_{10}(a_1 + a_2 - a_{10})]x^2 + 2(a_{02}y - 1)a_{10}x - 2(y - 1)(a_{02}y - 1) = 0$ and $\alpha_1 = \alpha_2 = \alpha_3 = -1$.

In the case ii): $\Phi = (4c^2 + 9)[(cx - 3y)^2 + 3cx] - 9y(5c^2 + 18) + 9(c^2 + 9)$ and $\alpha_1 = 3,\ \alpha_2 = -(10c^2 + 9)/(4c^2 + 9),\ \alpha_3 = -2(5c^2 + 18)/(4c^2 + 9)$.

In the case iii): $\Phi = (a - 1)(y - a_1x)(y + a_1x) - (ay - 1)$ and $\alpha_1 = \alpha_2 = (2a_1^2)/(2 - a),\ \alpha_3 = (2a_1^2 - 3a + 6)/(a - 2)$.

In the case iv): $\Phi = (a-1)(2ax^2 + cxy - 2x^2 - 2y^2) - cx + 2ay - 2$ and $\alpha_1 = \alpha_2 = \alpha_3 = -1$.

In the case v): $\Phi = [b(2gx+y) + 2g(gx-y)]^2 + (8g^3 - b^3)x + (b-2g)^2(1-2y)$ and $\alpha_1 = 3,\ \alpha_2 = [-2(b^2+bg+3g^2)]/[b(b+g)],\ \alpha_3 = [-3(b+2g)(b-g)]/[(b+g)b]$.

In the case vi): $\Phi = (a-1)[((b+g)gx + 2(b-g)y)(ab-b+g)g - (b^2+bg+2g^2)(b-g)]x - bg(b+g)(aby - by - b + gy - g)(y-1)$ and $\alpha_1 = 2,\ \alpha_2 = (b^2 - ab^2 - 2ag^2 - bg + 2g^2)/[b(ab-b+g)],\ \alpha_3 = (3b^2 - 3ab^2 + 2ag^2 - 3bg - 2g^2)/[b(ab-b+g)]$.

In the case vii): $\Phi = (a-2)(2(3a-4)(gx+y)-1)x + 2g(3ay - 4y - 2)(y-1)$ and $\alpha_1 = 3,\ \alpha_2 = [2(3-2a)]/(3a-4),\ \alpha_3 = (14-11a)/(3a-4)$.

In the case viii): $\Phi = ((2a_1 - b)x + 1)v + (2b - ab - 2v)y - (2b - v - ab)(a_1x - y)^2$ and $\alpha_1 = 3,\ \alpha_2 = [2(3a_1g^2 - 3bg^2 - 3g^3 - u)]/u,\ \alpha_3 = [3(2g^3 - 2a_1g^2 + 2bg^2 - u)]/u$, where $u = (4b-8g)a_1^2 - (4b^2 - 12bg - 10g^2)a_1 + b^3 - 4b^2g - 7bg^2 - 2g^3$.

In the case ix): $\Phi = (11874abx - 4620ay + 6725a - 15316bx + 5960y - 8675)x + 5b(338ay - 545a - 436y + 703)(y-1)$ and $\alpha_1 = 3,\ \alpha_2 = (667a - 860)/[2(17a - 22)],\ \alpha_3 = [27(40-31a)]/[2(17a-22)]$.

In the case x): $\Phi = [(3d+4)x + 18b(y-2)]x - 6(y-1)(y-3)$ and $\alpha_1 = (6bd - b - 3da_2)/(2a_2 - 3b),\ \alpha_2 = (b + 3bd - 3da_2)/(2a_2 - 3b),\ \alpha_3 = 3d - 2$.

In the case xi): $\Phi = (a-1)[(a-2)x - 2by]x + 2(ay - y - 1)(y-1)$ and $\alpha_1 = 1,\ \alpha_2 = 1,\ \alpha_3 = -4$.

In the case xii): $\Phi = (a-1)[2(ab+ac-3b-c)x - (5b-c)(b+c)y]x - (5b^2 - 6bc + c^2)x + 2(5b-c)(ay-y-1)(y-1)$ and $\alpha_1 = [2(7ab+ac-6b-2c)a_2 - 3ab^2 - 4abc - ac^2 + 4b^2 + 2bc + 2c^2]/[(b+c-4a_2)(1-a)(5b-c)],\ \alpha_2 = [2(7ab+ac-6b-2c)a_2 + 2b(b+3c-2ab-2ac)]/[(b+c-4a_2)(1-a)(5b-c)],\ \alpha_3 = (ac - 17ab + 16b)/[(a-1)(5b-c)]$.

In the case xiii): $\Phi = [(a_2^2 + 2ba_2 - 1)(a_2^2 - 1)x - 4(a_2^2 + ba_2 - 1)a_2y + 4a_2(a_2^2 - 1)]x + 4a_2^2(y-1)^2$ and $\alpha_1 = 1,\ \alpha_2 = (a_2^2 + 2ba_2 - 1)/2,\ \alpha_3 = (-a_2^2 - 2ba_2 - 5)/2$.

In the case xiv): $\Phi = (2a_1 - a_2 + 1)(2a_1 - a_2 - 1)(a_1x - a_2x + y)(a_1x - y) + a_2x + (4a_1^2 - 4a_1a_2 + a_2^2 - 2)y + 1$ and $\alpha_1 = 2,\ \alpha_2 = (2a_1^2)/(4a_1a_2 - 4a_1^2 - a_2^2 + 1),\ \alpha_3 = (12a_1a_2 - 10a_1^2 - 3a_2^2 + 3)/(4a_1^2 - 4a_1a_2 + a_2^2 - 1)$.

In the case xv): $\Phi = x^2(2a_2^2v - a_2^4v - 2a_2^2z - v + 2z) + 4xya_2(a_2^2v - v + z) + 4xa_2(a_2 + v) - 4(a_2vy + 1)(y-1)a_2$ and $\alpha_1 = 1,\ \alpha_2 = (a_2^3v^3 + a_2^2v^2 - a_2^2vz + a_2v^3 - 2a_2v^2z + v^2 - vz - 2z^2)/(2a_2v^2z),\ \alpha_3 = (2z^2 - a_2^3v^3 - a_2^2v^2 + a_2^2vz - a_2v^3 - 4a_2v^2z - v^2 + vz)/(2a_2v^2z)$.

In the case xvi): $\Phi = a_{11}(a_{11}^2 - 4)(a_{11}^2x^2 + 4a_{11}xy + 4(y-1)^2) - 4x(a_{11}^4 - 2a_{11}^2 + 8)$ and $\alpha_1 = (a_{11}^4 - 2a_{11}^2 + 8)/[2(a_{11}^2 - 4)],\ \alpha_2 = 2,\ \alpha_3 = (a_{11}^4 + 6a_{11}^2 - 24)/[2(4 - a_{11}^2)]$.

In the case xvii): $\Phi = (g^2 + h^2)(xg - y)^2 + 2g^3x - (2g^2 + h^2)y + g^2$ and $\alpha_1 = -3,\ \alpha_2 = (2g^2 + h)^2/[2(g^2 + h^2)],\ \alpha_3 = -(4g^4 + 4g^2h + g^2 + 2h^2)/[2(g^2 + h^2)]$.

In the case xviii): $\Phi = u(v^2 - 4)(ux - 2y)^2 + 8(uv - 2u^2 - v^2)x + 4(8u - uv^2 - 4v)y + 16(v - u)$ and $\alpha_1 = (4 - 2u^2 - v^2)/(v^2 - 4),\ \alpha_2 = 2,\ \alpha_3 = (2u^2 - 3v^2 + 12)/(v^2 - 4)$. □

Lemma 2. *The following three sets of conditions are sufficient conditions for the origin to be a center for system (5):*

i)
$$b = g = 0,\ \ d = (9 - 18a - 2c^2)/9,\ \ a_2 = c/3,\ \ a_1 = 0;$$

ii)
$$a = 1 - 6b^2,\ \ c = 11b,\ \ d = -(54b^2 + 5)/3,\ \ g = 0,$$
$$a_1 = 6b - a_2,\ \ 3a_2^2 - 18ba_2 + 36b^2 + 1 = 0;$$

iii)

$$a=(9-2b^2)/9,\ c=(5b)/3,\ d=(3-2b^2)/3,\ g=(-4b)/3,$$
$$a_1=(2b-3a_2)/3,\ 9a_2^2-6ba_2+4b^2+27=0.$$

Proof. In each of the cases **i)–iii)** the first Liapunov quantities vanish $L_1=0$. The system (5) along with invariant straight lines (3) has also one more invariant straight line and one invariant conic.

In the case i): $l_3=2(9a+c^2-9)y-9,\ \ \Phi=2(a-1)(cx-3y)^2+6cx+9y(1-2a)+9.$

In the case ii): $l_3=6bx-2y+3,\ \ \Phi=2b^2(36b^2+1)x^2-36b^3xy-5bx+(6b^2y+1)(y-1).$

In the case iii): $l_3=2bx-6y-3,\ \Phi=2(b^2+9)(2bx-3y)^2-27bx-9(2b^2+9)y-81.$

By Theorem 1 in each of these cases the origin is a center. □

Lemma 3. *The following four sets of conditions are sufficient conditions for the origin to be a center for system (5):*

i)

$$a=[(b^2+5bg+2g^2)(2b+g)]/u,\ \ c=[g(4g^2-3b^2+5bg)]/[u(2b+g)],$$
$$d=-(3b^3+11b^2g+16bg^2+6g^3)/u,\ \ (2b+g)u-g^2=0,$$
$$u=(b^2+4bg+2g^2)(2b+g),\ \ a_1=(g^2+bg-b^2)/g,\ \ a_2=3b+2g;$$

ii)

$$a=4/7,\ c=-6b,\ d=(-48)/7,\ g=-3b,\ 7b^2-9=0,$$
$$a_1=(-3a_2-14b)/3,\ 7ba_2^2+42a_2+45b=0;$$

iii)

$$a=-1,\ c=16b,\ d=(-37)/7,\ g=(-b)/4,\ 49b^2-8=0,$$
$$a_1=(29b-4a_2)/4,\ 196a_2^2-1421ba_2+380=0;$$

iv)

$$a=h+1,\ b=[2(h+u^2)^2(2-u^2)]/[u(h+2)(8h+7u^2+2)],$$
$$c=[2(h+u^2)(19u^2-4h^2-hu^2+10h-6)]/[u(h+2)(8h+7u^2+2)],$$
$$d=(u^6-52hu^2-88h-50u^4-84u^2-24)/[4(h+2)(8h+7u^2+2)],$$
$$a_1=[2(h+u^2)(13u^2-6h^2-4hu^2+8h-2)]/[u(h+2)(8h+7u^2+2)]-a_2,$$
$$g=[-2b(h+u^2)]/(u^2-2),\ u^4-2u^2+8h^2+8hu^2=0,$$
$$[2(8h+7u^2+2)(h+2)^2a_2^2+(5u^2-2h^2-hu^2+2h-2)(4h+u^2+6)(h+u^2)]u-4(13u^2-6h^2-4hu^2+8h-2)(h+u^2)(h+2)a_2=0.$$

Proof. In each of the cases **i)–iv)** the system (5) along with invariant straight lines (3) has also two more invariant straight lines and one invariant conic:

In the case i): $l_3=((2b+g)x-y)(b+g)+2b+g,\ \ l_4=2(b(2b+g)x+gy)(b+g)-g(2b+g),\ \Phi=((2a_1-b)x+1)(b+g)-(ab+2g)y-(b-g-ab)(a_1x-y)^2.$

In the case ii): $l_3=2bx+2y-1,\ l_4=4bx+12y-3,\ \Phi=144bx^2+288xy+112by^2-189x-161by+49b.$

In the case iii): $l_3=21bx-12y+28,\ l_4=42bx-6y+7,\ \Phi=27x^2-189bxy+35bx+54y^2-68y+14.$

In the case iv): $l_3=2(20h^2+17hu^2-10h-15u^2-2)(2hx+uy)+u(8h^2+7hu^2+18h+14u^2+4),\ l_4=2(34h^3+29h^2u^2-30h^2-36hu^2+8u^2)x+yu(20h^2+17hu^2-10h-15u^2-2)+u(8h^2+7hu^2+18h+14u^2+4),\ \Phi=a_{02}(ux-2y)^2+4(u-b)x-4(a_{02}+1)y+4,$

where $a_{02} = (8bh^2 - 2buv + 4hv + u^2v)/(v(4h + u^2))$, $v = [4(h + u^2)^2(u^2 - 2)]/[(8h + 7u^2 + 2)(h + 2)u]$.
By Theorem 1 in each of these cases the origin is a center. □

Theorem 4. ($l_j = 1 + a_jx - y$, $j = 1, 2$, $l_1 \cap l_2 \cap \Phi = (0, 1)$; $L = 4$), *where* $\Phi = 0$ *is an invariant conic of the form (7) is ILC for system (1), i.e. the order of a weak focus is at most four.*

Proof. To prove the theorem, we compute the first four Liapunov quantities L_j, $j = \overline{1, 4}$, in each of the following sets of conditions **1)**–**17)** using the algorithm described in [21]. In the expressions for L_j we will neglect denominators and non-zero factors.

In the case **1)** we calculate L_1. If $c = -b$, then $L_1 \equiv b^2 + (d + 4)^2 \neq 0$ and if $c \neq -b$, then $L_1 \equiv (a - 1)[4a^2 + (b + c)^2]I_3 \neq 0$. Therefore the origin is a focus.

In the case **2)** the first Liapunov quantity vanishes. We are in the conditions of Lemma 1, i).

In the case **3)** we have $L_1 = f_1f_2$, where $f_1 = (a_{02} - 1)^2 + (b - c + 2a_{10})^2 \neq 0$ and $f_2 = a_{02} + 2a_{10}^2 + 3ba_{10} - ca_{10} + b^2 - bc - 1$. If $f_2 = 0$, then $a = 1$. Therefore the origin is a focus.

In the case **4)** the vanishing of the first Liapunov quantity gives $c = 3b$. Then $L_2 \neq 0$ and the origin is a focus.

In the case **5)** the vanishing of the first Liapunov quantity gives $d = 1 - 2a_2^2 - 2a$ and we are in the conditions of Lemma 2, i).

In the case **6)** the vanishing of the first Liapunov quantity gives $a_2 = 0$. Then $L_2 = f_1f_2$, where $f_1 = a_{02} - 2$, $f_2 = (a_1^2 + 1)a_{02} - 4a_1^2 - 1$. If $f_1 = 0$, then Lemma 2, i) ($a = 2$), and if $f_2 = 0$, then Lemma 1, ii).

In the case **7)** the first Liapunov quantity is $L_1 = g_1g_2$, where $g_1 = a_1 + a_2$, $g_2 = a_1a_2 - 1 + a$. If $g_1 = 0$, then Lemma 1, iii) and if $g_2 = 0$, then Lemma 1, iv).

In the case **8)** the first Liapunov quantity is $L_1 = (5\beta^3 + 9\beta^2\gamma + 34\beta^2\delta + 3\beta\gamma^2 + 4\beta\gamma\delta + 64\beta\delta^2 - \gamma^3 + 2\gamma^2\delta + 32\delta^3)t^2 + 16(\beta - \gamma - 2\delta)$, where $\beta = b/t$, $\gamma = c/t$, $\delta = g/t$ and t is a real parameter. From $L_1 = 0$ we find t^2 and substituting into the expression for L_2, we obtain $L_2 = f_1f_2f_3$, where $f_1 = \beta^2 + \beta\gamma + 6\beta\delta - 2\gamma\delta + 8\delta^2$, $f_2 = \beta^2 + \beta\gamma + 3\beta\delta - \gamma\delta + 4\delta^2$, $f_3 = 11\beta - \gamma + 4\delta$. If $f_1 = 0$, then Lemma 1, v) and if $f_2 = 0$, then Lemma 1, xii).

Assume $f_1f_2 \neq 0$ and let $f_3 = 0$. Then $\gamma = 11\beta + 4\delta$ and $L_3 = h_1h_2$, where $h_1 = \beta + 2\delta$, $h_2 = 5\beta + \delta$. If $h_1 = 0$, then $\beta = -2\delta$ and $L_1 \equiv 18\delta^2 + 1 \neq 0$, therefore the origin is a focus. If $h_2 = 0$, then $L_3 = 0$, $L_4 \neq 0$ and the origin is a focus.

In the case **9)** we denote $b = \beta t$, $g = \gamma t$, $a_1 = \alpha t$ and calculate the first Liapunov quantity. From $L_1 = 0$ we find t^2 and substituting into the expression for L_2, we obtain $L_2 = f_1f_2f_3$, where $f_1 = \alpha\beta - \alpha\gamma + \beta\gamma + \gamma^2$, $f_2 = 2[(\beta + 3\gamma - (\beta + 2\gamma)a)(2\gamma - \beta)]\alpha^2 + [(\beta^2 - \beta\gamma - 18\gamma^2 - (\beta^2 - 4\beta\gamma - 10\gamma^2)a)(\beta + \gamma)]\alpha - [(\beta + 2\gamma)a - 6\gamma](\beta + \gamma)^2\gamma$, $f_3 = (a\beta + 2a\gamma - \beta - 3\gamma)\alpha - (3a\beta^2 + 5a\beta\gamma + 2a\gamma^2 - 3\beta^2 - 6\beta\gamma - 3\gamma^2)$.

If $f_1 = 0$, then Lemma 1, vi), if $f_2 = 0$ and $\alpha = -2\gamma$, then Lemma 1, vii); if $f_2 = 0$ and $\alpha \neq -2\gamma$, then Lemma 1, viii).

Assume $f_1 f_2 \neq 0$ and $f_3 = 0$. Then express α from $f_3 = 0$ and substituting in L_3 we obtain $L_3 = h_1 h_2$, where $h_1 = 5\beta(a-1)+\gamma,\ h_2 = a\beta^2 + 4a\beta\gamma + 2a\gamma^2 - \beta^2 - 5\beta\gamma - 2\gamma^2$. Let $h_1 = 0$, then $\gamma = 5\beta(1-a)$ and $L_4 = 5a^2 - 8a + 2$. If $L_4 = 0$, then Lemma 1, ix). Let now $h_1 \neq 0$ and $h_2 = 0$, then Lemma 3, i).

In the case **10)** the first Liapunov quantity is $L_1 = 7ab + ac + 5bd - 6b - cd - 2c$. If $L_1 = 0$ and $c = 5b$, then $a = 4/3$ and $L_2 = 36b^2 d + 12b^2 + 9d^2 + 12d + 4$. Let $L_2 = 0$, then Lemma 1, x).

Assume $L_1 = 0$ and $c \neq 5b$, then express d and calculate $L_2 = f_1 f_2 f_3$, where $f_1 = c - b,\ f_2 = 36a^2 - 10ab^2 - 8abc + 2ac^2 - 96a + 5b^2 + 14bc - 3c^2 + 64,\ f_3 = 11b - c$.

If $f_1 = 0$, then Lemma 1, xi) and if $f_2 = 0$, then Lemma 1, xii). Let $f_1 f_2 \neq 0$ and $f_3 = 0$, then $c = 11b$ and $L_3 = 1 - a - 6b^2$. If $L_3 = 0$, then Lemma 2, ii).

In the case **11)** we denote $a_1 = \gamma_1 t$, $a_2 = \gamma_2 t$, $b = \beta t$ and $a_{11} = \gamma_3 t$. From $F_{20} = 0$ we find t^2 and substituting into the expression for L_1, we obtain $L_1 = g_1 g_2 g_3$, where $g_1 = \gamma_1 + \gamma_3,\ g_2 = 2\gamma_1 + \gamma_3 + \beta,\ g_3 = 2\gamma_1^2 + 2\gamma_1\gamma_3 + \gamma_1\gamma_2 - \gamma_3\gamma_2 - \gamma_2^2 - \gamma_2\beta$.

If $g_1 = 0$, then $a = 1$; if $g_2 = 0$, then Lemma 1, xiii) and if $g_3 = 0$, then express β and substituting in L_2, we get $L_2 = 3\gamma_1 - 2\gamma_2$. If $L_2 = 0$, then $L_3 = 2\gamma_1^2 + 19\gamma_1\gamma_3 + 8\gamma_3^2$ and $F_{20} = (49\gamma_1^2 + 56\gamma_1\gamma_3 + 16\gamma_3^2)t^2 + 3$. The system of equations $\{L_3 = 0,\ F_{20} = 0\}$ has no real solutions. In this case the origin is a focus.

In the case **12)** the vanishing of the first Liapunov quantity gives $a = 1 - 2a_1^2 + 3a_1 a_2 - a_2^2$, then Lemma 1, xiv).

In the case **13)** the first Liapunov quantity vanishes, Lemma 1, xv).

In the case **14)** the first Liapunov quantity is $L_1 = g_1 g_2$, where $g_1 = (a_1 a_2 + 1)v - a_{02}(a_1 - a_2),\ g_2 = a_{02}^3 + (4a_1^2 - 4a_1 a_2 + 2va_1 + a_2^2 + va_2 - 1)a_{02}^2 + v(2a_1 a_2 v - 4a_1 - v)a_{02} - v^2(a_2^2 + a_2 v - 1)$. If $g_1 = 0$, then $a = 1$. Let $g_1 \neq 0,\ g_2 = 0$ and denote $a_{02} = \alpha t,\ v = \beta t$, then express t from $g_2 = 0$ and calculate $L_2 = 3(2\alpha a_1 - \alpha a_2 - \beta)^2 + (\beta a_2 + \alpha)^2$. The equation $L_2 = 0$ has no real solutions.

In the case **15)** the vanishing of the first Liapunov quantity gives $b = [2h^2(2ha_{11}^2 - a_{11}^2 + 4h^2)(a_{11}^2 + 4h)]/[a_{11}(a_{11}^4 - 2h^3 a_{11}^2 - 4h^2 a_{11}^2 + 6ha_{11}^2 - 16h^4)]$. The second one looks $L_2 = f_1 f_2 f_3$, where $f_1 = h - 1,\ f_2 = 2ha_{11}^2 - a_{11}^2 + 4h^2,\ f_3 = (8h^2 - 9h + 3)a_{11}^4 + 2h(5h^3 + 31h^2 - 29h + 9)a_{11}^2 + 16h^4(5h + 1)$.

If $f_1 = 0$, then Lemma 1, xvi) and if $f_1 \neq 0$, $f_2 = 0$, then $I_3 = 0$.

Assume $f_1 f_2 \neq 0$ and $f_3 = 0$. Then we calculate L_3 and the resultant of polynomials f_3 and L_3 by h. We obtain that $Res(f_3, L_3, h) = g_1 g_2 g_3$, where $g_1 = a_{11}^4 + 48a_{11}^2 + 144,\ g_2 = a_{11}^4 + 184a_{11}^2 + 16,\ g_3 = 12a_{11}^4 + 41a_{11}^2 + 36,\ g_4 = 100a_{11}^8 + 1225a_{11}^6 + 4380a_{11}^4 + 3440a_{11}^2 + 576$. It is easy to verify that $g_k = 0$, $k = \overline{1,4}$, have not real solutions and therefore the origin is a focus.

In the case **16)** the vanishing of the first Liapunov quantity gives $d = (-2g^4 - 2g^2 - h^2)/g^2$, then Lemma 1, xvii).

In the case **17)** the first Liapunov quantity is $L_1 = bf_1 - f_2$, where $f_1 = u(32h^3 u - 32h^3 v + 48h^2 v - 8hv^3 - u^2 v^3),\ f_2 = 4hv(4h^2 - 2huv - v^2)(4h + u^2)$.

Let $L_1 = 0$ and assume $f_1 = 0$. Then $L_1 \equiv f_2 = 0$ yields $u = (4h^2 - v^2)/(2hv)$ and $f_1 \equiv 64h^4 - 32h^3v^2 + 12h^2v^2 - v^4 = 0$. The equation $f_1 = 0$ admits the following parametrization $h = [(4\alpha^2 + \beta^2)(16\alpha^2 - \beta^2)]/(32\alpha^2\beta^2)$, $v = [(4\alpha^2 + \beta^2)(16\alpha^2 - \beta^2)]/(32\alpha^3\beta)$.

The vanishing of the second Liapunov quantity gives $b = [(768\alpha^6 - 432\alpha^4\beta^2 - 8\alpha^2\beta^4 + 5\beta^6)(96\alpha^4 - 4\alpha^2\beta^2 + \beta^4)(\beta^2 - 16\alpha^2)]/[256\alpha^3\beta(1152\alpha^6 - 136\alpha^4\beta^2 + 2\alpha^2\beta^4 - \beta^6)(4\alpha^2 - \beta^2)]$. Then $L_3 = g_1g_2$, where $g_1 = 28\alpha^2 - \beta^2$, $g_2 = 3840\alpha^6 - 304\alpha^4\beta^2 - 40\alpha^2\beta^4 + 5\beta^6$. If $g_1 = 0$, then we are in the conditions of Lemma 3, ii). The equation $g_2 = 0$ has not real solutions.

Let now $L_1 = 0$ and assume $f_1 \neq 0$, then $b = f_2/f_1$. The second Liapunov quantity is $L_2 = g_1g_2$, where $g_1 = 4h - uv$, $g_2 = 384h^4 + 96h^3(2u^2 - 3uv - 2) + 16h^2(21uv - 3u^3v - 6u^2 - 2v^2) + 4huv(6u^2 - 13uv - 10v^2 - 24) + u^2v^2(8 - 4u^2 - 5uv)$.

If $g_1 = 0$, then Lemma 1, xviii). Assume $g_1 \neq 0$ and calculate L_3. The resultant of the polynomials g_2 and L_3 by v is

$$Res(g_2, L_3, v) = h_1h_2h_3h_4h_5h_6h_7,$$

where $h_1 = 2h + u^2$, $h_2 = 8h + u^2$, $h_3 = 16h + u^2 - 4$, $h_4 = 8h^2 + 8hu^2 + u^4 - 2u^2$, $h_5 = 64h^2 - 24hu^2 - 96h - 4u^4 - 7u^2 + 36$, $h_6 = 1536h^6 + 768h^5(2u^2 - 1) + 8h^4u^2(63u^2 - 148) + 6h^3u^2(9u^4 - 84u^2 - 64) + 4h^2u^2(72 - 15u^4 - 2u^2) + 18hu^4(u^2 + 4) + u^6(u^2 + 4)$, $h_7 = 800h^6(3u^2 + 4) + 40h^5(15u^4 - 212u^2 - 96) + 8h^4(144 - 245u^4 + 1200u^2) - 2h^3u^2(15u^4 - 1160u^2 + 2352) + 8h^2u^2(10u^4 - 135u^2 + 108) + 2hu^4(84 - 25u^2) + u^6(4 - u^2)$.

Let $h_1 = 0$, then $h = (-u^2)/2$ and the system of equations $\{g_2 = 0,\ L_3 = 0\}$ has no real solutions.

Assume $h_1 \neq 0$, $h_2 = 0$, then $h = (-u^2)/8$ and $g_2 = e_1e_2$, where $e_1 = 3u + 2v$, $e_2 = 3u^3 - 32v$. If $e_1 = 0$, then Lemma 2, iii) and if $e_2 = 0$, then $L_3 \neq 0$.

Let $h_1h_2 \neq 0$, $h_3 = 0$, then $h = (4 - u^2)/16$ and $g_2 = e_1e_2$, where $e_1 \equiv 3(u^2 - 4)^2 + (8v)^2 \neq 0$, $e_2 = 7u^2 + 20uv + 4$. If $e_2 = 0$, then $v = (-7u^2 - 4)/(20u)$ and $L_3 \neq 0$.

Assume $h_1h_2h_3 \neq 0$ and $h_4 = 0$. If $h = -2$, then $h_4 = 0$ yields $u^2 = 2$. In this subcase the system of equations $\{g_2 = 0,\ L_3 = 0\}$ has solutions if $v = -8/(7u)$, then Lemma 3, iii). If $h \neq -2$, the equation $h_4 = 0$ admits the following parametrization $h = (-2\alpha\beta)/(\alpha^2 - 8\alpha\beta + 8\beta^2)$, $u^2 = (16\beta^2)/(\alpha^2 - 8\alpha\beta + 8\beta^2)$. In this case $g_2 = e_1e_2$, where $e_1 = (\alpha^2 - 8\alpha\beta + 8\beta^2)(\alpha - \beta)uv + 8\alpha\beta^2$, $e_2 = 5(\alpha^2 - 8\alpha\beta + 8\beta^2)^2v^2 + 12\alpha(\alpha^2 - 4\alpha\beta + 8\beta^2)(\alpha - 4\beta)$. If $e_1 = 0$, then Lemma 3, iv); if $e_1 \neq 0$, $e_2 = 0$, then reduce $L_3 = 0$ by v^2 from $e_2 = 0$. We express v from $L_3 = 0$, then the equation $e_2 = 0$ has no real solutions.

Let $h_1h_2h_3h_4 \neq 0$. The case $h_5 = 0$ or $h_6 = 0$ implies $f_1 = 0$, in contradiction with assumption that $f_1 \neq 0$. Therefore the origin is a focus.

Assume $h_1h_2h_3h_4h_5h_6 \neq 0$ and $h_7 = 0$. In this case from the system of equations $\{g_2 = 0,\ L_3 = 0\}$ we express v and calculate L_4. The resultant of the polynomials h_7 and L_4 by h is

$$Res(h_7, L_4, h) = e_1e_2e_3e_4e_5e_6e_7e_8e_9,$$

where $e_1 = u - 2$, $e_2 = u + 2$, $e_3 = 3u^2 - 4$, $e_4 = 5u^4 - 20u^3 - 80u^2 - 240u - 144$, $e_5 = 5u^4 + 20u^3 - 80u^2 + 240u - 144$, $e_6 = 5u^2 - 4u + 4$, $e_7 = 5u^2 + 4u + 4$, $e_8 = 15u^4 + 40u^2 + 128$, $e_9 = 300u^6 + 1105u^4 - 1080u^2 + 1296$.

If $e_1 = 0$, then $g_2 = L_3 = 0$ yields $15v^3 + 34v^2 + 36v - 72 = 0$ and $L_4 \neq 0$; if $e_2 = 0$, then $g_2 = L_3 = 0$ yields $15v^3 - 34v^2 + 36v + 72 = 0$ and $L_4 \neq 0$.

If $e_3 = 0$, then $b = 0$ and $a_{02} = 1$, in contradiction with assumption 3.4.2.

If $e_4 = 0$ or $e_5 = 0$, then $f_1 = 0$. The equations $e_6 = 0,\ e_7 = 0,\ e_8 = 0,\ e_9 = 0$ have no real solutions. □

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On cyclically-interval edge colorings of trees

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Abstract. For an undirected, simple, finite, connected graph G, we denote by $V(G)$ and $E(G)$ the sets of its vertices and edges, respectively. A function $\varphi : E(G) \to \{1, 2, \ldots, t\}$ is called a proper edge t-coloring of a graph G if adjacent edges are colored differently and each of t colors is used. An arbitrary nonempty subset of consecutive integers is called an interval. If φ is a proper edge t-coloring of a graph G and $x \in V(G)$, then $S_G(x, \varphi)$ denotes the set of colors of edges of G which are incident with x. A proper edge t-coloring φ of a graph G is called a cyclically-interval t-coloring if for any $x \in V(G)$ at least one of the following two conditions holds: a) $S_G(x, \varphi)$ is an interval, b) $\{1, 2, \ldots, t\} \setminus S_G(x, \varphi)$ is an interval. For any $t \in \mathbb{N}$, let $\mathfrak{M}_t$ be the set of graphs for which there exists a cyclically-interval t-coloring, and let $\mathfrak{M} \equiv \bigcup_{t \geq 1} \mathfrak{M}_t$. For an arbitrary tree G, it is proved that $G \in \mathfrak{M}$ and all possible values of t are found for which $G \in \mathfrak{M}_t$.

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1 Introduction

We consider undirected, simple, finite, and connected graphs. For a graph G we denote by $V(G)$ and $E(G)$ the sets of its vertices and edges, respectively. The set of edges of G incident with a vertex $x \in V(G)$ is denoted by $J_G(x)$. The set of vertices of G adjacent to a vertex $x \in V(G)$ is denoted by $I_G(x)$. For any $x \in V(G)$, $d_G(x)$ denotes the degree of the vertex x in G. For a graph G, we denote by $\Delta(G)$ and $\chi'(G)$ the maximum degree of a vertex of G and the chromatic index of G [32], respectively. The distance in a graph G between its vertices $x \in V(G)$ and $y \in V(G)$ is denoted by $\rho_G(x, y)$. For any vertex $x_0 \in V(G)$ and an arbitrary subset V_0 of the set $V(G)$, we define the distance $\rho_G(x_0, V_0)$ in a graph G between x_0 and V_0 as follows:

$$\rho_G(x_0, V_0) \equiv \min_{z \in V_0} \rho_G(x_0, z)$$

For any integer $n \geq 3$, we denote by C_n a simple cycle with n vertices. The terms and concepts that we do not define can be found in [35].

For an arbitrary finite set A, we denote by $|A|$ the number of elements of A. The set of positive integers is denoted by $\mathbb{N}$. An arbitrary nonempty subset of consecutive integers is called an interval. An interval with the minimum element p and the maximum element q is denoted by $[p, q]$. An interval D is called an h-interval if $|D| = h$.

For any $t \in \mathbb{N}$ and arbitrary integers i_1, i_2 satisfying the conditions $i_1 \in [1,t]$, $i_2 \in [1,t]$, we define [22] the sets $intcyc_1((i_1,i_2),t)$, $intcyc_1[(i_1,i_2),t]$, $intcyc_2((i_1,i_2),t)$, $intcyc_2[(i_1,i_2),t]$ and the number $dif((i_1,i_2),t)$ as follows:

$$intcyc_1[(i_1,i_2),t] \equiv [\min\{i_1,i_2\}, \max\{i_1,i_2\}],$$

$$intcyc_1((i_1,i_2),t) \equiv intcyc_1[(i_1,i_2),t] \backslash (\{i_1\} \cup \{i_2\}),$$

$$intcyc_2((i_1,i_2),t) \equiv [1,t] \backslash intcyc_1[(i_1,i_2),t],$$

$$intcyc_2[(i_1,i_2),t] \equiv [1,t] \backslash intcyc_1((i_1,i_2),t),$$

$$dif((i_1,i_2),t) \equiv \min\{|intcyc_1[(i_1,i_2),t]|, |intcyc_2[(i_1,i_2),t]|\} - 1.$$

If $t \in \mathbb{N}$ and Q is a non-empty subset of the set $\mathbb{N}$, then Q is called a t-cyclic interval if there exist integers i_1, i_2, j_0 satisfying the conditions $i_1 \in [1,t]$, $i_2 \in [1,t]$, $j_0 \in \{1,2\}$, $Q = intcyc_{j_0}[(i_1,i_2),t]$.

A function $\varphi : E(G) \to [1,t]$ is called a proper edge t-coloring of a graph G if adjacent edges are colored differently and each of t colors is used.

If φ is a proper edge t-coloring of a graph G and $E_0 \subseteq E(G)$, then $\varphi[E_0] \equiv \{\varphi(e)/e \in E_0\}$.

A proper edge t-coloring φ of a graph G is called an interval t-coloring of G [8,9,20] if for any $x \in V(G)$, the set $\varphi[J_G(x)]$ is a $d_G(x)$-interval. For any $t \in \mathbb{N}$, we denote by $\mathfrak{N}_t$ the set of graphs for which there exists an interval t-coloring. Let us also define the set $\mathfrak{N}$ of all interval colorable graphs:

$$\mathfrak{N} \equiv \bigcup_{t \geq 1} \mathfrak{N}_t.$$

For any $G \in \mathfrak{N}$, we denote by $w_{int}(G)$ and $W_{int}(G)$ the minimum and the maximum possible value of t, respectively, for which $G \in \mathfrak{N}_t$. For a graph G, let us set $\theta(G) \equiv \{t \in \mathbb{N}/G \in \mathfrak{N}_t\}$.

The problem of deciding whether a regular graph G belongs to the set $\mathfrak{N}$ is NP-complete [8,9,20]. Nevertheless, for graphs G of some classes the relation $G \in \mathfrak{N}$ was proved and investigations of the set $\theta(G)$ were fulfilled [8,9,19,20,26,27]. The concept of interval colorability of a graph represents an especially high interest for a bipartite graph, because in this case it can be used for mathematical modelling of timetable problems with compactness requirements (i.e. the lectures of each teacher and each group must be scheduled at consecutive periods) [1, 7, 20, 29]. Unfortunately, for an arbitrary bipartite graph G the problem keeps the complexity of a general case [3,13,31]. Some positive results were obtained for "small" bipartite graphs [14,15,25], for bipartite graphs with the "small" maximum degree of a vertex [13, 16, 28], and for biregular bipartite graphs [2–6, 11, 16–18, 24, 30, 36]. Very interesting approaches for biregular bipartite graphs were developed in [6, 11, 30]. The examples of interval non-colorable bipartite graphs were given in [7, 15, 18, 31].

Remark 1. It is not difficult to see that for any integer $k \geq 2$, $C_{2k} \in \mathfrak{N}$ and $\theta(C_{2k}) = [2, k+1]$.

A proper edge t-coloring φ of a graph G is called a cyclically-interval t-coloring of G if for any $x \in V(G)$, the set $\varphi[J_G(x)]$ is a t-cyclic interval. For any $t \in \mathbb{N}$, we denote by $\mathfrak{M}_t$ the set of graphs for which there exists a cyclically-interval t-coloring. Let us also define the set $\mathfrak{M}$ of all cyclically-interval colorable graphs:

$$\mathfrak{M} \equiv \bigcup_{t \geq 1} \mathfrak{M}_t.$$

For any $G \in \mathfrak{M}$, we denote by $w_{cyc}(G)$ and $W_{cyc}(G)$ the minimum and the maximum possible value of t, respectively, for which $G \in \mathfrak{M}_t$. For a graph G, let us set $\Theta(G) \equiv \{t \in \mathbb{N}/G \in \mathfrak{M}_t\}$.

Remark 2. The concept of cyclically-interval colorability of a graph generalizes that of interval colorability. Clearly, for an arbitrary graph $G \in \mathfrak{N}$, and for any $t \in \theta(G)$, an arbitrary interval t-coloring of the graph G is also a cyclically-interval t-coloring of G, therefore, for any $t \in \mathbb{N}$, $\mathfrak{N}_t \subseteq \mathfrak{M}_t$. $\mathfrak{N}_2 = \mathfrak{M}_2$. For any integer $t \geq 3$, $\mathfrak{N}_t \subset \mathfrak{M}_t$ (it is enough to consider the simple cycle C_t). $\mathfrak{N} \subset \mathfrak{M}$ (it is enough to consider the simple cycle C_3). For an arbitrary graph G, $\theta(G) \subseteq \Theta(G)$.

Remark 3. For any $G \in \mathfrak{N}$, the following inequality is true:

$$\Delta(G) \leq \chi'(G) \leq w_{cyc}(G) \leq w_{int}(G) \leq W_{int}(G) \leq W_{cyc}(G) \leq |E(G)|\,.$$

Remark 4. It is not difficult to note that there exist examples G_1 and G_2 of graphs from $\mathfrak{N}$ for which $w_{cyc}(G_1) < w_{int}(G_1)$, $W_{int}(G_2) < W_{cyc}(G_2)$. Let us set $G_1 = K_{3,2}$ and $G_2 = K_{2,2}$. In this case, evidently, $w_{cyc}(G_1) = 3$, $w_{int}(G_1) = 4$ [19], $W_{int}(G_2) = 3$ [19], $W_{cyc}(G_2) = 4$.

The problem of cyclically-interval colorability of a graph has been completely investigated as yet only for simple cycles [21, 23] and trees [22]. Some interesting results on this and related topics were obtained in [10, 12, 33, 34].

For a tree H with $V(H) = \{b_1, ..., b_p\}$, $p \geq 1$, we denote by $P(b_i, b_j)$ the simple path connecting the vertices b_i and b_j, $1 \leq i \leq p$, $1 \leq j \leq p$. The sets of vertices and edges of the path $P(b_i, b_j)$ are denoted by $VP(b_i, b_j)$ and $EP(b_i, b_j)$, respectively, $1 \leq i \leq p$, $1 \leq j \leq p$.

Let us also define:

$$intVP(b_i, b_j) \equiv VP(b_i, b_j) \backslash (\{b_i\} \cup \{b_j\});$$

$$\tilde{V}P(b_i, b_j) \equiv VP(b_i, b_j) \cup \left(\bigcup_{x \in intVP(b_i, b_j)} I_H(x) \right);$$

$$TP(b_i, b_j) \equiv \begin{cases} \bigcup\limits_{x \in intVP(b_i, b_j)} J_H(x), & \text{if } intVP(b_i, b_j) \neq \varnothing \\ EP(b_i, b_j), & \text{if } intVP(b_i, b_j) = \varnothing; \end{cases}$$

$$1 \leq i \leq p, 1 \leq j \leq p.$$

Assume:

$$M(H) \equiv \max\Big\{\big|TP(b_i, b_j)\big|/1 \le i \le p, 1 \le j \le p\Big\}.$$

In [19] the following result was obtained.

Theorem 1 (see [19])**.** *Let H be an arbitrary tree. Then*

1. $H \in \mathfrak{N}$,
2. $w_{int}(H) = \Delta(H)$,
3. $W_{int}(H) = M(H)$,
4. $\theta(H) = [\Delta(H), M(H)]$.

Corollary 1. *For any tree H, $H \in \mathfrak{M}$, $w_{cyc}(H) = \Delta(H)$, $W_{cyc}(H) \ge M(H)$, $[\Delta(H), M(H)] \subseteq \Theta(H)$.*

In this paper, for any tree H, we show that $W_{cyc}(H) = M(H)$ and $\Theta(H) = [\Delta(H), M(H)]$.

2 Results

Lemma 1. *If $Q_1, ..., Q_n$ $(n \ge 2)$ are t-cyclic intervals, and for any $j \in [1, n-1]$, $Q_j \cap Q_{j+1} \ne \varnothing$, then $\bigcup\limits_{i=1}^{n} Q_i$ is a t-cyclic interval.*

Proof can be easily accomplished by induction on n.

Lemma 2. *Let α be a cyclically-interval t-coloring of a graph G, and $P_0 = (x_0, e_1, x_1, ..., x_{k-1}, e_k, x_k)$ be a simple path connecting a vertex $x_0 \in V(G)$ with a vertex $x_k \in V(G)$, $k \ge 2$. Then $\alpha\left[\bigcup\limits_{i=1}^{k-1} J_G(x_i)\right]$ is a t-cyclic interval.*

Proof. If $k = 2$, then the statement follows from the definition of the cyclically-interval t-coloring. Now assume that $k \ge 3$. It is clear that the sets $\alpha[J_G(x_1)], ...,$ $\alpha[J_G(x_{k-1})]$ are t-cyclic intervals with

$$\alpha[J_G(x_j)] \cap \alpha[J_G(x_{j+1})] \ne \varnothing \text{ for any } j \in [1, k-2].$$

Lemma 1 implies that $\alpha\left[\bigcup\limits_{i=1}^{k-1} J_G(x_i)\right]$ is a t-cyclic interval. □

Lemma 3. *Let α be a cyclically-interval t-coloring of a graph G, and $P_0 = (x_0, e_1, x_1, ..., x_{k-1}, e_k, x_k)$ be a simple path connecting a vertex $x_0 \in V(G)$ with a vertex $x_k \in V(G)$, $k \ge 2$. Then at least one of the following statements is true:*

1. $intcyc_1((\alpha(e_1), \alpha(e_k)), t) \subseteq \alpha\left[\bigcup\limits_{i=1}^{k-1} J_G(x_i)\right]$,

2. $intcyc_2((\alpha(e_1), \alpha(e_k)), t) \subseteq \alpha\left[\bigcup\limits_{i=1}^{k-1} J_G(x_i)\right]$.

Proof. Without loss of generality we may assume that $dif((\alpha(e_1), \alpha(e_k)), t) \geq 2$.

Let us assume that none of the statements 1) and 2) is true. Then there are τ_1, τ_2 such that

$$\tau_1 \in intcyc_1((\alpha(e_1), \alpha(e_k)), t), \tau_1 \notin \alpha\left[\bigcup_{i=1}^{k-1} J_G(x_i)\right],$$

$$\tau_2 \in intcyc_2((\alpha(e_1), \alpha(e_k)), t), \tau_2 \notin \alpha\left[\bigcup_{i=1}^{k-1} J_G(x_i)\right],$$

therefore $\{\tau_1, \tau_2\} \cap \alpha\left[\bigcup\limits_{i=1}^{k-1} J_G(x_i)\right] = \varnothing$.

Lemma 2 implies that $\alpha\left[\bigcup\limits_{i=1}^{k-1} J_G(x_i)\right]$ is a t-cyclic interval with

$$\{\alpha(e_1), \alpha(e_k)\} \subseteq \alpha\left[\bigcup_{i=1}^{k-1} J_G(x_i)\right].$$

It is not hard to see that the relations

$$\{\alpha(e_1), \alpha(e_k)\} \subseteq \alpha\left[\bigcup_{i=1}^{k-1} J_G(x_i)\right] \text{ and } \{\tau_1, \tau_2\} \cap \alpha\left[\bigcup_{i=1}^{k-1} J_G(x_i)\right] = \varnothing$$

are incompatible. □

Lemma 4. *If α is a cyclically-interval t-coloring of a tree H, $t \in \Theta(H)$, $V(H) = \{b_1, ..., b_p\}$, $p \geq 1$, then there are vertices $b' \in V(H)$, $b'' \in V(H)$ such that $[1, t] = \alpha[TP(b', b'')]$.*

Proof. Assume the contrary. Suppose that for an arbitrary $b_i \in V(H)$, $b_j \in V(H)$, $\alpha[TP(b_i, b_j)] \subset [1, t]$. Set: $\max\left\{\left|\alpha[TP(b_i, b_j)]\right| / 1 \leq i \leq p, 1 \leq j \leq p\right\} \equiv m_0$. It is clear that $m_0 < t$. Without loss of generality we may assume that $m_0 \geq 2$. Consider the simple path $P_0 = (x_0, e_1, x_1, ..., x_{k-1}, e_k, x_k)$ of the tree H with $\left|\alpha[TP_0]\right| = m_0$. Clearly, without loss of generality, we may assume that $k \geq 2$.

Lemma 2 implies that there are $i' \in [1, t]$, $i'' \in [1, t]$, and $j' \in \{1, 2\}$, for which $\alpha\left[\bigcup\limits_{i=1}^{k-1} J_H(x_i)\right] = intcyc_{j'}[(i', i''), t]$. As $m_0 < t$, there is $\tau_0 \in [1, t]$ such that $\tau_0 \notin intcyc_{j'}[(i', i''), t]$.

Consider an edge $e^1 \in E(H)$ for which $\alpha(e^1) = \tau_0$, and assume that $e^1 = (u_0, u_1)$. Clearly, $e^1 \notin TP_0(x_0, x_k)$.

Without loss of generality we may assume that $\rho_H(u_1, \tilde{V}P_0(x_0, x_k)) < \rho_H(u_0, \tilde{V}P_0(x_0, x_k))$. Let $z_0 \in \tilde{V}P_0(x_0, x_k)$ be the vertex with $\rho_H(u_1, z_0) = \rho_H(u_1, \tilde{V}P_0(x_0, x_k))$. It is not hard to see that $z_0 \in \tilde{V}P_0(x_0, x_k) \backslash intVP_0(x_0, x_k)$ and for any $z' \in \tilde{V}P_0(x_0, x_k) \backslash intVP_0(x_0, x_k)$, $z' \neq z_0$, $\rho_H(u_1, z_0) < \rho_H(u_1, z')$.

Case 1. $z_0 = x_0$. Clearly, $\left|\alpha[TP(u_0, x_k)]\right| \geq m_0 + 1$, which contradicts the choice of P_0.

Case 2. $z_0 = x_k$. This case is considered similarly as the case 1.

Case 3. $z_0 \neq x_0$, $z_0 \neq x_k$.

Clearly, there is $\tilde{x} \in intVP_0(x_0, x_k)$ such that $z_0 \in I_H(\tilde{x})$. Suppose that $\alpha((z_0, \tilde{x})) = \tau'$. Clearly, $i' \neq i''$.

Case 3a. $\tau' = i'$.

Lemma 3, the equalities $\alpha(e^1) = \tau_0$, $\alpha((z_0, \tilde{x})) = i'$, and the definition of the path $P(u_0, \tilde{x})$ imply that $\exists j_1 \in \{1, 2\}$ such that $intcyc_{j_1}[(\tau_0, i'), t] \subseteq \alpha\left[\bigcup_{x \in intVP(u_0, \tilde{x})} J_H(x)\right]$. Consider the edge $\tilde{e} \in TP_0(x_0, x_k)$ with $\alpha(\tilde{e}) = i''$. Assume: $\tilde{e} = (x', x'')$. Without loss of generality we may assume that $\rho_H(z_0, x') < \rho_H(z_0, x'')$. It is not hard to check that $TP(z_0, x'') \subseteq TP_0(x_0, x_k)$, therefore, by the choice of τ_0, we have $\tau_0 \notin \alpha[TP(z_0, x'')]$. Lemma 2 implies that $\alpha[TP(z_0, x'')]$ is a t-cyclic interval.

Clearly, $\exists j_2 \in \{1, 2\}$ such that $\tau_0 \in intcyc_{j_2}((i', i''), t)$, and, therefore, $intcyc_{j_2}((i', i''), t) \nsubseteq \alpha[TP(z_0, x'')]$.

This conclusion, the equalities $\alpha((z_0, \tilde{x})) = i'$, $\alpha(\tilde{e}) = i''$, and Lemma 3 imply that $intcyc_{3-j_2}[(i', i''), t] \subseteq \alpha[TP(z_0, x'')]$, hence $\left|\alpha[TP(u_0, x'')]\right| \geq m_0 + 1$, which contradicts the choice of P_0.

Case 3b. $\tau' = i''$. This case is considered similarly as the case 3a with interchanging of the roles of i' and i''.

Case 3c. $\tau' \notin \{i', i''\}$.

Lemma 3, the equalities $\alpha(e^1) = \tau_0$, $\alpha((z_0, \tilde{x})) = \tau'$, and the definition of the path $P(u_0, \tilde{x})$ imply that $\exists j_1 \in \{1, 2\}$ such that $intcyc_{j_1}[(\tau_0, \tau'), t] \subseteq \alpha\left[\bigcup_{x \in intVP(u_0, \tilde{x})} J_H(x)\right]$. This implies that at least one of the following statements is true:

1. $i' \in intcyc_{j_1}[(\tau_0, \tau'), t]$,
2. $i'' \in intcyc_{j_1}[(\tau_0, \tau'), t]$.

Without loss of generality let us assume that the statement 1) is true. Consider the edge $\tilde{e} \in TP_0(x_0, x_k)$ with $\alpha(\tilde{e}) = i''$. Assume: $\tilde{e} = (x', x'')$. Without loss of generality we may assume that $\rho_H(z_0, x') < \rho_H(z_0, x'')$. It is not hard to check that

$TP(z_0, x'') \subseteq TP_0(x_0, x_k)$, therefore, by the choice of τ_0, we have $\tau_0 \notin \alpha[TP(z_0, x'')]$. Lemma 2 implies that $\alpha[TP(z_0, x'')]$ is a t-cyclic interval.

Clearly, $\exists j_2 \in \{1, 2\}$ such that $\tau_0 \in intcyc_{j_2}((\tau', i''), t)$, and, therefore, $intcyc_{j_2}((\tau', i''), t) \nsubseteq \alpha[TP(z_0, x'')]$. This conclusion, the equalities $\alpha((z_0, \tilde{x})) = \tau'$, $\alpha(\tilde{e}) = i''$, and Lemma 3 imply that $intcyc_{3-j_2}[(\tau', i''), t] \subseteq \alpha[TP(z_0, x'')]$, hence $\big|\alpha[TP(u_0, x'')]\big| \geq m_0 + 1$, which contradicts the choice of P_0. □

Corollary 2. *If α is a cyclically-interval t-coloring of a tree H, where $t \in \Theta(H)$, then there are vertices $x' \in V(H)$, $x'' \in V(H)$ such that $t \leq |TP(x', x'')|$.*

Proof. Since the inequality $\big|\alpha[TP(x, y)]\big| \leq |TP(x, y)|$ holds for arbitrary vertices $x \in V(H)$, $y \in V(H)$, it is not difficult to notice that our statement follows from Lemma 4. □

Corollary 3. *If α is a cyclically-interval $W_{cyc}(H)$-coloring of a tree H, then there are vertices $x' \in V(H)$, $x'' \in V(H)$ such that $W_{cyc}(H) \leq |TP(x', x'')|$.*

Corollary 4. *For any tree H, $W_{cyc}(H) \leq M(H)$.*

Theorem 2. *For any tree H, $W_{cyc}(H) = M(H)$.*

Proof follows from Corollaries 1 and 4.

Corollary 5. *[22] Let H be an arbitrary tree. Then*

1. $H \in \mathfrak{M}$,
2. $w_{cyc}(H) = \Delta(H)$,
3. $W_{cyc}(H) = M(H)$,
4. $\Theta(H) = [\Delta(H), M(H)]$.

Corollary 6. *For an arbitrary tree H and any positive integer t, $H \in \mathfrak{M}_t$ if and only if $H \in \mathfrak{N}_t$.*

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Post-optimal analysis of investment problem with Wald's ordered maximin criteria

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Abstract. We consider Markowitz's multicriteria portfolio optimization problem with Wald's ordered maximin criteria. We obtained lower and upper attainable bounds of the stability radius of lexicographically optimal portfolio.

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Keywords and phrases: Multicriteria optimization, investment portfolio, Wald's maximin efficient criteria, lexicographically optimal portfolio, stability radius.

In the papers [1, 2] we derived the bounds of the stability radius of a Pareto-optimal solution of Markowitz's investment problem with Savage's minimax criteria. In this paper we obtain lower and upper attainable bounds of the stability radius of lexicographical optimum for the Markowitz's multicriteria problem with Wald's maximin criteria.

1 Problem formulation and definitions

Let us consider the multicriterion variant of the investment managing problem based on Markowitz's classical portfolio theory [3]. As a portfolio efficiency criterion we use Wald's maximin criterion. We introduce the following notations: let $N_n = \{1, 2, \ldots, n\}$ be the set of investment projects (assets); N_m be the set of possible financial market states (situation); $x = (x_1, x_2, \ldots, x_n)^T \in X \subseteq \mathbf{E}^n \setminus \{\mathbf{0}\}$ be the investment portfolio, where $\mathbf{E} = \{0, 1\}$, $x_j = 1$ if project $j \in N_n$ is implemented, $x_j = 0$ otherwise. As usual $\mathbf{0}$ is the zero vector of the corresponding dimension.

There exist several approaches to the assessment of efficiency (utility) of investment projects (NPV, NFV, IRR et al.) which take into account the uncertainty and risk in different ways (see for example [4, 5]). Let N_s be the set of indicators of investment projects efficiency. An investment portfolio x is evaluated by $\sum\limits_{j \in N_n} a_{ijk} x_j$, where a_{ijk} is the efficiency indicator $k \in N_s$ of investment project $j \in N_n$ in the case when the market be in state $i \in N_m$. Therefore we may assume that the input data of the problem are determined by the three-dimensional matrix of investment project efficiency A of size $m \times n \times s$ with elements a_{ijk} from $\mathbf{R}$. Let us introduce the vector objective function

$$f(x, A) = (f_1(x, A_1), f_2(x, A_2), \ldots, f_s(x, A_s)),$$

whose partial objectives are well-known Wald's maximin criteria [6]

$$f_k(x, A_k) = \min_{i \in N_m} A_{ik}x = \min_{i \in N_m} \sum_{j \in N_n} a_{ijk}x_j \to \max_{x \in X}, \qquad k \in N_s,$$

where $A_k \in \mathbf{R}^{m \times n}$ is k-th cut of the matrix $A = [a_{ijk}] \in \mathbf{R}^{m \times n \times s}$, $A_{ik} = (a_{i1k}, a_{i2k}, ..., a_{ink})$ is i-th row of that cut. Thus, following Wald's criterion, the investor shows extreme caution when he/she optimizes the efficiency of the portfolio in assuming that the financial market is in the most unprofitable state, i. e. considering the uncertainty of the market state, the investor chooses the maximin strategy.

The problem of finding the set of lexicographically optimal portfolio $L^s(A)$ will be viewed as the multicriterion (s-criteriion) investment problem $Z^s(A)$ with Wald's ordered criteriion, $s \in \mathbf{N}$, where the set $L^s(A)$ is defined in the following traditional way [7–10]

$$L^s(A) = \{x \in X : \ \nexists x' \in X \ (x \underset{A}{\prec} x')\},$$

where

$$x \underset{A}{\prec} x' \quad \Leftrightarrow \ \exists p \in N_s \ (g_p(x, x', A_p) < 0 \quad \& \quad p = \max\{k \in N_s : \ g_k(x, x', A_k) \neq 0\}),$$

$$g_k(x, x', A_k) = f_k(x, A_k) - f_k(x', A_k) = \max_{i' \in N_m} \min_{i \in N_m} (A_{ik}x - A_{i'k}x'), \ k \in N_s. \quad (1)$$

Evidently, the set $L^s(A)$ is a non-empty subset of the Pareto set for any matrix $A \in \mathbf{R}^{m \times n \times s}$. It is also well-known (see e.g. [11]), that the lexicographic set $L^s(A)$ can be determined as a result of sequential solving of s scalar problems:

$$L^s_k(A) := \operatorname{Arg\,min}\{f_k(x, A_k) : \ x \in L^s_{k-1}(A)\}, \qquad k \in N_s,$$

where $L^s_0(A) = X$, $\operatorname{Arg\,min}\{\cdot\}$ is the set of all individual solutions of the corresponding scalar minimization problem. Thus, we have the chain of inclusions

$$X \supseteq L^s_1(A) \supseteq L^s_2(A) \supseteq \ldots \supseteq L^s_s(A) = L^s(A).$$

Therefore, the problem $Z^s(A)$ of fining the lexicographic set $L^s(A)$ can be seen as a problem of sequential minimization of partial objective functions $f_k(x, A_k)$, $k \in N_s$.

The following properties are obvious.

Property 1. *If for a portfolio $x^0 \in X$ it holds that*

$$\forall x \in X \setminus \{x^0\} \quad (g_1(x, x^0, A_1) > 0),$$

then $x^0 \in L^s(A)$.

Property 2. *If for a portfolio $x^0 \in X$ it holds that*

$$\exists x^* \in X \setminus \{x^0\} \quad (g_1(x^*, x^0, A_1) < 0),$$

then $x^0 \notin L^s(A)$.

In portfolio space $\mathbf{R}^n$, market state space $\mathbf{R}^m$ and efficiency (criteria) space $\mathbf{R}^s$, we define the linear metric l_1, i.e.

$$\|A_{ik}\| = \sum_{j \in N_n} |a_{ijk}|, \quad i \in N_m, \quad k \in N_s,$$

$$\|A_k\| = \sum_{i \in N_m} \|A_{ik}\| = \sum_{i \in N_m} \sum_{j \in N_n} |a_{ijk}|, \quad k \in N_s,$$

$$\|A\| = \sum_{k \in N_s} \|A_k\| = \sum_{i \in N_m} \sum_{j \in N_n} \sum_{k \in N_s} |a_{ijk}|.$$

The following inequalities are evident

$$\|A\| \geq \|A_k\| \geq \|A_{ik}\|, \quad i \in N_m, \ k \in N_s. \tag{2}$$

Apart from that, it is easy to see that for any x and x' the following inequalities hold

$$A_{ik}x - A_{i'k}x' \geq -\|A_k\|, \qquad i, \ i' \in N_m, \quad k \in N_s. \tag{3}$$

As usual [9, 13], the stability radius of portfolio $x^0 \in L^s(A)$ is defined as the number

$$\rho^s(x^0, A) = \begin{cases} \sup \ \Xi & \text{if } \ \Xi \neq \emptyset, \\ 0 & \text{if } \ \Xi = \emptyset, \end{cases}$$

where $\Xi = \{\varepsilon > 0 : \ \forall A' \in \Omega(\varepsilon) \quad (x^0 \in L^s(A + A'))\}$, $\Omega(\varepsilon) = \{A' \in \mathbf{R}^{m \times n \times s} : \|A'\| < \varepsilon\}$ is the set of perturbing matrices, $L^s(A + A')$ is the set of lexicographically optimal portfolios in the perturbed problem $Z^s(A + A')$.

Thus, the stability radius defines an extreme level of problem initial data perturbations (elements of matrix A) preserving lexicographic optimality of the portfolio.

2 Stability radius bounds

For $x^0 \in L^s(A)$ and $Z^s(A)$, denote

$$\varphi = \min_{x \in X \setminus \{x^0\}} \ \max_{i \in N_m} \ \min_{i' \in N_m} \ (A_{i'1}x^0 - A_{i1}x).$$

Evidently, $\varphi \geq 0$.

Theorem 1. *Given $Z^s(A)$, the stability radius $\rho^s(x^0, \ A)$, $\ s \geq 1$, of a lexicographically optimal portfolio x^0 has the following lower and upper bounds*

$$\varphi \leq \rho^s(x^0, A) \leq 2\varphi.$$

Proof. Let $x^0 \in L^s(A)$. First we will prove that $\rho^s(x^0, A) \geq \varphi$, which is evident if $\varphi = 0$. Let $\varphi > 0$. According to the definition of φ for every portfolio $x \neq x^0$ the following inequality holds

$$\max_{i \in N_m} \min_{i' \in N_m} (A_{i'1}x^0 - A_{i1}x) \geq \varphi. \tag{4}$$

Let A' be an arbitrary perturbing matrix belonging to $\Omega(\varphi)$. Then, taking into account (1)–(4), we obtain

$$g_1(x^0, x, A_1 + A_1') = \max_{i \in N_m} \min_{i' \in N_m} (A_{i'1}x^0 - A_{i1}x + A_{i'1}'x^0 - A_{i1}'x) \geq$$

$$\geq \max_{i \in N_m} \min_{i' \in N_m} (A_{i'1}x^0 - A_{i1}x) - \|A_1'\| \geq \varphi - \|A_1'\| \geq \varphi - \|A'\| > 0.$$

Therefore, due to Property 1, the portfolio x^0 preserves lexicographic optimality in any perturbed problem $Z^s(A + A')$, $A' \in \Omega(\varphi)$. Hence, $\rho^s(x^0, A) \geq \varphi$.

Further we show that $\rho^s(x^0, A) \leq 2\varphi$. Let $x^* \neq x^0$ be a portfolio such that the following equalities hold

$$g_1(x^0, x^*, A_1) = \max_{i \in N_m} \min_{i' \in N_m} (A_{i'1}x^0 - A_{i1}x^*) = \varphi. \tag{5}$$

The existence of such portfolio comes from the definition of φ.

Let us prove that

$$\forall \varepsilon > 2\varphi \quad \exists A^0 \in \Omega(\varepsilon) \qquad (x^0 \notin L^s(A + A^0)). \tag{6}$$

For this in accordance with Property 2 it is sufficient to construct a perturbing matrix A^0 with cut A_1^0 such that the following conditions hold

$$2\varphi < \|A^0\| < \varepsilon, \tag{7}$$

$$g_1(x^0, x^*, A_1 + A_1^0) < 0. \tag{8}$$

Let

$$i(x^0) = \arg\min\{A_{i1}x^0 : \ i \in N_m\}$$

and consider two possible cases.

Case 1. There exists an index $l \in N_n$ such that $x_l^0 = 1$ and $x_l^* = 0$. We define the elements of the cut $A_1^0 = [a_{ij1}^0] \in \mathbf{R}^{m \times n}$ of the perturbing matrix $A^0 = [a_{ijk}^0] \in \mathbf{R}^{m \times n \times s}$ as follows:

$$a_{ij1}^0 = \begin{cases} -\delta = & \text{if } \ i = i(x^0), \ j = l, \\ 0 & \text{otherwise,} \end{cases}$$

where $2\varphi < \delta < \varepsilon$. The elements of the remaining cuts A_k^0, $k \neq 1$, of the perturbing matrix A^0 set equal to zero. Hence we have

$$A^0_{i(x^0)1}x^0 = -\delta, \qquad A^0_{i(x^0)1}x^* = 0, \tag{9}$$

$$A^0_{i1}x^0 = A^0_{i1}x^* = 0, \qquad i \in N_m \setminus \{i(x^0)\}, \tag{10}$$

$$\|A^0\| = \|A^0_1\| = \delta.$$

Therefore, the inequality (7) is true.

As a result we have

$$f_1(x^0, A_1 + A^0_1) = \min\Big\{(A_{i(x^0)1} + A^0_{i(x^0)1})x^0, \min_{i \neq i(x^0)} (A_{i1} + A^0_{i1})x^0\Big\} = f_1(x^0, A_1) - \delta,$$

$$f_1(x^*, A_1 + A^0_1) = \min\Big\{(A_{i(x^0)1} + A^0_{i(x^0)1})x^*, \min_{i \neq i(x^0)} (A_{i1} + A^0_{i1})x^*\Big\} = f_1(x^*, A_1).$$

Thus, from (5) and $\delta > \varphi$ we verify the validity of the inequality (8).

Case 2. $x^0 \leq x^*$. Then in view of the inequalities $x^0 \neq x^* \neq \mathbf{0}$ there exists a pair of indexes $(p \times q) \in N_n \times N_n$ such that $x^0_p = 0$, $x^*_p = 1$, $x^0_q = x^*_q = 1$. The elements of the cut $A^0_1 = [a^0_{ij1}] \in \mathbf{R}^{m \times n}$ we define as follows:

$$a^0_{ij1} = \begin{cases} -\delta & \text{if } \ i = i(x^0), \ j = q, \\ \delta & \text{if } \ i = i(x^0), \ j = p, \\ 0 & \text{otherwise}, \end{cases}$$

where $2\varphi < 2\delta < \varepsilon$. The elements of the remaining cuts A^0_k, $k \neq 1$ of the perturbing matrix A^0 set equal to zero. Then the equations (9), (10) and $\|A^0_1\| = \|A^0\| = 2\delta$ hold, i.e. (7) holds. Further, repeating the reasoning of the case 1 and taking into account $\delta > \varphi$, we see that the inequality (8) is true.

As a result we construct in the first and second case the perturbing matrix A^0 such that the formula (6) is true. Hence, $\rho^s(x^0, A) \leq 2\varphi$. □

3 Lower bound attainability

We show that the lower bound of the stability radius $\rho^s(x^0, A)$, indicated in Theorem 1, is attainable.

Theorem 2. *There exists a class of investment problems $Z^s(A)$, $s \geq 1$, such that the stability radius of any lexicographically optimal portfolio x^0 is expressed by the formula $\rho^s(x^0, A) = \varphi$.*

Proof. To prove the equality $\rho^s(x^0, A) = \varphi$, where $\varphi > 0$, it is sufficient to identify a class of problems with $\rho^s(x^0, A) \leq \varphi$.

Assume x^* be such that the equality (5) holds. Since $x^0 \neq x^*$, there exists an index $l \in N_n$ such that $x^0_l \neq x^*_l$. We will assume that $x^0_l = 1$ and $x^*_l = 0$ (this is the actual specific of the class of problems we would like to identify).

Assuming $\varepsilon > \varphi$, we define the elements of the cut $A_1^0 = [a_{ij1}^0] \in \mathbf{R}^{m\times n}$ of the perturbing matrix $A^0 = [a_{ijk}^0] \in \mathbf{R}^{m\times n\times s}$ as follows

$$a_{ij1}^0 = \begin{cases} -\delta & \text{if } i = i(x^0),\ j = l, \\ 0 & \text{otherwise,} \end{cases}$$

$$\varphi < \delta < \varepsilon, \tag{11}$$

$$i(x^0) = \arg\min\{A_{i1}x^0 : \quad i \in N_m\}. \tag{12}$$

All elements in the remaining cuts A_k^0, $k \in N_s \setminus \{1\}$, of the perturbing matrix A^0 set equal to zero. As a result we get

$$A_{i(x^0)1}^0 x^0 = -\delta, \quad A_{i1}^0 x^* = 0, \quad i \in N_m,$$

$$A_{i1}^0 x^0 = 0, \quad i \in N_m \setminus \{i(x^0)\},$$

$$\|A^0\| = \|A_1^0\| = \delta, \quad A^0 \in \Omega(\varepsilon).$$

Now due to (12) it is easy to see that

$$f_1(x^*, A_1 + A_1^0) = \min_{i\in N_n}(A_{i1} + A_{i1}^0)x^* = \min_{i\in N_n} A_{i1}x^* = f_1(x^*, A_1),$$

$$f_1(x^0, A_1 + A_1^0) = \min\Big\{(A_{i(x^0)1} + A_{i(x^0)1}^0)x^0, \min_{i\neq i(x^0)}(A_{i1} + A_{i1}^0)x^0\Big\} =$$

$$= \min\Big\{f_1(x^0, A_1) - \delta, \min_{i\neq i(x^0)} A_{i1}x^0\Big\} = f_1(x^0, A_1) - \delta.$$

Therefore, based on (5) and (11), we obtain

$$g_1(x^0, x^*, A_1 + A_1^0) = g_1(x^0, x^*, A_1) - \delta = \varphi - \delta < 0.$$

The last together with Property 2 imply that for any $\varepsilon > \varphi$ there exists $A^0 \in \Omega(\varepsilon)$ such that $x^0 \notin L^s(A + A^0)$. Hence, $\rho^s(x^0, A) \le \varphi$. □

Consider a short numerical example illustrating Theorem 2.

Example. Let $m = 2$, $n = 3$, $s = 1$, $X = \{x^0, x^*\}$, $x^0 = (0, 1, 1)^T$, $x^* = (1, 1, 0)^T$,

$$A = \begin{pmatrix} -6 & 5 & -1 \\ 2 & -2 & 3 \end{pmatrix}.$$

Then $f(x^0, A) = 1$, $f(x^*, A) = -1$, i.e. x^0 is an optimal portfolio of $Z^1(A)$. Since $\varphi = 2$ then according to Theorem 1 $\rho^1(x^0, A) \ge 2$. If we define the perturbing matrix as follows

$$A^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\delta \end{pmatrix}, \quad \delta > 2,$$

then we have $\|A^0\| = \delta$ and $f(x^0, A + A^0) = 1 - \delta < -1 = f(x^*, A + A^0)$. Therefore, $x^0 \notin L^1(A + A^0)$, and hence $\rho^1(x^0, A) \le 2$. Finally, $\rho^1(x^0, A) = 2 = \varphi$.

4 Upper bound attainability

Before proving upper bound attainability 2φ we consider one of the properties of the matrixes by size $m \times 2$, $m \geq 2$.

When $\varphi > 0$ the matrix $W = [u, v] \in \mathbf{R}^{m\times 2}$, $m \geq 2$ with $u = (u_1, u_2, \ldots, u_m)^T$ and $v = (v_1, v_2, \ldots, v_m)^T$ is called φ-special if the inequality holds

$$\min_{i\in N_m}(u_i + v_i) - \min_{i\in N_m} u_i < \varphi.$$

Lemma. *The matrix $W = [u, v] \in \mathbf{R}^{m\times 2}$, $m \geq 2$, with the norm $\|W\| < 2\varphi$, where $\varphi > 0$, is φ-special.*

Proof. The proof is by induction on $m \geq 2$.

First we proof the lemma for $m = 2$. Let

$$W = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}.$$

Let us show that the inequality

$$\min\{u_1 + v_1, u_2 + v_2\} - \min\{u_1, u_2\} < \varphi \tag{13}$$

follows from the inequality $\|W\| < 2\varphi$, i.e from the inequality

$$|u_1| + |u_2| + |v_1| + |v_2| < 2\varphi. \tag{14}$$

Without loss of generality we assume that

$$u_1 + v_1 \leq u_2 + v_2. \tag{15}$$

We consider two possible cases.

Case 1. $u_1 \leq u_2$. Then the inequality (13) in view of (15) takes the form $\varphi > v_1$. We give the proof by contradiction. Let

$$\varphi \leq v_1. \tag{16}$$

From (15) and (16) we have

$$\varphi \leq -u_1 + u_2 + v_2,$$

and from (14) and (16) we derive

$$\varphi > |u_1| + |u_2| + |v_2|.$$

These inequalities lead to the contradiction

$$0 \leq |u_2| - u_2 + |v_2| - v_2 < -(u_1 + |u_1|) \leq 0.$$

Case 2. $u_1 > u_2$. Then the inequality (13) in view of (15) transform into inequality $\varphi > u_1 + v_1 - u_2$. Suppose the contrary

$$\varphi \leq u_1 + v_1 - u_2. \tag{17}$$

Therefore, taking into account (15) we have $\varphi \leq v_2$. Hence in view of (14) we find

$$\varphi > |u_1| + |u_2| + |v_1|.$$

This inequality with (17) leads to the contradiction

$$0 \leq |u_1| - u_1 + |v_1| - v_1 < -(u_2 + |u_2|) \leq 0.$$

Further we assume that the lemma is true for $m \geq 2$ and we show that the matrix $W = [u, v] \in \mathbf{R}^{(m+1)\times 2}$ with column $u = (u_1, u_2, \ldots, u_{m+1})^T$, $v = (v_1, v_2, \ldots, v_{m+1})^T$ and norm $\|W\| < 2\varphi$ is φ-special.

Let

$$i_1 = \arg\min\{u_i + v_i : \ i \in N_{m+1}\},$$

$$i_2 = \arg\min\{u_i : \ i \in N_{m+1}\}$$

and let the index $l \in N_{m+1}$ is such that

$$l \neq i_1 \ \ \& \ \ l \neq i_2. \tag{18}$$

Doped from the matrix W the l-th row, we have a matrix $W' \in \mathbf{R}^{m\times 2}$ with the norm $\|W'\| \leq \|W\| < 2\varphi$. Then by induction the matrix W' is φ-special, i.e. the following inequality is true:

$$\min_{i\in N_{m+1}\setminus\{l\}} (u_i + v_i) - \min_{i\in N_{m+1}\setminus\{l\}} u_i < \varphi.$$

In addition, according to (18) we have the equalities:

$$\min_{i\in N_{m+1}} (u_i + v_i) = u_{i_1} + v_{i_1} = \min_{i\in N_{m+1}\setminus\{l\}} (u_i + v_i),$$

$$\min_{i\in N_{m+1}} u_i = u_{i_2} = \min_{i\in N_{m+1}\setminus\{l\}} u_i.$$

Hence, the matrix W is φ-special. □

Theorem 3. *For $\varphi > 0$ there exists a class of investment problems $Z^s(A)$, $s \geq 1$, such that the stability radius of a lexicographically optimal portfolio x^0 is expressed by the formula*

$$\rho^s(x^0, A) = 2\varphi.$$

Proof. Due to Theorem 1 it is sufficient to identify a class of problems with $\rho^s(x^0, A) \geq 2\varphi$. Let us show that there exists a class when $m \geq 2$ and $X = \{x^0, x^*\}$, $x^0 \in L^s(A)$, $x^* \neq x^0$.

According to the definition of φ the following equality holds

$$\max_{i \in N_m} \min_{i' \in N_m} (A_{i'1}x^* - A_{i1}x^0) = \varphi. \tag{19}$$

Further we assume that the cut A_1 of the matrix A and portfolios x^0 and x^* satisfy the following conditions:

(a) $\forall i, i' \in N_m \quad \forall x \in X \qquad (A_{i1}x = A_{i'1}x)$,

(b) $x^0 \leq x^*$.

The condition *(a)* shows that $A_{i1}x$ for any portfolio $x \in X$ does not depend from index i. Denoting it by $\sigma(x)$ we have the following form of the equality (19)

$$\sigma(x^0) - \sigma(x^*) = \varphi.$$

From that equality for any matrix $A_1' \in \mathbf{R}^{m \times n}$ we derive

$$g_1(x^0, x^*, A_1 + A_1') = \min_{i \in N_m} (A_{i1} + A_{i1}')x^0 - \min_{i \in N_m} (A_{i1} + A_{i1}')x^* =$$

$$= \sigma(x^0) - \sigma(x^*) + \min_{i \in N_m} A_{i1}'x^0 - \min_{i \in N_m} A_{i1}'x^* = \varphi - \gamma, \tag{20}$$

where

$$\gamma = \min_{i \in N_m} (A_{i1}'x^0 + A_{i1}'(x^* - x^0)) - \min_{i \in N_m} A_{i1}'x^0.$$

Now let the perturbing matrix $A' = [a_{ijk}'] \in \Omega(2\varphi)$. Let us consider the matrix $W = [u, v] \in \mathbf{R}^{m \times 2}$ with column $u = A_1'x^0$ and $v = A_1'(x^* - x^0)$, where $A_1' = [a_{ij1}']$. Then for portfolio $x = (x_1, x_2, \ldots, x_n)^T$ we introduce the following notations: let

$$N(x) = \{j \in N_n : \quad x_j = 1\},$$

and also, taking into account (b) and $x^* \neq x^0$, we have

$$\|W\| = \|A_1'x^0\| + \|A_1'(x^* - x^0)\| = \sum_{i \in N_m} \mid \sum_{j \in N(x^0)} a_{ij1}' \mid + \sum_{i \in N_m} \mid \sum_{j \in N(x^* - x^0)} a_{ij1}' \mid \leq$$

$$\leq \sum_{i \in N_m} \sum_{j \in N(x^*)} |a_{ij1}'| \leq \|A_1'\| \leq \|A'\| < 2\varphi.$$

Therefore due to the lemma the matrix W is φ-special, i.e. the inequality $\gamma < \varphi$ holds, which with (20) gives us

$$g_1(x^0, x^*, A_1 + A_1') > 0.$$

Hence due to Property 1 we conclude that for any perturbing matrix $A' \in \Omega(2\varphi)$ the inclusion $x^0 \in L^s(A + A')$ holds, i.e. $\rho^s(x^0, A) \geq 2\varphi$. $\square$

5 Stability conditions

The portfolio $x^0 \in L^s(A)$ is called stable if $\rho^s(x^0, A) > 0$. Additionally, we introduce the set of strict lexicographically optimal portfolios of $Z^s(A)$:

$$S^s(A) = \{x \in X : \quad \forall x' \in X \setminus \{x\} \qquad (f_1(x, A_1) > f_1(x', A_1))\}.$$

Obviously, $S^s(A) \subseteq L^s(A)$ for any $A \in \mathbf{R}^{m \times n \times s}$. Apart from that it is clear that $S^s(A)$ can be empty.

Theorem 4. *For a lexicographically optimal portfolio x^0 of $Z^s(A)$ the following statements are equivalent:*

(i) $x^0 \in S^s(A)$,

(ii) portfolio x^0 is stable,

(iii) $\varphi > 0$.

Proof. $(i) \Rightarrow (ii)$. Let $x^0 \in L^s(A)$ be a strict lexicographically optimal portfolio, i. e. $x^0 \in S^s(A)$. Then for every $x \in X \setminus \{x^0\}$ we have

$$\xi(x) = \max_{i \in N_m} \min_{i' \in N_m} (A_{i'1}x^0 - A_{i1}x) = g_1(x^0, x, A_1) > 0.$$

Thus, due to Theorem 1 we conclude $\rho^s(x^0, A) \geq \varphi = \min\Big\{\xi(x) : \quad x \in X \setminus \{x^0\}\Big\} > 0$, i. e. $x^0 \in L^s(A)$ is stable.

$(ii) \Rightarrow (iii)$. Assume $x^0 \in L^s(A)$ be stable. Then according to Theorem 1 $2\varphi \geq \rho^s(x^0, A) > 0$, i. e. $\varphi > 0$.

$(iii) \Rightarrow (i)$. According to the definition of φ for any portfolio $x \neq x^0$ the inequality $\varphi \leq f_1(x, A_1) - f_1(x^0, A_1)$ is true. Hence from the inequality $\varphi > 0$ we have $x^0 \in S^s(A)$. □

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Matrix algorithm for Polling models with PH distribution

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Abstract. Polling systems provide performance evaluation criteria for a variety of demand-based, multiple-access schemes in computer and communication systems [1]. For studying this systems it is necessary to find their important characteristics. One of the important characteristics of these systems is the k-busy period [2]. In [3] it is showed that analytical results for k-busy period can be viewed as the generalization of classical Kendall functional equation [4]. A matrix algorithm for solving the generalization of classical Kendall functional equation is proposed. Some examples and numerical results are presented.

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Keywords and phrases: Polling Model, Kendall Equation, Generalization of Classical Kendall Functional Equation, k-Busy Period, Matrix Algorithm.

1 Introduction

In this paper we study one of the important characteristics for queueing system of Polling type, the k-busy period. A Polling model is a system of multiple queues accessed by a single server in cyclic order. We consider a queueing system of Polling type with semi-Markov switching. Handling mechanism for this system is given by Polling table $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, r\}$, where the function f shows that at the stage $j, j = \overline{1, n}$, the user number $k, k = \overline{1, r}$, is served. The items (messages) of the user k, arrive according to Poisson distribution with parameter $\tilde{\lambda}_k$. The service time for the items of class k is a random variable B_k with the distribution function $B_k(x) = P\{B_k < x\}$. Duration of the orientation from one user to another one is a random variable C_k with the distribution function $C_k(x) = P\{C_k < x\}$. In this paper, the matrix algorithm of determining the k-busy period for Polling systems is obtained, and some numerical examples are presented.

2 The k-busy period

Definition 2.1 *The k-busy period* is a measure of the time that expires from when a server begins to process, after an empty queue, to when the k-queue becomes empty again for the first time [3].

Denote by Π_k^δ the length of the k-busy period, and by

$$\Pi_k^\delta(x) = P\{\Pi_k^\delta < x\},$$

its distribution function. Consider that

$$\pi_k^\delta(s) = \int_0^\infty e^{-sx} d\Pi_k^\delta(x)$$

is the Laplace-Stieltjes transform of distribution function of k-busy period.
The following result is known [3]:
Theorem 2.1 *The function $\pi_k^\delta(s)$ is determined from the equation*

$$\pi_k^\delta(s) = c_k(s + \tilde{\lambda}_k - \tilde{\lambda}_k \pi_k(s))\pi_k(s), \tag{2.1}$$

where

$$\pi_k(s) = \beta_k(s + \tilde{\lambda}_k - \tilde{\lambda}_k \pi_k(s)), \tag{2.2}$$

and $c_k(s)$ and $\beta_k(s)$ denote the Laplace-Stieltjes transforms of distribution functions $C_k(x)$ and $B_k(x)$,

$$c_k(s) = \int_0^\infty e^{-sx} dC_k(x),$$

$$\beta_k(s) = \int_0^\infty e^{-sx} dB_k(x).$$

A matrix algorithm for solving the generalization of classical Kendall functional equation (2.1) is proposed. For this, the matrix algorithm for solving Kendall functional equation in Polling models [5] was used. It has no analytical solution, but it can be solved numerically with the accuracy required. Both distributions $B_k(x)$ and $C_k(x)$ were considered distributions of Phase Type (PH). All results were obtained in terms of the Laplace-Stieltjes transform.

3 Laplace-Stieltjes Transform of Phase Type distribution

Phase type distributions are getting to be very commonly used these days after Neuts [6] made them very popular and easily accessible. They are very often referred to as the PH distribution. The PH distribution has became very popular in stochastic modeling because it allows numerical tractability of some difficult problems and in addition, several distributions encountered in queueing seem to resemble the PH distribution.
Phase type distributions are distributions of the time until absorption in an absorbing CTMC (Continuous Time Markov Chain). Consider an $(n+1)$ absorbing CTMC

with the state space $\{0, 1, \ldots, n\}$ and let the state 0 be the absorbing state. The transition matrix Q of this absorbing Markov chain is given as

$$Q = \begin{pmatrix} T & T^0 \\ 0 & 0 \end{pmatrix}, \tag{3.1}$$

where the $n \times n$ matrix T satisfies $T_{ii} < 0$, for $1 \leq i \leq n$, and $T_{ij} \geq 0$, for $i \neq j$. $Te + T^0 = 0$, $\alpha^t e = 1$, and $(\alpha^t, 0)$ is the initial probability vector of Q. We suppose that all states $1, \ldots, n$ are transient.

The probability distribution $F(x)$ of the time until absorbtion in the state 0, corresponding to the initial probability vector $(\alpha^t, 0)$, is given by:

$$F(x) = 1 - \alpha^t e^{Tx} e, \text{ for } x \geq 0. \tag{3.2}$$

The phase type distribution with parameter α^t and T is usually written as PH distribution with representation (α^t, T). Let find the Laplace-Stieltjes transform of phase type distribution with representation (α^t, T):

$$\frac{dF(x)}{dx} = -\frac{d}{dx}\alpha^t e^{Tx} e = -\alpha^t [\frac{d}{dx} e^{Tx}] e,$$

where $e^{Tx} = \sum_{k=1}^{\infty} \frac{(Tx)^i}{i!}$.

$$\frac{de^{Tx}}{dx} = e^{Tx} \cdot \frac{d(Tx)}{dx} = e^{Tx} \cdot T,$$

$$\frac{dF(x)}{dx} = -\alpha^t e^{Tx} Te = \alpha^t e^{Tx}(-Te) = \alpha^t e^{Tx} T^0.$$

$$f(s) = \int_0^\infty e^{-sx} dF(x) = \int_0^\infty e^{-sx} \alpha^t e^{Tx} T^0 dx = \alpha^t \int_0^\infty e^{-sx} I e^{Tx} dx T^0$$

$$= \alpha^t \int_0^\infty e^{-(sI-T)x} dx T^0 = \alpha^t (sI - T)^{-1} T^0.$$

The Laplace-Stieltjes transform $f(s)$ of the PH distribution with representation (α^t, T), is:

$$f(s) = \alpha^t (sI - T)^{-1} T^0. \tag{3.3}$$

4 Matrix form for Kendall equation

We know that

$$\pi_k(s) = \beta_k(s + \tilde{\lambda}_k(1 - \pi_k(s))). \tag{4.1}$$

Suppose that $B_k(x)$ is a PH distribution with representation (α_k^t, T), where

$$T_k = \begin{pmatrix} -\lambda_k & \lambda_k & \dots & 0 & 0 \\ 0 & -\lambda_k & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -\lambda_k & \lambda_k \\ 0 & 0 & \dots & 0 & -\lambda_k \end{pmatrix},$$

and

$$T_k{}^0 = \begin{pmatrix} 0 & 0 & \dots & \lambda_k \end{pmatrix}^t.$$

The Laplace-Stieltjes transform $\beta_k(s)$ of probability distribution $B_k(x)$ of the time until absorbtion in the state 0 is

$$\beta_k(s) = \int_0^\infty e^{-sx} dB_k(x) = \alpha^t (sI - T_k)^{-1} T_k{}^0.$$

Then, from equation (4.1) we obtain:

$$\pi_k(s) = \alpha_k^t([s + \tilde{\lambda}_k(1 - \pi_k(s))]I + A_k)^{-1} A_k e,$$

where $A_k = -T_k$.

Denote $g_k(s) = s + \tilde{\lambda}_k(1 - \pi_k(s))$, then

$$a_k(s) = 1 - \pi_k(s) = 1 - \alpha_k^t (g_k(s)I + A_k)^{-1} A_k e = \alpha_k^t e - \alpha_k^t (g(s)_k I + A)^{-1} A_k e =$$
$$= \alpha_k^t [I - (g_k(s)I + A)^{-1} A_k] e = \alpha_k{}^t (g_k(s)I + A_k)^{-1} [g_k(s)I + A_k - A_k] e =$$
$$= \alpha_k^t g_k(s)(g_k(s)I + A_k)^{-1} e.$$

Denote $(g_k(s)I + A_k)^{-1} e = y_k(s)$, then the matrix form for Kendall equation is

$$a_k(s) = \alpha_k{}^t g_k(s) y_k(s), \tag{4.2}$$

where $y_k(s)$ can be found by solving these simultaneous linear equations

$$(g_k(s)I + A_k) y_k(s) = e. \tag{4.3}$$

5 Matrix Algorithm for Solving Kendall Equation

We have to calculate

$$a_k(s) = \alpha_k^t g_k(s) y_k(s), \tag{5.1}$$

where $g_k(s) = s + \lambda_k a_k(s)$ and $y_k(s) = (g_k(s)I + A_k)^{-1} e$.
For calculating $y_k(s)$ it is necessary to solve these simultaneous linear equations

$$(g_k(s)I + A_k) y_k(s) = e, \tag{5.2}$$

where $e = (11\dots1)^t$, $A_k = -T_k$ and

$$T_k = \begin{pmatrix} -\lambda_k & \lambda_k & \dots & 0 & 0 \\ 0 & -\lambda_k & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -\lambda_k & \lambda_k \\ 0 & 0 & \dots & 0 & -\lambda_k \end{pmatrix},$$

and

$$T_k^0 = \begin{pmatrix} 0 & 0 & \dots & \lambda_k \end{pmatrix}^t.$$

The simultaneous linear equations (5.2) have the analytical solution. Let write these simultaneous linear equations in explicit form:

$$\begin{pmatrix} g_k(s)+\lambda_k & -\lambda_k & \dots & 0 & 0 \\ 0 & g_k(s)+\lambda_k & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & g_k(s)+\lambda_k & -\lambda_k \\ 0 & 0 & \dots & 0 & g_k(s)+\lambda_k \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-2} \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}.$$

Then

$$\begin{aligned} y_{n-1} &= 1/(g_k(s)+\lambda_k) = \omega_k, \\ y_i &= (1+\lambda_k y_{i+1})\omega_k = \omega_k + \omega_k\lambda_k y_{i+1}, i = \overline{1, n-2}, \\ y_0 &= \omega_k + \omega_k\lambda_k y_1 = \frac{1-(\lambda_k\omega_k)^n}{g_k(s)}. \end{aligned} \tag{5.3}$$

First prove relation (5.3).

$$\begin{aligned} y_0 &= \omega_k + \omega_k\lambda_k y_1 = \omega_k + \omega_k\lambda_k(\omega_k + \omega_k\lambda_k y_2) = \\ &= \omega_k + \omega_k^2\lambda_k + (\omega_k\lambda_k)^2 y_2 = \omega_k + \omega_k^2\lambda_k + (\omega_k\lambda_k)^2(\omega_k + \omega_k\lambda_k y_3) = \\ &= \omega_k + \omega_k^2\lambda_k + \omega_k^3\lambda_k^2 + (\omega_k\lambda_k)^3 y_3 = \omega_k(1 + \omega_k\lambda_k + (\omega_k\lambda_k)^2 + (\omega_k\lambda_k)^3\frac{y_3}{\omega_k}) = \\ &= \dots = \omega_k(1 + \omega_k\lambda_k + (\omega_k\lambda_k)^2 + \dots + (\omega_k\lambda_k)^{n-1}\frac{y_{n-1}}{\omega_k}) = \omega_k\sum_{j=0}^{n-1}(\lambda_k\omega_k)^j = \\ &= \frac{\omega_k(1-(\lambda_k\omega_k)^n)}{1-\lambda_k\omega_k} = \frac{\omega_k(1-(\lambda_k\omega_k)^n)}{\omega_k(\frac{1}{\omega_k}-\lambda_k)} = \frac{1-(\lambda_k\omega_k)^n}{g_k(s)+\lambda_k-\lambda_k} = \frac{1-(\lambda_k\omega_k)^n}{g_k(s)}, \end{aligned}$$

because

$$|\omega_k\lambda_k| = |\frac{\lambda_k}{g_k(s)+\lambda_k}| < 1.$$

So, the solution of the simultaneous linear equations (5.2) is:

$$y_0 = \frac{1-(\lambda_k\omega_k)^n}{g_k(s)},$$

$$y_{i+1} = \frac{y_i - \omega_k}{\lambda_k\omega_k}, i = \overline{1, n-2},$$
$$y_{n-1} = \omega_k.$$

Then $a_k(s) = \alpha_k^t g_k(s) y_k(s)$ will be

$$a_k(s) = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \end{pmatrix} g_k(s) \begin{pmatrix} \frac{1-(\lambda_k\omega_k)^n}{g_k(s)} \\ y_1 \\ \vdots \\ y_{n-2} \\ y_{n-1} \end{pmatrix} = 1 - (\lambda_k\omega_k)^n.$$

So, $a_k(s) = 1 - (\lambda_k\omega_k)^n$, where $\alpha_k^t e = 1$, and we start with $a_k(s) = 1$ and $\alpha_k^t = (10\dots0)$, the remaining values we give by ourselves ($\lambda_k, \tilde{\lambda}_k$ and s).

6 Matrix Form for Generalization of Classical Kendall Functional Equation

It is known that analytical results for k-busy period can be viewed as a generalization of the classical Kendall functional equation

$$\pi_k^\delta(s) = c_k(s + \tilde{\lambda}_k(1 - \pi_k(s)))\pi_k(s). \tag{6.1}$$

Suppose that $C_k(x)$ is a PH distribution with representation (α_k^t, P_k), where

$$P_k = \begin{pmatrix} -\delta_k & \delta_k & \dots & 0 & 0 \\ 0 & -\delta_k & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -\delta_k & \delta_k \\ 0 & 0 & \dots & 0 & -\delta_k \end{pmatrix},$$

and

$$P_k^0 = \begin{pmatrix} 0 & 0 & \dots & \delta_k \end{pmatrix}^t.$$

The Laplace-Stieltjes transform is:

$$c_k(s) = \int_0^\infty e^{-sx} dB_k(x) = \alpha^t (sI - P_k)^{-1} {P_k}^0. \tag{6.2}$$

From equation (6.1) we obtain:

$$\pi_k^\delta(s) = \alpha_k^t([(s + \tilde{\lambda}_k(1 - \pi_k(s)))\pi_k(s)]I + D_k)^{-1} D_k e,$$

where $D_k = -P_k$. It is known from Section 4 that $\pi_k(s) = 1 - a_k(s)$ and $g_k(s) = s + \tilde{\lambda}_k(1 - \pi_k(s))$, then

$$b_k(s) = 1 - \pi_k^\delta(s) = 1 - \alpha_k^t(g_k(s)(1 - a_k(s))I + D_k)^{-1} D_k e =$$

$$= \alpha_k^t e - \alpha_k^t(g(s)_k(1 - a_k(s))I + D_k)^{-1} D_k e = \alpha_k^t[I - (g_k(s)(1 - a_k(s))I + D_k)^{-1} D_k]e =$$

$$= \alpha_k^t(g_k(s)(1 - a_k(s))I + D_k)^{-1}[g_k(s)(1 - a_k(s))I + D_k - D_k]e =$$

$$= \alpha_k^t g_k(s)(1 - a_k(s))(g_k(s)(1 - a_k(s))I + D_k)^{-1} e.$$

If we denote $(g_k(s)(1 - a_k(s))I + D_k)^{-1} e = \tilde{y}_k(s)$, then

$$b_k(s) = {\alpha_k}^t g_k(s)(1 - a_k(s))\tilde{y}_k(s), \tag{6.3}$$

where $\tilde{y}(s)$ can be found by solving these simultaneous linear equations

$$(g_k(s)(1 - a_k(s))(s)I + D_k)\tilde{y}_k = e. \tag{6.4}$$

7 Matrix Algorithm for Solving Generalization of Classical Kendall Functional Equation

We have to calculate

$$b_k(s) = \alpha_k^t g_k(s)(1 - a_k(s))\tilde{y}_k(s), \tag{7.1}$$

where $g_k(s) = s + \tilde{\lambda}_k a_k(s)$ and $\tilde{y}_k(s) = (g_k(s)(1 - a_k(s))I + D_k)^{-1} e$.
To calculate $y_k(s)$ it is necessary to solve these simultaneous linear equations

$$(g_k(s)(1 - a_k(s))I + D_k)\tilde{y}_k(s) = e, \tag{7.2}$$

where $e = (11 \dots 1)^t$, $D_k = -P_k$ and

$$P_k = \begin{pmatrix} -\delta_k & \delta_k & \dots & 0 & 0 \\ 0 & -\delta_k & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -\delta_k & \delta_k \\ 0 & 0 & \dots & 0 & -\delta_k \end{pmatrix},$$

and

$$P_k^0 = \begin{pmatrix} 0 & 0 & \dots & \delta_k \end{pmatrix}^t.$$

The simultaneous linear equations (7.2) have the analytical solution. Denote

$$\mathbf{A} = \begin{pmatrix} g_k(s)(1-a_k(s))+\delta_k & -\delta_k & \dots & 0 & 0 \\ 0 & g_k(s)(1-a_k(s))+\delta_k & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & g_k(s)(1-a_k(s))+\delta_k \end{pmatrix},$$

$$\mathbf{b} = \begin{pmatrix} y_0 & y_1 & \dots & y_{n-1} \end{pmatrix}^t,$$

$$\mathbf{e} = \begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix}^t.$$

These simultaneous linear equations have the matrix form:

$$\mathbf{A} \cdot \mathbf{b} = \mathbf{e}$$

Then

$$\begin{aligned} \tilde{y}_{n-1} &= 1/(g_k(s)(1-a_k(s))+\delta_k) = \gamma_k, \\ \tilde{y}_i &= (1+\delta_k\tilde{y}_{i+1})\gamma_k = \gamma_k + \gamma_k\delta_k\tilde{y}_{i+1}, i = \overline{1, n-2}, \\ \tilde{y}_0 &= \gamma_k + \gamma_k\delta_k\tilde{y}_1 = 1 - \frac{\delta_k}{g_k(s)(1-a_k(s))+\gamma_k}. \end{aligned} \tag{7.3}$$

First prove relation (7.3).

$$\begin{aligned} \tilde{y}_0 &= \gamma_k + \gamma_k\delta_k\tilde{y}_1 = \gamma_k + \gamma_k\delta_k(\gamma_k + \gamma_k\delta_k\tilde{y}_2) = \\ &= \gamma_k + \gamma_k^2\delta_k + (\gamma_k\delta_k)^2\tilde{y}_2 = \gamma_k + \gamma_k^2\delta_k + (\gamma_k\delta_k)^2(\gamma_k + \gamma_k\delta_k\tilde{y}_3) = \\ &= \gamma_k + \gamma_k^2\delta_k + \gamma_k^3\delta_k^2 + (\gamma_k\delta_k)^3\tilde{y}_3 = \gamma_k(1 + \gamma_k\delta_k + (\gamma_k\delta_k)^2 + (\gamma_k\delta_k)^3\frac{y_3}{\gamma_k}) = \\ &= \dots = \gamma_k(1 + \gamma_k\delta_k + (\gamma_k\delta_k)^2 + \dots + (\gamma_k\delta_k)^{n-1}\frac{\tilde{y}_{n-1}}{\gamma_k}) = \gamma_k\sum_{j=0}^{n-1}(\delta_k\gamma_k)^j = \\ &= \frac{\gamma_k(1-(\delta_k\gamma_k)^n)}{1-\delta_k\gamma_k} = \frac{\gamma_k(1-(\delta_k\gamma_k)^n)}{\gamma_k(\frac{1}{\gamma_k}-\delta_k)} = \\ &= \frac{1-(\delta_k\gamma_k)^n}{g_k(s)(1-a_k(s))+\delta_k-\delta_k} = \frac{1-(\delta_k\gamma_k)^n}{g_k(s)(1-a_k(s))}, \end{aligned}$$

because

$$|\gamma_k\delta_k| = |\frac{\delta_k}{g_k(s)(1-a_k(s))+\delta_k}| < 1.$$

In this case the solution of the simultaneous linear equations (4.1) is:

$$\tilde{y}_0 = \frac{1-(\delta_k\gamma_k)^n}{g_k(s)(1-a_k(s))},$$

$$\tilde{y}_{i+1} = \frac{y_i-\gamma_k}{\delta_k\gamma_k}, i = \overline{1, n-2},$$

$$\tilde{y}_{n-1} = \gamma_k.$$

Then $b_k(s) = \alpha_k^t g_k(s)(1-a_k(s))\tilde{y}_k(s)$ will be

$$b_k(s) = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \end{pmatrix} g_k(s)(1-a_k(s)) \begin{pmatrix} \dfrac{1-(\delta_k\gamma_k)^n}{g_k(s)(1-a_k(s))} \\ \tilde{y}_1 \\ \vdots \\ \tilde{y}_{n-2} \\ \tilde{y}_{n-1} \end{pmatrix} = 1-(\delta_k\gamma_k)^n.$$

So, $b_k(s) = 1-(\delta_k\gamma_k)^n$, where $\alpha_k^t e = 1$, and we start with $a_k(s) = 1$ and $\alpha_k^t = (10\dots0)$, the remaining values we give by ourselves ($\delta_k, \tilde{\lambda}_k$ and s).

8 Conclusion

The main purpose of research of the Polling system is to determine the characteristics of system development. But analytical formulas can not always be used directly, so great attention is paid to numerical algorithms. For finding numerical solutions for the k-busy period, in terms of Laplace-Stieltjes transform, PH distribution was used. A matrix algorithm for solving the generalization of classical Kendall functional equation was obtained. Some numerical examples are presented.

9 Examples

Example 1. The type of distribution function taken for $B_k(x)$ and $C_k(x)$ are PH distributions with representation (α^t, T_k), (α^t, P_k), so

$$B_k(x) = 1-\alpha_t e^{T_k x} e,\ x > 0,$$

$$C_k(x) = 1-\alpha_t e^{P_k x} e,\ x > 0,$$

with the following parameters:
$\lambda_k = \{0.5; 0.6; 0.3; 0.4; 0.5; 0.2; 0.6; 0.6; 0.2; 0.1\}$,
$\tilde{\lambda}_k = \{0.2; 0.3; 0.4; 0.2; 0.6; 0.7; 0.8; 0.4; 0.2; 0.3\}$,
$\delta_k = \{0.3; 0.4; 0.1; 0.2; 0.6; 0.8; 0.5; 0.4; 0.4; 0.8\}$,
$s = 0.5$.

The results of the program are presented in Table 1.

k	$\pi_k(s)$	$\pi_k^\delta(s)$	k	$\pi_k(s)$	$\pi_k^\delta(s)$
1	0.012692	0.864672	6	0.000060	0.999554
2	0.014688	0.865907	7	0.003161	0.959972
3	0.000978	0.957140	8	0.010383	0.891422
4	0.006395	0.895401	9	0.000542	0.995270
5	0.002997	0.973018	10	0.000017	0.999915

Table 1

Example 2. The type of distribution function taken for $B_k(x)$ and $C_k(x)$ are PH distributions with representation (α^t, T_k), (α^t, P_k), so

$$B_k(x) = 1 - \alpha_t e^{T_k x} e, x > 0,$$

$$C_k(x) = 1 - \alpha_t e^{P_k x} e, x > 0,$$

with the following parameters:
$\lambda_k = \{0.2; 0.3; 0.1; 0.5; 0.6; 0.7; 0.4; 0.8; 0.4; 0.5; 0.3; 0.7; 0.8; 0.4; 0.6; 0.9; 0.3; 0.4; 0.5; 0.4\}$,
$\tilde{\lambda}_k = \{0.2; 0.3; 0.5; 0.2; 0.3; 0.7; 0.8; 0.4; 0.3; 0.5; 0.1; 0.5; 0.8; 0.4; 0.3; 0.6; 0.4; 0.9; 0.4; 0.2\}$,
$\delta_k = \{0.5; 0.4; 0.8; 0.4; 0.4; 0.7; 0.4; 0.3; 0.8; 0.2; 0.1; 0.5; 0.4; 0.7; 0.5; 0.4; 0.7; 0.9; 0.4; 0.3\}$,
$s = 0.5$.
The results of the program are presented in Table 2.

k	$\pi_k(s)$	$\pi_k^\delta(s)$	k	$\pi_k(s)$	$\pi_k^\delta(s)$
1	0.002444	0.986440	11	0.012414	0.750670
2	0.005566	0.956771	12	0.029777	0.796091
3	0.000068	0.999659	13	0.021775	0.763410
4	0.030767	0.812228	14	0.009064	0.954890
5	0.034760	0.766861	15	0.034760	0.807550
6	0.018947	0.881203	16	0.043203	0.644660
7	0.003083	0.960978	17	0.003927	0.980092
8	0.051492	0.569886	18	0.002451	0.984919
9	0.012501	0.951740	19	0.016581	0.864629
10	0.012555	0.785024	20	0.017712	0.851134

Table 2

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On asymptotic representation of singular solutions of the model elliptic equation near boundary and formulation of singular boundary conditions

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Abstract. In the work the asymptotic representation of singular solution of the elliptic model Sobolev problem near components of arbitrary dimensions of boundary is specified. Using this asymptotical representation of solutions, the singular boundary conditions are formulated. The solvability of boundary problem with singular boundary conditions is proved.

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1 Introduction

This work is the continuation of [1], devoted to integral and asymptotic representation of singular solutions of elliptic equations near components of small dimensions of boundary. The problem of representation of solutions near boundary is interesting not only in itself, but also in connection with reduction of the boundary value problem to integral, integro-differential or differential equations on the boundary. In [2, 3] S. L. Sobolev for the first time formulated and studied the boundary value problem for polyharmonic equation in a domain with boundary, consisting of a submanifold of diverse dimensions (and afterwards this problem was named the Sobolev boundary problem).

Later the work [4] was published, where the Sobolev boundary value problem is studied for a general elliptic equation of order $2m$. In this work it is proved that the number of boundary conditions on the submanifold of boundary depends on the order of regularity of solutions $u(x)$ from Sobolev space $H^s(\Omega)$ near submanifold.

Moreover, it was proved that the solution of the elliptic equation admits asymptotic representation with respect to the power $p^{-\nu}$ and $\ln r$ (where $r = dist(x, \mathbb{R}^q)$), any explicit formulae to compute the coefficients have been done.

Using the integral representation of solution of the boundary value problem with Green function, in [1] the asymptotic representation of the components of the singular solutions generated by distributions with support on the submanifold $\mathbb{R}^q$ was obtained.

2 On elliptic model problem. Asymptotic representation of component of singular solution near boundary

Let $\mathbb{R}^n$ be a Euclidean n-dimensional space, $\mathbb{R}^q \subset \mathbb{R}^n$ a subspace of $\mathbb{R}^n$, $x = (x', x'') = (x_1, ..., x_q, x_{q+1}, ..., x_n)$ a point of $\mathbb{R}^n$, $D_x = (D_{x'}, D_{x''})$, $\Omega = \mathbb{R}^n \backslash \mathbb{R}^q$. By $C^\alpha(\Omega)$, $C_0^\alpha(\Omega), C^\alpha(\mathbb{R}^q)$, $C_0^\alpha(\mathbb{R}^q)$ we denote the usual Hölder spaces, spaces of functions with finite support in Ω and $\mathbb{R}^q$, respectively, $H^s(\Omega)$, $H^s(\mathbb{R}^q)$, $s \in \mathbb{R}^1$, are Hilbertian Sobolev spaces in Ω and $\mathbb{R}^q$, respectively [5,6].

Let $\mathcal{L}(D_x)$ be a homogeneous elliptic operator of order $2m$ with constant coefficients. In domain Ω we consider the elliptic equation

$$\mathcal{L}(D_x)\, u(x) = f(x), \tag{1}$$

where

$$u(x) \in H^s(\Omega), \ f(x) \in H^{s-2m}(\Omega), \ s \in \mathbb{R}^1. \tag{2}$$

First of all we consider the problem of asymptotic behavior of singular solutions $u(x) \in H^s(\Omega)$ near submanifold $\mathbb{R}^q$, and obtain the formulae of asymptotic representation of solutions, generated by distributions with support on the $\mathbb{R}^q$. For this we observe that it is known [5,6] that the non-zero element $f(x) \in H^{s-2m}(\mathbb{R}^n)$ is concentrated in $\mathbb{R}^q$ if and only if $s < 2m - \theta/2$ ($\theta = co\dim \mathbb{R}^q = n - q$) and there exist elements $f_\sigma(x') \in H^{s-2m+|\sigma|+\theta/2}(\mathbb{R}^q)$, $|\sigma| \le \tau = [2m - s - \theta/2]$ such that

$$f(x) = \sum_{|\sigma| \le \tau} D_\nu^\sigma \left(f_\sigma(x') \times \delta(x'') \right), \ \nu = x'', \tag{3}$$

where $[\alpha]$ is the integer part of number α, $D_\nu^\sigma = D_{x''}^\sigma = \dfrac{\partial^{\sigma_{q+1}}}{\partial x_{q+1}^{\sigma_{q+1}}} \cdots \dfrac{\partial^{\sigma_n}}{\partial x_n^{\sigma_n}}$, and $f_\sigma(x') \times \delta(x'')$ is the direct product of distributions, $|\sigma| = \sum_{i=q+1}^n \sigma_i$.

In [1], using the Green function of boundary value problem, the integral representation of solution of equation (1) near $\mathbb{R}^q$ is obtained, from which the asymptotic representation of singular part of solution $u(x)$ in Ω is obtained.

Really, let $G(x, y) = E(x - y) + g(x, y)$ be the Green function of homogeneous Dirichlet problem in the ball B_R of radius R (sufficiently large), where $E(x)$ is a fundamental solution of equation (1) in $\mathbb{R}^n$, and $g(x, y)$ is the solution of equation (1) in Ω, satisfying the condition $g(x, y)|_{B_R} = E(x - y)|_{\partial B_R}$.

Write the formulae of integral representation of solution of Dirichlet problem

$$u(x) = \int_{\mathbb{R}^n} G(x, y) f(y) dy$$

for $f(x) \in C_0^\infty$. After that approximate $f(x)$ with functions $f_\varepsilon(x) \in C_0^\infty(\mathbb{R}^n)$, $f_\varepsilon(x) \underset{\varepsilon \to 0}{\to} f(x)$ in $H^{s-2m}(\mathbb{R}^n)$, then integrating by parts with respect to variable x'',

and passing to the limit as $\varepsilon \to 0$, we obtain the integral representation of solution $u(x)$:

$$u(x) = \sum_{|\sigma|\le\tau} \int_{\mathbb{R}^q} \bar{D}^{\sigma}_{x''} E\left(x'-y', x''\right) f_\sigma\left(y'\right) dy' + \tilde{u}\left(x\right) =$$

$$= \sum_{|\sigma|\le\tau} \int_{\mathbb{R}^q} \bar{D}^{\sigma}_{x''} E\left(z', x''\right) f_\sigma\left(x'-z'\right) dz' + \tilde{u}\left(x\right) \equiv \sum_{|\sigma|\le\tau} v_\sigma\left(x\right) + \tilde{u}\left(x\right), \tag{4}$$

where $\tilde{u}(x)$ is a regular, bounded function, $\bar{D}_{x''} = -D_{x''}$.

It is known [7, 8] that

$$\left|D^{\sigma}_{x''} E\left(x'-y', x''\right)\right| \le c\left|x-y\right|^{2m-n-|\sigma|} \left|\ln\left|x-y\right|\right|,$$

where $\ln|x-y|$ is dropped for $2m-n-|\sigma|<0$. Moreover, if $2m-n-|\sigma|<0$, then $E^{(\sigma)}\left(z', x''\right)$ are homogeneous functions of degrees $2m-n-|\sigma|$ and if $n-2m+|\sigma|\ge q$, i.e. $n-q-2m+|\sigma| = \theta-2m+|\sigma| \overset{def}{\equiv} \alpha_\sigma \ge 0$, then the integrals $v_\sigma(x)$ are singular or hypersingular integrals with homogeneous kernels [7,8]. Now consider the singular and hypersingular integrals $v_\sigma(x)$. In [1], using the known procedure of regularization of divergent integrals (separation of the finite part in the Hadamard sense), by separating the singular and regular parts, the asymptotic representations of the divergent integrals $v_\sigma\left(x\right)$ near $\mathbb{R}^q$ are obtained. For convenience, here we shortly expose this known procedure [1].

Let $n\ge 3,\ r=|x''|,\ \rho=|x'|$. Denote by

$$\mathcal{P}_\alpha\left(x', z'\right) f\left(x'\right) = \sum_{\lambda=0}^{\alpha} \sum_{|k'|=\lambda} \frac{f^{(k')}\left(x'\right)}{k'!}\left(-z'\right)^{k'} \equiv \sum_{\lambda=0}^{\alpha} P_\lambda\left(x', z'\right) f\left(x'\right)$$

the segment of the Taylor expansion of the function $f\left(x'-z'\right)$ near the point $z'=0$, where $k'=(k_1,\ldots,k_q)$,

$$v_{\sigma 0}\left(x\right) = \int\limits_{\mathbb{R}^q} \bar{E}^{(\sigma)}\left(z', x''\right)\left(f_\sigma\left(x'-z'\right) - \mathcal{P}_{\alpha_\sigma-1}\left(x', z'\right) f_\sigma - \theta\left(z'\right) P_{\alpha_\sigma}\left(x', z'\right) f_\sigma\right) dz' \tag{5}$$

is the regularization (finite part) of the divergent integral $v_\sigma(x)$ at the point $z'=0$, $\theta\left(z'\right)=1$ for $|z'|\le 1$ and $\theta\left(z'\right)=0$ for $|z'|>1$. In [1] it is proved that the integrals $v_\sigma(x)$ could be presented in the form

$$v_\sigma\left(x\right) = v_{\sigma 0}\left(x\right) - \int\limits_{\mathbb{R}^q} \bar{E}^{(\sigma)}\left(z', x''\right) P_{\alpha_\sigma-1}\left(z', D'_x\right) f_\sigma\left(x'\right) dz' -$$

$$-\int\limits_{|z'|<1} P_{\alpha_\sigma}(z', D'_x)\, f_\sigma dz' \equiv v_{\sigma 0}\left(x\right) + \sum_{\lambda=0}^{\alpha_\sigma-1} \int\limits_{\mathbb{R}^q} \bar{E}^{(\sigma)}\left(z', x''\right) P_\lambda(z', D'_x)\, f_\sigma\left(z'\right) dz' + \tag{6}$$

$$+\int\limits_{|z'|<1} \bar{E}^{(\sigma)}\left(z', x''\right) P_{\alpha_\sigma}\left(z', D'_x\right) f_\sigma dz' \equiv v_{\sigma 0}\left(x\right) + \sum_{\lambda=0}^{\alpha_\sigma-1} \mathcal{I}_\lambda\left[f_\sigma\right] + \mathcal{I}_{\alpha_\sigma}\left[f_\sigma\right],$$

where

$$\mathcal{I}_\lambda\left[f_\sigma\right]=(-1)^\lambda\sum_{|k'|=\lambda}A_{\sigma k'}\left(\omega''\right)\frac{f_\sigma^{(k')}\left(x'\right)}{k'!}r^{-\alpha_\sigma+\lambda}\equiv Q_{\sigma\lambda}\left(\omega'',D_x'\right)f_\sigma\left(x'\right)r^{-\alpha_\sigma+\lambda},\quad(7)$$

$$A_{\sigma k'}\left(\omega''\right)=\int\limits_{\mathbb{R}^q}E^{(\sigma)}\left(\xi',\omega''\right)\xi'^{k'}d\xi',\ \ \omega''=x''/\left|x''\right|,\quad(8)$$

and

$$\mathcal{I}_{\alpha_\sigma}\left[f_\sigma\right]=-A_\sigma\left(D_x'\right)f_\sigma\left(x'\right)\ln r+B_\sigma\left(D_x'\right)f_\sigma\left(x'\right)+o\left(r\right),$$

with $o\left(r\right)\to 0$ as $r\to 0$,

$$A_\sigma\left(D_x'\right)f_\sigma\left(x'\right)=(-1)^{\alpha_\sigma}\sum_{|k'|=\alpha_\sigma}a_{\sigma k'}\frac{f_\sigma^{(k')}\left(x'\right)}{k'!},\ \ a_{\sigma k'}=\int\limits_{|\omega'|=1}E^{(\sigma)}\left(\omega',0\right)\left(\omega'\right)^{k'}d\omega',\quad(9)$$

$$B_\sigma\left(D_x'\right)f_\sigma\left(x'\right)=(-1)^{\alpha_\sigma}\sum_{|k'|=\alpha_\sigma}b_{\sigma k'}\left(\omega''\right)\frac{f_\sigma^{(k')}\left(x'\right)}{k'!},\quad(10)$$

and $b_{\sigma k'}\left(\omega''\right)$ is the integral

$$b_{\sigma k'}(\omega'')=\int\limits_{|\omega'|=1}\omega'^{k'}d\omega'\left(\int\limits_0^1E^{(\sigma)}(\rho\omega',\omega'')\rho^{|k'|+q-1}d\rho+\int\limits_1^\infty\left(E^{(\sigma)}(\rho\omega',\omega'')-E^{(\sigma)}(\omega',0)\right)\tfrac{1}{\rho}d\rho\right).$$

Hence, for divergent integrals $v_\sigma\left(x'\right)$ (singular and hypersingular) we obtain the representations

$$\begin{aligned}v_\sigma\left(x\right)=v_{\sigma_0}\left(x\right)+\sum_{\lambda=0}^{\alpha_\sigma-1}Q_{\sigma\lambda}\left(D_x'\right)f_\sigma(x')r^{-\alpha_\sigma+\lambda}-\\-A_\sigma\left(D_x'\right)f_\sigma(x')\ln r+B_\sigma\left(D_x'\right)f_\sigma(x')+o\left(r\right),\end{aligned}\quad(11)$$

where the functions $v_{\sigma_0}\left(x\right)$ and operators $Q_{\sigma\lambda}\left(D_x'\right),\ A_\sigma\left(D_x'\right),\ B_\sigma\left(D_x'\right)$ are defined by (5), (7), (9) and (10), $o(r)$ tends to zero as r tends to zero.

3 Asymptotic representation of singular part of integer solution near $\mathbb{R}^q$

Here, using the asymptotical representation of components $v_\sigma\left(x\right)$ of singular solution $u(x)$ near $\mathbb{R}^q$, generated by distribution $f(x)$, concentrated on the manifold $\mathbb{R}^q$, the asymptotic representation of integer solution $v(x)$ near $\mathbb{R}^q$ is obtained.

Really, summing the equality (11) ovwr σ for all σ such that $\alpha_\sigma \geq 0$, the asymptotic representation for the singular part $v(x)$ of the solution $u(x)$ near $\mathbb{R}^q$ is obtained:

$$v(x)=\sum_{\sigma:\,\alpha_\sigma\geq 0} v_{\sigma 0}\left(x\right)+\sum_{\sigma:\,\alpha_\sigma>0}\sum_{\lambda=0}^{\alpha_\sigma-1} Q_{\sigma\lambda}\left(D'_x\right) f_\sigma\left(x'\right) r^{-\alpha_\sigma+\lambda}+\sum_{\sigma:\,\alpha_\sigma\geq 0} A_\sigma\left(D'_x\right)\ln r+$$
$$+\sum_{\sigma:\,\alpha_\sigma\geq 0} B_\sigma\left(D'_x\right) f_\sigma\left(x'\right)+o\left(r\right)\equiv v_0(x)+w(x)+w_0\left(x\right)+B\left(D'_x\right) f\left(x'\right)+o\left(r\right). \tag{12}$$

Here by $v_0\left(x\right)$, $w\left(x\right)$ and $w_0\left(x\right)$ we denoted the first three sums of right hand side of equality (12), $o\left(r\right)\to 0$ when $r\to 0$. The equality (12) is the asymptotic representation of singular part of solution $u(x)$ near submanifold $\mathbb{R}^q$ with respect to the power $r^{-\nu}$ and $\ln r$. But in order to obtain an asymptotic ordered representation with respect to the ascending order of power $r^{-\nu}$ and $\ln r$ it is necessary to transform the equality (12). For this, at first, we consider the function $w(x)$ and transform it into an ordered sum with respect to the ascending order of power $r^{-\nu}$. Since $-\alpha_\mu+\lambda=-\left(\mu-\lambda+\theta-2m\right)$, the expression $\mu-\lambda+\theta-2m$ is constant on the any straight line $\mu-\lambda+\theta-2m=\nu$. Therefore, it is natural to denote $\mu-\lambda+\theta-2m=\nu$, and to obtain an ordered sum with respect to the ascending order of power $r^{-\nu}$ it remains to change the order of summing over λ, μ and ν. From the inequality $\nu=\mu-\lambda+\theta-2m\geq 1$ it follows that $\mu\geq 2m+\lambda-\theta+1\geq 2m-\theta+1$ and, since $\mu=|\sigma|\geq 0$, we have $\mu\geq\mu_1=\max\left(0,2m-\theta+1\right)$. Therefore, $\mu_1\leq|\sigma|=\mu\leq\tau$ and $\nu_1\leq\nu\leq\nu_2$, where $\nu_1=\mu_1+\theta-2m,\ \nu_2=\tau+\theta-2m$.

For $w(x)$ we obtain the representation

$$w(x)=\sum_{\mu=\mu_1}^{\tau}\sum_{\lambda=0}^{\alpha_\mu-1}\left(\sum_{|\sigma|=\mu}\sum_{|k'|=\lambda} A_{\sigma k'}\left(\omega''\right)\frac{f_\sigma^{(k')}\left(x'\right)}{k'!}\right) r^{-\alpha_\mu+\lambda}\equiv$$
$$\equiv\sum_{\mu=\mu_1}^{\tau}\sum_{\lambda=0}^{\alpha_\mu-1} P_{\mu\lambda}\left(\omega'',D'_x\right) f\left(x'\right) r^{\lambda-\alpha_\mu}= \tag{13}$$
$$=\sum_{\nu=\nu_1}^{\tau+\theta-2m}\sum_{\substack{\mu,\lambda:\\ \mu-\lambda=\nu+2m-\theta}} P_{\mu\lambda}\left(\omega'',D'_x\right) f\left(x'\right) r^{-\nu}=\sum_{\nu=\nu_1}^{\tau+\theta-2m} M_\nu\left(\omega'',D'_x\right) f\left(x'\right) r^{-\nu},$$

where by $P_{\mu\lambda}\left(\omega'',D'_x\right) f\left(x'\right)$ we denoted the double sum over σ and k' from right hand side of equality (13),

$$P_{\mu\lambda}\left(\omega'',D'_x\right) f\left(x'\right)=\sum_{|\sigma|=\mu}\sum_{|k'|=\lambda} A_{\sigma k'}\left(\omega''\right)\frac{f_\sigma^{(k')}\left(x'\right)}{k'!}, \tag{14}$$

where $A_{\sigma k'}\left(\omega''\right)=\int_{\mathbb{R}^q}\bar{E}^{(\sigma)}\left(\xi',\omega''\right)\xi'^{k'}d\xi'$. It remains to transform the expression

$w_0(x)$. Detailing the structure of sum, which defines the function $w_0(x)$, after ordered summation over σ, we obtain

$$w_0(x) = M_0\left(D'_x\right) f\left(x'\right) \ln r = \sum_{\mu,\lambda:\, \mu-\lambda=\nu+2m-\theta} P_{\mu\lambda}\left(D'_x\right) f\left(x'\right) \ln r, \tag{15}$$

where $P_{\mu\lambda}\left(D'_x\right) f\left(x'\right) = \sum\limits_{|\sigma|=\mu} \sum\limits_{|k'|=\lambda} a_{\sigma k'} \frac{f_\sigma^{(k')}(x')}{k'!}$, $a_{\sigma k'} = \int\limits_{|\omega'|=1} E^{(\sigma)}\left(\omega', 0\right) \omega'^{k'} d\omega'$.

Thus, substituting in (13) these functions $w(x)$ and $w_0(x)$ with their transformed expressions, we obtain the following

Theorem 1. *Let functions $f_\sigma\left(x'\right) \in C_0^{\alpha_\sigma+1}\left(\mathbb{R}^q\right)$. Then the singular part $v(x)$ of solution $u(x)$ near $\mathbb{R}^q$ is represented by*

$$\begin{aligned} v(x) = v_0(x) + \sum_{\nu=1}^{\tau+\theta-2m} M_\nu\left(\omega'', D'_x, f\right) r^{-\nu} + \\ + M_0\left(D'_x\right) f\left(x'\right) \ln r + \sum_{\sigma:\, \alpha_\sigma \geq 0} B_\sigma\left(\omega'', D'_x\right) f\left(x'\right) + o\left(r\right), \end{aligned} \tag{16}$$

where $M_\nu\left(\omega'', D'_x, f\right)$, $M_0\left(D'_x\right) f\left(x'\right)$ is defined by formulae (13), (15), respectively, $o\left(r\right) \to 0$ when $r \to 0$.

4 Formulation of the boundary value problem with singular boundary conditions

In the general theory of elliptic boundary value problems in domain Ω with smooth boundary $\partial\Omega$, the boundary problem is reduced to a system of pseudodifferential equations on the boundary $\partial\Omega$. This system is a system of regular integral (Fredholm) equations in the case of smooth solutions up to $\partial\Omega$ or a system of differential equations in the case of singular solutions.

Here, using the obtained formulae (16) of asymptotic representation of singular parts of solutions $u(x)$ near boundary $\mathbb{R}^q$, we formulate, firstly, the formal model boundary value problem with singular boundary conditions on the $\mathbb{R}^q$:

In the domain $\Omega = \mathbb{R}^n \backslash \mathbb{R}^q$ find the solutions of elliptic equation $L\left(D_x\right) u(x) = 0$ that have near $\mathbb{R}^q$ the given singular asymptotic representation:

$$z(x) = \sum_{\nu=1}^{\tau+\theta-2m} \Phi_\nu\left(\omega'', x'\right) r^{-\nu} + \Phi_0\left(\omega'', x'\right) \ln r + \tilde{z}\left(x\right), \tag{17}$$

where $\tilde{z}\left(x\right)$ is a regular bounded function.

Formally, equating the coefficients by the same power $r^{-\nu}$ and $\ln r$ from equalities (16) and (17), we obtain

$$M_\nu\left(\omega'', D'_x\right) f\left(x'\right) = \sum_{\substack{\mu,\lambda:\, \mu-\lambda+\theta-2m=\nu \\ \nu=\tau+\theta-2m,\ldots,1,}} P_{\mu\lambda}\left(\omega'', D'_x\right) f\left(x'\right) = \Phi_\nu\left(\omega'', x'\right), \tag{18}$$

$$M_0\left(\omega'', D'_x\right) f\left(x'\right) = \Phi_0\left(\omega'', x'\right). \tag{19}$$

The system of equations (18), (19) is a system of linear partial differential equations with unknown density $f_\sigma\left(x'\right)$, $\sigma:\ \mu_0 \le |\sigma| \le \tau$ and the solvability of boundary problem with singular boundary conditions is reduced to the solvability of system of differential equations (18), (19). This system is rather complicated, since the number of unknown densities $f_\sigma\left(x'\right)$, as well as the number of equations, depends on s and on the difference $\theta - 2m$, too.

Now we pass to the study of the structure of equations of system (18)-(19) depending on s, τ and θ. Denote by Π_m the linear space of all homogeneous polynomials of degree m. It is known [9] that the dimension of space Π_m $(\dim \Pi_m)$ is equal to $C_{m+\theta-1}^{\theta-1}$, where C_n^k are the binomial coefficients. Hence, the number of unknown functions $f_\sigma\left(x'\right)$ in the system (18)-(19) is equal to $\Pi = \sum_{m=\mu_0}^{\tau} \dim \Pi_m = \sum_{m=\mu_0}^{\tau} C_{m+\theta-1}^{\theta-1}$ which is greater (for $\theta > 1$) than the number of equations from system (18)-(19). Return to the system of equations (18)-(19). Since $f_\sigma\left(x'\right) \in H^{s-2m+|\sigma|+\theta/2}\left(\mathbb{R}^q\right)$ and $f_\sigma^{(k')}\left(x'\right) \in H^{s-2m+|\sigma|+\theta/2-|k'|}\left(\mathbb{R}^q\right)$, then for any multiindex σ and k' with $|\sigma| - |k'| = \mu - \lambda = \nu - \theta + 2m$ the left hand sides of equations (18), (19) belong to spaces $H^{s+\nu-\theta/2}\left(\mathbb{R}^q\right)$, $\nu = \tau + \theta - 2m, \ldots, 1, 0$. Therefore, the equalities (18), (19) define a bounded operator U from the space $E_1 = \prod\limits_{|\sigma|} H^{s-2m+|\sigma|+\theta/2}\left(\mathbb{R}^q\right)$, $|\sigma| \le \tau$, to the space $E_2 = \prod\limits_{\nu} H^{s+\nu-\theta/2}\left(\mathbb{R}^q\right), \nu = 0, 1, \ldots, \tau + \theta - 2m$.

Now we begin to investigate the system of equations (18), (19). At first, we will see that the number of equations of system (18)-(19), as well as the condition of solvability of this system, depends on the numbers $\theta - 2m$ and τ. Therefore, we consider two cases: a) $\theta - 2m \le 0$ and b) $\theta - 2m > 0$.

a) Assume that $\theta - 2m \le 0$. In this case the number of equations in the system (18)-(19) is $\tau + \theta - 2m$, which is no more than τ. The system of equations (18)-(19) takes the form

$$P_{\tau 0}\left(\omega'', D'_x\right) f\left(x'\right) = \sum_{|\sigma|=\tau} A_{\sigma 0}\left(\omega''\right) f_\sigma\left(x'\right) = \Phi_{\tau+\theta-2m}\left(\omega'', x'\right),$$

$$\begin{gathered} P_{\tau-10}\left(\omega'', D'_x\right) f\left(x'\right) + P_{\tau 1}\left(\omega'', D'_x\right) f\left(x'\right) = \Phi_{\tau+\theta-2m+1}\left(\omega'', x'\right), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ P_{2m+1-\theta 0}\left(\omega'', D'_x\right) f\left(x'\right) + \ldots + P_{\tau\tau-2m-1+\theta}\left(\omega'', D'_x\right) f\left(x'\right) = \Phi_{2m-\theta}\left(\omega'', x'\right), \end{gathered} \tag{20}$$

$$M_0\left(\omega'', D'_x\right) f\left(x'\right) = \Phi_0\left(\omega'', x'\right).$$

Now we see that in each of these equations the expressions $P_{\mu 0}(\omega'', f)$ are linear combinations of unknown functions $f_\sigma(x')$ with the coefficients $A_{\sigma 0}$ (the moments of fundamental solution $E(x)$). The system of equations (18)–(19) is of triangular form. Since $\sum_{|\sigma|=\tau} |A_{\sigma 0}(\omega'')| \neq 0$ (otherwise the first condition in (18) is absent), the first equation from (18) is solvable. Assume that functions $f_\sigma(x')$ with $|\sigma| = \tau$ are solutions to the first equation of (18). Substituting this functions $f_\sigma(x')$ with $|\sigma| = \tau$ in the other equations, for functions $f_\sigma(x')$ with $|\sigma| \leq \tau - 1$ we obtain also a triangular system. Continuing this procedure, we express all the functions $f_\sigma(x')$ with $\mu_0 \leq |\sigma| \leq \tau$ only through the functions $\Phi_\tau, \Phi_{\tau-1}, \ldots, \Phi_{\tau+\theta-2m}$. It means that the system of equations (18)–(19) is solvable.

b) Assume that $\theta > 2m$. In this case the system of equations (18)–(19) contains $\tau + \theta - 2m$ equations, their number is greater than τ. Repeating the above mentioned procedure, we express all the functions $f_\sigma(x')$ with $0 \leq |\sigma| \leq \tau$ by $\Phi_\tau(x'), \Phi_{\tau-1}(x'), \ldots, \Phi_{\tau+\theta-2m}(x')$. Substituting all functions $f_\sigma(x')$ in other equations, we obtain that the first $\theta - 2m$ equations of (18)-(19) become identities, and the functions $\Phi_0(x'), \ldots, \Phi_{\tau+\theta-2m}(x')$ are connected by (18), (19).

From what was mentioned above it follows that the formal model boundary value problem with singular boundary conditions is not solvable for any admissible right hand sides $\Phi_\nu(\omega', x')$. To obtain a solvable singular boundary value problem it is necessary to reformulate this problem in the following way:

In the domain $\Omega = \mathbb{R}^n \backslash \mathbb{R}^q$ find the solutions $u(x)$ of the model elliptic equation

$$L(D_x)\, u(x) = 0 \tag{21}$$

that have near $\mathbb{R}^q$ the asymptotic representation (16) with coefficients $M_\nu(\omega'', x')$, satisfying the conditions

$$M_\nu\left(\omega'', D'_x\right) f\left(x'\right) = \Phi_\nu\left(\omega'', x'\right), \quad \nu = \nu_1, \ldots, \nu_2 = \tau + \theta - 2m. \tag{22}$$

Repeating the similar reasons we obtain

Theorem 2. *For any admissible functions $\Phi_\nu(\omega'', x')$ the model boundary value problem with singular boundary conditions (18), (19) is solvable.*

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Moment analysis of the telegraph random process

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Abstract. We consider the Goldstein-Kac telegraph process $X(t),\ t > 0$, on the real line $\mathbb{R}^1$ performed by the random motion at finite speed c and controlled by a homogeneous Poisson process of rate $\lambda > 0$. Using a formula for the moment function $\mu_{2k}(t)$ of $X(t)$ we study its asymptotic behaviour, as c, λ and t vary in different ways. Explicit asymptotic formulas for $\mu_{2k}(t)$, as $k \to \infty$, are derived and numerical comparison of their effectiveness is given. We also prove that the moments $\mu_{2k}(t)$ for arbitrary fixed $t > 0$ satisfy the Carleman condition and, therefore, the distribution of the telegraph process is completely determined by its moments. Thus, the moment problem is completely solved for the telegraph process $X(t)$. We obtain an explicit formula for the Laplace transform of $\mu_{2k}(t)$ and give a derivation of the the moment generating function based on direct calculations. A formula for the semi-invariants of $X(t)$ is also presented.

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1 Preliminaries

Consider the one-dimensional stochastic process performed by a particle that starts at the time instant $t = 0$ from the origin $x = 0$ of the real line $\mathbb{R}^1$ and moves with some finite constant speed c. The initial direction of the motion (positive or negative) is taken on with equal probabilities $1/2$. The motion is driven by a homogeneous Poisson process of rate $\lambda > 0$ as follows. As a Poisson event occurs, the particle instantaneously takes on the opposite direction and keeps moving with the same speed c until the next Poisson event occurrence, then it takes on the opposite direction again independently of its previous motion, and so on. This random motion has first been studied by Goldstein [12] and Kac [16] and was called the *telegraph process* afterwards (the latter article [16] is a reprinting of an earlier 1956 work).

Let $X(t)$ denote the particle's position on $\mathbb{R}^1$ at an arbitrary time instant $t > 0$. Since the speed c is finite, then, at the time instant $t > 0$, the distribution $\Pr\{X(t) \in dx\}$ is concentrated in the finite interval $[-ct, ct]$ which is the support of the distribution of $X(t)$. The density $f(x,t),\ x \in \mathbb{R}^1,\ t \geq 0$, of the distribution $\Pr\{X(t) \in dx\}$ has the structure

$$f(x,t) = f_s(x,t) + f_{ac}(x,t),$$

where $f_s(x,t)$ and $f_{ac}(x,t)$ are the densities of the singular (with respect to the Lebesgue measure on the line) and of the absolutely continuous components of the distribution of $X(t)$, respectively.

The singular component of the distribution is, obviously, concentrated at two terminal points $\pm ct$ of the interval $[-ct, ct]$ and corresponds to the case when no one Poisson event occurs until the moment t and, therefore, the particle does not change its initial direction (the probability of this event is $e^{-\lambda t}$).

The density $f_{ac}(x,t)$ of the absolutely continuous components of the distribution corresponds to the case when at least one Poisson event occurs by moment t and, therefore, the particle changes its initial direction (the probability of this event is $1 - e^{-\lambda t}$). The support of this part of the distribution is the open interval $(-ct, ct)$.

The principal result by Goldstein [12] and Kac [16] states that the density $f = f(x,t),\ x \in [-ct, ct],\ t \geq 0$, satisfies the following hyperbolic partial differential equation

$$\frac{\partial^2 f}{\partial t^2} + 2\lambda \frac{\partial f}{\partial t} - c^2 \frac{\partial^2 f}{\partial x^2} = 0, \tag{1}$$

which is referred to as the *telegraph* or *damped wave* equation and can be found by solving (1) with the initial conditions

$$f(x,t)|_{t=0} = \delta(x), \qquad \left.\frac{\partial f(x,t)}{\partial t}\right|_{t=0} = 0,$$

where $\delta(x)$ is the Dirac delta-function. This means that the transition density $f(x,t)$ of the process $X(t)$ is the fundamental solution (i.e. the Green's function) of the telegraph equation (1).

The explicit form of the density $f(x,t)$ is given by the formula (see, for instance, [29, Section 0.4] or [27, Theorem 1]):

$$\begin{aligned} f(x,t) = {} & \frac{e^{-\lambda t}}{2}\left[\delta(ct - x) + \delta(ct + x)\right] + \\ & + \frac{e^{-\lambda t}}{2c}\left[\lambda I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2 - x^2}\right) + \frac{\partial}{\partial t} I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2 - x^2}\right)\right]\Theta(ct - |x|), \end{aligned} \tag{2}$$

where $\Theta(x)$ is the Heaviside step function

$$\Theta(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x \leq 0, \end{cases}$$

and $I_0(z)$ is the modified Bessel function of order zero (that is, the Bessel function with imaginary argument) given by

$$I_0(z) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2}\left(\frac{z}{2}\right)^{2k}.$$

The first term of (2)

$$f_s(x,t) = \frac{e^{-\lambda t}}{2}\left[\delta(ct - x) + \delta(ct + x)\right] \tag{3}$$

represents the density of the singular part of the distribution of $X(t)$ concentrated at two terminal points $\pm ct$ of the interval $[-ct, ct]$, while the second term of (2)

$$f_{ac}(x,t) = \frac{e^{-\lambda t}}{2c}\left[\lambda I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2 - x^2}\right) + \frac{\partial}{\partial t} I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2 - x^2}\right)\right]\Theta(ct - |x|), \quad (4)$$

is the density of the absolutely continuous part of the distribution of $X(t)$ concentrated in the open interval $(-ct, ct)$.

During last decades the Goldstein-Kac telegraph process $X(t)$ and its numerous generalizations have become the subject of intense researches provided both by great theoretical importance and fruitful applications in statistical physics, financial modeling, transport phenomena in physical and biological systems, hydrology and some other fields. Some properties of the solution space of the Goldstein-Kac telegraph equation (1) were studied by Bartlett [2]. The process of one-dimensional random motion at finite speed governed by a Poisson process with a time-depending parameter was considered by Kaplan [17]. The relationships between the Goldstein-Kac model and physical processes, including some emerging effects of the relativity theory, were thoroughly examined by Bartlett [1], Cane [5,6]. Formulas for the distributions of the first-exit time from a given interval and of the maximum displacement of the telegraph process were obtained by Pinsky [29, Section 0.5], Foong [10], Masoliver and Weiss [25, 26]. The behaviour of the telegraph process with absorbing and reflecting barriers was studied by Foong and Kanno [11], Orsingher [28]. A one-dimensional stochastic motion with an arbitrary number of velocities and governing Poisson processes was examined by Kolesnik [21]. The telegraph-type processes with random velocities were studied by Stadje and Zacks [32]. Probabilistic methods of solving the Cauchy problems for the telegraph equation (1) were developed by Kac [16], Kisynski [18], Kabanov [15], Turbin and Samoilenko [33]. A generalization of the Goldstein-Kac model for the case of a damped telegraph process with logistic stationary distributions was given by Di Crescenzo and Martinucci [8]. A random motion with velocities alternating at Erlang-distributed random times was studied by Di Crescenzo [7]. Formulas for the occupation time distributions of the telegraph process were recently obtained by Bogachev and Ratanov [4]. A generalization of the Goldstein-Kac telegraph process to the $\mathbb{R}^d$, $d \geq 1$, space with an arbitrary *finite number* of cyclically changing directions was thoroughly examined by Lachal [24]. A similar motion in the plane $\mathbb{R}^2$ with an arbitrary *finite number of directions* and uniform mechanism of their change was studied by Kolesnik and Turbin [23].

Moments of any stochastic process are one of the most interesting and useful objects both from theoretical and practical points of view. This especially concerns the telegraph process $X(t)$ which is the basis for many important models in financial mathematics, biology, physics and other fields. For example, the knowledge of moments enables to construct various moment-type estimators in statistics (see, for instance,[14]). However, despite the great variety of existing works on the subject and of the results obtained, the moment problem for the Goldstein-Kac telegraph process was not properly solved so far. In particular, it was not clear whether the distribution of $X(t)$ was completely determined by its moments.

The most enigmatic fact is that the transition density (2) of the one-dimensional telegraph process $X(t)$ has much more complicated form than the transition densities of its two- and four-dimensional counterparts with a *continuum number of directions* (for the transition density of the 2D and 4D-motions see [22, Theorem 2] and [20, Theorem 2], respectively). While the transition density (2) contains special functions, the densities of the 2D- and 4D-motions have very simple exponential form that enables to explicitly compute the moments (see [19, Theorems 1 and 3, respectively]). Note also that the moments of a special multidimensional random motion with a cyclic mechanism of choosing new directions were computed by Samoilenko [31].

In this article we give a detailed moment analysis of the Goldstein-Kac telegraph process $X(t)$. In Section 2 we study the asymptotic behaviour of the moment function as c, λ and t vary in different ways. In Section 3 we obtain an explicit formula for the Laplace transform of the moment function of $X(t)$. In Section 4 we give the complete solution of the moment problem for the telegraph process $X(t)$. We show that, for arbitrary $t > 0$, the moments of $X(t)$ satisfy the Carleman condition and, therefore, the distribution of $X(t)$ is completely determined by its moments. In Section 5 we derive the moment generating function by direct computations and give a formula for the semi-invariants of the telegraph process $X(t)$.

2 Asymptotic Behaviour of Moments

Consider the moment function of the Goldstein-Kac telegraph process $X(t)$ defined by the formula

$$\mu_n(t) = E[X(t)]^n, \quad n \geq 1,$$

where E means the expectation.

It is known (see, for instance,[14, Theorem 2.1]) that, for arbitrary $t > 0$, the moments of $X(t)$ are given by the formula

$$\begin{aligned} &\mu_{2k}(t) = e^{-\lambda t}\, c^{2k}\, 2^{k-1/2}\, \lambda^{-k+1/2}\, t^{k+1/2}\, \Gamma\left(k+\frac{1}{2}\right) \left[I_{k+1/2}(\lambda t) + I_{k-1/2}(\lambda t)\right], \\ &\mu_{2k+1}(t) = 0, \qquad\qquad k = 0, 1, 2, \ldots. \end{aligned} \tag{5}$$

where $I_\nu(z)$ is the modified Bessel function of order ν

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{2k+\nu},$$

and $\Gamma(x)$ is the Euler gamma-function. Note that formula (5) slightly differs from that of [14, Theorem 2.1]), however one can easily check that both these representations of the moment function $\mu_{2k}(t)$ are equivalent. For our purposes it is more convenient to use just the representation (5).

From (5) we can easily obtain the first and the second moments of the telegraph process $X(t)$:

$$\mu_1(t) = 0, \qquad \mu_2(t) = \frac{c^2 t}{\lambda} - \frac{c^2}{2\lambda^2}\,(1 - e^{-2\lambda t}), \tag{6}$$

and this coincides with [27, Formula (28)].

In this section we thoroughly study the asymptotic behaviour of the moment function given by (5). Clearly, we need to examine the behaviour of the even-order moments $\mu_{2k}(t),\ k = 1, 2, \dots,$ only.

2.1. *Asymptotic behaviour with respect to $c \to \infty,\ \lambda \to \infty$, ($t$ and k are fixed).* In this subsection we consider the case when, under fixed t and k, the speed of the motion c and the intensity of switching Poisson process λ both go to infinity in such a way that the following Kac condition holds:

$$c \to \infty, \qquad \lambda \to \infty, \qquad \frac{c^2}{\lambda} \to \rho, \quad \rho > 0. \tag{7}$$

Taking into account the well-known asymptotic formula for the modified Bessel function (see, for instance,[13, Formula 8.451(5)]):

$$I_\nu(z) \ \sim \ \frac{e^z}{\sqrt{2\pi z}}\,, \qquad z \to +\infty, \tag{8}$$

as well as the formula (see [13, Formula 8.339(2)])

$$\Gamma\left(k + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^k}\,(2k-1)!!, \qquad k \geq 0, \quad (-1)!! = 1, \tag{9}$$

we obtain

$$\begin{aligned}
\lim_{\substack{c,\,\lambda\to\infty \\ (c^2/\lambda)\to\rho}} \mu_{2k}(t) &= 2^{k-1/2} t^{k+1/2} \Gamma\left(k + \frac{1}{2}\right) \times \\
&\times \lim_{\substack{c,\,\lambda\to\infty \\ (c^2/\lambda)\to\rho}} \left[e^{-\lambda t} c^{2k} \lambda^{-k+1/2} \left(I_{k+1/2}(\lambda t) + I_{k-1/2}(\lambda t)\right)\right] \sim \\
&\sim 2^{k-1/2} t^{k+1/2} \Gamma\left(k + \frac{1}{2}\right) \lim_{\substack{c,\,\lambda\to\infty \\ (c^2/\lambda)\to\rho}} \left[e^{-\lambda t} c^{2k} \lambda^{-k+1/2}\, \frac{2e^{\lambda t}}{\sqrt{2\pi\lambda t}}\right] = \\
&= 2^k\, t^k\, \frac{1}{\sqrt{\pi}}\, \Gamma\left(k + \frac{1}{2}\right) \lim_{\substack{c,\,\lambda\to\infty \\ (c^2/\lambda)\to\rho}} \left(\frac{c^{2k}}{\lambda^k}\right) = \\
&= 2^k\, t^k\, \frac{1}{\sqrt{\pi}}\, \frac{\sqrt{\pi}}{2^k}\, (2k-1)!!\, \rho^k = \\
&= \rho^k\, t^k\, (2k-1)!!
\end{aligned}$$

and this coincides with the moment function of the one-dimensional homogeneous Brownian motion with zero drift and diffusion coefficient $\sigma^2 = \rho$.

2.2. *Asymptotic behaviour with respect to $t \to \infty,\ \lambda \to \infty$, ($c$ and k are fixed).* Similarly to the asymptotic analysis of Subsection 2.1 and by using (8) and (9) we

can easy show that for $t \to \infty$ or $\lambda \to \infty$ (or both t and λ tend to infinity), under fixed c and k, the following asymptotic formula holds:

$$\mu_{2k}(t) \sim \left(\frac{c^2 t}{\lambda}\right)^k (2k-1)!! \tag{10}$$

From (10) we see that the moments $\mu_{2k}(t)$ increase like t^k as $t \to \infty$ (for fixed c, λ and k). Conversely, the moments $\mu_{2k}(t)$ decrease like λ^{-k} as $\lambda \to \infty$ (for fixed c, t and k).

2.3. *Asymptotic behaviour with respect to $k \to \infty$, (c, t and λ are fixed).* Asymptotic analysis with respect to $k \to \infty$ is much more complicated due to the absence of general asymptotic formulas with respect to the index ν of the modified Bessel function $I_\nu(z)$ (except the very particular case when the argument z has a special form depending on index ν). Nevertheless, we are able to obtain asymptotic formulas for the moment function $\mu_{2k}(t)$, as $k \to \infty$, due to the special form of the indices of the modified Bessel functions in (5). This result is presented by the following theorem.

Theorem 1. *For any fixed c, λ and t the following asymptotic formula holds:*

$$\mu_{2k}(t) \sim e^{-\lambda t} (ct)^{2k} \left(1 + \frac{\lambda t}{2k+1}\right), \qquad k \to \infty. \tag{11}$$

The refined asymptotic formula has the form:

$$\mu_{2k}(t) \sim e^{-\lambda t} (ct)^{2k} \left(1 + \frac{\lambda t}{2k+1} + \frac{(\lambda t)^2}{4k+2} + \frac{(\lambda t)^3}{(4k+2)(2k+3)}\right), \qquad k \to \infty. \tag{12}$$

Proof. First we need to establish the following asymptotic formulas for the modified Bessel functions:

$$I_{k+1/2}(z) \sim \sqrt{\frac{2}{\pi}} \frac{z^{k+1/2}}{(2k+1)!!}, \qquad k \to \infty, \tag{13}$$

$$I_{k-1/2}(z) \sim \sqrt{\frac{2}{\pi}} \frac{z^{k-1/2}}{(2k-1)!!}, \qquad k \to \infty. \tag{14}$$

Let us prove (13). Using the series representation of the modified Bessel function

(see, for instance,[13, Formula 8.445]) we have

$$
\begin{aligned}
I_{k+1/2}(z) &= z^{k+1/2} \sum_{l=0}^{\infty} \frac{1}{l!\,\Gamma\left((l+k+1/2)+1\right)} \left(\frac{z}{2}\right)^{2l} = \\
&= z^{k+1/2} \sum_{l=0}^{\infty} \frac{1}{l!\,(l+k+1/2)\,\Gamma\left(l+k+1/2\right)} \left(\frac{z}{2}\right)^{2l} = \\
&\quad \text{(see formula(9))} \\
&= \frac{z^{k+1/2}}{\sqrt{\pi}} \sum_{l=0}^{\infty} \frac{z^{2l}\ 2^{l+k}}{l!\,(l+k+1/2)\,(2l+2k-1)!!\ 2^{2l+k+1/2}} = \\
&= \sqrt{\frac{2}{\pi}}\ z^{k+1/2} \sum_{l=0}^{\infty} \frac{z^{2l}}{l!\,(2l+2k+1)\,(2l+2k-1)!!\ 2^{l}} = \\
&= \sqrt{\frac{2}{\pi}}\ z^{k+1/2} \sum_{l=0}^{\infty} \frac{z^{2l}}{(2l)!!\,(2l+2k+1)!!} \sim \\
&\sim \sqrt{\frac{2}{\pi}}\ \frac{z^{k+1/2}}{(2k+1)!!}\,, \qquad k \to \infty,
\end{aligned}
$$

proving (13). Similarly, we have

$$
\begin{aligned}
I_{k-1/2}(z) &= z^{k-1/2} \sum_{l=0}^{\infty} \frac{1}{l!\,\Gamma(l+k+1/2)} \left(\frac{z}{2}\right)^{2l} = \\
&= \frac{z^{k-1/2}}{\sqrt{\pi}} \sum_{l=0}^{\infty} \frac{z^{2l}}{l!\,(2l+2k-1)!!\ 2^{l-1/2}} = \\
&= \sqrt{\frac{2}{\pi}}\ z^{k-1/2} \sum_{l=0}^{\infty} \frac{z^{2l}}{(2l)!!\,(2l+2k-1)!!} \sim \\
&\sim \sqrt{\frac{2}{\pi}}\ \frac{z^{k-1/2}}{(2k-1)!!}\,, \qquad k \to \infty,
\end{aligned}
$$

and (14) is also proved.

Therefore, by applying formulas (13) and (14) just now proved, we obtain:

$$
\begin{aligned}
\mu_{2k}(t) &= e^{-\lambda t}\ c^{2k}\ 2^{k-1/2}\ \lambda^{-k+1/2}\ t^{k+1/2}\ \Gamma\left(k+\frac{1}{2}\right)\ \left[I_{k+1/2}(\lambda t) + I_{k-1/2}(\lambda t)\right] \sim \\
&\sim e^{-\lambda t} c^{2k} 2^{k-1/2} \lambda^{-k+1/2} t^{k+1/2} \frac{\sqrt{\pi}}{2^k} (2k-1)!! \sqrt{\frac{2}{\pi}} \left[\frac{(\lambda t)^{k+1/2}}{(2k+1)!!} + \frac{(\lambda t)^{k-1/2}}{(2k-1)!!}\right] = \\
&= e^{-\lambda t}\ c^{2k}\ \lambda^{-k+1/2}\ t^{k+1/2}\ (2k-1)!!\ \frac{(\lambda t)^{k-1/2}}{(2k-1)!!} \left[1 + \frac{\lambda t}{2k+1}\right] = \\
&= e^{-\lambda t}\ (ct)^{2k}\ \left(1 + \frac{\lambda t}{2k+1}\right), \qquad k \to \infty,
\end{aligned}
$$

yielding (11).

Formula (12) can be proved in the same manner by applying, instead of (13) and (14), the refined asymptotic formulas for the modified Bessel function (see also Remark 1 below):

$$I_{k+1/2}(z) \sim \frac{z^{k+5/2} + (4k+6)z^{k+1/2}}{\sqrt{2\pi}\,(2k+3)!!}\,, \qquad k \to \infty, \tag{15}$$

$$I_{k-1/2}(z) \sim \frac{z^{k+3/2} + (4k+2)z^{k-1/2}}{\sqrt{2\pi}\,(2k+1)!!}\,, \qquad k \to \infty. \tag{16}$$

The theorem is thus completely proved. □

Remark 1. One can write down more accurate asymptotic formulas by taking arbitrary finite number of terms in the series expansions of the functions $I_{k+1/2}(z)$ and $I_{k-1/2}(z)$:

$$\begin{aligned} I_{k+1/2}(z) &= \sqrt{\frac{2}{\pi}}\, z^{k+1/2} \sum_{l=0}^{\infty} \frac{z^{2l}}{(2l)!!\,(2l+2k+1)!!}\,, \\ I_{k-1/2}(z) &= \sqrt{\frac{2}{\pi}}\, z^{k-1/2} \sum_{l=0}^{\infty} \frac{z^{2l}}{(2l)!!\,(2l+2k-1)!!}\,. \end{aligned} \tag{17}$$

Since the index k is presented in the denominators of (17) and, therefore, each term of these series tends to zero as $k \to \infty$, then for arbitrary integer $n \geq 0$ the following formulas hold:

$$\begin{aligned} I_{k+1/2}(z) &= \sqrt{\frac{2}{\pi}}\, z^{k+1/2} \sum_{l=0}^{n} \frac{z^{2l}}{(2l)!!\,(2l+2k+1)!!} + R^{+}_{k,n}(z), \\ I_{k-1/2}(z) &= \sqrt{\frac{2}{\pi}}\, z^{k-1/2} \sum_{l=0}^{n} \frac{z^{2l}}{(2l)!!\,(2l+2k-1)!!} + R^{-}_{k,n}(z), \end{aligned} \tag{18}$$

where the remainders $R^{\pm}_{k,n}(z) \to 0$, as $k \to \infty$, for any fixed z and $n \geq 0$. Note that formulas (13) and (14) follow, as $k \to \infty$, from (18) for $n = 0$, while (15) and (16) follow, as $k \to \infty$, from (18) for $n = 1$, respectively. One can also obtain the upper bounds for the remainders $R^{\pm}_{k,n}(z)$ and, therefore, to evaluate the rate of their convergence to zero, as $k \to \infty$, however this is not our concern here.

Remark 2. Asymptotic formulas (11) and (12) show that the behaviour of the moment function $\mu_{2k}(t)$ with respect to $k \to \infty$ depends on the factor ct as follows:

If $ct < 1$, then $\mu_{2k}(t) \to 0$, as $k \to \infty$;
If $ct = 1$, then $\mu_{2k}(t) \to e^{-\lambda t}$, as $k \to \infty$;
If $ct > 1$, then $\mu_{2k}(t) \to \infty$, as $k \to \infty$.

This enables us to make some interesting and somewhat unexpected conclusions concerning the asymptotic behaviour of the moment function $\mu_{2k}(t)$, as $k \to \infty$.

Since ct is the total length of an arbitrary sample path of the Goldstein-Kac telegraph process $X(t)$ at the time instant $t > 0$ whose distribution is concentrated in the interval $[-ct, ct]$, then $[-1, 1]$ is the critical interval in the following sense. If $[-ct, ct] \subset [-1, 1]$, then the moments $\mu_{2k}(t)$ are finite and tend to zero, as $k \to \infty$. If $[-ct, ct] = [-1, 1]$, then the moments $\mu_{2k}(t)$ are finite and tend to $e^{-\lambda t}$, as $k \to \infty$. Finally, if $[-ct, ct] \supset [-1, 1]$, then the moments $\mu_{2k}(t)$ tend to ∞, as $k \to \infty$. In terms of the time t this means that for $t < \frac{1}{c}$, the moments are finite and tend to zero, as $k \to \infty$; at the time instant $t = \frac{1}{c}$, the moments are finite and tend to $e^{-\lambda/c}$, as $k \to \infty$; for $t > \frac{1}{c}$, the moments tend to ∞, as $k \to \infty$.

Numerical computations of moments according to formula (5) and their approximations (for increasing k) by means of the asymptotic functions

$$g_0(t) = e^{-\lambda t}\,(ct)^{2k}\left(1 + \frac{\lambda t}{2k+1}\right),$$
$$g_1(t) = e^{-\lambda t}\,(ct)^{2k}\left(1 + \frac{\lambda t}{2k+1} + \frac{(\lambda t)^2}{4k+2} + \frac{(\lambda t)^3}{(4k+2)(2k+3)}\right),$$

obtained in Theorem 1 are given in the following table below (for the particular values of the parameters $c = 0.6,\ t = 1.5,\ \lambda = 2.5$):

k	$\mu_{2k}(1.5)$	$g_0(1.5)$	$g_1(1.5)$
100	$0.175030 \cdot 10^{-10}$	$0.169015 \cdot 10^{-10}$	$0.174926 \cdot 10^{-10}$
500	$0.415508 \cdot 10^{-47}$	$0.412600 \cdot 10^{-47}$	$0.415498 \cdot 10^{-47}$
1000	$0.722360 \cdot 10^{-93}$	$0.719826 \cdot 10^{-93}$	$0.722356 \cdot 10^{-93}$
5000	$0.626552 \cdot 10^{-459}$	$0.626113 \cdot 10^{-459}$	$0.626553 \cdot 10^{-459}$
10000	$0.166654 \cdot 10^{-916}$	$0.166597 \cdot 10^{-916}$	$0.166655 \cdot 10^{-916}$

We see that the second asymptotic function $g_1(t)$ yields a better approximation (for increasing k) of the moment function $\mu_{2k}(t)$ than the first asymptotic function $g_0(t)$. In particular, we see that the function $g_1(t)$ provides stabilization in the second digit already for $k = 100$, while the function $g_0(t)$ does so only for $k = 500$. Note also that in this example $ct = 0.6 \cdot 1.5 = 0.9 < 1$ and the moments $\mu_{2k}(1.5)$ tend to zero, as $k \to \infty$, very rapidly.

3 Laplace Transform of Moment Function

In this section we derive an explicit formula for the Laplace transform of the moment function $\mu_{2k}(t),\ k \geq 1$, given by (5). We show that, despite the fairly complicated form of the moment function (5), its Laplace transform has a very simple form. This result is presented by the following theorem.

Theorem 2. *The Laplace transform of moment function* (5) *is given by the formula:*

$$\mathcal{L}_t\left[\mu_{2k}(t)\right](s) = \frac{c^{2k}\,(2k)!}{s^{k+1}\,(s+2\lambda)^k}\,, \qquad \mathrm{Re}\, s > 0. \tag{19}$$

Proof. Applying the Laplace transformation to (5) we have:

$$\begin{aligned}\mathcal{L}_t\left[\mu_{2k}(t)\right](s) &= c^{2k}\, 2^{k-1/2}\, \lambda^{-k+1/2}\, \Gamma\left(k+\frac{1}{2}\right) \times \\ &\times \mathcal{L}_t\left[e^{-\lambda t}\, t^{k+1/2}\, \left(I_{k+1/2}(\lambda t)+I_{k-1/2}(\lambda t)\right)\right](s) = \\ &= c^{2k}\, 2^{k-1/2}\, \lambda^{-k+1/2}\, \Gamma\left(k+\frac{1}{2}\right) \times \\ &\times \mathcal{L}_t\left[t^{k+1/2}\, \left(I_{k+1/2}(\lambda t)+I_{k-1/2}(\lambda t)\right)\right](s+\lambda).\end{aligned} \tag{20}$$

According to [3, Table 4.16, Formulas 6 and 7]

$$\mathcal{L}_t\left[t^{k+1/2}\, I_{k+1/2}(\lambda t)\right](s) = \frac{1}{\sqrt{\pi}}\, 2^{k+1/2}\, \lambda^{k+1/2}\, k!\, \frac{1}{(s^2-\lambda^2)^{k+1}}\,,$$
$$\mathcal{L}_t\left[t^{k+1/2}\, I_{k-1/2}(\lambda t)\right](s) = \frac{1}{\sqrt{\pi}}\, 2^{k+1/2}\, \lambda^{k-1/2}\, k!\, \frac{s}{(s^2-\lambda^2)^{k+1}}\,.$$

Substituting these expressions into (20) we obtain

$$\begin{aligned}\mathcal{L}_t\left[\mu_{2k}(t)\right](s) &= c^{2k}\, 2^{2k}\, \Gamma\left(k+\frac{1}{2}\right)\, \frac{k!}{\sqrt{\pi}}\, \frac{s+2\lambda}{((s+\lambda)^2-\lambda^2)^{k+1}} = \\ &\quad \text{(see Formula (9))} \\ &= c^{2k}\, 2^{2k}\, \frac{\sqrt{\pi}}{2^k}(2k-1)!!\, \frac{k!}{\sqrt{\pi}}\, \frac{s+2\lambda}{((s+\lambda)^2-\lambda^2)^{k+1}} = \\ &= c^{2k}\, k!\, 2^k\, (2k-1)!!\, \frac{s+2\lambda}{(s(s+2\lambda))^{k+1}} = \\ &= c^{2k}\, (2k)!!\, (2k-1)!!\, \frac{s+2\lambda}{(s(s+2\lambda))^{k+1}} = \\ &= \frac{c^{2k}\, (2k)!}{s^{k+1}\, (s+2\lambda)^k}.\end{aligned}$$

The theorem is proved. □

In particular, for $k=1$, we obtain from (19) the formula for the Laplace transform of the second moment

$$\mathcal{L}_t\left[\mu_2(t)\right](s) = \frac{2c^2}{s^2\, (s+2\lambda)}. \tag{21}$$

On the other hand, applying Laplace transformation to (6) we have:

$$\begin{aligned}
\mathcal{L}_t\left[\mu_2(t)\right](s) &= \mathcal{L}_t\left[\frac{c^2 t}{\lambda} - \frac{c^2}{2\lambda^2}\,(1-e^{-2\lambda t})\right](s) = \\
&= \frac{c^2}{\lambda}\,\mathcal{L}_t[t](s) - \frac{c^2}{2\lambda^2}\,\Big(\mathcal{L}_t[1](s) - \mathcal{L}_t[e^{-2\lambda t}](s)\Big) = \\
&= \frac{c^2}{\lambda}\,\frac{1}{s^2} - \frac{c^2}{2\lambda^2}\left(\frac{1}{s} - \frac{1}{s+2\lambda}\right) = \\
&= \frac{c^2}{\lambda}\,\frac{1}{s^2} - \frac{c^2}{2\lambda^2}\,\frac{2\lambda}{s(s+2\lambda)} = \\
&= \frac{c^2}{\lambda}\left(\frac{1}{s^2} - \frac{1}{s(s+2\lambda)}\right) = \\
&= \frac{2c^2}{s^2\,(s+2\lambda)}
\end{aligned}$$

and this coincides with (21).

Remark 3. One can check that, under the Kac condition (7), function (19) turns into the Laplace transform of the moment function of Brownian motion. Really, for function (19) we have

$$\begin{aligned}
\lim_{\substack{c,\,\lambda\to\infty\\(c^2/\lambda)\to\rho}} \left\{\mathcal{L}_t\left[\mu_{2k}(t)\right](s)\right\} &= \frac{(2k)!}{s^{k+1}} \lim_{\substack{c,\,\lambda\to\infty\\(c^2/\lambda)\to\rho}} \left\{\frac{c^{2k}}{(s+2\lambda)^k}\right\} = \\
&= \frac{(2k)!!\,(2k-1)!!}{s^{k+1}} \lim_{\substack{c,\,\lambda\to\infty\\(c^2/\lambda)\to\rho}} \left\{\frac{c^{2k}}{(2\lambda)^k}\,\frac{1}{\left(\frac{s}{2\lambda}+1\right)^k}\right\} = \\
&= \frac{2^k\,k!\,(2k-1)!!}{s^{k+1}}\,\frac{1}{2^k} \lim_{\substack{c,\,\lambda\to\infty\\(c^2/\lambda)\to\rho}} \left\{\frac{c^{2k}}{\lambda^k}\right\} = \\
&= \frac{\rho^k\,k!\,(2k-1)!!}{s^{k+1}}.
\end{aligned}$$

On the other hand, for the Laplace transform of the moment function of the one-dimensional homogeneous Brownian motion with zero drift and diffusion coefficient $\sigma^2 = \rho$ derived in Subsection 2.1 above, we obtain the formula

$$\begin{aligned}
\mathcal{L}_t\left[\rho^k\,t^k\,(2k-1)!!\right](s) &= \rho^k\,(2k-1)!!\,\mathcal{L}_t[t^k](s) = \\
&= \rho^k\,(2k-1)!!\,\frac{\Gamma(k+1)}{s^{k+1}} = \\
&= \frac{\rho^k\,k!\,(2k-1)!!}{s^{k+1}}
\end{aligned}$$

exactly coinciding with the previous one.

4 Moment Problem

In this section we give the complete solution of the moment problem for the Goldstein-Kac telegraph process $X(t)$. We show that, for any fixed $t > 0$, the moments of $X(t)$ satisfy the Carleman condition and, therefore, the distribution of $X(t)$ is completely determined by its moments. This result is given by the following theorem.

Theorem 3. *For any fixed $t > 0$ the moments $\mu_{2k}(t)$ of the telegraph process $X(t)$, given by* (5), *satisfy the Carleman condition:*

$$\sum_{k=1}^{\infty} [\mu_{2k}(t)]^{-1/(2k)} = \infty. \tag{22}$$

Proof. To prove the theorem it suffices to show that the general term of the series on the left-hand side of (22) does not tend to zero, as $k \to \infty$. First, we prove that, for arbitrary $k \geq 1$, the following inequality holds:

$$\mu_{2k}(t) < (ct)^{2k}\ (1 + \lambda t)\ e^{\lambda^2 t^2/2}, \qquad k \geq 1. \tag{23}$$

By using formulas (9) and (17) we have:

$$\begin{aligned}
\mu_{2k}(t) &= e^{-\lambda t}\ c^{2k}\ 2^{k-1/2}\ \lambda^{-k+1/2}\ t^{k+1/2}\ \Gamma\left(k + \frac{1}{2}\right) \left[I_{k+1/2}(\lambda t) + I_{k-1/2}(\lambda t)\right] = \\
&= e^{-\lambda t}\ c^{2k}\ 2^{k-1/2}\ \lambda^{-k+1/2}\ t^{k+1/2}\ \frac{\sqrt{\pi}}{2^k}\ (2k-1)!!\ \left[I_{k+1/2}(\lambda t) + I_{k-1/2}(\lambda t)\right] = \\
&= e^{-\lambda t}\ c^{2k}\ \lambda^{-k+1/2}\ t^{k+1/2}\ \sqrt{\frac{\pi}{2}}\ (2k-1)!!\ \left[I_{k+1/2}(\lambda t) + I_{k-1/2}(\lambda t)\right] < \\
&< c^{2k}\ \lambda^{-k+1/2}\ t^{k+1/2}\ (2k-1)!! \left[(\lambda t)^{k+1/2} \sum_{l=0}^{\infty} \frac{(\lambda t)^{2l}}{(2l)!!\ (2l+2k+1)!!} + \right. \\
&\qquad \left. + (\lambda t)^{k-1/2} \sum_{l=0}^{\infty} \frac{(\lambda t)^{2l}}{(2l)!!\ (2l+2k-1)!!}\right] = \\
&= c^{2k}\ \lambda^{-k+1/2}\ t^{k+1/2} \left[(\lambda t)^{k+1/2} \sum_{l=0}^{\infty} \frac{(2k-1)!!}{(2l+2k+1)!!}\ \frac{(\lambda t)^{2l}}{(2l)!!} + \right. \\
&\qquad \left. + (\lambda t)^{k-1/2} \sum_{l=0}^{\infty} \frac{(2k-1)!!}{(2l+2k-1)!!}\ \frac{(\lambda t)^{2l}}{(2l)!!}\right] < \\
&< c^{2k}\ \lambda^{-k+1/2}\ t^{k+1/2} \left[(\lambda t)^{k+1/2} \sum_{l=0}^{\infty} \frac{(\lambda t)^{2l}}{(2l)!!} + (\lambda t)^{k-1/2} \sum_{l=0}^{\infty} \frac{(\lambda t)^{2l}}{(2l)!!}\right],
\end{aligned}$$

where in the last step we have used the fact that, for any $k \geq 1$, the following inequalities hold

$$\frac{(2k-1)!!}{(2l+2k+1)!!} < 1, \qquad \frac{(2k-1)!!}{(2l+2k-1)!!} \leq 1, \qquad \text{for any } \ l \geq 0.$$

Now taking into account that

$$\sum_{l=0}^{\infty} \frac{(\lambda t)^{2l}}{(2l)!!} = \sum_{l=0}^{\infty} \frac{(\lambda t)^{2l}}{2^l\, l!} = \sum_{l=0}^{\infty} \frac{1}{l!} \left(\frac{\lambda^2 t^2}{2}\right)^l = e^{\lambda^2 t^2/2}$$

we obtain

$$\begin{aligned} \mu_{2k}(t) &< c^{2k}\, \lambda^{-k+1/2}\, t^{k+1/2}\, (\lambda t)^{k-1/2}\, (1+\lambda t)\, e^{\lambda^2 t^2/2} = \\ &= (ct)^{2k}\, (1+\lambda t)\, e^{\lambda^2 t^2/2}, \end{aligned}$$

proving (23). From (23) we have the inequality:

$$[\mu_{2k}(t)]^{-1/(2k)} > (ct)^{-1}\, (1+\lambda t)^{-1/(2k)}\, e^{-\lambda^2 t^2/(4k)}, \qquad k \geq 1.$$

Then, by passing to the limit, as $k \to \infty$, in this last inequality, we obtain:

$$\lim_{k\to\infty} [\mu_{2k}(t)]^{-1/(2k)} \geq (ct)^{-1} > 0$$

for any c and $t > 0$. Hence, the sequence $[\mu_{2k}(t)]^{-1/(2k)}$ does not tend to zero as $k \to \infty$ and, therefore, the series (4.1) is divergent. The theorem is thus completely proved. □

5 Moment generating function

In this section we obtain a formula for the generating function of the moments $\mu_{2k}(t)$, $k \geq 1$, in an explicit form. Taking into account the well-know connection between the moments and the characteristic function of a stochastic process, this can be done by applying the known formula for the characteristic function of the Goldstein-Kac telegraph process (see, for instance,[9, Proposition 2.1] or [28, Theorem 2.3]). Instead, we give an alternative way of deriving the moment generating function based on direct computations and use of some properties of the modified Bessel functions.

For arbitrary complex number z such that

$$|z| < \frac{\lambda^2}{c^2},$$

introduce the function

$$\psi(z,t) = \sum_{k=0}^{\infty} z^k\, \frac{\mu_{2k}(t)}{(2k)!}. \tag{24}$$

The explicit form of function (24) is given by the following theorem.

Theorem 4. *For any $t > 0$ the moment generating function* (24) *has the form:*

$$\psi(z,t) = e^{-\lambda t} \left\{ \cosh\left(t\sqrt{\lambda^2 + c^2 z}\right) + \frac{\lambda}{\sqrt{\lambda^2 + c^2 z}}\, \sinh\left(t\sqrt{\lambda^2 + c^2 z}\right) \right\}. \tag{25}$$

Proof. First, we note that, in view of formula (9),

$$\begin{aligned}\frac{\mu_{2k}(t)}{(2k)!} &= e^{-\lambda t}\, c^{2k}\, 2^{k-1/2}\, \lambda^{-k+1/2}\, t^{k+1/2}\, \frac{\Gamma\left(k+\frac{1}{2}\right)}{(2k)!}\left[I_{k+1/2}(\lambda t)+I_{k-1/2}(\lambda t)\right] = \\ &= e^{-\lambda t} c^{2k} 2^{k-1/2} \lambda^{-k+1/2} t^{k+1/2} \frac{\sqrt{\pi}}{2^k} \frac{(2k-1)!!}{(2k)!!(2k-1)!!}\left[I_{k+1/2}(\lambda t)+I_{k-1/2}(\lambda t)\right] = \\ &= \sqrt{\frac{\pi\lambda t}{2}}\, e^{-\lambda t}\, \left(\frac{c^2 t}{\lambda}\right)^k \frac{1}{2^k\, k!}\, \left[I_{k+1/2}(\lambda t)+I_{k-1/2}(\lambda t)\right] = \\ &= \sqrt{\frac{\pi\lambda t}{2}}\, e^{-\lambda t}\, \left(\frac{c^2 t}{2\lambda}\right)^k \frac{1}{k!}\, \left[I_{k+1/2}(\lambda t)+I_{k-1/2}(\lambda t)\right].\end{aligned} \tag{26}$$

Substituting this into (24) we have:

$$\begin{aligned}\psi(z,t) &= e^{-\lambda t}\sqrt{\frac{\pi\lambda t}{2}}\, \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{c^2 t z}{2\lambda}\right)^k \left[I_{k+1/2}(\lambda t)+I_{k-1/2}(\lambda t)\right] = \\ &= e^{-\lambda t}\sqrt{\frac{\pi\lambda t}{2}}\left\{\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{c^2 t z}{2\lambda}\right)^k I_{k+1/2}(\lambda t) + \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{c^2 t z}{2\lambda}\right)^k I_{k-1/2}(\lambda t)\right\}.\end{aligned} \tag{27}$$

Consider separately the series on the right-hand side of (27). Applying the formula (see [30, page 694, Formula 6])

$$\sum_{k=0}^{\infty}\frac{\xi^k}{k!}\, I_{k-1/2}(x) = \sqrt{\frac{2}{\pi x}}\, \cosh\left(\sqrt{x^2+2\xi x}\right), \qquad |2\xi| < |x|,$$

we obtain for the second series in curl brackets of (27):

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{c^2 t z}{2\lambda}\right)^k I_{k-1/2}(\lambda t) &= \sqrt{\frac{2}{\pi\lambda t}}\, \cosh\left(\sqrt{\lambda^2 t^2 + 2\,\frac{c^2 t z}{2\lambda}\,\lambda t}\right) = \\ &= \sqrt{\frac{2}{\pi\lambda t}}\, \cosh\left(t\sqrt{\lambda^2 + c^2 z}\right).\end{aligned} \tag{28}$$

Similarly, by applying the formula (see [30, page 694, Formula 4 for $\nu = 1/2$])

$$\sum_{k=0}^{\infty}\frac{\xi^k}{k!}\, I_{k+1/2}(x) = \left(\frac{2\xi}{x}+1\right)^{-1/4} I_{1/2}\left(\sqrt{x^2+2\xi x}\right), \qquad |2\xi| < |x|,$$

and taking into account that (see [30, page 730])

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\, \sinh x,$$

we obtain for the first series in curl brackets of (27):

$$\sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{c^2 tz}{2\lambda}\right)^k I_{k+1/2}(\lambda t) = \left(1 + \frac{c^2}{\lambda^2}\, z\right)^{-1/4} I_{1/2}\left(\sqrt{\lambda^2 t^2 + c^2 t^2 z}\right) =$$
$$= \frac{\sqrt{\lambda}}{(\lambda^2 + c^2 z)^{1/4}} \sqrt{\frac{2}{\pi t \sqrt{\lambda^2 + c^2 z}}}\ \sinh\left(t\sqrt{\lambda^2 + c^2 z}\right) =$$
$$= \sqrt{\frac{2\lambda}{\pi t}}\ \frac{\sinh\left(t\sqrt{\lambda^2 + c^2 z}\right)}{\sqrt{\lambda^2 + c^2 z}}\ . \tag{29}$$

Substituting (28) and (29) into (27) we finally obtain:

$$\psi(z,t) = e^{-\lambda t} \sqrt{\frac{\pi\lambda t}{2}} \left\{ \sqrt{\frac{2}{\pi\lambda t}}\ \cosh\left(t\sqrt{\lambda^2 + c^2 z}\right) + \sqrt{\frac{2\lambda}{\pi t}}\ \frac{\sinh\left(t\sqrt{\lambda^2 + c^2 z}\right)}{\sqrt{\lambda^2 + c^2 z}} \right\} =$$
$$= e^{-\lambda t} \left\{ \cosh\left(t\sqrt{\lambda^2 + c^2 z}\right) + \frac{\lambda}{\sqrt{\lambda^2 + c^2 z}}\ \sinh\left(t\sqrt{\lambda^2 + c^2 z}\right) \right\},$$

proving (25). The theorem is proved. □

Remark 4. From (24) it follows that the $(2k)$-th moment $\mu_{2k}(t)$, $k \geq 1$, can be obtained by the k-time differentiation of the moment generating function $\psi(z,t)$ with respect to z and by setting then $z = 0$ in the expression obtained, that is,

$$\mu_{2k}(t) = (2k)!\ \left.\frac{\partial^k \psi(z,t)}{\partial z^k}\right|_{z=0}, \qquad k \geq 1.$$

Therefore, according to (25), we have for $k \geq 1$:

$$\mu_{2k}(t) = e^{-\lambda t}\ (2k)!\ \frac{\partial^k}{\partial z^k} \left\{ \cosh\left(t\sqrt{\lambda^2 + c^2 z}\right) + \lambda\ \frac{\sinh\left(t\sqrt{\lambda^2 + c^2 z}\right)}{\sqrt{\lambda^2 + c^2 z}} \right\}\Bigg|_{z=0}. \tag{30}$$

In particular, for $k = 1$, formula (30) yields:

$$\mu_2(t) = 2e^{-\lambda t}\ \frac{\partial}{\partial z} \left\{ \cosh\left(t\sqrt{\lambda^2 + c^2 z}\right) + \lambda\ \frac{\sinh\left(t\sqrt{\lambda^2 + c^2 z}\right)}{\sqrt{\lambda^2 + c^2 z}} \right\}\Bigg|_{z=0} =$$
$$= 2e^{-\lambda t} \left\{ \frac{c^2 t}{2}\ \frac{\sinh\left(t\sqrt{\lambda^2 + c^2 z}\right)}{\sqrt{\lambda^2 + c^2 z}} + \frac{\lambda}{\lambda^2 + c^2 z} \times \right.$$
$$\left. \times \left[\frac{c^2 t}{2}\ \cosh\left(t\sqrt{\lambda^2 + c^2 z}\right) - \frac{c^2}{2}\ \frac{\sinh\left(t\sqrt{\lambda^2 + c^2 z}\right)}{\sqrt{\lambda^2 + c^2 z}} \right] \right\}\Bigg|_{z=0} =$$

$$
\begin{aligned}
&= 2e^{-\lambda t}\ \left\{\frac{c^2 t}{2\lambda}\ \sinh(\lambda t) + \frac{1}{\lambda}\left[\frac{c^2 t}{2}\ \cosh(\lambda t) - \frac{c^2}{2\lambda}\ \sinh(\lambda t)\right]\right\} = \\
&= 2e^{-\lambda t}\ \left\{\frac{c^2 t}{2\lambda}\left[\sinh(\lambda t) + \cosh(\lambda t)\right] - \frac{c^2}{2\lambda^2}\ \sinh(\lambda t)\right\} = \\
&= 2e^{-\lambda t}\ \left\{\frac{c^2 t}{2\lambda}\ e^{\lambda t} - \frac{c^2}{2\lambda^2}\ \sinh(\lambda t)\right\} = \\
&= \frac{c^2 t}{\lambda} - \frac{c^2}{\lambda^2}\ e^{-\lambda t}\ \frac{e^{\lambda t} - e^{-\lambda t}}{2} = \\
&= \frac{c^2 t}{\lambda} - \frac{c^2}{2\lambda^2}\left(1 - e^{-2\lambda t}\right)
\end{aligned}
$$

and this exactly coincides with (6).

Note that the moment generating function is structurally similar to the characteristic function of the telegraph process $X(t)$ (see, for comparison, [9, Proposition 2.1] or [28, Theorem 2.3]).

Remark 5. We can use some formulas obtained above for deriving an expression for the semi-invariants of the Goldstein-Kac telegraph process $X(t)$. According to the general formula of probability theory, for any fixed $t > 0$, the semi-invariants $\eta_n(t),\ n \geq 1$, of $X(t)$ are expressed in terms of the moments $\mu_n(t),\ n \geq 1$, as follows:

$$
\eta_n(t) = n! \sum_{r=0}^{n} \sum_{j,\, l} \frac{(-1)^{j-1}\ (j-1)!}{j_1! \ldots j_r!}\ \left(\frac{\mu_{l_1(t)}}{l_1!}\right)^{j_1} \ldots \left(\frac{\mu_{l_r(t)}}{l_r!}\right)^{j_r}, \qquad n \geq 1, \tag{31}
$$

where the interior summation is doing with respect to all the non-negative integer numbers j and l such that

$$
l_1 j_1 + \cdots + l_r j_r = n, \qquad j_1 + \cdots + j_r = j.
$$

Since, according to (5), all the odd moments are equal to zero, then all the odd semi-invariants are equal to zero too, that is, $\eta_{2k+1}(t) = 0,\ k = 0, 1, 2, \ldots$. Therefore, formula (31) takes the form:

$$
\eta_{2k}(t) = (2k)! \sum_{r=0}^{2k} \sum_{j,\, l} \frac{(-1)^{j-1}\ (j-1)!}{j_1! \ldots j_r!}\ \left(\frac{\mu_{2l_1(t)}}{(2l_1)!}\right)^{j_1} \ldots \left(\frac{\mu_{2l_r(t)}}{(2l_r)!}\right)^{j_r}, \qquad k \geq 1, \tag{32}
$$

where

$$
l_1 j_1 + \cdots + l_r j_r = k, \qquad j_1 + \cdots + j_r = j. \tag{33}
$$

Each factor of the form $\mu_{2s}(t)/(2s)!$ in (32), according to (26), has the form:

$$
\frac{\mu_{2s}(t)}{(2s)!} = \sqrt{\frac{\pi\lambda t}{2}}\ e^{-\lambda t}\ \left(\frac{c^2 t}{2\lambda}\right)^{s}\ \frac{1}{s!}\ \left[I_{s+1/2}(\lambda t) + I_{s-1/2}(\lambda t)\right].
$$

Therefore, the product of such factors in (32), in view of (33), are given by

$$\left(\frac{\mu_{2l_1(t)}}{(2l_1)!}\right)^{j_1}\cdots\left(\frac{\mu_{2l_r(t)}}{(2l_r)!}\right)^{j_r}=$$
$$=\prod_{i=1}^{r}\left(\sqrt{\frac{\pi\lambda t}{2}}\,e^{-\lambda t}\,\left(\frac{c^2t}{2\lambda}\right)^{l_i}\,\frac{1}{l_i!}\,\left[I_{l_i+1/2}(\lambda t)+I_{l_i-1/2}(\lambda t)\right]\right)^{j_i}=$$
$$=\left(\frac{\pi\lambda t}{2}\right)^{j/2}\,e^{-\lambda tj}\,\left(\frac{c^2t}{2\lambda}\right)^{k}\,\prod_{i=1}^{r}\left(\frac{1}{l_i!}\,\left[I_{l_i+1/2}(\lambda t)+I_{l_i-1/2}(\lambda t)\right]\right)^{j_i}.$$

By substituting this into (32) we obtain the following formula for the semi-invariants:

$$\eta_{2k}(t)=(2k)!\,\left(\frac{c^2t}{2\lambda}\right)^{k}\,\sum_{r=0}^{2k}\,\sum_{j,\,l}\frac{(-1)^{j-1}\,(j-1)!}{j_1!\ldots j_r!}\,\left(\frac{\pi\lambda t}{2}\right)^{j/2}\,e^{-\lambda tj}\times$$
$$\times\prod_{i=1}^{r}\left(\frac{1}{l_i!}\,\left[I_{l_i+1/2}(\lambda t)+I_{l_i-1/2}(\lambda t)\right]\right)^{j_i}. \tag{34}$$

Formula (34) has a fairly complicated form and, apparently, cannot be simplified. Nevertheless, it can be used for computing the semi-invariants for small k.

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