Enhanced Generation Rate of the Coherent Entanglement Photon Pairs in Parametrical Down Conversion

N. A. Enaki and V. I. Ciornea
Institute of Applied Physics, Academy of Sciences of Moldova,
Academiei str. 5, Chisinau MD-2028, Moldova

Abstract

The effect of enhancing nonlinear generation of entangled photons in the process of interaction of the external coherent electromagnetic field with nonlinear dispersive medium is studied in this paper. Taking into account the second and third order of the susceptibility tensor of the crystal, it is demonstrated that in good cavity approximation the bistable behavior of the two photon generation coefficient as a function of intensity of the pump laser field is possible. This effect is stimulated by decreasing the detuning between the frequency of the cavity mode and pump frequency as a function of anharmonicity terms in polarization.

1 Introduction

The problem of quantum fluctuations and the generation of the nonclassical electromagnetic field (EMF) in two-photon and multiphoton processes has recently been the subject of a number of theoretical and experimental studies. The entanglement phenomenon between idler and signal photons generated in the parametric down conversion has been intensively studied in the last decade. For example such effects as quantum interference [1-5] and nonlocality [6,7] are possible thanks to the extremely short correlation time between the two photons produced in the large band of parametric down-conversion [8,9]. In one dimension approximation the broad band squeezed vacuum EMF consists of pairs of entanglement photons which can coherently excite the dipole forbidden transitions like coherent EMF [10]. The effects of coherent excitation arise in the problem of generating of more powerful broad band squeezed light in the parametrical down conversion [11].

The aim of this work is to study the process of generation of entanglement photon pairs in the nonlinear cavity which contains the second and third order nonliniarity driven by the strong external coherent laser field. Let's consider the situation when the parametric oscillator consists of a crystal in a double resonant cavity with mirrors which almost completely reflect the subharmonic light and reflect the pump light badly. If the mirrors only

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reflect the subharmonic light one can adiabatically eliminate the operators of the pump light. Father a large number of discrete modes for the subharmonic light inside the cavity which contains a nonlinear disperse medium is considered. If the double cavity frequency of entanglement photons $\omega_{2k_0} = \omega_k + \omega_{2k_0-k}$ is off the resonance with the pump field ω_p, the detuning $\Delta_0 = \omega_p - \omega_{2k_0}$ extinguishes the process of the generation of the entanglement photons in the cavity. In this situation, the third order nonlinearity can diminish the detuning factor in the nonlinear dispersive medium as a function of intensity of the pump field. In this critical point a more powerful enhancement of the generation rate of entanglement photons is observed.

A new master equation for the coupled subharmonic EMF with external driven coherent field is obtained. The coupled system obeys the $SU(1,1)$ symmetry and a Casimir pseudovector operator for $su(1,1)$ algebra is conserved. Using the generalized P-representation for $SU(1,1)$ symmetry Fokker-Planck equation for the proposed master equation is obtained. In order to obtain the steady state solution of master equation the two methods are proposed. The first method is based on the stationary solution of Fokker-Planck equation and the second is obtained representing the density matrix through antinormal products of creation and annihilation operators of $su(1,1)$ algebra. From the analytical and numerical results it follows that these two methods are not so equivalent, and the theory of stationary solutions for quantum master equations need more careful development. A similar problem was analyzed and solved in the case of a two-level system interacting with a coherent external field [12-14]. It is well known that such a two-level system obeys the $su(2)$ symmetry. However, in the last years it was realized that the $su(1,1)$ group plays an important role in many problems in Quantum Optics [15-19].

2 Master equation for subharmonic field

The Hamiltonian which describes the interaction of EMF with the nonlinear dispersive medium in the cavity can be obtained, following the Collett and Gardiner treatment [20]

$$H = H_e + H_i + H_c. \tag{1}$$

Here

$$H_e = \hbar \int_0^\infty d\omega \omega B_\omega^\dagger B_\omega \tag{2}$$

is the free Hamiltonian for the external field modes. B_ω and $B_\omega^\dagger$ are the annihilation and creation operators for the external field which satisfy the commutation relation $[B_\omega, B_{\omega'}^\dagger] = \delta(\omega - \omega')$.

$$H_i = i\hbar \int_{-\infty}^{+\infty} d\omega k(\omega)[B_\omega b^\dagger - B_\omega^\dagger b] \tag{3}$$

is the interaction Hamiltonian between the external coherent field with the frequency ω_p and the cavity field which contains the limited number of discrete field modes in the energy interval $(0, \hbar\omega_{2k_0})$. b and $b^\dagger$ are the annihilation and creation operators for the intracavity field with the frequency near the pump $(\Delta_0 \approx 0)$, $k(\omega)$ is coupling constantly. We consider that the cavity is good for the subharmonic field $(k(\omega_{k_0}) \approx 0)$ and for the high frequency field $\omega \approx \omega_p$ the coupling constant is large.

In order to obtain the intracavity Hamiltonian H_c, let us expand the polarization of the nonlinear medium to third order in the EMF strength,

$$P_\alpha = \chi^{(1)}_{\alpha\beta} E_\beta + \chi^{(2)}_{\alpha\beta\gamma} E_\beta E_\gamma + \chi^{(3)}_{\alpha\beta\gamma\delta} E_\beta E_\gamma E_\delta. \tag{4}$$

Here $\chi^{(n)}$ is a $(n+1)$th rank susceptibility tensor. After introducing this polarization in the density part of the interaction Hamiltonian $H_{int} = \int \vec{P}(\vec{E}) d\vec{E}$ one can obtain the following form of intracavity Hamiltonian [11]

$$H_c =: \int d^3\vec{r} \Big\{ \frac{|\vec{B}|^2}{2\mu_0} + \vec{E} \Big[\frac{1}{2}(\epsilon_0 + \chi^{(1)})\vec{E} + \frac{1}{3}\chi^{(2)}\vec{E}\vec{E} + \frac{1}{4}\chi^{(3)}\vec{E}\vec{E}\vec{E} \Big] \Big\} :. \tag{5}$$

As the external laser field pump is only the cavity mode $2k_0$ one can express the EMF strength $\vec{E}$ inside the cavity through the strength intracavity pump field $\vec{E}_p$ and the subharmonic mode components generated in the process of parametrical down conversion $\vec{E}_{sh}$,

$$\vec{E} = \vec{E}_p + \vec{E}_{sh}. \tag{6}$$

$\vec{E}_p$ can be expressed from the annihilation and creation operators in the following form

$$\vec{E}_p(\vec{r}, t) = i \Big(\frac{\bar{\omega}_p}{2\epsilon_0} \Big)^{1/2} \{ b\vec{u}(\vec{r}) \exp(-i\omega_p t) - b^\dagger \vec{u}^* \exp(i\omega_p t) \}, \tag{7}$$

where

$$\vec{u}(\vec{r}) = \frac{\vec{e}_\lambda}{\sqrt{V}} \exp[i(\vec{k}_p, \vec{r})].$$

Here V is the quantization volume, $\vec{e}_\lambda$ is the polarization vector of EMF. The electric-field operator for the cavity mode of subharmonic frequencies $\omega_k \leq \omega_{2k_0}$ can be written as

$$\vec{E}_{sh}(\vec{r}, t) = i \sum_{k=0}^{2k_0} \Big(\frac{\bar{\omega}_p}{2\epsilon_0} \Big)^{1/2} \{ a_k \vec{v}_k(\vec{r}) \exp(-i\omega_k t) - a_k^\dagger \vec{v}_k^*(\vec{r}) \exp(i\omega_k t) \}, \tag{8}$$

where

$$\vec{v}_k(\vec{r}) = \frac{\vec{e}_\lambda}{\sqrt{V}} \exp[i(\vec{k}, \vec{r})],$$

and a_k and $a_k^\dagger$ are the annihilation and creation photon operators in the cavity mode k. Taking into account that the losses of pump EMF in the cavity are larger than the losses of subharmonic EMF one can represent the intracavity Hamiltonian in the following way

$$\begin{aligned} H_c &= \hbar\tilde{\omega}_p b^\dagger b + \hbar\chi^0 {b^\dagger}^2 b^2 + \sum_{k=0}^{2k_0} \hbar\{\tilde{\omega}_k + \chi_k^0 b^\dagger b\} a_k^\dagger a_k \\ &+ \sum_{k_1,k_2=0}^{2k_0} \hbar\{\chi'_{k_1,k_2} a_{k_1}^\dagger a_{2k_0-k_1}^\dagger a_{k_2} a_{2k_0-k_2} + \chi''_{k_1,k_2} a_{k_1}^\dagger a_{k_1} a_{k_2}^\dagger a_{k_2}\} \\ &+ i\sum_{k-0}^{2k_0} g_k\{b^\dagger a_k a_{2k_0-k} - b a_k^\dagger a_{2k_0-k}^\dagger\}. \end{aligned} \tag{9}$$

Here the coefficients $\tilde{\omega}_a = \frac{1}{2}\left(1 + \frac{\chi^{(1)}}{\epsilon_0}\right)\omega_a$, where $a = (p, k)$,

$$\chi^0 = \frac{3}{8}\frac{\chi^{(3)}(\omega_p, \omega_p)\hbar\omega_p}{\epsilon_0^2}\omega_p, \qquad \chi_k^0 = \frac{3}{2}\frac{\chi^{(3)}(\omega_p, \omega_k)\hbar\omega_p}{\epsilon_0^2}\omega_k,$$

$$\chi'_{k_1,k_2} = \frac{3}{8}\frac{\chi^{(3)}(\omega_{k_1}, \omega_{k_2})\hbar}{\epsilon_0^2}\sqrt{\omega_{k_1}\omega_{2k_0-k_1}\omega_{k_2}\omega_{2k_0-k_2}}, \qquad \chi''_{k_1,k_2} = \frac{3}{8}\frac{\chi^{(3)}(\omega_{k_1}, \omega_{k_2})\hbar}{\epsilon_0^2}\omega_{k_1}\omega_{k_2}.$$

$$g_k = \left(\frac{\hbar}{2\epsilon_0}\right)^{1/2}\frac{\chi^{(2)}}{\epsilon_0}\hbar\sqrt{\omega_p\omega_k\omega_{2k_0-k}}$$

is the constant of the interaction of the intracavity pump and subharmonic fields with frequencies ω_p and ω_k respectively, obtained from the second order polarization expanding (1). Let us consider the operator $O(t)$ which belongs to the subharmonic EMF. Taking into account the Hamiltonian (1) one can write the following Haisenberg equation for this mean value of this operator

$$\begin{aligned} \frac{d\langle O(t)\rangle}{dt} &= i\sum_{k=0}^{2k_0} \hbar\{\tilde{\omega}_k + \chi_k^0 b^\dagger b\}\langle[a_k^\dagger a_k, O(t)]\rangle \\ &+ i\sum_{k_1,k_2=0}^{2k_0} \hbar\langle[\chi'_{k_1,k_2} a_{k_1}^\dagger a_{2k_0-k_1}^\dagger a_{k_2} a_{2k_0-k_2} + \chi'_{k_1,k_2} a_{k_1}^\dagger a_{k_1} a_{k_2}^\dagger a_{k_2}, O(t)]\rangle \\ &- \sum_{k-0}^{2k_0} \frac{g_k}{\hbar}\langle[b^\dagger a_k a_{2k_0-k} - b a_k^\dagger a_{2k_0-k}^\dagger, O(t)]\rangle. \end{aligned} \tag{10}$$

In this equation we must eliminate the operators of higher frequency cavity operators $b(t)$ and $b^\dagger(t)$. Using the system Hamiltonian one can obtain the following Heisenberg equation for these operators

$$\frac{db}{dt} = -i\{\tilde{\omega}_p + 2\chi^0 b^\dagger b + \sum_{k=0}^{2k_0} \chi_k^0 a_k^\dagger a_k\}b + \sum_k \frac{g_k}{\hbar} a_k a_{2k_0-k} + \int_{-\infty}^{+\infty} d\omega k(\omega) B_\omega. \tag{11}$$

As the external EMF is in the single mode coherent field $|in\rangle = \prod_{\omega_i \neq \omega_p} |0\rangle_{\omega_i} \exp(\beta B^{\dagger}_{\omega_p} - \beta^* B_{\omega_p})|0\rangle_{\omega_p}$, the solution of Haisenberg equation for external EMF can be represented in the following form

$$B_{\omega}(t) = B_{\omega}(0)e^{-i\omega t} - k(\omega)\int_0^t d\tau e^{-i\omega\tau} b(t-\tau), \tag{12}$$

where $B_{\omega}(0)$ is the free part of the EMF operator, which satisfies the identity $B_{\omega}(0)|in\rangle = \beta\delta_{\omega,\omega_p}|in\rangle$. After the substitution of this solution in Heisenberg equation (11) one can represent the solution of operator b_{2k_0} in the following form

$$\begin{aligned} b(t) &= b(0)e^{-i\omega_{2k_0}t} \\ &+ \int_0^t \Big(Te^{-i\int_{\tau_1}^t (\hat{\omega}(\tau)-i\Gamma)d\tau}\Big[\int_{-\infty}^{+\infty} k(\omega)B_{\omega}(0)e^{-i\omega\tau_1}d\omega + \sum_{k=0}^{2k_0}\frac{g_k}{\hbar}a_k(\tau_1)a_{2k_0-k}(\tau_1)\Big]\Big)d\tau_1. \end{aligned} \tag{13}$$

Here T representing the chronological product of operators, $\Gamma = \pi k^2(\omega_{2k_0})$ is the cavity losses at frequency ω_{2k_0}, The coupling between the cavity mode $2k_0$ and external EMF take places in the moment $t = 0$,

$$\hat{\omega}(t) = \bar{\omega}_p + 2\chi^0 b^{\dagger}b + \sum_{k=0}^{2k_0}\chi_k^0 a_k^{\dagger}a_k, \tag{14}$$

where $\bar{\omega}_p = \tilde{\omega}_p - \int d\omega k^2(\omega)P(\omega_{2k_0}-\omega)^{-1}$.

In order to eliminate the free field operators of external EMF and the cavity field at frequency ω_{2k_0} one can do the following approximation. Inside T we replace the operator $\hat{\omega}$ by a steady state value $\bar{\omega}$ in witch all the number state operators are replaced by their mean value $\langle b^{\dagger}b\rangle, \langle a_k^{\dagger}a_k\rangle$. Neglecting the subharmonic number term $\sum_{k=0}^{2k_0}\chi_k^0 a_k^{\dagger}a_k$ in comparison with the quasicoherent term $2\chi^0 b^{\dagger}b$ we obtain for $b^{\dagger}b$ the following equation

$$b^{\dagger}b = \frac{1}{\pi}\frac{\Gamma|\beta|^2}{(\omega_p - \bar{\omega})^2 + \Gamma^2}. \tag{15}$$

Here for $\bar{\omega}$ we used its second order approximation

$$\bar{\omega} = \bar{\omega}_p - 2\Delta_f,$$

where $\Delta_f = -\chi^0\Gamma|\beta|^2/[\pi(\omega_p - \bar{\omega}_p)^2 + \Gamma^2]$. After introducing equation (13) in (10) and using Born-Markoff approximation we obtain the following expression

$$\begin{aligned} \frac{d\langle O\rangle}{dt} &= \langle\Big[i\sum_{k=0}^{2k_0}\bar{\omega}_k a_k^{\dagger}a_k + i\sum_{k_1,k_2=0}^{2k_0}(\bar{\chi}'_{k_1,k_2}a_{k_1}^{\dagger}a_{2k_0-k_1}^{\dagger}a_{k_2}a_{2k_0-k_2} \\ &+ \chi''_{k_1,k_2}a_{k_1}^{\dagger}a_{k_1}a_{k_2}^{\dagger}a_{k_2}), O(t)\Big]\rangle \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{k=0}^{2k_0}(\Omega_k e^{-i\omega_p t}\langle[a_k^\dagger a_{2k_0-k}^\dagger, O(t)]\rangle - \Omega_k^* e^{i\omega_p t}\langle[a_k a_{2k_0-k}, O(t)]\rangle) \\
&+ \sum_{k_1,k_2=0}^{2k_0} \gamma_{k_1,k_2}\{\langle[a_{k_1}^\dagger a_{2k_0-k_1}^\dagger O(t), a_{k_2}a_{2k_0-k_2}]\rangle + \langle[a_{k_2}^\dagger a_{2k_0-k_2}^\dagger, O(t)a_{k_1}a_{2k_0-k_1}]\rangle\}, \quad (16)
\end{aligned}
$$

where $\bar{\omega}_k = \omega_k + \Delta_f$, $\bar{\chi}'_{k_1,k_2} = \chi'_{k_1,k_2} - \frac{g_{k_1}g_{k_2}(\bar{\omega}-2\omega_p)}{\hbar^2(\Gamma^2+(\bar{\omega}-2\omega_p)^2)}$ is the constant of interaction between the entanglement photon pairs stimulated by second and third order susceptibility, $\Omega_k^2 = \frac{g_k^2}{\hbar^2\chi_0}\Delta_f$, Ω is the analog of Rabi frequency for excitations of photon pairs in cavity, and $\gamma_{k_1,k_2} = \frac{g_{k_1}g_{k_2}\Gamma}{\hbar^2(\Gamma^2+(\bar{\omega}-2\omega_p)^2)}$ are the losses of coherent photon pairs in the cavity stimulated by losses of pump field in the cavity.

As $\langle O(t)\rangle = Tr\{\tilde{\rho}(t)O\} = Tr\{\tilde{\rho}O(t)\}$ in equation (15) one can pass from Heisenberg to Schrodinger picture. After the cyclic permutation under the $Tr\{...\}$ operation one can replace the commutators from operator O to density matrix of subharmonic fields $\rho(t) = e^{iH_0t/\hbar}\tilde{\rho}(t)e^{-iH_0t/\hbar}$ (here $H_0 = \sum_k \hbar\bar{\omega}_k a_k^\dagger a_k$)

$$
\begin{aligned}
\frac{d\rho}{dt} &= -i\sum_{k_1,k_2=0}^{2k_0}[\bar{\chi}'_{k_1,k_2}a_{k_1}^\dagger a_{2k_0-k_1}^\dagger a_{k_2}a_{2k_0-k_2} + \chi''_{k_1,k_2}a_{k_1}^\dagger a_{k_1}a_{k_2}^\dagger a_{k_2}), \rho(t)] \\
&- \sum_{k=0}^{2k_0}[\Omega e^{-i(\Delta_0-\Delta_f)t}a_k^\dagger a_{2k_0-k}^\dagger - \Omega^* e^{i(\Delta_0-\Delta_f)t}a_k a_{2k_0-k}, \rho(t)] \\
&+ \sum_{k_1,k_2=0}^{2k_0}\gamma_{k_1,k_2}\{[a_{k_1}a_{2k_0-k_1}, \rho(t)a_{k_2}^\dagger a_{2k_0-k_2}^\dagger] + [a_{k_2}a_{2k_0-k_2}\rho(t), a_{k_1}^\dagger a_{2k_0-k_1}^\dagger]\}. \quad (17)
\end{aligned}
$$

We observe that in the absorption and generation of pump photon in the cavity the pairs of photons with the summary energy $\hbar\omega_{k_1} + \hbar\omega_{k_2} = \hbar\omega_{2k_0}$ are generated. If we decomposer the density matrix of the coherent states of boson subharmonic operators one obtains the complicated Fokker-Planck equation due to the existence of a large number of subharmonic modes in the resonance with pump fields. In the case when we have only one cavity modes in this resonance $k = k_0$ the master equation (17) is reduced to the same equation studied in [21,22]. It is not difficult to observe that when the number of modes increases the Drummond decomposition becomes difficult and for the investigation of the behavior of photon pairs generation another coherent state decomposition for the density matrix is necessary. We observe that the coefficients in master equation (17) are smoothly dependent on the frequency of the subharmonic fields ω_k. In this situation it is conveniently to replace the frequency ω_k with ω_{k_0} in all the coefficients.

In this approximation one can introduce the collective cavity field operators [10,23]

$$
I^\dagger = \sum_{k=0}^{2k_0}\frac{a_k^\dagger a_{2k_0-k}^\dagger}{2}, \qquad I^- = \sum_{k=0}^{2k_0}\frac{a_k a_{2k_0-k}}{2}, \qquad I^z = \sum_{k=0}^{2k_0}\frac{1}{2}\Big(a_k^\dagger a_k + \frac{1}{2}\Big) \quad (18)
$$

which satisfy the following commutators for the operators of $su(1,1)$ algebra

$$[I^+, I^-] = -2I^z, \qquad [I^z, I^\pm] = \pm I^\pm. \tag{19}$$

Thus the equation density matrix $W(t) = \exp\{i(\omega_p - \bar{\omega}_{2k_0})I^z\}\rho(t)\exp\{-i(\omega_p - \bar{\omega}_{2k_0})I^z\}$ can be represented in following form

$$\frac{\partial W(t)}{\partial t} = -i[\chi I^\dagger I^- + \Delta I^z + i\{\Omega^* I^- - \Omega I^\dagger\}, W(t)] + \gamma\{[I^- W(t), I^\dagger] + [I^-, W(t)I^\dagger]\}, \tag{20}$$

where

$$\chi = 4(\bar{\chi}^{'}_{k_0,k_0} - \chi^{''}_{k_0,k_0}), \qquad \Delta = \Delta_0 - \Delta_f + \chi^{''}_{k_0,k_0},$$

$$\Omega = 2\Omega_{k_0}, \qquad \gamma = 4\gamma_{k_0,k_0}.$$

It is not difficult to observe that the Casimir operator

$$I^2 = (I^z)^2 - 1/2(I^+I^- + I^-I^+), \tag{21}$$

which satisfies $[I^2, I^\pm] = [I^2, I^z] = 0$ is conserved. The discrete representation of $su(1,1)$ Lie algebra is described by the state vectors $|m,j\rangle$ that satisfy [24]

$$\begin{aligned}
I^2|m,j\rangle &= j(j-1)|m,j\rangle \\
I^z|m,j\rangle &= (m+j)|m,j\rangle \\
I^+|m,j\rangle &= \sqrt{(m+1)(m+2j)}|m+1,j\rangle \\
I^-|m,j\rangle &= \sqrt{m(m+2j-1)}|m-1,j\rangle,
\end{aligned} \tag{22}$$

where $I^-|0,j\rangle = 0$. Here j is the Bargmann index and m is any nonnegative integer. The set $\{|m,j\rangle|m = 0,1,2,\ldots; j = const$ becomes the complete orthonormal basis

$$\begin{aligned}
\langle j,m|n,j\rangle &= \delta_{m,n}, \\
\sum_{m=0}^{\infty} |m,j\rangle\langle j,m| &= 1.
\end{aligned} \tag{23}$$

In analogy with the Dicke cooperating number $j = N/2$ for $su(2)$ algebra one can introduce the cooperative number j for distinguishing the conjugate mode pairs $2k_0 - k_i, k_i$, $i = 1,2,....,N$. Using the conservation vector $I^2 = j(j-1)$ one can find that the cooperative number for the pairs of photons is $j = \sum_{k=0}^{2k_0} 1/4 = N/4$. In the next section the stationary solution for master equation (20) will be analyzed. This solution gives us the possibility to obtain the mean value for the number of pairs of entanglement photons $\langle I^\dagger I^-\rangle$, number of photons $\langle I_z\rangle$ and their fluctuations $\delta^2 = \langle I_z^2\rangle - \langle I_z\rangle^2$.

3 Fokker-Planck equation and its steady state solution

Following the decomposition of the density matrix on non-diagonal generalized P representation for Bose algebra [21,22] one can introduce the following decomposition on coherent states for $SU(1,1)$ algebra

$$W = \int_D P(\alpha,\beta)\Big(\frac{|\alpha\rangle\langle\beta^*|}{\langle\beta^*|\alpha\rangle}\Big)d\mu(\alpha,\beta). \tag{24}$$

Here D is the integration domain, $d\mu(\alpha,\beta) = d\alpha d\beta$ is the integration measure,

$$|\alpha\rangle = (1-|\alpha|^2)^j \exp(\alpha I^\dagger)|j\rangle,$$

$$\langle\beta^*| = (1-|\beta|^2)^j \langle j| \exp(\beta I^-)$$

are the coherent states for the $SU(1,1)$ algebra,

$$\langle\beta^*|\alpha\rangle = \frac{(1-|\alpha|^2)^j(1-|\beta|^2)^j}{(1-\alpha\beta)^{2j}}$$

is the normalization coefficient for the projector operator $|\alpha\rangle\langle\beta^*|$. Using the following action of operators $I^\dagger$, I^-, I^z of $SU(1,1)$ algebra on the coherent state

$$I^\dagger|\alpha\rangle = (1-|\alpha|^2)^j \frac{\partial}{\partial\alpha}\exp(\alpha I^\dagger)|j\rangle,$$

$$I^-|\alpha\rangle = (1-|\alpha|^2)^j\Big(\alpha^2\frac{\partial}{\partial\alpha} + 2j\alpha\Big)\exp(\alpha I^\dagger)|j\rangle,$$

$$I^z|\alpha\rangle = (1-|\alpha|^2)^j\Big(\alpha\frac{\partial}{\partial\alpha} + j\Big)\exp(\alpha I^\dagger)|j\rangle$$

one can obtain the following Fokker-Planck equation

$$\begin{aligned}
\frac{\partial}{\partial t}P(\alpha,\beta) &= \frac{\partial}{\partial\alpha}\Big(2ij\chi\alpha\frac{1+\alpha\beta}{1-\alpha\beta} + i\Delta\alpha - g\Omega(\alpha^2-1) + 2j\gamma\alpha\Big)P(\alpha,\beta) \\
&+ \frac{\partial}{\partial\beta}\Big(-2ij\chi\beta\frac{1+\alpha\beta}{1-\alpha\beta} - i\Delta\beta - g\Omega(\beta^2-1) + 2j\gamma\beta\Big)P(\alpha,\beta) \\
&- \frac{\partial^2}{\partial\alpha^2}(\gamma+i\chi)\alpha^2P(\alpha,\beta) - \frac{\partial^2}{\partial\beta^2}(\gamma-i\chi)\beta^2P(\alpha,\beta) + 2\gamma\frac{\partial^2}{\partial\alpha\partial\beta}\alpha^2\beta^2P(\alpha,\beta)
\end{aligned} \tag{25}$$

For many problems in quantum optics it is sufficient to know the steady state solution of Fokker Planck equation. Representing the steady state solution in the potential form

$$P(\alpha,\beta) = N\exp(-\Phi(\alpha,\beta)), \tag{26}$$

one can obtain the following differential equations for potential $\Phi(\alpha,\beta)$

$$
\begin{aligned}
(\gamma+i\chi)\alpha^2\frac{\partial\Phi}{\partial\alpha} &- \gamma\alpha^2\beta^2\frac{\partial\Phi}{\partial\beta} \\
&= -2ij\chi\alpha\frac{1+\alpha\beta}{1-\alpha\beta}-i\Delta\alpha+\Omega(\alpha^2-1)-2j\gamma\alpha+2(\gamma+i\chi)\alpha-2\gamma\alpha^2\beta \\
-\gamma\alpha^2\beta^2\frac{\partial\Phi}{\partial\alpha} &+ (\gamma-i\chi)\beta^2\frac{\partial\Phi}{\partial\beta} \\
&= 2ij\chi\beta\frac{1+\alpha\beta}{1-\alpha\beta}-i\Delta\beta+\Omega(\beta^2-1)-2j\gamma\beta+2(\gamma-i\chi)\beta-2\gamma\alpha\beta^2 \quad (27)
\end{aligned}
$$

We observe that for arbitrary parameters Δ and χ the so called potential condition for $\Phi(\alpha,\beta)$ [11]

$$
\frac{\partial^2\Phi(\alpha,\beta)}{\partial\beta\partial\alpha}=\frac{\partial^2\Phi(\alpha,\beta)}{\partial\alpha\partial\beta} \quad (28)
$$

is not satisfied.

For solving equation (20) we can examine the case when $\chi=\Delta=0$. It is not difficult to observe that in this case the potential condition (28) is satisfied and the steady state solution can be written in the following form

$$
P(\alpha,\beta)=N(\alpha\beta)^{-2}\Big(\frac{1}{\alpha\beta}-1\Big)^{-2j}\exp\Big(-\frac{\Omega}{\gamma}\Big[\frac{1}{\alpha}+\frac{1}{\beta}\Big]\Big), \quad (29)
$$

where

$$
N=\Big[-4\pi^2\sum_{n=0}^{\infty}\Big(\frac{\Omega^2}{\gamma^2}\Big)^{2j+n-1}[n!\Gamma(2j)\Gamma(2j+n)]^{-1}\Big]^{-1}
$$

is the constant of normation.

Now we consider the situation when $\chi\neq 0$ and detuning $\Delta=0$. In this case the potential condition (28) remains unsatisfied, but it can be satisfied if one introduce the following two terms $-i\chi\partial^2/(\partial\alpha\partial\beta)[\alpha^2\beta^2P(\alpha,\beta)]$ and $+i\chi\partial^2/(\partial\beta\partial\alpha)[\alpha^2\beta^2P(\alpha,\beta)]$ in equation (25). As in deriving Fokker-Planck equation we have considered that $\partial^2/(\partial\beta\partial\alpha)P(\alpha,\beta)=\partial^2/(\partial\alpha\partial\beta)P(\alpha,\beta)$, these two terms in the right hand side of Fokker-Planck equation (25) give zero contribution. After this the equation (25) suffers some modification. The steady state solution of Fokker Planck equation can be obtained from equations

$$
\begin{aligned}
0&=\Big(-2ij\chi\alpha\frac{1+\alpha\beta}{1-\alpha\beta}-\Omega(\alpha^2-1)+2j\gamma\alpha-\frac{\partial}{\partial\alpha}(\gamma+i\chi)\alpha^2+(\gamma-i\chi)\frac{\partial}{\partial\beta}\alpha^2\beta^2\Big)P(\alpha,\beta)\\
0&=\Big(2ij\chi\beta\frac{1+\alpha\beta}{1-\alpha\beta}-\Omega(\beta^2-1)+2j\gamma\beta-\frac{\partial}{\partial\beta}(\gamma-i\chi)\beta^2+(\gamma+i\chi)\frac{\partial}{\partial\alpha}\alpha^2\beta^2\Big)P(\alpha,\beta). \quad (30)
\end{aligned}
$$

Using the representation of P function through potential $\Phi(\alpha,\beta)$ from equations (30) one can obtain the following differential equations which satisfy the potential condition (28)

$$\begin{aligned}\frac{\partial\Phi}{\partial\alpha} &= \frac{2j}{\alpha(\alpha\beta-1)} - \frac{\Omega}{\alpha^2(\gamma+i\chi)} + \frac{2}{\alpha},\\ \frac{\partial\Phi}{\partial\beta} &= \frac{2j}{\beta(\alpha\beta-1)} - \frac{\Omega}{\beta^2(\gamma-i\chi)} + \frac{2}{\beta}.\end{aligned} \tag{31}$$

After introducing of the potential $\Phi(\alpha\beta)$ determined from equations (31) in relation (26) we obtain the following relation from P function

$$P(\alpha\beta) = N^*(\alpha\beta)^{-2}\Big(\frac{1}{\alpha\beta}-1\Big)^{-2j}\exp\Big[-\Omega\Big(\frac{1}{\alpha(\gamma-i\chi)}+\frac{1}{\beta(\gamma+i\chi)}\Big)\Big]. \tag{32}$$

Here the constant of normation

$$N^* = \Big[-4\pi^2\sum_{n=0}^{\infty}\Big(\frac{\Omega^2}{\gamma^2+\chi^2}\Big)^{2j+n-1}[n!\Gamma(2j)\Gamma(2j+n)]^{-1}\Big]^{-1}$$

After introducing equation (32) in (24) we obtain the form of the density matrix operator and can calculate the values of different operators

$$\langle R\rangle = \sum_{n=0}^{\infty}\Big(\frac{\Omega^2}{\gamma^2+\chi^2}\Big)^{n}\frac{f_n(R)}{n!\Gamma(2j+n)}\Big[\sum_{n=0}^{\infty}\Big(\frac{\Omega^2}{\gamma^2+\chi^2}\Big)^{n}\frac{1}{n!\Gamma(2j+n)}\Big]^{-1}. \tag{33}$$

Here

$$f_{n+j}(R) = \begin{cases} n+j, & \text{if } R = I^z; \\ (n+j)^2, & \text{if } R = [I^z]^2; \\ n^2 + n(2j-1), & \text{if } R = I^\dagger I^-. \end{cases} \tag{34}$$

In this section we observe that the steady state solution is difficult to obtain in case when $\Delta \neq 0$. In the next section we proposed methods of representation of the density matrix through antinormal product of operators $I^\dagger$ and I^-. These methods give the possibility to solve the stationary master equation for $\Delta \neq 0$.

4 The antinormal representation of steady state solution of the master equation

In order to obtain the solution of master equation (20) for arbitrary detuning and arbitrary third order nonlinearity in this section we represent the density matrix of the steady-state master equation (20)

$$i[\chi I^\dagger I^- + \Delta I^z + ig\{\Omega^* I^- - \Omega I^\dagger\}, W_s] - \gamma\{[I^- W_s, I^\dagger] + [I^-, W_s I^\dagger]\} = 0, \tag{35}$$

through antinormal ordering operators I^+ and I^-. The same representation was used in the papers [12-14] for $su(2)$ algebra. Here we extend this method for SU(1,1) symmetry. Following the elegant method developed in [12-14] we are looking for the solution of equation (35) of the form

$$W_s = A^{-1} F(I^-) F^+(I^\dagger), \tag{36}$$

where $A = Tr[F(I^-)F^\dagger(I^\dagger)]$, $F(I^-)$ and $F^\dagger(I^\dagger)$ are operator functions of I^- and $I^\dagger$, respectively. Here the function $F(I^\pm)$ can be represented in a Taylor series

$$F(I^\pm) = \sum_{n=0}^{\infty} C_n \Big(I^\pm\Big)^n. \tag{37}$$

By using the commutation rules corresponding to $SU(1,1)$ symmetry, it is easy to demonstrate the following operator identities

$$I^z F(I^-) = -F(I^-) I^z - [I^\dagger, \int F(I^-) dI^-], \tag{38}$$

$$[I^\dagger I^-, F(I^-)F(I^\dagger)] = [I^\dagger, I^- F(I^-)] F(I^\dagger) - h.c. \tag{39}$$

where h.c. stands for the hermitic conjugate and

$$\int F(I^-) dI^- = \sum_{n=0}^{\infty} \frac{C_n}{(n+1)} \Big(I^-\Big)^{n+1}.$$

The operator equation (35) can be represented in the form

$$[I^+, G(I^-)]F^\dagger(I^\dagger) + h.c. = 0, \tag{40}$$

where

$$G(I^-) = I^- F(I^-)(-i\frac{\chi}{\gamma} - 1) - i\frac{\Delta}{\gamma}\int F(I^-)dI^- - \frac{\Omega}{\gamma}F(I^-). \tag{41}$$

¿From equation (40) it follows that the commutator $[I^\dagger, G(I^-)]$ must be proportional to $F(I^-)$, but in view of the commutation relations (19) this is not possible. Thus in order to satisfy equation (40) it is necessary that the commutator $[I^\dagger, G(I^-)]$ be zero. This is possible when $G(I^-) = const$, that is

$$I^- F(I^-)(1 + i\frac{\chi}{\gamma}) - i\frac{\Delta}{\gamma}\int F(I^-)dI^- + \frac{\Omega}{\gamma}F(I^-) = const. \tag{42}$$

The solution of equation (42) can be written in a compact form

$$F(I^-) = c(I^- - id)^{-(1+\xi)}, \tag{43}$$

where $d = \frac{i\Omega}{\gamma + i\chi}$ and $\xi = -\frac{i\Delta}{\gamma + i\chi}$. Finally the stationary density matrix can be represented in the form

$$\begin{aligned} W_s &= |c|^2(I^- - id)^{-(1+\xi)}(I^\dagger + id^*)^{-(1+\xi^*)} = \\ &= \lim_{n_0\to\infty} D^{-1}\sum_{k,l}^{n_0} i^{k-l}d^{-k}(d^*)^l \frac{\Gamma(1+\xi+k)\Gamma(1+\xi^*+l)}{k!l!\Gamma(1+\xi)\Gamma(1+\xi^*)}\Big(I^-\Big)^k\Big(I^\dagger\Big)^l \end{aligned} \tag{44}$$

where D is the normalization factor so that $Tr\{W_s\} = 1$, and $\Gamma(z)$ is the Γ-function. The normalization constant D is given by the limit $D = \lim_{n_0\to\infty} D(n_0)$, where

$$D(n_0) = \sum_{l=0}^{n_0} |d|^{-2l}\frac{\Gamma(1+\xi+l)\Gamma(1+\xi^*+l)}{\Gamma(1+\xi)\Gamma(1+\xi^*)}\sum_{p=j}^{n_0}\frac{\Gamma(l+p+j)\Gamma(l+p-j+1)}{\Gamma(p+j)\Gamma(p-j+1)} \tag{45}$$

In analogy with Fokker-Planck method one can obtain the following values for the mean number of operators $I^z, (I^z)^2$ and I^+I^-.

$$\begin{aligned} \langle R\rangle &= \lim_{n_0\to\infty} D^{-1}\sum_{l=0}^{n_0}\Big(\frac{\Omega^2}{\gamma^2+\chi^2}\Big)^{-l}\frac{\Gamma(1+\xi+l)\Gamma(1+\xi^*+l)}{\Gamma(1+\xi)\Gamma(1+\xi^*)} \\ &\times \sum_{p=j}^{n_0} f_p(R)\frac{\Gamma(l+p+j)\Gamma(l+p-j+1)}{\Gamma(p+j)\Gamma(p-j+1)}. \end{aligned} \tag{46}$$

We observed that these methods give the results, which slightly differ from expression (33). The difference consists in the representation of the mean value through two sums in expression (46). In the next section this differences is analyzed.

5 Results and Discussions

Let us consider the first case, when the detuning $\Delta = 0$. This situation corresponds to the strong resonance between the external pump coherent field and cavity mode $2k_0$. In order to neglect the field detuning Δ_f as compared to parameter χ in the master equation (20) one supposes that the third order susceptibility at frequency ω_p is less than the same susceptibility at frequency ω_{k_0} $[\chi^{(3)}(\omega_p, \omega_{k_0}) \ll \chi^{(3)}(\omega_{k_0}, \omega_{k_0})]$ and the value of the intensity of the external pump field does not affect the inequality $\Delta_f \ll \chi$. In this case one can use the solution obtained by Fokker-Planck method.

¿From Fokker-Planck and antinormal ordering methods it follows that the expressions for mean values of the physical quantity slightly differ. We observe that the Fokker-Planck methods do not give the possibility to find the steady state solution for the arbitrary detuning Δ and χ. The antinormal ordering method gives us this possibility, but it expresses the mean value for operator R through the ratio of the two double divergent sums and it is difficult to do numerical simulation of expression (46). It is interesting to find the approximate mathematical connections between expressions (46) and (33). For this we do some mathematical transformation of expression (46) in case $\Delta = 0$. We change the sum of the variables in expression (46) $n = p - j$ and $m = l + n$ in order to obtain the following expression for $\langle R \rangle$

$$\langle R \rangle = \frac{\sum\limits_{n=0}^{\infty} \left(\frac{\Omega^2}{\gamma^2+\chi^2}\right)^n f_n(R)(n!\Gamma(2j+n))^{-1} \sum\limits_{m=n}^{\infty} \left(\frac{\Omega^2}{\gamma^2+\chi^2}\right)^{-m} \Gamma(m+2j)\Gamma(m+1)}{\sum\limits_{n=0}^{\infty} \left(\frac{\Omega^2}{\gamma^2+\chi^2}\right)^n (n!\Gamma(2j+n))^{-1} \sum\limits_{m=n}^{\infty} \left(\frac{\Omega^2}{\gamma^2+\chi^2}\right)^{-m} \Gamma(m+2j)\Gamma(m+1)}. \tag{47}$$

This expression is more similar to expression (33), but under the sum on n we have the divergent sum $\sum\limits_{m=n}^{\infty} \left(\frac{\Omega^2}{\gamma^2+\chi^2}\right)^{-m} \Gamma(m+2j)\Gamma(m+1)$. If we change the summations in (47) to the integer parameter n_0 in the limit, when $n_0 \to \infty$, one can multiply the numerator and denominator of this expression by $\left(\frac{\Omega^2}{\gamma^2+\chi^2}\right)^{n_0}$. Making the change of variable $p = n_0 - m$ we obtain the following formula

$$\langle R \rangle = \lim_{n_0 \to \infty} \frac{\sum\limits_{n=0}^{n_0} \left(\frac{\Omega^2}{\gamma^2+\chi^2}\right)^n f_n(R)(n!\Gamma(2j+n))^{-1} S(n, n_0)}{\sum\limits_{n=0}^{n_0} \left(\frac{\Omega^2}{\gamma^2+\chi^2}\right)^n (n!\Gamma(2j+n))^{-1} S(n, n_0)}, \tag{48}$$

where $S(n, n_0) = \sum\limits_{p=0}^{n_0 - n} \left(\frac{\Omega^2}{\gamma^2+\chi^2}\right)^p \Gamma(n_0 - p + 2j)\Gamma(n_0 - p + 1)$. We observe that for large $n \ll n_0$ the sum $S(n, n_0)$ slowly depends on parameter n and in expression (48) one can simplify the numerator and the denumerator by $S(n_0)$. Under this supposition the equations (47) and (33) coincide.

Let us now discuss the behavior of the cavity subharmonic EMF, when the detuning Δ is different from zero. In order to obtain the convergent sums in (46) we divide the numerator and denominator $D(n_0)$ in (46) of expression $(n_0)^2$. In this case one obtains the convergent expressions of the numerator and denominator.

The main interesting effect in this case is described by the dependence of Δ on the input pumping coherent EMF

$$\Delta = \Delta_0 + \Delta_f$$

where $\Delta_0 \approx \omega_p - \omega_{2k_0}$ is the part of detuning which does not depend on the intensity of external EMF, $\Delta_f = \Gamma\chi^0\Omega^2/[\gamma(\Gamma^2+(\bar{\omega}-2\omega_p)^2)]^{-1}$ is the detuning part, which is proportional to the intensity of the external coherent field. If the sign of detuning Δ_0 is opposite to the field dependent detuning Δ_f in the process of increasing the external pump EMF these two detunings give the zero value for the summary detuning Δ. In this point the enhancement of the generation rate of biphotons takes place [see Fig.].

Going back to the definition of $\xi = -\frac{i\Delta}{\gamma+i\chi}$, in the case $\Delta_f = 0$ one observe that the expression

$$\frac{\Gamma(1+\xi+l)\Gamma(1+\xi^*+l)}{\Gamma(1+\xi)\Gamma(1+\xi^*)} = \prod_{k=0}^{l}[(Re\xi+k)^2+(Im\xi)^2]. \tag{49}$$

for $\gamma \ll \chi$, witch corresponds to $Im\xi \ll Re\xi$, the product (49) will be transformed into $\prod\limits_{k=0}^{l}(Re\xi+k)^2$. On the other hand the sums on l in (45) and (46) become truncated for $l^* = \Delta/\chi$. The infinity series become finite. It is clear that increasing the field strength the number of generated biphotons tends to the constant value due to the fact that the expression for the mean number is obtained from the ratio of two power polynoms of EMF strength. In Fig. we represent some numerical simulations of dependence $\{I^z\}$ as a function of Rubi frequency for different value of field detuning. We observe that with the increase of the field detuning for the large strength of EMF the mean number of generated photons in microcavity tends to zero.

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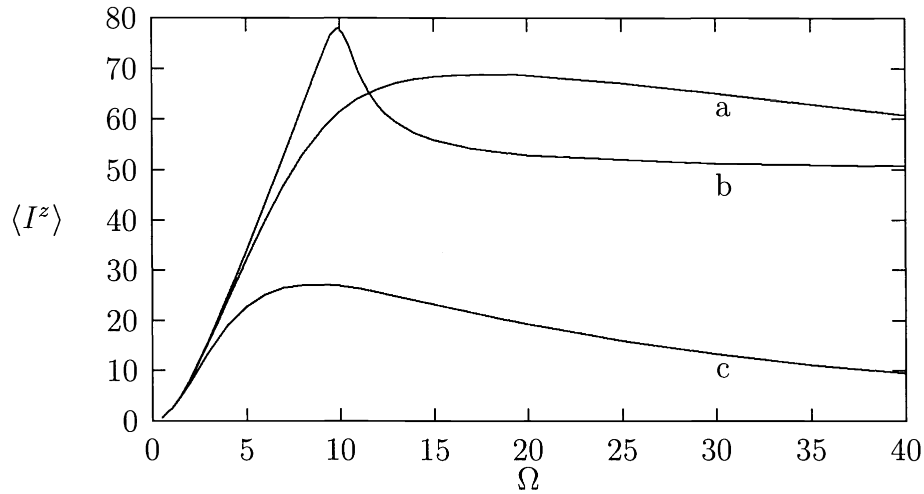

Fig. The dependence of the number of biphotons $\langle I^z \rangle$ as function of frequency Ω for $n_0 = 100$, $j = 20$, $\Delta_0 = 0.01$, $\gamma = 0.01$, $\chi = 0.1$ and *a)* $\Delta_f = 0.01\Omega^2$, *b)* $\Delta_f = 0$, *c)* $\Delta_f = 0.1\Omega^2$.