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Weak hyper residuated lattices

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Abstract. We introduced the notion of weak hyper residuated lattices which is a generalization of residuated lattices and prove some related results. Moreover, we introduce deductive systems, (positive) implicative and fantastic deductive systems and show the relations among them.

1. Introduction

The concept of hyperstructures was introduced by Marty [10] at 8th Congress of Scandinavian Mathematicians in 1934. Till now, the hyperstructures are studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics [1], [5]. Residuated lattices, introduced by Ward and Dilworth [11], are a common structure among algebras associated with logical systems. The main examples of residuated lattices are MV-algebras introduced by Chang [2] and BL-algebras introduced by Hájek [7]. Imai and Iséki introduced in [9] the notion of BCK-algebras. Borzooei et al. [2] introduced the concept of hyper K-algebras, which are a generalization of BCK-algebras. Also, they studied hyper K-ideals in hyper K-algebras. Recently, S. Ghorbani et al. [6], applied the hyperstructures to MV-algebras.

In this paper we want to construct a weak hyper residuated lattice as a generalization of the concept of residuated lattices that contain of the classes of MV-algebras, BL-algebras, and Heyting algebras.

A hyperoperation on a nonempty set A is a mapping $\circ : A \times A \to P^{\star}(A)$, where $P^{\star}(A)$ is the set of all the nonempty subsets of A and A with a hyperoperation is called a *hypergroupoid*.

Definition 1.1. A hypergroupoid $(A, *, 1)$ is called a *commutative semihypergroup* with 1 as the identity, if for all $x, y, z \in A$ we have:

(i) $x * (y * z) = (x * y) * z$,
(ii) $x * y = y * x$,
(iii) $x \in 1 * x$.

An element $a \in A$ is called a *scalar element* if for all $x \in A$ the set $a \odot x$ has only one element.

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Definition 1.2. By a *residuated lattice* we mean a structure $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that

$(RL1)$ $(L, \vee, \wedge, 0, 1)$ is a bounded lattice,
$(RL2)$ $(L, \odot, 1)$ is a commutative monoid,
$(RL3)$ the pair $(\odot, \rightarrow)$ is an adjoint pair, i.e., for any $x, y, z \in L$,

$$x * y \leqslant z \text{ if and only if } x \leqslant y \rightarrow z.$$

2. Weak hyper residuated lattices

Definition 2.1. By a *weak hyper residuated lattice* we mean a nonempty set L endowed with two binary operations $\vee$, $\wedge$ and two binary hyperoperations $\odot$, $\rightarrow$ and two constants 0 and 1 satisfying the following conditions:

$(WHRL1)$ $(L, \leqslant, \vee, \wedge, 0, 1)$ is a bounded lattice,
$(WHRL2)$ $(L, \odot, 1)$ is a commutative semihypergroup with 1 as the identity,
$(WHRL3)$ $a \odot c \ll b$ if and only if $c \ll a \rightarrow b$,

where $A \ll B$ means that $a \leqslant b$, for some $a \in A$ and $b \in B$; $A \leqslant B$ means that for any $a \in A$, there exists $b \in B$ such that $a \leqslant b$, where $\leqslant$ is the lattice ordering of L.

Example 2.2. Any residuated lattice is a weak hyper residuated lattice, too. □

Example 2.3. $L = [0, 1]$ with the natural ordering is a bounded lattice. Define the hyperoperations $\odot$, $\rightarrow$ and $\rightsquigarrow$ on L as follows:

$$a \odot b = a \times b, \qquad a \rightarrow b = \begin{cases} \{1\}, & a \leqslant b, \\ \{b\}, & a > b, \end{cases} \qquad a \rightsquigarrow b = \begin{cases} \{1\}, & a \leqslant b, \\ [b, 1], & a > b. \end{cases}$$

Then $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ and $(L, \vee, \wedge, \odot, \rightsquigarrow, 0, 1)$ are weak hyper residuated lattices. □

Example 2.4. Consider the chain $0 < a < b < 1$. Then $(L, \leqslant, 0, 1)$, where $L = \{0, a, b, 1\}$, is a bounded lattice. Putting $x \odot y = x \wedge y$ and defining the hyperoperations $\rightarrow$ and $\rightsquigarrow$ by the following two tables:

$\rightarrow$	0	a	b	1
0	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
a	$\{a, b, 1\}$	$\{1, a\}$	$\{1\}$	$\{1\}$
b	$\{a, 1\}$	$\{a\}$	$\{b, 1\}$	$\{1\}$
1	$\{0, 1\}$	$\{a\}$	$\{1, b\}$	$\{1\}$

$\rightsquigarrow$	0	a	b	1
0	$\{1\}$	$\{1, b\}$	$\{1, b\}$	$\{1, b\}$
a	$\{a, b, 1\}$	$\{1\}$	$\{1\}$	$\{1\}$
b	$\{a, b, 1\}$	$\{a\}$	$\{1, b\}$	$\{1, b\}$
1	$\{0, a, 1\}$	$\{1, a\}$	$\{1\}$	$\{1\}$

we obtain two hyper residuated lattices $(L, \leqslant, \odot, \rightarrow, 0, 1)$ and $(L, \leqslant, \odot, \rightsquigarrow, 0, 1)$. □

Proposition 2.5. *Let $\mathcal{L} = (L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ be a weak hyper residuated lattice. Then for nonempty subsets A, B, C of L and all $x, y, z \in L$ we have:*

(i) $1 \ll A \Leftrightarrow 1 \in A$ *and* $A \ll 0 \Leftrightarrow 0 \in A$, (*ii*) $x \leqslant y \Rightarrow 1 \in x \to y$ *and* $A \ll B \Rightarrow 1 \in A \to B$,

(*iii*) *if* 1 *is a scalar, then* $1 \in x \to y \Rightarrow x \leqslant y$ *and* $1 \in A \to B \Rightarrow A \ll B$,

(*iv*) $1 \in (x \to x) \cap (x \to 1) \cap (0 \to x)$,

(*v*) *if* 1 *is a scalar element of* L, *then* $x \in 1 \to x$,

(*vi*) $A \ll B \to C \Leftrightarrow A \odot B \ll C \Leftrightarrow B \ll A \to C$,

(*vii*) $x \odot y \ll x, y$ *and* $A \odot B \ll A, B$,

(*viii*) $x \ll y \to x$, $A \ll B \to A$ *and* $1 \in x \to (y \to x)$,

(*ix*) $x \to (y \to z) \leqslant (x \odot y) \to z \leqslant x \to (y \to z) \leqslant y \to (x \to z)$,

(*x*) $x \odot (x \to y) \ll x, y$,

(*xi*) $x \ll y \to (x \odot y)$ *and* $x \ll (y \to x) \to x$,

(*xii*) $x \leqslant y \Rightarrow x \odot z \ll y \odot z$, $z \to x \leqslant z \to y$ *and* $y \to z \leqslant x \to z$,

(*xiii*) $x \to y \leqslant (y \to z) \to (x \to z)$,

(*xiv*) $(x \to y) \odot (y \to z) \ll x \to z$ *and* $y \to z \ll (x \to y) \to (x \to z)$,

(*xv*) $0 \in x \odot y \Leftrightarrow x \ll \neg y$, *where* $\neg x = x \to 0$,

(*xvi*) $0 \in 0 \odot x$, $0 \in x \odot \neg x$, $1 \in \neg 0$ *and if* 1 *is a scalar, then* $0 \in \neg 1$,

(*xvii*) *if* $x \leqslant y$, *then* $\neg y \leqslant \neg x$,

(*xviii*) $x \to y \ll \neg y \to \neg x$,

(*xix*) $x \ll \neg\neg x$, $\neg\neg\neg x \ll \neg x \ll \neg\neg\neg x$ *and* $\neg x \ll x \to y$,

(*xx*) $\neg a, \neg b \ll \neg(a \wedge b)$ *and* $\neg(a \vee b) \ll \neg a$, $\neg b$,

(*xxi*) $x \to \neg y \ll \neg(x \odot y) \ll x \to \neg y$ *and* $y \to \neg x \ll \neg(x \odot y) \ll y \to \neg x$,

(*xxii*) *if* $\bigvee Y$ *exists, then* $\bigvee_{y \in Y}(x \odot y) \ll x \odot (\bigvee Y)$.

Proof. (*i*) Let $1 \ll A$. Then there exists $a \in A$ such that $1 \leqslant a$. Since, for any $x \in L$, $x \leqslant 1$, then $1 = a \in A$. The converse is obvious. Now, let $A \ll 0$. Then there exists $b \in A$ such that $b \leqslant 0$. Since, for any $x \in L$, $0 \leqslant x$, then $0 = b \in A$. The converse is clear.

(*ii*) Let $x \leqslant y$. Since $x \in x \odot 1$, then $x \odot 1 \ll y$. By ($WHRL3$), $1 \ll x \to y$ and so by (*i*), $1 \in x \to y$. Now, let $A \ll B$. Then there exist $a \in A$ and $b \in B$ such that $a \leqslant b$. So, by the above, $1 \in a \to b \subseteq A \to B$.

(*iii*) Let $1 \in x \to y$. Then $1 \leqslant x \to y$ and so $1 \odot x \ll y$. Now, since 1 is a scalar of L and $x \in 1 \odot x$, then $1 \odot x = x$ and so $x \leqslant y$. Similarly, $1 \in A \to B$ implies $A \ll B$.

(iv) Since, by the lattice ordering, $x \leqslant x$, $x \leqslant 1$ and $0 \leqslant x$, then $1 \in x \to x$, $1 \in x \to 1$ and $1 \in 0 \to x$. So we have (iv).

(v) Let 1 be a scalar of L. Then $x \odot 1 = x \leqslant x$ and by $(WHRL3)$, we get $x \ll 1 \to x$, i.e., there exists $a \in 1 \to x$ such that $x \leqslant a$. Since $a \in 1 \to x$, then $a = 1 \odot a \leqslant x \leqslant a$ and so $x = a \in 1 \to x$. Hence, for all $a \in A$, $a \in 1 \to a \subseteq 1 \to A$.

(vi) Let $A, B, C \subseteq L$. Then

$$A \ll B \to C \Leftrightarrow \exists a \in A, b \in B, c \in C \text{ such that } a \ll b \to c,$$
$$A \odot B \ll C \Leftrightarrow \exists a \in A, b \in B, c \in C \text{ such that } a \odot b \ll c,$$
$$B \ll A \to C \Leftrightarrow \exists a \in A, b \in B, c \in C \text{ such that } b \ll a \to c$$

and so, by $(WHRL3)$, we have (vi).

(vii) Since, for all $x, y \in L$, $y \leqslant 1 \in x \to x$ and $x \leqslant 1 \in y \to y$, then by $(WHRL3)$ $x \odot y = y \odot x \ll x, y$. By the similar way, we can prove that $A \odot B \ll A, B$.

$(viii)$ By (vii), $x \odot y \ll x$ and so, by $(WHRL3)$, $x \ll y \to x$. Hence, by (ii), $1 \in x \to (y \to x)$.

(ix) Let $u \in x \to (y \to z)$. Then

$$\begin{aligned} u \ll x \to (y \to z) &\Leftrightarrow (u \odot x) \ll y \to z, && \text{by } (vi) \\ &\Leftrightarrow (u \odot x) \odot y \ll z, && \text{by } (vi) \\ &\Leftrightarrow u \odot (x \odot y) \ll z \\ &\Leftrightarrow u \ll (x \odot y) \to z, && \text{by } (vi) \end{aligned}$$

and so, $x \to (y \to z) \leqslant (x \odot y) \to z$. By a similar way, we can prove that $(x \odot y) \to z \leqslant x \to (y \to z)$.

(x) It follows from (vi).

(xi) It follows from (vi) and $x \odot y \ll x \odot y$. Also, by (vi), $x \odot A \ll x$, where $A = y \to x$.

(xii) By the first part of (xi), $y \ll z \to (y \odot z)$. Now, since $x \leqslant y$, then $x \ll z \to (y \odot z)$. Hence, by (vi), we get $x \odot z \ll y \odot z$.

Now, let $u \in z \to x$. Since $u \ll z \to x$, then by $(WHRL3)$, $u \odot z \ll x$ and so by $x \leqslant y$, we get $u \odot z \ll y$. Hence, by $(WHRL3)$, $u \ll z \to y$ and so $z \to x \leqslant z \to y$.

Now, let $t \in y \to z$. Since, $t \ll y \to z$, then by (vi), $y \ll t \to z$ and so by $x \leqslant y$, we get that $x \ll t \to z$. Hence, by (vi), we get $t \ll x \to z$ and so $y \to z \leqslant x \to z$.

$(xiii)$ Let $u \in y \to z$. Then by (vi), $u \ll y \to z$ implies $y \ll u \to z$. So there exists $t \in u \to z$ such that $y \leqslant t$. Now, by (xii) and (ix), we have

$$x \to y \leqslant x \to t \subseteq x \to (u \to z) \leqslant u \to (x \to z) \subseteq (y \to z) \to (x \to z).$$

Hence, $x \to y \leqslant (y \to z) \to (x \to z)$.

(xiv) Those follow from (vi) and $(xiii)$.

(xv) $x \ll \neg y = y \to 0$, if and only if $x \odot y \ll 0$ if and only if $0 \in x \odot y$.

(xvi) We know that $x \leqslant 1 \in 0 \to 0$. Thus $x \odot 0 \ll 0$ and so by (i), we get $0 \in x \odot 0$. Also, it is clear that $x \to 0 \ll x \to 0$. Now, by (vi), we get $x \odot (x \to 0) \ll 0$ and so, by (i), $0 \in x \odot \neg x$.

$(xvii)$ It follows from (xii).

$(xviii)$ Since, by (xiv), $(x \to y) \odot (y \to 0) \ll x \to 0$, then by (vi), we get $x \to y \ll \neg y \to \neg x$.

(xix) By (xv), $x \odot (x \to 0) \ll 0$ and so by (vi), we get $x \ll (x \to 0) \to 0 = \neg\neg x$. Also, by (xii), we get $\neg\neg\neg x \ll \neg x$. On the other hand, if we put $A = x \to 0$ then by (xv), $A \odot (A \to 0) \ll 0$. Now, we conclude $A \ll (A \to 0) \to 0$ by (vi), i.e., $\neg x \ll \neg\neg\neg x$. (Note that, we do not have anti-symmetry for $\ll$.)

(xx) Those follow from (xii).

(xxi) It is conclude by (ix) and $(xiii)$.

$(xxii)$ If $\bigvee Y$ exists, then $y \leqslant \bigvee Y$ for all $y \in Y$. So, by (xii), $x \odot y \ll x \odot (\bigvee Y)$. Thus there exists $b_y \in x \odot (\bigvee Y)$ such that $x \odot y \ll b_y$ for any $y \in Y$. Hence, we get $\bigvee_{y \in Y}(x \odot y) \ll \bigvee_{y \in Y} b_y \leqslant \bigvee x \odot (\bigvee Y)$. □

Theorem 2.6. *Any weak hyper residuated lattice of order n can be extend to a weak hyper residuated lattice of order $n+1$.*

Proof. Let L be a weak hyper residuated lattice of order n, and $\overline{L} = L \cup \{e\}$ for some $e \notin L$. Putting

$$z \leqslant' y \Leftrightarrow z \leqslant y, \text{ for all } z, y \in L \text{ and } \; x \leqslant' e, \text{ for all } x \in L',$$

$$a \odot' b = \begin{cases} a \odot b & \text{if } a, b \in L, \\ \{a\} & \text{if } a \in L \text{ and } b = e, \\ \{b\} & \text{if } b \in L \text{ and } a = e, \\ \{e\} & \text{if } a = b = e, \end{cases}$$

$$a \to' b = \begin{cases} (a \to b) \cup \{e\} & \text{if } a, b \in L, 1 \in a \to b \\ a \to b & \text{if } a, b \in L, 1 \notin a \to b, \\ \{e\} & \text{if } b = e, \\ \{b\} & \text{if } a = e, \end{cases}$$

we see that $(\overline{L}, \leqslant')$ is a bounded lattice with 0 as the minimum and e as the maximum elements of $\overline{L}$. The proof of $(WHRL1)$ and $(WHRL2)$ are clear. Now, we prove the $(WHRL3)$. Let $x, y, z \in \overline{L}$. We consider the following cases:

Case 1. For $x = y = z = e$, the proof is obvious.

Case 2. Let $x = z = e$ and $y \in L$. Then $x \odot' y = \{y\}$ and $y \to' z = \{e\}$. Therefore, $x \odot' y \ll' z$ if and only if $x \ll' y \to' z$. By the similar way, we have for $y = z = e$ and $x = y = e$.

Case 3. Let $x, y \in L$ and $z = e$. Since $y \to' z = \{e\}$ and $u \ll' e$, for all $u \in L'$, then $x \odot' y \ll' z$ implies $x \ll' y \to' z$. Now, let $x \ll' y \to' z$. Since $z = e$, then $x \odot' y \ll' z$.

Case 4. Let $x, z \in L$ and $y = e$. Then $x \odot' y = \{x\}$ and $y \rightarrow' z = \{z\}$. Therefore, $x \odot' y \ll' z$ if and only if $x \ll' y \rightarrow' z$.

Case 5. Let $y, z \in L$ and $x = e$. Then $x \odot' y = \{y\}$. If $x \odot' y = \{y\} \ll' z$, then $y \ll' z$. Since $y, z \in L$ we get $y \ll z$ and so $1 \in y \rightarrow z$. Hence $e \in y \rightarrow' z$ and so $x \ll' y \rightarrow' z$. Now, let $x \ll' y \rightarrow' z$. Then by definition of $\leqslant'$, we have $e \in y \rightarrow' z$ and so $1 \in y \rightarrow z$ or $z = e$. Since $y \in L$, then $y \neq e$ and so $1 \in y \rightarrow z$. Therefore, $x \odot' y \ll' z$.

Case 6. Let $x, y, z \in L$ and $1 \in y \rightarrow z$. If $x \odot' y \ll' z$, then by definition of $\rightarrow'$, $e \in y \rightarrow' z$ and so $x \ll' y \rightarrow' z$. Now, let $x \ll' y \rightarrow' z$. Since $1 \in y \rightarrow z$, then $x \ll y \rightarrow z$ and so $x \odot y \ll z$. Hence $x \odot' y = x \odot y \ll' z$.

Case 7. Let $x, y, z \in L$ and $1 \notin y \rightarrow z$. Then by definitions of $\odot'$ and $\leqslant'$, we get

$$x \odot' y \ll' z \Leftrightarrow x \odot y \ll z \Leftrightarrow x \ll y \rightarrow z \Leftrightarrow x \ll' y \rightarrow' z.$$

Hence, $(\overline{L}, \leqslant', \odot', \rightarrow', 0, e)$ is a weak hyper residuated lattice of order $n+1$. □

Definition 2.7. A subset D of $\mathcal{L}$ containing 1 is called a *deductive system* (shortly: $\mathcal{DS}$) if $x \in D$ and $(x \rightarrow y) \subseteq D$ imply $y \in D$, for all $x, y \in L$.

Example 2.8. (i) Clearly, L is a $\mathcal{DS}$ of $\mathcal{L}$. If 1 is an scalar element, then $\{1\}$ is a $\mathcal{DS}$ of $\mathcal{L}$, too.

(ii) Let $([0,1], \vee, \wedge, \odot, \rightsquigarrow, 0, 1)$ be a weak hyper residuated lattice as in Example 2.3. It is easy to shows that $D = [\frac{1}{2}, 1]$ is its $\mathcal{DS}$.

(iii) In Example 2.4, $\{1\}$ is a $\mathcal{DS}$ and $\{1, b\}$ is not a $\mathcal{DS}$ of $\mathcal{L}$.

Definition 2.9. A nonempty subset D of $\mathcal{L}$ is called

- an *upset* if $x \in D$ and $x \leqslant y$, then $y \in D$, for all $x, y \in L$,
- an $S_{\rightarrow}$*reflexive* if $(A \rightarrow B) \cap D \neq \emptyset$ implies $(A \rightarrow B) \subseteq D$, for all $A, B \subseteq L$.

Proposition 2.10. *Every $S_{\rightarrow}reflexive$ $\mathcal{DS}$ of $\mathcal{L}$ is an upset.*

Proof. Let D be an $S_{\rightarrow}$-reflexive $\mathcal{DS}$, $x \in D$ and $x \leqslant y$, for some $y \in L$. By Proposition 2.5(ii), $1 \in x \rightarrow y$ and so $(x \rightarrow y) \cap D \neq \emptyset$. Since D is $S_{\rightarrow}$-reflexive, then $x \rightarrow y \subseteq D$ and so by DS, we have $y \in D$. □

Proposition 2.11. *Let D be an $S_{\rightarrow}reflexive$ $\mathcal{DS}$ of $\mathcal{L}$. Then*

(i) $D \ll A \rightarrow B \Leftrightarrow (A \rightarrow B) \cap D \neq \emptyset \Leftrightarrow A \rightarrow B \subseteq D$,

(ii) $A \rightarrow B \subseteq D$ *and* $A \rightarrow B \ll A' \rightarrow B'$ *imply* $D \ll A' \rightarrow B'$,

(iii) $D \ll A \rightarrow B \ll A' \rightarrow B'$ *implies* $D \ll A' \rightarrow B'$.

Proof. (i) If $D \ll A \rightarrow B$, then there exist $a \in A$ and $b \in B$ such that $D \ll a \rightarrow b$. So there exists $d \in D$ and $t \in a \rightarrow b$ such that $d \leqslant t$. Since D is an $S_{\rightarrow}reflexive$ $\mathcal{DS}$, then by Proposition 2.10, D is an upset and so $t \in D \cap (a \rightarrow b)$. Hence $(A \rightarrow B) \cap D \neq \emptyset$. Conversely, let $(A \rightarrow B) \cap D \neq \emptyset$. Then there exist $a \in A$ and

$b \in B$ such that $(a \to b) \cap D \neq \emptyset$. So there exists $t \in (a \to b) \cap D$ and since $t \leqslant t$, then $D \ll a \to b$. Hence $D \ll A \to B$.

(ii) Let $A \to B \subseteq D$. Since $A \to B \ll A' \to B'$, then there exist $a \in A$, $b \in B$, $a' \in A'$ and $b' \in B'$ such that $a \to b \ll a' \to b'$. So there exist $t \in a \to b$ and $t' \in a' \to b'$ such that $t \leqslant t'$. Now, we have $t \in a \to b \subseteq A \to B \subseteq D$ and so $t \in D$. Since D is an upset, then $t' \in D$. Therefore, $t' \in D \cap A' \to B'$ and so by (i), we get $D \ll A' \to B'$.

(iii) If $D \ll A \to B$, then by (i), $A \to B \subseteq D$ and so by (ii), we conclude $D \ll A' \to B'$. □

Example 2.12. Let $(L, \leqslant, 0, 1)$ be as in Example 2.4. Consider the following hyperoperations:

$\odot$	0	a	b	1
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{a, 0\}$	$\{a\}$	$\{a\}$
b	$\{0\}$	$\{a\}$	$\{b\}$	$\{b\}$
1	$\{0\}$	$\{a\}$	$\{b\}$	$\{1\}$

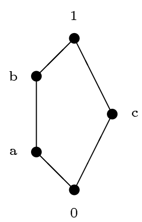
$\to$	0	a	b	1
0	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
a	$\{0, a\}$	$\{1\}$	$\{1\}$	$\{1\}$
b	$\{0\}$	$\{0, a\}$	$\{1\}$	$\{1\}$
1	$\{0\}$	$\{a\}$	$\{b\}$	$\{1\}$

Then $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a weak hyper residuated lattice and $D_1 = \{1\}$, $D_2 = \{1, b\}$ are its $S_{\to}$-reflexive deductive systems. □

3. Implicative deductive systems

Definition 3.1. A subset D of $\mathcal{L}$ containing 1 is called an *implicative* deductive system (shortly: $\mathcal{IDS}$), if $(x \to y) \subseteq D$ and $x \to (y \to z) \subseteq D$ imply $(x \to z) \subseteq D$.

Example 3.2. Let $L = \{a, b, c, 0, 1\}$ be the lattice with the following diagram.



Consider the following hyperoperations:

$\to$	0	a	b	c	1
0	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
a	$\{c\}$	$\{1\}$	$\{1\}$	$\{c\}$	$\{1\}$
b	$\{c\}$	$\{a, b, c\}$	$\{1\}$	$\{c\}$	$\{1\}$
c	$\{a, b\}$	$\{a, b\}$	$\{b, a\}$	$\{1\}$	$\{1\}$
1	$\{0\}$	$\{a\}$	$\{b, a\}$	$\{c\}$	$\{1\}$

$\odot$	0	a	b	c	1
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{a\}$	$\{a\}$	$\{0\}$	$\{a\}$
b	$\{0\}$	$\{a\}$	$\{b, a\}$	$\{0\}$	$\{a, b\}$
c	$\{0\}$	$\{0\}$	$\{0\}$	$\{c\}$	$\{c\}$
1	$\{0\}$	$\{a\}$	$\{b, a\}$	$\{c\}$	$\{1\}$

It is easy to show that $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a weak hyper residuated lattice. Moreover, easy calculations show that $\{1, a\}$ is an $\mathcal{IDS}$ of $\mathcal{L}$ and $\{1, b, c\}$ is not an $\mathcal{IDS}$. Since $(b \rightarrow 0) = \{c\} \subseteq \{1, b, c\}$ and $(b \rightarrow (0 \rightarrow a)) = \{1\} \subseteq \{1, b, c\}$ but $(b \rightarrow a) = \{a, b, c\} \nsubseteq \{1, b, c\}$. □

Theorem 3.3. *Let D be a nonempty subset of $\mathcal{L}$ containing 1. Then*

(i) *if D is an $\mathcal{IDS}$, then D is a $\mathcal{DS}$,*

(ii) *D is an $\mathcal{IDS}$ if and only if each $D_a = \{x \in L | a \rightarrow x \subseteq D\}$ is a $\mathcal{DS}$ of $\mathcal{L}$,*

(iii) *D is an $\mathcal{IDS}$ if and only if $(x \rightarrow (y \rightarrow z)) \cap D \neq \emptyset$ and $(x \rightarrow y) \cap D \neq \emptyset$ imply $(x \rightarrow z) \cap D \neq \emptyset$, for all $x, y, z \in L$.*

Proof. (i) Let $x \in D$ and $x \rightarrow y \subseteq D$. Since by Proposition 2.5(v), $x \in (1 \rightarrow x) \cap D$ and $(x \rightarrow y) \subseteq (1 \rightarrow (x \rightarrow y)) \cap D$ and D is an $\mathcal{IDS}$, then $y \in (1 \rightarrow y) \subseteq D$. Hence D is a $\mathcal{DS}$.

(ii) Let $a \in D$. Since, by Proposition 2.5(iv), $1 \in (a \rightarrow 1)$, then $1 \in D_a$. Suppose that $x \in D_a$ and $(x \rightarrow y) \subseteq D_a$. Then $(a \rightarrow x) \subseteq D$ and $(a \rightarrow (x \rightarrow y)) \subseteq D$. Hence $(a \rightarrow y) \subseteq D$ i.e., $y \in D_a$. Therefore, D_a is a $\mathcal{DS}$ of $\mathcal{L}$.

(iii) The proof is clear. □

Theorem 3.4. *For a nonempty subset D of $\mathcal{L}$ the following are equivalent:*

(i) *D is an $\mathcal{IDS}$,*

(ii) *D is a $\mathcal{DS}$ and $(y \rightarrow (y \rightarrow x)) \subseteq D$ implies $(y \rightarrow x) \subseteq D$, for any $x, y \in L$,*

(iii) *D is a $\mathcal{DS}$ and $(z \rightarrow (y \rightarrow x)) \subseteq D$ implies $((z \rightarrow y) \rightarrow (z \rightarrow x)) \subseteq D$, for any $x, y, z \in L$,*

(iv) *$1 \in D$ and $(z \rightarrow (y \rightarrow (y \rightarrow x))) \subseteq D$ and $z \in D$ imply $(y \rightarrow x) \subseteq D$, for any $x, y, z \in L$,*

(v) *$(x \rightarrow (x \odot x)) \subseteq D$, for any $x \in L$.*

Proof. $(i) \Rightarrow (ii)$ By Theorem 3.3, D is a $\mathcal{DS}$ of $\mathcal{L}$. Now, let $(y \rightarrow (y \rightarrow x)) \subseteq D$, for any $x, y \in L$. Since $1 \in (y \rightarrow y) \cap D$ and D is an $\mathcal{IDS}$ of $\mathcal{L}$, then $(y \rightarrow x) \subseteq D$.

$(ii) \Rightarrow (iii)$ Let $(z \rightarrow (y \rightarrow x)) \subseteq D$, for any $x, y \in L$. Then by Proposition 2.5(xiv),

$$y \rightarrow x \ll (z \rightarrow y) \rightarrow (z \rightarrow x), \tag{1}$$

and by Proposition 2.5(ix),

$$(z \rightarrow y) \rightarrow (z \rightarrow x) \leqslant z \rightarrow ((z \rightarrow y) \rightarrow x) \tag{2}$$

So, by Proposition 2.5(xii) and (1), we get $z \rightarrow (y \rightarrow x) \ll z \rightarrow ((z \rightarrow y) \rightarrow (z \rightarrow x))$, and by Proposition 2.5$(xii)$ and (2), we get $z \rightarrow ((z \rightarrow y) \rightarrow (z \rightarrow x)) \leqslant z \rightarrow (z \rightarrow ((z \rightarrow y) \rightarrow x))$. Hence, $z \rightarrow (y \rightarrow x) \ll z \rightarrow (z \rightarrow (z \rightarrow y) \rightarrow x))$. By

Proposition 2.11(i) and assumption, we have $D \ll z \to ((z \to y) \to x) \leqslant (z \to y) \to (z \to x)$ and so, we get $(z \to y) \to (z \to x) \subseteq D$.

$(iii) \Rightarrow (iv)$ Let $z \to (y \to (y \to x)) \subseteq D$ and $z \in D$, for any $x, y, z \in L$. Since D is a $\mathcal{DS}$, then $y \to (y \to x) \subseteq D$. Now, by (iii), $(y \to y) \to (y \to x) \subseteq D$. Also,by Proposition 2.5(iv, v), we get $y \to x \subseteq 1 \to (y \to x) \subseteq (y \to y) \to (y \to x) \subseteq D$ and so $y \to x \subseteq D$.

$(iv) \Rightarrow (i)$ Let $z \to (y \to x) \subseteq D$ and $z \to y \subseteq D$, for any $x, y, z \in L$. Then, by Proposition 2.5, we get $z \to (y \to x) \leqslant y \to (z \to x) \ll (z \to y) \to (z \to (z \to x))$, and so, by Proposition 2.11, we conclude that $(z \to y) \to (z \to (z \to x)) \subseteq D$. Now, by (iv), $z \to x \subseteq D$.

$(ii) \Rightarrow (v)$ Let $x \in A$ and $u \in x \odot x$. Then $u \in x \odot x$ and so $x \odot x \ll u$. Now, by $(WHRL3)$, $x \ll x \to u$ and so by Proposition 2.5(ii), $1 \in D \cap x \to (x \to u)$. Hence, by Proposition 2.11, $x \to (x \to u) \subseteq D$. Therefore, by (ii), $x \to u \subseteq D$.

$(v) \Rightarrow (ii)$ Put $A = y \to (y \to x) \subseteq D$. By using two times of Proposition 2.5(ix), we get

$$1 \in A \to A = A \to (y \to (y \to x)) \leqslant y \to (A \to (y \to x)) \leqslant y \to (y \to (A \to x)).$$

Hence, $1 \in y \to (y \to (A \to x))$ i.e., $\exists t \in A \to x$ such that $1 \in y \to (y \to t)$. Then $1 \ll y \to (y \to t)$ and so by $(WHRL3)$, $y = 1 \odot y \ll y \to t$. Since, by $(WHRL3)$, $y \odot y \ll t$, then $\exists a \in y \odot y$ such that $a \leqslant t$ and so by Proposition 2.5(xii), $y \to a \leqslant y \to t$. On the other hand, $y \to a \subseteq y \to (y \odot y) \subseteq D$. So, by Proposition 2.5(ix), $D \ll y \to t \subseteq y \to (A \to x) \leqslant A \to (y \to x)$. Now, by Proposition 2.11, $A \to (y \to x) \subseteq D$ and since D is a $\mathcal{DS}$, then $y \to x \subseteq D$. □

Corollary 3.5. *If $\{1\}$ is an $\mathcal{IDS}$, then $x \leqslant x \odot x$, for any $x \in L$.*

Proof. Since, for any $u \in x \odot x$ and $x \in L$, $x \to u \subseteq \{1\}$, then $1 \in x \to u$. Now, by Proposition 2.5(iii), we get $x \leqslant u$, for any $u \in x \odot x$, i.e. $x \leqslant x \odot x$. □

Theorem 3.6. *Let D be an $\mathcal{IDS}$ and E be a $\mathcal{DS}$ of L such that $D \subseteq E$. Then E is an $\mathcal{IDS}$, too.*

Proof. Put $A = z \to (y \to x) \subseteq E$. Now, by using two times of Proposition 2.5(ix), we have

$$1 \in A \to A = A \to (z \to (y \to x)) \leqslant z \to (A \to (y \to x)) \leqslant z \to (y \to (A \to x)).$$

So $1 \in D \cap z \to (y \to (A \to x))$. By Proposition 2.11($iii$), we get $z \to (y \to (A \to x)) \subseteq D$ and so by Theorem 3.4(iii), $(z \to y) \to (z \to (A \to x)) \subseteq D \subseteq E$. Also, by Proposition 2.5(ix),

$$(z \to y) \to (z \to (A \to x)) \leqslant (z \to y) \to (A \to (z \to x)) \leqslant A \to ((z \to y) \to (z \to x).$$

Therefore, $A \to ((z \to y) \to (z \to x)) \subseteq E$. Since E is a $\mathcal{DS}$ and $A \subseteq E$, then $(z \to y) \to (z \to x) \subseteq E$. Hence, by Theorem 3.3, E is an $\mathcal{IDS}$. □

Corollary 3.7. *The deductive system $\{1\}$ is an $\mathcal{IDS}$ if and only if every $\mathcal{DS}$ of L is an $\mathcal{IDS}$.*

4. Positive implicative deductive systems

Definition 4.1. A subset D of $\mathcal{L}$ containing 1 is a *positive implicative deductive system* (shortly: $\mathcal{PIDS}$), if $x \to ((y \to z) \to y) \subseteq D$ and $x \in D$ imply $y \in D$.

Example 4.2. Let $\mathcal{L}$ be as in the Example 3.2. Then easy calculations show that $\{1, a, b\}$ is a $\mathcal{PIDS}$ of $\mathcal{L}$ and $\{1, a\}$ is an $\mathcal{IDS}$ but not a $\mathcal{PIDS}$ of $\mathcal{L}$. Since we have $a \to ((b \to b) \to b) = a \to (\{1\} \to \{b\}) = a \to \{a, b\} = \{1\} \subseteq \{1, a\}$ and $a \in \{1, a\}$ but $b \notin \{1, a\}$. $\square$

Theorem 4.3. *Every $\mathcal{PIDS}$ is an $\mathcal{IDS}$.*

Proof. Let D be a $\mathcal{PIDS}$ and $y \to (y \to x) \subseteq D$. Then by Proposition 2.5($v, xiii$),

$$y \to (y \to x) \leqslant ((y \to x) \to x) \to (y \to x) \subseteq 1 \to (((y \to x) \to x) \to (y \to x)),$$

So, by Proposition 2.11, we get $1 \to (((y \to x) \to x) \to (y \to x)) \subseteq D$. Now, since D is a $\mathcal{PIDS}$ and $1 \in D$, then $y \to x \subseteq D$ and so, by Theorem 3.3, D is an $\mathcal{IDS}$. $\square$

Corollary 4.4. *Every $\mathcal{PIDS}$ is a $\mathcal{DS}$.* $\square$

Theorem 4.5. *Let D be a $\mathcal{DS}$ of $\mathcal{L}$. Then the following are equivalent:*

(i) *D is a $\mathcal{PIDS}$,*

(ii) *if $(x \to y) \to x \subseteq D$, then $x \in D$, for any $x, y \in L$,*

(iii) *$(\neg x \to x) \to x \subseteq D$, for any $x \in L$.*

Proof. (i) $\Rightarrow$ (ii) Let D be a $\mathcal{PIDS}$ and take $A = (x \to y) \to x \subseteq D$. Since $A \subseteq (1 \to A) \cap D$, then by Proposition 2.11, $1 \to A = 1 \to ((x \to y) \to x) \subseteq D$. So, by assumption, $x \in D$.

(ii) $\Rightarrow$ (i) Let $x \to ((y \to z) \to y) \subseteq D$ and $x \in D$. Since D is a $\mathcal{DS}$, then $(y \to z) \to y \subseteq D$ and so, by assumption, we get $y \in D$ i.e., D is a $\mathcal{PIDS}$.

(i) $\Rightarrow$ (iii) Let D be a $\mathcal{PIDS}$. By Proposition 2.5 (xi), $x \ll (y \to x) \to x$, for any $y \in L$. Now, take $y \in \neg x$. Hence $x \ll (\neg x \to x) \to x$ and we get

$$\begin{aligned}
1 &\in x \to ((\neg x \to x) \to x), && \text{by Proposition 2.5}(ii)\\
&\le (((\neg x \to x) \to x) \to 0) \to (x \to 0), && \text{by Proposition 2.5}(xiii)\\
&\le (\neg x \to x) \to ((((\neg x \to x) \to x) \to 0) \to x), && \text{by Proposition 2.5}(xiii)\\
&\le (\underbrace{((\neg x \to x) \to x)}_{A} \to 0) \to \underbrace{((\neg x \to x) \to x)}_{A}, && \text{by Proposition 2.5}(ix)\\
&= (A \to 0) \to A.
\end{aligned}$$

Then $1 \in D \cap ((A \to 0) \to A)$. Hence, by Proposition 2.11, $(A \to 0) \to A \subseteq D$. Therefore, by ($ii$), we have $A \subseteq D$ i.e., $(\neg x \to x) \to x \subseteq D$.

$(iii) \Rightarrow (i)$ Let $D \ll (x \to y) \to x$. It is enough to show that $x \in D$. Since $0 \leqslant y$, for any $y \in L$, then by using two times of Proposition 2.5$(xiii)$, we get $(x \to y) \to x \ll (x \to 0) \to x$. By Proposition 2.11, we get $\neg x \to x = (x \to 0) \to x \subseteq D$ and by assumption $(\neg x \to x) \to x \subseteq D$. Now, since D is a $\mathcal{DS}$, then $x \in D$. □

In the following proposition we give a condition that an $\mathcal{IDS}$ is a $\mathcal{PIDS}$.

Proposition 4.6. *Let D be an $\mathcal{IDS}$. Then D is a $\mathcal{PIDS}$ if and only if*
$(x \to y) \to y \subseteq D$ implies $(y \to x) \to x \subseteq D$, for any $x, y \in L$.

Proof. Let D be a $\mathcal{PIDS}$ and $(x \to y) \to y \subseteq D$. By Proposition 2.5$(xi)$, we have $x \ll (y \to x) \to x$ and so by Proposition 2.5$(xiii)$, $((y \to x) \to x) \to y \ll x \to y$. Since,

$$\begin{aligned}(x \to y) \to y &\le (y \to x) \to ((x \to y) \to x), && \text{by Proposition 2.5}(xiii)\\ &\le (x \to y) \to ((y \to x) \to x), && \text{by Proposition 2.5}(ix)\\ &\ll \underbrace{(((y \to x) \to x) \to y) \to ((y \to x) \to x)}_{A},\end{aligned}$$

by Proposition 2.5$(xiii)$ and Proposition 4.5, we have, $D \ll (x \to y) \to y \ll A \subseteq 1 \to A$. So, by Proposition 2.11, we get $(1 \to A) \subseteq D$. Hence $(1 \to A) = 1 \to ((((y \to x) \to x) \to y) \to ((y \to x) \to x)) \subseteq D$. Moreover, since $1 \in D$ and D is a $\mathcal{DS}$, then

$$(\underbrace{((y \to x) \to x)}_{X} \to y) \to \underbrace{((y \to x) \to x)}_{X} \subseteq D.$$

Since D is a $\mathcal{PIDS}$, then by Proposition 4.5 we obtain $(y \to x) \to x = X \subseteq D$.

Conversely, by Proposition 4.5, it is enough to show that $(x \to y) \to x \subseteq D$ implies $x \in D$. For this let $(x \to y) \to x \subseteq D$. Since, by Proposition 2.5(xii), $(x \to y) \to x \leqslant (x \to y) \to ((x \to y) \to y)$, then by Proposition 2.11, we have $(x \to y) \to ((x \to y) \to y) \subseteq D$. Since D is an $\mathcal{IDS}$, then by Theorem 3.4(ii), we get $(x \to y) \to y \subseteq D$. Now, by assumption, we have $(y \to x) \to x \subseteq D$.

On the other hand, since $y \odot x \ll y$, then $y \ll x \to y$ and, by Proposition 2.5(xii), we get $(x \to y) \to x \ll y \to x$. Now, by assumption, $(x \to y) \to x \subseteq D$. So, by Proposition 2.11, we get $y \to x \subseteq D$. Since $(y \to x) \to x \subseteq D$, $y \to x \subseteq D$ and D is a $\mathcal{DS}$, then $x \in D$. □

Theorem 4.7. *Let D be a $\mathcal{PIDS}$ and E be a $\mathcal{DS}$ of L such that $D \subseteq E$. Then E is a $\mathcal{PIDS}$, too.*

Proof. Let D be a $\mathcal{PIDS}$ and E be a $\mathcal{DS}$ such that $D \subseteq E$. Since, by Theorem 4.3, D is an $\mathcal{IDS}$, then by Theorem 3.6. E is an $\mathcal{IDS}$, too. Now, take $A = (x \to y) \to y \subseteq E$. By Proposition 4.6, it is enough to show that $(y \to x) \to x \subseteq E$. Since $1 \in A \to A = A \to ((x \to y) \to y)$, then $A \to ((x \to y) \to y) \subseteq D$. Also, by Theorem 3.4$(iii)$, $(A \to (x \to y)) \to (A \to y) \subseteq D$. Therefore, by Proposition 2.5(ix), $(x \to (A \to y)) \to (A \to y) \subseteq D$ and so, by Proposition 4.6,

$((A \to y) \to x) \to x \subseteq D \subseteq E$. Now, we get $(A \to y) \to x) \to x \subseteq E$. On the other hand, we have

$$\begin{aligned}(x \to y) \to y &\ll (\underbrace{((x \to y) \to y)}_{A} \to y) \to y, &&\text{by Proposition 2.5}(xi)\\ &\ll (y \to x) \to ((A \to y) \to x), &&\text{by Proposition 2.5}(xii)\\ &\ll (((A \to y) \to x) \to x) \to ((y \to x) \to x) \subseteq E,\end{aligned}$$

by Proposition 2.5(xii) and Proposition 2.11. This implies $(y \to x) \to x \subseteq E$ since E is a $\mathcal{D}S$. □

5. Fantastic deductive systems

Definition 5.1. A subset D of $\mathcal{L}$ containing 1 is called a *fantastic deductive system* (shortly: $\mathcal{FDS}$) if $z \to (y \to x) \subseteq D$ and $z \in D$ imply $((x \to y) \to y) \to x \subseteq D$.

Example 5.2. Let $\mathcal{L}$ be as in Example 3.2. Then $\{1, a, b\}$ is a $\mathcal{FDS}$ of $\mathcal{L}$.

Proposition 5.3. *Any $\mathcal{F}DS$ is a $\mathcal{D}S$.*

Proof. Let D be a $\mathcal{F}DS$, $x \to y \subseteq D$ and $x \in D$. Since by Proposition 2.5(v), $y \in 1 \to y$, then $x \to y \subseteq x \to (1 \to y) \cap D$ and so, by $x \in D$ and definition of a $\mathcal{F}DS$, $((y \to 1) \to 1) \to y \subseteq D$. Now, by Proposition 2.5(xi) and (i), we conclude that $1 \in (y \to 1) \to 1$. So

$$1 \to y \subseteq \bigcup_{a \in (y \to 1) \to 1} (a \to y) = ((y \to 1) \to 1) \to y \subseteq D.$$

Hence, $1 \to y \subseteq D$. Since, by Proposition 2.5(v), $y \in 1 \to y$, then $y \in D$. Thus D is a $\mathcal{D}S$. □

Proposition 5.4. *Let D be a $\mathcal{D}S$ of $\mathcal{L}$. Then D is a $\mathcal{F}DS$ if and only if*
$y \to x \subseteq D$ implies $((x \to y) \to y) \to x \subseteq D$.

Proof. Let D be a $\mathcal{F}DS$ and $y \to x \subseteq D$. By Proposition 2.5(v), $y \to x \subseteq 1 \to (y \to x)$, and so by Proposition 2.11, $1 \to (y \to x) \subseteq D$. Since $1 \in D$ and D is a $\mathcal{F}DS$, then $((x \to y) \to y) \to x \subseteq D$. Conversely, let $z \to (y \to x) \subseteq D$ and $z \in D$. Since D is a $\mathcal{D}S$, then we conclude $y \to x \subseteq D$. Now, by assumption, $((x \to y) \to y) \to x \subseteq D$. □

Theorem 5.5. *Let D be a $\mathcal{F}DS$ and E be a $\mathcal{D}S$ of $\mathcal{L}$ such that $D \subseteq E$. Then E is a $\mathcal{F}DS$, too.*

Proof. Let $y \to x \subseteq E$. Since, by Proposition 2.5(iv) and (ix), $1 \in (y \to x) \to (y \to x) \leqslant y \to ((y \to x) \to x)$, then $1 \in D \cap y \to ((y \to x) \to x)$ and so, by Proposition 2.11, $y \to ((y \to x) \to x) \subseteq D$. Now, take $X = (y \to x) \to x$. Since D

is a $\mathcal{F}DS$, then by Proposition 5.4, $y \to X \subseteq D$ implies $((X \to y) \to y) \to X \subseteq D$. Also, by Proposition 2.5(ix), we have

$$((X \to y) \to y) \to X \ll \underbrace{(y \to x) \to (((X \to y) \to y) \to x)}_{A},$$

which shows that $D \cap A \neq \emptyset$. Therefore, by Proposition 2.11, $A \subseteq D$ and since $D \subseteq E$, then $A \subseteq E$. On the other hand, $y \to x \subseteq E$ and E is a $\mathcal{D}S$ imply that $\underbrace{((X \to y) \to y) \to x}_{B} \subseteq E$. Moreover, using Proposition (ii), (iv), (ix) and $(xiii)$, from

$$\begin{aligned} 1 \in (y \to x) \to 1 &\subseteq (y \to x) \to (x \to x), \\ &\leq x \to ((y \to x) \to x), \\ &\leq (((y \to x) \to x) \to y) \to (x \to y), \\ &\leq ((x \to y) \to y) \to ((((y \to x) \to x) \to y) \to y), \\ &\leq \underbrace{(((((y \to x) \to x) \to y) \to y) \to x)}_{B} \to \underbrace{(((x \to y) \to y) \to x)}_{C}, \end{aligned}$$

we get $1 \in E \cap (B \to C)$. Now, since E is a $\mathcal{D}S$, then by Proposition 2.11, $C = ((x \to y) \to y) \to x \subseteq E$. Hence, by Proposition 5.4, E is a $\mathcal{F}DS$. □

Corollary 5.6. *$\{1\}$ is a $\mathcal{F}DS$ of $\mathcal{L}$ if and only if any $\mathcal{D}S$ of L is a $\mathcal{F}DS$.*

Theorem 5.7. *If D is a $\mathcal{P}IDS$ of $\mathcal{L}$, then it is a $\mathcal{F}DS$.*

Proof. Let D be a $\mathcal{P}IDS$ and $y \to x \subseteq D$. Then by Proposition 2.5$(xiii)$ and (ix), we have

$$y \to x \leqslant ((x \to y) \to y) \to ((x \to y) \to x) \ll (x \to y) \to \underbrace{(((x \to y) \to y) \to x)}_{A}.$$

Since $y \to x \subseteq D$, then by Proposition 2.11, $(x \to y) \to A \subseteq D$. Also, by Proposition (2.5)(vii), $x \odot ((x \to y) \to y) \ll x$. Therefore, by Proposition 2.5(vi), $x \ll ((x \to y) \to y) \to x$. Now, by Proposition 2.5$(xiii)$, we conclude $(((x \to y) \to y) \to x) \to y \ll x \to y$. So, by another using of Proposition 2.5$(xiii)$, we get

$$(x \to y) \to \underbrace{((x \to y) \to y) \to x}_{A} \ll \underbrace{((((x \to y) \to y) \to x) \to y)}_{B} \to \underbrace{(((x \to y) \to y) \to x)}_{C}.$$

Therefore, by Proposition 2.11, $B \to C \subseteq D$. Indeed, we have

$$B \to C = (\underbrace{(((x \to y) \to y) \to x)}_{X} \to y) \to \underbrace{(((x \to y) \to y) \to x)}_{X} \subseteq D.$$

Since D is a $\mathcal{P}IDS$, then by Theorem 4.5, $X = ((x \to y) \to y) \to x \subseteq D$. Thus D is a $\mathcal{F}DS$. □

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