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Bipolar fuzzy soft Lie algebras

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Abstract. We introduce the notion of bipolar fuzzy soft Lie subalgebras and investigate some of their properties. We also introduce the concept of an $(\in, \in \vee q)$-bipolar fuzzy (soft) Lie subalgebra and present some of its properties.

1. Introduction

In 1994, Zhang [13] initiated the concept of bipolar fuzzy sets as a generalization of fuzzy sets. Bipolar fuzzy sets are an extension of fuzzy sets [12] whose membership degree range is $[-1,1]$. In a bipolar fuzzy set, the membership degree 0 of an element means that the element is irrelevant to the corresponding property, the membership degree $(0,1]$ of an element indicates that the element somewhat satisfies the property, and the membership degree $[-1,0)$ of an element indicates that the element somewhat satisfies the implicit counter-property. Although bipolar fuzzy sets and intuitionistic fuzzy sets look similar to each other, they are essentially different sets. In many domains, it is important to be able to deal with bipolar information. It is noted that positive information represents what is granted to be possible, while negative information represents what is considered to be impossible. This domain has recently motivated new research in several directions. In particular, fuzzy and possibilistic formalisms for bipolar information have been proposed, because when we deal with spatial information in image processing or in spatial reasoning applications, this bipolarity also occurs. For instance, when we assess the position of an object in a space, we may have positive information expressed as a set of possible places and negative information expressed as a set of impossible places. As another example, let us consider the spatial relations. Human beings consider "left" and "right" as opposite directions. But this does not mean that one of them is the negation of the other. The semantics of "opposite" captures a notion of symmetry rather than a strict complementation. In particular, there may be positions which are considered neither to the right nor to the left of some reference object, thus leaving some room for indetermination. This corresponds to the idea that the union of positive and negative information does not cover the whole space.

In 1999, Molodtsov [8] initiated the novel concept of soft set theory to deal with uncertainties which can not be handled by traditional mathematical tools. He

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successfully applied the soft set theory into several disciplines, such as game theory, Riemann integration, Perron integration, measure theory etc. Applications of soft set theory in real life problems are now catching momentum due to the general nature parametrization expressed by a soft set. Yang and Li [10] introduced the notion of bipolar fuzzy soft sets. Recently, Akram and Feng introduced the notion of soft Lie subalgebras of Lie algebras in [11] and studied some of their results.

In this paper, we introduce the notion of bipolar fuzzy soft Lie subalgebras and investigate some of their properties. We introduce the concept of an $(\in, \in \vee q)$-bipolar fuzzy Lie subalgebra and present some of its properties. We also introduce the notion of an $(\in, \in \vee q)$-bipolar fuzzy soft Lie subalgebra and discuss some of its related properties.

2. Preliminaries

A *Lie algebra* is a vector space L over a field F (equal to $\mathbf{R}$ or $\mathbf{C}$) on which $L \times L \to L$ denoted by $(x, y) \to [x, y]$ is defined satisfying the following axioms:

(L1) $[x, y]$ is bilinear,

(L2) $[x, x] = 0$ for all $x \in L$,

(L3) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in L$ (Jacobi identity).

Throughout this paper, L is a Lie algebra and F is a field. We note that the operation $[.,.]$ is not associative, but it is *anticommutative*, i.e., $[x, y] = -[y, x]$. A subspace H of L closed under $[\cdot, \cdot]$ will be called a *Lie subalgebra*.

Let X be a nonempty set. A fuzzy subset μ of X is defined as a mapping from X into $[0, 1]$, where $[0, 1]$ is the usual interval of real numbers. We denote by $\mathbb{F}(X)$ the set of all fuzzy subsets of X.

A fuzzy set μ in a set X of the form

$$\mu(y) = \begin{cases} t \in (0, 1], & \text{if } \ y = x, \\ 0, & \text{if } \ y \neq x, \end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t. For a fuzzy point x_t and a fuzzy set μ in a set X, Pu and Liu [9] gave meaning to the symbol $x_t \alpha \mu$, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$. A fuzzy point x_t is called *belong to* a fuzzy set μ, written as $x_t \in \mu$, if $\mu(x) \geqslant t$. A fuzzy point x_t is said to be *quasicoincident with* a fuzzy set μ, written as $x_t q \mu$, if $\mu(x) + t > 1$. To say that $x_t \in \vee q \mu$ (resp. $x_t \in \wedge q \mu$) means that $x_t \in \mu$ or $x_t q \mu$ (resp. $x_t \in \mu$ and $x_t q \mu$). $x_t \overline{\alpha} \mu$ means that $x_t \alpha \mu$ does not hold, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$.

Molodtsov [8] defined the notion of soft set in the following way: Let U be an initial universe and E be a set of parameters. Let $P(U)$ denotes the power set of U and let A be a nonempty subset of E. Then a pair (F, A) is called a *soft set* over U, where F is a mapping given by $F : A \to P(U)$.

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set (F, A). Clearly, a soft set is not just a subset of U.

Definition 2.1. [13] Let X be a nonempty set. A *bipolar fuzzy set* B in X is an object having the form

$$B = \{(x,\, \mu^P(x),\, \mu^N(x)) \,|\, x \in X\},$$

where $\mu^P : X \to [0,\, 1]$ and $\mu^N : X \to [-1,\, 0]$ are mappings.

We use the positive membership degree $\mu^P(x)$ to denote the satisfaction degree of an element x to the property corresponding to a bipolar fuzzy set B, and the negative membership degree $\mu^N(x)$ to denote the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar fuzzy set B. If $\mu^P(x) \neq 0$ and $\mu^N(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for B. If $\mu^P(x) = 0$ and $\mu^N(x) \neq 0$, it is the situation that x does not satisfy the property of B but somewhat satisfies the counter property of B . It is possible for an element x to be such that $\mu^P(x) \neq 0$ and $\mu^N(x) \neq 0$ when the membership function of the property overlaps that of its counter property over some portion of X.

For the sake of simplicity, we shall use the symbol $B = (\mu^P,\, \mu^N)$ for the bipolar fuzzy set $B = \{(x,\, \mu^P(x),\, \mu^N(x)) \,|\, x \in X\}$.

Definition 2.2. Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set on X and let $\alpha \in [0, 1]$. α-cut A_α of A can be defined as

$$A_\alpha = A_\alpha^P \cup A_\alpha^N, \quad A_\alpha^P = \{x \mid \mu_\alpha^P(x) \geq \alpha\}, \quad A_\alpha^N = \{x \mid \mu_\alpha^N(x) \leq -\alpha\}.$$

We call A_α^P as positive α-cut and A_α^N as negative α-cut.

Definition 2.3. [13] For every two bipolar fuzzy sets $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B^P, \mu_B^N)$ in X, we define

- $(A \bigcap B)(x) = (\min(\mu_A^P(x), \mu_B^P(x)), \max(\mu_A^N(x), \mu_B^N(x)))$,
- $(A \bigcup B)(x) = (\max(\mu_A^P(x), \mu_B^P(x)), \min(\mu_A^N(x), \mu_B^N(x)))$.

The concept of bipolar fuzzy soft set was originally proposed in [10]. Let $BF(U)$ denote the family of all bipolar fuzzy sets in U.

Definition 2.4. [10] Let U be an initial universe and $A \subseteq E$ be a set of parameters. A pair (f, A) is called an *bipolar fuzzy soft set* over U, where f is a mapping given by $f : A \to BF(U)$. A bipolar fuzzy soft set is a parameterized family of bipolar fuzzy subsets of U. For any $\varepsilon \in A$, f_ε is referred to as the set of a-approximate elements of the bipolar fuzzy soft set (f, A), which is actually a bipolar fuzzy set on U and can be written as

$$f_\varepsilon = \{(\mu_{f_\varepsilon}^P(x), \mu_{f_\varepsilon}^N(x)) \mid x \in U\},$$

where $\mu^P_{f_\varepsilon}(x)$ denotes the degree of x keeping the parameter ε, $\mu^N_{f_\varepsilon}(x)$ denotes the degree of x keeping the non-parameter ε.

Definition 2.5. [10] Let (f, A) and (g, B) be two bipolar fuzzy soft sets over U. We say that (f, A) is a *bipolar fuzzy soft subset* of (g, B) and write $(f, A) \Subset (g, B)$ if $A \subseteq B$ and $f(\varepsilon) \subseteq g(\varepsilon)$ for $\varepsilon \in A$. (f, A) and (g, B) are said to be *bipolar fuzzy soft equal sets* and write $(f, A) = (g, B)$ if $(f, A) \Subset (g, B)$ and $(g, B) \Subset (f, A)$.

According to [10] for any two bipolar fuzzy soft sets (f, A) and (g, B) over U we define

- the *extended intersection* $(h, C) = (f, A)\widetilde{\cap}(g, B)$, where $C = A \cup B$ and

$$h(\varepsilon) = \begin{cases} f_\varepsilon & \text{if } \varepsilon \in A - B, \\ g_\varepsilon & \text{if } \varepsilon \in B - A, \\ f_\varepsilon \cap g_\varepsilon & \text{if } \varepsilon \in A \cap B, \end{cases}$$

- the *extended union* $(h, C) = (f, A)\widetilde{\cup}(g, B)$, where $C = A \cup B$ and

$$h(\varepsilon) = \begin{cases} f_\varepsilon & \text{if } \varepsilon \in A - B, \\ g_\varepsilon & \text{if } \varepsilon \in B - A, \\ f_\varepsilon \cup g_\varepsilon & \text{if } \varepsilon \in A \cap B, \end{cases}$$

- the operation $(f,A) \wedge (g,B) = (h, A \times B)$, where $h(a, b) = h(a) \cap g(b)$ for all $(a,b) \in A \times B$.

3. Bipolar fuzzy soft Lie algebras

Definition 3.1. Let (f, A) be a bipolar fuzzy soft set over L. Then (f, A) is said to be a *bipolar fuzzy soft Lie subalgebra* over L if $f(x)$ is a bipolar fuzzy Lie subalgebra of L for all $x \in A$, that is, a bipolar fuzzy soft set (f, A) over L is called a *bipolar fuzzy soft Lie subalgebra* of L if the following conditions are satisfied:

(1) $\mu^P_{f_\varepsilon}(x + y) \geqslant \min\{\mu^P_{f_\varepsilon}(x), \mu^P_{f_\varepsilon}(y)\}$,

(2) $\mu^N_{f_\varepsilon}(x + y) \leqslant \max\{\mu^N_{f_\varepsilon}(x), \mu^N_{f_\varepsilon}(y)\}$,

(3) $\mu^P_{f_\varepsilon}(mx) \geqslant \mu^P_{f_\varepsilon}(x)$, $\quad \mu^N_{f_\varepsilon}(mx) \leqslant \mu^N_{f_\varepsilon}(x)$,

(4) $\mu^P_{f_\varepsilon}([x, y]) \geqslant \min\{\mu^P_{f_\varepsilon}(x), \mu^P_{f_\varepsilon}(y)\}$,

(5) $\mu^N_{f_\varepsilon}([x, y]) \leqslant \max\{\mu^N_{f_\varepsilon}(x), \mu^N_{f_\varepsilon}(y)\}$

for all $x, y \in L$ and $m \in K$.

Example 3.2. The real vector space $\Re^2$ with $[x, y] = x \times y$ is a real Lie algebra. Let $\mathbb{N}$ and $\mathbb{Z}$ denote the set of all natural numbers and the set of all integers, respectively. By routine computations, we can easily check that $(f, \mathbb{Z})$, where $f : \mathbb{Z} \to ([0,1] \times [-1,0])^{\Re^2}$ with $f(n) = (\mu^P_{f_n}, \mu^N_{f_n}) : \Re^2 \to [0,1] \times [-1,0]$ for all $n \in \mathbb{Z}$,

$$\mu^P_{f_n}(x) = \begin{cases} 0.6 & \text{if } x = (0,0) = \underline{0}, \\ 0.2 & \text{if } x = (0,a),\ a \neq 0, \\ 0 & \text{otherwise,} \end{cases} \qquad \mu^N_{f_n}(x) = \begin{cases} -0.3 & \text{if } x = (0,0) = \underline{0}, \\ -0.2 & \text{if } x = (0,a),\ a \neq 0, \\ -1 & \text{otherwise,} \end{cases}$$

is a bipolar fuzzy soft Lie subalgebra of $\Re^2$. □

We state the following propositions without their proofs.

Proposition 3.3. *Let (f, A) be a bipolar fuzzy soft Lie subalgebra on L, then*

(i) $\mu^P_{f_\varepsilon}(\underline{0}) \geqslant \mu^P_{f_\varepsilon}(x), \quad \mu^N_{f_\varepsilon}(\underline{0}) \leqslant \mu^N_{f_\varepsilon}(x)$,

(ii) $\mu^P_{f_\varepsilon}([x,y]) = \mu^P_{f_\varepsilon}(-[y,x]) = \mu^P_{f_\varepsilon}([y,x])$,

(iii) $\mu^N_{f_\varepsilon}([x,y]) = \mu^N_{f_\varepsilon}(-[y,x]) = \mu^N_{f_\varepsilon}([y,x])$

for all $x, y \in L$. □

Proposition 3.4. *Let (f, A) and (g, B) be bipolar fuzzy soft Lie subalgebras over L, then $(f, A)\widetilde{\cap}(g, B)$ and $(f, A) \wedge (g, B)$ are bipolar fuzzy soft Lie subalgebras over L. If $A \cap B = \emptyset$, then also $(f, A)\widetilde{\cup}(g, B)$ is a bipolar fuzzy soft Lie subalgebra .* □

Proposition 3.5. *Let (f, A) be a bipolar fuzzy soft Lie subalgebra over L and let $\{(h_i, B_i) \mid i \in I\}$ be a nonempty family of bipolar fuzzy soft Lie subalgebras of (f, A). Then*

(a) $\widetilde{\cap}_{i \in I}(h_i, B_i)$ *is a bipolar fuzzy soft Lie subalgebra of (f, A),*

(b) $\bigwedge_{i \in I}(h_i, B_i)$ *is a bipolar fuzzy soft Lie subalgebra of $\bigwedge_{i \in I}(f, A)$,*

(c) *If $B_i \cap B_j = \emptyset$ for all $i, j \in I$, $i \neq j$, then $\widetilde{\bigvee}_{i \in I}(H_i, B_i)$ is a bipolar fuzzy soft Lie subalgebra of $\widetilde{\bigvee}_{i \in I}(f, A)$.* □

Definition 3.6. Let (f, A) be a bipolar fuzzy soft set over U. For each $s \in [0,1]$, $t \in [-1,0]$, the set $(f, A)^{(s,t)} = (f^{(s,t)}, A)$, where

$$(f, A)^{(s,t)}_\varepsilon = \{x \in U \mid \mu^P_{f_\varepsilon}(x) \geqslant s, \ \ \mu^N_{f_\varepsilon}(x) \leqslant t\} \ \text{ for all } \varepsilon \in A,$$

is called an (s,t)-*level soft set* of (f, A). Clearly, $(f, A)^{(s,t)}$ is a soft set over U.

Theorem 3.7. *Let (f, A) be a bipolar fuzzy soft set over L. (f, A) is a bipolar fuzzy soft Lie subalgebra if and only if $(f, A)^{(s,t)}$ is a soft Lie subalgebra over L for each $s \in [0,1]$, $t \in [-1,0]$.*

Proof. Suppose that (f, A) is a bipolar fuzzy soft Lie subalgebra. Then for each $s \in [0,1]$, $t \in [-1,0]$, $\varepsilon \in A$ and $x_1, x_2 \in (f, A)_\varepsilon^{(s,t)}$ we have $\mu_{f_\varepsilon}^P(x_1) \geqslant s$, $\mu_{f_\varepsilon}^P(x_2) \geqslant s$ and $\mu_{f_\varepsilon}^N(x_1) \leqslant t$, $\mu_{f_\varepsilon}^N(x_2) \leqslant t$. From Definition 3.1, it follows that $(f, A)_\varepsilon^{(s,t)}$ is a bipolar fuzzy Lie subalgebra over L. Thus $\mu_{f_\varepsilon}^P(x_1 + x_2) \geqslant \min(\mu_{f_\varepsilon}^P(x_1), \mu_{f_\varepsilon}^P(x_2))$, $\mu_{f_\varepsilon}^P(x_1 + x_2) \geqslant s$, $\mu_{f_\varepsilon}^N(x_1 + x_2) \leqslant \max(\mu_{f_\varepsilon}^N(x_1), \mu_{f_\varepsilon}^N(x_2))$, $\mu_{f_\varepsilon}^N(x_1 + x_2) \leqslant t$. This implies that $x_1 + x_2 \in f_\varepsilon^s$. The verification for other conditions is similar. Thus $(f, A)^{(s,t)}$ is a soft Lie subalgebra over L for each $s \in [0,1], t \in [-1,0]$.

Conversely, assume that $(f, A)^{(s,t)}$ is a soft Lie subalgebra over L for each $s \in [0,1], t \in [-1,0]$. For each $\varepsilon \in A$ and $x_1, x_2 \in G$, let $s = \min\{\mu_{f_\varepsilon}^P(x_1), \mu_{f_\varepsilon}^P(x_2)\}$ and let $t = \max\{\mu_{f_\varepsilon}^N(x_1), \mu_{f_\varepsilon}^N(x_2)\}$, then x_1, $x_2 \in (f, A)_\varepsilon^{(s,t)}$. Since $(f, A)_\varepsilon^{(s,t)}$ is a Lie subalgebra over L, then $x_1 + x_2 \in (f, A)_\varepsilon^{(s,t)}$. This means that $\mu_{f_\varepsilon}^P(x_1 + x_2) \geqslant \min(\mu_{f_\varepsilon}^P(x_1), \mu_{f_\varepsilon}^P(x_2))$ and $\mu_{f_\varepsilon}^N(x_1 + x_2) \leqslant \max(\mu_{f_\varepsilon}^N(x_1), \mu_{f_\varepsilon}^N(x_2))$. The verification for other conditions is similar. Thus according to Definition 3.1, (f, A) is a bipolar fuzzy soft Lie subalgebra over L. This completes the proof. □

Definition 3.8. Let $\phi : L_1 \to L_2$ and $\psi : A \to B$ be two functions, A and B are parametric sets from the crisp sets L_1 and L_2, respectively. Then the pair (ϕ, ψ) is called a *bipolar fuzzy soft function* from L_1 to L_2.

Definition 3.9. Let (f, A) and (g, B) be two bipolar fuzzy soft sets over L_1 and L_2, respectively and let (ϕ, ψ) be a bipolar fuzzy soft function from L_1 to L_2.

The *image of* (f, A) under the bipolar fuzzy soft function (ϕ, ψ), denoted by $(\phi, \psi)(f, A)$, is the bipolar fuzzy soft set on $\mathcal{K}_2$ defined by $(\phi, \psi)(f, A) = (\phi(f), \psi(A))$, where for all $k \in \psi(A), y \in L_2$

$$\mu_{\phi(f)_k}^P(y) = \begin{cases} \bigvee_{\phi(x)=y} \bigvee_{\psi(a)=k} f_a(x) & \text{if } x \in \psi^{-1}(y), \\ 1 & \text{otherwise,} \end{cases}$$

$$\mu_{\phi(f)_k}^N(y) = \begin{cases} \bigwedge_{\phi(x)=y} \bigwedge_{\psi(a)=k} f_a(x) & \text{if } x \in \psi^{-1}(y), \\ -1 & \text{otherwise.} \end{cases}$$

The *preimage of* (g, B) under the bipolar fuzzy soft function (ϕ, ψ), denoted by $(\phi, \psi)^{-1}(g, B)$, is the bipolar fuzzy soft set over $\mathcal{K}_1$ defined by $(\phi, \psi)^{-1}(g, B) = (\phi^{-1}(g), \psi^{-1}(B))$, where for all $a \in \psi^{-1}(A)$ for all $x \in L_1$,

$$\mu_{\phi^{-1}(g)_a}^P(x) = \mu_{g_{\psi(a)}}^P(\phi(x)), \quad \mu_{\phi^{-1}(g)_a}^N(x) = \mu_{g_{\psi(a)}}^N(\phi(x)).$$

Definition 3.10. Let (ϕ, ψ) be a bipolar fuzzy soft function from L_1 to L_2. If ϕ is a homomorphism from L_1 to L_2 then (ϕ, ψ) is said to be a *bipolar fuzzy soft homomorphism*. If ϕ is a isomorphism from L_1 to L_2 and ψ is one-to-one mapping from A onto B then (ϕ, ψ) is said to be a *bipolar fuzzy soft isomorphism*.

Theorem 3.11. *Let (g, B) be a bipolar fuzzy soft Lie subalgebra over L_2 and let (ϕ, ψ) be a bipolar fuzzy soft homomorphism from L_1 to L_2. Then $(\phi, \psi)^{-1}(g, B)$ is a bipolar fuzzy soft Lie subalgebra over L_1.*

Proof. Let $x_1, x_2 \in L_1$, then

$$\begin{aligned}\phi^{-1}(\mu^P_{g_\varepsilon})(x_1+x_2) &= \mu^P_{g_{\psi(\varepsilon)}}(\phi(x_1+x_2)) = \mu^P_{g_{\psi(\varepsilon)}}(\phi(x_1)+\phi(x_2))\\ &\geqslant \min\{\mu^P_{g_{\psi(\varepsilon)}}(\phi(x_1)), \mu^P_{g_{\psi(\varepsilon)}}(\phi(x_2))\}\\ &= \min\{\phi^{-1}(\mu^P_{g_\varepsilon})(x_1), \phi^{-1}(\mu^P_{g_\varepsilon})(x_2)\},\end{aligned}$$

$$\begin{aligned}\phi^{-1}(\mu^N_{g_\varepsilon})(x_1+x_2) &= \mu^N_{g_{\psi(\varepsilon)}}(\phi(x_1+x_2)) = \mu^N_{g_{\psi(\varepsilon)}}(\phi(x_1)+\phi(x_2))\\ &\leqslant \max\{\mu^N_{g_{\psi(\varepsilon)}}(\phi(x_1)), \mu^N_{g_{\psi(\varepsilon)}}(\phi(x_2))\}\\ &= \max\{\phi^{-1}(\mu^N_{g_\varepsilon})(x_1), \phi^{-1}(\mu^N_{g_\varepsilon})(x_2)\}.\end{aligned}$$

The verification for other conditions is similar and hence we omit the detail. Hence $(\phi,\psi)^{-1}(g,B)$ is a bipolar fuzzy soft Lie subalgebra over L_1. □

Note that $(\phi,\psi)(f,A)$ may not be a bipolar fuzzy soft Lie subalgebra over L_2.

4. $(\in, \in \vee q)$- bipolar fuzzy soft Lie algebras

Let $c \in G$ be fixed. If $\gamma \in (0,1]$ and $\delta \in [-1,0)$ are two real numbers, then $c(\gamma,\delta) = \langle x, c_\gamma, c_\delta\rangle$ is called a ***bipolar fuzzy point*** in G, where γ(resp, δ) is the positive degree of membership (resp, negative degree of membership) of $c(\gamma,\delta)$ and $c \in G$ is the support of $c(\gamma,\delta)$. Let $c(\gamma,\delta)$ be a bipolar fuzzy in G and let $A = \langle x, \mu^P_A, \mu^N_A\rangle$ be a bipolar fuzzy in G. Then $c(\gamma,\delta)$ is said to belong to A, written $c(\gamma,\delta) \in A$ if $\mu^P_A(c) \geqslant \gamma$ and $\mu^N_A(c) \leqslant \delta$. We say that $c(\gamma,\delta)$ is quasicoincident with A, written $c(\gamma,\delta)qA$, if $\mu^P_A(c)+\gamma > 1$ and $\mu^N_A(c)+\delta < -1$. To say that $c(\gamma,\delta) \in \vee qA$ (resp, $c(\gamma,\delta) \in \wedge qA$) means that $c(\gamma,\delta) \in A$ or $c(\gamma,\delta)qA$ (resp, $c(\gamma,\delta) \in A$ and $c(\gamma,\delta)qA$) and $c(\gamma,\delta)\overline{\in \vee q}A$ means that $c(\gamma,\delta) \in \vee qA$ does not hold.

Definition 4.1. A bipolar fuzzy set $A = (\mu^P_A, \mu^N_A)$ in L is called an $(\in, \in\vee q)$-*bipolar fuzzy Lie subalgebra* of L if it satisfies the following conditions:

(a) $x(s_1,t_1), y(s_2,t_2) \in A \Rightarrow (x+y)(\min(s_1,s_2), \max(t_1,t_2)) \in \vee qA$,

(b) $x(s,t) \in A \Rightarrow (mx)(s,t) \in \vee qA$,

(c) $x(s_1,t_1), y(s_2,t_2) \in A \Rightarrow ([x,y])(\min(s_1,s_2), \max(t_1,t_2)) \in \vee qA$

for all $x, y \in L$, $m \in K$, $s, s_1, s_2 \in (0,1]$, $t, t_1, t_2 \in [-1,0)$.

Example 4.2. Let $\Re^2$ be as in Example 3.2. We define a bipolar fuzzy set $A : G \to [0,1]\times[-1,0]$ by

$$\mu^P_A(x) = \begin{cases} 1 & \text{if } x = e,\\ 0.4 & \text{otherwise,}\end{cases} \qquad \mu^N_A(x) = \begin{cases} 0 & \text{if } x = e,\\ -0.2 & \text{otherwise.}\end{cases}$$

By routine computations, it is easy to see that A is not an $(\in, \in\vee q)$-bipolar fuzzy Lie subalgebra of L. □

Theorem 4.3. *A bipolar fuzzy set A in a Lie algebra L is an $(\in, \in \vee q)$-bipolar fuzzy Lie subalgebra of L if and only if*

- $\mu_A^P(x+y) \geqslant \min(\mu_A^P(x), \mu_A^P(y), 0.5)$, $\mu_A^N(x+y) \leqslant \max(\mu_A^N(x), \mu_A^N(y), -0.5)$,
- $\mu_A^P(mx) \geqslant \min(\mu_A^P(x), 0.5)$, $\mu_A^N(mx) \leqslant \max(\mu_A^N(x), -0.5)$,
- $\mu_A^P([x,y]) \geqslant \min(\mu_A^P(x), \mu_A^P(y), 0.5)$, $\mu_A^N([x,y]) \leqslant \max(\mu_A^N(x), \mu_A^N(y), -0.5)$

hold for all $x, y \in L$, $m \in K$. □

Theorem 4.4. *A bipolar fuzzy set A of a Lie algebra of L is an $(\in, \in \vee q)$-bipolar fuzzy Lie subalgebra of L if and only if for all $s \in (0.5, 1]$, $t \in [-1, -0.5)$ each nonempty $A_{(s,t)}$ is a Lie subalgebra of L.*

Proof. Assume that A is an $(\in, \in \vee q)$-bipolar fuzzy Lie subalgebra of L and let $s \in (0.5, 1]$, $t \in [-1, -0.5)$. If $x, y \in A_{(s,t)}$, then $\mu_A^P(x) \geq s$ and $\mu_A^P(y) \geq s$, $\mu_A^N(x) \leq t$ and $\mu_A^N(y) \leq t$. Thus, $\mu_A^P(x+y) \geqslant \min(\mu_A^P(x), \mu_A^P(y), 0.5) \geqslant \min(s, 0.5) = s$ and $\mu_A^N(x+y) \leqslant \max(\mu_A^N(x), \mu_A^N(y), -0.5) \leqslant \max(t, -0.5) = t$, so $x + y \in A_{(s,t)}$. The verification for other conditions is similar. The proof of converse part is obvious. □

Theorem 4.5. *If A is a bipolar fuzzy set in a Lie algebra L, then $A_{(s,t)}$ is a Lie subalgebra of L if and only if*

- $\max(\mu_A^P(x+y), 0.5) \geqslant \min(\mu_A^P(x), \mu_A^P(y))$,
  $\min(\mu_A^N(x+y), -0.5) \leqslant \max(\mu_A^N(x), \mu_A^N(y))$,
- $\max(\mu_A^P(mx), 0.5) \geqslant \min(\mu_A^P(x))$,
  $\min(\mu_A^N(mx), -0.5) \leqslant \max(\mu_A^N(x))$,
- $\max(\mu_A^P([x,y]), 0.5) \geqslant \min(\mu_A^P(x), \mu_A^P(y))$,
  $\min(\mu_A^N([x,y]), -0.5) \leqslant \max(\mu_A^N(x), \mu_A^N(y))$

for all $x, y \in L$, $m \in K$. □

Definition 4.6. Let (f, A) be a bipolar fuzzy soft set over a Lie algebra L. Then (f, A) is called an *$(\in, \in\vee q)$-bipolar fuzzy soft Lie subalgebra* if $f(\alpha)$ is an $(\in, \in\vee q)$-bipolar fuzzy Lie subalgebra of L for all $\alpha \in A$.

Theorem 4.7. *Let (f, A) and (g, B) be two $(\in, \in\vee q)$-bipolar fuzzy soft Lie subalgebras over a Lie algebra L. Then $(f, A) \wedge (g, B)$ is an $(\in, \in\vee q)$-bipolar fuzzy soft Lie subalgebra over L.*

Proof. By the definition, we can write $(f, A) \wedge (g, B) = (h, C)$, where $C = A \times B$ and $h(\alpha, \beta) = f(\alpha) \cap g(\beta)$ for all $(\alpha, \beta) \in C$. Now for any $(\alpha, \beta) \in C$, since (f, A) and (g, B) are $(\in, \in\vee q)$-bipolar fuzzy soft Lie subalgebras over L, we have both $f(\alpha)$ and $g(\beta)$ are $(\in, \in\vee q)$-bipolar fuzzy Lie subalgebras of L. Thus $h(\alpha, \beta) = f(\alpha) \cap g(\beta)$ is an $(\in, \in\vee q)$-bipolar fuzzy Lie subalgebra of L. Hence, $(f, A) \wedge (g, B)$ is an $(\in, \in\vee q)$-bipolar fuzzy soft Lie subalgebra over L. □

Theorem 4.8. *Let (f, A) and (g, B) be two $(\in, \in\vee q)$-bipolar fuzzy soft Lie subalgebras over a Lie algebra L. Then $(f, A)\widetilde{\cap}(g, B)$ is an $(\in, \in\vee q)$-bipolar fuzzy soft Lie subalgebra over L.*

Proof. We have $(f, A)\widetilde{\cap}(g, B) = (h, C)$, where $C = A \cup B$ and

$$h(\varepsilon) = \begin{cases} f(\varepsilon) & \text{if } \varepsilon \in A - B, \\ g(\varepsilon) & \text{if } \varepsilon \in B - C, \\ f(\varepsilon) \cap g(\varepsilon) & \text{if } \varepsilon \in A \cap B \end{cases}$$

for all $\alpha \in C$.

Now for any $\alpha \in C$, we consider the following cases.

1. $\alpha \in A - B$. Then $h(\alpha) = f(\alpha)$ is an $(\in, \in\vee q)$-bipolar fuzzy Lie subalgebra of L since (f, A) is an $(\in, \in\vee q)$-bipolar fuzzy soft Lie subalgebra over L.

2. $\alpha \in B - A$. Then $h(\alpha) = g(\alpha)$ is an $(\in, \in\vee q)$-bipolar fuzzy Lie subalgebra of L since (g, B) is an $(\in, \in\vee q)$-bipolar fuzzy soft Lie subalgebra over L.

3. $\alpha \in A \cap B$. Then $h(\alpha) = f(\alpha) \cap g(\alpha)$ is an $(\in, \in\vee q)$-bipolar fuzzy Lie subalgebra of L by the assumption. Thus, in any case, $h(\alpha)$ is an $(\in, \in\vee q)$-bipolar fuzzy Lie subalgebra of L. Therefore, $(f, A)\widetilde{\cap}(g, B)$ is an $(\in, \in\vee q)$-bipolar fuzzy soft Lie subalgebra over L. □

Theorem 4.9. *Let (f, A) and (g, B) be two $(\in, \in\vee q)$-bipolar fuzzy soft Lie subalgebras over a Lie algebra L. If $A\cap B \neq \emptyset$, then $(f, A)\widetilde{\cap}(g, B)$ is an $(\in, \in\vee q)$-bipolar fuzzy soft Lie subalgebra over L.*

Proof. $(f, A)\widetilde{\cap}(g, B) = (h, C)$, where $C = A \cap B$ and $h(\alpha) = f(\alpha) \cap g(\alpha)$ for all $\alpha \in C$. Now for any $\alpha \in C$, since (f, A) and (g, B) are $(\in, \in\vee q)$-bipolar fuzzy soft Lie subalgebras over L, we have both $f(\alpha)$ and $g(\alpha)$ are $(\in, \in\vee q)$-bipolar fuzzy Lie subalgebras of L. Thus $h(\alpha) = f(\alpha) \cap g(\alpha)$ is an $(\in, \in\vee q)$-bipolar fuzzy Lie subalgebra of L. Therefore, $(f, A)\widetilde{\cap}(g, B)$ is an $(\in, \in\vee q)$-bipolar fuzzy soft Lie subalgebra over L. □

Theorem 4.10. *Let (f, A) be an $(\in, \in\vee q)$- bipolar fuzzy soft Lie subalgebra over L and let $\{(h_i, B_i) \mid i \in I\}$ be a nonempty family of $(\in, \in\vee q)$-bipolar fuzzy soft Lie subalgebras of (f, A). Then*

(a) $\widetilde{\cap}_{i\in I}(h_i, B_i)$ *is an $(\in, \in\vee q)$- bipolar fuzzy soft Lie subalgebra of (f, A),*

(b) $\bigwedge_{i\in I}(h_i, B_i)$ *is an $(\in, \in\vee q)$- bipolar fuzzy soft Lie subalgebra of $\bigwedge_{i\in I}(f, A)$,*

(c) *If $B_i \cap B_j = \emptyset$ for all i, $j \in I$, then $\widetilde{\bigvee}_{i\in I}(H_i, B_i)$ is an $(\in, \in\vee q)$- bipolar fuzzy soft Lie subalgebra of $\widetilde{\bigvee}_{i\in I} f, A)$.* □

Theorem 4.11. *Let (f, A) and (g, B) be two $(\in, \in\vee q)$-bipolar fuzzy soft Lie subalgebras over a Lie algebra L. If A and B are disjoint, then $(f, A)\widetilde{\cup}(g, B)$ is an $(\in, \in\vee q)$-bipolar fuzzy soft Lie subalgebra over L.* □

References

[1] **M. Akram**, *Intuitionistic (S,T)-fuzzy Lie ideals of Lie algebras*, Quasigroups and Related Systems **15** (2007), 201 − 218.

[2] **M. Akram**, *Bipolar fuzzy $\mathcal{L}$-Lie algebras*, World Appl. Sci. J. **14** (2011), 1908−1913.

[3] **M. Akram and N. O. Al-Shehrie**, *Bipolar fuzzy Lie ideals*, Utilitas Math. **87** (2012), 265 − 278.

[4] **M. Akram, W. Chen and Y. Lin**, *Bipolar fuzzy Lie superalgebras*, Quasigroups and Related Systems **20** (2012), 139 − 156.

[5] **M. Akram and F. Feng**, *Soft intersection Lie algebras*, Quasigroups and Related Systems **21** (2013), 11 − 18.

[6] **C. G. Kim and D. S. Lee**, *Fuzzy Lie ideals and fuzzy Lie subalgebras*, Fuzzy Sets Syst. **94** (1998), 101 − 107.

[7] **P. K. Maji, R. Biswas and R. Roy**, *Soft set theory*, Comput. Math. Appl. **45** (2003), 555 − 562.

[8] **D. Molodtsov**, *Soft set theory first results*, Comput. Math. Appl. **37** (1999), 19−31.

[9] **P. M. Pu and Y. M. Liu**, *Fuzzy topology, I. Neighborhood structure of a fuzzy point and Moore-Smith convergence*, J. Math. Anal. Appl. **76** (1980), 571 − 599.

[10] **W. Yang and S. Li**, *Bipolar-value fuzzy soft sets*, Computer Engineering Appl. **48(35)** (2012), 15 − 18.

[11] **S. E. Yehia**, *Fuzzy ideals and fuzzy subalgebras of Lie algebras*, Fuzzy Sets Syst. **80** (1996), 237 − 244.

[12] **L. A. Zadeh**, *Fuzzy sets*, Information and Control **8** (1965), 338 − 353.

[13] **W.-R. Zhang**, *Bipolar fuzzy sets*, Proc. of FUZZ-IEEE, 1998, 835 − 840.

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Soft intersection Lie algebras

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Abstract. In 1999, Molodtsov introduced the concept of soft set theory as a general mathematical tool for dealing with uncertainty and vagueness, and many researchers have created some models to solve problems in decision making and medical diagnosis. In this paper, we introduce the concept of soft Lie subalgebras (resp. soft Lie ideals) and state some of their fundamental properties. We also introduce the concept of soft intersection Lie subalgebras (resp. soft intersection Lie ideals) and investigate some of their properties.

1. Introduction

The theory of Lie algebras is an area of mathematics in which we can see a harmonious between the methods of classical analysis and modern algebra. This theory, a direct outgrowth of a central problem in the calculus, has today become a synthesis of many separate disciplines, each of which has left its own mark. Theory of Lie groups were developed by the Norwegian mathematician Sophus Lie in the late nineteenth century in connection with his work on systems of differential equations. Lie algebras were also discovered by Sophus Lie when he first attempted to classify certain smooth subgroups of general linear groups. The groups he considered are called Lie groups. The importance of Lie algebras for applied mathematics and for applied physics has also become increasingly evident in recent years. In applied mathematics, Lie theory remains a powerful tool for studying differential equations, special functions and perturbation theory. Lie theory finds applications not only in elementary particle physics and nuclear physics, but also in such diverse fields as continuum mechanics, solid-state physics, cosmology and control theory. Lie algebra is also used by electrical engineers, mainly in the mobile robot control. For the basic information of Lie algebras, the readers are refereed to [7, 10, 12].

Most of the problems in engineering, medical science, economics, environments, and so forth, have various uncertainties. The problems in system identification involve characteristics which are essentially non probabilistic in nature. In response to this situation Zadeh [19] introduced fuzzy set theory as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a more general framework, the approach to uncertainty is outlined by Zadeh [20]. Molodtsov [16] initiated the concept of soft set theory as a new mathematical

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tool for dealing with uncertainties and vagueness. In soft set theory, the problem of setting the membership function does not arise, which makes the theory easily applied to many different fields including game theory, operations research, Riemann-integration, Perron integration. At present, work on soft set theory is progressing rapidly. After Molodtsov's work, some operations and application of soft sets were studied by many researchers including Ali et al. [6], Aktas et al. [5], Chen et al. [9] and Maji et al. [15]. Maji et al. [15] gave first practical application of soft sets in decision making problems. The algebraic structure of soft set theories has been studied increasingly in recent years. Aktas and Cagman [5] defined the notion of soft groups. Feng et al. [13] initiated the study of soft semirings and soft rings were defined by Acar et al. [1], Cağman et al. [8] introduced the concept of soft int-groups, Yamark et al. [17] introduced soft hyperstructure. In this paper, we introduce the concept of soft Lie subalgebras (resp. soft Lie ideals) and state some of their fundamental properties. We also introduce the concept of soft intersection Lie subalgebras (resp. soft intersection Lie ideals) and investigate some of their properties.

2. Review of literature

In this paper by L will be a Lie algebra. We note that the multiplication in a Lie algebra is not associative, but it is *anti commutative*, i.e., $[x, y] = -[y, x]$ for all $x, y \in L$. A subspace H of L closed under $[\cdot, \cdot]$ will be called a *Lie subalgebra*.

In 1999, Molodtsov [16] initiated soft set theory as a new approach for modelling uncertainties. Later on, Maji et al.[14] expanded this theory to fuzzy soft set theory. Based on the idea of parameterization, a soft set gives a series of approximate descriptions of a complicate object from various different aspects. Each approximate description has two parts, namely predicate and approximate value set. A soft set can be determined by a set-valued mapping assigning to each parameter exactly one crisp subset of the universe. More specifically, we can define the notion of soft set in the following way. Let U be an initial universe and E be a set of parameters. Let $P(U)$ denote the power set of U and let A be a non-empty subset of E.

Definition 2.1. A pair $F_A = (F, A)$ is called a *soft set* over U, where $A \subseteq E$ and $F\colon A \to P(U)$ is a set-valued mapping, called the *approximate function* of the soft set F_A. It is easy to represent a soft set F_A by a set of ordered pairs as follows:

$$F_A = (F, A) = \{(x, F(x)) \mid x \in A\}.$$

It is clear that a soft set is a parameterized family of subsets of the set U.

Definition 2.2. Let F_A and G_B be two soft sets over a common universe U. F_A is a said to be *soft subset* of G_B, denoted by $F_A \tilde{\subset} G_B$, if $F(x) \subseteq G(x)$ for all $x \in E$.

We refer the readers to the papers [2-4, 6, 9, 12, 13, 15, 16, 18] for further information regarding soft set theory and the theory of fuzzy Lie algebras.

3. Soft Intersection Lie algebras

Definition 3.1. Let $F_A = (F, A)$ be a soft set over L. Then F_A is called a *soft Lie subalgebra* (resp. *soft Lie ideal*) over L if $F(x)$ is a Lie subalgebra (resp. Lie ideal) of a Lie algebra L for all $x \in A$.

Example 3.2. The real vector space $\mathbb{R}^3$ with the bracket $[.,.]$ defined as the cross product, i.e., $[x, y] = x \times y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)$ forms a real Lie algebra over the field R. Now we define a soft set $\langle F, \mathbb{R}^3\rangle$ as $F : \mathbb{R}^3 \longrightarrow P(\mathbb{R}^3)$ by $F((0, 0, 0)) = \{(0, 0, 0)\}$, $F(x, 0, 0) = \{(0, 0, 0), (x, 0, 0) : x \neq 0\}$ and $F(x, y, z) = \mathbb{R}^3$. By routine computations, it is easy to see that $(F, \mathbb{R}^3)$ is soft Lie subalgebra but not soft Lie ideal of $\mathbb{R}^3$.

Example 3.3. Let $\{e_1, e_2, \ldots, e_5\}$ be a basis of a vector space over V over a field F with Lie brackets as follows:

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = e_5, \quad [e_1, e_4] = e_5, \quad [e_1, e_5] = 0,$$

$$[e_2, e_3] = e_5, \quad [e_2, e_4] = 0, \quad [e_2, e_5] = 0, \quad [e_3, e_4] = 0,$$

$$[e_3, e_5] = 0, \quad [e_4, e_5] = 0, \quad [e_i, e_j] = -[e_j, e_i]$$

and $[e_i, e_j] = 0$ for all $i = j$. Then V is a Lie algebra over F. Let (F, V) be soft over V and define by

$$F(x) = \begin{cases} \langle e_7\rangle \text{ if } x = e_1 \\ \langle e_8\rangle \text{ if } x = e_2, e_3 \\ \langle e_7, e_8\rangle \text{ if } x = e_4, e_5 \\ V \quad \text{otherwise} \end{cases}$$

Routine computations show that (F, V) is a soft Lie ideal over V.

Definition 3.4. [9] Let F_A and G_B be two soft sets over a common universe U. Then the *intersection* $F_A \widetilde{\cap} G_B$, is defined as $F_A \widetilde{\cap} G_B(x) = F(x) \cap G(x)$ for all $x \in E$. The $\widetilde{\wedge}$*-product* $F_A \widetilde{\wedge} G_B$, is defined by $F_A \widetilde{\wedge} G_B(x, y) = F(x) \cap G(y)$ for all $(x, y) \in E \times E$.

Proposition 3.5. *If F_A and F_B are soft Lie subalgebras (resp. soft Lie ideals) over L. Then $F_A \widetilde{\wedge} F_B$ and $F_A \widetilde{\cap} F_B$ are soft Lie subalgebras (resp. soft Lie ideals) over L.* □

Definition 3.6. A soft Lie subalgebra (resp. soft Lie ideal) F_A over L is called *trivial* over L if $F(x) = \{\underline{0}\}$ for all $x \in A$, and *whole* over L if $F(x) = L$ for all $x \in A$.

Definition 3.7. Let L_1, L_2 be two Lie algebras and $\varphi : L_1 \to L_2$ a mapping of Lie algebras. If F_A and G_B are soft sets over L_1 and L_2 respectively, then $\varphi(F_A)$ is a soft set over L_2 where $\varphi(F) : E \to P(L_2)$ is defined by $\varphi(F)(x) = \varphi(F(x))$ for all $x \in E$ and $\varphi^{-1}(G_B)$ is a soft set over L_1 where $\varphi^{-1}(G) : E \to P(L_1)$ is defined by $\varphi^{-1}(G)(y) = \varphi^{-1}(G(y))$ for all $y \in E$.

Proposition 3.8. *Let $\varphi : L_1 \to L_2$ be an onto homomorphism of Lie algebras.*

(*i*) *If F_A is a soft Lie algebra over L_1, then $\varphi(F_A)$ is a soft Lie algebra over L_2.*

(*ii*) *If F_B is a soft Lie algebra over L_2, then $\varphi^{-1}(F_B)$ is a soft Lie algebra over L_1 if it is non-null.* $\square$

Theorem 3.9. *Let $f : L_1 \to L_2$ be a homomorphism of Lie algebras. Let F_A and G_B be two soft Lie algebras over L_1 and L_2, respectively.*

(a) *If $F(x) = \ker(\varphi)$ for all $x \in A$, then $\varphi(F_A)$ is the trivial soft Lie algebra over L_2.*

(b) *If φ is onto and F_A is whole, then $\varphi(F_A)$ is the whole soft Lie algebra over L_2.*

(c) *If $G(y) = \varphi(L_1)$ for all $y \in B$, then $\varphi^{-1}(G_B)$ is the whole soft Lie algebra over L_1.*

(d) *If φ is injective and G_B is trivial, then $\varphi^{-1}(G_B)$ is the trivial soft Lie algebra over L_1.* $\square$

We now introduce the concept of soft intersection Lie subalgebras (resp. soft intersection Lie ideals).

Definition 3.10. Let $L = E$ be a Lie algebra and let A be a subset of L. Let F_A be a soft set over U. Then, F_A is called a *soft intersection Lie subalgebra* over U if it satisfies the following conditions:

(a) $F(x + y) \supseteq F(x) \cap F(y)$,

(b) $F(mx) \supseteq F(x)$,

(c) $F([x, y]) \supseteq F(x) \cap F(y)$

for all $x, y \in A$, $m \in K$. A soft set F_A is called a *soft intersection Lie ideal* over U if it satisfies (a), (b) and

(d) $F([x, y]) \supseteq F(x)$

for all x, $y \in A$.

Example 3.11. Assume that $U = \mathbb{Z}$ is the universal set. The vector space $E = \mathbb{R}^2$ with the bracket $[.,.]$ defined as the usual cross product, i.e., $[x, y] = x \times y = xy - yx$ forms a real Lie algebra. Let $A = \{(0, 0), (0, x), x \neq 0\}$ be a subset of E. Let F_A be a soft set over U. Then $F(0, 0) = \mathbb{Z}$ and $F(0, x) = \{-2, -1, 0, 1, 2\}$. It is easy to see that F_A is a soft intersection Lie algebra (resp. soft intersection Lie ideal) over U.

From now on, we will always assume $L = E$ unless otherwise specified.

Proposition 3.12. *Let L be a Lie algebra and let A be Lie subalgebra (resp. Lie ideal) of L. If F_A is a soft intersection Lie subalgebra (resp. soft intersection Lie ideal) over U. Then $F(\underline{0}) \supseteq F(x)$ and $F(-x) = F(x)$ for all $x \in A$.* $\square$

Proposition 3.13. *Let L be a Lie algebra and let A and B be Lie subalgebras (resp. Lie ideals) of L. If F_A and G_B are soft intersection Lie subalgebras (resp. soft intersection Lie ideals) over U. Then $F_A \widetilde{\wedge} G_B$ is a soft intersection Lie subalgebras (resp. soft intersection Lie ideal) over U.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in A \times B$ and $m \in K$. Then

$$\begin{aligned}
(F_A \widetilde{\wedge} G_B)((x_1, y_1) + (x_2, y_2)) &= (F_A \widetilde{\wedge} G_B)((x_1 + x_2, y_1 + y_2)) \\
&= F(x_1 + x_2) \cap G(y_1 + y_2) \\
&\supseteq (F(x_1) \cap F(x_2)) \cap (G(y_1) \cap G(y_2)) \\
&= (F(x_1) \cap G(y_1)) \cap (F(x_2) \cap G(y_2)) \\
&= (F_A \widetilde{\wedge} G_B)(x_1, y_1) \cap (F_A \widetilde{\wedge} G_B)(x_2, y_2), \\
(F_A \widetilde{\wedge} G_B)(m(x_1, y_1)) &= (F_A \widetilde{\wedge} G_B)(mx_1, my_1) \\
&= F(mx_1) \cap G(my_1) \\
&\supseteq F(x_1) \cap G(y_1) = (F_A \widetilde{\wedge} G_B)(x_1, y_1), \\
(F_A \widetilde{\wedge} G_B)([(x_1, y_1), (x_2, y_2)]) &= (F_A \widetilde{\wedge} G_B)([x_1, x_2], [y_1, y_2]) \\
&= F([x_1, x_2]) \cap G([y_1, y_2]) \\
&\supseteq ([F(x_1), F(x_2)] \cap [G(y_1), G(y_2)]) \\
&= [F(x_1), G(y_1)] \cap [F(x_2), G(y_2)] \\
&= (F_A \widetilde{\wedge} G_B)[x_1, y_1] \cap (F_A \widetilde{\wedge} G_B)[x_2, y_2], \\
(F_A \widetilde{\wedge} G_B)([(x_1, y_1), (x_2, y_2)]) &= (F_A \widetilde{\wedge} G_B)([x_1, x_2], [y_1, y_2]) \\
&= F([x_1, x_2]) \cap G([y_1, y_2]) \\
&\supseteq F(x_1) \cap G(y_1) = (F_A \widetilde{\wedge} G_B)[x_1, y_1].
\end{aligned}$$

Hence $F_A \widetilde{\wedge} G_B$ is a soft intersection Lie subalgebra (resp. soft intersection Lie ideal) over U. $\square$

Theorem 3.14. *Let $\{(F_i)_{A_i} \mid i \in \Lambda\}$ be a family of soft intersection Lie subalgebras (resp. soft intersection Lie ideals) over U. Then $\widetilde{\bigwedge}_{i \in \Lambda} (F_i)_{A_i}$ a soft intersection Lie subalgebra (resp. soft intersection Lie ideal) over U.* $\square$

Proposition 3.15. *Let L be a Lie algebra and let A be a Lie subalgebra (resp. Lie ideal) of L. If F_A and G_A are soft intersection Lie subalgebras (resp. soft intersection Lie ideals) over U. Then $F_A \widetilde{\cap} G_A$ is a soft intersection Lie algebra (resp. soft intersection Lie ideal) over U.*

Proof. Similarly as Proposition 3.13. $\square$

Theorem 3.16. *Let $\{(F_i)_{A_i} \mid i \in \Lambda\}$ be a family of soft intersection Lie subalgebras (resp. soft intersection Lie ideals) over U. Then $\widetilde{\bigcap}_{i\in\Lambda}(F_i)_{A_i}$ a soft intersection Lie subalgebra (resp. soft intersection Lie ideal) over U.* □

In the same manner we can prove

Proposition 3.17. *Let A and B be Lie subalgebras (resp. Lie ideals) of a Lie algebra L. If F_A and G_B are soft intersection Lie subalgebras (resp. soft intersection Lie ideals) over U. Then $F_A\widetilde{\times}G_B$ defined by $F_A\widetilde{\times}G_B(x,y) = F(x)\times G(y)$ for all $(x,y)\in A\times B$, is a soft intersection Lie algebra (resp. soft intersection Lie ideal) over U.* □

Theorem 3.18. *Let $\{(F_i)_{A_i} \mid i \in \Lambda\}$ be a family of soft intersection Lie subalgebras (resp. soft intersection Lie ideals) over U. Then $\widetilde{\prod}_{i\in\Lambda}(F_i)_{A_i}$ a soft intersection Lie subalgebra (resp. soft intersection Lie ideal) over U.* □

Proposition 3.19. *Let L be a Lie algebra and let A, B and C be Lie subalgebras (resp. Lie ideals) of L. If F_A, G_B and F_C are soft intersection Lie subalgebras (resp. soft intersection Lie ideals) over U such that $F_A\widetilde{\leq}G_B$ and $F_C\widetilde{\leq}G_B$, then $F_A\widetilde{\cap}F_C\widetilde{\leq}G_B$ over U.*

Proof. Straightforward. □

Definition 3.20. Let F_A and G_B be two soft sets over the common universe U and let φ be a function from A to B. The *soft image* $\varphi(F_A)$ of F_A under φ is a soft set over U defined by

$$\varphi(F)(y) = \begin{cases} \bigcup\{F(x)\,|\,x\in A \text{ and } \varphi(x)=y\} & \text{if } \varphi^{-1}(y)\neq\emptyset,\\ \emptyset & \text{otherwise}\end{cases}$$

for all $y\in B$. The *soft pre-image* (or *soft inverse image*) $\varphi^{-1}(G_B)$ of G_B under φ is a soft set over U such that $\varphi^{-1}(G)(x) = G(\varphi(x))$ for all $x\in A$.

Proposition 3.21. *Let L be a Lie algebra and let A be Lie ideal of L. If F_A is a soft intersection Lie subalgebra (resp. soft intersection Lie ideal) over U, then $A_F = \{x\in A\,|\,F(x) = F(\underline{0})\}$ is a soft intersection Lie subalgebra (resp. soft intersection Lie ideal) over U.* □

Theorem 3.22. *Let A and B be Lie ideals of a Lie algebra L and φ be a Lie homomorphism from A to B. If G_B is a soft intersection Lie subalgebra (resp. soft intersection Lie ideal) over U, then $\varphi^{-1}(G_B)$ is a soft intersection Lie subalgebra (resp. soft intersection Lie ideal) over U.*

Proof. Straightforward. □

Theorem 3.23. *Let A and B be Lie ideals of a Lie algebra L. If $\varphi : A\to B$ is a surjective Lie homomorphism and F_A is a soft intersection Lie subalgebra (resp. soft intersection Lie ideal) over U, then $\varphi(F_A)$ is a soft intersection Lie subalgebra (resp. soft intersection Lie ideal) over U.*

Proof. Since φ is surjective, for all $a, b \in B$ there exist $x, y \in A$ such that $a = \varphi(x)$ and $b = \varphi(y)$. Then

$$\begin{aligned}
\varphi(F_A)(x+y) &= \cup\{F(z) \mid z \in A, \varphi(z) = a+b\} \\
&= \cup\{F(x+y) \mid x, y \in A, \varphi(x) = a,\ \varphi(y) = b\} \\
&\supseteq \cup\{F(x) \cap F(y) \mid x, y \in A, \varphi(x) = a,\ \varphi(y) = b\} \\
&= (\cup\{F(x) \mid x \in A, \varphi(x) = a\}) \cap (\cup\{F(y) \mid y \in A, \varphi(y) = b\}) \\
&= \varphi(F_A)(a) \cap \varphi(F_A)(b), \\
\varphi(F_A)(mx) &= \cup\{F(z) \mid z \in A, \varphi(z) = ma\} \\
&= \cup\{F(mx) \mid x \in A, \varphi(x) = a\} \\
&\supseteq \cup\{F(x) \mid x \in A, \varphi(x) = a\} \\
&= \varphi(F_A)(a), \\
\varphi(F)([x,y]) &= \cup\{F(z) \mid z \in A, \varphi(z) = [a,b]\} \\
&= \cup\{F_A([x,y]) \mid x, y \in A, \varphi(x) = a,\ \varphi(y) = b\} \\
&\supseteq \cup\{F(x) \cap F(y) \mid x, y \in A, \varphi(x) = a,\ \varphi(y) = b\} \\
&= (\cup\{F(x) \mid x \in A, \varphi(x) = a\}) \cap (\cup\{F(y) \mid y \in A, \varphi(y) = b\}) \\
&= \varphi(F_A)(a) \cap \varphi(F_A)(b), \\
\varphi(F_A)([x,y]) &= \cup\{F(z) \mid z \in A, \varphi(z) = [a,b]\} \\
&= \cup\{F([x,y]) \mid x, y \in A, \varphi(x) = a,\ \varphi(y) = b\} \\
&\supseteq \cup\{F(x) \mid x \in A, \varphi(x) = a\} \\
&= \cup\{F(x) \mid x \in A, \varphi(x) = a\} \\
&= \varphi(F_A)(a).
\end{aligned}$$

Hence $\varphi(F_A)$ is a soft intersection Lie subalgebra (resp. soft intersection Lie ideal) over U. $\square$

References

[1] **U. Acar, F. Koyuncu and B. Tanay**, *Soft sets and soft rings*, Comput. Math. Appl. **59** (2010), 3458 – 3463.

[2] **M. Akram**, *Intuitionistic (S,T)-fuzzy Lie ideals of Lie algebras*, Quasigroups and Related Systems **15** (2007), 201 – 218.

[3] **M. Akram**, *Generalized fuzzy Lie subalgebras*, J. Generalized Lie Theory Appl. **2** (2008), 261 – 268.

[4] **M. Akram and K. P. Shum**, *Intuitionistic fuzzy Lie algebras*, Southeast Asian Bull. Math. **31** (2007), 843 – 855.

[5] **H. Aktas and N. Cagman**, *Soft sets and soft groups*, Information Sci. **177** (2007), 2726 – 2735.

[6] **M. I. Ali, F. Feng, X. Y. Liu, W. K. Min and M. Shabir**, *On some new operations in soft set theory*, Comput. Math. Appl. **57** (2009), 1547 − 1553.

[7] **J. G. F. Belinfante and B. Kolman**, *A survey of Lie groups and Lie algebras with applications and computational methods*, Second Edition, 1989.

[8] **N. Çağman, F. Çitak and K. Aktaş**, *Soft int-groups and its applications to group theory*, Neural Comput Appl. **21** (2012), $S151 - S158$.

[9] **D. Chen, E. C. C. Tsang, D. S. Yeung and X. Wang**, *The parameterization reduction of soft sets and its applications*, Comput. Math. Appl. **49** (2005), 757−763.

[10] **P. Coelho and U. Nunes**, *Lie algebra application to mobile robot control: a tutorial*, Robotica **21** (2003), 483 − 493.

[11] **F. Feng, Y. B. Jun and X. Z. Zhao**, *Soft semirings*, Comput. Math. Appl. **56** (2008), 2621-2628.

[12] **J. E. Humphreys**, *Introduction to Lie algebras and representation theory*, Springer, New York 1972.

[13] **C. G. Kim and D. S. Lee**, *Fuzzy Lie ideals and fuzzy Lie subalgebras*, Fuzzy Sets Systems **94** (1998), 101 − 107.

[14] **P. K. Maji, R. Biswas and R. Roy**, *Fuzzy soft sets*, J. Fuzzy Math. **9** (2001), 589 − 602.

[15] **P. K. Maji, R. Biswas and R. Roy**, *Soft set theory*, Comput. Math. Appl. **45**(2003), 555 − 562.

[16] **D. Molodtsov**, *Soft set theory first results*, Comput. Math. Appl. **37** (1999), 19−31.

[17] **S. Yamak, O. Kazanci and B. Davvaz**, *Soft hyperstructure*, Comput. Math. Appl. **62** (2011), 797 − 803.

[18] **S. E. Yehia**, *Fuzzy ideals and fuzzy subalgebras of Lie algebras*, Fuzzy Sets Systems **80** (1996), 237 − 244.

[19] **L. A. Zadeh**, *Fuzzy sets*, Inform. and Control **8** (1965), 338 − 353.

[20] **L. A. Zadeh**, *The concept of a linguistic and application to approximate reasoning I*, Information Sci. **8** (1975), 199 − 249.

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A study of n-subracks

Guy R. Biyogmam

Abstract. In this paper, we introduce the notion of n-subracks ($n > 2$) and provide a characterization that enables us to obtain several results on n-racks. We also define a cohomology theory on n-racks.

1. Introduction

The category of n-racks [2] has been introduced as a generalization of the category of left distributive left quasigroups [9], or simply racks [6], and was shown to be associated to the category of Leibniz n-algebras [5]. In the pursue of studying the structure of this new category, we study in this paper the notion of n-subracks and explore several classical examples such as the normalizer, the center of a n-rack, and the components of a decomposable n-rack. In section 4, we provide several properties of decomposable n-racks.

In [8], Fenn, Rourke and Sanderson introduced a cohomology theory for racks which was modified in [4] by Carter, Jelsovsky, Kamada, Landford and Saito to obtain quandle cohomology, and several results have been recently established. In section 5, we use these cohomology theories to define cohomology theories on n-racks and n-quandles.

Let us recall a few definitions.

A *pointed rack* $(R, \circ, 1)$ is a set R with a binary operation $\circ$ and a specific element $1 \in R$ such that the following conditions are satisfied:

(R1) $x \circ (y \circ z) = (x \circ y) \circ (x \circ z)$.

(R2) For each $x, y \in R$, there exits a unique $a \in R$ such that $x \circ a = y$.

(R3) $1 \circ x = x$ and $x \circ 1 = 1$ for all $x \in R$.

A rack R is *decomposable* [1] if there are disjoints subracks X and Y of R such that $R = X \cup Y$. R is *indecomposable* if otherwise.

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2. n-racks

For the remaining of this paper, we assume $n \geqslant 2$, integer.

Definition 2.1. [2] A *n-rack* $(R, [\ldots])$ is a set R endowed with an n-ary operation $[\ldots] : R^n \longrightarrow R$ such that

(NR1) $\big[x_1, \ldots, x_{n-1}, [y_1, \ldots, y_{n-1}]\big] = \big[[x_1, \ldots, x_{n-1}, y_1], \ldots, [x_1, \ldots, x_{n-1}, y_n]\big]$ (This is the *left distributive property* of n-racks.)

(NR2) For $a_1, \ldots, a_{n-1}, b \in R$, there exists a unique $x \in R$ such that $[a_1, \ldots, a_{n-1}, x] = b$.

If in addition there is a distinguish element $1 \in R$, such that

(NR3) $[1, \ldots, 1, y] = y$ and $[x_1, \ldots, x_{n-1}, 1] = 1$ for all $x_1, \ldots, x_{n-1} \in R$,

then $(R, [\ldots], 1)$ is said to be a *pointed n-rack*.

An n-rack in which $[x_1, \ldots, x_{n-1}, y] = y$ if $x_i = y$ for some $i \in \{1, \ldots, n-1\}$, is an *n-quandle*.

Definition 2.2. A n-rack R is *involutive* if

$$\big[x_1, \ldots, x_{n-1}, [x_1, \ldots, x_{n-1}, y]\big] = y \quad \text{for all } x_1, \ldots, x_{n-1}, y \in R.$$

Note that an involutive n-quandle is an n-kei [2].

A n-rack R is *trivial* if it satisfies $[x_1, x_2, \ldots, x_{n-1}, y] = y$ for all $x_i, y \in R$.

For $n = 2$, one recovers involutive racks [1] and trivial racks [3].

Definition 2.3. Let K be a ring and M a K-module. Then M endowed with the n-ary operation $[\ldots]$ defined by

$$[x_1, \ldots, x_n] = q_1x_1 + q_2x_2 + \ldots + q_nx_n \quad \text{with} \quad \sum_{i=1}^{n} q_i = 1$$

is a n-rack called an *affine n-rack* associated to the K-module M.

Example 2.4. A $\mathbb{Z}_4$-module M endowed with the operation $[\ldots]_M$ defined by

$$[x_1, \ldots, x_n]_M = 2x_1 + 2x_2 + \ldots + 2x_{n-1} + x_n$$

is an affine n-rack if n is odd.

Proposition 2.5. [2] *Any pointed rack $(R, \circ, 1)$ has a pointed n-rack structure under the n-ary operation defined by*

$$[x_1, x_2, \ldots, x_n] = x_1 \circ (x_2 \circ (\ldots (x_{n-1} \circ x_n) \ldots)).$$

This process determines a functor $\mathfrak{G} : prack \longrightarrow {}_nprack$, which has as left adjoint, the functor $\mathfrak{G}' : {}_nprack \longrightarrow prack$ defined as follows:

Given a pointed n-rack $(R, [\ldots], 1)$, then R^{n-1} endowed with the binary operation

$$(x_1, \ldots, x_{n-1}) \circ (y_1, \ldots, y_{n-1}) = \big([x_1, \ldots, x_{n-1}, y_1], \ldots, [x_1, \ldots, x_{n-1}, y_{n-1}]\big) \tag{2.1}$$

is a rack pointed at $(1, 1, \ldots, 1)$.

Proposition 2.6. *Let m, n be nonnegative integers with $m = 2n - 1$. Then any pointed n-rack $(R, [\ldots], 1)$ has a pointed m-rack structure under the operation $\langle \ldots \rangle$ defined by*

$$\langle x_1, \ldots, x_m \rangle = \big[x_1, \ldots, x_{n-1}, [x_n, \ldots, x_m]\big].$$

Proof. To show (NR1), let $\{x_i\}_{i=1,\ldots,m-1}, \{y_i\}_{i=1,\ldots,m} \subseteq R$. We have by definition

$$\begin{aligned}\langle x_1, \ldots, x_{m-1}, \langle y_1, \ldots, y_m \rangle\rangle &= \langle x_1, \ldots, x_{m-1}, [y_1, \ldots, y_{n-1}, [y_n, \ldots, y_m]]\rangle \\ &= \big[x_1, \ldots, x_{n-1}, [x_n, \ldots, x_{m-1}, [y_1, \ldots, y_{n-1}, [y_n, \ldots, y_m]]]\big],\end{aligned}$$

then use consecutively (NR1) on $(R, [\ldots], 1)$ from inside out to obtain

$$\begin{aligned}&= \langle [x_1, \ldots, x_{n-1}, [x_n, \ldots, x_{m-1}, y_1]], \ldots, [x_1, \ldots, x_{n-1}, [x_n, \ldots, x_{m-1}, y_m]]\rangle \\ &= \langle \langle x_1, \ldots, x_{m-1}, y_1 \rangle \ldots, \langle x_1, \ldots, x_{m-1}, y_m \rangle\rangle.\end{aligned}$$

To show (NR2), let $\{x_i\}_{i=1,\ldots,m-1} \subseteq R$ and $y \in R$. Then by (NR2) on $(R, [\ldots], 1)$, there are unique $t, z \in R$ such that $y = [x_1, \ldots, x_{n-1}, t]$ and $t = [x_n, \ldots, x_{m-1}, z]$, i.e.,

$$y = \big[x_1, \ldots, x_{n-1}, [x_n, \ldots, x_{m-1}, z]\big] = \langle x_1, \ldots, x_{m-1}, z \rangle.$$

To show (NR3), we have by (NR3) on $(R, [\ldots], 1)$,

$$\langle 1, \ldots, 1, y \rangle = \big[1, \ldots, 1, [1, \ldots, 1, y]\big] = [1, \ldots, 1, y] = y \quad \text{for all } y \in R,$$

and for all $\{x_i\}_{i=1,\ldots,x_{m-1}} \subseteq R$,

$$\langle x_1, \ldots, x_{m-1}, 1 \rangle = \big[x_1, \ldots, x_{n-1}, [x_n, \ldots, x_{m-1}, 1]\big] = [x_1, \ldots, x_{n-1}, 1] = 1,$$

which completes the proof. □

3. n-subracks

Let $(R, [\ldots])$ be a n-rack (resp. pointed n-rack). A nonempty subset $S \subseteq R$ is called a *n-semisubrack* of R if S is closed under the n-rack operation. $(S, [\ldots])$ is called a *n-subrack* of R if it has a n-rack structure (resp. pointed n-rack structure).

In particular, $\{1\}$ and R are n-subracks of R.

Example 3.1. Let S be a $\mathbb{Z}_4$-submodule of M (the n-rack of Example 2.4) annihilated by 2. Then S has a trivial n-rack structure when endowed with the operation $[\ldots]$ of M. Therefore S is a n-subrack of M when n is odd.

The following theorem provides a characterization of n-subracks in a pointed n-rack.

Theorem 3.2. *A n-semisubrack S of a pointed n-rack $(R, [\ldots], 1)$ is a n-subrack if and only if for all $b \in R$, $[a_1, a_2, \ldots, a_{n-1}, b] \in S$ and $\{a_i\}_{i=1,\ldots,n-1} \subseteq S$ implies $b \in S$.*

Proof. Assume that S is a n-subrack and let $\{a_i\}_{i=1,\ldots,n-1} \subseteq S$ and $b \in R$ with $[a_1, \ldots, a_{n-1}, b] \in S$. Then by (NR2), there is a unique $u \in S$ with $[a_1, \ldots, a_{n-1}, b] = [a_1, a_2, \ldots, a_{n-1}, u]$. Thus $b = u \in S$ by uniqueness. For the converse, it is enough to establish (NR2) for the n-semisubrack S. Let $a_1, a_2, \ldots, a_{n-1}, x \in S \subseteq R$. Then there is a unique $b \in R$ with $x = [a_1, a_2, \ldots, a_{n-1}, b]$, and thus $b \in S$ by hypothesis. $\square$

Proposition 3.3. *Let R, R' be pointed n-racks and $\phi : R \longrightarrow R'$ be a homomorphism. Let $K = \{x \in R : \phi(x) = 1_{R'}\}$ be the kernel of ϕ. Then K and $I = \phi(R)$ are n-subracks of R and R' respectively.*

Proof. $\phi(1_R) = 1_{R'}$. So $1_R \in K$ and $1_{R'} \in I$. Let $\{a_i\}_{i=1,\ldots,n} \subseteq K$. Then $[a_1, \ldots, a_n]_R \in K$ since $\phi([a_1, \ldots, a_n]_R) = [\phi(a_1), \ldots, \phi(a_n)]_{R'} = [1_{R'}, \ldots, 1_{R'}]_{R'} = 1_{R'}$. Now let $b \in R$ and $\{a_i\}_{i=1,\ldots,n-1} \subseteq K$ with $[a_1, \ldots, a_{n-1}, b]_R \in K$. Then

$$\begin{aligned}\phi(b) &= [1_{R'}, \ldots, 1_{R'}, \phi(b)]_{R'} = [\phi(a_1), \ldots, \phi(a_{n-1}), \phi(b)]_{R'} \\ &= \phi([a_1, \ldots, a_{n-1}, b]_R) = 1_{R'}.\end{aligned}$$

Thus $b \in K$. Hence K is a n-subrack of R by Theorem 3.2. To show that I is an n-subrack, notice that $[\phi(x_1), \ldots, \phi(x_n))]_{R'} = \phi([x_1, \ldots, x_n]_R)$ for all $\{x_i\}_{i=1,\ldots,n} \subseteq R$. Now let $y \in R'$ such that $[\phi(x_1), \ldots, \phi(x_{n-1}), y]_{R'} = \phi(d)$ for some $d \in R$. We have by (NR2) on R that $[x_1, \ldots, x_{n-1}, c]_R = d$ for some unique $c \in R$. So $[\phi(x_1), \ldots, \phi(x_{n-1}), \phi(c)]_{R'} = \phi(d)$, and thus $y = \phi(c)$ by uniqueness. Hence I is a n-subrack of R' by Theorem 3.2. $\square$

Proposition 3.4. *Every pointed n-rack has a trivial n-subrack.*

Proof. Let R be a pointed n-rack and consider the subset

$$Z(R) = \big\{a \in R \,|\, [x_1, \ldots, x_{n-1}, a] = a, \ \forall \{x_i\}_{i=1,\ldots,n-1} \subseteq R\big\}.$$

Clearly, $1 \in Z(R)$ by (NR3). Let $\{x_i\}_{i=1,\ldots,n-1} \subseteq R$ and $\{a_i\}_{i=1,\ldots,n} \subseteq Z(R)$. Then by (NR1),

$$[x_1, \ldots, x_{n-1}, [a_1, \ldots, a_n]] = [[x_1, \ldots, x_{n-1}, a_1], \ldots, [x_1, \ldots, x_{n-1}, a_n]] = [a_1, \ldots, a_n].$$

Now, for $y \in R$ such that $[a_1, \dots, a_{n-1}, y] \in Z(R)$, we have

$$\begin{aligned}
[a_1, \dots, a_{n-1}, y] &= \big[x_1, \dots, x_{n-1}, [a_1, \dots, a_{n-1}, y]\big] \\
&= \big[[x_1, \dots, x_{n-1}, a_1], \dots, [x_1, \dots, x_{n-1}, a_{n-1}], [x_1, \dots, x_{n-1}, y]\big] \\
&= \big[a_1, \dots, a_{n-1}, [x_1, \dots, x_{n-1}, y]\big].
\end{aligned}$$

By uniqueness, $[x_1, \dots, x_{n-1}, y] = y$ and thus $y \in Z(R)$. The result follows by Theorem 3.2. □

Definition 3.5. The n-subrack $Z(R)$ is called the *center* of R.

Proposition 3.6. *For every pointed n-rack, there is an involutive subrack of R^{n-1}.*

Proof. Recall by Proposition 2.5 that R^{n-1} has a pointed rack structure and denote the operation $\circ$ by $[-,-]$. Now consider the subset

$$\mathfrak{I}_R = \big\{(a_1, a_2, \dots, a_{n-1}) \in R^{n-1} \,|\, [a_1, \dots, a_{n-1}, [a_1, \dots, a_{n-1}, y]] = y, \quad \forall y \in R\big\}.$$

Clearly, $(1, \dots, 1) \in \mathfrak{I}_R$ by (NR3).

Now let $a = (a_1, \dots, a_{n-1}), b = (b_1, \dots, b_{n-1}) \in \mathfrak{I}_R$ and $x = (x_1, \dots, x_{n-1}) \in R^{n-1}$. Then

$$\begin{aligned}
\big[[a,b], [[a,b], x]\big] &= \big[[a,b], [[a,b], [a, [a, x]]]\big] \\
&= \big[[a,b], [a, [b, [a, x]]]\big] \\
&= \big[a, [b, [b, [a, x]]]\big] \\
&= \big[a, [a, x]\big] = x.
\end{aligned}$$

So $\mathfrak{I}_R$ is closed under the rack operation. Moreover, this implies that for $a = (a_1, \dots, a_{n-1}) \in \mathfrak{I}_R$ and $y = (y_1, \dots, y_{n-1}) \in R^{n-1}$, we have

$$\begin{aligned}
\big[a, [a, y]\big] &= \big[(a_1, \dots, a_{n-1}), [(a_1, \dots, a_{n-1}), (y_1, \dots, y_{n-1})]\big] \\
&= \big[(a_1, \dots, a_{n-1}), \big([a_1, \dots, a_{n-1}, y_1], \dots, [a_1, \dots, a_{n-1}, y_{n-1}]\big)\big] \\
&= \big(\big[a_1, \dots, a_{n-1}, [a_1, \dots, a_{n-1}, y_1]\big], \dots, \big[a_1, \dots, a_{n-1}, [a_1, \dots, a_{n-1}, y_{n-1}]\big]\big) \\
&= (y_1, y_2, \dots, y_{n-1}) = y.
\end{aligned}$$

The result follows by Theorem 3.2. □

Proposition 3.7. *Let S be a n-semisubrack of a pointed n-rack R. Let*

$$\mathfrak{N}(S) = \big\{a \in R \,|\, [u_1, \dots, u_{n-1}, a] \in S, \ \forall \ \{u_i\}_{i=1,\dots,n-1} \subseteq S\big\}.$$

Then

(1) $1 \in S$ *iff* $1 \in \mathfrak{N}(S)$.

(2) $\mathfrak{N}(S) \subseteq J$ *for any n-subrack J of R containing S as a n-semisubrack.*

(3) $S \subseteq \mathfrak{N}(S)$. *The equality holds (thus* $\mathfrak{N}(R)$ *is a* n*-subrack of* R*) if* S *is a* n*-subrack of* R*.*

Proof. (1). By (NR3), $1 = [u_1, \ldots, u_{n-1}, 1]$ for all $\{u_i\}_{i=1,\ldots,n-1} \subseteq S$. Thus $1 \in S$ iff $1 \in \mathfrak{N}(S)$.

(2). Let J be a n-subrack of R containing S as a n-semisubrack, and let $a \in \mathfrak{N}(S)$. Then $[u_1, \ldots, u_{n-1}, a] \in S \subseteq J$, for all $\{u_i\}_{i=1,\ldots,n-1} \subseteq S \subseteq J$. This implies that $a \in J$ as J is a n-subrack. Hence $\mathfrak{N}(S) \subseteq J$.

(3). It is clear that $S \subseteq \mathfrak{N}(S)$ as S is closed under the n-rack operation. Now let $\{a_i\}_{i=1,\ldots,n} \subseteq \mathfrak{N}(S)$. Then by (NR1) on S,

$$\big[u_1, \ldots, u_{n-1}, [a_1, \ldots, a_n]\big] = \big[[u_1, \ldots, u_{n-1}, a_1], \ldots, [u_1, \ldots, u_{n-1}, a_n]\big] \in S$$

for all $\{u_i\}_{i=1,\ldots,n-1} \subseteq S$. So $[a_1, \ldots, a_n] \in \mathfrak{N}(S)$ and thus $\mathfrak{N}(S)$ is closed under the n-rack operation. In addition, for $y \in R$ such that $[a_1, \ldots, a_{n-1}, y] \in \mathfrak{N}(S)$, we have $[u_1, \ldots, u_{n-1}, [a_1, \ldots, a_{n-1}, y]] \in S$, i.e.,

$$\big[[u_1, \ldots, u_{n-1}, a_1], \ldots, [u_1, \ldots, u_{n-1}, a_{n-1}], [u_1, \ldots, u_{n-1}, y]\big] \in S.$$

So $[u_1, \ldots, u_{n-1}, y] \in S$ if S is a n-subrack, and thus $y \in \mathfrak{N}(S)$. Hence $\mathfrak{N}(S)$ is a n-subrack of R. □

$\mathfrak{N}(S)$ is called *normalizer* of S. The right normalizer of the n-semisubrack S is dually defined by

$$\mathfrak{N}_r(S) = \{a \in R \,|\, [a, u_1, \ldots, u_{n-1}] \subseteq S, \text{ for all } \{u_i\}_{i=1\ldots,n-1} \subseteq S\}$$

and does not appear to be of interest for left n-racks. However $\mathfrak{N}_r(S)$ satisfies the same properties above for right n-racks.

4. Decomposition of n-racks

In this section we assume that the n-rack R is not pointed.

Let ${}_nAut(R)$ be the set of all automorphisms of the n-rack R, i.e., bijective maps $\xi : R \longrightarrow R$ such that $\xi([x_1, \ldots, x_n]) = [\xi(x_1), \ldots, \xi(x_n)]$.

It is not difficult to see that for all $x_1, \ldots, x_{n-1} \in R$ the map

$$\phi(x_1, ..., x_{n-1})(y) = [x_1, ..., x_{n-1}, y]$$

is an automorphism of R. So, we can consider the map

$$\phi : R^{n-1} \longrightarrow {}_nAut(R) \quad \text{such that} \quad \phi : (x_1, \ldots, x_{n-1}) \mapsto \phi(x_1, \ldots, x_{n-1}).$$

If ϕ is injective, then R is called *faithful*.

Definition 4.1. A n-rack R is *decomposable* if there are two disjoint n-subracks of R such that $R = X_1 \cup X_2$.

Proposition 4.2. *If R is a decomposable n-rack, then the following statements are true:*

(1) $[X_1, \ldots, X_1, X_2] \subseteq X_2$, (4.1)

(2) $(X_1)^{n-1}$ *and* $(X_2)^{n-1}$ *are subracks of the rack* R^{n-1} *satisfying* $[(X_1)^{n-1}, (X_2)^{n-1}]_{R^{n-1}} \subseteq (X_2)^{n-1}$ *and* $[(X_2)^{n-1}, (X_1)^{n-1}]_{R^{n-1}} \subseteq (X_1)^{n-1}$,

(3) $\phi((X_1)^{n-1}) \in {}_nAut(X_2)$ *and* $\phi((X_2)^{n-1}) \in {}_nAut(X_1)$.

Proof. (1). Let $\{x_i\}_{i=1,\ldots,n-1} \subseteq X_1$ and $y \in X_2$ with $[x_1 \ldots, x_{n-1}, y] \notin X_2$, i.e., $[x_1, \ldots, x_{n-1}, y] \in X_1$. Then by Theorem 3.3, $y \in X_1$ as X_1 is a n-subrack, and thus $y \in X_1 \cap X_2$. A contradiction.

(2). Recall that the rack operation on R^{n-1} is given by the equality (2.1). So $(X_1)^{n-1}$ is closed under this operation and satisfies (R2) as X_1 is a n-subrack of R. Moreover, it is clear by (4.1) that each coordinate of the right hand side of the equality above is in X_2 for $\{x_i\}_{i=1,\ldots,n-1} \subseteq X_1$ and $\{y_i\}_{i=1,\ldots,n-1} \subseteq X_2$. Thus $[(X_1)^{n-1}, (X_2)^{n-1}]_{R^{n-1}} \subseteq (X_2)^{n-1}$. The other inclusion is obtained similarly.

(3). Let $\{x_i\}_{i=1,\ldots,n-1} \subseteq X_1$. The restriction of the map $\phi(x_1, \ldots, x_{n-1})$ to X_2 together with (4.1) completes the proof. The proof that $\phi((X_2)^{n-1}) \in {}_nAut(X_1)$ is similar. □

Proposition 4.3. *If R is a decomposable rack, then R is decomposable as a n-rack for all integer $n > 2$.*

Proof. Let $n > 2$ (integer), and $R = X_1 \cup X_2$ be a decomposition of the rack $(R, \circ)$. It is enough to show that X_1 and X_2 are n-subracks. Indeed, for $\{x_i\}_{i=1,\ldots,n}$ from X_1, we have, by Proposition 2.5, $[x_1, x_2, \ldots, x_n] = x_1(x_2(\ldots(x_{n-1} \circ x_n)\ldots)) \in X_1$ as X_1 is closed under $\circ$. Also for $y \in X_1$, there is by $(R2)$ a unique $t_1 \in X_1$ with $y = x_1 \circ t_1$. Repeating the process, there exists uniquely $t_2, t_3, \ldots, t_{n-1}, z \in X_1$ with $t_i = x_{i+1} \circ t_{i+1}$ and $t_{n-2} = x_{n-1} \circ z$ such that

$$y = x_1 \circ t_1 = x_1 \circ (x_2 \circ t_2) = \ldots = x_1 \circ (x_2(\ldots(x_{n-1} \circ z)\ldots)) = [x_1, x_2, \ldots x_{n-1}, z].$$

Hence X_1 is a n-subrack. The proof that X_2 is a n-subrack is similar. □

Proposition 4.4. *If R is a decomposable n-rack, then R is decomposable as a $(2n-1)$-rack.*

Proof. The proof is similar to the proof of Proposition 4.3 and follows by Proposition 2.6. □

5. A homology theory on n-racks

Recall that for a rack $(X, \circ)$, one defines (see [4] for the right rack version) the rack homology $H_*^R(X)$ of X as the homology of the chain complex $\{C_k^R(X), \partial_k\}$ where

$C_k^R(X)$ is the free abelian group generated by k-uples $(x_1, x_2, \dots, x_k)$ of elements of X and the boundary maps $\partial_k : C_k^R(X) \longrightarrow C_{k-1}^R(X)$ are defined by

$\partial_k(x_1, x_2, \dots, x_k) =$

$\sum_{i=2}^{k}(-1)^i[(x_1, \dots, x_{i-1}, \widehat{x_i}, x_{i+1}, \dots, x_k) - (x_i \circ x_1, \dots, x_i \circ x_{i-1}, \widehat{x_i}, x_{i+1}, \dots, x_k)]$

for $k \geqslant 2$ and $\partial_k = 0$ for $k \leqslant 1$, where $\widehat{x_i}$ means that x_i is deleted. If X is a quandle, the subgroups $C_k^D(X)$ of $C_k^R(X)$ generated by k-tuples $(x_1, x_2, \dots, x_k)$ with $x_i = x_{i+1}$ for some i, $1 \leqslant i < k$ form a subcomplex $C_*^D(X)$ of $C_*^R(X)$ whose homology $H_*^D(X)$ is called the *degeneration homology* of X. The homology $H_*^Q(X)$ of the quotient complex $\{C_k^Q(X) = C_k^R(X)/C_k^D(X), \partial_k\}$ is called the *quandle homology* of X.

Lemma 5.1. *Let $\mathcal{X}$ be a n-rack. Then $\mathcal{X}^{n-1}$ has a rack structure. $\mathcal{X}^{n-1}$ is a quandle if $\mathcal{X}$ is a n-quandle.*

Proof. Endow $\mathcal{X}^{n-1}$ with the binary operation

$$(x_1, \dots, x_{n-1}) \circ (y_1, \dots y_{n-1}) = ([x_1, \dots, x_{n-1}, y_1], \dots, [x_1, \dots, x_{n-1}, y_{n-1}]). \quad \square$$

We define the chain complexes ${}_nC_*^R(\mathcal{X}) := C_*^R(\mathcal{X}^{n-1})$ if $\mathcal{X}$ is an n-rack, ${}_nC_*^D(\mathcal{X}) := C_*^D(\mathcal{X}^{n-1})$ and ${}_nC_*^Q(\mathcal{X}) := C_*^Q(\mathcal{X}^{n-1})$ if $\mathcal{X}$ is a n-quandle.

Definition 5.2. Let $\mathcal{X}$ be an n-rack. The *kth n-rack homology group* of $\mathcal{X}$ with trivial coefficients is defined by

$${}_nH_k^R(\mathcal{X}) = H_k({}_nC_*^R(\mathcal{X})).$$

Definition 5.3. Let $\mathcal{X}$ be a n-quandle.

1. The *kth n-degeneration homology group* of $\mathcal{X}$ with trivial coefficients is defined by

$${}_nH_k^D(\mathcal{X}) = H_k({}_nC_*^D(\mathcal{X})).$$

2. The *kth n-quandle homology group* of $\mathcal{X}$ with trivial coefficients is defined by

$${}_nH_k^Q(\mathcal{X}) = H_k({}_nC_*^Q(\mathcal{X})).$$

Definition 5.4. Let A be a abelian group, we define the chain complexes

$${}_nC_*^W(\mathcal{X}; A) = {}_nC_*^W(\mathcal{X}) \otimes A, \quad \partial = \partial \otimes id \ \text{ with } \ W = D, R, Q.$$

1. The *kth n-rack homology group* of $\mathcal{X}$ with coefficients in A is defined by

$${}_nH_k^R(\mathcal{X}; A) = H_k({}_nC_*^R(\mathcal{X}; A)).$$

2. The *kth n-degenerate homology group* of $\mathcal{X}$ with coefficients in A is defined by

$${}_nH_k^D(\mathcal{X}; A) = H_k({}_nC_*^D(\mathcal{X}; A)).$$

3. The *kth n-quandle homology group* of $\mathcal{X}$ with coefficients in A is defined by

$${}_nH_k^Q(\mathcal{X}; A) = H_k({}_nC_*^Q(\mathcal{X}; A)).$$

One defines the cohomology theory of n-racks and n-quandles by duality. Note that for $n = 2$, one recovers the homology and cohomology theories defined by Carter, Jelsovsky, Kamada, Landford and Saito [4].

Proposition 5.5. *Let $\mathcal{X}$ be a n-quandle and $S \subset \mathcal{X}$ a n-subquandle. The following diagram of long exact sequences commutes:*

$$\begin{array}{ccccccc}
{}_nH_k^D(S) & \longrightarrow & {}_nH_k^R(S) & \longrightarrow & {}_nH_k^Q(S) & \longrightarrow & {}_nH_{k+1}^D(S) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
{}_nH_k^D(\mathcal{X}) & \longrightarrow & {}_nH_k^R(\mathcal{X}) & \longrightarrow & {}_nH_k^Q(\mathcal{X}) & \longrightarrow & {}_nH_{k+1}^D(\mathcal{X}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
{}_nH_k^D(\mathcal{X}_S) & \longrightarrow & {}_nH_k^R(\mathcal{X}_S) & \longrightarrow & {}_nH_k^Q(\mathcal{X}_S) & \longrightarrow & {}_nH_{k+1}^D(\mathcal{X}_S)
\end{array}$$

where ${}_nH_k^W(\mathcal{X}_S)$ stands for the homology of the complex

$$\{{}_nC_k^W(X_S) =_n C_k^W(\mathcal{X})/{}_nC_k^W(S), \partial_k\}, \quad W = R, D, Q.$$

Proof. The diagram above is induced by the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & {}_nC_*^D(S) & \longrightarrow & {}_nC_*^R(S) & \longrightarrow & {}_nC_*^Q(S) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & {}_nC_*^D(\mathcal{X}) & \longrightarrow & {}_nC_*^R(\mathcal{X}) & \longrightarrow & {}_nC_*^Q(\mathcal{X}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & {}_nC_*^D(\mathcal{X}_S) & \longrightarrow & {}_nC_*^R(\mathcal{X}_S) & \longrightarrow & {}_nC_*^Q(\mathcal{X}_S) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

□

Remark. Since $\mathcal{X}^{n-1}$ carries most of the properties of $\mathcal{X}$, several results established on racks are valid on n-racks. For instance; if $\mathcal{X}$ is finite, then $\mathcal{X}^{n-1}$ is also finite. Cohomology of finite racks were studied by Etingof and Graña in [7].

Proposition 5.6. *Let $\mathcal{X}$ be a trivial n-rack. Then we have the following isomorphisms:*

$${}_nH_*^R(\mathcal{X}) \cong \left(\mathbb{Z}\mathcal{X}^{n-1}\right)^*$$

Proof. It is easy to check with Lemma 2.1 that $\mathcal{X}^{n-1}$ is a trivial rack. That all chains are cycles follows by definition. □

References

[1] **N. Andruskiewitsch and M. Graña**, *From racks to pointed Hopf algebras*, Adv. Math., **178** (2003), 177 – 243.

[2] **G. R. Biyogmam**, *Lie n-rack*, C. R. Acad. Sci. Paris, Ser. I, **349** (2011), 957 – 960.

[3] **M. E. Bozhüyük**, *Topics in knot theory*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993.

[4] **J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford and M. Saito**, *Quandle cohomology and state-sum invariants of knotted curves and surfaces*, Trans. Amer. Math. Soc. **355** (2003), 3947 – 3989.

[5] **J. M. Casas, J. L. Loday and T. Pirashvili**, *Leibniz n-algebras*, Forum Math. **14** (2002), 189 – 207.

[6] **J. Conway and G. Wraith**, Unpublished correspondence (1958).

[7] **P. Etingof and M. Graña**, *On rack cohomology*, J. Pure Appl. Algebra **177** (2003), 49 – 59.

[8] **R. Fenn, C. Rourke and B. Sanderson**, *James bundles and applications*, Proc. London Math. Soc. **89** (2004), 217 – 240.

[9] **D. Larue**, *Left-distributive idempotent algebras*, Commun. Alg., **27** (1999), 2003 – 2009.

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Weak hyper residuated lattices

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Abstract. We introduced the notion of weak hyper residuated lattices which is a generalization of residuated lattices and prove some related results. Moreover, we introduce deductive systems, (positive) implicative and fantastic deductive systems and show the relations among them.

1. Introduction

The concept of hyperstructures was introduced by Marty [10] at 8th Congress of Scandinavian Mathematicians in 1934. Till now, the hyperstructures are studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics [1], [5]. Residuated lattices, introduced by Ward and Dilworth [11], are a common structure among algebras associated with logical systems. The main examples of residuated lattices are MV-algebras introduced by Chang [2] and BL-algebras introduced by Hájek [7]. Imai and Iséki introduced in [9] the notion of BCK-algebras. Borzooei et al. [2] introduced the concept of hyper K-algebras, which are a generalization of BCK-algebras. Also, they studied hyper K-ideals in hyper K-algebras. Recently, S. Ghorbani et al. [6], applied the hyperstructures to MV-algebras.

In this paper we want to construct a weak hyper residuated lattice as a generalization of the concept of residuated lattices that contain of the classes of MV-algebras, BL-algebras, and Heyting algebras.

A hyperoperation on a nonempty set A is a mapping $\circ : A \times A \to P^{\star}(A)$, where $P^{\star}(A)$ is the set of all the nonempty subsets of A and A with a hyperoperation is called a *hypergroupoid*.

Definition 1.1. A hypergroupoid $(A, *, 1)$ is called a *commutative semihypergroup* with 1 as the identity, if for all $x, y, z \in A$ we have:

(i) $x * (y * z) = (x * y) * z$,
(ii) $x * y = y * x$,
(iii) $x \in 1 * x$.

An element $a \in A$ is called a *scalar element* if for all $x \in A$ the set $a \odot x$ has only one element.

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Definition 1.2. By a *residuated lattice* we mean a structure $(L, \vee, \wedge, \odot, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that

$(RL1)$ $(L, \vee, \wedge, 0, 1)$ is a bounded lattice,
$(RL2)$ $(L, \odot, 1)$ is a commutative monoid,
$(RL3)$ the pair $(\odot, \to)$ is an adjoint pair, i.e., for any $x, y, z \in L$,

$$x * y \leqslant z \text{ if and only if } x \leqslant y \to z.$$

2. Weak hyper residuated lattices

Definition 2.1. By a *weak hyper residuated lattice* we mean a nonempty set L endowed with two binary operations $\vee$, $\wedge$ and two binary hyperoperations $\odot$, $\to$ and two constants 0 and 1 satisfying the following conditions:

$(WHRL1)$ $(L, \leqslant, \vee, \wedge, 0, 1)$ is a bounded lattice,
$(WHRL2)$ $(L, \odot, 1)$ is a commutative semihypergroup with 1 as the identity,
$(WHRL3)$ $a \odot c \ll b$ if and only if $c \ll a \to b$,

where $A \ll B$ means that $a \leqslant b$, for some $a \in A$ and $b \in B$; $A \leqslant B$ means that for any $a \in A$, there exists $b \in B$ such that $a \leqslant b$, where $\leqslant$ is the lattice ordering of L.

Example 2.2. Any residuated lattice is a weak hyper residuated lattice, too. $\square$

Example 2.3. $L = [0, 1]$ with the natural ordering is a bounded lattice. Define the hyperoperations $\odot$, $\to$ and $\rightsquigarrow$ on L as follows:

$$a \odot b = a \times b, \qquad a \to b = \begin{cases} \{1\}, & a \leqslant b, \\ \{b\}, & a > b, \end{cases} \qquad a \rightsquigarrow b = \begin{cases} \{1\}, & a \leqslant b, \\ [b, 1], & a > b. \end{cases}$$

Then $(L, \vee, \wedge, \odot, \to, 0, 1)$ and $(L, \vee, \wedge, \odot, \rightsquigarrow, 0, 1)$ are weak hyper residuated lattices. $\square$

Example 2.4. Consider the chain $0 < a < b < 1$. Then $(L, \leqslant, 0, 1)$, where $L = \{0, a, b, 1\}$, is a bounded lattice. Putting $x \odot y = x \wedge y$ and defining the hyperoperations $\to$ and $\rightsquigarrow$ by the following two tables:

$\to$	0	a	b	1
0	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
a	$\{a, b, 1\}$	$\{1, a\}$	$\{1\}$	$\{1\}$
b	$\{a, 1\}$	$\{a\}$	$\{b, 1\}$	$\{1\}$
1	$\{0, 1\}$	$\{a\}$	$\{1, b\}$	$\{1\}$

$\rightsquigarrow$	0	a	b	1
0	$\{1\}$	$\{1, b\}$	$\{1, b\}$	$\{1, b\}$
a	$\{a, b, 1\}$	$\{1\}$	$\{1\}$	$\{1\}$
b	$\{a, b, 1\}$	$\{a\}$	$\{1, b\}$	$\{1, b\}$
1	$\{0, a, 1\}$	$\{1, a\}$	$\{1\}$	$\{1\}$

we obtain two hyper residuated lattices $(L, \leqslant, \odot, \to, 0, 1)$ and $(L, \leqslant, \odot, \rightsquigarrow, 0, 1)$. $\square$

Proposition 2.5. *Let $\mathcal{L} = (L, \vee, \wedge, \odot, \to, 0, 1)$ be a weak hyper residuated lattice. Then for nonempty subsets A, B, C of L and all $x, y, z \in L$ we have:*

(i) $1 \ll A \Leftrightarrow 1 \in A$ *and* $A \ll 0 \Leftrightarrow 0 \in A$, (*ii*) $x \leqslant y \Rightarrow 1 \in x \to y$ *and* $A \ll B \Rightarrow 1 \in A \to B$,

(*iii*) *if* 1 *is a scalar, then* $1 \in x \to y \Rightarrow x \leqslant y$ *and* $1 \in A \to B \Rightarrow A \ll B$,

(*iv*) $1 \in (x \to x) \cap (x \to 1) \cap (0 \to x)$,

(*v*) *if* 1 *is a scalar element of* L, *then* $x \in 1 \to x$,

(*vi*) $A \ll B \to C \Leftrightarrow A \odot B \ll C \Leftrightarrow B \ll A \to C$,

(*vii*) $x \odot y \ll x, y$ *and* $A \odot B \ll A, B$,

(*viii*) $x \ll y \to x$, $A \ll B \to A$ *and* $1 \in x \to (y \to x)$,

(*ix*) $x \to (y \to z) \leqslant (x \odot y) \to z \leqslant x \to (y \to z) \leqslant y \to (x \to z)$,

(*x*) $x \odot (x \to y) \ll x, y$,

(*xi*) $x \ll y \to (x \odot y)$ *and* $x \ll (y \to x) \to x$,

(*xii*) $x \leqslant y \Rightarrow x \odot z \ll y \odot z$, $z \to x \leqslant z \to y$ *and* $y \to z \leqslant x \to z$,

(*xiii*) $x \to y \leqslant (y \to z) \to (x \to z)$,

(*xiv*) $(x \to y) \odot (y \to z) \ll x \to z$ *and* $y \to z \ll (x \to y) \to (x \to z)$,

(*xv*) $0 \in x \odot y \Leftrightarrow x \ll \neg y$, *where* $\neg x = x \to 0$,

(*xvi*) $0 \in 0 \odot x$, $0 \in x \odot \neg x$, $1 \in \neg 0$ *and if* 1 *is a scalar, then* $0 \in \neg 1$,

(*xvii*) *if* $x \leqslant y$, *then* $\neg y \leqslant \neg x$,

(*xviii*) $x \to y \ll \neg y \to \neg x$,

(*xix*) $x \ll \neg\neg x$, $\neg\neg\neg x \ll \neg x \ll \neg\neg\neg x$ *and* $\neg x \ll x \to y$,

(*xx*) $\neg a$, $\neg b \ll \neg(a \wedge b)$ *and* $\neg(a \vee b) \ll \neg a$, $\neg b$,

(*xxi*) $x \to \neg y \ll \neg(x \odot y) \ll x \to \neg y$ *and* $y \to \neg x \ll \neg(x \odot y) \ll y \to \neg x$,

(*xxii*) *if* $\bigvee Y$ *exists, then* $\bigvee_{y \in Y}(x \odot y) \ll x \odot (\bigvee Y)$.

Proof. (*i*) Let $1 \ll A$. Then there exists $a \in A$ such that $1 \leqslant a$. Since, for any $x \in L$, $x \leqslant 1$, then $1 = a \in A$. The converse is obvious. Now, let $A \ll 0$. Then there exists $b \in A$ such that $b \leqslant 0$. Since, for any $x \in L$, $0 \leqslant x$, then $0 = b \in A$. The converse is clear.

(*ii*) Let $x \leqslant y$. Since $x \in x \odot 1$, then $x \odot 1 \ll y$. By $(WHRL3)$, $1 \ll x \to y$ and so by (*i*), $1 \in x \to y$. Now, let $A \ll B$. Then there exist $a \in A$ and $b \in B$ such that $a \leqslant b$. So, by the above, $1 \in a \to b \subseteq A \to B$.

(*iii*) Let $1 \in x \to y$. Then $1 \leqslant x \to y$ and so $1 \odot x \ll y$. Now, since 1 is a scalar of L and $x \in 1 \odot x$, then $1 \odot x = x$ and so $x \leqslant y$. Similarly, $1 \in A \to B$ implies $A \ll B$.

(iv) Since, by the lattice ordering, $x \leqslant x$, $x \leqslant 1$ and $0 \leqslant x$, then $1 \in x \to x$, $1 \in x \to 1$ and $1 \in 0 \to x$. So we have (iv).

(v) Let 1 be a scalar of L. Then $x \odot 1 = x \leqslant x$ and by $(WHRL3)$, we get $x \ll 1 \to x$, i.e., there exists $a \in 1 \to x$ such that $x \leqslant a$. Since $a \in 1 \to x$, then $a = 1 \odot a \leqslant x \leqslant a$ and so $x = a \in 1 \to x$. Hence, for all $a \in A$, $a \in 1 \to a \subseteq 1 \to A$.

(vi) Let $A, B, C \subseteq L$. Then

$$\begin{aligned} A \ll B \to C &\Leftrightarrow \exists a \in A, b \in B, c \in C \text{ such that } a \ll b \to c, \\ A \odot B \ll C &\Leftrightarrow \exists a \in A, b \in B, c \in C \text{ such that } a \odot b \ll c, \\ B \ll A \to C &\Leftrightarrow \exists a \in A, b \in B, c \in C \text{ such that } b \ll a \to c \end{aligned}$$

and so, by $(WHRL3)$, we have (vi).

(vii) Since, for all $x, y \in L$, $y \leqslant 1 \in x \to x$ and $x \leqslant 1 \in y \to y$, then by $(WHRL3)$ $x \odot y = y \odot x \ll x, y$. By the similar way, we can prove that $A \odot B \ll A, B$.

$(viii)$ By (vii), $x \odot y \ll x$ and so, by $(WHRL3)$, $x \ll y \to x$. Hence, by (ii), $1 \in x \to (y \to x)$.

(ix) Let $u \in x \to (y \to z)$. Then

$$\begin{aligned} u \ll x \to (y \to z) &\Leftrightarrow (u \odot x) \ll y \to z, && \text{by } (vi) \\ &\Leftrightarrow (u \odot x) \odot y \ll z, && \text{by } (vi) \\ &\Leftrightarrow u \odot (x \odot y) \ll z \\ &\Leftrightarrow u \ll (x \odot y) \to z, && \text{by } (vi) \end{aligned}$$

and so, $x \to (y \to z) \leqslant (x \odot y) \to z$. By a similar way, we can prove that $(x \odot y) \to z \leqslant x \to (y \to z)$.

(x) It follows from (vi).

(xi) It follows from (vi) and $x \odot y \ll x \odot y$. Also, by (vi), $x \odot A \ll x$, where $A = y \to x$.

(xii) By the first part of (xi), $y \ll z \to (y \odot z)$. Now, since $x \leqslant y$, then $x \ll z \to (y \odot z)$. Hence, by (vi), we get $x \odot z \ll y \odot z$.

Now, let $u \in z \to x$. Since $u \ll z \to x$, then by $(WHRL3)$, $u \odot z \ll x$ and so by $x \leqslant y$, we get $u \odot z \ll y$. Hence, by $(WHRL3)$, $u \ll z \to y$ and so $z \to x \leqslant z \to y$.

Now, let $t \in y \to z$. Since, $t \ll y \to z$, then by (vi), $y \ll t \to z$ and so by $x \leqslant y$, we get that $x \ll t \to z$. Hence, by (vi), we get $t \ll x \to z$ and so $y \to z \leqslant x \to z$.

$(xiii)$ Let $u \in y \to z$. Then by (vi), $u \ll y \to z$ implies $y \ll u \to z$. So there exists $t \in u \to z$ such that $y \leqslant t$. Now, by (xii) and (ix), we have

$$x \to y \leqslant x \to t \subseteq x \to (u \to z) \leqslant u \to (x \to z) \subseteq (y \to z) \to (x \to z).$$

Hence, $x \to y \leqslant (y \to z) \to (x \to z)$.

(xiv) Those follow from (vi) and $(xiii)$.

(xv) $x \ll \neg y = y \to 0$, if and only if $x \odot y \ll 0$ if and only if $0 \in x \odot y$.

(xvi) We know that $x \leqslant 1 \in 0 \to 0$. Thus $x \odot 0 \ll 0$ and so by (i), we get $0 \in x \odot 0$. Also, it is clear that $x \to 0 \ll x \to 0$. Now, by (vi), we get $x \odot (x \to 0) \ll 0$ and so, by (i), $0 \in x \odot \neg x$.

$(xvii)$ It follows from (xii).

$(xviii)$ Since, by (xiv), $(x \to y) \odot (y \to 0) \ll x \to 0$, then by (vi), we get $x \to y \ll \neg y \to \neg x$.

(xix) By (xv), $x \odot (x \to 0) \ll 0$ and so by (vi), we get $x \ll (x \to 0) \to 0 = \neg\neg x$. Also, by (xii), we get $\neg\neg\neg x \ll \neg x$. On the other hand, if we put $A = x \to 0$ then by (xv), $A \odot (A \to 0) \ll 0$. Now, we conclude $A \ll (A \to 0) \to 0$ by (vi), i.e., $\neg x \ll \neg\neg\neg x$. (Note that, we do not have anti-symmetry for $\ll$.)

(xx) Those follow from (xii).

(xxi) It is conclude by (ix) and $(xiii)$.

$(xxii)$ If $\bigvee Y$ exists, then $y \leqslant \bigvee Y$ for all $y \in Y$. So, by (xii), $x \odot y \ll x \odot (\bigvee Y)$. Thus there exists $b_y \in x \odot (\bigvee Y)$ such that $x \odot y \ll b_y$ for any $y \in Y$. Hence, we get $\bigvee_{y \in Y}(x \odot y) \ll \bigvee_{y \in Y} b_y \leqslant \bigvee x \odot (\bigvee Y)$. □

Theorem 2.6. *Any weak hyper residuated lattice of order n can be extend to a weak hyper residuated lattice of order $n+1$.*

Proof. Let L be a weak hyper residuated lattice of order n, and $\overline{L} = L \cup \{e\}$ for some $e \notin L$. Putting

$$z \leqslant' y \Leftrightarrow z \leqslant y, \text{ for all } z, y \in L \text{ and } \ x \leqslant' e, \text{ for all } x \in L',$$

$$a \odot' b = \begin{cases} a \odot b & \text{if } a, b \in L, \\ \{a\} & \text{if } a \in L \text{ and } b = e, \\ \{b\} & \text{if } b \in L \text{ and } a = e, \\ \{e\} & \text{if } a = b = e, \end{cases}$$

$$a \to' b = \begin{cases} (a \to b) \cup \{e\} & \text{if } a, b \in L, 1 \in a \to b \\ a \to b & \text{if } a, b \in L, 1 \notin a \to b, \\ \{e\} & \text{if } b = e, \\ \{b\} & \text{if } a = e, \end{cases}$$

we see that $(\overline{L}, \leqslant')$ is a bounded lattice with 0 as the minimum and e as the maximum elements of $\overline{L}$. The proof of $(WHRL1)$ and $(WHRL2)$ are clear. Now, we prove the $(WHRL3)$. Let $x, y, z \in \overline{L}$. We consider the following cases:

Case 1. For $x = y = z = e$, the proof is obvious.

Case 2. Let $x = z = e$ and $y \in L$. Then $x \odot' y = \{y\}$ and $y \to' z = \{e\}$. Therefore, $x \odot' y \ll' z$ if and only if $x \ll' y \to' z$. By the similar way, we have for $y = z = e$ and $x = y = e$.

Case 3. Let $x, y \in L$ and $z = e$. Since $y \to' z = \{e\}$ and $u \ll' e$, for all $u \in L'$, then $x \odot' y \ll' z$ implies $x \ll' y \to' z$. Now, let $x \ll' y \to' z$. Since $z = e$, then $x \odot' y \ll' z$.

Case 4. Let $x, z \in L$ and $y = e$. Then $x \odot' y = \{x\}$ and $y \rightarrow' z = \{z\}$. Therefore, $x \odot' y \ll' z$ if and only if $x \ll' y \rightarrow' z$.

Case 5. Let $y, z \in L$ and $x = e$. Then $x \odot' y = \{y\}$. If $x \odot' y = \{y\} \ll' z$, then $y \ll' z$. Since $y, z \in L$ we get $y \ll z$ and so $1 \in y \rightarrow z$. Hence $e \in y \rightarrow' z$ and so $x \ll' y \rightarrow' z$. Now, let $x \ll' y \rightarrow' z$. Then by definition of $\leqslant'$, we have $e \in y \rightarrow' z$ and so $1 \in y \rightarrow z$ or $z = e$. Since $y \in L$, then $y \neq e$ and so $1 \in y \rightarrow z$. Therefore, $x \odot' y \ll' z$.

Case 6. Let $x, y, z \in L$ and $1 \in y \rightarrow z$. If $x \odot' y \ll' z$, then by definition of $\rightarrow'$, $e \in y \rightarrow' z$ and so $x \ll' y \rightarrow' z$. Now, let $x \ll' y \rightarrow' z$. Since $1 \in y \rightarrow z$, then $x \ll y \rightarrow z$ and so $x \odot y \ll z$. Hence $x \odot' y = x \odot y \ll' z$.

Case 7. Let $x, y, z \in L$ and $1 \notin y \rightarrow z$. Then by definitions of $\odot'$ and $\leqslant'$, we get

$$x \odot' y \ll' z \Leftrightarrow x \odot y \ll z \Leftrightarrow x \ll y \rightarrow z \Leftrightarrow x \ll' y \rightarrow' z.$$

Hence, $(\overline{L}, \leqslant', \odot', \rightarrow', 0, e)$ is a weak hyper residuated lattice of order $n+1$. □

Definition 2.7. A subset D of $\mathcal{L}$ containing 1 is called a *deductive system* (shortly: $\mathcal{DS}$) if $x \in D$ and $(x \rightarrow y) \subseteq D$ imply $y \in D$, for all $x, y \in L$.

Example 2.8. (i) Clearly, L is a $\mathcal{DS}$ of $\mathcal{L}$. If 1 is an scalar element, then $\{1\}$ is a $\mathcal{DS}$ of $\mathcal{L}$, too.

(ii) Let $([0,1], \vee, \wedge, \odot, \rightsquigarrow, 0, 1)$ be a weak hyper residuated lattice as in Example 2.3. It is easy to shows that $D = [\frac{1}{2}, 1]$ is its $\mathcal{DS}$.

(iii) In Example 2.4, $\{1\}$ is a $\mathcal{DS}$ and $\{1, b\}$ is not a $\mathcal{DS}$ of $\mathcal{L}$.

Definition 2.9. A nonempty subset D of $\mathcal{L}$ is called

- an *upset* if $x \in D$ and $x \leqslant y$, then $y \in D$, for all $x, y \in L$,
- an $S_{\rightarrow}$*reflexive* if $(A \rightarrow B) \cap D \neq \emptyset$ implies $(A \rightarrow B) \subseteq D$, for all $A, B \subseteq L$.

Proposition 2.10. *Every $S_{\rightarrow}reflexive$ $\mathcal{DS}$ of $\mathcal{L}$ is an upset.*

Proof. Let D be an $S_{\rightarrow}$-reflexive $\mathcal{DS}$, $x \in D$ and $x \leqslant y$, for some $y \in L$. By Proposition 2.5(ii), $1 \in x \rightarrow y$ and so $(x \rightarrow y) \cap D \neq \emptyset$. Since D is $S_{\rightarrow}$-reflexive, then $x \rightarrow y \subseteq D$ and so by DS, we have $y \in D$. □

Proposition 2.11. *Let D be an $S_{\rightarrow}reflexive$ $\mathcal{DS}$ of $\mathcal{L}$. Then*

(i) $D \ll A \rightarrow B \Leftrightarrow (A \rightarrow B) \cap D \neq \emptyset \Leftrightarrow A \rightarrow B \subseteq D$,

(ii) $A \rightarrow B \subseteq D$ *and* $A \rightarrow B \ll A' \rightarrow B'$ *imply* $D \ll A' \rightarrow B'$,

(iii) $D \ll A \rightarrow B \ll A' \rightarrow B'$ *implies* $D \ll A' \rightarrow B'$.

Proof. (i) If $D \ll A \rightarrow B$, then there exist $a \in A$ and $b \in B$ such that $D \ll a \rightarrow b$. So there exists $d \in D$ and $t \in a \rightarrow b$ such that $d \leqslant t$. Since D is an $S_{\rightarrow}reflexive$ $\mathcal{DS}$, then by Proposition 2.10, D is an upset and so $t \in D \cap (a \rightarrow b)$. Hence $(A \rightarrow B) \cap D \neq \emptyset$. Conversely, let $(A \rightarrow B) \cap D \neq \emptyset$. Then there exist $a \in A$ and

$b \in B$ such that $(a \to b) \cap D \neq \emptyset$. So there exists $t \in (a \to b) \cap D$ and since $t \leqslant t$, then $D \ll a \to b$. Hence $D \ll A \to B$.

(ii) Let $A \to B \subseteq D$. Since $A \to B \ll A' \to B'$, then there exist $a \in A$, $b \in B$, $a' \in A'$ and $b' \in B'$ such that $a \to b \ll a' \to b'$. So there exist $t \in a \to b$ and $t' \in a' \to b'$ such that $t \leqslant t'$. Now, we have $t \in a \to b \subseteq A \to B \subseteq D$ and so $t \in D$. Since D is an upset, then $t' \in D$. Therefore, $t' \in D \cap A' \to B'$ and so by (i), we get $D \ll A' \to B'$.

(iii) If $D \ll A \to B$, then by (i), $A \to B \subseteq D$ and so by (ii), we conclude $D \ll A' \to B'$. □

Example 2.12. Let $(L, \leqslant, 0, 1)$ be as in Example 2.4. Consider the following hyperoperations:

$\odot$	0	a	b	1
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{a, 0\}$	$\{a\}$	$\{a\}$
b	$\{0\}$	$\{a\}$	$\{b\}$	$\{b\}$
1	$\{0\}$	$\{a\}$	$\{b\}$	$\{1\}$

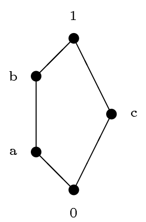
$\to$	0	a	b	1
0	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
a	$\{0, a\}$	$\{1\}$	$\{1\}$	$\{1\}$
b	$\{0\}$	$\{0, a\}$	$\{1\}$	$\{1\}$
1	$\{0\}$	$\{a\}$	$\{b\}$	$\{1\}$

Then $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a weak hyper residuated lattice and $D_1 = \{1\}$, $D_2 = \{1, b\}$ are its $S_{\to}$-reflexive deductive systems. □

3. Implicative deductive systems

Definition 3.1. A subset D of $\mathcal{L}$ containing 1 is called an *implicative* deductive system (shortly: $\mathcal{IDS}$), if $(x \to y) \subseteq D$ and $x \to (y \to z) \subseteq D$ imply $(x \to z) \subseteq D$.

Example 3.2. Let $L = \{a, b, c, 0, 1\}$ be the lattice with the following diagram.



Consider the following hyperoperations:

$\to$	0	a	b	c	1
0	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
a	$\{c\}$	$\{1\}$	$\{1\}$	$\{c\}$	$\{1\}$
b	$\{c\}$	$\{a, b, c\}$	$\{1\}$	$\{c\}$	$\{1\}$
c	$\{a, b\}$	$\{a, b\}$	$\{b, a\}$	$\{1\}$	$\{1\}$
1	$\{0\}$	$\{a\}$	$\{b, a\}$	$\{c\}$	$\{1\}$

$\odot$	0	a	b	c	1
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{a\}$	$\{a\}$	$\{0\}$	$\{a\}$
b	$\{0\}$	$\{a\}$	$\{b, a\}$	$\{0\}$	$\{a, b\}$
c	$\{0\}$	$\{0\}$	$\{0\}$	$\{c\}$	$\{c\}$
1	$\{0\}$	$\{a\}$	$\{b, a\}$	$\{c\}$	$\{1\}$

It is easy to show that $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a weak hyper residuated lattice. Moreover, easy calculations show that $\{1, a\}$ is an $\mathcal{IDS}$ of $\mathcal{L}$ and $\{1, b, c\}$ is not an $\mathcal{IDS}$. Since $(b \to 0) = \{c\} \subseteq \{1, b, c\}$ and $(b \to (0 \to a)) = \{1\} \subseteq \{1, b, c\}$ but $(b \to a) = \{a, b, c\} \nsubseteq \{1, b, c\}$. □

Theorem 3.3. *Let D be a nonempty subset of $\mathcal{L}$ containing 1. Then*

(i) *if D is an $\mathcal{IDS}$, then D is a $\mathcal{DS}$,*

(ii) *D is an $\mathcal{IDS}$ if and only if each $D_a = \{x \in L | a \to x \subseteq D\}$ is a $\mathcal{DS}$ of $\mathcal{L}$,*

(iii) *D is an $\mathcal{IDS}$ if and only if $(x \to (y \to z)) \cap D \neq \emptyset$ and $(x \to y) \cap D \neq \emptyset$ imply $(x \to z) \cap D \neq \emptyset$, for all $x, y, z \in L$.*

Proof. (i) Let $x \in D$ and $x \to y \subseteq D$. Since by Proposition 2.5(v), $x \in (1 \to x) \cap D$ and $(x \to y) \subseteq (1 \to (x \to y)) \cap D$ and D is an $\mathcal{IDS}$, then $y \in (1 \to y) \subseteq D$. Hence D is a $\mathcal{DS}$.

(ii) Let $a \in D$. Since, by Proposition 2.5(iv), $1 \in (a \to 1)$, then $1 \in D_a$. Suppose that $x \in D_a$ and $(x \to y) \subseteq D_a$. Then $(a \to x) \subseteq D$ and $(a \to (x \to y)) \subseteq D$. Hence $(a \to y) \subseteq D$ i.e., $y \in D_a$. Therefore, D_a is a $\mathcal{DS}$ of $\mathcal{L}$.

(iii) The proof is clear. □

Theorem 3.4. *For a nonempty subset D of $\mathcal{L}$ the following are equivalent:*

(i) *D is an $\mathcal{IDS}$,*

(ii) *D is a $\mathcal{DS}$ and $(y \to (y \to x)) \subseteq D$ implies $(y \to x) \subseteq D$, for any $x, y \in L$,*

(iii) *D is a $\mathcal{DS}$ and $(z \to (y \to x)) \subseteq D$ implies $((z \to y) \to (z \to x)) \subseteq D$, for any $x, y, z \in L$,*

(iv) *$1 \in D$ and $(z \to (y \to (y \to x))) \subseteq D$ and $z \in D$ imply $(y \to x) \subseteq D$, for any $x, y, z \in L$,*

(v) *$(x \to (x \odot x)) \subseteq D$, for any $x \in L$.*

Proof. $(i) \Rightarrow (ii)$ By Theorem 3.3, D is a $\mathcal{DS}$ of $\mathcal{L}$. Now, let $(y \to (y \to x)) \subseteq D$, for any $x, y \in L$. Since $1 \in (y \to y) \cap D$ and D is an $\mathcal{IDS}$ of $\mathcal{L}$, then $(y \to x) \subseteq D$.

$(ii) \Rightarrow (iii)$ Let $(z \to (y \to x)) \subseteq D$, for any $x, y \in L$. Then by Proposition 2.5(xiv),

$$y \to x \ll (z \to y) \to (z \to x), \tag{1}$$

and by Proposition 2.5(ix),

$$(z \to y) \to (z \to x) \leqslant z \to ((z \to y) \to x) \tag{2}$$

So, by Proposition 2.5(xii) and (1), we get $z \to (y \to x) \ll z \to ((z \to y) \to (z \to x))$, and by Proposition 2.5$(xii)$ and (2), we get $z \to ((z \to y) \to (z \to x)) \leqslant z \to (z \to ((z \to y) \to x))$. Hence, $z \to (y \to x) \ll z \to (z \to (z \to y) \to x))$. By

Proposition 2.11(i) and assumption, we have $D \ll z \rightarrow ((z \rightarrow y) \rightarrow x) \leqslant (z \rightarrow y) \rightarrow (z \rightarrow x)$ and so, we get $(z \rightarrow y) \rightarrow (z \rightarrow x) \subseteq D$.

$(iii) \Rightarrow (iv)$ Let $z \rightarrow (y \rightarrow (y \rightarrow x)) \subseteq D$ and $z \in D$, for any $x, y, z \in L$. Since D is a $\mathcal{DS}$, then $y \rightarrow (y \rightarrow x) \subseteq D$. Now, by (iii), $(y \rightarrow y) \rightarrow (y \rightarrow x) \subseteq D$. Also,by Proposition 2.5(iv, v), we get $y \rightarrow x \subseteq 1 \rightarrow (y \rightarrow x) \subseteq (y \rightarrow y) \rightarrow (y \rightarrow x) \subseteq D$ and so $y \rightarrow x \subseteq D$.

$(iv) \Rightarrow (i)$ Let $z \rightarrow (y \rightarrow x) \subseteq D$ and $z \rightarrow y \subseteq D$, for any $x, y, z \in L$. Then, by Proposition 2.5, we get $z \rightarrow (y \rightarrow x) \leqslant y \rightarrow (z \rightarrow x) \ll (z \rightarrow y) \rightarrow (z \rightarrow (z \rightarrow x))$, and so, by Proposition 2.11, we conclude that $(z \rightarrow y) \rightarrow (z \rightarrow (z \rightarrow x)) \subseteq D$. Now, by (iv), $z \rightarrow x \subseteq D$.

$(ii) \Rightarrow (v)$ Let $x \in A$ and $u \in x \odot x$. Then $u \in x \odot x$ and so $x \odot x \ll u$. Now, by $(WHRL3)$, $x \ll x \rightarrow u$ and so by Proposition 2.5(ii), $1 \in D \cap x \rightarrow (x \rightarrow u)$. Hence, by Proposition 2.11, $x \rightarrow (x \rightarrow u) \subseteq D$. Therefore, by (ii), $x \rightarrow u \subseteq D$.

$(v) \Rightarrow (ii)$ Put $A = y \rightarrow (y \rightarrow x) \subseteq D$. By using two times of Proposition 2.5(ix), we get

$$1 \in A \rightarrow A = A \rightarrow (y \rightarrow (y \rightarrow x)) \leqslant y \rightarrow (A \rightarrow (y \rightarrow x)) \leqslant y \rightarrow (y \rightarrow (A \rightarrow x)).$$

Hence, $1 \in y \rightarrow (y \rightarrow (A \rightarrow x))$ i.e., $\exists t \in A \rightarrow x$ such that $1 \in y \rightarrow (y \rightarrow t)$. Then $1 \ll y \rightarrow (y \rightarrow t)$ and so by $(WHRL3)$, $y = 1 \odot y \ll y \rightarrow t$. Since, by $(WHRL3)$, $y \odot y \ll t$, then $\exists a \in y \odot y$ such that $a \leqslant t$ and so by Proposition 2.5(xii), $y \rightarrow a \leqslant y \rightarrow t$. On the other hand, $y \rightarrow a \subseteq y \rightarrow (y \odot y) \subseteq D$. So, by Proposition 2.5(ix), $D \ll y \rightarrow t \subseteq y \rightarrow (A \rightarrow x) \leqslant A \rightarrow (y \rightarrow x)$. Now, by Proposition 2.11, $A \rightarrow (y \rightarrow x) \subseteq D$ and since D is a $\mathcal{DS}$, then $y \rightarrow x \subseteq D$. □

Corollary 3.5. *If $\{1\}$ is an $\mathcal{IDS}$, then $x \leqslant x \odot x$, for any $x \in L$.*

Proof. Since, for any $u \in x \odot x$ and $x \in L$, $x \rightarrow u \subseteq \{1\}$, then $1 \in x \rightarrow u$. Now, by Proposition 2.5(iii), we get $x \leqslant u$, for any $u \in x \odot x$, i.e. $x \leqslant x \odot x$. □

Theorem 3.6. *Let D be an $\mathcal{IDS}$ and E be a $\mathcal{DS}$ of L such that $D \subseteq E$. Then E is an $\mathcal{IDS}$, too.*

Proof. Put $A = z \rightarrow (y \rightarrow x) \subseteq E$. Now, by using two times of Proposition 2.5(ix), we have

$$1 \in A \rightarrow A = A \rightarrow (z \rightarrow (y \rightarrow x)) \leqslant z \rightarrow (A \rightarrow (y \rightarrow x)) \leqslant z \rightarrow (y \rightarrow (A \rightarrow x)).$$

So $1 \in D \cap z \rightarrow (y \rightarrow (A \rightarrow x))$. By Proposition 2.11($iii$), we get $z \rightarrow (y \rightarrow (A \rightarrow x)) \subseteq D$ and so by Theorem 3.4(iii), $(z \rightarrow y) \rightarrow (z \rightarrow (A \rightarrow x)) \subseteq D \subseteq E$. Also, by Proposition 2.5(ix),

$$(z \rightarrow y) \rightarrow (z \rightarrow (A \rightarrow x)) \leqslant (z \rightarrow y) \rightarrow (A \rightarrow (z \rightarrow x)) \leqslant A \rightarrow ((z \rightarrow y) \rightarrow (z \rightarrow x).$$

Therefore, $A \rightarrow ((z \rightarrow y) \rightarrow (z \rightarrow x)) \subseteq E$. Since E is a $\mathcal{DS}$ and $A \subseteq E$, then $(z \rightarrow y) \rightarrow (z \rightarrow x) \subseteq E$. Hence, by Theorem 3.3, E is an $\mathcal{IDS}$. □

Corollary 3.7. *The deductive system $\{1\}$ is an $\mathcal{IDS}$ if and only if every $\mathcal{DS}$ of L is an $\mathcal{IDS}$.*

4. Positive implicative deductive systems

Definition 4.1. A subset D of $\mathcal{L}$ containing 1 is a *positive implicative deductive system* (shortly: $\mathcal{PIDS}$), if $x \to ((y \to z) \to y) \subseteq D$ and $x \in D$ imply $y \in D$.

Example 4.2. Let $\mathcal{L}$ be as in the Example 3.2. Then easy calculations show that $\{1, a, b\}$ is a $\mathcal{PIDS}$ of $\mathcal{L}$ and $\{1, a\}$ is an $\mathcal{IDS}$ but not a $\mathcal{PIDS}$ of $\mathcal{L}$. Since we have $a \to ((b \to b) \to b) = a \to (\{1\} \to \{b\}) = a \to \{a, b\} = \{1\} \subseteq \{1, a\}$ and $a \in \{1, a\}$ but $b \notin \{1, a\}$. $\square$

Theorem 4.3. *Every $\mathcal{PIDS}$ is an $\mathcal{IDS}$.*

Proof. Let D be a $\mathcal{PIDS}$ and $y \to (y \to x) \subseteq D$. Then by Proposition 2.5($v, xiii$),

$$y \to (y \to x) \leqslant ((y \to x) \to x) \to (y \to x) \subseteq 1 \to (((y \to x) \to x) \to (y \to x)),$$

So, by Proposition 2.11, we get $1 \to (((y \to x) \to x) \to (y \to x)) \subseteq D$. Now, since D is a $\mathcal{PIDS}$ and $1 \in D$, then $y \to x \subseteq D$ and so, by Theorem 3.3, D is an $\mathcal{IDS}$. $\square$

Corollary 4.4. *Every $\mathcal{PIDS}$ is a $\mathcal{DS}$.* $\square$

Theorem 4.5. *Let D be a $\mathcal{DS}$ of $\mathcal{L}$. Then the following are equivalent:*

(i) *D is a $\mathcal{PIDS}$,*

(ii) *if $(x \to y) \to x \subseteq D$, then $x \in D$, for any $x, y \in L$,*

(iii) *$(\neg x \to x) \to x \subseteq D$, for any $x \in L$.*

Proof. (i) $\Rightarrow$ (ii) Let D be a $\mathcal{PIDS}$ and take $A = (x \to y) \to x \subseteq D$. Since $A \subseteq (1 \to A) \cap D$, then by Proposition 2.11, $1 \to A = 1 \to ((x \to y) \to x) \subseteq D$. So, by assumption, $x \in D$.

(ii) $\Rightarrow$ (i) Let $x \to ((y \to z) \to y) \subseteq D$ and $x \in D$. Since D is a $\mathcal{DS}$, then $(y \to z) \to y \subseteq D$ and so, by assumption, we get $y \in D$ i.e., D is a $\mathcal{PIDS}$.

(i) $\Rightarrow$ (iii) Let D be a $\mathcal{PIDS}$. By Proposition 2.5 (xi), $x \ll (y \to x) \to x$, for any $y \in L$. Now, take $y \in \neg x$. Hence $x \ll (\neg x \to x) \to x$ and we get

$$\begin{aligned}
1 \in\ & x \to ((\neg x \to x) \to x), && \text{by Proposition 2.5}(ii)\\
\le\ & (((\neg x \to x) \to x) \to 0) \to (x \to 0), && \text{by Proposition 2.5}(xiii)\\
\le\ & (\neg x \to x) \to ((((\neg x \to x) \to x) \to 0) \to x), && \text{by Proposition 2.5}(xiii)\\
\le\ & (\underbrace{((\neg x \to x) \to x)}_{A} \to 0) \to \underbrace{((\neg x \to x) \to x)}_{A}, && \text{by Proposition 2.5}(ix)\\
=\ & (A \to 0) \to A.
\end{aligned}$$

Then $1 \in D \cap ((A \to 0) \to A)$. Hence, by Proposition 2.11, $(A \to 0) \to A \subseteq D$. Therefore, by ($ii$), we have $A \subseteq D$ i.e., $(\neg x \to x) \to x \subseteq D$.

$(iii) \Rightarrow (i)$ Let $D \ll (x \to y) \to x$. It is enough to show that $x \in D$. Since $0 \leqslant y$, for any $y \in L$, then by using two times of Proposition 2.5$(xiii)$, we get $(x \to y) \to x \ll (x \to 0) \to x$. By Proposition 2.11, we get $\neg x \to x = (x \to 0) \to x \subseteq D$ and by assumption $(\neg x \to x) \to x \subseteq D$. Now, since D is a $\mathcal{DS}$, then $x \in D$. $\square$

In the following proposition we give a condition that an $\mathcal{IDS}$ is a $\mathcal{PIDS}$.

Proposition 4.6. *Let D be an $\mathcal{IDS}$. Then D is a $\mathcal{PIDS}$ if and only if*
$(x \to y) \to y \subseteq D$ *implies* $(y \to x) \to x \subseteq D$, *for any* $x, y \in L$.

Proof. Let D be a $\mathcal{PIDS}$ and $(x \to y) \to y \subseteq D$. By Proposition 2.5$(xi)$, we have $x \ll (y \to x) \to x$ and so by Proposition 2.5$(xiii)$, $((y \to x) \to x) \to y \ll x \to y$. Since,

$$\begin{aligned}(x \to y) \to y &\leq (y \to x) \to ((x \to y) \to x), && \text{by Proposition 2.5}(xiii)\\ &\leq (x \to y) \to ((y \to x) \to x), && \text{by Proposition 2.5}(ix)\\ &\ll \underbrace{(((y \to x) \to x) \to y) \to ((y \to x) \to x)}_{A},\end{aligned}$$

by Proposition 2.5$(xiii)$ and Proposition 4.5, we have, $D \ll (x \to y) \to y \ll A \subseteq 1 \to A$. So, by Proposition 2.11, we get $(1 \to A) \subseteq D$. Hence $(1 \to A) = 1 \to ((((y \to x) \to x) \to y) \to ((y \to x) \to x)) \subseteq D$. Moreover, since $1 \in D$ and D is a $\mathcal{DS}$, then

$$(\underbrace{((y \to x) \to x)}_{X} \to y) \to \underbrace{((y \to x) \to x)}_{X} \subseteq D.$$

Since D is a $\mathcal{PIDS}$, then by Proposition 4.5 we obtain $(y \to x) \to x = X \subseteq D$.

Conversely, by Proposition 4.5, it is enough to show that $(x \to y) \to x \subseteq D$ implies $x \in D$. For this let $(x \to y) \to x \subseteq D$. Since, by Proposition 2.5(xii), $(x \to y) \to x \leqslant (x \to y) \to ((x \to y) \to y)$, then by Proposition 2.11, we have $(x \to y) \to ((x \to y) \to y) \subseteq D$. Since D is an $\mathcal{IDS}$, then by Theorem 3.4(ii), we get $(x \to y) \to y \subseteq D$. Now, by assumption, we have $(y \to x) \to x \subseteq D$.

On the other hand, since $y \odot x \ll y$, then $y \ll x \to y$ and, by Proposition 2.5(xii), we get $(x \to y) \to x \ll y \to x$. Now, by assumption, $(x \to y) \to x \subseteq D$. So, by Proposition 2.11, we get $y \to x \subseteq D$. Since $(y \to x) \to x \subseteq D$, $y \to x \subseteq D$ and D is a $\mathcal{DS}$, then $x \in D$. $\square$

Theorem 4.7. *Let D be a $\mathcal{PIDS}$ and E be a $\mathcal{DS}$ of L such that $D \subseteq E$. Then E is a $\mathcal{PIDS}$, too.*

Proof. Let D be a $\mathcal{PIDS}$ and E be a $\mathcal{DS}$ such that $D \subseteq E$. Since, by Theorem 4.3, D is an $\mathcal{IDS}$, then by Theorem 3.6. E is an $\mathcal{IDS}$, too. Now, take $A = (x \to y) \to y \subseteq E$. By Proposition 4.6, it is enough to show that $(y \to x) \to x \subseteq E$. Since $1 \in A \to A = A \to ((x \to y) \to y)$, then $A \to ((x \to y) \to y) \subseteq D$. Also, by Theorem 3.4$(iii)$, $(A \to (x \to y)) \to (A \to y) \subseteq D$. Therefore, by Proposition 2.5(ix), $(x \to (A \to y)) \to (A \to y) \subseteq D$ and so, by Proposition 4.6,

$((A \to y) \to x) \to x \subseteq D \subseteq E$. Now, we get $(A \to y) \to x) \to x \subseteq E$. On the other hand, we have

$$\begin{aligned}(x \to y) \to y &\ll (\underbrace{((x \to y) \to y)}_{A} \to y) \to y, && \text{by Proposition 2.5}(xi)\\ &\ll (y \to x) \to ((A \to y) \to x), && \text{by Proposition 2.5}(xii)\\ &\ll (((A \to y) \to x) \to x) \to ((y \to x) \to x) \subseteq E,\end{aligned}$$

by Proposition 2.5(xii) and Proposition 2.11. This implies $(y \to x) \to x \subseteq E$ since E is a $\mathcal{D}S$. □

5. Fantastic deductive systems

Definition 5.1. A subset D of $\mathcal{L}$ containing 1 is called a *fantastic deductive system* (shortly: $\mathcal{FDS}$) if $z \to (y \to x) \subseteq D$ and $z \in D$ imply $((x \to y) \to y) \to x \subseteq D$.

Example 5.2. Let $\mathcal{L}$ be as in Example 3.2. Then $\{1, a, b\}$ is a $\mathcal{FDS}$ of $\mathcal{L}$.

Proposition 5.3. *Any $\mathcal{F}DS$ is a $\mathcal{D}S$.*

Proof. Let D be a $\mathcal{F}DS$, $x \to y \subseteq D$ and $x \in D$. Since by Proposition 2.5(v), $y \in 1 \to y$, then $x \to y \subseteq x \to (1 \to y) \cap D$ and so, by $x \in D$ and definition of a $\mathcal{F}DS$, $((y \to 1) \to 1) \to y \subseteq D$. Now, by Proposition 2.5(xi) and (i), we conclude that $1 \in (y \to 1) \to 1$. So

$$1 \to y \subseteq \bigcup_{a \in (y \to 1) \to 1} (a \to y) = ((y \to 1) \to 1) \to y \subseteq D.$$

Hence, $1 \to y \subseteq D$. Since, by Proposition 2.5(v), $y \in 1 \to y$, then $y \in D$. Thus D is a $\mathcal{D}S$. □

Proposition 5.4. *Let D be a $\mathcal{D}S$ of $\mathcal{L}$. Then D is a $\mathcal{F}DS$ if and only if*
$y \to x \subseteq D$ implies $((x \to y) \to y) \to x \subseteq D$.

Proof. Let D be a $\mathcal{F}DS$ and $y \to x \subseteq D$. By Proposition 2.5(v), $y \to x \subseteq 1 \to (y \to x)$, and so by Proposition 2.11, $1 \to (y \to x) \subseteq D$. Since $1 \in D$ and D is a $\mathcal{F}DS$, then $((x \to y) \to y) \to x \subseteq D$. Conversely, let $z \to (y \to x) \subseteq D$ and $z \in D$. Since D is a $\mathcal{D}S$, then we conclude $y \to x \subseteq D$. Now, by assumption, $((x \to y) \to y) \to x \subseteq D$. □

Theorem 5.5. *Let D be a $\mathcal{F}DS$ and E be a $\mathcal{D}S$ of $\mathcal{L}$ such that $D \subseteq E$. Then E is a $\mathcal{F}DS$, too.*

Proof. Let $y \to x \subseteq E$. Since, by Proposition 2.5(iv) and (ix), $1 \in (y \to x) \to (y \to x) \leqslant y \to ((y \to x) \to x)$, then $1 \in D \cap y \to ((y \to x) \to x)$ and so, by Proposition 2.11, $y \to ((y \to x) \to x) \subseteq D$. Now, take $X = (y \to x) \to x$. Since D

is a $\mathcal{F}DS$, then by Proposition 5.4, $y \to X \subseteq D$ implies $((X \to y) \to y) \to X \subseteq D$. Also, by Proposition 2.5(ix), we have

$$((X \to y) \to y) \to X \ll \underbrace{(y \to x) \to (((X \to y) \to y) \to x)}_{A},$$

which shows that $D \cap A \neq \emptyset$. Therefore, by Proposition 2.11, $A \subseteq D$ and since $D \subseteq E$, then $A \subseteq E$. On the other hand, $y \to x \subseteq E$ and E is a $\mathcal{D}S$ imply that $\underbrace{((X \to y) \to y) \to x}_{B} \subseteq E$. Moreover, using Proposition (ii), (iv), (ix) and ($xiii$), from

$$\begin{aligned} 1 \in (y \to x) \to 1 &\subseteq (y \to x) \to (x \to x), \\ &\leq x \to ((y \to x) \to x), \\ &\leq (((y \to x) \to x) \to y) \to (x \to y), \\ &\leq ((x \to y) \to y) \to ((((y \to x) \to x) \to y) \to y), \\ &\leq \underbrace{(((((y \to x) \to x) \to y) \to y) \to x)}_{B} \to \underbrace{(((x \to y) \to y) \to x)}_{C}, \end{aligned}$$

we get $1 \in E \cap (B \to C)$. Now, since E is a $\mathcal{D}S$, then by Proposition 2.11, $C = ((x \to y) \to y) \to x \subseteq E$. Hence, by Proposition 5.4, E is a $\mathcal{F}DS$. $\square$

Corollary 5.6. *$\{1\}$ is a $\mathcal{F}DS$ of $\mathcal{L}$ if and only if any $\mathcal{D}S$ of L is a $\mathcal{F}DS$.*

Theorem 5.7. *If D is a $\mathcal{P}IDS$ of $\mathcal{L}$, then it is a $\mathcal{F}DS$.*

Proof. Let D be a $\mathcal{P}IDS$ and $y \to x \subseteq D$. Then by Proposition 2.5($xiii$) and (ix), we have

$$y \to x \leqslant ((x \to y) \to y) \to ((x \to y) \to x) \ll (x \to y) \to \underbrace{(((x \to y) \to y) \to x)}_{A}.$$

Since $y \to x \subseteq D$, then by Proposition 2.11, $(x \to y) \to A \subseteq D$. Also, by Proposition (2.5)(vii), $x \odot ((x \to y) \to y) \ll x$. Therefore, by Proposition 2.5(vi), $x \ll ((x \to y) \to y) \to x$. Now, by Proposition 2.5($xiii$), we conclude $(((x \to y) \to y) \to x) \to y \ll x \to y$. So, by another using of Proposition 2.5($xiii$), we get

$$(x \to y) \to \underbrace{((x \to y) \to y) \to x}_{A} \ll \underbrace{((((x \to y) \to y) \to x) \to y)}_{B} \to \underbrace{(((x \to y) \to y) \to x)}_{C}.$$

Therefore, by Proposition 2.11, $B \to C \subseteq D$. Indeed, we have

$$B \to C = (\underbrace{(((x \to y) \to y) \to x)}_{X} \to y) \to \underbrace{(((x \to y) \to y) \to x)}_{X} \subseteq D.$$

Since D is a $\mathcal{P}IDS$, then by Theorem 4.5, $X = ((x \to y) \to y) \to x \subseteq D$. Thus D is a $\mathcal{F}DS$. $\square$

References

[1] **R. A. Borzooei and M. Bakhshi**, (*Weak*) *implicative hyper* BCK*-ideals*, Quasigroups and Related Systems **12** (2004), 13 – 28.

[2] **R. A. Boorzooei, A. Hasankhani, M. M. Zahedi, Y. B. Jun**, *On hyper* K*-algebra*, J. Math. Japonica **52** (2000), 113 – 121.

[3] **C. C. Chang**, *Algebraic analysis of many valued logics*, Trans. Amer. Math. Soc. **88** (1958), 467 – 490.

[4] **P. Corsini**, *Prolegomena of hypergroup*, Aviani Editore, (1993).

[5] **P. Corsini and V. Leoreanu**, *Applications of hyperstructure theory*, Kluwer Academic Publishers, Dordrecht, (2003).

[6] **Sh. Ghorbani, A. Hasankhani and E. Eslami**, *Hyper* MV*-algebras*, Set-Valued Mathematics and Applications **1** (2008), 205 – 222.

[7] **P. Hájek**, *Metamathematics of fuzzy logic*, Kluwer Academic Publishers, Dordrecht, (1998).

[8] **M. Haveshki, A. Borumand Saeid and E. Eslami**, *Some types of filters in* BL*-algebra*, Soft Computing **10** (2006), 657 – 664.

[9] **Y. Imai and K. Iséki**, *On axiom system of propositional calculi* XIV, Proc. Japan Academy **42** (1966), 19 – 22.

[10] **F. Marty**, *Sur une generalization de la notion de groups*, 8th Congress Math. Scandinaves, Stockholm, (1934), 45 – 49.

[11] **M. Ward and R. P. Dilworth**, *Residuated lattices*, Trans. Amer. Math. Soc. **45** (1939), 335 – 354.

[12] **O. Zahiri, R. A. Borzooei and M. Bakhshi**, (*Quotient*) *hyper residuated lattices*, Quasigroups and Related Systems **20** (2012), 125 – 138.

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On classes of regularity in an ordered semigroup

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Abstract. Let m, n be nonnegative integers. An element a in an ordered semigroup $(S, \cdot, \leqslant)$ is said to be (m,n)-*regular* if there exists $x \in S$ such that $a \leqslant a^m x a^n$. This paper gives necessary and sufficient conditions for the set of all (m,n)-regular elements of S to be a subsemigroup of S. The results obtained extend the results on semigroups without order.

1. Introduction

Let S be a semigroup without order and m, n nonnegative integers. An element $a \in S$ is said to be (m,n)-*regular* [3] if there exists $x \in S$ such that $a = a^m x a^n$. Here, we let $a^0 x = x$ and $x a^0 = x$. In [4], the author investigated some sufficient conditions for classes of (m,n)-regularity to be subsemigroups of S. The purpose of this paper is to extend the results on semigroups without order to ordered semigroups.

The rest of this section we recall some definitions and results used throughout the paper.

A semigroup $(S, \cdot)$ together with a partial order $\leqslant$ (on S) that is *compatible* with the semigroup operation, meaning that for $x, y, z \in S$,

$$x \leqslant y \Rightarrow zx \leqslant zy, \quad xz \leqslant yz,$$

is called an *ordered semigroup* ([1], [5]). If A, B are nonempty subsets of S, we let

$$\begin{aligned} AB &= \{xy \in S \mid x \in A, y \in B\}, \\ (A] &= \{x \in S \mid x \leqslant a \text{ for some } a \in A\}. \end{aligned}$$

If $a \in S$, then we write Sa and aS instead of $S\{a\}$ and $\{a\}S$, respectively. It is well-known that the following conditions hold:

(1) $(S] = S$,

(2) $A \subseteq B$ implies $(A] \subseteq (B]$, and

(3) $((A]] = (A]$.

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Let $(S, \cdot, \leqslant)$ be an ordered semigroup and m, n nonnegative integers. An element $a \in S$ is said to be (m, n)-*regular* [10] if there exists $x \in S$ such that

$$a \leqslant a^m x a^n.$$

Here, we let $a^0 x = x$ and $x a^0 = x$. The set of all (m, n)-regular elements of S will be denoted by $R_S(m, n)$. The following conditions hold for nonnegative integers m, m_1, m_2, n, n_1, n_2:

(1) $R_S(0, 0) = S$.

(2) If $m_1 \geqslant m_2$ and $n_1 \geqslant n_2$, then $R_S(m_1, n_1) \subseteq R_S(m_2, n_2)$.

(3) If $m_1 \geqslant m_2 \geqslant 2$, then $R_S(m_1, n) = R_S(m_2, n)$.

(4) If $n_1 \geqslant n_2 \geqslant 2$, then $R_S(m, n_1) = R_S(m, n_2)$.

(5) $R_S(1, 2) = R_S(1, 1) \cap R_S(0, 2)$.

(6) $R_S(2, 1) = R_S(1, 1) \cap R_S(2, 0)$.

A nonempty subset A of an ordered semigroup $(S, \cdot, \leqslant)$ is called a *left* (respectively, *right*) *ideal* [7] of S if

(i) $SA \subseteq A$ (respectively, $AS \subseteq A$);

(ii) for $x \in A$ and $y \in S$, $y \leqslant x$ implies $y \in A$.

If A is both a left and a right ideal of S, then A is called a (two-sided) *ideal* of S. The principal left (respectively, right) ideal of S containing $a \in S$, denoted by $L(a)$, is of the form $(a \cup Sa] := (\{a\} \cup Sa]$. Similarly, the principal right ideal of S containing $a \in S$ is of the form $R(a) := (a \cup aS]$.

It is easy to see for an ordered semigroup $(S, \cdot, \leqslant)$ that the following hold:

(1) If $R_S(1, 0) \neq \emptyset$, then $R_S(1, 0)$ is a left ideal of S.

(2) If $R_S(0, 1) \neq \emptyset$, then $R_S(0, 1)$ is a right ideal of S.

A nonempty subset A of an ordered semigroup $(S, \cdot, \leqslant)$ is called a *subsemigroup* of S if $AA \subseteq A$. It is clear that every left (respectively, right, two-sided) ideals of S is a subsemigroup of S.

2. Main Results

In [2], a left (respectively, right, two-sided) ideal A of an ordered semigroup $(S, \cdot, \leqslant)$ is said to be *complete* if $(SA] = A$ (respectively, $(AS] = A$, $(SAS] = A$). Since if A is a left ideal of S then $(SA] \subseteq A$, it follows that A is complete if $A \subseteq (SA]$. For complete right ideals and complete two-sided ideals of S can be considered similarly.

Theorem 2.1. *Let $(S,\cdot,\leqslant)$ be an ordered semigroup. Then $R_S(1,0)$ (respectively, $R_S(0,1)$) is nonempty if and only if at least one of the principal right (left) ideal of S is complete.*

Proof. Assume that $R_S(1,0)$ is nonempty. Then there exists $a \in R_S(1,0)$, that is $a \in (aS]$. We have

$$(a \cup aS] \subseteq (aS] \subseteq ((a \cup aS]S],$$

hence $(a \cup aS]$ is complete.

Conversely, assume that there exists $a \in S$ such that $(a \cup aS]$ is complete. Then

$$a \in (a \cup aS] = ((a \cup aS]S] \subseteq ((aS]] = (aS].$$

This proves that $a \in R_S(1,0)$, and so $R_S(1,0)$ is nonempty.

The second statement can be proved similarly. □

A left (respectively, right, two-sided) ideal A of an ordered semigroup $(S,\cdot,\leqslant)$ is said to be *semiprime* [6] if for $a \in A$ and any positive integer k, $a^k \in A$ implies $a \in A$.

Theorem 2.2. *Let $(S,\cdot,\leqslant)$ be an ordered semigroup. If at least one principal right (left) ideal of S generated by a^2 for some $a \in S$ is semiprime, then $R_S(2,0)$ (respectively, $R_S(0,2)$) is nonempty.*

Proof. Let $a \in S$ be such that $(a^2 \cup a^2S]$ is semiprime. Since $a^2 \in (a^2 \cup a^2S]$, we obtain $a \in (a^2 \cup a^2S]$, and so $a \leqslant a^2$ or $a \in (a^2S]$. Each of the cases implies that $a \in R_S(2,0)$.

The second statement can be proved analogously. □

Theorem 2.3. *Let $(S,\cdot,\leqslant)$ be an ordered semigroup. The class of regularity $R_S(1,1)$ (also $R_S(2,1), R_S(1,2), R_S(2,2)$) is nonempty if and only if S contains an element a such that $a \leqslant a^2$.*

Proof. Assume that $R_S(1,1)$ is nonempty. Then there exists $a \in (aSa]$. If $x \in S$ such that $a \leqslant axa$, then $ax \leqslant axax = (ax)^2$.

The opposite direction is clear. □

An ordered semigroup $(S,\cdot,\leqslant)$ is said to be *left* (respectively, *right*) *simple* [8] if S has no left (respectively, right) proper ideal. It is easy to see that S is left (respectively, right) simple if and only if $S = (Sa]$ for all $a \in S$ (respectively, $S = (aS]$ for all $a \in S$). Note that if S is left simple then $S = R_S(0,1)$.

Theorem 2.4. *Let $(S,\cdot,\leqslant)$ be an ordered semigroup such that $R_S(1,1)$ is nonempty. If (1), (2) or (3) holds, then $R_S(1,1)$ is a subsemigroup of S.*

(1) *If $a, b \in R_S(1,1)$, then $ab \leqslant (ab)^2$.*

(2) $R_S(1,1) = R_S(1,0) \cap R_S(0,1)$.

(3) *For* $a, b \in S$, $a \leqslant a^2$ *and* $b \leqslant b^2$ *imply* $ab = ba$.

Proof. Clearly, if (1) holds then $R_S(1,1)$ is a subsemigroup of S. Since $R_S(1,0)$ is a left ideal of S and $R_S(0,1)$ is a right ideal of S, it follows that $R_S(1,0) \cap R_S(0,1)$ is an ideal of S, and so this is a subsemigroup of S. Hence (2) holds.

Assume that (3) holds. Let $a, b \in R_S(1,1)$. Then there exist $x, y \in S$ such that $a \leqslant axa$ and $b \leqslant byb$. Since $xa \leqslant (xa)^2$ and $by \leqslant (by)^2$, we have $(xa)(by) = (by)(xa)$, and so

$$ab \leqslant a(xa)(by)b = a(by)(xa)b = (ab)(yx)(ab).$$

Thus $ab \in R_S(1,1)$. □

If $(S, \cdot, \leqslant)$ is an ordered semigroup, then the *center* of S is defined by

$$Z = \{a \in S \mid ax = xa \text{ for all } x \in S\}.$$

Theorem 2.5. *Let* $(S, \cdot, \leqslant)$ *be an ordered semigroup such that* $R_S(2,0)$ *is nonempty. If* $(1), (2)$ *or* (3) *holds, then* $R_S(2,0)$ *is a subsemigroup of* S.

(1) *If* $a, b \in R_S(2,0)$, *then* $ab \leqslant (ab)^2$.

(2) *For* $a, b \in R_S(2,0)$, *if* $a \leqslant a^2x$ *and* $b \leqslant b^2y$ *for some* $x, y \in S$, *then* $ab \leqslant (ab)(ax)(by)$.

(3) *For* $a \in R_S(2,0)$, *if* $a \leqslant a^2x$ *for some* $x \in S$, *then* $ax \in Z$.

Proof. Let (1) hold. If $a, b \in R_S(2,0)$, then $ab \leqslant (ab)^2$, and so $ab \leqslant (ab)^3$. Thus $ab \in R_S(2,0)$.

Assume that (2) holds. Let $a, b \in R_S(2,0)$. Then $a \leqslant a^2x$ and $b \leqslant b^2y$ for some $x, y \in S$. We have $ab \leqslant ab(ax)(by)$ and $ba \leqslant ba(by)(ax)$. Since

$$ab \leqslant ab(ax)(by) = a(ba)(xby) \leqslant a(ba(by)(ax))(xby) = (ab)^2(yax)(xby),$$

we get $ab \in R_S(2,0)$.

Finally, we assume that (3) holds. Let $a, b \in R_S(2,0)$ be such that $a \leqslant a^2x$ and $b \leqslant b^2y$ for some $x, y \in S$. Then $ax \in Z$, and so

$$ab \leqslant (a^2x)(b^2y) = a(ax)b(by) = ab(ax)(by).$$

This shows that the condition (2) holds, hence $R_S(2,0)$ is a subsemigroup of S. □

Analogous to Theorem 2.5, we have:

Theorem 2.6. *Let* $(S, \cdot, \leqslant)$ *be an ordered semigroup such that* $R_S(0,2)$ *is nonempty. If* $(1), (2)$ *or* (3) *holds, then* $R_S(0,2)$ *is a subsemigroup of* S.

(1) *If* $a, b \in R_S(0,2)$, *then* $ab \leqslant (ab)^2$.

(2) *For* $a, b \in R_S(0,2)$, *if* $a \leqslant xa^2$ *and* $b \leqslant yb^2$ *for some* $x, y \in S$, *then* $ab \leqslant (xa)(yb)(ab)$.

(3) *For* $a \in R_S(0,2)$, *if* $a \leqslant xa^2$ *for some* $x \in S,$, *then* $xa \in Z$.

Lemma 2.7. *The following holds for an ordered semigroup* $(S, \cdot, \leqslant)$:

$$R_S(2,2) = R_S(2,1) \cap R_S(1,2).$$

Proof. It is clear that $R_S(2,2) \subseteq R_S(2,1) \cap R_S(1,2)$. For the reverse inclusion, let $a \in R_S(2,1) \cap R_S(1,2)$. Then there exist $x, y \in S$ such that $a \leqslant a^2xa$ and $a \leqslant aya^2$. Since $a \leqslant a^2xaya^2$, $a \in R_S(2,2)$. □

Theorem 2.8. *Let* $(S, \cdot, \leqslant)$ *be an ordered semigroup such that for* $a \in S$, *if* $a \leqslant a^2$ *then* $a \in Z$. *If* $R_S(1,1)$ *(respectively,* $R_S(2,1), R_S(1,2), R_S(2,2)$*) is nonempty, then* $R_S(1,1)$ *(respectively,* $R_S(2,1), R_S(1,2), R_S(2,2)$*) is a subsemigroup of* S.

Proof. If $R_S(1,1)$ is nonempty, then by Theorem 2.4 we have $R_S(1,1)$ is a subsemigroup of S.

Assume that $R_S(2,1)$ is nonempty. Let $a, b \in R_S(2,1)$. Then there exist $x, y \in S$ such that $a \leqslant a^2xa$ and $b \leqslant b^2yb$. Since $a^2x, b^2y \in Z$, we have

$$\begin{aligned}
ab &\leqslant a^2xab^2yb = (a^2x)a(b^2y)b = (a^2x)(b^2y)(ab) = a(ax)b(by)(ab) \\
&\leqslant (a^2xa)(ax)(b^2yb)(by)(ab) = a(a^2x)(axb)(b^2y)(by)(ab) \\
&= a(a^2x)(ax)(b^2y)(b^2y)(ab) = a(b^2y)(a^2x)(ax)(b^2y)(ab) \\
&= (ab)(by)(a^2x)(ax)(b^2y)(ab) = (ab)(a^2x)(by)(ax)(b^2y)(ab) \\
&= (ab)(a^2x)(b^2y)(by)(ax)(ab) = (ab)a(ax)(b^2y)(by)(ax)(ab) \\
&= (ab)a(b^2y)(ax)(by)(ax)(ab) = (ab)^2(by)(ax)(by)(ax)(ab).
\end{aligned}$$

Therefore, $ab \in R_S(2,1)$. Similarly, if $R_S(1,2)$ is nonempty, then $R_S(1,2)$ is a subsemigroup of S.

By Lemma 2.7, if $R_S(2,2)$ is nonempty then $R_S(2,2)$ is a subsemigroup of S. □

References

[1] **G. Birkhoff**, *Lattice theory*, Amer. Math. Soc. Coll. Publ., Vol. XXV, 1967.

[2] **T. Changphas**, *On complete ideals in ordered semigroups*, Far East J. Math. Sci., (accepted).

[3] **R. Croisot**, *Demi-groups inversifs et demi-groups reunions de demi-groups simples*, Ann. Sci. Ecole, **70** (1953), 361-379.

[4] **I. Fabrici**, *Classes of regularity in semigroups*, Mathematički Časopis, **19**(4) (1969), 299-304.

[5] **L. Fuchs**, *Partially Ordered Algebraic Systems*, Addison-Wesley Publ. Comp., 1963.

[6] **N. Kehayopulu**, *On prime, weakly prime ideals in ordered semigroups*, Semigroup Forum, **44** (1992), 341 − 346.

[7] **N. Kehayopulu**, *Remark on ordered semigroups*, Math. Japonica, **35** (1990), 1061–1063.

[8] **N. Kehayopulu**, *On regular duo ordered semigroups*, Math. Japonica, **37** (1992), 535 – 540.

[9] **B. Pondelicek**, *A note on classes of regularity in semigroups*, Mathematički Časopis, **21** (1971), 312 – 317.

[10] **J. Sanborisoot, T. Changphas**, *On characterizations of (m, n)-regular ordered semigroups*, Far East J. Math. Sci., **65**(1) (2012), 75 – 86.

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OD-Characterization of almost simple groups related to $U_3(17)$

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Abstract. We characterize groups with the same order and degree pattern as an almost simple groups related to $U_3(17)$.

1. Introduction

Let G be a finite group. For any group G, we denote by $\pi_e(G)$ the set of orders of its elements and by $\pi(G)$ the set of prime divisors of $|G|$. Let $\pi(G) = \{p_1, p_2, \ldots, p_k\}$. The *prime graph* $\Gamma(G)$ of a group G is the graph whose vertex set is $\pi(G)$ and two distinct primes p and q are joined by an edge (we write $p \sim q$) if and only if G contains an element of order pq ($pq \in \pi_e(G)$). For $p \in \pi(G)$, we put $deg(p) := |\{q \in \pi(G) | p \sim q\}|$, which is called the *degree* of p. If $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ we define $D(G) := (deg(p_1), deg(p_2), \ldots, deg(p_k))$, where $p_1 < p_2 < \ldots < p_k$, to be called the *degree pattern* of G. A group G is called *k-fold OD-characterizable* if there exist exactly k non-isomorphic finite groups having the same order and degree pattern as G. In particular, a 1-fold OD-characterizable group is simply called *OD-characterizable.* A group G is said to be an almost simple group related to S if and only if $S \trianglelefteq G \lesssim \mathrm{Aut}(S)$ for some non-abelian simple group S. In a series of articles, it has been proved, up to now, that many finite almost simple groups are OD-characterizable or k-fold OD-characterizable for $k \geqslant 2$, for instance see [2, 3, 5, 7, 8, 9]. In this paper $U := U_3(17)$ and $\mathrm{Aut}(U) \cong U : \mathbb{S}_3$ and we show that U and $U : 2$ are OD-characterizable, also $U : 3$ and $U : \mathbb{S}_3$ are 3-fold and 5-fold OD-characterizable respectively ($H.K$ means an extension of a group H by a group K and $H : K$ denotes split extension). We denote the socle of G by $\mathrm{Soc}(G)$, which is the subgroup generated by the set of all minimal normal subgroups of G. For $p \in \pi(G)$, we denote by G_p and $\mathrm{Syl}_p(G)$ a Sylow p-subgroup of G and the set of all Sylow p-subgroups of G respectively, all further unexplained notation are standard and can be found in [4].

Throughout this article, all groups under consideration are finite.

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2. Lemmas

It is well-known that $\mathrm{Aut}(U_3(17)) \cong U_3(17) : \mathbb{S}_3$, hence the following lemma follows from definition.

Lemma 2.1. *If G is an almost simple group related to $U := U_3(17)$, then G is isomorphic to one of the following groups: U, $U : 2$, $U : 3$ or $U : \mathbb{S}_3$.* □

G is said to be *completely reducible group* if and only if either $G = 1$ or G is the direct product of a finite number of simple groups. A completely reducible group will be called a *CR-group*. A CR-group has trivial center if and only if it is a direct product of non-abelian simple groups and in this case, it has been named a *centerless CR-group*. The following lemma determines the structure of the automorphism group of a centerless CR-group.

Lemma 2.2. ([4], Theorem 3.3.20) *Let R be a finite centerless CR-group and write $R = R_1 \times R_2 \times \ldots \times R_k$, where R_i is a direct product of n_i isomorphic copies of a simple group H_i, and H_i and H_j are not isomorphic if $i \neq j$. Then* $\mathrm{Aut}(R) = \mathrm{Aut}(R_1) \times \mathrm{Aut}(R_2) \times \ldots \times \mathrm{Aut}(R_k)$ *and* $\mathrm{Aut}(R_i) \cong \mathrm{Aut}(H_i) \wr \mathbb{S}_{n_i}$, *where in this wreath product* $\mathrm{Aut}(H_i)$ *appears in its right regular representation and the symmetric group $\mathbb{S}_{n_i}$ in its natural permutation representation. Moreover, these isomorphisms induce isomorphisms* $\mathrm{Out}(R) \cong \mathrm{Out}(R_1) \times \mathrm{Out}(R_2) \times \ldots \times \mathrm{Out}(R_k)$ *and* $\mathrm{Out}(R_i) \cong \mathrm{Out}(H_i) \wr \mathbb{S}_{n_i}$. □

Let $p \geqslant 5$ be a prime. We denote by $\mathfrak{S}_p$ the set of all simple groups with prime divisors at most p. Clearly, if $q \leqslant p$ then $\mathfrak{S}_q \subseteq \mathfrak{S}_p$. We list all the simple groups in class $\mathfrak{S}_{17}$ in Table 1 below, taken from [6].

Table 1: Simple groups in $\mathfrak{S}_p$, $p \leqslant 17$.

S	$\|S\|$	$\|\mathrm{Out}(S)\|$	S	$\|S\|$	$\|\mathrm{Out}(S)\|$
A_5	$2^2 \cdot 3 \cdot 5$	2	A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	2
A_6	$2^3 \cdot 3^2 \cdot 5$	4	$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	8
$S_4(3)$	$2^6 \cdot 3^4 \cdot 5$	2	$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2	$S_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	1
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3	$O_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	6
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2	$L_2(11)$	$2^2 \cdot 3 \cdot 5 \cdot 11$	2
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2	M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	1
$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	4	M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	2
$U_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	6	$U_5(2)$	$2^{10} \cdot 3^5 \cdot 5 \cdot 11$	2
$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	12	M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	2
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	A_{11}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	2
A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	2	M^cL	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	2
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2	HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	2
A_{12}	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$	2	A_{15}	$2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	2
$U_6(2)$	$2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	6	$L_6(3)$	$2^{11} \cdot 3^{15} \cdot 5 \cdot 7 \cdot 11^2 \cdot 13^2$	4

(continued)

S	$\|S\|$	$\|\mathrm{Out}(S)\|$	S	$\|S\|$	$\|\mathrm{Out}(S)\|$
$L_3(3)$	$2^4 \cdot 3^3 \cdot 13$	2	Suz	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2
$L_2(25)$	$2^3 \cdot 3 \cdot 5^2 \cdot 13$	4	A_{16}	$2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	2
$U_3(4)$	$2^6 \cdot 3 \cdot 5^2 \cdot 13$	4	Fi_{22}	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2
$S_4(5)$	$2^6 \cdot 3^2 \cdot 5^4 \cdot 13$	2	$L_2(17)$	$2^4 \cdot 3^2 \cdot 17$	2
$L_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 13$	4	$L_2(16)$	$2^4 \cdot 3 \cdot 5 \cdot 17$	4
${}^2F_4(2)'$	$2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$	2	$S_4(4)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 17$	4
$L_2(13)$	$2^2 \cdot 3 \cdot 7 \cdot 13$	2	He	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$	2
$L_2(27)$	$2^2 \cdot 3^3 \cdot 7 \cdot 13$	6	$O_8^-(2)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$	2
$G_2(3)$	$2^6 \cdot 3^6 \cdot 7 \cdot 13$	2	$L_4(4)$	$2^{12} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17$	4
${}^3D_4(2)$	$2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$	3	$S_8(2)$	$2^{16} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$	1
$Sz(8)$	$2^6 \cdot 5 \cdot 7 \cdot 13$	3	$O_{10}^-(2)$	$2^{20} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17$	2
$L_2(64)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	6	$F_4(2)$	$2^{24} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17$	2
$U_4(5)$	$2^7 \cdot 3^4 \cdot 5^6 \cdot 7 \cdot 13$	4	$U_4(4)$	$2^{12} \cdot 3^2 \cdot 5^3 \cdot 13 \cdot 17$	4
$L_3(9)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13$	4	$S_6(4)$	$2^{18} \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 13 \cdot 17$	2
$S_6(3)$	$2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	2	$O_8^+(4)$	$2^{24} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$	12
$O_7(3)$	$2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	2	$L_3(16)$	$2^{12} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$	24
$G_2(4)$	$2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$	2	$S_4(13)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13^4 \cdot 17$	2
$S_4(8)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13$	6	$L_2(13^2)$	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 13^2 \cdot 17$	4
$O_8^+(3)$	$2^{12} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13$	24	$U_3(17)$	$2^6 \cdot 3^4 \cdot 7 \cdot 13 \cdot 17^3$	6
$L_5(3)$	$2^9 \cdot 3^{10} \cdot 5 \cdot 11^2 \cdot 13$	2	A_{17}	$2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$	2
A_{13}	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2	A_{18}	$2^{15} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$	2
A_{14}	$2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13$	2			

Lemma 2.3. ([1], Theorem 10.3.1) *Let G be a Frobenius group with kernel K and complement H. Then*

(a) *K is a nilpotent group,*
(b) $|K| \equiv 1(\mathrm{mod}|H|)$. □

3. Almost simple groups related to $U_3(17)$

Theorem 3.1. *Let M be an almost simple group related to $U := U_3(17)$. If G is a finite group such that $D(G) = D(M)$ and $|G| = |M|$, then the following assertions hold:*

(a) *If $M = U$, then $G \cong U$.*
(b) *If $M = U : 2$, then $G \cong U : 2$.*
(c) *If $M = U : 3$, then $G \cong U : 3$, $\mathbb{Z}_3 \times U$ or $\mathbb{Z}_3.U$.*
(d) *If $M = U : \mathbb{S}_3$, then $G \cong U : \mathbb{S}_3$, $\mathbb{Z}_3 \times (U : 2)$, $\mathbb{Z}_3.(U : 2)$, $(\mathbb{Z}_3 \times U).\mathbb{Z}_2$ or $(\mathbb{Z}_3.U).\mathbb{Z}_2$.*

In particular, U and $U : 2$ are OD-characterizable, $U : 3$ is 3-fold OD-characterizable and $U : \mathbb{S}_3$ is 5-fold OD-characterizable.

Proof. We break the proof into a number of separate cases. Note that the set of order elements in each of the following cases is calculated using GAP.
Case 1. If $M = U$, then $G \cong U$.

By Table 1, $|G| = |U| = 2^6.3^4.7.13.17^3$ and we have $\pi_e(U) = \{1, 2, 3, 4, 6, 7, 8, 9, 12, 13, 16, 17, 18, 24, 32, 34, 48, 51, 91, 96, 102\}$, so by assumption, $D(G) = D(U) = (2, 2, 1, 1, 2)$. Therefore, there exist two possibilities for $\Gamma(G)$ are as follows:

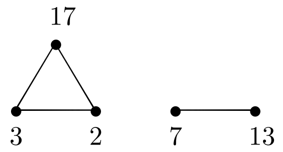

Figure 1-1

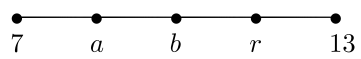

Figure 1-2

where a, b, r are distinct prime numbers that belong to $\{2, 3, 17\}$. We have to show that G is isomorphic to $U := U_3(17)$ and we break the proof into a sequence of steps.

Step 1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2, 3\}$-group. In particular, G is non-solvable.
We consider these two parts separately:

Part A. Consider Figure 1-1, and Figure 1-2 where $r \neq 17$.

First, we show that K is a $17'$-group. Assume the contrary and let $17 \in \pi(K)$. Then 13 does not divide the order of K (otherwise, we may suppose that T is a Hall $\{17, 13\}$-subgroup of K. It is seen that T is a nilpotent subgroup of order 13.17^i for $i = 1, 2$ or 3. Thus, $13.17 \in \pi_e(K) \subseteq \pi_e(G)$, a contradiction). Thus, $\{17\} \subseteq \pi(K) \subseteq \{2, 3, 7, 17\}$. Let $K_{17} \in \mathrm{Syl}_{17}(K)$ and $N := N_G(K_{17})$. By the Frattini argument, $G = KN$. Therefore, N contains an element of order 13, say σ. Since G has no element of order 13.17, $\langle\sigma\rangle$ should act fixed point freely on K_{17}, implying $\langle\sigma\rangle K_{17}$ is a Frobenius group. By Lemma 2.3(b), $|\langle\sigma\rangle| | (|K_{17}| - 1)$. It follows that $13 | 17^i - 1$, for $i = 1, 2$ or 3, which is a contradiction.
Next, we show that K is a p'-group for $p \in \{13, 7\}$. Let x be an element of K of order p and set

$$C := C_G(x), \quad N := N_G(< x >).$$

Let $p = 13$. According to Figure 1-1, C is a $\{7, 13\}$-group. Now, using (N/C)-Theorem, the factor group N/C is embedded in $\mathrm{Aut}(< x >) \cong \mathbb{Z}_{12}$. Hence N is a $\{2, 3, 7, 13\}$-group and by the Frattini argument, $G = KN$ then 17 must divide the order of K, which is a contradiction. According to Figure 1-2, C is a $\{r, 13\}$-group, where $r = 2$ or 3. Therefore, by the same argument, we conclude that N is a $\{2, 3, 13\}$-group and by the Frattini argument, 17 must divide the order of K, which is a contradiction, so K is a $\{2, 3, 7\}$-group.

Let $p = 7$. According to Figure 1-1, C is a $\{7, 13\}$-group. Now, using (N/C)-Theorem, the factor group N/C is embedded in $\mathrm{Aut}(< x >) \cong \mathbb{Z}_6$. Hence N is a $\{2, 3, 7, 13\}$-group and by the Frattini argument, $G = KN$ then 17 must divide the order of K, which is a contradiction. According to Figure 1-2, C is a $\{7, a\}$-

group, where $a = 2, 3$ or 17. Then by the same argument, we conclude that N is a $\{2, 3, 7\}$-group for $a = 2, 3$, and $\{2, 3, 7, 17\}$-group for $a = 17$. Now by the Frattini argument, $G = KN$ then 13 must divide the order of K, which is a contradiction. Therefore, K is a $\{2, 3\}$-group.

Part B. Consider Figure 1-2 where $r = 17$.

First, we show that K is a $17^{'}$-group. Assume the contrary and let $17 \in \pi(K)$. Then 7 does not divide the order of K (otherwise, we may suppose that T is a Hall $\{7, 17\}$-subgroup of K. It is seen that T is a nilpotent subgroup of order 7.17^i for $i = 1, 2$ or 3. Thus, $7.17 \in \pi_e(K) \subseteq \pi_e(G)$, a contradiction). Thus, $\{17\} \subseteq \pi(K) \subseteq \{2, 3, 13, 17\}$. Let $K_{17} \in \mathrm{Syl}_{17}(K)$ and $N := N_G(K_{17})$. By the Frattini argument, $G = KN$. Therefore, N contains an element of order 7, say σ. Since G has no element of order 7.17, $\langle\sigma\rangle$ should act fixed point freely on K_{17}, implying $\langle\sigma\rangle K_{17}$ is a Frobenius group. By Lemma 2.3(b), $|\langle\sigma\rangle| | (|K_{17}| - 1)$. It follows that $7|17^i - 1$, for $i = 1, 2$ or 3, which is a contradiction.
Next, we show that K is a $p^{'}$-group for $p \in \{13, 7\}$. Let x be an element of K of order p and set

$$C := C_G(x), \qquad N := N_G(< x >).$$

Let $p = 7$. By the prime graph of G, C is a $\{7, a\}$-group, where $a = 2$ or 3. Now, using (N/C)-Theorem, the factor group N/C is embedded in $\mathrm{Aut}(< x >) \cong \mathbb{Z}_6$. Hence N is a $\{2, 3, 7\}$-group and by the Frattini argument, $G = KN$, so 17 must divide the order of K, which is a contradiction. Therefore, K is a $\{2, 3, 13\}$-group.

Let $p = 13$. By the prime graph of G, C is a $\{13, 17\}$-group. Now, using (N/C)-Theorem, the factor group N/C is embedded in $\mathrm{Aut}(< x >) \cong \mathbb{Z}_{12}$. Hence N is a $\{2, 3, 13, 17\}$-group and by the Frattini argument, 7 must divide the order of K, which is a contradiction, so K is a $\{2, 3\}$-group. In addition since $G \neq K$, G is non-solvable, and this completes the proof of Step 1.

Step 2. The quotient G/K is an almost simple group. In fact, $S \leq G/K \lesssim \mathrm{Aut}(S)$, where S is a finite non-abelian simple group.

Let $\overline{G} = G/K$. Then $S := \mathrm{Soc}(\overline{G}) = P_1 \times P_2 \times \ldots \times P_m$, where P_i's are finite non-abelian simple groups and $S \leq G/K \lesssim \mathrm{Aut}(S)$, see [3, Proposition 3.1, Step 2]. In what follows, we will show that $m = 1$. Suppose that $m \geqslant 2$. We claim 13 does not divide $|S|$. Assume the contrary and let $13 \mid |S|$, on the other hand, $\{2, 3\} \subset \pi(P_i)$ (by Table 1), hence $2 \sim 13$ and $3 \sim 13$, which is a contradiction. Now, by Step 1, we observe that $13 \in \pi(\overline{G}) \subseteq \pi(\mathrm{Aut}(S))$. But $\mathrm{Aut}(S) = \mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2) \times \ldots \times \mathrm{Aut}(S_k)$, where the groups S_j are direct products of isomorphic P_i's such that $S = S_1 \times S_2 \times \ldots \times S_k$. Therefore, for some j, 13 divides the order of an automorphism group of a direct product S_j of t isomorphic simple groups P_i. Since $P_i \in \mathfrak{S}_{17}$, it follows that $|\mathrm{Out}(P_i)|$ is not divisible by 13 (see Table 1), so 13 does not divide the order of $\mathrm{Aut}(P_i)$. Now, by Lemma 2.2, we obtain $|\mathrm{Aut}(S_j)| = |\mathrm{Aut}(P_i)|^t.t!$. Therefore, $t \geqslant 13$ and so 2^{26} must divide the order of G, which is a contradiction. Therefore, $m = 1$ and $S = P_1$, so the proof is completed.

Step 3. G is isomorphic to $U_3(17)$.

By Table 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.7.13.17^3$, where $2 \leqslant \alpha \leqslant 6$ and $1 \leqslant \beta \leqslant 4$. Now, using the collected results contained in Table 1, we deduce that $S \cong U_3(17)$ and by Step 2, we conclude that $U \trianglelefteq G/K \lesssim \mathrm{Aut}(U)$. As $|G| = |U|$, we deduce $K = 1$, so $G \cong U$, and the proof is completed.

Case 2. If $M = U : 2$, then $G \cong U : 2$.

$|G| = 2|U| = 2^7.3^4.7.13.17^3$ and $\pi_e(U : 2) = \{1, 2, 3, 4, 6, 7, 8, 9, 12, 13, 16, 17, 18, 24, 32, 34, 36, 48, 51, 68, 91, 96, 102\}$, so $D(G) = D(U : 2) = (2, 2, 1, 1, 2)$, and therefore we conclude that the possibilities for $\Gamma(G)$ are as in Figure 1-1 and Figure 1-2, where a, b, r are distinct prime numbers that belong to $\{2, 3, 17\}$.

Step 1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2, 3\}$-group. In particular, G is non-solvable.

By an argument similar to that used in Case 1, we can obtain this assertion.

Step 2. The quotient G/K is an almost simple group. In fact, $S \leq G/K \lesssim \mathrm{Aut}(S)$, where S is a finite non-abelian simple group.

The proof is similar to Step 2, in Case 1.

By Table 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.7.13.17^3$, where $2 \leqslant \alpha \leqslant 7$ and $1 \leqslant \beta \leqslant 4$. Now, using the collected results contained in Table 1, we deduce that $S \cong U_3(17)$. Therefore by Step 2, $U \trianglelefteq G/K \lesssim \mathrm{Aut}(U)$, which implies that $|K| = 1$ or 2.

If $|K| = 1$, then $G \cong U : 2$.

If $|K| = 2$, then $K \leq Z(G)$ and so $deg(2) = 4$, which is a contradiction.

Case 3. If $M = U : 3$, then $G \cong U : 3$, $\mathbb{Z}_3 \times U$ or $\mathbb{Z}_3.U$.

$|G| = 3|U| = 2^6.3^5.7.13.17^3$ and $\pi_e(U : 3) = \{1, 2, 3, 4, 6, 7, 8, 9, 12, 13, 16, 17, 18, 21, 24, 32, 34, 36, 39, 48, 51, 72, 91, 96, 102, 144, 153, 273, 288, 306\}$. Thus, we get $D(G) = D(U : 3) = (2, 4, 2, 2, 2)$. Therefore we have two possibilities for $\Gamma(G)$:

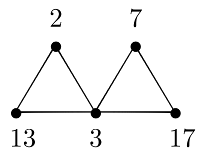

Figure 2-1

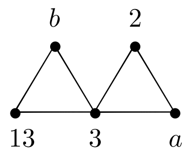

Figure 2-2

where a, b are distinct prime numbers which belong to $\{7, 17\}$.

Step 1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2, 3\}$-group. In particular, G is non-solvable.

We consider these two parts separately:

Part A. Consider Figure 2-1, and Figure 2-2 where $a = 17$ and $b = 7$.

First, we claim K is a $17'$-group. Assume the contrary and let $17 \in \pi(K)$. Then 13 does not divide the order of K (otherwise, we may suppose that T is a Hall $\{17, 13\}$-subgroup of K. It is seen that T is a nilpotent subgroup of order

13.17^i for $i = 1, 2$ or 3. Thus, $13.17 \in \pi_e(K) \subseteq \pi_e(G)$, a contradiction). Thus, $\{17\} \subseteq \pi(K) \subseteq \{2, 3, 7, 17\}$. Let $K_{17} \in \mathrm{Syl}_{17}(K)$ and $N := N_G(K_{17})$. By the Frattini argument, $G = KN$. Therefore, N contains an element of order 13, say σ. Since G has no element of order 13.17, $\langle\sigma\rangle$ should act fixed point freely on K_{17}, implying $\langle\sigma\rangle K_{17}$ is a Frobenius group. By Lemma 2.3(b), $|\langle\sigma\rangle| | (|K_{17}| - 1)$. It follows that $13 | 17^i - 1$, for $i = 1, 2$ or 3, which is a contradiction.

Next, we show that K is a p'-group for $p \in \{13, 7\}$. Let x be an element of K of order p and set

$$C := C_G(x), \qquad N := N_G(< x >).$$

Let $p = 13$. So C is a $\{2, 3, 13\}$ and $\{3, 7, 13\}$-group, in Figure 2-1 and Figure 2-2 respectively. Now, using (N/C)-Theorem, the factor group N/C is embedded in $\mathrm{Aut}(< x >) \cong \mathbb{Z}_{12}$. Hence N is a $\{2, 3, 13\}$-group in Figure 2-1, and $\{2, 3, 7, 13\}$-group in Figure 2-2. On the other hand, by the Frattini argument, $G = KN$. Then 17 must divide the order of K, which is a contradiction.

Let $p = 7$. According to Figure 2-1, C is a $\{3, 7, 17\}$-group. Now, using (N/C)-Theorem, the factor group N/C is embedded in $\mathrm{Aut}(< x >) \cong \mathbb{Z}_6$. Hence N is a $\{2, 3, 7, 17\}$-group and by the Frattini argument, $G = KN$ then 13 must divide the order of K, which is a contradiction. According to Figure 2-2, C is a $\{3, 7, 13\}$-group. Then by a same argument, we conclude that N is a $\{2, 3, 7, 13\}$-group. Now by the Frattini argument, $G = KN$ then 17 must divide the order of K, which is a contradiction. Therefore, K is a $\{2, 3\}$-group.

Part B. Consider Figure 2-2, where $a = 7$ and $b = 17$.

First, we claim K is a $17'$-group. Assume the contrary and let $17 \in \pi(K)$. Then 7 does not divide the order of K (otherwise, we may suppose that T is a Hall $\{7, 17\}$-subgroup of K. It is seen that T is a nilpotent subgroup of order 7.17^i for $i = 1, 2$ or 3. Thus, $7.17 \in \pi_e(K) \subseteq \pi_e(G)$, a contradiction). Thus, $\{17\} \subseteq \pi(K) \subseteq \{2, 3, 13, 17\}$. Let $K_{17} \in \mathrm{Syl}_{17}(K)$ and $N := N_G(K_{17})$. By the Frattini argument, $G = KN$. Therefore, N contains an element of order 7, say σ. Since G has no element of order 7.17, $\langle\sigma\rangle$ should act fixed point freely on K_{17}, implying $\langle\sigma\rangle K_{17}$ is a Frobenius group. By Lemma 2.3(b), $|\langle\sigma\rangle| | (|K_{17}| - 1)$. It follows that $7 | 17^i - 1$, for $i = 1, 2$ or 3, which is a contradiction. Therefore, K is a $17'$-group.

Next, we show that K is a p'-group for $p \in \{7, 13\}$. Let x be an element of K of order p and set

$$C := C_G(x), \qquad N := N_G(< x >).$$

Let $p = 7$. So C is a $\{2, 3, 7\}$-group. Now, using (N/C)-Theorem, the factor group N/C is embedded in $\mathrm{Aut}(< x >) \cong \mathbb{Z}_6$. Hence N is a $\{2, 3, 7\}$-group and by the Frattini argument, $G = KN$ then 17 must divide the order of K, which is a contradiction.

Let $p = 13$. Therefore, C is a $\{3, 13, 17\}$-group. Now, using (N/C)-Theorem, the factor group N/C is embedded in $\mathrm{Aut}(< x >) \cong \mathbb{Z}_{12}$. Hence N is a $\{2, 3, 13, 17\}$-group and by the Frattini argument, $G = KN$ then 7 must divide the order of K, which is a contradiction. So K is a $\{2,3\}$-group. In addition since $G \neq K$, G is non-solvable, and this completes the proof of Step 1.

Step 2. The quotient G/K is an almost simple group. In fact, $S \leq G/K \lesssim \mathrm{Aut}(S)$, where S is a finite non-abelian simple group.

Similar to Step 1, we consider two parts:

Part A. Consider Figure 2-1, and Figure 2-2 when $a = 17$ and $b = 7$.
Let $\overline{G} = G/K$. Then $S := \mathrm{Soc}(\overline{G}) = P_1 \times P_2 \times \ldots \times P_m$, where P_i's are finite non-abelian simple groups and $S \leq G/K \lesssim \mathrm{Aut}(S)$. In what follows, we will show that $m = 1$. Suppose that $m \geqslant 2$. We claim 7 does not divide $|S|$. Assume the contrary and let $7 \mid |S|$, on the other hand, $\{2,3\} \subset \pi(P_i)$ (by Table 1), hence $2 \sim 7$ and $3 \sim 7$, which is a contradiction. Now, by Step 1, we observe that $7 \in \pi(\overline{G}) \subseteq \pi(\mathrm{Aut}(S))$. But $\mathrm{Aut}(S) = \mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2) \times \ldots \times \mathrm{Aut}(S_k)$, where the groups S_j are direct products of isomorphic P_i's such that $S = S_1 \times S_2 \times \ldots \times S_k$. Therefore, for some j, 7 divides the order of an automorphism group of a direct product S_j of t isomorphic simple groups P_i. Since $P_i \in \mathfrak{S}_{17}$, it follows that $|\mathrm{Out}(P_i)|$ is not divisible by 7 (see Table 1), so 7 does not divide the order of $\mathrm{Aut}(P_i)$. Now, by Lemma 2.2, we obtain $|\mathrm{Aut}(S_j)| = |\mathrm{Aut}(P_i)|^t.t!$. Therefore, $t \geqslant 7$ and so 2^{14} must divide the order of G, which is a contradiction. Therefore, $m = 1$ and $S = P_1$, so the proof of this part is completed.

Part B. Consider Figure 2-2, when $a = 7$ and $b = 17$.
Let $\overline{G} = G/K$. Then $S := \mathrm{Soc}(\overline{G}) = P_1 \times P_2 \times \ldots \times P_m$, where P_i's are finite non-abelian simple groups and $S \leq G/K \lesssim \mathrm{Aut}(S)$. In what follows, we will show that $m = 1$. Suppose that $m \geqslant 2$. We claim 13 does not divide $|S|$. Assume the contrary and let $13 \mid |S|$, on the other hand, $\{2,3\} \subset \pi(P_i)$ (by Table 1), hence $2 \sim 13$ and $3 \sim 13$, which is a contradiction. Now, by Step 1, we observe that $13 \in \pi(\overline{G}) \subseteq \pi(\mathrm{Aut}(S))$. But $\mathrm{Aut}(S) = \mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2) \times \ldots \times \mathrm{Aut}(S_k)$, where the groups S_j are direct products of isomorphic P_i's such that $S = S_1 \times S_2 \times \ldots \times S_k$. Therefore, for some j, 13 divides the order of an automorphism group of a direct product S_j of t isomorphic simple groups P_i. Since $P_i \in \mathfrak{S}_{17}$, it follows that $|\mathrm{Out}(P_i)|$ is not divisible by 13 (see Table 1), so 13 does not divide the order of $\mathrm{Aut}(P_i)$. Now, by Lemma 2.2, we obtain $|\mathrm{Aut}(S_j)| = |\mathrm{Aut}(P_i)|^t.t!$. Therefore, $t \geqslant 13$ and so 2^{26} must divide the order of G, which is a contradiction. Therefore, $m = 1$ and $S = P_1$, so the proof is completed.

Now by Table 1 and Step 1, it is evident that $|S| = 2^\alpha.3^\beta.7.13.17^3$, where $2 \leqslant \alpha \leqslant 6$ and $1 \leqslant \beta \leqslant 5$. By using the collected results contained in Table 1, we deduce that $S \cong U_3(17)$ and by Step 2, we conclude that $U \trianglelefteq G/K \lesssim \mathrm{Aut}(U)$. Hence, $|K| = 1$ or 3.

If $|K| = 1$, then $G \cong U : 3$.

If $|K| = 3$, then $G/K \cong U$. In this case we have $G/C_G(K) \lesssim \mathrm{Aut}(K) \cong \mathbb{Z}_2$. Thus, $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, G is a central extension of $\mathbb{Z}_3$ by U. If G splits over K, we obtain $G \cong \mathbb{Z}_3 \times U$, otherwise, we have $G \cong \mathbb{Z}_3.U$. If $|G/C_G(K)| = 2$, then $K \subset C_G(K)$ and $1 \neq$

$C_G(K)/K \trianglelefteq G/K \cong U$. Thus, we obtain $G = C_G(K)$ because U is simple, which is a contradiction.

Case 4. If $M = U : \mathbb{S}_3$, then $G \cong U : \mathbb{S}_3$, $\mathbb{Z}_3 \times (U : 2)$, $\mathbb{Z}_3.(U : 2)$, $(\mathbb{Z}_3 \times U).\mathbb{Z}_2$ or $(\mathbb{Z}_3.U).\mathbb{Z}_2$.

$|G| = 6|U| = 2^7.3^5.7.13.17^3$ and $\pi_e(U : \mathbb{S}_3) = \{1, 2, 3, 4, 6, 7, 8, 9, 12, 13, 16, 17, 18, 21, 24, 32, 34, 36, 39, 48, 51, 68, 72, 91, 96, 102, 144, 153, 273, 288, 306\}$, so $D(G) = D(U : \mathbb{S}_3) = (2, 4, 2, 2, 2)$, and therefore we conclude that there exist two possibilities for the prime graph of G presented by Figure 2-1 and Figure 2-2, where a, b are distinct prime numbers which belong to $\{7, 17\}$.

Step 1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2, 3\}$-group. In particular, G is non-solvable.

We can prove this by the similar way to that in Case 3.

Step 2. The quotient G/K is an almost simple group. In fact, $S \leq G/K \lesssim \mathrm{Aut}(S)$, where S is a finite non-abelian simple group.

The proof is similar to Step 2, in Case 3.

Now by Table 1 and Step 1, it is evident that $|S| = 2^\alpha.3^\beta.7.13.17^3$, where $2 \leqslant \alpha \leqslant 7$ and $1 \leqslant \beta \leqslant 5$. By using the collected results contained in Table 1, we deduce that $S \cong U_3(17)$ and by Step 2, we conclude that $U \trianglelefteq G/K \lesssim \mathrm{Aut}(U)$. Hence, $|K| = 1, 2, 3$ or 6.

If $|K| = 1$, then $G \cong U : \mathbb{S}_3$.

If $|K| = 2$, then $K \leq Z(G)$. It follows that $deg(2) = 4$, which is a contradiction.

If $|K| = 3$, then $G/K \cong U : 2$. In this case, we have $G/C_G(K) \lesssim \mathrm{Aut}(K) \cong \mathbb{Z}_2$. Thus, $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, G is a central extension of $\mathbb{Z}_3$ by $U : 2$. If G splits over K, we obtain $G \cong \mathbb{Z}_3 \times (U : 2)$, otherwise, we have $G \cong \mathbb{Z}_3.(U : 2)$. If $|G/C_G(K)| = 2$, then $K \subset C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong U : 2$ and we obtain $C_G(K)/K \cong U$. Because $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of K by U. Thus, $C_G(K) \cong \mathbb{Z}_3 \times U$ or $\mathbb{Z}_3.U$. Therefore, $G \cong (\mathbb{Z}_3 \times U).\mathbb{Z}_2$ or $(\mathbb{Z}_3.U).\mathbb{Z}_2$.

If $|K| = 6$, then $G/K \cong U$ and $K \cong \mathbb{Z}_6$ or $\mathbb{S}_3$.

If $K \cong \mathbb{Z}_6$, then $G/C_G(K) \lesssim \mathbb{Z}_2$ and so $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$. It follows that $deg(2) = 4$, a contradiction. If $|G/C_G(K)| = 2$, then $K \subset C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong U$, which is a contradiction because U is simple.

If $K \cong \mathbb{S}_3$, then $K \cap C_G(K) = 1$ and $G/C_G(K) \lesssim \mathbb{S}_3$. Thus, $C_G(K) \neq 1$. Hence, $1 \neq C_G(K) \cong C_G(K)K/K \trianglelefteq G/K \cong U$. It follows that $U \cong G/K \cong C_G(K)$ because U is simple. Therefore, $G \cong \mathbb{S}_3 \times U$, which implies that $deg(2) = 4$, a contradiction. The proof of Theorem 3.1 is completed. □

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References

[1] **D. Gorenstein**, *Finite Groups*, New York: Harper and Row, 1980.

[2] **A. R. Moghaddamfar, A. R. Zokayi and M. R. Darafsheh**, *A characterization of finite simple groups by the degrees of vertices of their prime graphs*, Algebra Coll. **12** (2005), 431 − 442.

[3] **G. R. Rezaeezadeh, M. R. Darafsheh, M. Sajjadi and M. Bibak**, *OD-Characterization of almost simple groups related to* $L_3(25)$, Bull. Iranian Math. Soc. (2013), inprint.

[4] **J. S. Robinson Derek**, *A Course in the Theory of Groups*, Springer-Verlag, 2003.

[5] **Y. Yan, G. Chen and L. Wang**, *OD-Characterization of the automorphism groups of* $O_{10}^{\pm}(2)$, Indian J. Pure Appl. Math. **43** (2012), 183 − 195.

[6] **A. Zavarnitsin**, *Finite simple groups with narrow prime spectru*, Sibirskie Elektronnye Mat. Izv. **6** (2009), 1 − 12.

[7] **L. C. Zhang and W. J. Shi**, *OD-Characterization of almost simple groups related to* $L_2(49)$, Archivum Math. (Brno) **44** (2008), 191 − 199.

[8] **L. C. Zhang and W. J. Shi**, *OD-Characterization of almost simple groups related to* $U_3(5)$, Acta Math. Sinica, English Series **26** (2010), 161 − 168.

[9] **L. C. Zhang and W. J. Shi**, *OD-Characterization of almost simple groups related to* $U_6(2)$, Acta Math. Scientia **31B** (2011), 441 − 450.

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Zariski-topology for co-ideals of commutative semirings

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Abstract. Let R be a semiring and *co-spec*(R) be the collection of all prime strong co-ideals of R. In this paper, we introduce and study a generalization of the Zariski topology of ideals in rings to co-ideals of semirings. We investigate the interplay between the algebraic-theoretic properties and the topological properties of *co-spec*(R). Semirings whose Zariski topology is respectively T_1, Hausdorff or cofinite are studied, and several characterizations of such semirings are given.

1. Introduction

As a generalization of rings, semirings have been found useful for solving problems in different areas of applied mathematics and information sciences, since the structure of a semiring provides an algebraic framework for modelling and studying the key factors in these applied areas. Let R be a commutative ring with identity. The prime spectrum $spec(R)$ and the topological space obtained by introducing Zariski topology on the set of prime ideals of R play an important role in the ideals of commutative algebra, algebraic geometry and lattice theory. Also, recently the notion of prime submodules and Zariski topology on $spec(M)$, the set of all prime submodules of a module M over R, are studied by many authors. In this paper, we concentrate on Zariski topology for co-ideals of semirings and generalize the some well known results of Zariski topology on the sets of prime ideals of a commutative ring to the sets of prime strong co-ideals of a commutative semiring and investigate the basic properties of this topology. For example, we prove that if R is a $*$-semiring, then *co-spec*(R) is a T_0-space; it is a compact space; the quasi-compact open subsets of its are closed under finite intersection and it is a sober space. Consequently, it is a spectral space. Equivalently, it is homeomorphic to $spec(S)$, with the Zariski topology, for some commutative ring S.

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2. Preliminaries

In order to make this paper easier to follow, we recall in this section various notions from topology theory and co-ideals theory of commutative semirings which will be used in the sequel. A *commutative semiring* R is defined as an algebraic system $(R, +, .)$ such that $(R, +)$ and $(R, cdot)$ are commutative semigroups, connected by $a(b+c) = ab+ac$ for all $a, b, c \in R$, and there exists $0, 1 \in R$ such that $r+0 = r$ and $r0 = 0r = 0$ and $r1 = 1r = r$ for each $r \in R$. In this paper all semirings considered will be assumed to be commutative with non-zero identity.

Let R be a semiring. A non-empty subset I of R is called *co-ideal*, if it is closed under multiplication and satisfies the condition $r + a \in I$ for all $a \in I$ and $r \in R$. A co-ideal I in R is called *strong* provided that $1 \in I$. (Clearly, $0 \in I$ if and only if $I = R$) [4, 7, 8, 10]. A strong co-ideal I of R is called *subtractive* if $x, xy \in I$, then $y \in I$ [7]. A proper strong co-ideal P of R is *prime* if $x + y \in P$, then $x \in P$ or $y \in P$. The notation *co-spec*(R) denotes the set of all prime strong co-ideals of R. A proper strong co-ideal I of R is said to be *maximal* if J is a strong co-ideal in R with $I \subseteq J$ and $I \neq J$, then $J = R$. If D is an arbitrary nonempty subset of R, then the set $F(D)$ consisting of all elements of R of the form $d_1 d_2 \cdots d_n + r$ (with $d_i \in D$ for all $1 \leqslant i \leqslant n$ and $r \in R$) is a co-ideal of R containing D [8, 10].

We need the following propositions, proved in [7].

Proposition 2.1. *Let R be a semiring. Then any proper co-ideal of R is contained in a maximal co-ideal of R. Moreover, any maximal co-ideal of R is a prime and subtractive strong co-ideal of R.* □

A topological space X is called *irreducible* if $X \neq \emptyset$ and every finite intersection of non-empty open sets of X is non-empty. A (non-empty) subset Y of a topology space X is an *irreducible set* if the subspace Y of X is irreducible. For this to be so, it is necessary and sufficient that, for every pair of sets Y_1, Y_2 which are closed in X and satisfy $Y \subseteq Y_1 \cup Y_2$, then $Y \subseteq Y_1$ or $Y \subseteq Y_2$.

Let Y be a closed subset of a topological space. An element $y \in Y$ is called a *generic point* of Y if $Y = \overline{\{y\}}$. Note that a generic point of the irreducible closed subset Y of a topological space is unique if the topological space is a T_0-space.

The *cofinite topology* (sometimes called the *finite complement topology*) is a topology which can be defined on every set X. It has precisely the empty set and all cofinite subsets of X as open sets. As a consequence, in the cofinite topology, the only closed subsets are finite sets, or the whole of X. Then X is automatically compact in this topology, since every open set only omits finitely many points of X. Also, the cofinite topology is the smallest topology satisfying the T_1 axiom; i.e., it is the smallest topology for which every singleton set is closed. If X is not finite, then this topology is not Hausdorff.

Following Hochster [9], we say that a topological space X is a *spectral space* in case X is homeomorphic to *spec*(S), with the Zariski topology, for some commutative ring S. Spectral spaces have been characterized by Hochster [9] as the topological spaces X which is a quasi-compact T_0-space such that the quasi-compact open subsets of X are closed under finite intersection and each its irreducible closed subset has a generic point, i.e., X is a *sober space*.

3. Strong co-ideals and Zariski topology

Let R be a semiring with non-zero identity. For any subset E of R by $V(E)$ we mean the set of all prime strong co-ideals of R containing E.

Lemma 3.1. *Let R be a semiring. Then $V(R) = \emptyset$ and $V(F(\{1\})) = co - spec(R)$.*

Proof. This follows directly from definitions. □

Lemma 3.2. *Let P be a prime strong co-ideal of a semiring R. If I and J are co-ideals of R such that $I + J \subseteq P$, then $I \subseteq P$ or $J \subseteq P$.*

Proof. It suffices to show that if $I + J \subseteq P$ and $I \not\subseteq P$, then $J \subseteq P$. Let $b \in J$. By assumption, there exists $a \in I$ such that $a \notin P$. As $a + b \in P$, P prime gives $b \in P$, as needed. □

Proposition 3.3. *Let R be a semiring.*

(1) *If E is a subset of R, then $V(E) = V(F(E))$.*
(2) *If I and J are co-ideals of R with $I \subseteq J$, then $V(J) \subseteq V(I)$.*
(3) *If I and J are co-ideals of R, then $V(I + J) = V(J) \cup V(I)$.*
(4) *If $\{I_i\}_{i\in\Gamma}$ is a family of co-ideals of R, then $V(F(\bigcup_{i\in\Gamma} I_i)) = \bigcap_{i\in\Gamma} V(I_i)$.*

Proof. (1). Assume that $P \in V(E)$ (so $E \subseteq P$) and let $r + s_1 \cdots s_n \in F(E)$ where $s_1, \dots, s_n \in E$ and $r \in R$. Since $s_1, \dots, s_n \in E \subseteq P$, we must have $s_1 \cdots s_n \in P$; hence $r + s_1 \cdots s_n \in P$ since P is a co-ideal. Therefore $F(E) \subseteq P$, and so $P \in V(F(E))$. Thus $V(E) \subseteq V(F(E))$. For the reverse inclusion, assume that $P \in V(F(E))$. Since $E \subseteq F(E) \subseteq P$, we get $P \in V(E)$, and so we have equality.

(2). is clear.

(3). Let $P \in V(I + J)$. By Lemma 3.2, either $I \subseteq P$ or $J \subseteq P$. This implies that $P \in V(I) \cup V(J)$; hence $V(I+J) \subseteq V(J) \cup V(I)$. Since I and J are co-ideals, we have $I + J \subseteq I$ and $I + J \subseteq J$; thus $V(J) \cup V(I) \subseteq V(I + J)$ by (2). Therefore, $V(I + J) = V(J) \cup V(I)$.

(4). By (1), it suffices to show that $V(\bigcup_{i\in\Gamma} I_i) = \bigcap_{i\in\Gamma} V(I_i)$. Consider an arbitrary $P \in \bigcap_{i\in\Gamma} V(I_i)$. Then for each $i \in \Gamma$, $I_i \subseteq P$. Thus $\bigcup_{i\in\Gamma} I_i \subseteq P$. Therefore $P \in V(\bigcup_{i\in\Gamma} I_i)$. For the reverse inclusion, let $P \in V(\bigcup_{i\in\Gamma} I_i)$. From $I_i \subseteq \bigcup_{i\in\Gamma} I_i$ and $P \in V(\bigcup_{i\in\Gamma} I_i)$, we have $P \in V(I_i)$ for each $i \in \Gamma$. Therefore $V(\bigcup_{i\in\Gamma} I_i) \subseteq \bigcap_{i\in\Gamma} V(I_i)$. Hence $V(\bigcup_{i\in\Gamma} I_i) = \bigcap_{i\in\Gamma} V(I_i)$. □

Let R be a semiring. If $\xi(R)$ denotes the collection of all subsets $V(I)$ of *co-spec*(R), then $\xi(R)$ contains the empty set and *co-spec*$(R) = X$ and is closed under arbitrary intersection and finite union by Proposition 3.3. Thus $\xi(R)$ satisfies the axioms of closed subsets of a topological spaces, which is called the *Zariski-topology* for co-ideals of commutative semirings.

Let I be a co-ideal of R. Put

$$\textit{co-rad}(I) = \{x \in R \mid nx \in I \ for\ some\ n \in \mathbb{N}\}$$

and

$$\textit{co-rad}(R) = \{x \in R \mid nx \in F(\{1\}) \ for\ some\ n \in \mathbb{N}\}.$$

We will denote the closure of Y in *co-spec*(R) by $\overline{Y}$, and intersections of elements of Y by $\mathcal{T}(Y)$.

Proposition 3.4. *Let R be a semiring.*

(1) *If I is a co-ideal of R, then $V(I) = V(\textit{co-rad}(I))$.*

(2) *If I is a co-ideal of R, then $V(I) = V(\mathcal{T}(V(I)))$.*

(3) *If I and J are co-ideals of R with $V(I) \subseteq V(J)$, then $J \subseteq \mathcal{T}(V(I))$.*

(4) *$V(I) = V(J)$ if and only if $\mathcal{T}(V(I)) = \mathcal{T}(V(J))$ for each co-ideals I and J of R.*

Proof. (1). Since $I \subseteq$*co-rad*(I), $V(\textit{co-rad}(I)) \subseteq V(I)$ by Proposition 3.3. For the reverse inclusion, assume that $P \in V(I)$. If $x \in$*co-rad*(I), then $nx \in I$ for some $n \in \mathbb{N}$. Since $I \subseteq P$, $nx \in P$, consequently $x \in P$. Thus *co-rad*$(I) \subseteq P$ and so $V(I) \subseteq V(\textit{co-rad}(I))$. Hence $V(\textit{co-rad}(I)) = V(I)$.

(2). As $I \subseteq \mathcal{T}(V(I))$, we have $V(\mathcal{T}(V(I))) \subseteq V(I)$. Conversely, let $P \in V(I)$, hence $\mathcal{T}(V(I)) = \bigcap_{Q \in V(I)} Q \subseteq P$. Therefore we have $V(I) \subseteq V(\mathcal{T}(V(I)))$, and so $V(\mathcal{T}(V(I))) = V(I)$.

(3). Let I and J be co-ideals of R and $V(I) \subseteq V(J)$. Therefore we obtain $\mathcal{T}(V(J)) \subseteq \mathcal{T}(V(I))$. Since $J \subseteq \mathcal{T}(V(J))$, $J \subseteq \mathcal{T}(V(I))$.

(4). Let $V(I) = V(J)$. By (2), we have $V(I) = V(\mathcal{T}(V(J)))$; hence we get $\mathcal{T}(V(J)) \subseteq \mathcal{T}(V(I))$. Similarly, the reverse inclusion is hold. The converse implication is clear. $\square$

Let $X =$*co-spec*(R). For each subset E of R, by $D(E)$ we mean $X - V(E) = \{P \in X \mid E \nsubseteq P\}$. If $E = \{f\}$, then by X_f we denote the set $\{P \in X \mid f \notin P\}$.

Theorem 3.5. *Let R be a semiring. Then $\mathcal{A} = \{X_f \mid f \in R\}$ forms a base for Zariski topology for co-ideals of R.*

Proof. Let U be an open set. Then $U = X - V(I)$ for some co-ideal I of R. Let $P \in U$. Then $I \nsubseteq P$, so there exists $f \in I$ such that $f \notin P$; hence $P \in X_f$. We claim that $X_f \subseteq U$. Let $Q \in X_f$. Then $f \notin Q$, so $I \nsubseteq Q$; thus $Q \in U$. Hence $X_f \subseteq U$. Therefore $\mathcal{A}$ is a base for Zariski topology on X. $\square$

Proposition 3.6. *Let R be a semiring and $X = \bigcup_{i \in \Gamma} X_{a_i}$. If $I = F(\{a_i\}_{i \in \Gamma})$, then $I = R$.*

Proof. Suppose that $I \neq R$. Then there exists a maximal co-ideal P of R such that $I \subseteq P$ by Proposition 2.1. Since $P \in X$, there exists $i \in \Gamma$ such that $a_i \notin P$, a contradiction with $I \subseteq P$. Hence $I = R$. □

Theorem 3.7. *Let R be a semiring. Then the following statements are hold.*

(1) $X_f \cap X_g = X_{f+g}$ *for each* $f, g \in R$.

(2) $X_f = X$ *if and only if* f^n *has additive inverse for some* $n \in \mathbb{N}$.

(3) $X_f = \emptyset$ *if and only if* $f \in P$ *for each* $P \in$*co-spec*(R) (*or equivalently,* $f \in \mathcal{T}(V(\{1\}))$).

Proof. (1). If $P \in X_f \cap X_g$, then $f \notin P$ and $g \notin P$; hence $f + g \notin P$. Thus $X_f \cap X_g \subseteq X_{f+g}$. For the reverse inclusion, let $P \in X_{f+g}$. Then $f + g \notin P$, so $f \notin P$ and $g \notin P$. Therefore $P \in X_f \cap X_g$, and we have equality.

(2). Let $X_f = X$. By Proposition 3.6, $R = F(\{f\})$. Therefore $f^n + r = 0$ for some $n \in \mathbb{N}$ and $r \in R$. Conversely, assume that f^n has inverse for some $n \in \mathbb{N}$. We show that $X_f = X$. If $P \in X$ and $P \notin X_f$, then $f \in P$. It follows that $0 \in P$; hence $P = R$, which is a contradiction. Thus $X = X_f$.

(3). It is clear that $X_f = \emptyset$ if and only if $f \in P$ for each $P \in$*co-spec*(R). □

Proposition 3.8. *Let I be a strong co-ideal of semiring R. Then* $D(I) = \bigcup_{a \in I} X_a$. *In particular, if* $I = F(\{a_1, \dots, a_n\})$, *then* $D(I) = \bigcup_{i=1}^{n} X_{a_i}$.

Proof. Let $P \in D(I)$. So $I \nsubseteq P$. Thus there exists $a \in I$ such that $a \notin P$; hence $P \in X_a$. Therefore, $P \in \bigcup_{a \in I} X_a$, and so $D(I) \subseteq \bigcup_{a \in I} X_a$. Conversely, assume that $P \in \bigcup_{a \in I} X_a$. Then $P \in X_a$ for some $a \in I$. Since $a \notin P$, $I \nsubseteq P$. Hence $P \in D(I)$ and so the equality is hold. The "in particular" statement is clear. □

Theorem 3.9. *Let R be a semiring. Then $X =$co-spec(R) is a compact space.*

Proof. Let $X = \bigcup_{i \in \Gamma} X_{a_i}$. By Proposition 3.6, $F(\{a_i\}_{i \in \Gamma}) = R$; hence $0 = r + a_1 \cdots a_n$ for some $a_1, \dots, a_n \in \{a_i\}_{i \in \Gamma}$. We claim that $X \subseteq \bigcup_{i=1}^{n} X_{a_i}$. Let $P \in X$. If for each $1 \leqslant i \leqslant n$, $a_i \in P$, then $a_1 \cdots a_n \in P$, and so $0 = r + a_1 \cdots a_n \in P$ which is a contradiction. Therefore there exists $1 \leqslant i \leqslant n$ such that $a_i \notin P$. Hence $P \in X_{a_i}$, as desired. □

Definition 3.10. A semiring R is called $*$*-semiring* if *co-rad*$(I) = \mathcal{T}(V(I))$ for each proper strong co-ideal I of R.

Example 3.11. (1) Let $R = (\mathbb{Z}^+, +, \times)$. Then the only strong co-ideals of R is $I_1 = \{n \in \mathbb{Z}^+ \mid 1 \leqslant n\}$ and $\mathbb{Z}^+$. Also the only prime strong co-ideals of R is I_1. Therefore, R is a $*$-semiring.

(2) Let Y={a,b,c} and $S = (P(Y), \cup, \cap)$ a semiring, where $P(Y)$ is the family of all subsets of Y. An inspection will show that S is a $*$-semiring.

(3) Let $T = (\mathbb{Z}^+ \cup \{\infty\}, \max, \min)$. An inspection will show that the list of strong co-ideals of T are T, $I_n = \{k \mid k \geqslant n\}$. It is clear that each proper strong co-ideal of T is prime and T is a $*$-semiring. □

The following example shows that a semiring need not be a $*$-semiring.

Example 3.12. Let $R = \{0,1,2,3,4,5\}$. Define

$$a+b = \begin{cases} 5 & \text{if } a \neq 0,\ b \neq 0,\ a \neq b, \\ a & \text{if } a = b, \\ b & \text{if } a = 0, \\ a & \text{if } b = 0, \end{cases}$$

and

$$a*b = \begin{cases} 0 & \text{if } a = 0 \text{ or } b = 0, \\ 2 & \text{if } a = b = 3, \\ b & \text{if } a = 1, \\ a & \text{if } b = 1, \\ 5 & \text{otherwise.} \end{cases}$$

Then $(R,+,*)$ is easily checked to be a commutative semiring. Suppose that $I = \{1,4,5\}$. It is clear that I is a strong co-ideal of R and $V(I) = \{P_1, P_2\}$, where

$$P_1 = \{1,2,4,5\}, \quad P_2 = \{1,2,3,4,5\}.$$

Hence $\mathcal{T}(V(I)) = P_1$. It can be seen $\mathcal{T}(V(I)) \neq$*co-rad*(I) because $2 \in \mathcal{T}(V(I))$ and $2 \notin$ *co-rad*(I). Therefore R is not $*$-semiring. □

Theorem 3.13. *Let R be a $*$-semiring. For every $a \in R$, the set X_a is compact. Specifically, the whole space $X_0 = X$ is compact.*

Proof. Assume that $X_a \subseteq \bigcup_{i\in\Gamma} X_{b_i}$ and let $I = F(\{b_i\}_{i\in\Gamma})$. We claim that $V(I) \subseteq V(\{a\})$. Assume that $P \in V(I)$, so $I \subseteq P$; hence $P \notin \bigcup_{i\in\Gamma} X_{b_i}$. Since $X_a \subseteq \bigcup_{i\in\Gamma} X_{b_i}$, $P \notin X_a$. This implies that $a \in P$. Therefore $V(I) \subseteq V(\{a\})$. It follows that $a \in \mathcal{T}(V(I))$. As R is $*$-semiring, $a \in$*co-rad*(I). Therefore $na \in I$ for some $n \in \mathbb{N}$. Hence $na = b_{i_1} \cdots b_{i_n} + r$ for some $b_{i_j} \in \{b_i\}_{i\in\Gamma}$, $r \in R$. We show that $X_a \subseteq \bigcup_{j=1}^{n} X_{b_{i_j}}$. Let $P \in X_a$ (so $a \notin P$). If for each $1 \leqslant j \leqslant n$, $b_{i_j} \in P$, then $na = b_{i_1} \cdots b_{i_n} + r \in P$, consequently $a \in P$, a contradiction. Therefore there exists $1 \leqslant j \leqslant n$ such that $b_{i_j} \notin P$. Hence $P \in \bigcup_{j=1}^{n} X_{b_{i_j}}$. Thus $X_a \subseteq \bigcup_{j=1}^{n} X_{b_{i_j}}$. □

Corollary 3.14. *Let R be a $*$-semiring. Then an open subset of $X =$ co-spec(R) is compact if and only if it is a finite union of basic open sets.*

Proof. Apply Theorem 3.5 and Theorem 3.13. □

Theorem 3.15. *Let R be a semiring. Then the toplologic space $X =$ co-spec(R) is a T_0-space.*

Proof. Let $P, Q \in X$ and $P \neq Q$. We note that the set X_a is a neighborhood of P if and only if $a \notin P$. Suppose that $Q \in X_b$ for all $b \notin P$. Then we conclude that $b \in Q$ implies that $b \in P$; hence $Q \subset P$. Now let $c \in P - Q$. Then $c \notin Q$ gives X_c is a neighborhood of Q, but $c \in P$, so $P \notin X_c$. This completes the proof. □

Definition 3.16. A semiring R is called *p-subtractive* if every prime strong co-ideal of R is subtractive.

Example 3.17. (1) Let Y={a,b,c} and $R = (P(Y), \cup, \cap)$ a semiring, where $P(Y)$= the set of all subsets of Y. An inspection will shows that *co-spec*$(R) = \{P_1, P_2, P_3\}$, where

$$P_1 = \{\{a\}, \{a,b\}, \{a,c\}, X\},$$
$$P_2 = \{\{b\}, \{a,b\}, \{b,c\}, X\},$$
$$P_3 = \{\{c\}, \{a,c\}, \{b,c\}, X\}.$$

Since P_1, P_2 and P_3 are maximal co-ideal, they are subtractive by Proposition 2.1. Hence R is a p-subtractive semiring.

(2) Let $S = (\mathbb{Z}^+, +, \times)$. Then $P = S - \{0\}$ is the only prime co-ideal of S which is subtractive. Hence S is a p-subtractive semiring. □

Theorem 3.18. *Let R be a p-subtractive semiring. If the only elements of R such that $a+b \in P$ and $ab \notin P$ for each $P \in$ co-spec(R) are $0, 1$, then $X =$ co-spec(R) is connected.*

Proof. Suppose that X is not connected. Let $X = X_a \cup X_b$ and $X_a \cap X_b = \emptyset$ for some $a, b \in R$. Since $X_a \cap X_b = \emptyset$, $X_{a+b} = \emptyset$ by Theorem 3.7. Thus $a + b \in P$ for all $P \in$ *co-spec*(R) by Theorem 3.7. We claim that $X_{ab} = X$. Let $P \in X$ and $ab \in P$. Since $X_{a+b} = \emptyset$, $a+b \in P$, therefore $a \in P$ or $b \in P$. As P is subtractive and $ab \in P$, $P \notin X_a \cup X_b$. This contradicts our hypothesis that $X = X_a \cup X_b$. Therefore $ab \notin P$ and $X_{ab} = X$. Hence $ab \notin P$ for all $P \in X$ by Theorem 3.7. Hence $\{a, b\} = \{0, 1\}$. Thus X is connected. □

Example 3.19. (1) Let Y={a,b,c} and $R = (P(Y), \cup, \cap)$ be a semiring, where $P(Y)$ is the collection of all subsets of Y. Then *co-spec*$(R) = X_{\{a\}} \bigcup X_{\{b,c\}}$ and $X_{\{a\}} \bigcap X_{\{b,c\}} = \emptyset$. Therefore *co-spec*$(R)$ is not connected.

(2) Let $T = (\mathbb{Z}^+ \cup \{\infty\}, \max, \min)$ and $I_i = \{n \in T \mid n \geqslant i\}$. It is clear that I_i is a prime strong co-ideal of T for each $i \in \mathbb{N}$. Then for each $n \in T$, $X_n = \{I_i \mid i \geqslant n+1\}$. Therefore $X_0 \supseteq X_1 \supseteq \cdots \supseteq X_\infty$. This implies that *co-spec*(T) is connected. □

Theorem 3.20. *Let R be a semiring. Then co-spec(R) is irreducible if and only if $\mathcal{T}(V(\{1\}))$ is a prime strong co-ideal.*

Proof. Let *co-spec*(R) be irreducible, and $a + b \in \mathcal{T}(V(\{1\}))$ for some $a, b \in R$. Then $X_{a+b} = X_a \cap X_b = \emptyset$ by Theorem 3.7. Since *co-spec*(R) is irreducible, $X_a = \emptyset$ or $X_b = \emptyset$. Thus $a \in \mathcal{T}(V(\{1\}))$ or $b \in \mathcal{T}(V(\{1\}))$. Therefore $\mathcal{T}(V(\{1\}))$ is prime.

Conversely, let $\mathcal{T}(V(\{1\}))$ be prime; we show that *co-spec*(R) is irreducible. If $X_a \cap X_b = \emptyset$, then by Theorem 3.7, $X_{a+b} = \emptyset$. Hence $a + b \in \mathcal{T}(V(\{1\}))$. As $\mathcal{T}(V(\{1\}))$ is prime, $a \in \mathcal{T}(V(\{1\}))$ or $b \in \mathcal{T}(V(\{1\}))$. Thus $X_a = \emptyset$ or $X_b = \emptyset$. Therefore, *co-spec*(R) is irreducible. □

Proposition 3.21. *Let R be a semiring and $P, Q \in X =$ co-spec(R). Then:*

(1) $\overline{\{P\}} = V(P)$ *for each* $P \in$ *co-spec*(R),

(2) $Q \in \overline{\{P\}}$ *if and only if* $P \subseteq Q$,

(3) $\{P\}$ *is closed in* X *if and only if* P *is a maximal co-ideal of* R.

Proof. (1). As $\overline{\{P\}} = \bigcap_{P \in V(I)} V(I)$, and $P \in V(P)$, we have $\overline{\{P\}} \subseteq V(P)$. On the other hand, if $Q \in V(P)$, then $P \subseteq Q$. Thus $Q \in V(I)$ for each $I \subseteq P$. Hence $Q \in \overline{\{P\}}$. Therefore $\overline{\{P\}} = V(P)$.

(2) is a consequence of (1), (3) is a consequence of (2). □

Theorem 3.22. *Let R be a semiring. Then X is a T_1-space if and only if each prime strong co-ideal is maximal.*

Proof. Let X be a T_1-space, then for each $P \in X$, $\{P\}$ is closed in X. Hence P is maximal strong co-ideal by Proposition 3.21. Conversely, assume that each prime strong co-ideal of R is maximal, then using Proposition 3.21 we see that each singleton $\{P\}$ is closed in X, for each $P \in X$. Hence X is a T_1-space. □

Let R be a semiring with $|$*co-spec*$(R)| \leqslant 1$. Then *co-spec*(R) is the trivial space and so it is a Hausdorff space. The following theorem gives a relation between Hausdorff axiom and T_1 axiom for Zariski-topology for co-ideals of semirings.

Theorem 3.23. *Let R be a semiring. If $X =$ co-spec(R) is a Hausdorff space, then it is a T_1-space.*

Proof. Let $P_1, P_2 \in X$. Since X is a Hausdorff space, there exist $a, b \in R$ such that $P_1 \in X_a$ and $P_2 \in X_b$ and $X_a \cap X_b = \emptyset$. Hence $X_{a+b} = \emptyset$. Therefore, $a + b \in P_1$ and $a + b \in P_2$. This implies that $a \in P_2$ and $b \in P_2$. Consequently, $P_1 \not\subseteq P_2$ and $P_2 \not\subseteq P_1$. Hence each prime strong co-ideal is maximal. Therefore, X is a T_1-space. □

It is well-known that if X is a finite space, then X is a T_1-space if and only if X is the discrete space. Thus we have the following Proposition.

Proposition 3.24. *For a semiring R with a finite $X =$ co-spec(R) the following conditions are equivalent:*

(1) *X is a Hausdorff space,*

(2) *X is a T_1-space,*

(3) *X has a cofinite topology,*

(4) *X is discrete,*

(5) *every prime co-ideal is maximal.* □

Lemma 3.25. *Let R be a semiring. Then for each $P \in$ co-spec(R), $V(P)$ is irreducible.*

Proof. Let $V(P) \subseteq Y_1 \cup Y_2$, where Y_1 and Y_2 are closed sets; so $P \in V(P)$ gives, $P \in Y_1$ or $P \in Y_2$. Let $P \in Y_1$. As $V(P) = \overline{\{P\}}$ by Proposition 3.21, we have $V(P) = \cap\{Y |\ P \in Y,\ Y is\ closed\ set\} \subseteq Y_1$. Similarly, if $P \in Y_2$, then $V(P) \subseteq Y_2$. Hence $V(P)$ is irreducible. □

Theorem 3.26. *Let R be a semiring. Then $Y \subseteq$ co-spec(R) is irreducible if and only if $\mathcal{T}(Y)$ is a prime strong co-ideal.*

Proof. Let Y be irreducible and $a+b \in \mathcal{T}(Y)$. We claim that $Y \subseteq V(\{a\}) \cup V(\{b\})$. Let $P \in Y$. Since $Y \subseteq V(\mathcal{T}(Y))$ and $a+b \in \mathcal{T}(Y)$, $a+b \in P$. Hence $a \in P$ or $b \in P$. Therefore $Y \subseteq V(\{a\}) \cup V(\{b\})$. As Y is irreducible, $Y \subseteq V(\{a\})$ or $Y \subseteq V(\{b\})$. If $Y \subseteq V(\{a\})$, then $a \in \mathcal{T}(Y)$. Similarly, If $Y \subseteq V(\{b\})$, then $b \in \mathcal{T}(Y)$. Hence $\mathcal{T}(Y)$ is prime. Conversely, assume that $\mathcal{T}(Y)$ is a prime strong co-ideal. We show that Y is irreducible. Let $Y \subseteq Y_1 \cup Y_2$ for some closed subset Y_1 and Y_2 of *co-spec*(R). Thus $Y_1 = V(I_1)$ and $Y_2 = V(I_2)$ for some strong co-ideals I_1 and I_2. As $Y \subseteq V(I_1) \cup V(I_2)$, for each $P \in Y$, $I_1 \subseteq P$ or $I_2 \subseteq P$. Hence $I_1 + I_2 \subseteq P$ for each $P \in Y$. Thus $I_1 + I_2 \subseteq \mathcal{T}(Y)$. Since $\mathcal{T}(Y)$ is prime $I_1 \subseteq \mathcal{T}(Y)$ or $I_2 \subseteq \mathcal{T}(Y)$ by Lemma 3.2. Therefore $Y \subseteq Y_1$ or $Y \subseteq Y_2$, as needed. □

Theorem 3.27. *For every $*$-semiring R, co-spec(R) is spectral.*

Proof. Let R be a $*$-semiring. We show that $X =$ *co-spec*(R) is spectral in four steps.

1. X is a T_0-space by Theorem 3.15.
2. X is quasi-compact by Theorem 3.9.
3. The quasi-compact open subsets of X are closed under finite intersection by Corollary 3.14.
4. Let Y be an irreducible closed subset of X. Then $Y = V(I)$ for some strong co-ideal I of R. By Theorem 3.26, $P = \mathcal{T}(Y)$ is a prime strong co-ideal of R. An inspection will show that $V(P) = Y$. Since $\overline{\{P\}} = V(P) = Y$, $\{P\}$ is a generic point of Y. Thus X is spectral. □

Corollary 3.28. *Let R be a $*$-semiring, then $X =$ co-spec(R) is a T_1-space if and only if it is a Hausdorff space.*

Proof. By Theorem 3.27, *co-spec*(R) is homeomorphic to *spec*(S), with the Zariski topology, for some commutative ring S. By [1], *spec*(S) is a Hausdorff space if and only if it is T_1. Therefore X is Hausdorff if and only if it is T_1. □

References

[1] **M. F. Atiyah and I. G. Macdonald**, *Introduction to commutative lgebra*, Addison Wesley Publishing Company, 1969.

[2] **M. Behboodi and M. R. Haddadi**, *Classical Zariski topology of modules and spectral spaces I*, Int. Electron. J. Algebra **4** (2008), 104 – 130.

[3] **N. Bourbaki**, *Commutative algebra*, Chap, 1.2, Hermann, Paris, 1961.

[4] **J. N. Chaudhari and K. J. Ingale**, *Prime avoidance theorem for co-ideals in semirings*, Research J. Pure Algebra **1** (2011), 213 – 216.

[5] **S. Ebrahimi Atani**, *The ideal theory in quotients of commutative semirings*, Glas. Math. **42** (2007), 301 – 308.

[6] **S. Ebrahimi Atani and R. Ebrahimi Atani**, *A Zariski topology for k-semirings*, Quasigroups and Related Systems **20** (2012), 29 – 36.

[7] **S. Ebrahimi Atani, S. Dolati Pish Hesari and M. Khoramdel**, *Strong co-ideal theory in quotients of semirings*, J. Adv. Research Pure Math. **5** (2013), 19 – 32.

[8] **J. S. Golan**, *Semirings and their applications*, Kluwer Academic Publisher Dordrecht, 1999.

[9] **M. Hochster**, *Prime ideal structure in commutative rings*, Trans. Amer. Math. Soc. **137** (1969), 43 – 60.

[10] **H. Wang**, *On rational series and rational language*, Theor. Comput. Sci. **205** (1998), 329 – 336.

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Shortest single axioms with neutral element for groups of exponent 2 and 3

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Abstract. In this note, we study identities in product and a constant e only that are valid in all groups of exponent 2 (3) with neutral element e and that imply that a groupoid satisfying one of them is a group of exponent 2 (3) with neutral element e. Such an identity will be called a single axiom with neutral element for groups of exponent 2 (3). We utilize the automated reasoning software Prover9 and Mace4 to attempt to find all shortest single axioms with neutral element for groups of exponent 2 (3). Beginning with a list of 1323 (1716) candidate identities that contains all shortest possible single axioms with neutral element for groups of exponent 2 (3), we find 173 (148) single axioms with neutral element for groups of exponent (2) 3 and eliminate all but 5 (119) of the remaining identities as not being single axioms with neutral element for groups of exponent 3. We also prove that a finite model of any of these 5 (119) identities must be a group of exponent 2 (3) with neutral element e.

1. Introduction

We assume the reader is familiar with the definitions of groupoids, semigroups, and groups. The variables v, w, x, y, and z will always be universally quantified. The letter e will always denote a constant and will denote the neutral element if in the context of a group. The letter n will always denote a natural number. We write x^{n+1} for xx^n, where $x^1 = x$. The capital letters S and T will always denote terms in product or in product and e unless otherwise stated and $T\backslash e$ will denote the corresponding term with all occurrences of e deleted. We denote by $V(T)$ the number of variable occurrences in T. Identities are always in product only or in product and e only.

Definition 1.1. A *group of exponent* n is a group such that $x^n = e$.

Strictly speaking, a group of exponent n is a group such that n is the *smallest* integer for which $x^n = e$. Therefore, our "groups of exponent n" are actually groups of exponent dividing n. Nevertheless, we will refer to any group satisfying the condition in Definition 1.1 as a group of exponent n.

Therefore, groups of exponent n can be axiomatized in terms of product only by

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(1) $xy \cdot z = x \cdot yz$,

(2) $x^n = y^n$, and

(3) $xy^n = x$

or in terms of product and e only by

(1′) $xy \cdot z = x \cdot yz$,

(2′) $x^n = e$, and

(3′) $xe = x$.

Definition 1.2. By an *identity* (*identity with neutral element*), we shall mean an identity in product only (identity in product and e only) unless otherwise stated. We say that an identity (identity with neutral element) $S = T$ is a *single axiom for groups of exponent* n (*single axiom with neutral element for groups of exponent* n) if and only if $S = T$ is true in all groups of exponent n (groups of exponent n with neutral element e) and every model of $S = T$ satisfies (1), (2), and (3) ((1′), (2′), and (3′)). In either case, it is clear that we must have S or T being just a single variable occurrence, otherwise the identity would be valid in any zero semigroup. We sometimes refer to identities and identities with neutral element simply as identities (Note that we do not assume that e is neutral, only that it is a constant. An identity must imply that e is neutral for it to be a single axiom.).

Neumann [12] proved that the variety of groups that satisfy $S = e$, where S is any term in product and inverse only, can be axiomatized by the single identity $T = x$, where T is the term in product and inverse only

$$w(((x^{-1} \cdot w^{-1}y)^{-1}z \cdot (xz)^{-1})(SS'^{-1})^{-1})^{-1}.$$

In the above, w, x, y, and z are variables not occurring in S and S' is a renaming of S using different variables. Taking $S = v^n$ and replacing all occurrences of -1 by $n-1$ in the above identity, we obtain a single axiom for groups of exponent n of the form $T = x$, where $V(T) = n^4 - 2n^2 + n + 1$. This leaves open the problem of finding shorter and simpler single axioms (with neutral element) for groups of exponent n. No variety of groups can be axiomatized by a single identity in product, inverse, and neutral element [12], [15].

For example, in [11], Meredith and Prior proved that

$$(yx \cdot z) \cdot yz = x$$

is a single axiom for groups of exponent 2 (*Boolean groups*), in [10], Mendelsohn and Padmanabhan proved that

$$x \cdot (xy \cdot z)y = z$$

and

$$(xy \cdot xz)y = z$$

are also single axioms for Boolean groups, and, in [9], McCune and Wos proved that

$$x((y \cdot ez) \cdot xz) = y$$

is a single axiom with neutral element for Boolean groups.

As another example, in [9], it is proved that

$$y \cdot (y \cdot y(x \cdot zz))z = x,$$

$$y(y(yx \cdot z) \cdot zz) = x,$$

and

$$y((yy \cdot xz)z \cdot z) = x$$

are single axioms for groups of exponent 3 and that

$$x(x(xy \cdot z) \cdot (e \cdot zz)) = y$$

is a single axiom with neutral element for groups of exponent 3.

As a final example, in [5], Kunen proved that

$$y(y(yy \cdot xz) \cdot (z \cdot zz)) = x,$$

$$(yy \cdot y) \cdot ((y \cdot xz) \cdot zz)z = x,$$

and

$$y(((yy \cdot y) \cdot xz)z) \cdot zz = x$$

are single axioms for groups of exponent 4 and, in [9], it is proved that

$$x(x(x(e(xy \cdot z) \cdot z) \cdot z) \cdot z) = y$$

along with nine others are single axioms with neutral element for groups of exponent 4.

In the present note, we endeavor to find all shortest (with respect to the number of variable and constant occurrences) single axioms with neutral element for groups of exponent 2 (3) using the automated theorem-prover Prover9 and the model-finder Mace4. Beginning with a list of 1323 (1716) candidate identities that contains all shortest possible single axioms with neutral element for groups of exponent 3, we find 173 (148) single axioms with neutral element for groups of exponent 2 (3) and eliminate all but 5 (119) of the remaining identities as not being single axioms with neutral element for groups of exponent 2 (3). We also prove that a finite model of any of these 5 (119) identities must be a group of exponent 2 (3) with neutral element e, hence obtaining the same type of classification as was achieved in [10] ([4]) for shortest single axioms (without neutral element) for groups of exponent 2 (3).

2. Preliminary Results

In this section, we present some preliminary results that will be needed in the subsequent sections. We begin with an obvious observation.

Definition 2.1. Define the *mirror* of T, denoted $M(T)$, as follows: $M(T'T'') = M(T'')M(T')$ for subterms T' and T'' of T, $M(x) = x$ for variables x occurring in T, and $M(e) = e$ for constants e occurring in T.

Theorem 2.2. *The identity* (*with neutral element*) $T = x$ *is a single axiom* (*with neutral element*) *for groups of exponent* n *if and only if* $M(T) = x$ *is a single axiom* (*with neutral element*) *for groups of exponent* n.

The next result demonstrates that the structure of the single axioms (with neutral element) for groups of exponent n presented in Section is no accident.

Theorem 2.3. [4], [5] *Suppose* $T = x$ *is a single axiom* (*with neutral element*) *for groups of exponent* n, $n \geq 2$. *Then* $V(T) \geq 2n + 1$. *If* $n = 2$ *and* $V(T) = 5$, *then a renaming of* $T\backslash e$ *is an association of an arrangement of* y^2xz^2 *If* $n \geq 3$ *and* $V(T) = 2n + 1$, *then a renaming of* $T\backslash e$ *is an association of* y^nxz^n. (*in the latter case, it is clear that we must not have* x *being the left-most* (*right-most*) *symbol in* T, *otherwise the identity would be valid in any left-zero* (*right-zero*) *semigroup*).

In light of Theorem 2.3, the single axioms (with neutral element) for groups of exponent n presented in Section are as short as possible. In the case of exponent 3, in [4], it is proved that the three examples from Section are the only shortest single axioms (up to renaming, mirroring, and symmetry), with the possible exceptions of

$$y \cdot y((y \cdot xz) \cdot zz) = x$$

and

$$yy \cdot (y(xz \cdot z) \cdot z) = x.$$

The status of these two identities is unknown. It is known that a non-group model of either identity must be infinite [4]. In the case of exponent 4, it is proved in [5] that the three examples from Section are the only shortest single axioms (up to mirroring, renaming, and symmetry). It is known that there are shortest possible single axioms (with neutral element) for groups of exponent n, n odd [4]. It is unknown if there are shortest possible single axioms (with neutral element) for groups of exponent n, $n \geq 6$ even. An exhaustive search for shortest possible single axioms for groups of exponent 6 was attempted in [3]. The search failed to find any single axioms but did reduce the number of candidates to 204.

We need two more results.

Theorem 2.4. [4], [5] *Suppose a renaming of* $T\backslash e$ *is an association of an arrangement of* y^2xz^2 *with* T *containing at most one occurrence of* e. *Then any associative and commutative model of* $T = x$ *is a group of exponent* 2 (*in particular, if all* *models of $T = x$ are associative and commutative, then $T = x$ is a single axiom (with neutral element) for groups of exponent 2). Suppose a renaming of $T\backslash e$ is an association of $y^n x z^n$, $n \geq 3$, with T containing at most one occurrence of e. Then any associative model of $T = x$ is a group of exponent n (in particular, if all models of $T = x$ are associative, then $T = x$ is a single axiom (with neutral element) for groups of exponent n).*

Theorem 2.5. [4] *If G is a non-trivial group, then there exists a non-associative groupoid H such that H satisfies every identity that contains at most two distinct variables and that is valid in G.*

3. Prover9 and Mace4

In this section, we briefly describe the software Prover9 and Mace4.

Prover9 [8] is a resolution-style [1], [13] automated theorem-prover for first-order logic with equality that was developed by McCune at Argonne National Laboratory. Prover9 is the successor to the well-known OTTER [7] theorem-prover and, like OTTER, utilizes the *set of support strategy* [1], [16].

The language of Prover9 is the language of *clauses*, a clause being a disjunction of (possible one or zero) literals in which all variables whose names begin with u, v, w, x, y, or z are implicitly universally quantified and all other symbols represent constants, functions, or predicates (relations). An axiom may also be given to Prover9 as an explicitly quantified first-order formula which is immediately transformed by Prover9 into a set of clauses by a *Skolemization* [1], [2] procedure. The conjunction of these clauses is not necessarily logically equivalent to the formula, but they will be *equisatisfiable* [1], [2]. Therefore, the set of clauses can be used by Prover9 in place of the formula in proofs by contradiction.

Prover9 can be asked to prove a potential theorem by giving it clauses or formulas expressing the hypotheses and a clause or formula expressing the negation of the conclusion. Prover9 finds a proof when it derives the *empty clause*, a contradiction.

Prover9 has an *autonomous mode* [8] in which all inference rules, settings, and parameters are automatically set based upon a syntactic analysis of the input clauses (the mechanisms of inference for purely equational problems being *demodulation* and *paramodulation* [1], [14]).

One very important parameter used by Prover9 is the *maximum weight* [8] of a clause. By default, the weight of a literal is the number of occurrences of constants, variables, functions, and predicates in the literal and the weight of a clause is the sum of the weights of its literals. Prover9 discards derived clauses whose weight exceeds the maximum weight specified. By specifying a maximum weight, we sacrifice *refutation-completeness* [1], [13], although in practice it is frequently necessary. We will use the autonomous mode throughout this paper, sometimes overriding Prover9's assignment to the maximum weight parameter.

A useful companion to Prover9 is Mace4 [6], also developed by McCune. Mace4 is a finite first-order model-finder. With possibly some minor modifications, the same input can be given to Mace4 as to Prover9, Prover9 searching for a proof by contradiction and Mace4 searching for counter-examples of specified sizes (a groupoid found by Mace4 would be returned as an $n \times n$ Cayley table with the elements of the structure assumed to be $0, 1, \ldots, n-1$ and the element in the ith row and jth column being ij).

Remark 3.1. The reader should note that Mace4 interprets non-negative integers as *distinct* constants and other constants as not necessarily distinct unless otherwise stated. This is in contrast to Prover9 which interprets all constants as not necessarily distinct unless otherwise stated. The use of non-negative integers for constants in Mace4 can have the advantage of speeding up the search for a model.

The scripting language Perl was also used to further automate the process.

4. The Search

In this section, we describe our search for shortest single axioms with neutral element for groups of exponent 3.

First, all identities $T = x$ such that T contains exactly one occurrence of e and $T\backslash e$ is an association of an arrangement of y^3xz^3 were generated up to renaming and mirroring. This resulted in 1716 identities.

We then sent the negation of each identity (stored in the Perl variable `$negated_identity`) to Prover9 and ran

```
assign(max_seconds, 1).  % one second time limit per identity
formulas(sos).           % set of support clauses
e * x = x.
x * e = x.
x * y = y * x.
(x * x) * x = e.
x * (x * x) = e.
(x * x) * (x * y) = y.
x * (x * (x * y)) = y.
((x * y) * y) * y = x.
(x * y) * (y * y) = x.   % one and two distinct variable identities
                         % valid in Z_3
$negated_identity.       % negation of candidate identity
end_of_list.             % end of set of support clauses
```

to search for a proof that the identity is derivable from the set of one and two distinct variable identities that are valid in $\mathbb{Z}_3$. If this is the case, then by Theorem 2.5, there is a non-associative model for the identity and it can be eliminated. This reduced the number of candidate identities to 546.

Remark 4.1. We determine whether or not Prover9 has found a proof by observing its exit status. Prover9 outputs an exit code of 0 if and only if it finds a proof.

Next, we sent each identity (stored in the variable `$identity`) to Mace4 (`e` will always be renamed `0` in Mace4 input) and ran

```
assign(max_seconds, 60).                % one minute time limit
                                        % per identity
formulas(theory).                       % theory clauses
x * y != x * z | y = z.                 % left cancellative
y * x != z * x | y = z.                 % right cancellative
(x * y) * (z * u) = (x * z) * (y * u).  % medial
0 * 0 = 0.                              % e idempotent
$identity.                              % candidate identity
(a * b) * c != a * (b * c).             % non-associative
end_of_list.                            % end of theory clauses
```

to search for a non-associative, *cancellative* (*left cancellative* ($xy = xz$ implies $y = z$) and *right cancellative* ($yx = zx$ implies $y = z$)), *medial* ($xy \cdot zu = xz \cdot yu$) groupoid with e *idempotent* ($ee = e$) that satisfies the identity. Any identity for which an example was found was eliminated. This reduced the number of candidate identities to 267.

Remark 4.2. We determine whether or not Mace4 has found a model by observing its exit status. Mace4 outputs an exit code of 0 if and only if it finds a model.

We then sent each remaining identity to Prover9 and ran

```
assign(max_seconds, 300).          % five minute time limit
                                   % per weight per identity
assign(max_weight, $max_weight).   % maximum clause weight
formulas(sos).                     % set of support clauses
$identity.                         % candidate identity
a * b = a * c.
b != c.                            % not left cancellative
end_of_list.                       % end of set of support clauses
```

to search for a proof that the identity implies left cancellativity. We made a run for every value of `$max_weight` from 20 to 150 in steps of 5. A proof was found for 186 identities. The mirror of each identity for which a proof was not found was then sent back to Prover9 to search for a proof that it implies left cancellativity. An additional 57 proofs were found.

Next, we sent each of these 243 identities back to Prover9 and ran

```
assign(max_seconds, 300).          % five minute time limit
                                   % per weight per identity
assign(max_weight, $max_weight).   % maximum clause weight
formulas(sos).                      % set of support clauses
```

```
$identity.                          % candidate identity
x * y != x * z | y = z.             % left cancellative
b * a = c * a.
b != c.                             % not right cancellative
end_of_list.                        % end of set of support clauses
```

to search for a proof that the identity implies right cancellativity. We made a run for every value of `$max_weight` from 20 to 150 in steps of 5. A proof was found for 148 identities.

We then sent each of these 148 identities back to Prover9 and ran

```
assign(max_seconds, 600).           % 10 minute time limit
                                    % per weight per identity
assign(max_weight, $max_weight).    % maximum clause weight
formulas(sos).                       % set of support clauses
$identity.                          % candidate identity
x * y != x * z | y = z.             % left cancellative
y * x != z * x | y = z.             % right cancellative
e * e != e.                         % e not idempotent
end_of_list.                        % end of set of support clauses
```

to search for a proof that the identity implies that e is idempotent. We made a run for every value of `$max_weight` from 18 to 150 in steps of 2. A proof was found for all 148 identities.

Finally, we sent each of these 148 identities back to Prover9 and ran

```
assign(max_seconds, 600).           % 10 minute time limit
                                    % per weight per identity
assign(max_weight, $max_weight).    % maximum clause weight
formulas(sos).                       % set of support clauses
$identity.                          % candidate identity
x * y != x * z | y = z.             % left cancellative
y * x != z * x | y = z.             % right cancellative
e * e = e.                          % e idempotent
(a * b) * c != a * (b * c).         % non-associative
end_of_list.                        % end of set of support clauses
```

to search for a proof that the identity implies associativity. We made a run for every value of `$max_weight` from 18 to 150 in steps of 2. A proof was found for all 148 identities. By Theorem 2.4, these 148 identities are all single axioms with neutral element for groups of exponent 3.

5. Finite Models

In this section, we show that a finite model of any of the 119 remaining candidate identities must be a group of exponent 3 with neutral element e.

Consider the following identity (one of the 119 remaining candidate identities).

$$(y \cdot ey)(y(x \cdot zz) \cdot z) = x$$

Let G be a finite groupoid satisfying this identity. Define $L_x, R_x : G \longrightarrow G$ by $L_x(y) = xy$ and $R_x(y) = yx$. Therefore,

$$L_{y \cdot ey} \circ R_z \circ L_y \circ R_{zz} = Id$$

where Id is the identity mapping on G. Thus, R_{zz} is injective and $L_{y \cdot ey}$ is surjective. Since G is finite, R_{zz} is surjective and $L_{y \cdot ey}$ is injective.

Running the third block of code in Section for every value of `$max_weight` from 20 to 150 in steps of 5 with this candidate identity and with the additional lines

```
R(z,u) = u * (z * z).          % R_zz definition
L(y,u) = (y * (e * y)) * u.   % L_y(ey) definition
R(z,f(z,u)) = u.               % R_zz surjective
L(y,u) != L(y,v) | u = v.     % L_y(ey) injective
```

in the set of support, Prover9 finds a proof that this identity implies that G is left cancellative. Running the fourth block of code in Section for every value of `$max_weight` from 20 to 150 in steps of 5 with this candidate identity and with these additional lines, Prover9 finds a proof that this identity implies that G is right cancellative. Running the fifth block of code in Section for every value of `$max_weight` from 18 to 150 in steps of 2 with this candidate identity and with these additional lines, Prover9 finds a proof that this identity implies that e is idempotent in G. Running the sixth block of code in Section for every value of `$max_weight` from 18 to 150 in steps of 2 with this candidate identity and with these additional lines, Prover9 finds a proof that this identity implies that G is associative. By Theorem 2.4, G must be a group of exponent 3 with neutral element e.

The above procedure was automated and carried out for each of the 119 remaining candidate identities and each one was shown to imply that a finite model of it must be a group of exponent 3 with neutral element e. Therefore, if any one of these 119 identities fails to be a single axiom with neutral element for groups of exponent 3 (the authors feel that it is likely that most if not all of them fail to be), then it can only be eliminated from contention through the construction of an *infinite* non-associative model.

6. Conclusion

In this section, we summarize our results.

Theorem 6.1. *The following* 148 *identities with neutral element* (*and their mirrors*) *are single axioms with neutral element for groups of exponent* 3.

$$\begin{gathered}
ey \cdot ((yy \cdot xz)z \cdot z) = x \\
(yy \cdot ((ye \cdot x)z \cdot z))z = x \\
(ey \cdot ((yy \cdot x)z \cdot z))z = x \\
(yy \cdot (ey \cdot xz)z)z = x \\
yy \cdot ((ye \cdot xz)z \cdot z) = x \\
y(y \cdot (y \cdot (e \cdot xz)z)z) = x \\
(ey \cdot (yy \cdot x)z) \cdot zz = x \\
(ye \cdot ((yy \cdot x) \cdot zz))z = x \\
(y \cdot (y \cdot y(ex \cdot z))z)z = x \\
((yy \cdot e)(yx \cdot z) \cdot z)z = x \\
y(e(y \cdot (yx \cdot z)z) \cdot z) = x \\
(y \cdot y((e \cdot yx)z \cdot z))z = x \\
((e \cdot yy) \cdot (y \cdot xz)z)z = x \\
(y \cdot e(y(y \cdot xz) \cdot z))z = x \\
y(y \cdot (y(e \cdot xz) \cdot z)z) = x \\
(y \cdot ((y \cdot ye) \cdot xz)z)z = x \\
y(((e \cdot yy) \cdot xz)z \cdot z) = x \\
e((y \cdot y(yx \cdot z))z \cdot z) = x \\
(y \cdot ((e \cdot yy)x \cdot z)z)z = x \\
y((y \cdot y(ex \cdot z))z \cdot z) = x \\
(y \cdot ((ye \cdot y) \cdot xz)z)z = x \\
(yy \cdot e)((y \cdot xz)z \cdot z) = x \\
y((y \cdot y(e \cdot xz)) \cdot zz) = x \\
y(y \cdot e(yx \cdot z)) \cdot zz = x \\
y(ey \cdot (yx \cdot zz)) \cdot z = x \\
(y \cdot y((ye \cdot x) \cdot zz))z = x \\
y(e(yy \cdot xz) \cdot zz) = x \\
e(yy \cdot (yx \cdot z)) \cdot zz = x \\
y \cdot (ey \cdot (yx \cdot zz))z = x \\
(yy \cdot e)(y(x \cdot zz) \cdot z) = x \\
(ey \cdot y)(yx \cdot zz) \cdot z = x \\
y((ey \cdot y)x \cdot z) \cdot zz = x \\
(yy \cdot e)((y \cdot xz) \cdot zz) = x \\
y \cdot (yy \cdot e(x \cdot zz))z = x \\
yy \cdot (e \cdot y(x \cdot zz))z = x \\
y((ye \cdot y)(x \cdot zz) \cdot z) = x \\
(y \cdot y((ey \cdot x) \cdot zz))z = x \\
y(e \cdot (yy \cdot x)z) \cdot zz = x \\
y((y \cdot e(yx \cdot z)) \cdot zz) = x \\
e((y \cdot (y \cdot yx)z) \cdot zz) = x \\
ey \cdot (y \cdot y(x \cdot zz))z = x \\
(y \cdot ye)(yx \cdot z) \cdot zz = x \\
y(y((e \cdot yx)z \cdot z) \cdot z) = x \\
y(y \cdot (y \cdot (ex \cdot z)z)z) = x
\end{gathered}$$

$$\begin{gathered}
y(e(y(yx \cdot z) \cdot z) \cdot z) = x \\
e(y(y \cdot (y \cdot xz)z) \cdot z) = x \\
y(e(y(y \cdot xz) \cdot z) \cdot z) = x \\
((yy \cdot e) \cdot (y \cdot xz)z)z = x \\
y(((yy \cdot e) \cdot xz)z \cdot z) = x \\
(e \cdot (y \cdot y(yx \cdot z))z)z = x \\
(ey \cdot ((yy \cdot x) \cdot zz))z = x \\
(yy \cdot (ye \cdot x)z) \cdot zz = x \\
(yy \cdot ((ye \cdot x) \cdot zz))z = x \\
e(y(y \cdot (yx \cdot z)z) \cdot z) = x \\
(y \cdot e((y \cdot yx)z \cdot z))z = x \\
e(y \cdot (y(y \cdot xz) \cdot z)z) = x \\
((ey \cdot y) \cdot (y \cdot xz)z)z = x \\
((y \cdot ey) \cdot (y \cdot xz)z)z = x \\
(y \cdot y(y(e \cdot xz) \cdot z))z = x \\
(y \cdot ye)((y \cdot xz)z \cdot z) = x \\
y(((ey \cdot y) \cdot xz)z \cdot z) = x \\
y((y \cdot e(yx \cdot z))z \cdot z) = x \\
(y \cdot ((ey \cdot y)x \cdot z)z)z = x \\
y(y \cdot (e(y \cdot xz) \cdot z)z) = x \\
(y \cdot ((yy \cdot e) \cdot xz)z)z = x \\
((ye \cdot y) \cdot (yx \cdot z)z)z = x \\
ey \cdot (y(yx \cdot z) \cdot zz) = x \\
y(y(ey \cdot (x \cdot zz)) \cdot z) = x \\
y((ye \cdot (yx \cdot z)) \cdot zz) = x \\
(ye \cdot (y \cdot yx)z) \cdot zz = x \\
(ey \cdot y)((y \cdot xz) \cdot zz) = x \\
yy \cdot (y(ex \cdot zz) \cdot z) = x \\
ye \cdot (y(yx \cdot z) \cdot zz) = x \\
y(y \cdot (ey \cdot x)z) \cdot zz = x \\
(yy \cdot e)(yx \cdot z) \cdot zz = x \\
y((yy \cdot e)x \cdot zz) \cdot z = x \\
(ye \cdot y)((y \cdot xz) \cdot zz) = x \\
y((y \cdot (ye \cdot x)z) \cdot zz) = x \\
(ey \cdot y)(y(x \cdot zz) \cdot z) = x \\
(ye \cdot y(yx \cdot zz))z = x \\
e((y \cdot (yy \cdot x)z) \cdot zz) = x \\
e((y \cdot y(yx \cdot z)) \cdot zz) = x \\
y \cdot e(y(yx \cdot z) \cdot zz) = x \\
y(y \cdot e(yx \cdot zz)) \cdot z = x \\
y \cdot (y \cdot e(yx \cdot zz))z = x \\
y((e \cdot y(y \cdot xz))z \cdot z) = x \\
(y \cdot e(y \cdot (y \cdot xz)z))z = x \\
y((y \cdot e(y \cdot xz))z \cdot z) = x
\end{gathered}$$

$$\begin{gathered}
y(y(e(yx \cdot z) \cdot z) \cdot z) = x \\
e(y(y(y \cdot xz) \cdot z) \cdot z) = x \\
(ye \cdot (yy \cdot xz)z)z = x \\
y(y(y \cdot (e \cdot xz)z) \cdot z) = x \\
y(y \cdot (e \cdot (y \cdot xz)z)z) = x \\
(y \cdot y(e(yx \cdot z) \cdot z))z = x \\
(ye \cdot (yy \cdot x)z) \cdot zz = x \\
(y \cdot y(y(ex \cdot z) \cdot z))z = x \\
(y \cdot ((yy \cdot e)x \cdot z)z)z = x \\
(e \cdot y((y \cdot yx)z \cdot z))z = x \\
y(y(e \cdot (yx \cdot z)z) \cdot z) = x \\
(e \cdot y(y(y \cdot xz) \cdot z))z = x \\
y(e \cdot (y(y \cdot xz) \cdot z)z) = x \\
(y \cdot y(e(y \cdot xz) \cdot z))z = x \\
(y \cdot y((y \cdot ex)z \cdot z))z = x \\
y(y(y \cdot (ex \cdot z)z) \cdot z) = x \\
y(((y \cdot ey) \cdot xz)z \cdot z) = x \\
((y \cdot ye) \cdot (yx \cdot z)z)z = x \\
(y \cdot ((y \cdot ey)x \cdot z)z)z = x \\
y((e \cdot y(yx \cdot z))z \cdot z) = x \\
(ye \cdot y)((y \cdot xz)z \cdot z) = x \\
((yy \cdot e) \cdot (yx \cdot z)z)z = x \\
y \cdot e(y(yx \cdot zz) \cdot z) = x \\
e(y((y \cdot yx) \cdot zz) \cdot z) = x \\
ye \cdot (y \cdot y(x \cdot zz))z = x \\
ey \cdot (y(y \cdot xz) \cdot zz) = x \\
y(((yy \cdot e) \cdot xz) \cdot zz) = x \\
y(y(ye \cdot (x \cdot zz)) \cdot z) = x \\
y(ye \cdot (yx \cdot zz)) \cdot z = x \\
y((ey \cdot y)(x \cdot zz) \cdot z) = x \\
(e \cdot yy)(yx \cdot zz) \cdot z = x \\
y(((ey \cdot y) \cdot xz) \cdot zz) = x \\
y \cdot (ye \cdot y(x \cdot zz))z = x \\
y((yy \cdot (ex \cdot z)) \cdot zz) = x \\
y((yy \cdot e)(x \cdot zz) \cdot z) = x \\
(yy \cdot e(yx \cdot zz))z = x \\
e \cdot (yy \cdot (yx \cdot zz))z = x \\
e \cdot (y \cdot y(yx \cdot zz))z = x \\
y(y(y \cdot e(x \cdot zz)) \cdot z) = x \\
y((ey \cdot (yx \cdot z)) \cdot zz) = x \\
y((y \cdot ye)x \cdot zz) \cdot z = x \\
(e \cdot y(y \cdot (y \cdot xz)z))z = x \\
e(y \cdot (y \cdot (yx \cdot z)z)z) = x \\
(y \cdot y(e \cdot (y \cdot xz)z))z = x
\end{gathered}$$

$y(y((y\cdot ex)z\cdot z)\cdot z)=x$	$e(y\cdot(y(yx\cdot z)\cdot z)z)=x$	$y((y\cdot y(e\cdot xz))z\cdot z)=x$
$(y\cdot(e\cdot y(y\cdot xz))z)z=x$	$(e\cdot y(y\cdot(yx\cdot z)z))z=x$	$y(e\cdot(y\cdot(yx\cdot z)z)z)=x$
$y(y\cdot(y(ex\cdot z)\cdot z)z)=x$	$e(y((y\cdot yx)z\cdot z)\cdot z)=x$	$y((y\cdot(e\cdot yx)z)z\cdot z)=x$
$(y\cdot(y\cdot e(y\cdot xz))z)z=x$	$(y\cdot e(y\cdot(yx\cdot z)z))z=x$	$y((y\cdot(y\cdot ex)z)z\cdot z)=x$
$y(e\cdot(y(yx\cdot z)\cdot z)z)=x$	$(y\cdot y(e\cdot(yx\cdot z)z))z=x$	$e((y\cdot(y\cdot yx)z)z\cdot z)=x$
$(y\cdot(y\cdot y(e\cdot xz))z)z=x$		

A finite model of any of the following 119 *identities with neutral element* (*or their mirrors*) *is a group of exponent* 3 *with neutral element* e.

$y\cdot y(y(x\cdot zz)\cdot ez)=x$	$yy\cdot(y(x\cdot ze)\cdot zz)=x$	$y((yy\cdot(x\cdot ze))\cdot zz)=x$
$y\cdot y(y(x\cdot ze)\cdot zz)=x$	$yy\cdot(y\cdot(x\cdot ez)z)z=x$	$y\cdot(yy\cdot(xe\cdot z)z)z=x$
$y(y\cdot(y(xe\cdot z)\cdot z)z)=x$	$y(y\cdot(y\cdot(xe\cdot z)z)z)=x$	$y(y\cdot(y\cdot x(ez\cdot z))z)=x$
$y(y(y(xz\cdot z)\cdot z)\cdot e)=x$	$y(y\cdot(y(xz\cdot z)\cdot z)e)=x$	$y(y(y(xz\cdot z)\cdot e)\cdot z)=x$
$y(y\cdot(y(xz\cdot z)\cdot e)z)=x$	$y(y\cdot(y(xz\cdot e)\cdot z)z)=x$	$ey\cdot((yy\cdot xz)\cdot zz)=x$
$y\cdot e(y(y\cdot xz)\cdot zz)=x$	$ye\cdot((yy\cdot xz)\cdot zz)=x$	$ye\cdot(yy\cdot(xz\cdot z))z=x$
$y(ye\cdot((y\cdot xz)\cdot zz))=x$	$y((ye\cdot y)(xz\cdot z)\cdot z)=x$	$yy\cdot((ey\cdot xz)\cdot zz)=x$
$(yy\cdot e)(y(xz\cdot z)\cdot z)=x$	$y\cdot y(y(e\cdot xz)\cdot zz)=x$	$y((y\cdot ye)(xz\cdot z)\cdot z)=x$
$yy\cdot((ye\cdot xz)\cdot zz)=x$	$ey\cdot(yy\cdot(x\cdot zz))z=x$	$ye\cdot(yy\cdot(x\cdot zz))z=x$
$yy\cdot(ey\cdot(x\cdot zz))z=x$	$y(y\cdot(y\cdot e(x\cdot zz))z)=x$	$yy\cdot(ye\cdot(x\cdot zz))z=x$
$yy\cdot(ey\cdot(xz\cdot z))z=x$	$y\cdot(yy\cdot(x\cdot zz)e)z=x$	$e(yy\cdot(y(x\cdot zz)\cdot z))=x$
$yy\cdot(y\cdot(x\cdot ze)z)z=x$	$y(y\cdot(y\cdot x(z\cdot ez))z)=x$	$y(y\cdot(y\cdot x(e\cdot zz))z)=x$
$y(y\cdot(y\cdot xz)(zz\cdot e))=x$	$y(y\cdot(y\cdot xz)(z\cdot ze))=x$	$y(y\cdot(y\cdot xz)(ze\cdot z))=x$
$e(y\cdot y(y(xz\cdot z)\cdot z))=x$	$(e\cdot yy)(y(xz\cdot z)\cdot z)=x$	$(ey\cdot y)(y(xz\cdot z)\cdot z)=x$
$y(e\cdot y(y(xz\cdot z)\cdot z))=x$	$y(e\cdot(y\cdot y(xz\cdot z))z)=x$	$ye\cdot y((y\cdot xz)\cdot zz)=x$
$(y\cdot ey)(y(xz\cdot z)\cdot z)=x$	$y(y\cdot e((y\cdot xz)\cdot zz))=x$	$y(y\cdot(e\cdot y(xz\cdot z))z)=x$
$y\cdot y((ye\cdot xz)\cdot zz)=x$	$y(y\cdot(y\cdot e(xz\cdot z))z)=x$	$y((yy\cdot e)(xz\cdot z)\cdot z)=x$
$e(y\cdot(y\cdot y(xz\cdot z))z)=x$	$y\cdot(yy\cdot x(zz\cdot e))z=x$	$y(ey\cdot((y\cdot xz)\cdot zz))=x$
$yy\cdot e((y\cdot xz)\cdot zz)=x$	$yy\cdot(y\cdot x(z\cdot ez))z=x$	$y\cdot(yy\cdot x(z\cdot ez))z=x$
$y((e\cdot yy)(x\cdot zz)\cdot z)=x$	$yy\cdot(y\cdot xz)(zz\cdot e)=x$	$yy\cdot(y\cdot xz)(z\cdot ze)=x$
$y\cdot(yy\cdot xz)(z\cdot ez)=x$	$y(((e\cdot yy)\cdot xz)\cdot zz)=x$	$y\cdot y((ey\cdot xz)\cdot zz)=x$
$yy\cdot(y(e\cdot xz)\cdot zz)=x$	$e\cdot y((yy\cdot xz)\cdot zz)=x$	$y(y\cdot e(y(x\cdot zz)\cdot z))=x$
$y(y\cdot(ey\cdot(x\cdot zz))z)=x$	$(ye\cdot y)(y(x\cdot zz)\cdot z)=x$	$yy\cdot(y(xe\cdot zz)\cdot z)=x$
$y((yy\cdot(xz\cdot z))\cdot ze)=x$	$yy\cdot(y(xz\cdot ez)\cdot z)=x$	$yy\cdot(y(xz\cdot e)\cdot zz)=x$
$(e\cdot yy)((y\cdot xz)\cdot zz)=x$	$yy\cdot(y\cdot x(zz\cdot e))z=x$	$y\cdot(yy\cdot x(z\cdot ze))z=x$
$yy\cdot(y\cdot x(e\cdot zz))z=x$	$(e\cdot yy)(y(x\cdot zz)\cdot z)=x$	$yy\cdot(y(xz\cdot z)\cdot ze)=x$
$yy\cdot(y\cdot xz)(z\cdot ez)=x$	$y\cdot(yy\cdot xz)(e\cdot zz)=x$	$y(((ye\cdot y)\cdot xz)\cdot zz)=x$
$y((yy\cdot(x\cdot zz))z\cdot e)=x$	$y(e\cdot(yy\cdot(x\cdot zz))z)=x$	$y((yy\cdot(xe\cdot z))\cdot zz)=x$
$yy\cdot(y(xe\cdot z)\cdot z)z=x$	$y((yy\cdot(xe\cdot z))z\cdot z)=x$	$yy\cdot(y(xz\cdot z)\cdot e)z=x$
$y(e\cdot(y\cdot y(x\cdot zz))z)=x$	$yy\cdot(y\cdot(xe\cdot z)z)z=x$	$y(y\cdot((y\cdot xz)z\cdot z)e)=x$
$y(y(y\cdot(xz\cdot z)e)\cdot z)=x$	$y(y\cdot((y\cdot xz)z\cdot e)z)=x$	$y(y\cdot(y\cdot(xz\cdot z)e)z)=x$
$y(y(y\cdot(xz\cdot e)z)\cdot z)=x$	$y(y\cdot((y\cdot xz)e\cdot z)z)=x$	$y(y\cdot(y\cdot(xz\cdot e)z)z)=x$
$e(y\cdot y((y\cdot xz)z\cdot z))=x$	$y(e\cdot y((y\cdot xz)z\cdot z))=x$	$y(e(y\cdot y(xz\cdot z))\cdot z)=x$
$y(y\cdot e(y(xz\cdot z)\cdot z))=x$	$y(y\cdot e((y\cdot xz)z\cdot z))=x$	$y(y(e\cdot y(xz\cdot z))\cdot z)=x$
$y(y(y\cdot e(xz\cdot z))\cdot z)=x$	$y((yy\cdot(xz\cdot z))e\cdot z)=x$	$y(((y\cdot ye)\cdot xz)\cdot zz)=x$

$$
\begin{array}{rrr}
(y \cdot ye)(y(x \cdot zz) \cdot z) = x & yy \cdot (y \cdot x(ez \cdot z))z = x & y((y \cdot ye)(x \cdot zz) \cdot z) = x \\
(y \cdot ye)((y \cdot xz) \cdot zz) = x & y \cdot (yy \cdot xz)(ez \cdot z) = x & (y \cdot ey)((y \cdot xz) \cdot zz) = x \\
y \cdot (yy \cdot x(ze \cdot z))z = x & y((y \cdot ey)(x \cdot zz) \cdot z) = x & y(((y \cdot ey) \cdot xz) \cdot zz) = x \\
yy \cdot (y \cdot x(ze \cdot z))z = x & y \cdot (yy \cdot xz)(ze \cdot z) = x & (y \cdot ey)(y(x \cdot zz) \cdot z) = x \\
yy \cdot (y \cdot xz)(ze \cdot z) = x & y(y(y \cdot (xe \cdot z)z) \cdot z) = x &
\end{array}
$$

Any additional single axioms with neutral element for groups of exponent 3 *are among these* 119 *identities with neutral element* (*up to renaming, mirroring, and symmetry*).

A similar search for shortest single axioms with neutral element for Boolean groups was also carried out with the following results.

Theorem 6.2. *The following* 173 *identities with neutral element* (*and their mirrors*) *are single axioms with neutral element for Boolean groups.*

$$
\begin{array}{rrr}
e(x(xy \cdot z) \cdot y) = x & ex \cdot (xy \cdot zy) = x & e(x \cdot y(x \cdot zy)) = x \\
e(xy \cdot (x \cdot zy)) = x & ex \cdot y(x \cdot zy) = x & (e \cdot xy)(y \cdot zx) = x \\
e((x \cdot yz) \cdot xy) = x & ex \cdot y(z \cdot xy) = x & e(x \cdot y(z \cdot yx)) = x \\
e((x \cdot yz)y \cdot x) = x & ex \cdot y(z \cdot yx) = x & e(x \cdot yz) \cdot yx = x \\
(e \cdot (x \cdot yz)z)x = x & x(ey \cdot (x \cdot zy)) = x & xe \cdot y(x \cdot zy) = x \\
xe \cdot (yx \cdot z)y = x & (xe \cdot y)(x \cdot zy) = x & (x \cdot (ey \cdot x)z)y = x \\
(x \cdot (ey \cdot x)z)z = x & (xe \cdot (yx \cdot z))z = x & xe \cdot y(yz \cdot x) = x \\
x(ey \cdot (z \cdot xy)) = x & x((e \cdot yz) \cdot xy) = x & x(ey \cdot z) \cdot xy = x \\
x((e \cdot yz)x \cdot z) = x & xe \cdot (y \cdot zx)z = x & xe \cdot (yz \cdot x)z = x \\
((xe \cdot y) \cdot zx)z = x & ((x \cdot ey)z \cdot x)z = x & ((xe \cdot y)z \cdot x)z = x \\
x(e \cdot y(zy \cdot x)) = x & x((ey \cdot z) \cdot yx) = x & (xe \cdot y)(zy \cdot x) = x \\
x(ey \cdot z) \cdot yx = x & (x(e \cdot yz) \cdot y)x = x & ((xe \cdot y)z \cdot y)x = x \\
(x(e \cdot yz) \cdot z)x = x & (xe \cdot yz)z \cdot x = x & x(x(ey \cdot z) \cdot y) = x \\
x \cdot (xe \cdot yz)y = x & x(x(y \cdot ez) \cdot y) = x & x(y \cdot e(x \cdot zy)) = x \\
x(ye \cdot (x \cdot zy)) = x & xy \cdot (ex \cdot z)y = x & x(y \cdot (e \cdot yz)x) = x \\
x(ye \cdot (yz \cdot x)) = x & xy \cdot (ey \cdot z)x = x & (x \cdot ye)(y \cdot zx) = x \\
(xy \cdot (e \cdot yz))x = x & ((x \cdot ye) \cdot yz)x = x & x(y \cdot e(z \cdot xy)) = x \\
(x \cdot ye)(zx \cdot y) = x & x(y \cdot ez) \cdot xy = x & ((x \cdot ye)z \cdot x)y = x \\
(xy \cdot ez) \cdot xz = x & (x \cdot ye)z \cdot xz = x & (xy \cdot ez)x \cdot z = x \\
x(y \cdot ez) \cdot yx = x & ((x \cdot ye) \cdot zy)x = x & x((y \cdot ez) \cdot zx) = x \\
(xy \cdot ez) \cdot zx = x & (x(y \cdot ez) \cdot z)x = x & e(x \cdot (x \cdot yz)y) = x \\
(ex \cdot (xy \cdot z))y = x & e(x \cdot (yx \cdot z)y) = x & e(xy \cdot (xz \cdot y)) = x \\
ex \cdot (yx \cdot zy) = x & (e \cdot xy)(x \cdot zy) = x & (e \cdot x(y \cdot xz))y = x \\
e(xy \cdot xz) \cdot y = x & (ex \cdot (yx \cdot z))y = x & ((e \cdot xy) \cdot xz)y = x \\
(ex \cdot (y \cdot xz))z = x & (ex \cdot (yx \cdot z))z = x & e(xy \cdot (yz \cdot x)) = x \\
ex \cdot y(yz \cdot x) = x & (ex \cdot y)(yz \cdot x) = x & ((e \cdot xy) \cdot yz)z = x \\
e(x \cdot (yz \cdot x)y) = x & ex \cdot (yz \cdot xy) = x & (e \cdot x(y \cdot zx))y = x \\
e((xy \cdot z)x \cdot z) = x & ex \cdot (yz \cdot xz) = x & ex \cdot (yz \cdot x)z = x \\
(e(xy \cdot z) \cdot x)z = x & e(x \cdot y(zy \cdot x)) = x & e((x \cdot yz) \cdot yx) = x
\end{array}
$$

$$ex \cdot (yz \cdot yx) = x$$
$$e(xy \cdot z) \cdot yz = x$$
$$(e(x \cdot yz) \cdot y)z = x$$
$$e(xy \cdot z) \cdot zy = x$$
$$x((e \cdot yx)z \cdot y) = x$$
$$x(ey \cdot xz) \cdot y = x$$
$$xe \cdot (y \cdot xz)z = x$$
$$x(e \cdot y(yz \cdot x)) = x$$
$$x((ey \cdot z) \cdot xy) = x$$
$$((xe \cdot y) \cdot zx)y = x$$
$$x \cdot e(yz \cdot yx) = x$$
$$x((x \cdot ey) \cdot zy) = x$$
$$x(xy \cdot (e \cdot zy)) = x$$
$$x((xy \cdot e)z \cdot y) = x$$
$$xy \cdot e(xz \cdot y) = x$$
$$x(ye \cdot xz) \cdot y = x$$
$$x(y \cdot (ey \cdot z)x) = x$$
$$(xy \cdot e)(y \cdot zx) = x$$
$$(x \cdot y(e \cdot zx))y = x$$
$$x((y \cdot ez)x \cdot z) = x$$
$$x(ye \cdot zx) \cdot z = x$$
$$x(y \cdot (ez \cdot y)x) = x$$
$$e(x \cdot (xy \cdot z)y) = x$$
$$e(x(yx \cdot z) \cdot z) = x$$
$$(e(x \cdot yz) \cdot y)x = x$$
$$x(e(x \cdot yz) \cdot y) = x$$
$$x \cdot e(xy \cdot zy) = x$$
$$xe \cdot (xy \cdot z)y = x$$
$$(ex \cdot yz) \cdot yx = x$$
$$e(xy \cdot zy) \cdot z = x$$
$$(ex \cdot yz)z \cdot x = x$$
$$x \cdot e(yx \cdot zy) = x$$
$$(x \cdot ey)(xz \cdot y) = x$$
$$(xe \cdot (y \cdot xz))y = x$$
$$(x \cdot e(yx \cdot z))z = x$$
$$x \cdot e(yz \cdot xy) = x$$
$$x((e \cdot yz)x \cdot y) = x$$
$$x(e \cdot (yz \cdot x)z) = x$$
$$x(ey \cdot (z \cdot yx)) = x$$
$$x((xe \cdot y)z \cdot y) = x$$
$$x(x(ye \cdot z) \cdot y) = x$$
$$x(xy \cdot ez) \cdot y = x$$
$$(xy \cdot e)(x \cdot zy) = x$$
$$(xy \cdot (e \cdot xz))y = x$$
$$xy \cdot e(y \cdot zx) = x$$
$$((xy \cdot e) \cdot yz)x = x$$
$$x((y \cdot ez) \cdot xz) = x$$
$$(x \cdot y(e \cdot zx))z = x$$
$$((xy \cdot e) \cdot zx)z = x$$
$$(x(y \cdot ez) \cdot y)x = x$$
$$e(x(yx \cdot z) \cdot y) = x$$
$$((e \cdot xy) \cdot yz)x = x$$
$$x(ye \cdot z) \cdot xz = x$$
$$x(e(xy \cdot z) \cdot y) = x$$
$$x \cdot (ex \cdot yz)y = x$$
$$(x \cdot e(xy \cdot z))y = x$$
$$e((x \cdot yz)y \cdot z) = x$$
$$((e \cdot xy) \cdot zy)z = x$$
$$e((x \cdot yz)z \cdot y) = x$$
$$x((ey \cdot x) \cdot zy) = x$$
$$(x \cdot e(yx \cdot z))y = x$$
$$((xe \cdot y) \cdot xz)y = x$$
$$((xe \cdot y) \cdot xz)z = x$$
$$x(e \cdot (yz \cdot x)y) = x$$
$$(x \cdot e(y \cdot zx))y = x$$
$$(xe \cdot yz) \cdot xz = x$$
$$(xe \cdot yz) \cdot zx = x$$
$$(x \cdot (x \cdot ey)z)y = x$$
$$x \cdot (xy \cdot ez)y = x$$
$$xy \cdot e(x \cdot zy) = x$$
$$(x \cdot y(e \cdot xz))y = x$$
$$((xy \cdot e) \cdot xz)y = x$$
$$xy \cdot e(yz \cdot x) = x$$
$$x((y \cdot ez)x \cdot y) = x$$
$$x \cdot (ye \cdot zx)z = x$$
$$(x \cdot y(ez \cdot x))z = x$$
$$x(y \cdot e(zy \cdot x)) = x$$
$$x(ye \cdot z) \cdot zx = x$$
$$(ex \cdot (y \cdot xz))y = x$$
$$(ex \cdot yz) \cdot xy = x$$
$$e((xy \cdot z) \cdot yz) = x$$
$$xe \cdot (x \cdot yz)y = x$$
$$x((e \cdot xy)z \cdot y) = x$$

A finite model of any of the following 5 *identities with neutral element* (*or their mirrors*) *is a Boolean group with neutral element* e.

$$(ex \cdot y)z \cdot xz = x$$
$$(ex \cdot yz)y \cdot z = x$$
$$((ex \cdot y)z \cdot x)z = x$$
$$(ex \cdot yz)z \cdot y = x$$
$$(ex \cdot y)z \cdot yz = x$$

Any additional single axioms with neutral element for Boolean groups are among these 5 *identities with neutral element* (*up to renaming, mirroring, and symmetry*).

References

[1] **C. Chang and R. Lee**, *Symbolic Logic and Mechanical Theorem Proving*, Academic Press, San Diego, CA, 1973.

[2] **M. Davis and H. Putnam**, *A computing procedure for quantification theory*, J. Assoc. Comput. Mach. **7** (1960), 201–215.

[3] **N. C. Fiala and K. M. Agre**, *Searching for shortest single axioms for groups of exponent* 6, J. Automat. Reason. **36** (2006), 241–257

[4] **J. Hart and K. Kunen**, *Single axioms for odd exponent groups*, J. Automat. Reason. **14** (1995), 383–412.

[5] **K. Kunen**, *The shortest single axioms for groups of exponent* 4, Computers and Math. with Appl. **29** (1995), 1–12.

[6] **W. McCune**, *Mace4* (`http://www.cs.unm.edu/~mccune/prover9/`).

[7] **W. McCune**, *OTTER* (`http://www.cs.unm.edu/~mccune/otter/`).

[8] **W. McCune**, *Prover9* (`http://www.cs.unm.edu/~mccune/prover9/`).

[9] **W. McCune and L. Wos**, *Applications of automated deduction to the search for single axioms for exponent groups*, in: Logic Programming and Automated Reasoning, Springer-Verlag, Berlin, 1992, 131–136.

[10] **N. S. Mendelsohn and R. Padmanabhan**, *Minimal identities for Boolean groups*, J. Algebra **34** (1975), 451–457.

[11] **C. A. Merdith and A. N. Prior**, *Equational logic*, Notre Dame J. Formal Logic **9** (1968), 212–226.

[12] **B. H. Neumann**, *Another single law for groups*, Bull. Australian Math. Soc. **23** (1981), 81–102.

[13] **J. A. Robinson**, *A machine-oriented logic based on the resolution principle*, J. Assoc. Comput. Mach. **12** (1965), 23–41.

[14] **G. A. Robinson and L. Wos**, *Paramodulation and theorem-proving in first-order theories with equality*, in: Machine Intelligence Vol. 4, Elsevier, New York, 1969, 135–150.

[15] **A. Tarski**, *Equational logic and equational theories of algebras*, in: Contributions to Math. Logic, North-Holland, Amsterdam, 275–288.

[16] **L. Wos, G. A. Robinson, and D. F. Carson**, *Efficiency and completeness of the set of support strategy in theorem proving*, J. Assoc. Comput. Mach. **12** (1965), 536–541.

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On 0-minimal $(0,2)$-bi-ideals in ordered semigroups

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Abstract In this paper, we study $(0,2)$-ideals, $(1,2)$-ideals and 0-minimal $(0,2)$-ideals in ordered semigroups. The notions of $(0,2)$-bi-ideals in ordered semigroups and 0-$(0,2)$-bisimple ordered semigroups are introduced and described. The results obtained extend the results on semigroups without order.

1. Introduction

In [5], the notion of (m,n)-ideals in semigroups was introduced by S. Lajos as a generalization of ideals in semigroups. In [4], D. N. Krgović described $(0,2)$-ideals, $(1,2)$-ideals and 0-minimal $(0,2)$-ideals. The author also introduced the notions of $(0,2)$-bi-ideals and 0-$(0,2)$-bisimple semigroups; and showed that a semigroup S with a zero element 0 is 0-$(0,2)$-bisimple if and only if S is left 0-simple.

In the present paper, using the concept of (m,n)-ideals in ordered semigroups defined by J. Sanborisoot and T. Changphas in [7], we extend the results in [4], mentioned above, to ordered semigroups. We begin with investigation $(0,2)$-ideals, $(1,2)$-ideals and 0-minimal $(0,2)$-ideals in ordered semigroups. The notions of $(0,2)$-bi-ideals in ordered semigroups and 0-$(0,2)$-bisimple ordered semigroups will be introduced.

The rest of this section let us recall some definitions and results used throughout the paper.

Definition 1.1. [1] A semigroup $(S,\cdot)$ together with a partial order $\leqslant$ (on S) that is compatible with the semigroup operation, meaning that for $x,y,z\in S$,

$$x\leqslant y\Rightarrow zx\leqslant zy \quad \& \quad xz\leqslant yz,$$

is called an *ordered semigroup*.

Let $(S,\cdot,\leqslant)$ be an ordered semigroup. If A,B are nonempty subsets of S, we let

$$AB=\{xy\in S\mid x\in A, y\in B\},$$
$$(A]=\{x\in S\mid x\leqslant a \text{ for some } a\in A\}.$$

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Let $(S, \cdot, \leqslant)$ be an ordered semigroup and let A, B be nonempty subsets of S. The following was proved in [2]:

(1) $(A](B] \subseteq (AB]$;

(2) $A \subseteq B \Rightarrow (A] \subseteq (B]$;

(3) $((A]] = (A]$.

Definition 1.2. [2] Let $(S, \cdot, \leqslant)$ be an ordered semigroup. A nonempty subset A of S is called a *left* (respectively, *right*) *ideal* of S if

(i) $SA \subseteq A$ (respectively, $AS \subseteq A$);

(ii) for $x \in A$ and $y \in S$, $y \leqslant x$ implies $y \in A$.

If A is both a left and a right ideal of S, then A is called a *(two-sided) ideal* of S.

It is clear that every left, right and (two-sided) ideals of an ordered semigroup S is a subsemigroup of S.

Definition 1.3. [7] Let $(S, \cdot, \leqslant q)$ be an ordered semigroup and let m, n be non-negative integers. A subsemigroup A of S is called an (m, n)-*ideal* of S if the following hold:

(i) $A^m S A^n \subseteq A$;

(ii) for $x \in A$ and $y \in S$, $y \leqslant qx$ implies $y \in A$.

Here, let $A^0 S = S$ and $SA^0 = S$.

From Definition 1.3, if $m = 1, n = 1$ then A is called a *bi-ideal* of S.
Note that if A is a nonempty subset of an ordered semigroup S, then the set $(A^2 \cup ASA^2]$ is a bi-ideal of S. Indeed: we have $((A^2 \cup ASA^2]] = (A^2 \cup ASA^2]$ and

$$\begin{aligned}
&(A^2 \cup ASA^2]S(A^2 \cup ASA^2] \\
&= (A^2 \cup ASA^2](S](A^2 \cup ASA^2] \\
&\subseteq (A^2SA^2 \cup A^2SASA^2 \cup ASA^2SA^2 \cup ASA^2SASA^2] \\
&\subseteq (ASA^2] \\
&\subseteq (A^2 \cup ASA^2].
\end{aligned}$$

Therefore, $(A^2 \cup ASA^2]$ is a bi-ideal of S.
We define $(0, 2)$-bi-ideals in an ordered semigroup analogue to [4] as follows:

Definition 1.4. A subsemigroup A of an ordered semigroup $(S, \cdot, \leqslant)$ is called a $(0, 2)$-*bi-ideal* of S if A is both a bi-ideal and a $(0, 2)$-ideal of S.

2. Main Results

We give a characterization of $(0,2)$-ideals of an ordered semigroup in term of left ideals as follows:

Lemma 2.1. *Let $(S,\cdot,\leqslant)$ be an ordered semigroup and let $A \subseteq S$. Then A is a $(0,2)$-ideal of S if and only if A is a left ideal of some left ideal of S.*

Proof. If A is a $(0,2)$-ideal of S, then

$$(A \cup SA]A \subseteq (A^2 \cup SA^2] \subseteq (A] = A$$

and $((A]] = (A]$. Hence A is a left ideal of the left ideal $(A \cup SA]$ of S.

Conversely, assume that A is a left ideal of a left ideal L of S. Then

$$SA^2 \subseteq SLA \subseteq LA \subseteq A.$$

Let $x \in A$ and $y \in S$ be such that $y \leqslant x$. Since $x \in L$, we have $y \in L$. The assumption applies $y \in A$. □

The following result give some characterizations of $(1,2)$-ideals of an ordered semigroup.

Theorem 2.2. *Let $(S,\cdot,\leqslant)$ be an ordered semigroup and let $A \subseteq S$. The following statements are equivalent:*

(i) *A is a $(1,2)$-ideal of S;*

(ii) *A is a left ideal of some bi-ideal of S;*

(iii) *A is a bi-ideal of some left ideal of S;*

(iv) *A is a $(0,2)$-ideal of some right ideal of S;*

(v) *A is a right ideal of some $(0,2)$-ideal of S.*

Proof. (i) $\Rightarrow$ (ii). If A is a $(1,2)$-ideal of S, then

$$(A^2 \cup ASA^2]A = (A^2 \cup ASA^2](A] \subseteq (A^3 \cup ASA^3] \subseteq (A^2 \cup ASA^2] \subseteq (A] = A.$$

Clearly, if $x \in A, y \in (A^2 \cup ASA^2]$ such that $y \leqslant x$ then $y \in A$. Hence A is a left ideal of the bi-ideal $(A^2 \cup ASA^2]$ of S.

(ii) $\Rightarrow$ (iii). Let A be a left ideal of a bi-ideal B of S. Note that $(A \cup SA]$ is a left ideal of S. By assumption, we have

$$A(A \cup SA]A \subseteq (A](A \cup SA](A] \subseteq (A^3 \cup ASA^2] \subseteq (A \cup BSBA] \subseteq (A \cup BA] \subseteq (A] = A.$$

Let $x \in A, y \in (A \cup SA]$ such that $y \leqslant qx$. Since $x \in A$, $x \in B$. Thus $y \in B$, so $y \in A$. Therefore, A is a bi-ideal of the left ideal $(A \cup SA]$ of S.

(iii) $\Rightarrow$ (iv). Assume that A is a bi-ideal of a left ideal L of S. Note that $(A \cup AS]$ is a right ideal of S. We have

$$(A \cup AS]A^2 \subseteq (A \cup AS](A^2] \subseteq (A^3 \cup ASA^2] \subseteq (A \cup ASLA] \subseteq (A \cup ALA] \subseteq (A] = A.$$

Let $x \in A, y \in (A \cup AS]$ such that $y \leqslant x$, then $x \in L$. Thus $y \in L$, so $y \in A$. Hence A is a $(0,2)$-ideal of the right ideal $(A \cup AS]$ of S.

(iv) $\Rightarrow$ (v). If A is a $(0,2)$-ideal of a right ideal R of S, then $(A \cup SA^2]$ is a $(0,2)$-ideal of S and

$$A(A \cup SA^2] \subseteq (A](A \cup SA^2] \subseteq (A^2 \cup ASA^2] \subseteq (A \cup RSA^2] \subseteq (A \cup RA^2] \subseteq (A] = A.$$

Assume that $x \in A, y \in (A \cup SA^2]$ such that $y \leqslant x$. Then $x \in R$, so $y \in R$, thus $y \in A$. Hence (v) holds.

(v) $\Rightarrow$ (i). If A is a right ideal of a $(0,2)$-ideal R of S, then

$$ASA^2 \subseteq ASR^2 \subseteq AR \subseteq A.$$

Assume that $x \in A, y \in S$ such that $y \leqslant x$. Since $x \in R$, so $y \in R$, thus $y \in A$. Hence A is a $(1,2)$-ideal of S. □

The following characterize $(1,2)$-ideals in term of left ideals and right ideals of an ordered semigroup.

Lemma 2.3. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup and let A be a subsemigroup of S such that $A = (A]$. Then A is a $(1,2)$-ideal of S if and only if there exist a $(0,2)$-ideal L of S and a right ideal R of S such that $RL^2 \subseteq A \subseteq R \cap L$.*

Proof. Assume that A is a $(1,2)$-ideal of S. We have $(A \cup SA^2]$ and $(A \cup AS]$ are $(0,2)$-ideal and right ideal of S, respectively. Setting $L = (A \cup SA^2]$ and $R = (A \cup AS]$, we obtain

$$RL^2 \subseteq (A^3 \cup A^2SA^2 \cup ASA^2 \cup ASASA^2] \subseteq (A^3 \cup ASA^2] \subseteq (A] = A.$$

It is clear that $A \subseteq R \cap L$.

Conversely, let R be a right ideal of S and L be a $(0,2)$-ideal of S such that $RL^2 \subseteq A \subseteq R \cap L$. Then

$$ASA^2 \subseteq (R \cap L)S(R \cap L)(R \cap L) \subseteq RSL^2 \subseteq RL^2 \subseteq A.$$

Hence A is a $(1,2)$-ideal of S. □

Definition 2.4. A $(0,2)$-bi-ideal A of an ordered semigroup $(S, \cdot, \leqslant)$ with a zero element 0 will be said to be 0-*minimal* if $A \neq \{0\}$ and $\{0\}$ is the only $(0,2)$-bi-ideal of S properly contained in A.

Assume that $(S, \cdot, \leqslant)$ is an ordered semigroup with a zero element 0. It is easy to see that every left ideal of S is a $(0,2)$-ideal of S. Hence if L is a 0-minimal $(0,2)$-ideal of S and A is a left ideal of S contained in L then $A = \{0\}$ or $A = L$. What can we say about $(0,2)$-ideals contained in some 0-minimal left ideal of S ? The answer to the same question for a semigroup without order was given in [4].

Lemma 2.5. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup with a zero element 0. Suppose that L is a 0-minimal left ideal of S and A is a subsemigroup of L such that $A = (A]$. Then A is a $(0,2)$-ideal of S contained in L if and only if $(A^2] = \{0\}$ or $A = L$.*

Proof. Assume that A is a $(0,2)$-ideal of S contained in L. Then $(SA^2] \subseteq L$. Since $(SA^2]$ is a left ideal of S, we have $(SA^2] = \{0\}$ or $(SA^2] = L$. If $(SA^2] = L$, then $L = (SA^2] \subseteq (A]$. Hence $A = L$. Let $(SA^2] = \{0\}$. Since $S(A^2] \subseteq (SA^2] = \{0\} \subseteq (A^2]$, it follows that $(A^2]$ is a left ideal of S contained in L. By the minimality of L, $(A^2] = \{0\}$ or $(A^2] = L$. If $A^2 = L$, then $A = L$. The opposite direction is clear. □

Lemma 2.6. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup with a zero element 0 and let L be a 0-minimal $(0,2)$-ideal of S. Then $(L^2] = \{0\}$ or L is a 0-minimal left ideal of S.*

Proof. We have

$$S(L^2]^2 = S(L^2](L^2] \subseteq (SL^2](L^2] \subseteq (L](L^2] \subseteq (L^2].$$

Then $(L^2]$ is a $(0,2)$-ideal of S contained in L, hence $(L^2] = \{0\}$ or $(L^2] = L$. Suppose that $(L^2] = L$. Since $SL = S(L^2] \subseteq (SL^2] \subseteq (L] = L$, we obtain L is a left ideal of S. Let B be a left ideal of S contained in L. It follows that $SB^2 \subseteq B^2 \subseteq B \subseteq L$. This shows that B is a $(0,2)$- ideal of S contained in L, so $B = \{0\}$ or $B = L$. □

The following corollary follows from Lemma 2.5 and Lemma 2.6:

Corollary 2.7. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup without zero. Then L is a minimal $(0,2)$-ideal of S if and only if L is a minimal left ideal of S.*

Lemma 2.8. *Let $(S, \cdot, \leqslant q)$ be an ordered semigroup without zero and let A be a nonempty subset of S. Then A is a minimal $(2,1)$-ideal of S if and only if A is a minimal bi-ideal of S.*

Proof. Assume that A is a minimal $(2,1)$-ideal of S. Then $(A^2SA]$ is a $(2,1)$-ideal of S contained in A, and hence $(A^2SA] = A$. Since

$$ASA = (A^2SA]SA \subseteq (A^2SASA] \subseteq (A^2SA] = A,$$

it follows that A is a bi-ideal of S. Suppose that there exits a bi-ideal B of S contained in A. Then $B^2SB \subseteq B^2 \subseteq B \subseteq A$, so B is a $(2,1)$-ideal of S contained in A. Using the minimality of A we get $B = A$.

Conversely, assume that A is a minimal bi-ideal of S. Then A is a $(2,1)$-ideal of S. Let D be a $(2,1)$-ideal of S contained in A. Since $(D^2SD]S(D^2SD] \subseteq (D^2(SDSD^2S)D] \subseteq (D^2SD]$, we have $(D^2SD]$ is a bi-ideal of S. This implies that $(D^2SD] = A$. Since $A = (D^2SD] \subseteq (D] = D$, $A = D$. Therefore A is a minimal $(2,1)$-ideal of S. □

Lemma 2.9. *Let $(S,\cdot,\leqslant)$ be an ordered semigroup and let $A \subseteq S$. Then A is a $(0,2)$-bi-ideal of S if and only if A is an ideal of some left ideal of S.*

Proof. Assume that A is a $(0,2)$-bi-ideal of S. Then

$$S(A^2 \cup SA^2] \subseteq (SA^2 \cup S^2A^2] \subseteq (SA^2] \subseteq (A^2 \cup SA^2],$$

hence $(A^2 \cup SA^2]$ is a left ideal of S. Since

$$A(A^2 \cup SA^2] \subseteq (A^3 \cup ASA^2] \subseteq (A] = A,\ (A^2 \cup SA^2]A \subseteq (A^3 \cup SA^3] \subseteq (A] = A$$

we obtain A is an ideal of $(A^2 \cup SA^2]$.

Conversely, if A is an ideal of a left ideal L of S then $ASA \subseteq ASL \subseteq AL \subseteq A$. Hence, by Lemma 2.1, A is a $(0,2)$-bi-ideal of S. □

Theorem 2.10. *Let $(S,\cdot,\leqslant)$ be an ordered semigroup with a zero element 0. If A is a 0-minimal $(0,2)$-bi-ideal of S, then exactly one of the following cases occurs:*

(i) $A = \{0\},\ \ \big(aS^1a\big] = \{0\}$;

(ii) $A = (\{0,a\}],\ \ a^2 = 0,\ \ (aSa] = A$;

(iii) $\forall a \in A\backslash\{0\},\ \ \big(Sa^2\big] = A$.

Proof. Assume that A is a 0-minimal $(0,2)$-bi-ideal of S. Let $a \in A\backslash\{0\}$. Then $(Sa^2] \subseteq A$. Moreover, $(Sa^2]$ is a $(0,2)$-bi-ideal of S. Hence $(Sa^2] = \{0\}$ or $(Sa^2] = A$.

Suppose that $(Sa^2] = \{0\}$. Since $a^2 \in A$, we have either

$$a^2 = a \text{ or } a^2 = 0 \text{ or } a^2 \in A\backslash\{0,a\}.$$

If $a^2 = a$, then $a = 0$. This is a contradiction. Suppose that $a^2 \in A\backslash\{0,a\}$. We have

$$S^1(\{0\} \cup a^2]^2 \subseteq (\{0\} \cup Sa^2] = (\{0\}] \cup (Sa^2] = \{0\} \subseteq (\{0\} \cup a^2],$$

$$(\{0\} \cup a^2]S(\{0\} \cup a^2] \subseteq (a^2Sa^2] \subseteq (Sa^2] = \{0\} \subseteq \{0,a^2\}.$$

Then $(\{0\} \cup a^2]$ is a $(0,2)$-bi-ideal of S contained in A. We observe that $(\{0\} \cup a^2] \neq \{0\}$ and $(\{0\} \cup a^2] \neq A$. This is a contradiction because A is 0-minimal $(0,2)$-bi-ideal of S. Therefore, $a^2 = 0$, hence, by Lemma 2.9, $A = (\{0,a\}]$. Now, using $(aSa]$ is a $(0,2)$-bi-ideal of S contained in A we obtain $(aSa] = \{0\}$ or $(aSa] = A$. Therefore, $(Sa^2] = \{0\}$ implies either $A = \{0,a\}$ and $(aS^1a] = \{0\}$ or $A = \{0,a\}$, $a^2 = \{0\}$ and $(aSa] = A$. If $(Sa^2] \neq \{0\}$, then $(Sa^2] = A$. □

Corollary 2.11. *Let A be a 0-minimal $(0,2)$-bi-ideal of an ordered semigroup $(S,\cdot,\leqslant)$ with a zero element 0. If $(A^2] \neq \{0\}$, then $A = (Sa^2]$ for every $a \in A\backslash\{0\}$.*

Definition 2.12. An ordered semigroup $(S,\cdot,\leqslant)$ with a zero element 0 is said to be 0-$(0,2)$-*bisimple* if $(S^2] \neq \{0\}$ and $\{0\}$ is the only proper $(0,2)$-bi-ideal of S.

Corollary 2.13. *Let* $(S, \cdot, \leqslant)$ *be an ordered semigroup with zero* 0. *Then* S *is* 0-$(0,2)$*-bisimple if and only if* $(Sa^2] = S$ *for every* $a \in S \backslash \{0\}$.

Proof. Assume that $(Sa^2] = S$ for all $a \in S \backslash \{0\}$. Let A be a $(0,2)$-bi-ideal of S such that $A \neq \{0\}$. Let $a \in A \backslash \{0\}$. Since $S = (Sa^2] \subseteq (SA^2] \subseteq (A] = A$, so $S = A$. Since $S = (Sa^2] \subseteq (SS] = (S^2]$ we have $(S^2] = S \neq \{0\}$. Therefore S is 0-$(0,2)$-bi-simple.

The converse statement follows from Corollary 2.11. □

Theorem 2.14. *Let* $(S, \cdot, \leqslant)$ *be an ordered semigroup with zero* 0. *Then* S *is* 0-$(0,2)$*-bisimple if and only if* S *is left* 0*-simple.*

Proof. Assume that S is 0-$(0,2)$-bisimple. If A is a left ideal of S, then A is a $(0,2)$-bi-ideal of S, and so $A = \{0\}$ or $A = S$.

Conversely, assume that S is left 0-simple. Let $a \in S \backslash \{0\}$. Then $(Sa] = S$, hence

$$S = (Sa] = ((Sa]a] \subseteq ((Sa^2]] = (Sa^2].$$

By Corollary 2.13, S is 0-$(0,2)$-bisimple. □

Theorem 2.15. *Let* $(S, \cdot, \leqslant)$ *be an ordered semigroup with a zero element* 0. *If* A *is a* 0*-minimal* $(0,2)$*-bi-ideal of* S*, then either* $(A^2] = \{0\}$ *or* A *is left* 0*-simple.*

Proof. Assume that $(A^2] \neq \{0\}$. Using Corollary 2.11, $(Sa^2] = A$ for every $a \in A \backslash \{0\}$. Since $a^2 \in A \backslash \{0\}$ for every $a \in A \backslash \{0\}$, we have $a^4 = (a^2)^2 \in A \backslash \{0\}$ for every $a \in A \backslash \{0\}$. Let $a \in A \backslash \{0\}$. Since

$$(Aa^2]S^1(Aa^2] \subseteq (AAa^2] \subseteq (Aa^2],$$

$$S(Aa^2]^2 \subseteq (SAa^2Aa^2] \subseteq (SA^2a^2] \subseteq (Aa^2],$$

we obtain $(Aa^2]$ is a $(0,2)$-bi-ideal of S contained in A. Hence $(Aa^2] = \{0\}$ or $(Aa^2] = A$. Since $a^4 \in Aa^2 \subseteq (Aa^2]$ and $a^4 \in A \backslash \{0\}$, we get $(Aa^2] = A$. We conclude by Corollary 2.13 that A is 0-$(0,2)$-bisimple. Theorem 2.14 applies A is left 0-simple. □

References

[1] **G. Birkhoff**, *Lattice theory*, New York: Amer. Math. Soc. Coll. Publ., Vol. 25, Providence, 1967.

[2] **N. Kehayopulu and M. Tsingelis**, *On left regular ordered semigroups*, Southeast Asian Bull. Math. **25** (2002), 609 – 615.

[3] **D. N. Krgović**, *On* (m,n)*-regular semigroups*, Publ. Inst. Math. (Beograd) **18 (32)** (1975), 107 – 110.

[4] **D. N. Krgović**, *On* 0*-minimal* $(0,2)$*-bi-ideal of semigroups*, Publ. Inst. Math. (Beograd) **31(45)** (1982), 103 – 107.

[5] **S. Lajos**, *Generalized ideals in semigroups*, Acta Sci. Math. **22** (1961), 217 − 222.

[6] **S. Lajos**, *Theorems on* $(1,1)$*-ideals in semigroups*, K. Marx Univ. Econ., Dep. Math., Budapest, (1972).

[7] **J. Sanborisoot and T. Changphas**, *On characterizations of* (m,n)*-regular ordered semigroups*, Far East J. Math. Sci. **65** (2012), 75 − 86.

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Intra-regular, left quasi-regular and semisimple fuzzy ordered semigroups

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Abstract. We characterize the ordered semigroup which are both intra-regular and left quasi-regular also the ordered semigroups which are both intra-regular and semisimple in terms of fuzzy sets.

In this paper we prove that an ordered semigroup S is intra-regular and left quasi-regular if and only if for every fuzzy subset f of S we have $f \preceq 1 \circ f^2 \circ 1 \circ f$. It is intra-regular and semisimple if and only if for every fuzzy subset f of S we have $f \preceq 1 \circ f^2 \circ 1 \circ f \circ 1$. Moreover, the property $f \preceq f \circ 1 \circ f^2 \circ 1$ characterizes the ordered semigroups which are intra-regular and right quasi-regular. An ordered semigroup $(S, \cdot, \leqslant)$ is called left (resp. right) quasi-regular if $a \in (SaSa]$ (resp. $a \in (aSaS]$) for every $a \in S$. In other words, S is left (resp. right) quasi-regular if for every $a \in S$ there exist $x, y \in S$ such that $a \leqslant xaya$ (resp. $a \leqslant axay$). An ordered semigroup S is called semisimple if $a \in (SaSaS]$ for every $a \in S$. That is, if for every $a \in S$ there exist $x, y, z \in S$ such that $a \leqslant xayaz$ [2]. Intra-regular ordered semigroups are well known. These are the ordered semigroups in which $a \in (Sa^2S]$ for each $a \in S$. We remind that for a subset H of S, $(H]$ is the set $\{t \in S \mid t \leqslant h \text{ for some } h \in H\}$. As always, denote by 1 the fuzzy subset of S defined by $1(x) = 1$ for every $x \in S$. Recall that if S is an intra-regular ordered semigroup, then $1 \circ 1 = 1$. If f, g are fuzzy subsets of S such that $f \preceq g$, then for any fuzzy subset h of S we have $f \circ h \preceq g \circ h$ and $h \circ f \preceq h \circ g$. Denote $f^2 := f \circ f$, and by f_a the characteristic function on the set S defined by $f_a(x) = 1$ if $x = a$ and $f_a(x) = 0$ if $x \neq a$ $(a \in S)$. Denote by A_a the subset of $S \times S$ defined by $A_a := \{(x, y) \in S \times S \mid a \leqslant xy\}$ [3]. The paper in a continuation of our papers in [1,5], for information not given in the present paper we refer to those papers. Exactly as in [1,5], our aim is to present a proof which is drastically simplified than the usual one.

Lemma 1. *Let $(S, \cdot, \leqslant)$ be an ordered groupoid, f, g fuzzy subsets of S and $a \in S$. The following are equivalent:*

(1) $(f \circ g)(a) \neq 0$.

(2) *There exists $(x, y) \in A_a$ such that $f(x) \neq 0$ and $g(y) \neq 0$.*

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Lemma 2. *Let $(S, \cdot, \leqslant)$ be an ordered groupoid, f a fuzzy subset of S and $a \in S$. The following are equivalent:*

(1) $(f \circ 1)(a) \neq 0$.

(2) *There exists $(x, y) \in A_a$ such that $f(x) \neq 0$.*

Lemma 3. *Let $(S, \cdot, \leqslant)$ be an ordered groupoid, g a fuzzy subset of S and $a \in S$. The following are equivalent:*

(1) $(1 \circ g)(a) \neq 0$.

(2) *There exists $(x, y) \in A_a$ such that $g(y) \neq 0$.*

Theorem 4. *An ordered semigroup S is intra-regular and left quasi-regular if and only if for every fuzzy subset f of S we have*

$$f \preceq 1 \circ f^2 \circ 1 \circ f.$$

Proof. $(\Rightarrow)$. Let $a \in S$. By hypothesis, there exist $x, y, z, t \in S$ such that $a \leqslant xa^2y$ and $a \leqslant zata$. Then we have $a \leqslant z(xa^2y)ta$. Since $(zxa^2yt, a) \in A_a$, we have $A_a \neq \emptyset$ and

$$\begin{aligned}(1 \circ f^2 \circ 1 \circ f)(a) &= \bigvee_{(u,v) \in A_a} \min\{(1 \circ f^2 \circ 1)(u), f(v)\} \\ &\geqslant \min\{(1 \circ f^2 \circ 1)(zxa^2yt), f(a)\}.\end{aligned}$$

Since $(zxa^2, yt) \in A_{zxa^2yt}$, we have $A_{zxa^2yt} \neq \emptyset$ and

$$\begin{aligned}(1 \circ f^2 \circ 1)(zxa^2yt) &= \bigvee_{(u,v) \in A_{zxa^2yt}} \min\{(1 \circ f^2)(u), 1(v)\} \\ &\geqslant \min\{(1 \circ f^2)(zxa^2), 1(yt)\} \\ &= (1 \circ f^2)(zxa^2).\end{aligned}$$

Since $(zxa, a) \in A_{zxa^2}$, we have $A_{zxa^2} \neq \emptyset$ and

$$\begin{aligned}(1 \circ f^2)(zxa^2) &= \bigvee_{(u,v) \in A_{zxa^2}} \min\{(1 \circ f)(u), f(v)\} \\ &\geqslant \min\{(1 \circ f)(zxa), f(a)\}.\end{aligned}$$

Since $(zx, a) \in A_{zxa}$, we have $A_{zxa} \neq \emptyset$ and

$$\begin{aligned}(1 \circ f)(zxa) &= \bigvee_{(u,v) \in A_{zxa}} \min\{1(u), f(v)\} \\ &\geqslant \min\{1(zx), f(a)\} \\ &= f(a).\end{aligned}$$

Thus we have

$$\begin{aligned}(1 \circ f^2 \circ 1 \circ f)(a) &\geqslant \min\{(1 \circ f^2 \circ 1)(zxa^2yt), f(a)\}\\ &\geqslant \min\{(1 \circ f^2)(zxa^2), f(a)\}\\ &\geqslant \min\{\min\{(1 \circ f)(zxa), f(a)\}, f(a)\}\\ &\geqslant \min\{f(a), f(a)\}\\ &= f(a).\end{aligned}$$

($\Leftarrow$). Let $a \in S$. Since f_a is a fuzzy set in S, by hypothesis, we have $1 = f_a(a) \leqslant (1 \circ f_a^2 \circ 1 \circ f_a)(a)$. Since $1 \circ f_a^2 \circ 1 \circ f_a$ is a fuzzy set in S, we have $(1 \circ f_a^2 \circ 1 \circ f_a)(a) \leqslant 1$. Thus we have $(1 \circ f_a^2 \circ 1 \circ f_a)(a) = 1$. By Lemma 1, there exists $(x, y) \in A_a$ such that $(1 \circ f_a^2)(x) \neq 0$ and $(1 \circ f_a)(y) \neq 0$. Since $(1 \circ f_a^2)(x) \neq 0$, by Lemma 3, there exists $(z, t) \in A_x$ such that $f_a^2(t) \neq 0$. Since $(1 \circ f_a)(y) \neq 0$, by Lemma 3, there exists $(u, v) \in A_y$ such that $f_a(v) \neq 0$. Since $f_a^2(t) \neq 0$, by Lemma 1, there exists $(w, h) \in A_t$ such that $f_a(w) \neq 0$ and $f_a(h) \neq 0$. Since $f_a(v) \neq 0$, we have $f_a(v) = 1$. Similarly $f_a(w) = 1$, $f_a(h) = 1$. Hence we obtain

$$a \leqslant xy \leqslant (zt)(uv) \leqslant z(wh)uv \text{ and } v = w = h = a.$$

Then $a \leqslant za^2ua \in Sa^2S \cap SaSa$. Then $a \in (Sa^2S]$ and $a \in (SaSa]$, that is, S is intra-regular and left quasi-regular. □

In an analogous way we prove the next theorem.

Theorem 5. *An ordered semigroup S is intra-regular and right quasi-regular if and only if for every fuzzy subset f of S we have*

$$f \preceq f \circ 1 \circ f^2 \circ 1.$$

Theorem 6. *An ordered semigroup S is intra-regular and semisimple if and only if for every fuzzy subset f of S we have*

$$f \preceq 1 \circ f^2 \circ 1 \circ f \circ 1.$$

Proof. ($\Rightarrow$). Let $a \in S$. By hypothesis, there exist $x, y, z, t, h \in S$ such that $a \leqslant xa^2y$ and $a \leqslant zatah$, then $a \leqslant z(xa^2y)tah$. Since $(zxa^2yta, h) \in A_a$, we have $A_a \neq \emptyset$ and

$$\begin{aligned}(1 \circ f^2 \circ 1 \circ f \circ 1)(a) &= \bigvee_{(u,v) \in A_a} \min\{(1 \circ f^2 \circ 1 \circ f)(u), 1(v)\}\\ &\geqslant \min\{(1 \circ f^2 \circ 1 \circ f)(zxa^2yta), 1(h)\}\\ &= (1 \circ f^2 \circ 1 \circ f)(zxa^2yta).\end{aligned}$$

Since $(zxa^2yt, a) \in A_{zxa^2yta}$, we have $A_{zxa^2yta} \neq \emptyset$ and

$$\begin{aligned}(1 \circ f^2 \circ 1 \circ f)(zxa^2yta) &= \bigvee_{(u,v) \in A_{zxa^2yta}} \min\{(1 \circ f^2 \circ 1)(u), f(v)\}\\ &\geqslant \min\{(1 \circ f^2 \circ 1)(zxa^2yt), f(a)\}.\end{aligned}$$

Since $(zxa, ayt) \in A_{zxa^2yt}$, we have $A_{zxa^2yt} \neq \emptyset$ and

$$\begin{aligned}(1 \circ f^2 \circ 1)(zxa^2yta) &= \bigvee_{(u,v)\in A_{zxa^2yt}} \min\{(1 \circ f^2)(u), 1(v)\}\\ &\geqslant \min\{(1 \circ f^2)(zxa), 1(ayt)\}\\ &= (1 \circ f^2)(zxa).\end{aligned}$$

Since $(zxa, a) \in A_{zxa^2}$, we have $A_{zxa^2} \neq \emptyset$ and

$$\begin{aligned}(1 \circ f^2)(zxa) &= \bigvee_{(u,v)\in A_{zxa^2}} \min\{(1 \circ f)(u), f(v)\}\\ &\geqslant \min\{(1 \circ f)(zxa), f(a)\}.\end{aligned}$$

Since $(zx, a) \in A_{zxa}$, we have $A_{zxa} \neq \emptyset$ and

$$\begin{aligned}(1 \circ f)(zxa) &= \bigvee_{(u,v)\in A_{zxa}} \min\{1(u), f(v)\}\\ &\geqslant \min\{1(zx), f(a)\}\\ &= f(a).\end{aligned}$$

Thus we have

$$\begin{aligned}(1 \circ f^2 \circ 1 \circ f \circ 1)(a) &\geqslant (1 \circ f^2 \circ 1 \circ f)(zxa^2yta)\\ &\geqslant \min\{(1 \circ f^2 \circ 1)(zxa^2yt), f(a)\}\\ &\geqslant \min\{(1 \circ f^2)(zxa), f(a)\}\\ &\geqslant \min\{\min\{(1 \circ f)(zxa), f(a)\}, f(a)\}\\ &= \min\{f(a), f(a)\}\\ &= f(a).\end{aligned}$$

$(\Leftarrow)$. Let $a \in S$. Since f_a and $1 \circ f_a^2 \circ 1 \circ f_a \circ 1$ are fuzzy sets in S, by hypothesis, we have $1 = f_a(a) \leqslant (1 \circ f_a^2 \circ 1 \circ f_a \circ 1)(a) \leqslant 1$, then $(1 \circ f_a^2 \circ 1 \circ f_a \circ 1)(a) = 1$. By Lemma 1, there exists $(x, y) \in A_a$ such that $(1 \circ f_a^2)(x) \neq 0$ and $(1 \circ f_a \circ 1)(y) \neq 0$. Since $(1 \circ f_a^2)(x) \neq 0$, by Lemma 3, there exists $(z, t) \in A_x$ such that $f_a^2(t) \neq 0$. Since $(1 \circ f_a \circ 1)(y) \neq 0$, by Lemma 3, there exists $(u, v) \in A_y$ such that $(f_a \circ 1)(v) \neq 0$. Since $f_a^2(t) \neq 0$, by Lemma 1, there exists $(h, k) \in A_t$ such that $f_a(h) \neq 0$, $f_a(k) \neq 0$. Since $(f_a \circ 1)(v) \neq 0$, by Lemma 2, there exists $(g, w) \in A_v$ such that $f_a(g) \neq 0$. We have

$$a \leqslant xy \leqslant (zt)(uv) \leqslant z(hk)uv \leqslant z(hk)u(gw) \text{ and } h = k = g = a.$$

Then $a \leqslant zhkugw = za^2uaw \in Sa^2S \cap SaSaS$, so $a \in (Sa^2S]$ and $a \in (SaSaS]$ which means that S is intra-regular and semisimple. $\square$

For a second proof of Theorems 4 and 6 we need the following lemmas.

Lemma 7. [4] *An ordered semigroup S is intra-regular if and only if for any fuzzy subset f of S we have $f \preceq 1 \circ f^2 \circ 1$.*

Lemma 8. [2] *An ordered semigroup S is left (resp. right) quasi-regular if and only if for any fuzzy subset f of S we have $f \preceq 1 \circ f \circ 1 \circ f$ (resp. $f \preceq f \circ 1 \circ f \circ 1$).*

Lemma 9. [2] *An ordered semigroup S is semisimple if and only if for any fuzzy subset f of S we have $f \preceq 1 \circ f \circ 1 \circ f \circ 1$.*

Proof of Theorem 4. $(\Rightarrow)$. Let f be a fuzzy subset of S. Since S is intra-regular, by Lemma 7, we have $f \preceq 1 \circ f^2 \circ 1$. Since S is left quasi-regular, by Lemma 8, we have $f \preceq 1 \circ f \circ 1 \circ f$. Thus we have

$$f \preceq 1 \circ f \circ 1 \circ f \preceq 1 \circ (1 \circ f^2 \circ 1) \circ 1 \circ f = 1 \circ f^2 \circ 1 \circ f.$$

$(\Leftarrow)$. By hypothesis, for any fuzzy subset f of S, we have

$$f \preceq 1 \circ f^2 \circ 1 \circ f \preceq 1 \circ f^2 \circ 1,\ \ 1 \circ f \circ 1 \circ f.$$

By Lemmas 7 and 8, S is intra-regular and left quasi-regular. □

Proof of Theorem 6. $(\Rightarrow)$. By Lemmas 7 and 9, for any fuzzy subset f of S, we have $f \preceq 1 \circ f^2 \circ 1$ and $f \preceq 1 \circ f \circ 1 \circ f \circ 1$, then

$$f \preceq 1 \circ (1 \circ f^2 \circ 1) \circ 1 \circ f \circ 1 = 1 \circ f^2 \circ 1 \circ f \circ 1.$$

$(\Leftarrow)$. For any fuzzy subset f of S, by hypothesis, we have

$$f \preceq 1 \circ f^2 \circ 1 \circ f \circ 1 \preceq 1 \circ f^2 \circ 1,\ \ 1 \circ f \circ 1 \circ f \circ 1.$$

By Lemmas 7 and 9, S is intra-regular and semisimple. □

References

[1] **N. Kehayopulu,** *Left quasi-regular and intra-regular ordered semigroups using fuzzy ideals*, Quasigroups and Related Systems **20** (2012), 239 – 249.

[2] **N. Kehayopulu,** *Characterization of left quasi-regular and semisimple ordered semigroups in terms of fuzzy sets*, Int. J. Algebra **6** (2012), 747 – 755.

[3] **N. Kehayopulu,** *On fuzzy ordered semigroups*, Quasigroups and Related Systems **20** (2012), 251 – 256.

[4] **N. Kehayopulu and M. Tsingelis,** *Characterization of some types of ordered semigroups in terms of fuzzy sets*, Lobachevskii J. Math. **29** (2008), 14 – 20.

[5] **N. Kehayopulu and M. Tsingelis,** *On fuzzy ordered semigroups*, Quasigroups and Related Systems **20** (2012), 61 – 70.

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Nuclei and commutants of C-loops

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Abstract. C-loops are loops that satisfy the identity $x(y(yz)) = ((xy)y)z$. In this note we use the order of nuclei of C-loops to show that (1) nonassociative C-loops of order $2p$, where p is prime, are Steiner loops, (2) nonassociative C-loops of order $3n$ are non-simple and non-Steiner, (3) no nonassociative C-loop of order $2{\cdot}3^t$, $t \geqslant 1$ exists, and (4) if every element of the commutant of a C-loop is of odd order the commutant forms a subloop.

1. Introduction

C-*loops* are loops satisfying the identity $x(y(yz)) = ((xy)y)z$. The nature of the identity, where unlike in other Bol-Moufang identities the repeated variable is not separated by either of the other variables, makes them a difficult target of study. Nevertheless they have been investigated in [1, 2, 3, 4, 6, 9, 10, 12, 13, 14, 15].

In this note we extend some results of [14], in particular [14, Proposition 3.1] that states that only even order nonassociative C-loops exist. Investigating this result further using the order of nuclei of C-loops, we prove that (1) all nonassociative C-loops of order $2p$, where p is prime, are Steiner loops, (2) all nonassociative C-loops of order $3n$ are non-simple and non-Steiner, (3) there exists no nonassociative C-loop of order $2 \cdot 3^t$, $t \geqslant 1$, and (4) if $C(L)$ is the commutant of a C-loop L and every element of $C(L)$ is of odd order, then $C(L)$ is a subloop of L.

All examples presented in this paper have been computed by FINDER [16] and verified by GAP [11].

2. Preliminaries

In this paper we are concerned exclusively with finite loops. Let L be a loop we then define *left nucleus* N_λ, *middle nucleus* N_μ, and *right nucleus* N_ρ of L as the sets

$$\begin{aligned} N_\lambda &= \{x \in L; x(yz) = (xy)z \text{ for every } y, z \in L\}, \\ N_\mu &= \{x \in L; y(xz) = (yx)z \text{ for every } y, z \in L\}, \\ N_\rho &= \{x \in L; y(zx) = (yz)x \text{ for every } y, z \in L\}. \end{aligned}$$

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The *nucleus* N of L is the defined as $N = N_\lambda \cap N_\mu \cap N_\rho$. N is subgroup of L and, in particular, for C-loops we have $N = N_\lambda = N_\mu = N_\rho$.

We also define the *commutant* $C(L)$ of a loop L to be the set

$$C(L) = \{c \in L : cx = xc \text{ for every } x \in L\}.$$

The following hold for a C-loop L with commutant $C(L)$ and nucleus N.

(i) There is no C-loop with nucleus of index 2 [14, Lemma 2.9].

(ii) $C(L)$ is a normal subgroup of L [14, Proposition 2.7].

(iii) If L is nonassociative, of order n and N of order m. Then

(a) $n/m \equiv 2(\text{mod } 6)$ or $n/m \equiv 4(\text{mod } 6)$,

(b) n is even, and

(c) if $n = pk$ for some prime p and positive integer k, then $p = 2$ and $k > 3$ [14, Proposition 3.1].

Moreover, there is a nonassociative non-Steiner C-loop of order $2k$ for every $k > 3$.

3. Nucleus of C-loops

We start our considerations with a corollary to [14, Proposition 3.1].

Corollary 3.1. *Let L be a nonassociative C-loop of order n with nucleus N of order m. Then*

(i) $n/m \equiv 1(\text{mod}3)$ *or* $n/m \equiv 2(\text{mod}3)$,

(ii) $(n/2)/m$ *is an integer of the form* $3k - 1$ *or* $3k + 1$,

(iii) $(n/m)^2 \equiv 4(\text{mod}6)$ *or* $n/m \equiv 4(\text{mod}6)$,

(iv) n/m *is of the form* $2(3k - 1)$ *or* $(n/m)^2$ *is of the form* $2(3k - 1)$.

Proof. (i) and (iii) are straightforward.
(ii) We have

$$\begin{aligned}
n/m &\equiv 2(\text{mod}6) \text{ or } n/m \equiv 4(\text{mod}6)\\
n/m &= 6k + 2 \text{ or } n/m = 6k + 4 \text{ for some positive integer } k\\
n/m &= 2(3k + 1) \text{ or } n/m = 2(3k + 2)\\
n/2m &= 3k + 1 \text{ or } n/2m = 3k + 2\\
(n/2)/m &= 3k + 1 \text{ or } (n/2)/m = 3k + 2. \text{ But every integer of the form}\\
&\quad 3k + 2 \text{ is also of the form } 3k - 1.
\end{aligned}$$

Thus $(n/2)/m = 3k + 1$ or $(n/2)/m = 3k - 1$.
(iv) By part (iii), we have

$$\begin{aligned}
(n/m)^2 &\equiv 4(\text{mod}6) \text{ or } n/m \equiv 4(\text{mod}6)\\
(n/m)^2 &= 6k + 4 \text{ or } n/m = 6k + 4 \text{ for some positive integer } k\\
(n/m)^2 &= 2(3k + 2) \text{ or } n/m = 2(3k + 2)\\
(n/m)^2 &= 2(3k - 1) \text{ or } n/m = 2(3k - 1).
\end{aligned}$$

□

Proposition 3.2. *A nonassociative C-loop L of order $3n$ is non-simple and non-Steiner.*

Proof. $L/N(L)$ is Steiner, hence $3n/m$ is congruent to 2 or 4 mod 6. So $3n/m$ is not divisible by 3, thus m is divisible by 3. Therefore, $N(L)$ is a group containing an element of order 3 and hence L is not Steiner. Since $N(L)$ is nontrivial and since $N(L)$ is normal in L by [14], it follows that L is not simple. $\square$

The following example illustrates the above proposition.

Example 3.3. A nonassociative, noncommutative, non-Steiner non-simple C-loop of order 12 (size of nucleus = 3) is given in Table 1.

·	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	2	0	4	5	3	7	8	6	10	11	9
2	2	0	1	5	3	4	8	6	7	11	9	10
3	3	4	5	0	1	2	9	10	11	6	7	8
4	4	5	3	1	2	0	10	11	9	7	8	6
5	5	3	4	2	0	1	11	9	10	8	6	7
6	6	7	8	10	11	9	0	1	2	5	3	4
7	7	8	6	11	9	10	1	2	0	3	4	5
8	8	6	7	9	10	11	2	0	1	4	5	3
9	9	10	11	8	6	7	3	4	5	2	0	1
10	10	11	9	6	7	8	4	5	3	0	1	2
11	11	9	10	7	8	6	5	3	4	1	2	0

Table 1:

·	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	0	3	2	5	4	9	8	7	6
2	2	3	0	1	6	8	4	9	5	7
3	3	2	1	0	7	9	8	4	6	5
4	4	5	6	7	0	1	2	3	9	8
5	5	4	8	9	1	0	7	6	2	3
6	6	9	4	8	2	7	0	5	3	1
7	7	8	9	4	3	6	5	0	1	2
8	8	7	5	6	9	2	3	1	0	4
9	9	6	7	5	8	3	1	2	4	0

Table 2:

Corollary 3.4. *Let L be a nonassociative C-loop of order n with nucleus N of order m, then if for some positive integer t, 3^t divides n, then 3^t also divides m.* $\square$

The next proposition confirms that there are indeed some even orders for which no nonassociative C-loop exists.

Proposition 3.5. *There is no nonassociative C-loop of order $2 \cdot 3^t$ for $t \geqslant 1$.*

Proof. n/m is not divisible by 3, hence $L/N(L)$ is of index at most 2, which is impossible by [14]. $\square$

The following proposition states that there exist orders for which all nonassociative C-loops will be Steiner.

Proposition 3.6. *A nonassociative C-loop L of order $2p$ with p prime, is Steiner.*

Proof. Since L is nonassociative, $p > 2$. Let m be the order of $N(L)$. Since $N(L)$ is normal in L by [14], m divides $2p$. If $m = 2p$, $L = N(L)$ is a group. If $m = p$ then $N(L)$ is of index 2 in L, which is impossible by [14]. Similarly, by [14] $L/N(L)$ is Steiner. If $m = 2$ then $L/N(L)$ is Steiner of order p, which again is impossible. Thus $m = 1$ and L is Steiner. $\square$

Example 3.7. The smallest nonassociative C-loop (size of nucleus = 1) is given in table 2. Since its order is $n = 10 = 2 \cdot 5$, it is also Steiner.

It is well known that there are two nonassociative C-loops of order 14. Being of order of the form $2p$ both are Steiner with nucleus of order 1.

Remark 3.8. Exploiting the results of Propositions 3.2, 3.5, and 3.6 can speed up automatic enumeration of C-loops. For example, we know by 3.2 that there is no nonassociative C-loop of order 18 , by 3.6 that C-loops of order 24 are all non-Steiner and by 3.5 that C-loops of order 22 are all Steiner.

Next we give the general forms of the nuclei of the nonassociative C-loops. Here p is an odd prime other than 3.

Order of C-loop	Admissible order of nucleus
$2 \cdot 3^k p,\ k \geqslant 1$	3^k
$2p$	1
$2^l,\ l \geqslant 4$	$1, 2, 2^2, \ldots, 2^{l-2}$
$2^l \cdot 3^k,\ l \geqslant 1, k \geqslant 1$	$2^h \cdot 3^k, 0 \leqslant h \leqslant l-2$
$2^2 p$	$1, 2, p$
$2p^2$	$1, p$
$2^k p,\ k > 2$	$2^h, 2^l p, 0 \leqslant h \leqslant k-1, 0 \leqslant l \leqslant k-2$
$2p^k,\ k > 2$	$p^l, 0 \leqslant l \leqslant k-1$
$2^2 p^2$	$1, 2, p, p^2, 2p$
$2^2 \cdot 3 \cdot p$	$3, 6, 3p$

As application of the above table we can give the orders of C-loops and the admissible orders of their corresponding nuclei in the following table.

C-loop	Nucleus	C-loop	Nucleus	C-loop	Nucleus
10	1	42	3	74	1
12	3	44	$1, 2, 11$	76	$1, 2, 19$
14	1	46	1	78	3
16	$1, 2, 4$	48	$3, 6, 12$	80	$1, 2, 4, 5, 8, 10, 20$
20	$1, 2, 5$	50	$1, 5$	82	1
22	1	52	$1, 2, 13$	84	$3, 6, 21$
24	$3, 6$	56	$1, 2, 4, 7, 14$	86	1
26	1	58	1	88	$1, 2, 4, 11$
28	$1, 2, 7$	60	$3, 6, 15$	90	$9, 18, 45$
30	3	62	1	92	$1, 2, 23$
32	$1, 2, 4, 8$	64	$1, 2, 4, 8, 16$	94	1
34	1	66	3	96	$3, 6, 12$
36	9	68	$1, 2, 7$	98	$1, 7$
38	1	70	$1, 5, 7$	100	$1, 2, 5$
40	$1, 2, 4, 5, 10$	72	$9, 18$		

4. Commutant of C-loops

The commutant of a loop is also known as the centrum, Moufang center or semi-center [8]. As discussed in [8], in a group, or even a Moufang loop, the commutant is a subloop, but this does not need to be the case in general. In [8], it has been proved that the commutant of a Bol loop of odd order is a subloop. In the following we discuss such a special case for the commutant of C-loops, which is not necessarily a subloop as the following example demonstrates:

Example 4.1. Consider the following nonassociative flexible C-loop of order 20, which has a commutant as $\{0, 1, 2, 3, 4, 5\}$ that is not a subloop.

$\cdot$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14	17	16	19	18
2	2	3	1	0	6	7	5	4	10	11	9	8	18	19	16	17	15	14	13	12
3	3	2	0	1	7	6	4	5	11	10	8	9	19	18	17	16	14	15	12	13
4	4	5	6	7	1	0	3	2	12	13	16	17	9	8	18	19	11	10	15	14
5	5	4	7	6	0	1	2	3	13	12	17	16	8	9	19	18	10	11	14	15
6	6	7	5	4	3	2	0	1	14	15	18	19	16	17	8	9	12	13	10	11
7	7	6	4	5	2	3	1	0	15	14	19	18	17	16	9	8	13	12	11	10
8	8	9	10	11	12	13	15	14	0	1	2	3	4	5	7	6	18	19	16	17
9	9	8	11	10	13	12	14	15	1	0	3	2	5	4	6	7	19	18	17	16
10	10	11	9	8	16	17	19	18	2	3	1	0	15	14	12	13	5	4	6	7
11	11	10	8	9	17	16	18	19	3	2	0	1	14	15	13	12	4	5	7	6
12	12	13	18	19	9	8	17	16	4	5	14	15	1	0	11	10	6	7	3	2
13	13	12	19	18	8	9	16	17	5	4	15	14	0	1	10	11	7	6	2	3
14	14	15	16	17	18	19	9	8	6	7	13	12	10	11	1	0	3	2	5	4
15	15	14	17	16	19	18	8	9	7	6	12	13	11	10	0	1	2	3	4	5
16	16	17	15	14	11	10	13	12	18	19	5	4	7	6	3	2	0	1	8	9
17	17	16	14	15	10	11	12	13	19	18	4	5	6	7	2	3	1	0	9	8
18	18	19	13	12	15	14	11	10	16	17	7	6	3	2	5	4	8	9	0	1
19	19	18	12	13	14	15	10	11	17	16	6	7	2	3	4	5	9	8	1	0

We now investigate a condition under which the commutant of C-loop will be a subloop.

Proposition 4.2. *Let $C(L)$ be the commutator of a C-loop L. If every element in $C(L)$ has odd order then $C(L)$ is a subloop of L.*

Proof. Since $C(L)$ is has odd order by [14], then in fact, C(L) = Z(L). By [14] L is power-alternative, thus $C(L)$ is closed under powers. Now, let $a, b \in C(L)$ with $|a| = 2k + 1$. Then $a = a^{2k+2}$ is a square, hence in $N(L)$ again by [14]. The rest of the proof is clear from this observation. $\square$

References

[1] **A. Beg**, *A Theorem on C-loops*, Kyungpook Math. J. **17** (1977), 91–94.

[2] **A. Beg**, *On LC-, RC-, and C-loops*, Kyungpook Math. J. **20**(2) (1980), 211–215.

[3] **F. Fenyves**, *Extra Loops I*, Publ. Math. Debrecen **15** (1968), 235–238.

[4] **F. Fenyves**, *Extra Loops II*, Publ. Math. Debrecen **16** (1969), 187–192.

[5] **E. G. Goodaire, E. Jespers and C. P. Milies**, *Alternative Loop Rings*, North-Holland Math. Studies, **184**, Elsevier.

[6] **M. K. Kinyon, K. Kunen and J. D. Philips**, *A generalization of Moufang and Steiner loops*, Algebra Universalis **48** (2002), 81–101.

[7] **M. K. Kinyon, Kyle Pula and P. Vojtechovsky**, *Admissible orders Of Jordan loops*, J. Combinatorial Designs **17** (2009), 103–118.

[8] **M. K. Kinyon, J. D. Phillips**, *Commutants of Bol loops of odd order*, Proc. Amer. Math. Soc. **132** (2004), 617–619.

[9] **M. K. Kinyon, J. D. Philips and P. Vojtechovsky**, *Loops of Bol-Moufang type with a subgroup of index* 2, Bul. Acad. Stiinte Repub. Mold. Mat. **3** (2005), 71–87.

[10] **M. K. Kinyon, J. D. Philips and P. Vojtechovsky**, *C-loops: extensions and constructions*, J. Algebra Appl. **6** (2007), 1–20.

[11] **G. P. Nagy and P. Vojtechovsky**, *LOOPS: Computing with quasigroups and loops in GAP, version 1.0.0*, http://www.math.du.edu/loops.

[12] **J. D. Phillips and P. Vojtechovsky**, *The varieties of loops of Bol-Moufang type*, Algebra Universalis **54** (2005), 259–271.

[13] **J. D. Phillips and P. Vojtechovsky**, *The varieties of quasigroups of Bol-Moufang type: An equational reasoning approach*, J. Algebra **293** (2005), 17–33.

[14] **J. D. Philips and P. Vojtechovsky**, *C-loops: an introduction*, Publ. Math. Debrecen **68** (2006), 115–137.

[15] **V. S . Ramamurthi and A. R. T. Solarin**, *On finite right central loops*, Publ. Math.Debrecen **35** (1988), 260–264.

[16] **J. Slaney**, *FINDER, finite domain enumerator: System description.* In Proc. of CADE-12, pp. 798–801, Springer, 1994.

[17] **J. Slaney and A. Ali**, *Generating loops with the inverse property.* In Proc. of ESARM 2008, pp. 55–66.

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Simple ternary semigroups

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Abstract. Simple ternary semigroups are studied using idempotent pairs. The concept of primitive idempotent pairs is introduced and the connection between them and minimal left (right) ideals are studied. An example of ternary semigroups containing primitive idempotent pairs is given. Some simple ternary semigroups containing a primitive idempotent pair are characterized.

1. Introduction

Investigation of ideals is an essential part of the study of any algebraic system. Investigation of ideals and radicals in ternary semigroups was initiated by Sioson [16]. The study has been continued by many authors for ternary semigroups and more generally for n-ary semigroups [8, 9, 10]. Cyclic ternary groups are described by Dörnte [3]. The n-ary power was introduced by Post [12]. The notion of minimal (maximal) left and right ideals in a ternary semigroups has been studied in [9] and a characterization has been obtained. In this paper we study some aspects of ternary semigroups such as Green's relations and simplicity. The definition of $\mathcal{D}$- and $\mathcal{H}$-equivalences given here are more general than those defined in [2]. In this paper a 0-t-simple ternary semigroup is defined and a characterization is obtained. Primitive idempotent pairs in a ternary semigroup are defined. Some results for 0-t-simple ternary semigroup which contains primitive idempotent pairs are proved. A connection between primitive idempotents and minimal (left and right) ideals are established. Completely 0-t-simple ternary semigroups are introduced and characterized.

2. (0)-simple ternary semigroups

A ternary semigroup is called (*right, left*) *simple* if it does not contains any proper (right, left) ideals. A ternary semigroup T is called t-*simple* if it does not contain any proper two-sided ideal. A t-simple ternary semigroup is simple. A simple ternary semigroup is *surjective*, i.e., $T = T^{<1>} = [TTT]$.

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For other definitions we refer [8, 9, 16].

We start with the following simple lemma proved in [8].

Lemma 2.1. *A ternary semigroup T is a right (left) simple if and only if $[aTT] = T$ (respectively, $[TTa] = T$) for all $a \in T$.* □

From this lemma we deduce

Corollary 2.2. *A ternary semigroup T is a right (left) simple if and only if for given $a, b \in T$ there exist $u, v \in T$ such that $[auv] = b$.* □

The following facts are almost obvious

Lemma 2.3. *A ternary semigroup T is a right (left) simple if and only if $[abT] = T$ (resp. $[Tab] = T$) for all $a, b \in T$.* □

Corollary 2.4. *A ternary semigroup T is a right (left) simple if and only if for given $a, b, c \in T$ there exist $x \in T$ such that $[abx] = c$ (resp. $[xab] = c$).* □

Lemma 2.5. *A ternary semigroup T is simple if and only if $T = [TaT] \cup [TTaTT]$ for any $a \in T$.* □

Lemma 2.6. *A ternary semigroup T is t-simple if and only if $[TTaTT] = T$ for any $a \in T$.* □

An element $z \in T$ is called a *zero element* if $[abz] = [zab] = [azb] = z$ for all $a, b \in T$. A zero element is uniquely determined and is denoted by 0. If T has no zero element, then a zero element can be adjoined by putting $[abc] = 0$ if any of a, b, c is a zero. We denote this fact by $T^0 = T \cup \{0\}$. If a ternary semigroup has a zero, then clearly $\{0\}$ is an ideal of T. It is denoted by (0). A ternary semigroup T with 0 is called a *null ternary semigroup* if $[abc] = 0$ for all $a, b, c \in T$. It is clear that a ternary semigroup with 0 has at least two ideals: 0 and T. If it has no other ideals (two-sided ideals) and $T^{<1>} \neq (0)$, then it is called 0-*simple* (resp. 0-t-*simple*.

Lemma 2.7. *If a ternary semigroup T with 0 has only one two-sided ideal $A \neq T$, then either T is 0-t-simple or T is the null ternary semigroup of order 2.*

Proof. Clearly $A = (0)$. Since $T^{<1>}$ is an ideal of T, we have $T^{<1>} = T$ or $T^{<1>} = (0)$. In the first case $T = (0)$, which means that T is 0-t-simple. In the second case for any non-zero element $t \in T$ the set $\{0, t\}$ is a non-zero two-sided ideal of T and so $\{0, t\} = T$. Thus T is a null ternary semigroup of order 2. □

Lemma 2.8. *A ternary semigroup T is 0-simple if and only if for every non-zero $a \in T$ we have $T = [TaT] \cup [TTaTT]$.*

Proof. Suppose that T is 0-simple. Then $T^{<1>}$ is a non-zero ideal of T and so $T^{<1>} = T$. Hence $T = T^{<1>} = T^{<2>}$. For any non-zero element $a \in T$ the subset $[TaT] \cup [TTaTT]$ is an ideal of T. Hence we have either $[TaT] \cup [TTaTT] = (0)$ or $[TaT] \cup [TTaTT] = T$. Suppose $[TaT] = (0)$. Then the set $M = \{m \in T : [TmT] = (0)\}$ contains the nonzero element a. M is a non-zero ideal and so $M = T$. This means that $T^{<1>} = (0)$, a contradiction. Therefore $[TaT] \cup [TTaTT] = T$ for every non-zero element $a \in T$. The converse is obvious. □

Lemma 2.9. *A ternary semigroup T is 0-t-simple if and only if $T = [TTaTT]$ for all $a \neq 0 \in T$.*

Proof. Suppose that T is 0-t-simple. Then $T^{<1>} \neq 0$ and $T^{<1>}$ is an ideal of T. Hence $T = T^{<1>} = T^{<2>}$. For any non-zero element $a \in T$, the subset $[TTaTT]$ of T is a two-sided ideal. Thus we have either $[TTaTT] = T$ or $[TTaTT] = (0)$. If $[TTaTT] = (0)$, then as in Lemma 2.8 we obtain a contradiction. Thus $T = [TTaTT]$. □

In a similar way we can prove

Lemma 2.10. *If a ternary semigroup T is 0-t-simple, then $T = [TaT]$ for all $a \neq 0 \in T$.* □

3. Green's equivalence on ternary semigroups

The Green's equivalence relation $\mathcal{L}$ and $\mathcal{R}$ on a ternary semigroup T are defined as follows (see [2]):

$$a\mathcal{L}b \Longleftrightarrow a \cup [TTa] = b \cup [TTb],$$
$$a\mathcal{R}b \Longleftrightarrow a \cup [aTT] = b \cup [bTT],$$
$$\mathcal{H} = \mathcal{L} \cap \mathcal{R}.$$

In other words $a\mathcal{L}b$ if and only if a and b generate the same left ideal, i.e., $a = b$ or $a = [xyb]$, $b = [uva]$ for some $x, y, u, v \in T$. Similarly, $a\mathcal{R}b$ if and only if a and b generate the same right ideal, i.e., $a = b$ or $a = [bpq]$, $b = [ars]$ for some $p, q, r, s \in T$.

Note that our definition of $\mathcal{H}$ is different from that found in [2].

Lemma 3.1. *$\mathcal{L}$ is a right congruence and $\mathcal{R}$ is a left congruence.* □

Proposition 3.2. *In ternary semigroups $\mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$.*

Proof. Let $(a, b) \in \mathcal{L} \circ \mathcal{R}$. Then there exists $c \in T$ such that $a\mathcal{L}c$ and $c\mathcal{R}b$ so, there exist $x, y, u, v, p, q, r, s \in T$ such that $a = [xyc]$, $c = [uva]$ and $c = [bpq], b = [crs]$. Put $d = [[xyc]rs]$. Then $[ars] = [[xyc]rs] = d$, and, $[dpq] = [[xyc]rs]pq] = [[xy[crs]]pq] = [[xyb]pq] = [xy[bpq]] = [xyc] = a$. Therefore $a\mathcal{R}d$. Also $[xyb] = [xy[crs]] = d$ and $[uvd] = [[uva]rs] = [crs] = b$, and so $d\mathcal{L}b$. Hence $(a, b) \in \mathcal{R} \circ \mathcal{L}$. Thus $\mathcal{L} \circ \mathcal{R} \subseteq \mathcal{R} \circ \mathcal{L}$. Similarly we can prove $\mathcal{R} \circ \mathcal{L} \subseteq \mathcal{L} \circ \mathcal{R}$. Therefore $\mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$ is an equivalence relation on T. □

Proposition 3.3. *$\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$ is the smallest equivalence on T containing $\mathcal{R}$ and $\mathcal{L}$.* □

The equivalence $\mathcal{D}$ defined by us is contained in the equivalence $\mathcal{D}$ defined by Dixit and Dewan [2].

Recall [13, 14] that an element $t \in T$ is called *regular* if $[tut] = t$ for some $t \in T$. If $[tut] = t$ and $[utu] = u$, then u and t are *inverses* of one another.

Proposition 3.4. *If a $\mathcal{D}$-class D of T contains a regular element, then every element of D is regular.*

Proof. Let D be a $\mathcal{D}$-class in T and $a \in T$ be a regular element in D. Let b be an arbitrary element of D. Since $b \in D$, for some $c \in T$ we have $a\mathcal{L}c\mathcal{R}b$. From $a\mathcal{L}c$ we obtain either $a = c$ or $a = [efc]$, $c = [uva]$ for some $e, f, u, v \in T$. Similarly, $c\mathcal{R}b$ gives either $c = b$ or $c = [bpq]$, $b = [crs]$ for some $p, q, r, s \in T$. Let x be an inverse of a. Then $[axa] = a$, $[xax] = x$. Take $y = [[pqx]ef]$ we get $[byb] = [b[[pqx]ef]b] = [[bpq]x[efb]] = [cx[efb]] = [cx[ef[crs]]] = [cx[efc]rs] = [[uva]xa]rs] = [uv[axa]rs] = [uv[ars]] = [[uva]rs] = [crs] = b$. Similarly if $a = c$, or $c = b$, then by taking $y = [pqx]$ $(y = [xef])$ we can show that b is regular element. □

Let $\mathrm{L_a}$ $(\mathrm{R_a},\ \mathrm{D_a},\ \mathrm{H_a})$ be the $\mathcal{L}$ $(\mathcal{R},\ \mathcal{D},\ \mathcal{H})$-class containing $a \in T$.

The following Lemmas are found in [2].

Lemma 3.5. *Let a, b be $\mathcal{R}$-equivalent elements in a ternary semigroup T and let $p, q, r, s \in T$ be such that $a = [brs]$, $b = [apq]$. Then the right translations $\rho_{pq}|L_a$, $\rho_{rs}|L_b$ are mutually inverse $\mathcal{R}$-class preserving bijections from L_a onto L_b and from L_b onto L_a respectively.* □

Lemma 3.6. *Let a, b be $\mathcal{L}-$equivalent elements in a ternary semigroup T and let $x, y, u, v \in T$ be such that $a = [xyb]$, $b = [uva]$. Then the left translations $\lambda_{xy}|R_a$, $\lambda_{rs}|R_b$ are mutually inverse $\mathcal{L}$- class preserving bijections from R_a onto R_b and from R_b onto R_a respectively.* □

Using the above maps the following lemma can be proved.

Lemma 3.7. *Let a, b be $\mathcal{D}$-equivalent elements in a ternary semigroup T. Then $|H_a| = |H_b|$.*

Proof. If c is such that $a\mathcal{R}c$, $c\mathcal{L}b$, then there exists $p, q, r, s, x, y, u, v \in T$ such that $a = [crs]$, $c = [apq]$ and $b = [xyc]$, $c = [uvb]$. Then by Lemmas 3.5 and 3.6 we see that $\rho_{pq}|H_a$ is a bijection onto H_c and $\lambda_{xy}|H_c$ is a bijection onto H_b. Thus $\rho_{pq}\lambda_{xy}$ is a bijection from H_a onto H_b. Therefore, $|H_a| = |H_b|$. □

Corollary 3.8. *If $x, y, z \in T$ are such that $[xyz] \in H_x$, then ρ_{yz} is a bijection of H_x onto itself. If $[xyz] \in H_z$, then λ_{xy} is a bijection of H_z onto itself.* □

Theorem 3.9. *If H is an $\mathcal{H}$-class of a ternary semigroup T, then we have either $H^{<1>} \cap H = \emptyset$ or $H^{<1>} = H$ and H is a ternary subgroup of T.*

Proof. Suppose that $H^{<1>} \cap H \neq \emptyset$. Then there exists $a, b, c \in H$ such that $[abc] \in H$. By Corollary 3.8, the right translation ρ_{bc} and the left translation λ_{ab} are bijections of H onto itself. Hence $[hbc] \in H$ and $[abh] \in H$ for every $h \in H$. Also ρ_{bh} and λ_{hb} are bijections of H onto itself. Therefore $[hbH] = H$ and $[Hbh] = H$. By Lemma 2.1, H is a right and left simple ternary semigroup. Therefore by Theorem 1.1 in [13], H is a ternary group. □

4. Minimal ideals

If A, B are two-sided ideals of a ternary semigroup T, then A and B both contain the product $[ATB]$. Therefore there can be almost one minimal two-sided ideal of T. Similarly we see that if A and B are ideals of a ternary semigroup, then $[ATB] \cup [TATBT]$ is an ideal of T contained in A and B. Therefore a minimal ideal (if it exists) is unique. If $T = [a] = \{a^{<n>}, n \geqslant 0\}$ is a cyclic ternary semigroup, then $[a] = (a) \supset (a)^3 \supset \ldots$ is an infinite descending chain of ideals of T and T does not have a minimal two-sided ideal. If T is a finite cyclic ternary semigroup, then $T = \{a^{<n>} : a^{<m>} = a^{<m+r>} : m = \text{index},\ r = \text{period}\}$ and $T = (a) \supset (a)^3 \supset \cdots \supset K_a$, where $K_a = \{a^{<m>}, \ldots, a^{<m+r-1>}\}$ is the unique minimal ideal of T.

If a non-zero ideal M of a ternary semigroup T with 0 ,is said to be 0-*minimal* if $M \neq (0)$ and (0) is the only ideal of T contained in M. Similarly 0-minimal left (right, two-sided) ideals are defined.

Lemma 4.1. *Let L be a minimal left ideal of a ternary semigroup T and let $x, y \in T$. Then $[Lxy]$ is a minimal left ideal of T.*

Proof. $[Lxy]$ is a left ideal of T. Let M be a left ideal of T contained in $[Lxy]$. Consider the set $N = \{n \in L : [nxy] \in M\}$. Then $[Nxy] = M$. For $t_1, t_2 \in T$, and $n \in N$ $[[t_1t_2n]xy] = [t_1t_2[nxy]] \in [TTM] \subseteq M$. Therefore $[t_1t_2n] \in N$ and so N is a left ideal of T contained in L. From the minimality of L we obtain $N = L$. Therefore $M = [Lxy]$ and so $[Lxy]$ is minimal. □

Theorem 4.2. *Let M be a minimal two-sided ideal of a ternary semigroup T. Then M is a t-simple ternary subsemigroup of T.*

Proof. $M^{<1>}$ is a two-sided ideal of T contained in M. Therefore $M^{<1>} = M$. For any $a \in M$, $(a)_t = a \cup [TTa] \cup [aTT] \cup [TTaTT]$ is a two-sided ideal of T contained in M. Therefore $(a)_t = M$. Consequently, $M = M^{<1>} = M^{<2>} = [MM(a)_tMM] = [MM(a \cup [TTa] \cup [aTT] \cup [TTaTT])MM] = [MMaMM] \subseteq M^{<2>} = M$. Thus, $M = [MMaMM]$ for all $a \in M$ and so M is a t-simple ternary semigroup by Lemma 2.9. □

Let K denote the intersection of all two-sided ideals and K^* the intersection of all ideals of a ternary semigroup T. Clearly $K \subset K^*$. Suppose $K \neq \emptyset$.

Lemma 4.3. *K is a t-simple ternary semigroup.*

Proof. For $a \in K$, $(a)_t = a \cup [TTa] \cup [aTT] \cup [TTaTT]$ is a two-sided ideal of T contained in K and so $K = (a)_t$ for all $a \in K$. Thus K is the unique minimal two-sided ideal of T and so K is t-simple by Theorem 4.2. □

Lemma 4.4. *K^* is a simple ternary semigroup.*

Proof. For $a \in K^*$, $(a) = a \cup [TTa] \cup [aTT] \cup [TaT] \cup [TTaTT] \subset K$. Therefore $K^* = (a)$ for all $a \in K^*$. Hence K^* is a simple ternary semigroup. □

Lemma 4.5. $K^* = [TKT]$.

Proof. Since $K \subseteq K^*$, we have $[TKT] \subseteq [TK^*T] \subseteq K^*$. Thus $[TT(K \cup [TKT])] = [TTK] \cup [TTTKT] = [TTK] \cup [TKT] \subseteq K \cup [TKT]$. Therefore $K \cup [TKT]$ is a left ideal of T. Similarly, $K \cup [TKT]$ is an ideal and so $K^* \subseteq K \cup [TKT]$. Since $K \subseteq K^*$ we have $K^* \subseteq [TKT]$. Therefore $K^* = [TKT]$. □

Theorem 4.6. $K = K^*$.

Proof. Put $M = [KK^*K]$. Then $M \subset K$; $M \subseteq K^*$. M is an ideal and so $K^* \subset M$. Therefore $K^* = M$. Similarly $K = M$. Hence $K = K^*(= M)$. □

Definition 4.7. If $K = K^*$ is nonempty, then it is called the *kernel* of T.

Lemma 4.8. *If L is a 0-minimal left ideal of a ternary semigroup T with 0 such that $L^{<1>} \neq (0)$, then $L = [TTa]$ for every element $a \neq 0$ of L.*

Proof. For any $a \neq (0)$ in L, $[TTa]$ is clearly a left ideal of T contained in L. If $[TTa] = (0)$ then $a^{<1>} = (0)$ and $\{0, a\}$ is a non-zero left ideal of T contained in L and so $\{0, a\} = L$ and $L^{<1>} = (0)$, a contradiction. Hence $[TTa] \neq (0)$ and so $[TTa] = L$. □

Lemma 4.9. *Let L be a 0-minimal left ideal of a ternary semigroup T with 0 and let $x, y \in T$. Then $[Lxy]$ is either (0) or a 0-minimal left ideal of T.*

Proof. Assume that $[Lxy] \neq (0)$. Then $[Lxy]$ is a left ideal of T. Let M be a left ideal of T contained in $[Lxy]$. Let $N = \{n \in L : [nxy] \in M\}$. Then $[Nxy] = M$. Recalling the proof of Lemma 4.1, it can be shown that N is a left ideal of T so that $N = (0)$ or $N = L$. Therefore either $M = (0)$ or $M = [Lxy]$ proving that $[Lxy]$ is a 0-minimal left ideal. □

Theorem 4.10. *Let M be a 0-minimal two-sided ideal of a ternary semigroup with zero 0. Then either $M^{<1>} = (0)$ or M is a 0-t-simple ternary subsemigroup of T.*

Proof. $M^{<1>}$ is an two-sided ideal of T contained in M. Therefore $M^{<1>} = (0)$ or $M^{<1>} = M$. Suppose $M^{<1>} \neq (0)$. Then $M = M^{<1>} = M^{<2>}$. As in the proof of Theorem 4.2, we can show that for every $a \in M$, $a \neq 0$ $M = [MMaMM]$. Thus M is a 0-t-simple ternary semigroup. □

Theorem 4.11. *Let T be a ternary semigroup with 0. If a 0-minimal two-sided ideal M of T contains at least one 0-minimal left ideal of T, then M is the union of all the 0-minimal left ideals of T contained in M.*

Proof. Let N be the union of all the 0-minimal left ideal of T contained in M. Clearly N is a left ideal of T. We prove that N is a right ideal. Let $n \in N$ and $x, y \in T$. By the definition, $n \in L$ for some 0- minimal left ideal L of T contained in M. By Lemma 4.9, $[Lxy] = (0)$ or $[Lxy]$ is a 0-minimal left ideal of T. Moreover, $[Lxy] \subseteq [Mxy] \subseteq M$ and hence $[Lxy] \subseteq N$. Therefore $[nxy] \in N$, for all $n \in N$. Hence, $N \neq (0)$ since it contains at least one 0-minimal left ideal of T. Thus N is a non-zero two-sided ideal of T contained in M. Therefore $N = M$, by the 0-minimality of M. □

Lemma 4.12. *Let M be a 0-minimal two-sided ideal of a ternary semigroup T with 0 such that $M^{<1>} \neq (0)$. Then also $L^{<1>} \neq (0)$ L for any non-zero left ideal of T contained in M.*

Proof. Since $[LTT]$ is two-sided ideal of T contained in M we have either $[LTT] = M$ or $[LTT] = (0)$. If $[LTT] = (0)$, then L is an ideal of T whence $L = M$, and so $M^{<1>} = [LMM] \subset [LTT] = (0)$, contrary to our hypothesis on M. Hence $[LTT] = M$ and so $M = M^{<1>} = [[LTT][LTT][LTT]] = [L[TTL][TTL]TT] \subseteq [[LLL]TT]$. Therefore $L^{<1>} \neq (0)$. □

Theorem 4.13. *Let M be a 0-minimal two-sided ideal of a ternary semigroup T with 0 such that $M^{<1>} \neq (0)$, and assume that M contains at least one 0-minimal left ideal of T. Then every left ideal of M is also a left ideal of T.*

Proof. Let L be a non-zero left ideal of M and $0 \neq a \in L$. By Theorem 4.10, M is 0-t-simple and so $M = [MMaMM]$. Hence $[MMa] \neq (0)$. By Theorem 4.10, there is 0-t-minimal left ideal L_1 of T such that $a \in L_1 \subseteq M$. Since $[MMa]$ is a non-zero left ideal of T contained in L_1, $[MMa] = L_1$. Therefore $a \in [MMa]$. Hence $L = \bigcup\{[MMa] : a \in L\}$ is a left ideal of T. □

Similar results can be proved for right ideals and also for 0-minimal ideals.

5. Completely 0-t-simple ternary semigroups

We recall [13, 14] that a pair of elements (a, b) of a ternary semigroup T is said to be an *idempotent pair* if $[ababt] = [abt]$ and $[tabab] = [tab]$. Two idempotent pairs (a, b) and (c, d) are said to be *equivalent* if $[abt] = [cdt]$ and $[tab] = [tcd]$. $\langle a, b\rangle$ denotes the equivalence class containing the idempotent pair (a, b). If (a, b),

(c, d) are idempotent pairs of T, then $(a, b) \leqslant (c, d)$ if $[abcdt] = [cdabt] = [abt]$ and $[tabcd] = [tcdab] = [tab]$. Then $\leqslant$ is a partial order on the set E of equivalence classes of idempotent pair of T. If S contains 0, then the class $\langle 0, 0\rangle$ is the least element of E. An idempotent pair (a, b) is said to be non-zero if (a, b) does not belong to $\langle 0, 0\rangle$. If T contains zero, a non-zero idempotent pair (u, v) is called *primitive* if $(a, b) \leqslant (u, v)$ for any idempotent pair (a, b) implies either $(a, b) = \langle 0, 0\rangle$ or $\langle a, b\rangle = \langle u, v\rangle$. If T does not contain zero, a primitive idempotent pair is similarly defined. A *completely* 0-*t-simple ternary semigroup* is a 0-t-simple ternary semigroup T containing a primitive idempotent pair.

Lemma 5.1. *If L is a* 0*-minimal left ideal of a ternary semigroup T, then $L\backslash\{0\}$ is an $\mathcal{L}$-class.*

Proof. For every $x \in L$, $[TTx]$ is a left ideal of T contained in L so that $[TTx] = (0)$ or $[TTx] = L$. Suppose $[TTx] = L$ for every $x \in L\backslash\{0\}$. Then $x \cup [TTx] = L = y \cup [TTy]$ for every $x, y \in L\backslash\{0\}$ and so $L\backslash\{0\}$ is contained in the $\mathcal{L}$-class L_x. If $y \in L_x$, then $y \in x \cup [TTx] = L$ so that $L_x \subseteq L\backslash\{0\}$. Therefore $L\backslash\{0\}$ is an $\mathcal{L}$-class of T. Suppose $[TTx] = (0)$ for some $x \in L$. Then $\{0, x\}$ is a non-zero left ideal of T contained in L so that $\{0, x\} = L$. Then $x \cup [TTx] = L$ and $x\mathcal{L}y$ implies $x = y$. Hence in this case also $L\backslash\{0\}$ is a $\mathcal{L}$-class of T. □

A similar result can be proved for 0-minimal right ideals.

Lemma 5.2. *Let T be a* 0*-t-simple ternary semigroup containing a* 0*-minimal left ideal and a* 0*-minimal right ideal. Then to each* 0*-minimal left ideal L of T there exists a* 0*-minimal right ideal R of T such that $[LRT] \neq (0)$ and $[LRT] = T$. Also $[LTR] \neq (0)$ and $[LTR] = T$.*

Proof. $[LTT]$ is a two-sided ideal of T so that $[LTT] = (0)$ or $[LTT] = L$. If $[LTT] = (0)$, then $L^{<1>} = (0)$ and L is a two-sided ideal of T so that $T = L$ and $T^{<1>} = L^{<1>} = (0)$ contrary to the hypothesis. Therefore $[LTT] = L$. Then for some $x \in T$ $[LxT] \neq (0)$. Since T is the union of all the 0-minimal right ideals of T (by the dual of Theorem 4.11), $x \in R$ for some 0-minimal right ideal R of T. Hence $[LRT] \neq (0)$, $[LRT]$ is a non-zero two sided ideal of T and so $[LRT] = T$. Similarly it can be shown that $[LTR] = T$. □

Lemma 5.3. *Let L and R be* 0*-minimal left and right ideals respectively of a* 0*-t-simple ternary semigroup T. Then $[LRT] \neq (0)$ if and only if $[TLR] \neq (0)$. In this case $[LRT] = T = [TLR]$.*

Proof. By Lemma 5.2, if $[LRT] \neq (0)$, then $[LRT] = T$. Then $T = T^{<1>} = [LRTTLRT]$, whence $[TLR] \neq (0)$. Then $[TLR] = T$. Conversely, if $[TLR] \neq (0)$, then we can show that $[TLR] = T$. Further, $T = T^{<1>} = [TLRTT]$. Therefore, $[LRT] \neq (0)$ and $[LRT] = T$. □

Lemma 5.4. *Let L (resp. R) be a* 0*-minimal left (right) ideal of a* 0*-t-simple ternary semigroup and $a \in L\backslash\{0\}$ (resp. $R\backslash\{0\}$). Then $[TTa] = L$ (resp. $[aTT] = R$).*

Proof. Since T is a 0-t-simple, by Lemma 2.9, $T = [TTaTT]$, so $[TTa] \neq (0)$. Since $[TTa]$ is a non-zero left ideal contained in L, $[TTa] = L$. Similarly we can show that $[aTT] = R$ for $a \in R \backslash \{0\}$. $\square$

Let T be a 0-t-simple ternary semigroup and L and R are 0-minimal left and right ideals of T such that $[LRT] \neq (0)$. Then we have the following result.

Lemma 5.5. *$[RTL]$ is a ternary group with* 0.

Proof. Since $[LRT] \neq (0)$ by Lemma 5.3 $[LRT] = T = [TLR]$. Then $T = T^{<1>} = [LRTLRTT]$ and so $[RTL] \neq (0)$. Choose $a \in [RTL]$, $a \neq 0$. Then $a \in R \cap L$. Then by Lemma 2.9, $T = [TTaTT]$ and so $[aTT] \neq (0)$. Therefore $[aTT] = R$. Similarly $[TTa] = L$ and $T = [TLR] = [TLaTT]$. Therefore $[TLa] \neq (0)$, so $[TLa] = L$. $[RTL] = [RTTLa] = [RT[LRT]La] = [[RTL][RTL]a]$ proving that $[RTL]$ is left simple. Similarly $[aRT] = R$ and $[a[RTL][RTL]] = [aR[LRT]TL] = [aRTTL] = [RTL]$. Therefore $[RTL]$ is right simple. Hence by Theorem 1.1 in [13], $[RTL]$ is a ternary group with 0. $\square$

Lemma 5.6. $[RTL] = R \cap L$.

Proof. Clearly $[RTL] \subset R \cap L$. By Lemma 5.1, $L \backslash \{0\}$ is a $\mathcal{L}$-class of T. Similarly $R \backslash \{0\}$ is a $\mathcal{R}$-class of T. Therefore $H = R \backslash \{0\} \cap L \backslash \{0\}$ is a $\mathcal{H}$-class of T. Since $[RTL]$ is a ternary group with 0, for every $a \in [RTL]$, $a \neq 0$ there exists the ternary group inverse a^{-1} of a in $[RTL]$. Thus (a, a^{-1}) is an idempotent pair in $[RTL]$ and for every $z \in [RTL]$, $z \neq 0$ $z = [za^{-1}a]$. Since $a, a^{-1}, z \in [RTL]$, $a, a^{-1}, z \in H$ and $z = [za^{-1}a] \in H^{<1>} \cap H$. Hence, by Theorem 3.9, H is a ternary group. Therefore $R \cap L$ is a ternary group with 0. If $z \in R \cap L$, $z \neq 0$, then, by Lemma 5.1, z and a are in some $\mathcal{L}$-class and so for some $u, v \in T$, we have $z = [uva] = [uvaa^{-1}a] = [za^{-1}a] \in [RTL]$. Therefore $[RTL] = R \cap L$. $\square$

Lemma 5.7. *For every non-zero idempotent pair (a, b) in $[RTL]$, $R = [abT]$, $L = [Tab]$ and $[RTL] = [abTab]$.*

Proof. Let (a, b) be a non-zero idempotent pair in $[RTL]$. If $[aba] = 0$, then $[abx] = [ababx] = 0$ for every $x \in [RTL]$. Similarly $[xab] = 0$. Therefore (a, b) is equivalent to the zero idempotent pair, contrary to the hypothesis that (a, b) is a non-zero idempotent pair. Therefore $[aba] \neq 0$ and $[bab] \neq 0$. Then $[Tab] \neq (0)$ and $[abT] \neq (0)$. If $L = [Tab]$ and $R = [abT]$, then $[RTL] = [abTTTab] = [abTab]$. In particular for every $a \in [RTL]$, $a \neq 0$, $[RTL] = [aa^{-1}Taa^{-1}]$. $\square$

Lemma 5.8. *Every idempotent pair in $[RTL]$ is primitive in T.*

Proof. Let (a, b) be an idempotent pair in $[RTL]$. Then $[aba]$ is regular with $[bab]$ as the inverse in $[RTL]$ and (a, b) and $([aba], [bab])$ are equivalent to $[RTL]$. Therefore $[abz] = [abababz] = z$ for all $z \in [RTL]$. Similarly $[zab] = z$. Since $[aba]$ is regular, $([aba], [bab])$ is an idempotent pair in T. Therefore for any $t \in T$, $[abt] = [[abababa]bt] = [ababt]$. Similarly $[tab] = [tabab]$. Thus (a, b) is an idempotent pair in T and $(a,b) \sim ([aba],[bab])$. Hence without loss of generality we can take an idempotent pair (a,a^{-1}) in $[RTL]$. $\square$

Lemma 5.9. *Let T be a completely 0-t-simple ternary semigroup and (a,b) a primitive idempotent pair in T. Then $[Tab]$ and $[abT]$ are 0-minimal left and right ideals of T, respectively.*

Proof. Since (a,b) is a primitive idempotent pair, as in Lemma 5.7 we see that $L=[Tab]$ is a non-zero left ideal of T and $R=[abT]$ is a non-zero right ideal. Let A be a non-zero right ideal of T contained in R. Let $x \neq 0$, $x \in A$. Then $x \in R$ and $[abx]=x$. Since T is 0-t-simple, $T=[TTxTT]$ (Lemma 2.9). Hence for some $u_i,v_i,w_i,z_i \in T$, $i=1,2$, $a=[u_1v_1xw_1z_1]$, $b=[u_2v_2xw_2z_2]$. Put $c_1=[abau_2v_2ab]$, $d_1=[w_2z_2aba]$, $c_2=[babu_1v_1ab]$, $d_2=[w_1z_1bab]$. We can easily show that $[c_1xd_1]=[aba]$ and $[c_2xd_2]=[bab]$. Clearly $c_i,d_i \neq 0$, $i=1,2$. So, $([c_1xd_1],[c_2xd_2])$ is an idempotent pair equivalent to (a,b). Also, $[abc_1]=c_1=[c_1ab]$, $[bac_2]=c_2=[c_2ab]$. Put $f_1=[xd_1c_2]$, $f_2=[xd_2c_1]$, $[c_1f_1xd_2a]=[aba]$ and $[c_2f_2xd_1b]=[bab]$. Therefore $f_1 \neq 0, f_2 \neq 0$. Further $[f_1f_2f_1]=f_1$ and $[f_2f_1f_2]=f_2$. Therefore (f_1,f_2) is a non-zero idempotent pair in T. Moreover, also $(f_1,f_2) \leqslant (a,b)$. Since (a,b) is a primitive idempotent pair, $(f_1,f_2) \sim (a,b)$. Therefore $R=[abT]=[f_1f_2T]=[xd_1c_2xd_2c_1T]=[xTT] \subseteq A$. Thus $R=A$ and R is 0-minimal. Let B be a non-zero left ideal of T contained in L. Let $x \in B$, $x \neq 0$. Since T is 0-t-simple, $T=[TTxTT]$. Hence we can find elements $u_i,v_i,w_i,z_i \in T$, $i=1,2$ such that $a=[u_1v_1xw_1z_1]$ and $b=[u_2v_2xw_2z_2]$. Put $c_1=[abau_2v_2]$, $d_1=[abw_2z_2aba]$, $c_2=[babu_1v_1]$, $d_2=[abw_1z_1bab]$. Then $[c_1xd_1]=[aba]$ and $[c_2xd_2]=[bab]$. Put $f_1=[d_1c_2x]$, $f_2=[d_2c_1x]$. As before we can show that (f_1,f_2) is a non-zero idempotent pair such that $(f_1,f_2) \leqslant (a,b)$. Therefore $(f_1,f_2) \sim (a,b)$. Hence $L=[Tab]=[Tf_1f_2]=[Td_1c_2xd_2c_1x] \subseteq [TTx] \subseteq B$. Therefore $L=B$ and L is 0-minimal. $\square$

Theorem 5.10. *Let T be 0-t-simple. T is completely 0-t-simple if and only if T contains at least one 0-minimal left ideal and at least one 0-minimal right ideal.*

Proof. If T is completely 0-t-simple, then T contains a primitive idempotent pair (a,b). By Lemma 5.9, $[Tab]$ and $[abT]$ are 0-minimal left ideal and 0-minimal right ideal, respectively. Conversely, assume that T contains at least one 0-minimal left ideal and one 0-minimal right ideal. Let L be a 0-minimal left ideal of T. Then by Lemma 5.2, there exists a 0-minimal right ideal R of T such that $[LTR] \neq (0)$. Then by Lemma 5.4, T contains a primitive idempotent pair and so T is completely 0-t-simple. $\square$

Corollary 5.11. *A completely 0-t-simple ternary semigroup is union of its 0-minimal left (right) ideals.*

Proof. Follows from the above Theorem and Lemma 5.4. $\square$

Corollary 5.12. *Let M be a 0-minimal two-sided ideal of a ternary semigroup T such that $M^{<1>} \neq (0)$. If M contains at least one 0-minimal left ideal and at least one 0-minimal right ideal, then M is a completely 0-t-simple ternary subsemigroup of T.* □

Theorem 5.13. *Let T be a completely 0-t-simple ternary semigroup. Then non-zero elements of T form a $\mathcal{D}$-class and T is regular.*

Proof. Let T be a completely 0-t-simple ternary semigroup. Let a, b be non-zero elements of T. Then a lies in some 0-minimal left ideal L and b lies in some 0-minimal right ideal R of T. Thus $L = [TTa]$ and $R = [bTT]$. By Lemma 5.1 $L\backslash\{0\}$ is the $\mathcal{L}$-class containing a, $R\backslash\{0\}$ is the $\mathcal{R}$-class containing b and $[bTa] \subseteq R \cap L$. Since T is 0-t-simple, $T = [TTaTT]$ and $T = [TTbTT]$. Hence $T = [TTT] = [TTbTTTTTaTT] = [TTbTaTT]$. Therefore $[bTa] \neq (0)$ and, by Lemma 5.1, its dual $[bTa] \subset \mathrm{R_b} \cap \mathrm{L_a}$. If $c \in \mathrm{R_b} \cap \mathrm{L_a}$, then $a\mathcal{L}c$, $c\mathcal{R}b$ so, $a\mathcal{D}b$. Since a completely 0-t-simple ternary semigroup T containing a primitive idempotent pair (u, v) then (u, v) and $[uvu]$ both belongs to $\mathcal{D}$, and $[uvu]$ is a regular element in $\mathcal{D}$. Therefore by Proposition 3.4 the $\mathcal{D}$-class $T\backslash\{0\}$ is regular. Hence T is regular. □

6. $\mathcal{M}$-ternary semigroups

Below we introduce the concept of $\mathcal{M}$-ternary semigroups generalizing the notion of Rees matrix ternary semigroups.

Let G be a ternary group. We consider $G \cup \{0\}$, where we extend the ternary multiplication in G to $G \cup \{0\}$ by putting $[abc] = 0$ whenever any of a, b, c is zero. Let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix with entries in $G \cup \{0\}$. P is said to be *regular* if for every $i \in I$ there exists $\lambda \in \Lambda$ such that $p_{\lambda i} \neq 0$ and for every $\lambda \in \Lambda$ there exists $i \in I$ such that $p_{\lambda i} \neq 0$. Consider the set

$$\mathcal{M}^0(G; I, \Lambda; P) = \{(a)_{i\lambda} : a \in G \cup \{0\},\ i \in I,\ \lambda \in \Lambda\},$$

where $(a)_{i\lambda}$ denotes the $\Lambda \times I$ matrix with entries a in (i, λ) position and 0 in other places. The $(0)_{i\lambda}$ is written as 0 and is independent of i and λ. We see that $(a)_{i\lambda} = (b)_{j\mu}$ if and only if $a = b$, $i = j$, $\lambda = \mu$. A ternary multiplication is introduced on this set as follows:

$$[(a)_{i\lambda}(b)_{j\mu}(c)_{k\nu}] = ([ap_{\lambda j}bp_{\mu k}c])_{i\nu}.$$

Lemma 6.1. *$\mathcal{M}^0(G; I, \Lambda; P)$ is a ternary semigroup.* □

Definition 6.2. The ternary semigroup $\mathcal{M}^0(G; I, \Lambda; P)$ is called a *$\mathcal{M}$-ternary semigroup* (Matrix ternary semigroup).

Lemma 6.3. *If P is regular, then $\mathcal{M}^0(G; I, \Lambda; P)$ is a regular ternary semigroup.*

Proof. For given $(a)_{i\lambda}$ consider the elements $([p_{\lambda j}^{-1}a^{-1}p_{\mu i}^{-1}])_{j\mu}$ for every (j,μ). The set $\{([p_{\lambda j}^{-1}a^{-1}p_{\mu i}^{-1}])_{j\mu}\}$ of non-zero element is the set $I((a)_{i\lambda})$ of all inverses of $(a)_{i\lambda}$. □

Corollary 6.4. *If P is regular, then the pair $((a)_{i\lambda},([p_{\lambda j}^{-1}a^{-1}p_{\mu i}^{-1}])_{j\mu})$ is an idempotent pair.* □

Lemma 6.5. *If P is regular, then the idempotent pairs $((a)_{i\lambda},([p_{\lambda j}^{-1}a^{-1}p_{\mu i}^{-1}])_{j\mu})$ and $((b)_{k\nu},([p_{\nu l}^{-1}b^{-1}p_{\omega k}^{-1}])_{l\omega})$ are equivalent if and only if $k=i$ and $\mu=\omega$.*

Proof. Suppose that $((a)_{i\lambda},([p_{\lambda j}^{-1}a^{-1}p_{\mu i}^{-1}])_{j\mu})$ and $((b)_{k\nu},([p_{\nu l}^{-1}b^{-1}p_{\omega k}^{-1}])_{l\omega})$ are idempotent pairs. Then for all $z=(z)_{m\delta}$ we have $[xx_1z]=[(a)_{i\lambda}([p_{\lambda j}^{-1}a^{-1}p_{\mu i}^{-1}])_{j\mu})(z)_{m\delta}]$ $=([p_{\mu i}^{-1}p_{\mu m}z])_{i\delta}$ and $[yy_1z]=[(b)_{k\nu}([p_{\nu l}^{-1}b^{-1}p_{\omega k}^{-1}])_{l\omega})(z)_{m\delta}]=([p_{\omega k}^{-1}p_{\omega m}z])_{k\delta}$. They are equivalent if and only if $i=k$ and $\omega=\mu$. In the same manner we obtain $[zxx_1]=[(z)_{m\delta}(a)_{i\lambda}([p_{\lambda j}^{-1}a^{-1}p_{\mu i}^{-1}])_{j\mu})]=([zp_{\delta i}p_{\mu i}^{-1}])_{m\mu}$, and analogously, $[zyy_1]=[(z)_{m\delta}(b)_{k\nu}([p_{\nu l}^{-1}b^{-1}p_{\omega k}^{-1}])_{l\omega})]=([zp_{\delta k}p_{\omega k}^{-1}])_{m\omega}$. Therefore, $[zxx_1]=[zyy_1]$ if and only if $k=i$ and $\omega=\mu$. □

Theorem 6.6. *If P is regular, then $\mathcal{M}^0(G;I,\Lambda;P)$ is a 0-t-simple ternary semigroup.*

Proof. For $(a)_{i\lambda}$, $(b)_{j\mu}$ we have $[(a^{-1})_{j\gamma}([p_{\gamma i}^{-1}ap_{\gamma i}^{-1}])_{i\gamma}(a)_{i\lambda}([p_{\lambda k}^{-1}a^{-1}p_{\lambda k}^{-1}])_{k\lambda}(b)_{k\mu}]$ $=([a^{-1}p_{\gamma i}p_{\gamma i}^{-1}ap_{\gamma i}^{-1}p_{\gamma i}ap_{\lambda k}p_{\lambda k}^{-1}a^{-1}p_{\lambda k}^{-1}bp_{\lambda k}])_{j\mu}=[(b)_{j\mu}]$. Hence $\mathcal{M}^0(G;I,\Lambda;P)$ is a 0-t-simple ternary semigroup by Lemma 2.9. □

Theorem 6.7. *If P is regular, then in $\mathcal{M}^0(G;I,\Lambda;P)$ every idempotent pair is primitive.*

Proof. Suppose that $((a)_{i\lambda},([p_{\lambda j}^{-1}a^{-1}p_{\mu i}^{-1}])_{j\mu})$ and $((b)_{k\nu},([p_{\nu l}^{-1}b^{-1}p_{\omega k}^{-1}])_{l\omega})$ are idempotent pairs. If $(x,x_1)\leqslant(y,y_1)$ for some $x=(a)_{i\lambda}$, $x_1=([p_{\lambda j}^{-1}a^{-1}p_{\mu i}^{-1}])_{j\mu})$, $y=(b)_{k\nu}$ and $y_1=([p_{\nu l}^{-1}b^{-1}p_{\omega k}^{-1}])_{l\omega}$, then for any $z=(t)_{m\alpha}\in\mathcal{M}^0(G;I,\Lambda;P)$ we have $[xx_1yy_1z]=[(a)_{i\lambda}([p_{\lambda j}^{-1}a^{-1}p_{\mu i}^{-1}])_{j\mu})(b)_{k\nu}([p_{\nu l}^{-1}b^{-1}p_{\omega k}^{-1}])_{l\omega})(t)_{m\alpha}]$ and $[yy_1xx_1z]=$ $[(b)_{k\nu}([p_{\nu l}^{-1}b^{-1}p_{\omega k}^{-1}])_{l\omega})(a)_{i\lambda}([p_{\lambda j}^{-1}a^{-1}p_{\mu i}^{-1}])_{j\mu})(t)_{m\alpha}]$, which obviously implies that $[ap_{\lambda j}p_{\lambda j}^{-1}a^{-1}p_{\mu j}^{-1}p_{\mu k}bp_{\nu l}p_{\nu l}^{-1}b^{-1}p_{\omega k}^{-1}p_{\omega\alpha}t]_{i\alpha}=[bp_{\nu l}p_{\nu l}^{-1}b^{-1}p_{\omega k}^{-1}p_{\omega i}ap_{\lambda j}p_{\lambda j}^{-1}a^{-1}p_{\mu j}^{-1}p_{\mu\alpha}t]_{k\alpha}$ $=([p_{\mu i}^{-1}p_{\mu\alpha}])_{i\alpha}$ Therefore $i=k$. Using the same method we can see that $[zxx_1yy_1]$ $=[(t)_{m\alpha}(a)_{i\lambda}([p_{\lambda j}^{-1}a^{-1}p_{\mu i}^{-1}])_{j\mu})(b)_{k\nu}([p_{\nu l}^{-1}b^{-1}p_{\omega k}^{-1}])_{l\omega})]$. Analogously, $[zyy_1xx_1]=$ $[(t)_{m\alpha}(b)_{k\nu}([p_{\nu l}^{-1}b^{-1}p_{\omega k}^{-1}])_{l\omega})(a)_{i\lambda}([p_{\lambda j}^{-1}a^{-1}p_{\mu i}^{-1}])_{j\mu})]$. From the above we obtain that $[tp_{\alpha i}ap_{\lambda j}p_{\lambda j}^{-1}a^{-1}p_{\mu j}^{-1}p_{\mu k}bp_{\nu l}p_{\nu l}^{-1}b^{-1}p_{\omega k}^{-1}]_{m\omega}=[tp_{\alpha k}bp_{\nu l}p_{\nu l}^{-1}b^{-1}p_{\omega k}^{-1}p_{\omega i}ap_{\lambda j}p_{\lambda j}^{-1}a^{-1}p_{\mu j}^{-1}]_{m\mu}$ Therefore, (x,x_1) is primitive if and only if $k=i$, $\omega=\mu$. This, by Lemma 6.5, means that (x,x_1) and (y,y_1) are equivalent. Thus every idempotent pair is primitive. □

As a consequence of Theorem 6.6 and Theorem 6.7 we obtain the following corollary.

Corollary 6.8. *If P is regular, then $\mathcal{M}^0(G; I, \Lambda; P)$ is a completely 0-t-simple semigroup.* □

Lemma 6.9. *If P is regular, then in $\mathcal{M}^0(G; I, \Lambda; P)$*

$$(a)_{i\lambda}\mathcal{L}(b)_{j\mu} \Longleftrightarrow \lambda = \mu,$$

$$(a)_{i\lambda}\mathcal{R}(b)_{j\mu} \Longleftrightarrow i = j.$$

Corollary 6.10. *If P is regular, then non-zero elements of $\mathcal{M}^0(G; I, \Lambda; P)$ form a single $\mathcal{D}$-class in G.*

Proof. Indeed, $(a)_{i\lambda}\mathcal{L}(c)_{j\lambda}\mathcal{R}(b)_{j\mu}$ for any $c \in G$. □

It is clear that the set of non-zero $\mathcal{L}$-classes in $\mathcal{M}^0(G; I, \Lambda; P)$ is $\{L_\lambda; \lambda \in \Lambda\}$, where $L_\lambda = \{(a)_{i\lambda} : a \in G,\ i \in I\}$. Similarly, the set of non-zero $\mathcal{R}$-classes is $\{R_i : i \in I\}$, where $R_i = \{(a)_{i\lambda} : a \in G, \lambda \in \Lambda\}$.

Corollary 6.11. *If P is regular, then $H_{i\lambda} = L_\lambda \cap R_i = \{(a)_{i\lambda} : a \in G\}$.* □

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References

[1] **A. H. Clifford and G. B. Preston**, *The algebraic theory of semigroups*, vol. *I*, A.M.S. Providence (1961).

[2] **V. N. Dixit and S. Dewan**, *Congruence and Green's equivalence relation on ternary semigroups*, Commun. Fac. Sci. Univ. Ank. Series A1, **46** (1997), 103 – 117.

[3] **W. Dörnte**, *Untersuchungen über einen verallgemeinerten Gruppenbegriff*, Math. Zeitschr. **29** (1928), 1 – 19.

[4] **W. A. Dudek**, *Idempotents in n-ary semigroups*, Southeast Asian Bull. Math. **25** (2001), 97 – 104.

[5] **W. A. Dudek and K. Glazek**, *Around the Hosszú-Gluskin Theorem for n-ary groups*, Discrete Math. **308** (2008), 4861 – 4876.

[6] **W. A. Dudek and I. Grozdzinska**, *On ideals in regular n-semigroups*, Mat. Bilten (Skopje) **3-4(29-30)** (1979-1980), 35 – 44.

[7] **J. M. Howie**, *An introduction to semigroup theory*, Academic Press (1976).

[8] **A. Iampan**, *Lateral ideals of ternary semigroups*. Ukrainian Math. Bull. **4** (2007), 517526.

[9] **A. Iampan**, *The minimality and maximality of left (right) ideals in ternary semigroups*, Int. J. Contemp. Math. Sci. **5** (2010), 49 – 52.

[10] **S. Kar and B. K. Maity**, *Congruences on ternary semigroups*, J. Chungcheong Math. Soc. **20** (2007), 191 – 201.

[11] **S. Kar and B. K. Maity**, *Some ideals of ternary semigroups*, Anales Sti. Univ. "Al. I. Cuza" ser. Math. **57** (2011), 247 – 258.

[12] **E. L. Post**, *P*olyadic groups, Trans. Amer. Math. Soc. **48** (1940), 208 – 350.

[13] **M. L. Santiago**, *Regular ternary semigroups*, Bull. Calcutta Math. Soc. **82** (1990), 67 – 71.

[14] **M. L. Santiago and S. Sri Bala**, *Ternary semigroups*, Semigroup Forum **81** (2010), 380 – 388.

[15] **M. L. Santiago and S. Sri Bala**, *Completely regular ternary semigroups*, J. Calcutta Math. Soc. **7** (2011), 105 – 120.

[16] **F. M. Sioson**, *Ideal theory in ternary semigroups*, Math. Japon. **10** (1965), 63 – 84.

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