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Spectra of semimodules

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Abstract. The purpose of this paper is to investigate possible structures and useful properties of prime subsemimodules of a semimodule M over a semiring R and show various applications of the properties. The main part of this work is to introduce a new class of semimodules over R called strong primeful R-semimodules. It is shown that every non-zero strong primeful semimodule possesses the non-empty prime spectrum with the surjective natural map. Also, it is proved that this class contains the family of finitely generated R-semimodules properly.

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1 Introduction

Semimodules over semirings also appear naturally in many areas of mathematics. For example, semimodules are useful in the area of theoretical computer science as well as in the solution of problems in the graph theory and cryptography [13, 18]. This paper generalizes some well know results on prime submodules in commutative rings to commutative semirings. The main difficulty is figuring out what additional hypotheses the ideal or subsemimodule must satisfy to get similar results. The two new key notions are that of a "strong ideal" and a "strong subsemimodule". Moreover, quotient semimodules are determined by equivalence relations rather than by subsemimodules as in the module case. Allen [1] has presented the notion of a partitioning ideal (= Q-ideal) I in the semiring R and constructed the quotient semiring R/I. Quotient semimodules over a semiring R have already been introduced and studied by present authors in [10]. Chaudhari and Bonde extended the definition of Q_M-subsemimodule of a semimodule and some results given in Section 2 in [10] to a more general quotient semimodules case in [3]. Of course "quotient semimodule" is a natural extension of "quotient semiring" and, hence, ought to be in the literature. So quotient semimodules are particularly important in the study of the representation theory of semimodules over semiring. The representation theory of semimodules over semirings has developed greatly in the recent years. One of the aims of the modern representation theory of semimodules is to generalize the properties of modules over rings to semimodules over semirings. The aim of present paper is to extend some basic results of C. P. Lu [15, 16, 17] to semimodules over semirings. We know (at least as far as we are aware) of no systematic study of the topological space Spec(M) in the semimodule over semiring context. Our results is particularly important in the topological space Spec(M) equipped with a topology called the Zariski topology in the semimodule context and, we hope to address in a later paper.

2 Preliminaries

In order to make this paper easier to follow, we recall in this section various notions from semimodule theory which will be used in the sequel. For the definitions of monoid, semirings, semimodules and subsemimodules we refer to [4, 9, 10, 13, 14]. All semiring in this paper are commutative with non-zero identity.

Definition 1. (a) A semiring R is said to be semidomain whenever $a, b \in R$ with $ab = 0$ implies that either $a = 0$ or $b = 0$.

(b) A semifield is a semiring in which non-zero elements form a group under multiplication.

(c) An R-semimodule M is said to be semivector space if R is a semifield.

(d) Let M be a semimodule over a semiring R. A subtractive subsemimodule (= k-subsemimodule) N is a subsemimodule of M such that if $x, x + y \in N$, then $y \in N$ (so $\{0_M\}$ is a k-subsemimodule of M).

(e) A prime subsemimodule (resp. primary subsemimodule) of M is a proper subsemimodule N of M in which $x \in N$ or $rM \subseteq N$ (resp. $x \in N$ or $r^n M \subseteq N$ for some positive integer n) whenever $rx \in N$. The collection of all prime (resp. maximal) subsemimodules of M is called the spectrum (resp. the maximal spectrum) of M and denoted by Spec(M) (resp. Max(M)). Similarly, the collection of all P-prime subsemimodules of M for any prime k-ideal P of R is designated by $\mathrm{Spec}_P(M)$. We define k-ideals and prime ideals of a semiring R in a similar fashion.

(f) We say that $r \in R$ is a zero-divisor for a semimodule M if $rm = 0$ for some non-zero element m of M. The set of zero-divisors of M is written $Z_R(M)$.

(g) An R-semimodule M is called multiplication semimodule provided that for every subsemimodule N of M there exists an ideal I of R such that $N = IM$.

(h) We say that M is a torsion-free R-semimodule whenever $r \in R$ and $m \in M$ with $rm = 0$ implies that either $m = 0$ or $r = 0$ (so every semivector space over a semifield R is a torsion-free R-semimodule).

(i) A proper ideal I of a semiring R is said to be strong ideal (or strongly zero-sum ideal) if for each $a \in I$ there exists $b \in I$ such that $a + b = 0$ (see [11, Example 2.3] and [8]).

A subsemimodule N of a semimodule M over a semiring R is called a partitioning subsemimodule (= Q_M-subsemimodule) if there exists a subset Q_M of M such that $M = \cup\{q + N : q \in Q_M\}$ and if $q_1, q_2 \in Q_M$ then $(q_1 + N) \cap (q_2 + N) \neq \emptyset$ if and only if $q_1 = q_2$. Let N be a Q_M-subsemimodule of M and let $M/N = \{q + N : q \in Q_M\}$. Then M/N forms an R-semimodule under the operations $\oplus$ and $\odot$ defined as follows: $(q_1 + N) \oplus (q_2 + N) = q_3 + N$, where $q_3 \in Q_M$ is the unique element such that $q_1 + q_2 + N \subseteq q_3 + N$ and $r \odot (q_1 + N) = q_4 + I$, where $r \in R$ and $q_4 \in Q_M$ is the unique element such that $rq_1 + N \subseteq q_4 + N$. This R-semimodule M/N is called the quotient semimodule of M by N [3]. By [3, Lemma 2.3], there exists a unique element $q_0 \in Q_M$ such that $q_0 + N = N$. Thus $q_0 + N$ is the zero element of M/N. Also, [3, Theorem 2.4] show that the structure $(M/N, \oplus, \odot)$ is essentially independent of Q_M (see [3, Example 2.6]).

3 Spec(M)

In this section we extend some results of C. P. Lu [15] to semimodules over semirings.

Remark 1. (Change of semirings.) Assume that I is a Q-ideal of a semiring R and let N be a Q_M-subsemimodule of an R-semimodule M. We show now how M/N can be given a natural structure as a semimodule over R/I. Let $q_1, q_2 \in Q$ be such that $q_1 + I = q_2 + I$, and let $m_1, m_2 \in Q_M$ be such that $m_1 + N = m_2 + N$. Then $q_1m_1 + N = q_2m_2 + N$. By assumption, there exist the unique elements $t_1, t_2 \in Q_M$ such that $q_1m_1 + N \subseteq t_1 + N$ and $q_2m_2 + N \subseteq t_2 + N$; so $t_1 = t_2$. Hence we can unambiguously define a mapping $R/I \times M/N$ into M/N (sending $(q + I, m + N)$ to $t + N$), where $qm + N \subseteq t + N$ for some unique element $t \in Q_M$, and it is routine to check that this turns the commutative additive semigroup with a zero element M/N into an R/I-semimodule.

Definition 2. A proper subsemimodule N of a semimodule M over a semiring R is said to be strong subsemimodule if for each $x \in N$ there exists $y \in N$ such that $x + y = 0$.

Example 1. Let that E_0^+ be the set of all non-negative integers. The monoid $M = (Z_6, +_6)$ is a semimodule over $(E_0^+, +, .)$ (see [13, p. 151]). An inspection will show that $N = \{\bar{0}, \bar{2}, \bar{4}\}$ is a strong Q_M-subsemimodule of M, where $Q_M = \{\bar{0}, \bar{1}\}$.

Lemma 1. *Let N be a strong Q_M-subsemimodule of a module M over a semiring R. Then the following hold:*

(i) If $q_0 \in Q_M$ and $q_0 + N$ is the zero in M/N, then $q_0 \in N$.

(ii) If $q \in N \cap Q_M$ and $q_0 + N$ is the zero in M/N, then $q = q_0$.

(iii) If $q_0 + N$ is the zero in M/N, then $m \in N$ if and only if $m + N = \{m + a : a \in N\}$ and $N + m = N$ are equal as sets.

Proof. (i) By [3, Lemma 2.3], $q_0 + N = N$; hence $q_0 \in N$ since every Q_M-subsemimodule is a k-subsemimodule of M by [3, Theorem 3.2].

(ii) Since $q + q_0 \in (q + N) \cap (q_0 + N)$, we must have $q = q_0$. (iii) follows from (i) and (ii).

(iii) Let $m \in N$. Since the inclusion $m + N \subseteq N$ is clear, we will prove the reverse inclusion. Assume that $x \in N$. There exist $a, b, b' \in N$ such that $x = q_0 + a$, $m = q_0 + b$ and $b + b' = 0$; so $x = m + a + b' \in m + N$, and so we have equality. The other implication is obvious. $\square$

Theorem 1. *Let N be a proper strong Q_M-subsemimodule of a semimodule M over a semiring R with $(N : M) = P$ a Q-ideal of R. Then the following statements are equivalent:*

(i) N is a prime subsemimodule of M;
(ii) M/N is a torsion-free R/P-semimodule;
(iii) $(N :_M < r >) = N$ for every $r \in R - P$;
(iv) $(N :_M J) = N$ for every ideal $J \nsubseteq P$;
(v) $(N :_R < m >) = P$ for every $m \in M - N$;
(vi) $(N :_R L) = P$ for every subsemimodule L of M properly containing N;
(vii) $Z_R(M/N) = P$.

Proof. $(i) \Rightarrow (ii)$ Note that M/N is an R/P-semimodule by Remark 1. Assume that q_0 is the unique element in Q_M such that $q_0 + N$ is the zero in M/N and let $(q + P)(m + N) = q_0 + N$, where $qm + N \subseteq q_0 + N$ for some $q \in Q$ and $m \in Q_M$, so $qm \in N$ since N is a k-subsemimodule of M. Therefore, N prime gives either $q \in P$ or $m \in N$. If $q \in P$, then $q + P$ is the zero in R/P by [6, Lemma 2.3]. If $m \in N$, then $m + N$ is the zero in M/N by Lemma 1. Thus M/N is torsion-free semimodule as an R/P-semimodule.

$(ii) \Rightarrow (iii)$ Assume that $q_0 + P$ is the zero element in R/P. It suffices to show that $(N :_M < r >) \subseteq N$. Let $m \in (N :_M < r >)$. Then $rm \in N$, $r = q + a$ and $m = t + x$ for some $q \in Q$, $a \in P$, $t \in Q_M$ and $x \in N$ (so $q \notin P$); hence $qt \in N$ since N is a k-subsemimodule. Since $(q + P)(t + N) = q_0 + N$ by Lemma 1 and $q + P \neq q_0 + P$, we must have $t + N = q_0 + N$; hence $t = q_0 \in N$. Therefore, $m = t + x \in N$, and so we have equality.

$(iii) \Rightarrow (iv)$ Clearly, $N \subseteq (N :_M J)$. For the reverse inclusion, assume that $m \in (N :_M J)$. By assumption, there exists $r \in J$ such that $r \in R - P$ and $rm \in N$; so $(N :_M < r >) = N$ by (iii). This completes the proof.

$(iv) \Rightarrow (v)$ Since $PM \subseteq N$, we conclude that $P \subseteq (N :_R < m >)$ for every $m \in M - N$. For the other containment, assume that $m \in M - N$ and $r \in (N :_R < m >)$; we show that $r \in P$. Suppose not. Then $J =< r > \nsubseteq P$, and so $m \in (N :_M J) = N$ by (iv), which is a contradiction, as required.

$(v) \Rightarrow (vi)$ If $a \in P$, then $aL \subseteq aM \subseteq N$; so $P \subseteq (N :_R L)$. Now suppose that $b \in (N :_R L)$. By assumption, there exists $m \in L$ such that $m \in M - N$. Then $b \in (N :_R < m >) = P$ by (v), as needed.

$(vi) \Rightarrow (vii)$ Let $r \in Z_R(M/N)$. Then there exists $t \in Q_M - N$ such that $r(t + N) = q_0 + N$, where $rt + N \subseteq q_0 + N$, so $rt \in N$ since N is a k-subsemimodule; hence $r \in (N :_R Rt + N) = P$ by (vi). Thus $Z_R(M/N) \subseteq P$. For the reverse conclusion, assume that $a \in P$. By assumption, there is an element $m \in M - N$ such that $am \in N$. There exist $s \in Q_M - N$ and $y \in N$ such that $m = s + y$ (so

$s \notin N$) such that $as \in N$; hence $a(s+N) = q_0+N$ by Lemma 1. Thus $a \in Z_R(M/N)$. This completes the proof.

$(vii) \Rightarrow (i)$ Let $rm \in N$ for some $r \in R$ and $m \in M - N$; we show that $r \in P$. By assumption, there are elements $t \in Q_M - N$ and $z \in N$ such that $m = t + z$, so $rt \in N$. Then $r(t + N) = q_0 + N$ by Lemma 1; hence $r \in Z_R(M/N) = P$ by (vii), as required. □

Proposition 1. *Let N be a proper strong Q_M-subsemimodule of a semimodule M over a semiring R with $(N : M) = P$ a maximal Q-ideal of R. Then N is a prime subsemimodule. In particular, $P'M$ is a prime subsemimodule of an R-semimodule M for every maximal Q-ideal P' of R such that $P'M \neq M$.*

Proof. By [4, Theorem 2.10], R/P is a semifield, so M/N is a semivector space over the semifield R/P by Remark 1; hence it is a torsion-free R/P-semimodule. Thus N is prime by Theorem 1. Finally, suppose that $(P'M : M) = J \neq R$. Then $P' \subseteq J$, so $J = P'$ since P' is maximal, as required. □

Theorem 2. *Let N be a proper strong Q_M-subsemimodule of a semimodule M over a semiring R with $(N : M) = P$ a Q-ideal of R and let P be a maximal ideal of R. Then N is P-prime if and only if $PM \subseteq N$. In particular, if N is a P-prime subsemimodule of M, then so is every proper subsemimodule of M containing N.*

Proof. It suffices to show that if $PM \subseteq M$, then N is P-prime. Let $p \in P$. Then $p \in (N : M)$, so $P = (N : M)$ by maximality of P. Now apply Proposition 1. □

Proposition 2. *Let M be a finitely generated semimodule over a semiring R and let I be a strong k-ideal of R such that $I = \mathrm{rad}(I)$. Then $(IM : M) = I$ if and only if $\mathrm{ann}(M) \subseteq I$.*

Proof. The necessity is clear. Assume that $\mathrm{ann}(M) \subseteq I$ and let $x \in (IM : M)$. First we show that if M is generated by n elements, then there exists a $y \in I$ such that $x^n+y \in \mathrm{ann}(M)$. To see that, we use induction on n. Consider first the case in which $n = 1$. Here we have $x < m > \subseteq I < m >$. So $xm = sm$ for some $s \in I$; hence there is an element $s' \in I$ such that $(x+s')m = sm+s'm = 0$. It follows that $(x+s')M = 0$. We now turn to the inductive step. Assume, inductively, that $n = k + 1$, where $k \geq 1$, and that the result has been proved in the case where $n = k$. Then we must have $(x + a)(x^k + b)M = (x^{k+1} + ax^k + bx + ab)(< m_1, ..., m_k > + < m_{k+1} >) = 0$ for some $a, b \in I$, so $(x^{k+1}+c)M = 0$, where $ax^k+bx+ab = c \in J$. Thus $x^n+y \in I$. Since I is a k-ideal, we must have $x^n \in I$ and, therefore, $(IM : M) \subseteq \mathrm{rad}(I) = I$. Now we can see easily that $(IM : M) = I$. □

Theorem 3. *If M is a finitely generated semimodule over a semiring R and P is a strong maximal Q-ideal of R containing $\mathrm{ann}(M)$, then $PM \neq M$ so that PM is a prime subsemimodule of M. In particular, if M is a finitely generated faithful R-semimodule, then PM is a prime subsemimodule of M for every strong maximal Q-ideal P of R.*

Proof. Apply Proposition 1 and Proposition 2 (note that every Q-ideal is a k-ideal). □

4 Spec(M_S)

Assume that S is a multiplicatively closed subset of the commutative semiring R and let M be an R-semimodule. We introduce a useful relationship between Spec(M) and Spec(M_S) (Theorem 6) and exhibit its application through the remaining of the paper.

Lemma 2. *Let R be a semiring. If N is a primary subsemimodule of an R-semimodule M, then $(N : M)$ (or equivalently* $\mathrm{ann}(M/N)$*) is a primary ideal.*

Proof. Since $M \nsubseteq N$, the ideal $(N; M)$ is a proper ideal. Now suppose that $a, b \in R$ such that $ab \in (N : M)$, $b \notin (N : M)$. Since $b \notin (N : M)$, there exists $m \in M$ such that $bm \notin N$. But N is a primary submodule, consequently $a^s M \subseteq N$ for some integer s. This completes the proof. □

If N is a primary subsemimodule of an R-semomodule M, then Lemma 2 shows that $P' = (N : M)$ is a primary ideal. Consequently, $P = \mathrm{rad}(P')$ is a prime ideal. In this case, we shall say that N is P-primary.

Lemma 3. *Let R be a semiring. A primary subsemimodule N of any R-semimodule M is prime if and only if $(N : M)$ is a prime ideal. In particular, if K is a P-primary subsemimodule of M containing a P-prime subsemimodule, then K is prime.*

Proof. The proof is straightforward. □

Definition 3. Let S be a multiplicatively closed subset of the commutative semiring R, and let M be an R-semimodule. M is called a S-cancellative semimodule whenever $am = an$ for some $0 \neq a \in S$ and $m, n \in M$, then $m = n$. A semiring is called a S-cancellative semiring if it is a S-cancellative semimodule over itself.

Example 2. Assume that E_0^+ is the set of all non-negative integers and let $S = E_0^+ - \{0\}$. Then $(E_0^+, +, .)$ is a S-cancellative semiring. Let $M = (E_0^+, \mathrm{gcd})$. Clearly, M is a commutative monoid in which every element is idempotent. Moreover, M is a S-cancellative semimodule over E_0^+ with scalar multiplication defined by $rm = 0$ if $r = 0$ and $rm = m$ if $r > 0$ for all $r \in E_0^+$ and $m \in M$ [13, p. 151].

Let R be a S-cancellative semiring. Define a relation $\sim$ on $R \times S$ as follows: for $(a, s), (b, t) \in R \times S$, we write $(a, s) \sim (b, t)$ if and only if $ad = bc$. Then $\sim$ is an equivalence relation on $R \times S$. For $(a, s) \in R \times S$, denote the equivalence class of $\sim$ which contains (a, s) by a/s, and denote the set of all equivalence classes of $\sim$ by R_S. Then R_S can be given the structure of a commutative semiring under operations for which $a/s + b/t = (ta + sb)/st$, $(a/s)(b/t) = (ab)/st$ for all $a, b \in r$ and $s, t \in S$.

This new semiring R_S is called the semiring of fractions of R with respect to S; its zero element is $0/1$, its multiplicative identity element is $1/1$ and each element of S has a multiplicative inverse in R_S (see [9, 13, 19]). Assume that R is a semidomain and let $S = R - \{0_R\}$. Then R_S is a semifield. The semifield F constructed from the semidomain R is referred to as the semifield of fractions of the semidomain R. Moreover, assume that P is a prime ideal of R. Then $S = R - P$ is a multiplicatively closed subset of R. In this case we set $R_S = R_P$ and $I_S = IR_P$, where I is an ideal of R.

Let M be a S-cancellative semimodule over a S-cancellative semiring R. The relation $\sim'$ on $M \times S$ defined by, for $(m, s), (n, t) \in M \times S$, $(m, s) \sim' (n, t)$ if and only if $tm = sn$ is an equivalence relation on $M \times S$; for $(m, s) \in M \times S$, the equivalence class of $\sim'$ which contains (m, s) is denoted by m/s. Similarly, a simple argument will show that the set M_S of all equivalence classes of $\sim'$ has the structure of a semimodule over the semiring R_S of fractions of R with respect to S under operations for which $m/s + n/t = (tm + sn)/st$, $(r/s)(n/t) = (rn)/st$ for all $m, n \in M$, $s, t \in S$ and $r \in R$. The R_S-semimodule M_S is called the semimodule of fractions of M with respect to S; its zero element is $0_M/1$, and this is equal to $0_M/s$ for all $s \in S$.

Convention. Throughout this section we shall assume unless otherwise stated, that R denotes a commutative S-cancellative semiring with an identity element and S a non-empty multiplicatively closed subset of R. M will designate a fixed S-cancellative semimodule over R. If N is a subsemimodule of M, then N_S will be regarded as an R_S-subsemimodule of M_S.

Proposition 3. *Let N be a P-primary subsemimodule of M. If $P \cap S \neq \emptyset$, then $N_S = M_S$. On the other hand if $P \cap S = \emptyset$, then N_S is a P_S-primary subsemimodule of M_S and $N = N_S \cap M = \{m \in M : m/1 \in N_S\}$.*

Proof. First suppose that there is an element s which is common to P and S. Since $P = \mathrm{rad}(N : M)$, there is an integer n such that $s^n M \subseteq N$. Suppose now that $m/s' \in M_S$. Then $m/s' = (s^n m)/(s^n s') \in N_S$. This shows that $M_S = N_S$ and the first assertion follows. From here on we assume that P does not meet S. Since the inclusion $N \subseteq M \cap N_S$ is clear, we will prove the reverse inclusion. Let $m \in N_S \cap M$. Then there are elements $n \in N$ and $t \in S$ such that $m/1 = n/t$; hence $tm \in N$. Using the facts that N is P-primary in M and $t \notin P$, we conclude that $m \in N$, and so we have equality. This shows, in particular, that N_S is a proper subsemimodule of M_S. Assume next that $(r/s)(m/t) \in N_S$ and $m/t \notin N_S$. Then $m \notin N$. Multiplying $(rm)/(st)$ by $(s^2t^2)/(st)$, we obtain $(rm)/1 = (s^2t^2rm)/(s^2t^2) \in N_S$. Thus $rm \in N$. It now follows that $r^v M \subseteq N$ for some integer v, which in turn implies that $(r/s)^v M_S \subseteq N_S$. This establishes that N_S is a primary subsemimodule of M_S. By [5, Lemma 2.3] it must be P'_S-primary, where P' is a prime ideal of R with $P' \cap S = \emptyset$. Let $a \in P$. Then $a^n M \subseteq N$ for some integer n; hence if $s \in S$, then $((sa)/s)^n M_S \subseteq N_S$. It follows that $(sa)/s \in P'_S$ and therefore $a \in P'_S \cap R = P'$ by

[5, Lemma 2.3] again; thus $P \subseteq P'$. For the other containment, assume that $b \in P'$. If now $t \in S$, then there exists an integer u such that $((tb)/t)^u M_S \subseteq N_S$. Select $m \in M$ so that $m \notin N$. Then $(t^{u-1}b^u m)/t^{u+1} = ((tb)/t)^u (tm)/t \in N_S$ and therefore $t^u m \in N_S \cap M = N$; hence $t \in P$, and so we have equality, as required. □

Let N be a subsemimodule of a semimodule M. An inspection will show that $N_S \cap M = \{m \in M : sm \in N \text{ for some } s \in S\}$. Let P be a prime ideal of R. The saturation $S_P(N) = N_P \cap M$ of N with respect to P is known in the literature as the S-component of N in M for the multiplicatively closed subset $S = R - P$, but designated in various way. A subsemimodule K of M is said to be saturated with respect to P if $S_P(K) = K$.

Theorem 4. *Every P-primary subsemimodule of a semimodule M is a saturated subsemimodule.*

Proof. Apply Proposition 3. □

Theorem 5. *Let P be a prime ideal of R with $P \cap S = \emptyset$, and let M be an R-semimodule. Then there is a one-to-one correspondence between the P-primary subsemimodules N of M and the P_S-primary subsemimodules L of M_S. This is such that, when N and L correspond, $L = N_S$ and $N = L \cap M$.*

Proof. Let L be a P_S-primary subsemimodule of M_S. By Proposition 3, it is enough to show that there is a P-primary subsemimodule N of M such that $N = L \cap M$. Suppose that $L \cap M = N$; we show that $N_S = L$. Since the inclusion $N_S \subseteq L$ is clear, we will prove the reverse inclusion. Let $x \in L$. Then $x = m/s$ for some $m \in M$ and $s \in S$, so $(s^2/s)(m/s) = m/1 \in L$ and therefore $m \in N$. It follows that $m/s \in N_S$. This shows that $L = N_S$ and N is a proper subsemimodule of M. Now assume that $rm \in N$, where $r \in R$, $m \in M$ and $m \notin N$. If s is an arbitrary element of S, then $(rs)/s)((sm)/s^2) = (rs^2m)/s^2 \in N_S = L$. On the other hand, $(sm)/s \notin L$ for the contrary assumption would imply that $m \in N$. So there exists an integer w such that $((rs)/s)^w M_S \subseteq L$ since L is primary. Let $m' \in M$ then $(s^{w+1}r^w m')/s^{w+1} = ((rs)/s)^w (sm')/s \in L$, whence $r^w m' \in N$. As this holds for every $m' \in M$, we may conclude that $r^w M \subseteq N$. This proves that N is a primary subsemimodule of M. Let it be P'-primary. Since $N_S = L$ and $L \neq M_S$, Proposition 3 shows that $P' \cap S \neq \emptyset$. The same proposition shows that $N_S = L$ is P'_S-primary. Thus $P'_S = P_S$ and therefore $P = P'$ by [4, Lemma 2.3]. This completes the proof. □

Lemma 4. *Let M be an R-semimodule. Then the following hold:*

(i) If $N_1, N_2, ..., N_k$ are subsemimodules of M, then $(N_1 + N_2 + ... + N_k)_S = (N_1)_S + (N_2)_S + ... + (N_k)_S$ and $(N_1 \cap N_2 \cap ... \cap N_k)_S = (N_1)_S \cap (N_2)_S \cap ... \cap (N_k)_S$.

(ii) If $m \in M$ and N is a subsemimodule of M, then $(N : m)_S = (N_S : m/1)$.

(iii) If $m_1, m_2, ..., m_n$ are elements which generate M, then the R_S-semimodule generated by $m_1/1, m_2/2, ..., m_n/1$ is just M_S.

(iv) If I is an ideal of R, then $I_S = R_S$ if and only if $I \cap S \neq \emptyset$.

(v) If M is finitely generated and N a subsemimodule of M, then $(N : M)_S = (N_S : M_S)$. In particular, $(\mathrm{ann}(M))_S = \mathrm{ann}(M_S)$.

(vi) If M is finitely generated and N a subsemimodule of M, then $N_S = M_S$ if and only if $(N : M) \cap S \neq \emptyset$.

Proof. The proofs of (i), (ii), (iii) and (iv) are straightforward. To see that (v), let $m_1, m_2, ..., m_k$ be elements which generate M. Then $(N : M) = (N : m_1) \cap ... \cap (N : m_k)$ and therefore, by (i) and (ii), $(N : M)_S = (N : m_1)_S \cap ... \cap (N : m_k)_S = (N_S : R_S m/1 + ... + R_S m_k/1)$. This completes the proof by (iii).

(vi) We have $N_S = M_S$ if and only if $(N_S : M_S) = R_S$. By (v), $(N : M)_S = (N_S : M_S)$ and, by (iv), this equals R_S if and only if S meets $(N : M)$, as required. □

Theorem 6. *Let P be a prime ideal of R with $P \cap S = \emptyset$, and let M be an R-semimodule. Then there is a one-to-one correspondence between the P-prime subsemimodules N of M and the P_S-prime subsemimodules L of M_S. This is such that, when N and L correspond, $L = N_S$ and $N = L \cap M$.*

Proof. By Theorem 5, we need to show that, under this correspondence of primary subsemimodules, N is prime if and only if $L = N_S$ is prime. By Lemma 4, it suffices to show that $(N :_R M) = P$ if and only if $(N_S :_{R_S} M_S) = P_S$ provided that $P = \mathrm{rad}(N : M)$ and $P_S = \mathrm{rad}(N_S : M_S)$ as N and N_S are, respectively, P-primary and P_S-primary. If $P = (N : M)$, then $P_S = (N : M)_S \subseteq (N_S : M_S) \subseteq \mathrm{rad}(N_S : M_S) = P_S$ whence $(N_S : M_S) = P_S$. Conversely, if $(N_S : M_S) = P_S$, then $P_S M_S \subseteq N_S$ so that $(p/s)(m/t) \in P_S$ for every $p \in P$, $m \in M$, and $s, t \in S$. Since $(pm/st)(s^2t^2/st) \in N_S$, $pm \in N$ for every $m \in M$. Thus $p \in (N : M)$ for every $p \in P = \mathrm{rad}(N : M)$. Therefore, $(N : M) = \mathrm{rad}(N : M) = P$. □

Corollary 1. *If N is a prime subsemimodule of an R-semimodule M, then $(N : M)_S = (N_S : M_S)$.*

Proof. In the proof of Theorem 6 we have seen that if $(N : M) \cap S = \emptyset$, then $(N : M)_S = (N_S : M_S)$. On the other hand if $(N : M) \cap S \neq \emptyset$, then $N_S = M_S$ by Proposition 9 so that $(N : M)_S = (N_S : M_S) = R_S$. □

Corollary 2. *Let M be an R-semimodule and P a prime ideal of R. Then the prime subsemimodules of the R_P-semimodule M_P are in a one-to-one correspondence with those prime subsemimodules N of M with $(N : M) \subseteq P$.*

Proof. Set $S = R - P$ and apply Theorem 6. □

Proposition 4. *Let R be a semiring and N a subsemimodule of an R-semimodule M. If $N_S \neq M_S$, then $(N : M) \cap S = \emptyset$. Conversely, if $(N : M) \cap S = \emptyset$, then $N_S \neq M_S$ provided that either i) M is finitely generated or ii) N is a primary subsemimodule.*

Proof. Assume that $(N : M) \cap S \neq \emptyset$ and $r \in (N : M) \cap S$. Let $m/s \in M_S$. Then $rm \in N$ so that $m/s = (rm)/(rs) \in N_S$, which proves that $N_S = M_S$, a contradiction. Thus $(N : M) \cap S = \emptyset$. Conversely, assume that $(N : M) \cap S = \emptyset$. Note that $(N : M) \cap S = \emptyset$ if and only if $\mathrm{rad}(N : M) \cap S = \emptyset$. Now the assertion follows from Lemma 4 (vi) and Proposition 3. □

Proposition 5. *Let R be a semiring and N a subsemimodule of an R-semimodule M such that $(N : M)$ is a primary ideal (resp. $(N : M) = P$) for some prime ideal P of R. Then N is a P-primary (resp. P-prime) subsemimodule of M if and only if $N_P \cap M = N$.*

Proof. The necessity is due to Proposition 3 (resp. Theorem 6). To see the sufficiency, suppose that $rm \in N$ such that $m \in M - N$ and $r \in R$; we show that $r \in P$. Suppose not. Then $m/1 = (rm)/r \in N_P$, so $m \in N_P \cap M = N$, which is a contradiction. Thus, $r \in P$ so that N is a P-primary (resp. P-prime) subsemimodule of M. □

Theorem 7. *Let R be a semidomain which is not a semifield and F the field of fractions of R. Then the R-semimodule F has* $\mathrm{Spec}(F) = \{0\}$.

Proof. Let N be a proper subsemimodule of F. Then $(N : F) = 0$ since $aF = F$ for every non-zero element a of R. Let $r.a/b = (ra)/b \in \{0/1\}$ such that $r \in R$ and $a/b \neq 0/1$. Then $a \neq 0_R$ and $ra = 0_R$; so $r = 0$ since R is a semidomain. It follows that $\{0/1\}$ is a 0_F-prime subsemimodule of F. To show that $\{0/1\}$ is the only prime subsemimodule of F, we assume the contrary and let L be a non-zero prime subsemimodule of F. Since L is a non-zero subsemimodule, there exists $0/1 \neq x = c/d \in L$, where $c, d \in R$, such that $(d/1)x = c/1 \in L$. On the other hand, there exists $0 \neq y \in R$ such that $1/y \notin L$ since $L \neq F$. Now we have $(c/1)(y/1) = (cy)/1 \notin (L : F)$ and $1/y \notin L$, but $(cy/1)(1/y) = c/1 \in L$, which is a contradiction. Thus $\mathrm{Spec}(F) = \{0\}$. □

5 Strong primeful semimodules

In this section we extend some definitions and results of C. P. Lu [16, 17] to semimodules over semirings. Let M be a semimodule over a semiring R with $\mathrm{ann}(M)$ a Q-ideal of R. The map $\psi : \mathrm{Spec}(M) \to \mathrm{Spec}(R/\mathrm{ann}(M))$ defined by $\psi(N) = (N : M)/\mathrm{ann}(M)$ for every $N \in \mathrm{Spec}(M)$ will be called the natural map of $\mathrm{Spec}(M)$. The surjectivity of the natural map ψ is particularly important in the topological space $\mathrm{Spec}(M)$ equipped with a topology called the Zariski topology. An R-semimodule M is called primeful if either $M = 0$ or the natural map of $\mathrm{Spec}(M)$ is surjective [17].

We continue to use the notation already established, so R denotes a commutative S-cancellative semiring with an identity element and S a non-empty multiplicatively closed subset of R. M will designate a fixed S-cancellative semimodule over R.

Moreover, assume that P is a prime ideal of R. Then $S = R - P$ is a multiplicatively closed subset of R. In this case we set $R_S = R_P$ and $I_S = IR_P$, where I is an ideal of R.

Lemma 5. *Assume that P is a prime k-ideal of a semiring R and let R be a S-cancellative semiring, where $S = R - P$. Then R_P is a local semiring with unique maximal k-ideal of PR_P.*

Proof. By [8, Lemma 5 and Theorem 2], it suffices to show that PR_P is exactly the set of non-semi-units of R_P. Let $y \in R_P - PR_P$, and take any representation $y = a/s$ with $a \in R$, $s \in S$. We must have $a \notin P$, so that a/s is a unit of R_P with inverse s/a (so a/s is a semi-unit by [11, Remark 2.4]. On the other hand, if y is a semi-unit of R_P, and $y = b/t$ for some $b \in R$, $t \in S$, then there exist $c, d \in R$ and $u, w \in S$ such that $1/1 + (bc)/(tu) = (bd)/(tw)$. It follows that $t^2uw + bctw = tubc$; hence $b \notin P$ since P is a k-ideal, and since this reasoning applies to every representation $y = b/t$ with $b \in R$, $t \in S$, of y as a formal fraction, it follows that $y \notin PR_S$, and so the proof is complete (see [9, Theorem 3]). □

Example 3. The monoid $M = (Z_6, +_6)$ is a semimodule over $(E_0^+, +, .)$ (see [13, p. 151]) with $\mathrm{ann}(M) = \{60k : k \in E_0^+\}$. It is easy to see that $\mathrm{ann}(M)$ is a Q-ideal of E_0^+ with respect to $Q = \{1, 2, ..., 59\}$.

Proposition 6. *Let M be a non-zero semimodule over a semiring R with* $\mathrm{ann}(M)$ *a Q-ideal of R. Then the following hold:*

(i) M is a primeful semimodule if and only if for every prime k-ideal P with $\mathrm{ann}(M) \subseteq P$*, there exists a prime subsemimodule N of M such that $(N : M) = P$.*

(ii) If M is a primeful semimodule, then $PM_P \neq M_P$ for every prime k-ideal P with $\mathrm{ann}(M) \subseteq P$.

Proof. (i) Assume that M is primeful and let P be a prime k-ideal of R with $\mathrm{ann}(M) \subseteq P$. Then $P/\mathrm{ann}(M)$ is a prime ideal of $R/\mathrm{ann}(M)$ by [4, Theorem 2.5]. By assumption, there exists a prime subsemimodule N of M such that $\psi(N) = (N : M)/\mathrm{ann}(M) = P/\mathrm{ann}(M)$; hence $(N : M) = P$ by [4, Lemma 2.13]. The reverse implication is clear.

(ii) For any prime k-ideal P of R with $\mathrm{ann}(M) \subseteq P$, let N be a P-prime subsemimodule of M. Then $PM \subseteq N$ with $N \neq M$ so that N_P is a PR_P-prime subsemimodule of M_P by Theorem 6. Since $PM_P \subseteq N_P$ with $N_P \neq M_P$, $M_P \neq PM_P$. □

We begin this section by proving the following fundamental theorems of this paper:

Theorem 8. *Let M be a non-zero semimodule over a semiring R with* $\mathrm{ann}(M)$ *a Q-ideal of R. Then the following are equivalent:*

(i) M is primeful;

(ii) Let P be a prime partitioning ideal of R such that $\mathrm{ann}(M) \subseteq P$ *and PR_P is a partitioning ideal of R_P. Then there exists a prime subsemimodule N of M such that $(N : M) = P$;*

(iii) Let P be a prime partitioning ideal of R such that $\mathrm{ann}(M) \subseteq P$ *and* PR_P *is a partitioning ideal of* R_P*. Then* $PM_P \neq M_P$*;*

(iv) Let P be a prime partitioning ideal of R such that $\mathrm{ann}(M) \subseteq P$ *and* PR_P *is a partitioning ideal of* R_P*. Then* $S_P(PM)$ *is a P-prime subsimimodule;*

(v) Let P be a prime partitioning ideal of R such that $\mathrm{ann}(M) \subseteq P$ *and* PR_P *is a partitioning ideal of* R_P*. Then* $\mathrm{Spec}_P(M) \neq \emptyset$.

Proof. By Proposition 6, we prove $(iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (i)$. $(iii) \Rightarrow (iv)$: Since by assumption and Lemma 5, PR_P is a maximal partitioning ideal of R_P and $PM_P \neq M_P$, $(PR_P)M_P$ is a PR_P-prime subsemimodule of M_P by Proposition 1. Hence $S_P(PM) = PM_P \cap M$ is a P-prime subsemimodule of M by Theorem 6. Thus (iv) follows. $(iv) \Rightarrow (v)$ and $(v) \Rightarrow (ii)$ are clear. □

Let M be a semimodule over a semiring R with $\mathrm{ann}(M)$ a Q-ideal of R. The collection of all prime (resp. maximal) k-subsemimodules of M with $(N : M)$ a strong ideal of R (resp. with $(N : M)$ a strong Q-ideal of R) is called the k-spectrum (resp. the maximal k-spectrum) of M and denoted by $\mathrm{Spec}_k(M)$ (resp. $\mathrm{Max}_k(M)$). Set

$$\mathrm{Spec}_k(R/\mathrm{ann}(M)) = \{P/\mathrm{ann}(M) \in \mathrm{Spec}(R/\mathrm{ann}(M)) : P \text{ is a strong } k\text{-ideal of } R\}.$$

Definition 4. Let M be a semimodule over a semiring R with $\mathrm{ann}(M)$ a Q-ideal of R:

(i) M is called strong primeful if either $M = 0$ or the natural map $\psi : \mathrm{Spec}_k(M) \to \mathrm{Spec}_k(R/\mathrm{ann}(M))$ defined by $\psi(N) = (N : M)/\mathrm{ann}(M)$ for every $N \in \mathrm{Spec}_k(M)$ is surjective.

(ii) M is called strong fulmaximal if either $M = 0$ or the natural map $\psi : \mathrm{Max}_k(M) \to \mathrm{Max}_k(R/\mathrm{ann}(M))$ defined by $\psi(N) = (N : M)/\mathrm{ann}(M)$ for every $N \in \mathrm{Max}_k(M)$ is surjective.

Theorem 9. *Let M be a non-zero semimodule over a semiring R with* $\mathrm{ann}(M)$ *a Q-ideal of $R \neq \{0\}$. Then the following hold:*

(i) If M is finitely generated, then M is a strong primeful semimodule and, similarly, M is a strong fulmaximal semimodule. Consequently, $\mathrm{Spec}_k(M) \neq \emptyset$ *and* $\mathrm{Max}_k(M) \neq \emptyset$*.*

(ii) If M is multiplication, then M is a strong primeful semimodule. Consequently, $\mathrm{Spec}_k(M) \neq \emptyset$*.*

Proof. (i) Let $P/\mathrm{ann}(M) \in \mathrm{Spec}_k(R/\mathrm{ann}(M))$. Then by assumption and [4, Theorem 2.5], P is a strong prime k-ideal containing $\mathrm{ann}(M)$. Since M is a non-zero finitely generated R-semimodule, M_P is a non-zero finitely generated R_P-semimodule with $\mathrm{ann}(M_P) = (\mathrm{ann}(M))_P$ and $\mathrm{ann}(M_P) \subseteq PR_P$ by Lemma 4. If $a/s \in PR_P$ for some $a \in P$ and $s \notin P$, then $a + b = 0$ for some $b \in P$, and so $a/s + b/s = 0/1$; hence

PR_P is a strong prime k-ideal of R_P (see [9, Lemma 5]). According to Proposition 2 and Theorem 3, $PM_P \neq M_P$ so that PM_P is a PR_P-prime subsemimodule of M_P. Applying Theorem 6, we can conclude that $N = PM_P \cap M$ is a prime subsemimodule of M; hence $\psi(N) = (N : M)/\text{ann}(M) = P/\text{ann}(M)$. This proves that ψ is surjective. Finally, assume that $P/\text{ann}(M) \in \text{Max}_k(R/\text{ann}(M))$. Then by assumption and [4, Theorem 2.14], P is a strong maximal k-ideal containing $\text{ann}(M)$. Let $T(P)$ be the set of all P-prime k-subsemimodules N of M with $(N : M)$ a strong Q-ideal. In the proof above, we have seen that $T(P) \neq \emptyset$. With the aid of Zorn's lemma, we can see that there exists a maximal element L in $T(P)$. Since $(L : M) = P$ is a maximal Q-ideal, L is a maximal subsemimodule. (ii) follows from [12, Theorem 3.8]. □

References

[1] Allen P. J. *A fundamental theorem of homomorphisms for semirings.* Proc. Amer. Math. Soc., 1969, **21**, 412–416.

[2] El-Bast Z., Smith P. F. *Multiplication modules.* Comm. Algebra, 1988, **16**, 755–779.

[3] Chaudhari J. N., Bonde D. *On Partitioning and Subtractive Subsemimodules of Semimodules over Semirings.* Kyungpook Math. J., 2010, **50**, 329–336.

[4] Ebrahimi Atani S. *The ideal theory in quotients of commutative semirings.* Glas. Math., 2007, **42**, 301–308.

[5] Ebrahimi Atani S. *On primal and weakly primal ideals over commutative semirings.* Glas. Math., 2008, **43**, 13–23.

[6] Ebrahimi Atani S. *The zero-divisor graph with respect to ideals of a commutative semiring.* Glas. Math., 2008, **43**, 309–320.

[7] Ebrahimi Atani S. *An ideal-based zero-divisor graph of a commutative semiring.* Glas. Math., 2009, **44(1)**, 141–153.

[8] Ebrahimi Atani S., Ebrahimi Atani R. *Very strong multiplication ideals and the ideal $\theta(I)$.* Glas. Math., 2010, **45(2)**, 395–406.

[9] Ebrahimi Atani R., Ebrahimi Atani S. *Ideal theory in commutative semirings.* Bul. Acad. Stiinte Rep. Mold. Matematica, 2008, No. 2(57), 14–23.

[10] Ebrahimi Atani R., Ebrahimi Atani S. *On subsemimodules of semimodules.* Bul. Acad. Stiinte Rep. Mold., Matematica, 2010, No. 2(63), 20–30.

[11] Ebrahimi Atani S., Ebrahimi Atani R. *Some remarks on partitioning semirings.* An. St. Univ. Ovidius Constanta, 2010, **18(1)**, 49–62.

[12] Ebrahimi Atani S., Shajari Kohan M. *A note on finitely generated multiplication semimodules over commutative semirings.* Inter. J. of Algebra, 2010, **4(8)**, 389–396.

[13] Golan J. S. *Semirings and their applications.* Kluwer Academic Publisher, Dordrecht, 1999.

[14] Hebisch U., Weinert U. J. *Halbringe – Algebraische Theorie und Anwendungen in der Informatik.* Teubner, Stuttgart 1993.

[15] Lu C. P. *Prime submodules of modules.* Comment. Math. Univ. St. Paul, 1984, **33**, 61–69.

[16] Lu C. P. *Spectra of modules.* Comm. Algebra, 1995, **23(10)**, 3741–3752.

[17] Lu C. P. *A module whose prime spectrum has the surjective natural map.* Houston J. of Math., 2007, **33(1)**, 125–143.

[18] Maze M., Monico C., Rosenthal J. *Public key cryptography based on semigroup actions.* Adv. Mathematics of Communication, 2007, **1(4)**, 489–281.

[19] Mosher J. R. *Generalized quotients of hemirings.* Compositio Math., 1970, **22**, 275–281.

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