Telegraph Processes and Option Pricing

Nikita Ratanov • Alexander D. Kolesnik

Telegraph Processes and Option Pricing

Second Edition

Springer

Nikita Ratanov
Chelyabinsk State University
Chelyabinsk, Russia

Alexander D. Kolesnik
Institute of Mathematics and Computer Science
Chisinau, Moldova

ISBN 978-3-662-65826-0 ISBN 978-3-662-65827-7 (eBook)
https://doi.org/10.1007/978-3-662-65827-7

This Springer imprint is published by the registered company Springer-Verlag GmbH, DE, part of Springer Nature.
The registered company address is: Heidelberger Platz 3, 14197 Berlin, Germany

Preface to the Second Edition

This monograph is a revised and greatly expanded version of our previous book [128], *Telegraph Processes and Option Pricing*, Springer, 2013.

While the general structure of the book is kept, the changes are fairly substantial and related mostly to new results obtained in recent years.

Chapter 2 is supplemented by a section concerning the probability distribution function of the Goldstein-Kac telegraph process, which is expressed in term of Gauss hypergeometric functions.

The rest of the book has been heavily rewritten. Chapter 3, on asymmetric jump-telegraph processes, has been greatly expanded. A number of sections have been added on telegraph processes with alternating deterministic and random jumps. In addition, this chapter includes rather new results related to piecewise linear processes with arbitrary sequences of velocities, jump amplitudes and switching intensities. A very detailed new Chap. 4 on jump-diffusion processes with regime switching is presented. With financial applications in mind, we study there, among other things, martingales and Girsanov's transformations, and introduce the concept of entropy.

Chapter 5, devoted to the functionals of the telegraph process, is considerably expanded by adding the sections related to exponential functionals, telegraphic meanders and running extrema, first passage times for the telegraph processes with alternating random jumps, as well as the distribution of the Euclidean distance between two independent telegraph processes.

The new Chap. 6, related to the multidimensional counterparts of the telegraph processes, is included into the monograph. The method of integral transforms for studying the finite-velocity stochastic motions in the Euclidean space $\mathbb{R}^m$ of arbitrary dimension $m \geq 2$ is presented. Application of this method to the stochastic motions in the spaces $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^4$, $\mathbb{R}^6$ enables to obtain their basic characteristics. Moreover, for the stochastic motions in the even-dimensional spaces $\mathbb{R}^2$, $\mathbb{R}^4$, $\mathbb{R}^6$ one managed to derive closed-form expressions for their distributions, while in the space $\mathbb{R}^3$ the distribution is obtained only in an asymptotic form on small time intervals.

Chapter 7 is devoted to a more or less complete presentation of financial models based on various versions of the telegraph processes. Since the publication of paper Ratanov [182] and book Kolesnik and Ratanov [128], many works have appeared on the topic, including several monographs, see [36, 169, 220]. However, some of the results presented in recent works do not seem to us completely justified and raise doubts. One of our goals is to correct these misinterpretations and to formulate an accurate understanding of such models.

The market model, built on the telegraph process with constant jump amplitudes, has already become standard. It is worth noting that this model continues to evolve in various directions. First, such an approach to modelling the financial market can rely on short-memory processes, when the future development of the market depends on the time spent by the process in the current state. Second, the standard model can be modified by adding a new source of stochasticity. For instance, we have included in the second edition a model based on a double stochastic telegraph process with a double jump component, where the switching rates depend on external (environmental) factors. This model is followed by a model with Poisson-modulated switching intensities. The striking effect is that in all cases, for certain combinations of coefficients, the market model is complete.

Detailed diffusion-telegraph market models are also presented, Sect. 7.12. This model is incomplete, and the Esscher transform is typically used for option pricing. We introduce the concept of relative entropy and correct the common mistake that the Esscher transform always provides the minimum relative entropy.

Despite the fact that the book has been significantly expanded, many aspects have not been included in it. The main loss is the rapidly developing fractional approach to telegraph processes with wide applications, see, among others, e.g. [32, 138, 162, 163]. Hopefully, a detailed presentation of these issues awaits us in the near future.

Kishinev, Moldova
Chelyabinsk, Russia
November 2021

Alexander D. Kolesnik
Nikita Ratanov

Preface to the First Edition

This book gives an introduction to the contemporary mathematical theory of non-interacting particles moving at finite velocity in one dimension with alternating directions, so-called the telegraph (or telegrapher's) stochastic processes. The main objective is to give the basic properties of the one-dimensional telegraph processes and to present their applications to option pricing. The book contains both the well known results and the most recent achievements in this field.

The model of a mass-less particle that moves at infinite speed on the real line and alternates at random two possible directions of motion infinitely many times per unit of time is of great interest to physicists and mathematicians beginning with the classical works of A. Einstein (1905) [66] and M. Smoluchowski (1906) [209]. First, A. Einstein determined the transition density of such a kind of motion as the fundamental solution to the heat equation. Then, M. Smoluchowski described this as a limit of random walks. This interpretation is used by physicists as an instrument for mathematical modelling the physical processes of mass and heat transfer. Later this stochastic process, called afterward the *Brownian motion*, was applied to explain the motion first observed by the botanist Robert Brown in 1828.

It is curious that in 1900, i.e. five years before Einstein, Louis Bachelier proposed and analysed the model of financial contracts based on what is now called "Brownian motion" [7] (see also [8]). A. Einstein was completely unaware of the work of R. Brown, as well as of the work of L. Bachelier. Nevertheless, Einstein's paper has had incredible influence on the science of the XXth century, but the unusual and outstanding work of L. Bachelier has been lost from scientific interchange and it was only rediscovered in 1964. In the textbook of W. Feller [75] Brownian motion is named as the process of Wiener-Bachelier.

The crucial point in studying the Brownian motion was a work by N. Wiener (1923) [219] in which he was able to introduce a Gaussian measure in the space of continuous functions. Thus, he had given the opportunity of rigorous axiomatic constructing an extremely important stochastic process that was afterward called the Wiener process. After the appearance of the Einstein-Smoluchowski's model governed by the heat equation, Brownian motion has been extensively used to describe various real phenomena in statistical physics, optics, biology, hydrodynamics, financial markets and other fields of science and technology. It was discovered that the theoretical calculations based on this model agree well with experimental data if the speed of the process is sufficiently big. If the speed is small, this agreement becomes worse. This fact, however, is not too surprising if we take into account the infinite-velocity nature of the Wiener process.

That is why many attempts were made to suggest alternative models in which the finiteness of both the speed of motion and the intensity of changes of directions per unit of time, could be assumed. Such a model was first introduced in 1922 by G. I. Taylor [213] in describing the turbulent diffusion, (see also the discussion between Prof. Karl Pearson and Lord Rayleigh in 1905, [165, 166], [202]). In 1926 V. Fock [76] suggested the use of a hyperbolic partial differential equation (called the *telegraph*, or *damped wave equation*) to describe the process of the diffusion of a light ray passing through a homogeneous medium. Later the time-grid approach was developed at length by S. Goldstein [85]. This naturally led to the telegraph equation describing the spatio-temporal dynamics of the potential in a transmitting cable, (without leakage) [217]. In his 1956 lecture notes, M. Kac, (see [106]), considered a continuous-time version of the telegraph model. Since then, the telegraph process and its various generalisations have been studied in great detail with numerous applications in physics, and, more recently, in financial market modelling. The telegraph process is the simplest example of the so-called *random evolution* (see, e.g., [74, Ch. 12] and [168, Ch. 2]).

An efficient conventional approach to the analytical study of the telegraph process, similar to that for diffusion processes, is based on pursuing a fundamental link relating various expected values of the process with initial value and/or boundary value problems for certain partial differential equations. One should note that the telegraph equation first has appeared more than 150 years ago in a work by W. Thomson (Lord Kelvin) in an attempt to describe the propagation of electric signals on the transatlantic cable [214]. At present, it is one of the classical equations of mathematical physics.

The main objective of the book is to give a modern systematic treatment of the telegraph stochastic processes theory with an accent on the financial markets applications. These applications are rather new in the literature, but we believe that our approach is quite natural if we take into account the finite-velocity market motions joint with abrupt jumps (deterministic or of random values) that naturally produce heavy tails in such models.

In the book we develop an unified approach based on integral and differential equations. This approach might seem somewhat unusual for the specialists who mostly use the stochastic calculus methods in their research. We, however, believe that our approach is quite natural and could be exploited as a fruitful addition to the classical methodology.

The book consists of five chapters and is organised as follows.

In Chap. 1, for the reader's convenience and in order to make the book more self-contained, we recall some mathematical preliminaries needed for further analysis.

Chapter 2 deals with the general definition and basic properties of a two-state telegraph process on the real line performed by a stochastic motion at finite speed driven by a homogeneous Poisson process. We derive the finite-velocity counterparts of the classical Kolmogorov equations for the joint transition densities of the process and its direction representing a hyperbolic system of two first-order partial differential equations with constant coefficients. Basing on this system, we derive a second-order telegraph equation for the transition density of the process. The explicit formulae are obtained for the transition density of the process and its characteristic function as the solutions of respective Cauchy problems. It is also shown that, under the standard Kac's condition, the transition density of the telegraph process tends to the transition density of the one-dimensional Brownian motion. The formulae for the Laplace transforms of the transition density and of the characteristic function of the telegraph process are also obtained.

In Chap. 3 we consider some important functionals of telegraph processes. We describe the distributions of the telegraph process in the presence of absorbing and reflecting barriers. First passage times and spending times of the telegraph processes are considered also. This presentation corrects some stable inaccuracies in the field.

The applications of telegraph processes to financial modelling presented in Chap. 5, require studying the asymmetric telegraph processes. Moreover, it is crucial to add jumps to the asymmetric telegraph process. In Chap. 4 we introduce the reader to this new situation.

Chapter 5 is devoted to some contemporary applications of the telegraph processes to financial modelling. We modify the classical Black-Scholes market model exploiting a telegraph process instead of Brownian motion. As is easy to see, the simple substitution of a telegraph process instead of Brownian motion in the framework of Black-Scholes-Merton model leads to arbitrage opportunities. To get an arbitrage-free model we add a jump component to the telegraph process.

The huge literature on the mathematical modelling of financial markets began from two fundamental papers of F. Black and M. Scholes [23] and of R. C. Merton [154] in 1973. In this classical model the price of risky asset is assumed to follow a geometric Brownian motion. This assumption permits one to obtain nice closed formulae for option prices and hedging strategies.

Nevertheless, the famous Black-Scholes formula has well known shortages. It is commonly accepted that Black-Scholes pricing formula distorts some option prices. Typically, it substantially underprices deep-in-the-money and out-of-the-money options and overprices at-the-money options, but downward (or upward) slopes are possible. To accord the Black-Scholes formula with market prices of standard European options different volatilities for different strikes and maturities are used. This trick is referred to as the *volatility smile*. In the "typical" comportment the implied volatility of deeply in-the-money and out-of-the-money options is higher than at-the-money options. Modern fearful markets are afraid of large downward movements and crashes. A smile pattern of these markets more resembles a "skew", where implied volatility increases with shortening the maturity time.

These observations provoke a growing interest in the construction of more and more complicated extensions of the Black-Scholes model. Stochastic volatility models are based on the stochastic dynamics of the Black-Scholes implied volatility. Various patterns of smiles and skews can be constructed depending on the correlation and the parameters of the volatility process. These models have some empirically approved evidences of their realistic and unrealistic features, but we believe that such an approach proposes the quantity sophistication instead of fundamental explanation of problem.

Another approach, that adds a pure jump process to Black-Scholes diffusion, can capture many volatility smiles and skews. This idea of jump-diffusion model have been proposed for better adequacy by Merton [155], and nowadays is applied to handle option pricing, especially when options are close to maturity. Similarly to stochastic volatility models, jump-diffusion models increase Markov dimension of the market and form incomplete market models.

We suggest here a new model to explain market's movements. Suppose that the log-returns are driven by a telegraph process, i.e. they move with pair of constant velocities alternating one to another at Poisson times. To make the model more adequate and to avoid arbitrage opportunities the log-return movement should be supplied with jumps occurring at times of the tendency switchings.

Such a model looks attractive due to finite propagation velocity and the intuitively clear comportment. The jump-telegraph model captures bullish and bearish trends using velocity values, and it describes crashes and spikes by means of jump values. This model describes adequately the processes on oversold and overbought markets, when changes on the market tendencies accumulate in course of time.

At the same time, the model is analytically tractable. It allows us to get solutions for hedging and investment problems in closed form. Jumps are used in the model to avoid arbitrage opportunities, but not solely for adequacy.

The model based on the telegraph processes with jumps of deterministic values is complete as well as in the classical Cox-Ross-Rubinstein and Black-Scholes cases. It is attractive mathematically and allows us to freely modify the model to meet the needs of applications.

Under respective rescaling, the jump-telegraph model converges to Black-Scholes model. It permits us to define naturally a volatility of the jump-telegraph model depending on the velocities and jump values as well as on the switching intensities. The model based on jump-telegraph processes is characterised by volatility smiles of various shapes including frowns and skews depending on the parameters' values.

We hope that this book will be interesting to the specialists in the area of diffusion processes with finite speed of propagation and in financial modelling. We expect that the book will also be useful for the students and postgraduates who make their first steps in these intriguing and attractive fields.

Finally, we would like to thank the staff of Springer, Editorial Statistics Europe, for the invitation to write this book.

Kishinev, Moldova — Alexander D. Kolesnik
Bogota, Colombia — Nikita Ratanov
March 2013

Contents

About the Authors

Alexander Kolesnik is a professor, former head of laboratory (2015–2019) and principal researcher (since 2020) at the Institute of Mathematics and Computer Science "Vladimir Andrunachievici", Kishinev (Chişinău), Moldova. He graduated from Moldova State University (1975–1980) and pursued his postgraduate studies (1987–1991) in the Institute of Mathematics of the National Academy of Sciences of Ukraine. He earned his PhD (1991) and PhD habilitation (2010) degrees in mathematics and physics with a specialization in stochastic processes, probability and statistics, conferred by the Institute of Mathematics of the National Academy of Sciences of Ukraine. His research interests include probability and statistics, stochastic processes, random evolutions, stochastic dynamic systems, random flights, diffusion processes, transport processes, random walks, stochastic processes in random environments, partial differential equations in stochastic models, statistical physics, and wave processes. Dr Kolesnik has authored more than 70 scientific publications, predominantly in leading international journals, and 2 monographs. He has also served as an external reviewer for many outstanding international journals in mathematics and physics, garnering him, for example, a "Certificate of Outstanding Contribution in Reviewing" from the journal Stochastic Processes and their Applications. He has been a visiting professor and scholarship holder at several universities in Italy and Germany and served on the Board of Global Advisors of the International Federation of Nonlinear Analysts (IFNA), USA.

Nikita Ratanov graduated from Moscow State University (Lomonosov) in 1976. He subsequently earned his PhD (1984) at the same institution, followed by a doctorate in PDE and stochastics (1999).

He has published over 100 papers, 2 textbooks (in Russian and Spanish) and a monograph in these areas. For nearly the past two decades, he has been a professor at Universidad del Rosario (Bogotá, Colombia).

He is currently affiliated with Chelyabinsk State University (Russia) as a leading research fellow.

Dr Ratanov has a wide range of research interests, including stochastic analysis with applications in statistical physics, financial market modelling and some aspects of neural modelling.