NORMAL FAMILIES AND NORMAL FUNCTIONS

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I will talk about a book that we wrote together with Professor S.G. Krantz. This is new unpublished joint work.

The idea of normal family of holomorphic functions is usually associated with the name Montel. Although Montel was the first to use the phrase "normal family", the concept was in use well before Montel defined it. The fact is that Montel's work was preceded by related ideas of Weierstrass, Stieltjes, Osgood and others.

The idea of compactness had emerged as a fundamental concept in analysis during the nineteenth century: provided a set is bounded in $\mathbf{R}^n$, it is possible to define for any sequence $\{x_j\}$ of points of the set a subsequence $\{x_{j_k}\}$ which converges to a point of $\mathbf{R}^n$ (the Bolzano-Weierstrass theorem).

Riemann had sought to extend this extremely useful property to sets $\mathcal{E}$ of functions of real variables, but it soon appeared that boundedness of $\mathcal{E}$ was not sufficient.

Around 1880 Ascoli introduced the additional condition of equicontinuity of $\mathcal{E}$, which implies that $\mathcal{E}$ has again the Bolzano-Weierstrass property.

At the beginning of the twentieth century, Ascoli's theorem had very few applications, and it was Montel who made it popular by showing how useful it could be for analytic functions of a complex variable.

His fundamental concept is what he called a normal family, which is a set $\mathcal{H}$ of functions defined in a domain $\Omega \subset \mathbb{C}$, taking their values in the Riemann sphere and meromorphic in Ω, and satisfying the following condition: from any sequence of functions of $\mathcal{H}$ it is possible to extract a subsequence that, in every compact subset of Ω, converges uniformly either to a holomorphic function or to the point ∞ of the Riemann sphere.

Most of Montel's mathematical papers are concerned with the theory of analytic functions of one complex variable, a very active field among French mathematicians between 1880 and 1940.

Montel's central observation is that if $\mathcal{H}$ consists of uniformly bounded holomorphic functions in Ω, it is a normal family; this is a consequence of the Cauchy integral formula and of Ascoli's theorem. From this criterion follow many others; for instance, if the values of the functions of a set $\mathcal{H}$ belong to a domain Ω that can be mapped conformally on a bounded domain, then $\mathcal{H}$ is a normal family. This is the case in particular when Ω is the complement of a set of two points in the complex plane C.

Montel showed how the introduction of normal families may bring substantial simplifications in the proofs of many classical results of function theory such as the mapping theorem of Riemann and Hadamard's characterization of entire functions of finite order.

An ingenious application is to the proof of Picard's theorem on essential singularities: suppose 0 is an essential singularity of a function f holomorphic in $\Omega := \{0 < |z| \leq 1\}$. Then Picard's theorem asserts that $f(z)$ takes on all finite complex values, with one possible exception, as z ranges through Ω. This can be proved by observing that if there are two values that f does not take in Ω, then the family of functions $f_n(z) = f\left(z/2^n\right)$ in the ring $\Gamma : \frac{1}{2} \leq |z| \leq 1$ would be a normal family, and there would be either a subsequence (f_{mk}) with $|f_m(z)| \leq M$ in Γ, or a subsequence with $|f_{m_k}(z)| \geq 1/M$ in Γ, contradicting the assumption that 0 is an essential singularity of f.

Montel's works solidified the circle of ideas, and he provided the name "normal family". This led to work of Valiron that was later developed by Lohwater–Pommerenke and Zalcman. Montel also proved the very elegant theorem—characterizing normal families—that every mathematics student learns.

Today the theory of normal families has blossomed into a subject area all its own. Normal families are a powerful tool in function theory, differential equations, geometry, and many other parts of mathematics.

For 65 years after the publication of Montel's treatise on the subject, *Leçons sur les Familles Normales de Fonctions Analytiques et leur Applications*, Recueillies et rédigées par J. Barbotte, Gauthier-Villars, Paris, 1927 (reissued in 1974 by Chelsea, New York), which appeared 20 years after the subject began., no book devoted to normal families appeared. And only in 1993, two books on normal families of one complex variable have appeared—the book by Joel L. Schiff [6] and one by Chi Tai Chuang [1]. In 2017 there appeared the book of Steinmetz [8]. In Chapter 4 the author gives the alternative proof of the famous Zalcman Rescaling Lemma. In the theory of normal families, Steinmetz played an important role by making use of the rescaling method, treated in detail in [8]. See also the Chapter 4 of the recent book of Schiff [7] where an overview of the theory of normal families in one complex variables is given.

In Chapter 9 of his treatise Montel deals with the normal families of holomorphic functions of two complex variables. Since the publication of this book, many interesting results have been obtained in the theory of normal holomorphic functions (maps) and normal families of several complex variables but there had not appeared a single book on the subject. Therefore a book like the one by Krantz and this author has been long awaited.

We do not try to cover all developments in normal families since Montel's book, but concentrate mainly only on our own results. This manuscript contains a number of our results concerning the theory of normal families and normal functions of several complex variables. We have somewhat revised and supplemented them with some of our later investigations.

In our manuscript Professor S.G. Krantz gives a new proof of the Riemann mapping theorem. By the way, a proofs of the Big Picard Theorem and the Schottky Theorem (with the evaluation of universal constant) are given not only without tears, but with a smile.

The conventional wisdom is that all the theorems on the theory of normal families for holomorphic functions of one variable may be transferred without essential changes to the case of functions of several complex variables. Actually, it's not exactly like that! Sometimes a great deal of initial work is needed.

Definition 0.1. A family $\mathbf{F}$ of meromorphic functions from a domain $\Omega \subset \mathbb{C}^n$ to the Riemann sphere $\widehat{\mathbb{C}}$ is normal in Ω if every sequence of functions $\{f_j\} \subseteq \mathbf{F}$ contains a subsequence which converges uniformly on each compact subset of Ω.

This elegant definition of normal family is from Cima and Krantz 1983 article [2, Definition 1.2, p. 306]. We think not of holomorphic functions but of meromorphic functions. Given that, one thinks of the functions as taking values in the Riemann sphere. If we do it that way then we don't have to talk about the concept of compactly divergent anymore. We can still say that a family is normal if there is a subsequence that converges uniformly on compact sets. Because, since the range is now the Riemann sphere, convergence to ∞ is a possibility.

Now let $\mathcal{C}^2(\{z\})$ denote the functions which are defined and twice continuously differentiable on a neighbourhood of the point z in $\mathbb{C}^n$. We often call $\mathcal{C}^2(\{z\})$ the space of *germs* at z. For $\varphi \in \mathcal{C}^2(\{z\})$ we define the Levi form of φ at z, $L_z(\varphi, v)$, by

$$L_z(\varphi, v) := \sum_{k,l=1}^{n} \frac{\partial^2 \varphi}{\partial z_k \partial \overline{z}_l} v_k \overline{v}_l \qquad (v \in \mathbf{C}^n),$$

where $(z; v) = (z; (v_1, \ldots, v_n)) \in T_z(\Omega)$.

For a holomorphic function f on Ω, set

$$f^\sharp(z) := \sup_{|v|=1} \sqrt{L_z(\log(1+|f|^2), v)}. \tag{0.1}$$

It should be noted that the quantity $f^\sharp(z)$ is well defined since the Levi form $L_z(\log(1+|f|^2), v)$ is nonnegative for all $z \in \Omega$.

In particular, for $n = 1$ the formula (0.1) takes the form

$$f^\sharp(z) := \frac{|f'(z)|}{1+|f(z)|^2}$$

and $z^\sharp$ coincides with the spherical metric on $\widehat{\mathbb{C}}$.

In keeping with the usual practice in mathematics, we have attributed results to the author (s) who first proved them in one complex variable. But the form in which the results are presented in the manuscript is different from the original ones; at least all one-dimensional results are valid for meromorphic functions. The reader is advised to consult the relevant sources.

Theorem 0.2 (Marty)**.** *A family* $\mathbf{F}$ *of functions holomorphic on* Ω *is normal on* $\Omega \subset \mathbf{C}^n$ *if and only if for each compact subset* $K \subset \Omega$ *there exists a constant* $M(K)$ *such that at each point* $z \in K$

$$f^\sharp(z) \le M(K) \tag{0.2}$$

for all $f \in \mathbf{F}$.

Marty's Normality Criterion has a host of applications. Here we generalize the famous Zalcman's rescalling lemma to the theory of holomorphic functions in $\mathbf{C}^n$. The provided proof is fairly short and elementary; it uses only Marty's normality criterion.

Theorem 0.3. *A family* $\mathbf{F}$ *of functions holomorphic on* $\Omega \subset \mathbb{C}^n$ *is not normal at some point* $z_0 \in \Omega$ *if and only if there exist sequences*

$f_j \in \mathbf{F}$, $z_j \to z_0$, $\rho_j = 1/f_j^\sharp(z_j) \to 0$, such that the sequence

$$g_j(z) := f_j(z_j + \rho_j z) \tag{0.3}$$

converges locally uniformly in $\mathbb{C}^n$ to a non-constant entire function g satisfying $g^\sharp(z) \le g^\sharp(0) = 1$.

As Royden [5] pointed out, although Marty's result is necessary and sufficient for the relative compactness of a family of holomorphic or meromorphic functions, it may not be easy to apply in certain instances. For example, it is not obvious how Theorem 0.2 can be applied to establish normality of the family

$$\mathbf{F} = \left\{ f \in \mathcal{O}(\Omega) : (1 + |f(z)|^2) f^\sharp(z) \le e^{|f(z)|} \right\}.$$

Zalcman's rescalling lemma can be used to prove the following strengthening of the sufficiency part of Marty's normality criterion:

Theorem 0.4 (Royden, Schwick)**.** *Let $\mathbf{F}$ be a family of holomorphic functions on a domain $\Omega \subset \mathbb{C}^n$ with the property that for each compact set $K \subset \Omega$ there is a function $h_K : (0, \infty) \to (0, \infty)$, which is bounded in some neighborhood of each $x_0 \in (0, \infty)$, such that*

$$(1 + |f(z)|^2) f^\sharp(z) \le h_K(|f(z)|)$$

for all $f \in \mathbf{F}$ and $z \in K$. Then $\mathbf{F}$ is normal in Ω.

And, in the past fifty years, new ideas have grown out of Montel's concept. The idea of normal functions is one of the best and most fruitful of these. As we show in this text, normal functions (a natural generalization of bounded holomorphic functions) are the natural context in which to study normal families.

The genesis of the idea of normal function began with a paper of Kosaku Yosida in 1934. In 1939 Noshiro introduce the notion of normal function (although the notion of normal family existed at this time he did not use that name; he says, after Yosida, that such function belongs to class (A)). This name was given to these functions much later in another pioneering paper in this vein: the 1957 paper of Lehto and Virtanen. In that article Lehto and Virtanen showed that the notion of a normal meromorphic function is closely related to some of the most important problems of the boundary behavior of meromorphic functions. The 1957 paper of Lehto and Virtanen really formalized the concept of normal function and is the standard reference today.

Theorem 0.5 (Noshiro, Lehto-Virtanen)**.** *A non-constant function $f(z)$ is a normal function in the unit disc $D = \{z \in \mathbf{C}\colon\ |z| < 1\}$ if and only if*

$$\frac{|f'(z)|}{1+|f(z)|^2} \leq C\frac{1}{1-|z|^2} \tag{0.4}$$

is satisfied at every point of D where $\mathbf{C}$ is a fixed finite constant.

We see at once that inequality (0.4) can be rewritten in the following form

$$L_z(\log 1 + |f|^2, 1) \leq C ds^2_\Delta(z, 1).$$

One can give the following:

Definition 0.6. Let Ω be a homogeneous bounded domain in $\mathbb{C}^n$ and let N be a complex manifold. We say that a holomorphic mapping $f : \Omega \to N$ is *normal* if the family

$$\mathbf{F} = \{f \circ g : g \in \ \mathrm{Aut}(\Omega)\}$$

is normal.

Repeating the proof of Theorem 0.5, one can obtain at once the following result.

Proposition 0.7. *Let F_K^D be the Kobayashi metric of a homogeneous bounded domain D in $\mathbb{C}^n$ and $\left(N, ds_N^2\right)$ be a compact Hermitian manifold. If a holomorphic mapping $f : D \to N$ satisfies*

$$f^* ds_N^2 \leq C \cdot (F_K^D)^2$$

for a finite constant $\mathbf{C}$*, then f is a normal holomorphic mapping.*

The point here is that a generic domain in $\mathbb{C}^n$ has no biholomorphic self-maps except the identity. So a different approach is needed to develop a theory of normal functions in this more general context.

In view of Proposition 0.7 the definition of normal mappings is extended in the obvious way to arbitrary complex manifolds.

Definition 0.8. Let M be a hyperbolic complex manifolds and (N, ds_N^2) be a compact Hermitian complex manifold. Let F_M be the Kobayashi metrics on M. A holomorphic map $f\colon M \to N$ is called *normal* if there exists a positive constant C such that

$$ds_N^2 f(z, v) \leq C(F_M(z, v))^2 \tag{0.5}$$

for all $(z, v) \in T_z(M)$.

What we try to do in our book is to put the study of normal families and normal functions into a natural, geometric context. Inspired by ideas of H. H. Wu, we treat normal families and functions on complex manifolds. We describe the relationship of these ideas with invariant metrics.

Pioneering work on the Lindelöf principle in several complex variables was done by Cirka [3]. In the book we also introduce new techniques that address the shortcomings of Cima and Krantz article [2] and produce a sharp version of the Lindelöf principle. In common with [2], we shall be able to prove our result not only for bounded holomorphic functions but also for normal functions.

The most important work related to normal functions was published by MacLane [4], who considered the general question of the asymptotic values of holomorphic functions. It should be noted here that there is a natural division in the study of normal functions into normal meromorphic functions (which may not have asymptotic values) and normal holomorphic functions; the study of the latter is more fruitful, since a holomorphic function always has at least one asymptotic value. Using Lindelöf's theorem in $\mathbb{C}^n$, we give example showing that some of these results do not hold in the multidimensional case. Therefore the conventional wisdom that all theorems on the theory of normal holomorphic functions of one variable may be transferred without essential changes to the case of functions of several complex variables is not true.

We hope that our book will inspire other mathematicians to take up the gauntlet of normal families.

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