Some subsets in bitopological spaces and bioperations continuity associated

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Abstract. The aim of this paper is to introduce and study some new types of sets via bioperations and to define and characterize some new forms of bioperations continuity using these sets and their relationships.

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1 Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity, separation axioms etc. by utilizing generalized open sets. Kasahara [1] defined the concept of an operation on topological spaces. Umehara et al. [6] introduced the notion of $\tau(\gamma, \gamma')$, which is the collection of all $(\gamma, \gamma')$-open sets in a topological space $(X, \tau)$. Carpintero et al. (see [3, 4] and [5]) obtained new decomposition forms of bioperation continuity. The aim of this paper is to introduce and study some new types of sets via bioperations and introducing new associated bioperation continuity with these new sets and relationships between them.

2 Preliminaries

The closure and the interior of a subset $A$ of $(X, \tau)$ are denoted by $(\gamma, \gamma')\text{Cl}(A)$ and $(\gamma, \gamma')\text{Int}(A)$, respectively.

Definition 2.1. [1] Let $(X, \tau)$ be a topological space. An operation $\gamma$ on the topology $\tau$ is a function from $\tau$ onto power set $\mathcal{P}(X)$ of $X$ such that $V \subset V^{\gamma}$ for each $V \in \tau$, where $V^{\gamma}$ denotes the value of $\tau$ at $V$. It is denoted by $\gamma : \tau \rightarrow \mathcal{P}(X)$.

Definition 2.2. [6] A topological space $(X, \tau)$ equipped with two operations, say $\gamma$ and $\gamma'$, defined on $\tau$ is called a bioperation topological space, it is denoted by $(X, \tau, \gamma, \gamma')$. 

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Definition 2.3. [6] A subset $A$ of a bioperation topological space $(X, \tau, \gamma, \gamma')$ is said to be $(\gamma, \gamma')$-open if for each $x \in A$ there exist open neighborhoods $U$ and $V$ of $x$ such that $U^\tau \cup V^\tau \subseteq A$. The complement of a $(\gamma, \gamma')$-open set is called a $(\gamma, \gamma')$-closed set. $T_{(\gamma, \gamma')}$ denotes the set of all $(\gamma, \gamma')$-open sets in $(X, \tau, \gamma, \gamma')$.

Definition 2.4. [6] For a subset $A$ of a bioperation topological space $(X, \tau, \gamma, \gamma')$, $(\gamma, \gamma')$-Cl$(A)$ denotes the intersection of all $(\gamma, \gamma')$-closed sets containing $A$, that is, $(\gamma, \gamma')$-Cl$(A) = \cap \{F : A \subseteq F, X \setminus F \in T_{(\gamma, \gamma')}\}$.

Definition 2.5. Let $A$ be any subset of $X$. The $(\gamma, \gamma')$-Int$(A)$ is defined as $(\gamma, \gamma')$-Int$(A) = \bigcup \{U : U \text{ is a } (\gamma, \gamma')\text{-open set and } U \subseteq A\}$.

Definition 2.6. Let $(X, \tau)$ be a topological space and $A$ be a subset of $X$ and $\gamma$ and $\gamma'$ be operations on $\tau$. Then $A$ is said to be

1. $(\gamma, \gamma')$-\alpha\text{-open if } $A \subseteq (\gamma, \gamma')$-Int$(\{\gamma, \gamma\text{-}Cl((\gamma, \gamma')\text{-Int}(A))\}$,
2. $(\gamma, \gamma')$-\preopen [2] if $A \subseteq (\gamma, \gamma')$-Int$(\{\gamma, \gamma'\text{-}Cl(A)\}$,
3. $(\gamma, \gamma')$-\preclosed if $(\gamma, \gamma')$-Cl$(\{\gamma, \gamma'\text{-}Int(A)\}) \subseteq A$,
4. $(\gamma, \gamma')$-\semi\text{-open if $A \subseteq (\gamma, \gamma')$-Cl$(\{\gamma, \gamma'\text{-}Int(A)\}$,
5. $(\gamma, \gamma')$-\semi\text{-closed if $(\gamma, \gamma')$-Int$(\{\gamma, \gamma'\text{-}Cl(A)\}) \subseteq A$,
6. $(\gamma, \gamma')$-\semi\text{-\preopen if $A \subseteq (\gamma, \gamma')$-Cl$(\{\gamma, \gamma'\text{-}Int((\gamma, \gamma')\text{-Cl}(A))\})$,
7. $(\gamma, \gamma')$-\semi\text{-\preclosed if $(\gamma, \gamma')$-Int$(\{\gamma, \gamma'\text{-}Cl((\gamma, \gamma')\text{-Int}(A))\}) \subseteq A$,
8. $(\gamma, \gamma')$-\regular\text{-open if $A = (\gamma, \gamma')$-Int$(\{\gamma, \gamma'\text{-}Cl(A)\}$,
9. $(\gamma, \gamma')$-\regular\text{-closed if $A = (\gamma, \gamma')$-Cl$(\{\gamma, \gamma'\text{-}Int(A)\}$,
10. $(\gamma, \gamma')$-t\text{-set if $(\gamma, \gamma')$-Int$(\{\gamma, \gamma'\text{-}Cl(\{S\})\} = (\gamma, \gamma')$-Int$(S)$,
11. $(\gamma, \gamma')$-\alpha\text{-*\text{-set if $(\gamma, \gamma')$-Int$(\{\gamma, \gamma'\text{-}Cl((\gamma, \gamma')\text{-Int}(S))\} = (\gamma, \gamma')$-Int$(S)$,
12. $(\gamma, \gamma')$-A\text{-set if $A = G \cap H$, where $G \in \tau_{(\gamma, \gamma')}$ and $H$ is a $(\gamma, \gamma')$-\regular\text{-closed set},
13. $(\gamma, \gamma')$-B\text{-set if $A = G \cap H$, where $G \in \tau_{(\gamma, \gamma')}$ and $H$ is a $(\gamma, \gamma')$-t\text{-set},
14. $(\gamma, \gamma')$-C\text{-set if $A = G \cap H$, where $G \in \tau_{(\gamma, \gamma')}$ and $H$ is a $(\gamma, \gamma')$-\alpha\text{-*\text{-set.}
15. $(\gamma, \gamma')$-\text{locally closed set if $A = G \cap H$, where $G \in \tau_{(\gamma, \gamma')}$ and $H$ is a $(\gamma, \gamma')$-\text{closed set.}

The family of all $(\gamma, \gamma')$-\alpha\text{-open (resp. $(\gamma, \gamma')$-\semi\text{-open, $(\gamma, \gamma')$-\preopen, $(\gamma, \gamma')$-\semi\text{-\preopen, $(\gamma, \gamma')$-\regular\text{-open, $(\gamma, \gamma')$-\regular\text{-closed, $(\gamma, \gamma')$-A, $(\gamma, \gamma')$-B, $(\gamma, \gamma')$-C, $(\gamma, \gamma')$-\text{locally closed}) sets of $(X, \tau, \gamma, \gamma')$ is denoted by $\alpha O(X)$ (resp. $SO(X), PO(X), SPO(X)$, $RO(X), RC(X), A(X), B(X), C(X), LC(X)$).
Definition 2.7. A function \( f : (X, \tau, \gamma, \gamma') \rightarrow (Y, \sigma, \beta, \beta') \) is said to be

1. \((\gamma, \gamma')-(\beta, \beta')\)-continuous if \( f^{-1}(V) \) is \((\gamma, \gamma')\)-open in \( X \) for every \( V \in \sigma_{(\beta, \beta)} \),
2. \((\gamma, \gamma')-(\beta, \beta')\)-\(\alpha\)-continuous if \( f^{-1}(V) \) is \((\gamma, \gamma')\)-\(\alpha\)-open in \( X \) for every \( V \in \sigma_{(\beta, \beta)} \),
3. \((\gamma, \gamma')-(\beta, \beta')\)-precontinuous if \( f^{-1}(V) \) is \((\gamma, \gamma')\)-preopen in \( X \) for every \( V \in \sigma_{(\beta, \beta)} \),
4. \((\gamma, \gamma')-(\beta, \beta')\)-semi-continuous if \( f^{-1}(V) \) is \((\gamma, \gamma')\)-semi-open in \( X \) for every \( V \in \sigma_{(\beta, \beta)} \),
5. \((\gamma, \gamma')-(\beta, \beta')\)-semi-precontinuous if \( f^{-1}(V) \) is \((\gamma, \gamma')\)-semi-preopen in \( X \) for every \( V \in \sigma_{(\beta, \beta)} \),
6. \((\gamma, \gamma')-(\beta, \beta')\)-completely continuous if \( f^{-1}(V) \) is \((\gamma, \gamma')\)-regular open in \( X \) for every \( V \in \sigma_{(\beta, \beta)} \),
7. \((\gamma, \gamma')-(\beta, \beta')\)-\(\mathcal{A}\)-continuous if \( f^{-1}(V) \) is \((\gamma, \gamma')\)-\(\mathcal{A}\)-set in \( X \) for every \( V \in \sigma_{(\beta, \beta)} \),
8. \((\gamma, \gamma')-(\beta, \beta')\)-\(\mathcal{B}\)-continuous if \( f^{-1}(V) \) is \((\gamma, \gamma')\)-\(\mathcal{B}\)-set in \( X \) for every \( V \in \sigma_{(\beta, \beta)} \),
9. \((\gamma, \gamma')-(\beta, \beta')\)-\(\mathcal{C}\)-continuous if \( f^{-1}(V) \) is \((\gamma, \gamma')\)-\(\mathcal{C}\)-set in \( X \) for every \( V \in \sigma_{(\beta, \beta)} \),
10. \((\gamma, \gamma')-(\beta, \beta')\)-\(\mathcal{LC}\)-continuous if \( f^{-1}(V) \) is \((\gamma, \gamma')\)-\(\mathcal{LC}\)-set in \( X \) for every \( V \in \sigma_{(\beta, \beta)} \).

3 Some subsets in bioperation spaces

Definition 3.1. A subset \( F \) of a bioperation topological space \((X, \tau, \gamma, \gamma')\) is said to be:

1. a \((\gamma, \gamma')\)-\(A_1\)-set if \( F = G \cap H \), where \( G \) is a \((\gamma, \gamma')\)-\(\alpha\)-open set and \( H \) is a \((\gamma, \gamma')\)-regular closed set,
2. a \((\gamma, \gamma')\)-\(A_2\)-set if \( F = G \cap H \), where \( G \) is a \((\gamma, \gamma')\)-\(\alpha\)-open set and \( H \) is a \((\gamma, \gamma')\)-locally closed set,
3. a \((\gamma, \gamma')\)-\(A_3\)-set if \( F = G \cap H \), where \( G \) is a \((\gamma, \gamma')\)-\(\alpha\)-open set and \( H \) is a \((\gamma, \gamma')\)-\(t\)-set,
4. a \((\gamma, \gamma')\)-\(A_4\)-set if \( F = G \cap H \), where \( G \) is a \((\gamma, \gamma')\)-\(\alpha\)-open set and \( H \) is a \((\gamma, \gamma')\)-pre-closed set,
5. a \((\gamma, \gamma')\)-\(A_5\)-set if \( F = G \cap H \), where \( G \) is a \((\gamma, \gamma')\)-\(\alpha\)-open set and \( H \) is a \((\gamma, \gamma')\)-semi-preclosed set.
The families of all \((\gamma, \gamma')-\alpha A_i\) subsets of \((X, \tau, \gamma, \gamma')\) are denoted by \(\alpha A_i(X)\), where \(i = 1, 2, 3, 4, 5\).

**Proposition 3.2.** For a bipoeration topological space \((X, \tau, \gamma, \gamma')\), we have the following:

1. Every \((\gamma, \gamma')-\alpha A_1\)-set is a \((\gamma, \gamma')-\alpha A_2\)-set.
2. Every \((\gamma, \gamma')-\alpha A_2\)-set is a \((\gamma, \gamma')-\alpha A_3\)-set.
3. Every \((\gamma, \gamma')-\alpha A_3\)-set is a \((\gamma, \gamma')-\alpha A_4\)-set.
4. Every \((\gamma, \gamma')-\alpha A_2\)-set is a \((\gamma, \gamma')-\alpha A_1\)-set.
5. Every \((\gamma, \gamma')-\alpha A_4\)-set is a \((\gamma, \gamma')-\alpha A_5\)-set.
6. The notions of \((\gamma, \gamma')-\alpha A_3\)-sets and \((\gamma, \gamma')-\alpha A_4\)-sets are independent.

**Proof.** (1). Let \(F = G \cap H \in \alpha A_1(X)\), where \(G \in \alpha O(X)\) and \((\gamma, \gamma')-\text{Cl}((\gamma, \gamma')-\text{Int}(H)) = H\). Since \((\gamma, \gamma')-\text{Cl}((\gamma, \gamma')-\text{Int}(H)) = (\gamma, \gamma')-\text{Cl}(H) = H\), we obtain \(F \in \alpha A_2(X)\).

The converses of items (2), (3), (4) and (5) are similar. In order to prove (6) it is necessary to find counterexamples and we will write them later.

The converses of Proposition 3.2 are not true as shown in the following examples.

**Example 3.3.** Let \(X = \{a, b, c\}\) and \(\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}\). We define the operations \(\gamma, \gamma' : \tau \to \mathcal{P}(X)\) as follows

\[
A^\gamma = \left\{ \begin{array}{ll} A & \text{if } a \in A, \\ A \cup \{a\} & \text{if } a \notin A, \end{array} \right. \quad \text{and} \quad A^{\gamma'} = \left\{ \begin{array}{ll} A & \text{if } a \in A, \\ \text{Cl}(A) & \text{if } a \notin A. \end{array} \right.
\]

Observe that:

1. \(\tau(\gamma, \gamma') = \emptyset, \{a\}, \{a, b\}, \{a, c\}, X\).
2. \((\gamma, \gamma')-\alpha A_1\)-set = \(\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\).
3. \((\gamma, \gamma')-\alpha A_2\)-set = \(\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\).

Clearly, \(\{b\} \in \alpha A_2(X) \setminus \alpha A_1(X)\).

**Example 3.4.** Let \(X = \{a, b, c\}\) and \(\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}\). We define the operations \(\gamma, \gamma' : \tau \to \mathcal{P}(X)\) as follows

\[
A^\gamma = \left\{ \begin{array}{ll} A & \text{if } b \notin A, \\ \text{Cl}(A) & \text{if } b \in A, \end{array} \right. \quad \text{and} \quad A^{\gamma'} = \left\{ \begin{array}{ll} \text{Cl}(A) & \text{if } b \notin A, \\ A & \text{if } b \in A. \end{array} \right.
\]

Observe that:

1. \(\tau(\gamma, \gamma') = \emptyset, \{b\}, \{a, b\}, \{a, c\}, X\).
2. \((\gamma, \gamma')-\alpha A_2\)-set = \(\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\).
3. \((\gamma, \gamma')-\alpha A_4\)-set = \(\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\)
Clearly, \( \{b, c\} \in \alpha A_4(X) \setminus \alpha A_2(X) \).

**Example 3.5.** Let \( X = \{a, b, c, d\} \) and \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\} \), \( \gamma \) and \( \gamma' \) be the identity operations. Then observe that:

1. \( \tau(\gamma, \gamma') = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\} \).
2. \( (\gamma, \gamma')-\alpha A_3\)-set = \( \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\} \).
3. \( (\gamma, \gamma')-\alpha A_4\)-set = \( \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\} \).
4. \( (\gamma, \gamma')-\alpha A_5\)-set = \( \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\} \).

Clearly:

1. \( \{a, c, d\} \in \alpha A_3(X) \setminus \alpha A_4(X) \).
2. \( \{a, c, d\} \in \alpha A_5(X) \setminus \alpha A_4(X) \).

**Example 3.6.** Let \( X = \{a, b, c, d\} \) and \( \tau = \{\emptyset, \{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, X\} \). We define the operations \( \gamma = \gamma' : \tau \to \mathcal{P}(X) \) as follows

\[
A^\gamma = \begin{cases} 
A & \text{if } c \notin A, \\
X & \text{if } c \in A,
\end{cases}
\]

and \( A^\gamma' = \begin{cases} 
A & \text{if } A \neq \{a\}, \\
X & \text{if } A = \{a\},
\end{cases} \)

Observe that:

1. \( \tau(\gamma, \gamma') = \{\emptyset, \{d\}, \{a, b\}, \{a, b, d\}, X\} \).
2. \( (\gamma, \gamma')-\alpha A_3\)-set = \( \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\} \).
3. \( (\gamma, \gamma')-\alpha A_4\)-set = \( \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\} \).
4. \( (\gamma, \gamma')-\alpha A_5\)-set = \( \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\} \).

Clearly,

1. \( \{b, c, d\} \in \alpha A_4(X) \setminus \alpha A_3(X) \).
2. \( \{b, c, d\} \in \alpha A_5(X) \setminus \alpha A_3(X) \).

We have the following implication diagram.

**Remark 3.7.** For a bioperation topological space \( (X, \tau, \gamma, \gamma') \), we have the following implication diagram:
Definition 3.8. A subset $A$ of bioperation topological space $(X, \tau, \gamma, \gamma')$ is said to be $(\gamma, \gamma')$-dense if $(\gamma, \gamma')$-Cl$(A) = X$.

Definition 3.9. A bioperation topological space $(X, \tau, \gamma, \gamma')$ is said to be
1. $(\gamma, \gamma')$-submaximal if each $(\gamma, \gamma')$-dense set is a $(\gamma, \gamma')$-open set.
2. $(\gamma, \gamma')$-extremally disconnected if the $(\gamma, \gamma')$-closure of every $(\gamma, \gamma')$-open set is $(\gamma, \gamma')$-open.

Theorem 3.10. If $(X, \tau, \gamma, \gamma')$ is a $(\gamma, \gamma')$-extremally disconnected space, then $\alpha A_5(X) = \alpha A_4(X)$.

Proof. Let $F = G \cap H \in \alpha A_5(X)$, where $G \in \alpha O(X)$ and $(\gamma, \gamma')$-Int$((\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(H))) \subset H$. Since $(X, \tau, \gamma, \gamma')$ is extremally disconnected, $(\gamma, \gamma')$-Int$((\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(H))) = (\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(H))$ and so $(\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(H)) \subset H$. As a consequence, we obtain $\alpha A_5(X) \subset \alpha A_4(X)$ and hence $\alpha A_5(X) = \alpha A_4(X)$.

Proposition 3.11. Let $(X, \tau, \gamma, \gamma')$ be a bioperation topological space. Then
1. $RO(X) \subset LC(X)$.
2. $LC(X) \subset B(X)$.
3. $B(X) \subset C(X)$.

Proof. (1). Let $F = G \cap H \in A(X)$, where $G \in \alpha O(X)$ and $(\gamma, \gamma')$-Int$((\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(H))) = H$. Then we obtain $(\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(H))) = (\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Cl$((\gamma, \gamma')$-Int$(H))) = (\gamma, \gamma')$-Cl$(H) = (\gamma, \gamma')$-$t'$ set.

Proposition 3.12. For a bioperation topological space $(X, \tau, \gamma, \gamma')$, we have the following:
1. Every $(\gamma, \gamma')$-regular open set is a $(\gamma, \gamma')$-$\alpha A_1$-set.
2. Every $(\gamma, \gamma')$-locally closed set is a $(\gamma, \gamma')$-$\alpha A_2$-set.
3. Every $(\gamma, \gamma')$-$C$-set is a $(\gamma, \gamma')$-$\alpha A_5$-set.
Example 3.13. In Example 3.3, RO(X) = \{\emptyset, X\} and then \{a\} \in \alpha A_1(X) \setminus RO(X).

Example 3.14. In Example 3.3, the (\gamma, \gamma')-locally closed set = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, X\}, then \{a, c\} is a (\gamma, \gamma')-\alpha A_2(X)-set that is not a (\gamma, \gamma')-locally closed set.

Example 3.15. In Example 3.5, C = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, X\}, and then \{a, b, d\} \in \alpha A_5(X) \setminus C(X).

Theorem 3.16. Let (X, \tau, \gamma, \gamma') be a (\gamma, \gamma')-submaximal space. Then the following hold:

1. \(A(X) = \alpha A_1(X)\).
2. \(\mathcal{LC}(X) = \alpha A_2(X)\).
3. \(\mathcal{B}(X) = \alpha A_3(X)\).
4. \(\mathcal{C}(X) = \alpha A_5(X)\).

Proof. It is clear that in a (\gamma, \gamma')-submaximal space, \(\tau_{(\gamma, \gamma')} = \alpha O(X)\).

Proposition 3.17. Let (X, \tau, \gamma, \gamma') be a bioperation topological space.

1. If \(F \in SO(X)\) and \(G \in \alpha O(X)\), then \(F \cap G \in SO(X)\).
2. If \(F \in PO(X)\) and \(G \in \alpha O(X)\), then \(F \cap G \in PO(X)\).

Theorem 3.18. A subset \(F\) of a bioperation topological space \((X, \tau, \gamma, \gamma')\) is a (\gamma, \gamma')-regular open set if and only if it is a (\gamma, \gamma')-semi-open set and a (\gamma, \gamma')-locally closed set.

Proof. Let \(F = G \cap H \in A(X)\), where \(G\) is (\gamma, \gamma')-open and (\gamma, \gamma')-Cl((\gamma, \gamma')-Int(H)) = H. Then we obtain (\gamma, \gamma')-Cl((\gamma, \gamma')-Cl((\gamma, \gamma')-Int(H))) = (\gamma, \gamma')-Cl((\gamma, \gamma')-Int(H)) = (\gamma, \gamma')-Cl(H) = H. Also (\gamma, \gamma')-Int(F) = G \cap (\gamma, \gamma')-Int(H) and \(F = G \cap (\gamma, \gamma')-Cl((\gamma, \gamma')-Int(H)) = (\gamma, \gamma')-Cl(G \cap (\gamma, \gamma')-Int(H))\). Hence \(F \subset (\gamma, \gamma')-Cl((\gamma, \gamma')-Int(F))\). As a consequence, \(F\) is (\gamma, \gamma')-semi-open set. Conversely, let \(F\) be (\gamma, \gamma')-semi-open set and (\gamma, \gamma')-locally closed. Then \(F = G \cap H\), where \(G\) is (\gamma, \gamma')-open and (\gamma, \gamma')-Cl(H) = H. Since \(F\) is (\gamma, \gamma')-semi-open, \(F \subset (\gamma, \gamma')-Cl((\gamma, \gamma')-Int(B)) \subset (\gamma, \gamma')-Cl((\gamma, \gamma')-Int(F))\) and so (\gamma, \gamma')-Cl(F) \subset (\gamma, \gamma')-Cl((\gamma, \gamma')-Int(F)). Thus we obtain (\gamma, \gamma')-Cl(F) = (\gamma, \gamma')-Cl((\gamma, \gamma')-Int(F)). Also since \(F = G \cap H\) and \(F \subset G \subset (\gamma, \gamma')-Cl(F)\), we obtain \(F \subset G \cap (\gamma, \gamma')-Cl(F) \subset G \cap (\gamma, \gamma')-Cl(G \cap H) \subset (\gamma, \gamma')-Cl(G \cap H) = G \cap (\gamma, \gamma')-Cl((\gamma, \gamma')-Int(F))\). As a consequence, \(F = G \cap (\gamma, \gamma')-Cl(F) = G \cap (\gamma, \gamma')-Cl((\gamma, \gamma')-Int(F))\) and so \(F\) is a (\gamma, \gamma')-regular open set.
Theorem 3.19. For a bioperation topological space \((X, \tau, \gamma, \gamma')\), 
\(\alpha O(X) = PO(X) \cap \alpha A_3(X)\).

Proof. Let \(F\) be a \((\gamma, \gamma')\)-\(\alpha\)-open set. Put \(G = F\) and \(H = X \in \tau(\gamma, \gamma')\). Then \(F = G \cap H\), where \(G \in \alpha O(X)\) and \((\gamma, \gamma')\)-\(\text{Int}((\gamma, \gamma')\text{-Cl}(H)) = (\gamma, \gamma')\)-\(\text{Int}(H)\). Since \(F\) is \((\gamma, \gamma')\)-\(\alpha\)-open, \(\alpha O(X) \subset PO(X) \cap \alpha A_3(X)\). Let \(F\) be a \((\gamma, \gamma')\)-\(\alpha\)-preopen set and a \((\gamma, \gamma')\)-\(\alpha\)-\(A_3\)-set. Since \(F\) is a \((\gamma, \gamma')\)-\(\alpha\)-preopen set, \(F \subset (\gamma, \gamma')\)-\(\text{Int}((\gamma, \gamma')\text{-Cl}(F))\). Also we have \(F = G \cap H\), where \(G\) is a \((\gamma, \gamma')\)-\(\alpha\)-open set and \((\gamma, \gamma')\)-\(\text{Int}((\gamma, \gamma')\text{-Cl}(H)) = (\gamma, \gamma')\)-\(\text{Int}(H)\) since \(F\) is a \((\gamma, \gamma')\)-\(\alpha\)-\(A_3\)-set. Then \(F \subset (\gamma, \gamma')\)-\(\text{Int}((\gamma, \gamma')\text{-Cl}(F)) = (\gamma, \gamma')\)-\(\text{Int}((\gamma, \gamma')\text{-Cl}(G)) \cap (\gamma, \gamma')\)-\(\text{Int}((\gamma, \gamma')\text{-Cl}(H))\). Also since \(F \subset (\gamma, \gamma')\)-\(\text{Int}((\gamma, \gamma')\text{-Cl}(F))\), the following holds: \(\alpha O(X) \subset PO(X) \cap \alpha A_2(X)\).

Lemma 3.20. A subset of a bioperation topological space \((X, \tau, \gamma, \gamma')\) is \((\gamma, \gamma')\)-\(\alpha\)-open if and only if it is \((\gamma, \gamma')\)-\(\alpha\)-\(semi\)-open and \((\gamma, \gamma')\)-\(\alpha\)-\(pre\)-open.

Theorem 3.21. For a bioperation topological space \((X, \tau, \gamma, \gamma')\), the following holds: \(\alpha O(X) = PO(X) \cap \alpha A_2(X)\).

Proof. By Lemma 3.20, \(\alpha O(X) \subset PO(X)\) since \(\alpha O(X) = PO(X) \cap SO(X)\). Also since \(\alpha O(X) \subset \alpha A_2(X)\), we obtain that \(\alpha O(X) \subset PO(X) \cap \alpha A_2(X)\). By Theorem 3.19, we have \(\alpha O(X) = PO(X) \cap \alpha A_3(X)\). Also since \(\alpha A_2(X) \subset \alpha NA_3(X)\), we obtain that \(PO(X) \cap \alpha A_2(X) \subset PO(X) \cap \alpha A_3(X) = \alpha O(X)\).

4 Decomposition of bioperation continuity

Definition 4.1. A function \(f : (X, \tau, \gamma, \gamma') \rightarrow (Y, \sigma, \beta, \beta')\) is said to be \((\gamma, \gamma')\)-(\(\beta, \beta'\))-\(\alpha\)-\(A_i\)-continuous if \(f^{-1}(G) \in \alpha A_i(X)\) for every \(G \in \sigma(\beta, \beta')\), where \(i = 1, 2, 3, 4, 5\).

Proposition 4.2. For a function \(f : (X, \tau, \gamma, \gamma') \rightarrow (Y, \sigma, \beta, \beta')\), we have the following:

1. Every \((\gamma, \gamma')\)-(\(\beta, \beta'\))-\(\alpha\)-\(A_1\)-continuous function is a \((\gamma, \gamma')\)-(\(\beta, \beta'\))-\(\alpha\)-\(A_2\)-continuous function.
2. Every \((\gamma, \gamma')\)-(\(\beta, \beta'\))-\(\alpha\)-\(A_2\)-continuous function is a \((\gamma, \gamma')\)-(\(\beta, \beta'\))-\(\alpha\)-\(A_3\)-continuous function.
3. Every \((\gamma, \gamma')\)-(\(\beta, \beta'\))-\(\alpha\)-\(A_3\)-continuous function is a \((\gamma, \gamma')\)-(\(\beta, \beta'\))-\(\alpha\)-\(A_5\)-continuous function.
4. Every \((\gamma, \gamma')-(\beta, \beta')-\alpha A_2\)-continuous function is a \((\gamma, \gamma')-(\beta, \beta')-\alpha A_4\)-continuous function.

5. Every \((\gamma, \gamma')-(\beta, \beta')-\alpha A_4\)-continuous function is a \((\gamma, \gamma')-(\beta, \beta')-\alpha A_4\)-continuous function.

**Proof.** The proof follows from Proposition 3.2.

The converse of the above Proposition is easily to construct using the Examples 3.3, 3.4, 3.5, 3.6.

**Proposition 4.3.** For a function \(f : (X, \tau, \gamma, \gamma') \rightarrow (Y, \sigma, \beta, \beta')\), we have the following

1. Every \((\gamma, \gamma')-(\beta, \beta')\)-completely continuous function is \((\gamma, \gamma')-(\beta, \beta')-\alpha A_1\)-continuous.

2. Every \((\gamma, \gamma')-(\beta, \beta')-\text{LC}\)-continuous function is \((\gamma, \gamma')-(\beta, \beta')-\alpha A_2\)-continuous.

3. Every \((\gamma, \gamma')-(\beta, \beta')-\text{B}\)-continuous function is \((\gamma, \gamma')-(\beta, \beta')-\alpha A_3\)-continuous.

4. Every \((\gamma, \gamma')-(\beta, \beta')-\text{C}\)-continuous function is \((\gamma, \gamma')-(\beta, \beta')-\alpha A_5\)-continuous.

**Proof.** It is a direct consequence of Proposition 3.12.

**Theorem 4.4.** For a function \(f : (X, \tau, \gamma, \gamma') \rightarrow (Y, \sigma, \beta, \beta')\), where \((X, \tau, \gamma, \gamma')\) is a \((\gamma, \gamma')\)-submaximal space. Then the following hold:

1. \((\gamma, \gamma')-(\beta, \beta')-\text{A}\)-continuous if and only if \((\gamma, \gamma')-(\beta, \beta')-\alpha A_1\)-continuous.

2. \((\gamma, \gamma')-(\beta, \beta')-\text{LC}\)-continuous if and only if \((\gamma, \gamma')-(\beta, \beta')-\alpha A_2\)-continuous.

3. \((\gamma, \gamma')-(\beta, \beta')-\text{B}\)-continuous if and only if \((\gamma, \gamma')-(\beta, \beta')-\alpha A_3\)-continuous.

4. \((\gamma, \gamma')-(\beta, \beta')-\text{C}\)-continuous if and only if \((\gamma, \gamma')-(\beta, \beta')-\alpha A_5\)-continuous.

**Proof.** The proof follows from Theorem 3.16.

**Theorem 4.5.** A function \(f : (X, \tau, \gamma, \gamma') \rightarrow (Y, \sigma, \beta, \beta')\) is \((\gamma, \gamma')-(\beta, \beta')-\alpha\)-continuous if and only if \(f\) is \((\gamma, \gamma')-(\beta, \beta')-\text{precontinuous}\) and \((\gamma, \gamma')-(\beta, \beta')-\alpha A_3\)-continuous.

**Proof.** The proof follows from Theorem 3.19.

**Theorem 4.6.** A function \(f : (X, \tau, \gamma, \gamma') \rightarrow (Y, \sigma, \beta, \beta')\) is \((\gamma, \gamma')-(\beta, \beta')-\alpha\)-continuous if and only if \(f\) is \((\gamma, \gamma')-(\beta, \beta')-\text{precontinuous}\) and \((\gamma, \gamma')-(\beta, \beta')-\alpha A_2\)-continuous.

**Proof.** The proof follows from Theorem 3.21.
Conclusions

1. The \((\gamma, \gamma')-\alpha A_i\)-sets, for \(i \in \{1, 2, 3, 4, 5\}\), are characterized and the relationships between them and also with other well known in the literature \((\gamma, \gamma')\)-sets are studied.

2. Under what conditions some of the studied sets are the same.

3. The \((\gamma, \gamma')-(\beta, \beta')-\alpha A_i\)-continuous functions, for \(i \in \{1, 2, 3, 4, 5\}\), are studied and the relationships between them are studied.

References


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