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Integrals and phase portraits of planar quadratic differential systems with invariant lines of total multiplicity four

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Abstract. In this article we consider the class $\mathbf{QSL_4}$ of all real quadratic differential systems $\dfrac{dx}{dt} = p(x,y)$, $\dfrac{dy}{dt} = q(x,y)$ with $\gcd(p,q) = 1$, having invariant lines of total multiplicity four and a finite set of singularities at infinity. We first prove that all the systems in this class are integrable having integrating factors which are Darboux functions and we determine their first integrals. We also construct all the phase portraits for the systems belonging to this class. The group of affine transformations and homotheties on the time axis acts on this class. Our Main Theorem gives necessary and sufficient conditions, stated in terms of the twelve coefficients of the systems, for the realization of each one of the total of 69 topologically distinct phase portraits found in this class. We prove that these conditions are invariant under the group action.

Mathematics subject classification: 34A26, 34C40, 34C14.
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1 Introduction

We consider here real planar differential systems of the form

$$(S) \qquad \frac{dx}{dt} = p(x,y), \qquad \frac{dy}{dt} = q(x,y), \tag{1}$$

where $p,\ q \in \mathbb{R}[x,y]$, i.e. $p,\ q$ are polynomials in $x,\ y$ over $\mathbb{R}$, their associated vector fields

$$\tilde{D} = p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y} \tag{2}$$

and differential equations

$$q(x,y)dx - p(x,y)dy = 0. \tag{3}$$

We call *degree* of a system (1) (or of a vector field (2) or of a differential equation (3)) the integer $\deg(S) = \max(\deg p, \deg q)$. In particular we call *quadratic* a differential system (1) with $\deg(S) = 2$.

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A system (1.1) is said to be integrable on an open set U of $\mathbb{R}^2$ if there exists a C^1 function $F(x,y)$ defined on U which is a first integral of the system, i.e. such that $\tilde{D}F(x,y)=0$ on U and which is nonconstant on any open subset of U. The cases of integrable systems are rare but as Arnold said in [2, p. 405] "...these integrable cases allow us to collect a large amount of information about the motion in more important systems...". In particular we indicate below how integrable systems play a role in the second part of Hilbert's 16th problem for polynomial differential systems.

There are several hard open problems on the class of all quadratic differential systems (1). Among them the most famous one is the second part of Hilbert's 16th problem which asks for the determination of the so called Hilbert number $H(2)$ for this class where

$$H(n)=\max\{LC(S)\,\big|\,\deg(S)=n\}$$

and $LC(S)$ is the number of limit cycles of the system (S). It is known that for any polynomial system (S), $LC(S)$ is finite. This is the so called individual finiteness theorem which was proved independently by Ilyashenko and Ecalle (see [12, 15]).

The class of quadratic differential systems possessing a singularity which is a center is formed by integrable systems on open sets of $\mathbb{R}^2$ which are complements of real invariant algebraic curves. These systems do not possess limit cycles but they turn out to be very important in the determination of $H(2)$ as perturbations of such systems could produce limit cycles. Furthermore we have evidence indicating that $H(2)$ could be linked to the number of limit cycles occurring in perturbations of the most degenerate ones of all quadratic systems with a center (which happen to have a rational first integral) as we explain below.

In [3] the authors studied the class of all quadratic systems possessing a second order weak focus. It is known that the maximum number of limit cycles occurring in systems in this class is two (see [32, 33]). In the bifurcation diagram drawn in [3] for this three parameter family of systems, modulo the action of the affine group and time rescaling, the maximum number of two limit cycles which one has for this class, occurs in perturbations of an quadratic system (S_0) with a center, which has a rational first integral foliating the plane into conic curves. In addition this system (S_0) has three invariant affine lines and its line at infinity is filled up with singularities. Although other systems in this class having this maximum number of two limit cycles could be far away in the parameter space from the particular degenerate system (S_0), their phase portraits are topologically equivalent with a small perturbations of (S_0). This indicates the importance of integrable systems having invariant algebraic curves (see Definition 3), even with a rational first integral, in the second part of Hilbert's 16th problem and adds to the motivation for studying such systems. However, such a study is interesting for its own sake being at crossroads of differential equations and algebraic geometry.

The simplest class of integrable quadratic systems due to the presence of invariant algebraic curves is the class of integrable quadratic systems due to the presence of invariant lines. The study of this class was initiated in articles [25, 27–29]. In particular it was shown in [29] that the above mentioned system (S_0) possesses invariant affine lines of total multiplicity three.

In this article we study the class $\mathbf{QSL_4}$ of all quadratic differential systems possessing invariant lines of total multiplicity four (including the line at infinity and including multiplicities of the lines). The study of $\mathbf{QSL_4}$ was initiated in [27] where we proved a theorem of classification for this class. This classification, which is taken modulo the action of the group of real affine transformations and time rescaling, is given in terms of algebraic invariants and comitants and also geometrically, using cycles on the complex projective plane and on the line at infinity. An important ingredient in this classification is the notion of configuration of invariant lines of a polynomial differential system.

Definition 1. We call configuration of invariant lines of a system (1) the set of all its (complex) invariant lines (which may have real coefficients), each endowed with its own multiplicity [25] and together with all the real singular points of this system located on these lines, each one endowed with its own multiplicity.

The goal of this article is to complete the study we began in [27]. More precisely in this work we

- prove that all systems in this class $\mathbf{QSL_4}$ are integrable. We show this by using the geometric method of integration of Darboux. We construct explicit Darboux integrating factors and we give the list of first integrals for each system in this class;
- construct all topologically distinct phase portraits of the systems in this class;
- give invariant (under the action of the group $\mathrm{Aff}\,(2,\mathbb{R})\times\mathbb{R}^*$)) necessary and sufficient conditions, in terms of the twelve coefficients of the systems, for the realization of each specific phase portrait.

This article is organized as follows:

In Section 2 we give the preliminary definitions and results needed in this article. These are mainly of a differential-algebraic nature.

In Section 3 we associate to each real quadratic system (1) possessing invariant lines with corresponding multiplicities, a divisor on the complex projective plane which encodes this information. We also define several integer-valued affine invariants of such systems using divisors on the line at infinity or zero-cycles on $\mathbb{P}_2(\mathbb{C})$ defined in [25] and [27], which encode the multiplicities of the singularities of the systems. We also state Theorem 5 which was proved in [27] illustrating how these cycles are useful for classification purposes. This theorem lists all possible configurations of invariant lines of total multiplicity four of the systems under study.

In Section 4 we prove the integrability of the systems in this class by using their invariant lines with their multiplicities. The main result in this Section states that all these systems have either a polynomial inverse integrating factor which splits into linear factors over $\mathbb{C}$ or a Darboux integrating factor which is a product of powers of the polynomials defining their invariant lines and an exponential factor $e^{G(x,y)}$ with

G a rational function over $\mathbb{C}$. The result is summed up in Table 1 where all these integrating factors are listed along with the first integrals, some of which but not all are Darboux functions.

In Section 5 we construct the phase portraits of the systems in this class and state our Main Theorem which gives necessary and sufficient conditions, invariant under the group action, for the realization of each one of the total of 69 topologically distinct phase portraits obtained for this class, in terms of the twelve coefficients of the systems.

2 Preliminaries

In this Section we give the basic notions and results needed in this paper. We are concerned here with the integrability in the sense of Darboux [10] of systems (1) possessing invariant straight lines of total multiplicity four. We work with the notion of multiplicity of an invariant line introduced by us in [25].

In [10] Darboux gave a geometric method of integration of planar complex differential equations (3) using invariant algebraic curves of the equations (see Definition 3). Each real differential system (1) generates a complex differential system when the variables range over $\mathbb{C}$. For this reason the method of Darboux can be applied also for real systems.

Poincaré was enthusiastic about the work of Darboux [10], which he called "admirable" in [19]. This method of integration was applied to give unified proofs of integrability for several families of systems (1). For example in [24] it was applied to show in a unified way (unlike previous proofs which used ad hoc methods) the integrability of planar quadratic systems possessing a center.

A brief and easily accessible exposition of the method of Darboux can be found in the survey article [23].

The topic of Darboux' paper [10] is best treated using the language of differential algebra, subject which started to be developed in the work of Ritt [1893–1951], long after Darboux wrote his paper [10]. The term *"Differential Algebra"* was introduced by Ellis Kolchin, who as Buium and Cassidy said in [6], "deepened and modernized differential algebra and developed differential algebraic geometry and differential algebraic groups". According to Ritt, differential algebra began to be developed in the 1930's (e.g.[21]) under the influence of Emmy Noether's work of the 1920's in algebra. (In his book [22] Ritt said: "the form in which the results of differential algebra are presented has been deeply influenced by her teachings".)

Whenever a definition below is given for a system (1) or equivalently for a vector field (2), this definition could also be given for an equation (3) and viceversa. For brevity we sometimes state only one of the possibilities.

An integrating factor of an equation (3) on an open subset U of $\mathbb{R}^2$ is usually defined as a C^1 function $R(x,y) \not\equiv 0$ such that the 1-form

$$\omega = R\,q(x,y)dx - R\,p(x,y)dy$$

is exact, i.e. there exist a C^1 function $F : U \longrightarrow \mathbb{K}$ on U such that

$$\omega = dF. \tag{4}$$

If R is an integrating factor on U of (3) then the function F such that $\omega = Rqdx - Rpdy = dF$ is a first integral of the equation $w = 0$ (or a system (1)). In this case we necessarily have on U:

$$\frac{\partial(Rq)}{\partial y} = -\frac{\partial(Rp)}{\partial x} \tag{5}$$

and developing the above equality we obtain $\dfrac{\partial R}{\partial x}p + \dfrac{\partial R}{\partial y}q = -R\Big(\dfrac{\partial p}{\partial x} + \dfrac{\partial q}{\partial y}\Big)$ or equivalently

$$\tilde{D}R = -R \operatorname{div} \tilde{D}. \tag{6}$$

In view of Poincaré's lemma (see for example [31]), if $R(x, y)$ is a C^1 function on a star-shaped open set U of $\mathbb{R}^2$, then $R(x, y)$ is an integrating factor of (3) if and only if (5) (or equivalently (6)) holds on U. So for star shaped open sets U (6) can be taken as a definition of an integrating factor on U. This is sufficient for our purpose. We note that this last definition is much simpler than the one usually used in textbooks as it no longer involves an existential quantifier.

In this work we shall apply to our real quadratic system (1) the method of integration of Darboux which was developed for complex differential equations (3). This method uses multiple-valued complex functions of the form:

$$F = e^{G(x,y)} f_1(x, y)^{\lambda_1} \cdots f_s(x, y)^{\lambda_s}, \quad G \in \mathbb{C}(x, y), \quad f_i \in \mathbb{C}[x, y], \quad \lambda_i \in \mathbb{C}, \tag{7}$$

$G = G_1/G_2$, $G_i \in \mathbb{C}[x, y]$, f_i irreducible over $\mathbb{C}$. It is clear that in general an expression (7) makes sense only for $G_2 \neq 0$ and for points $(x, y) \in \mathbb{C}^2 \setminus (\{G_2(x, y) = 0\} \cup \{f_1(x, y) = 0\} \cup \cdots \cup \{f_s(x, y) = 0\})$.

The above expression (7) yields a multiple-valued function on

$$\mathcal{U} = \mathbb{C}^2 \setminus (\{G_2(x, y) = 0\} \cup \{f_1(x, y) = 0\} \cup \cdots \cup \{f_s(x, y) = 0\}).$$

The function F in (7) belongs to a differential field extension of $\Big(\mathbb{C}(x, y), \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}\Big)$ obtained by adjoining to $\mathbb{C}(x, y)$ a finite number of algebraic and of transcendental elements over $\mathbb{C}(x, y)$. For example $f(x, y)^{1/2}$ is an expression of the form (7), when $f \in \mathbb{C}[x, y] \setminus \{0\}$. This function belongs to the algebraic differential field extension $\Big(\mathbb{C}(x, y)[u], \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}\Big)$ of $\Big(\mathbb{C}(x, y), \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}\Big)$ obtained by adjoining to $\mathbb{C}(x, y)$ a root of the equation $u^2 - f(x, y) = 0$. In general, the expression (7) belongs to a differential field extension which is not necessarily algebraic. Indeed, for example this occurs if $G(x, y)$ is not a constant.

Definition 2. A function F in a differential field extension K of $\left(\mathbb{C}(x,y), \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}\right)$ which is finite over $\mathbb{C}(x,y)$, is a first integral (integrating factor, respectively inverse integrating factor) of a complex differential system (1) or a vector field (2) or a differential equation (3) if $\tilde{D}F = 0$ ($\tilde{D}F = -F \operatorname{div} \tilde{D}$, respectively $\tilde{D}F = F \operatorname{div} \tilde{D}$).

In 1878 Darboux introduced the notion of invariant algebraic curve for differential equations on the complex projective plane. This notion can be adapted for equations (3) on $\mathbb{C}^2$ or equivalently for systems (1) or vector fields (2).

Definition 3 (Darboux [10])**.** An affine algebraic curve $f(x,y) = 0$, $f \in \mathbb{C}[x,y]$, $\deg f \geq 1$ is invariant for an equation (3) or for a system (1) if and only if $f \mid \tilde{D}f$ in $\mathbb{C}[x,y]$, i.e. $k = \dfrac{\tilde{D}f}{f} \in \mathbb{C}[x,y]$. In this case k is called the cofactor of f.

Definition 4 (Darboux [10])**.** An algebraic solution of an equation (3) (respectively (1), (2)) is an invariant algebraic curve $f(x,y) = 0$, $f \in \mathbb{C}[x,y]$ ($\deg f \geq 1$) with f an irreducible polynomial over $\mathbb{C}$.

Darboux showed that if an equation (3) or (1) or (2) possesses a sufficient number of such invariant algebraic solutions $f_i(x,y) = 0$, $f_i \in \mathbb{C}[x,y]$, $i = 1, 2, \ldots, s$ then the equation has a first integral of the form (7).

Definition 5. An expression of the form $F = e^{G(x,y)}$, $G(x,y) \in \mathbb{C}(x,y)$, i.e. G is a rational function over $\mathbb{C}$, is an exponential factor[1] for a system (1) or an equation (3) if and only if $k = \dfrac{\tilde{D}F}{F} \in \mathbb{C}[x,y]$. In this case k is called the cofactor of the exponential factor F.

Proposition 1 (Christopher [8])**.** *If an equation* (3) *admits an exponential factor* $e^{G(x,y)}$ *where* $G(x,y) = \dfrac{G_1(x,y)}{G_2(x,y)}$, $G_1, G_2 \in \mathbb{C}[x,y]$ *then* $G_2(x,y) = 0$ *is an invariant algebraic curve of* (3)*.*

Definition 6. We say that a system (1) or an equation (3) has a Darboux first integral (respectively Darboux integrating factor) if it admits a first integral (respectively integrating factor) of the form $e^{G(x,y)} \prod_{i=1}^{s} f_i(x,y)^{\lambda_i}$, where $G(x,y) \in \mathbb{C}(x,y)$ and $f_i \in \mathbb{C}[x,y]$, $\deg f_i \geq 1$, $i = 1, 2, \ldots, s$, f_i irreducible over $\mathbb{C}$ and $\lambda_i \in \mathbb{C}$. A system (1) or an equation (3) has a Liouvillian first integral (respectively a Liouvillian integrating factor) if it admits a first integral (respectively integrating factor) which is a Liouvillian function, i.e. a function which is built up from rational functions over $\mathbb{C}$ using exponentiation, integration and algebraic functions.

[1] Under the name *degenerate invariant algebraic curve* this notion was introduced by Christopher in [8].

Proposition 2 (Darboux [10])**.** *If an equation* (3) *(or* (1)*, or* (2)*) has an integrating factor (or first integral) of the form* $F = \prod_{i=1}^{s} f_i^{\lambda_i}$ *then* $\forall i \in \{1, \ldots, s\}$*,* $f_i = 0$ *is an algebraic invariant curve of* (3) *(*(1)*,* (2)*).*

In [10] Darboux proved the following theorem of integrability using invariant algebraic solutions of differential equation (3):

Theorem 3 (Darboux [10])**.** *Consider a differential equation* (3) *with* $p, q \in \mathbb{C}[x, y]$*. Let us assume that* $m = \max(\deg p, \deg q)$ *and that the equation admits* s *algebraic solutions* $f_i(x, y) = 0$*,* $i = 1, 2, \ldots, s$ *(*$\deg f_i \geq 1$*). Then we have:*

I. If $s = m(m+1)/2$ *then there exists* $\lambda = (\lambda_1, \ldots, \lambda_s) \in \mathbb{C}^s \setminus \{0\}$ *such that* $R = \prod_{i=1}^{s} f_i(x, y)^{\lambda_i}$ *is an integrating factor of* (3)*.*

II. If $s \geq m(m+1)/2 + 1$ *then there exists* $\lambda = (\lambda_1, \ldots, \lambda_s) \in \mathbb{C}^s \setminus \{0\}$ *such that* $F = \prod_{i=1}^{s} f_i(x, y)^{\lambda_i}$ *is a first integral of* (3)*.*

Remark 1. We stated the theorem for the equation (3) but clearly we could have stated it for the vector field $\tilde{D}$ (2) or for the polynomial differential system (1). We recall that Darboux's work was done for differential equations in the complex projective plane. The above formulation is an adaptation of his theorem for the complex affine plane.

In [16] Jouanolou proved the following theorem which improves part II of Darboux's Theorem.

Theorem 4 (Jouanolou [16])**.** *Consider a polynomial differential equation* (3) *over* $\mathbb{C}$ *and assume that it has* s *algebraic solutions* $f_i(x, y) = 0$*,* $i = 1, 2, \ldots, s$ *(*$\deg f_i \geq 1$*). Suppose that* $s \geq m(m+1)/2 + 2$*. Then there exists* $(n_1, \ldots, n_s) \in \mathbb{Z}^s \setminus \{0\}$ *such that* $F = \prod_{i=1}^{s} f_i(x, y)^{n_i}$ *is a first integral of* (3)*. In this case* $F \in \mathbb{C}(x, y)$*, i.e.* F *is rational function over* $\mathbb{C}$*.*

The above mentioned theorem of Darboux gives us sufficient conditions for integrability via the method of Darboux using algebraic solutions of systems (1). However these conditions are not necessary as it can be seen from the following example. The system

$$dx/dt = -y - x^2 - y^2, \quad dy/dt = x + xy$$

has two invariant algebraic curves: the invariant line $1 + y = 0$ and a conic invariant curve $f = 6x^2 + 3y^2 + 2y - 1 = 0$. This system is integrable having as first integral $F = (1 + y)^2 f$ but here $s = 2 < 3 = m(m+1)/2$.

Other sufficient conditions for Darboux integrability were obtained by Christopher and Kooij in [17] and Zoladek in [34]. Their theorems say that if a system has s invariant algebraic solutions in "generic position" (with "generic" as defined in their papers) such that $\sum_{i=1}^{s} \deg f_i = m + 1$ then the system has an inverse integrating factor of the form $\prod_{i=1}^{s} f_i$. But their theorem does not cover the above case as the two curves are not in "generic position". Indeed, the line $1 + y = 0$ is tangent to the curve $f = 0$ at $(0, -1)$. For similar reasons the above example is not covered by the more general result: Theorem 7.1 in [9]. Other sufficient conditions for integrability covering the example above were given in [7]. However we do not have necessary and sufficient conditions for Darboux integrability and the search is on for finding such conditions.

Problem resulting from the work [10] of Darboux: *Give necessary and sufficient conditions for a polynomial system (1.1) to have: (i) a polynomial inverse integrating factor; (ii) an integrating factor of the form* $\prod_{i=1}^{s} f_i(x,y)^{\lambda_i}$*; (iii) a Darboux integrating factor (or a Darboux first integral); (iv) a rational first integral.*

The last problem *(iv)* above, was stated in 1891 in the articles [19] and [20] of Poincaré where it was called the problem of algebraic integrability of the equations. In recent years there has been much activity in this area of research.

One of the goals of this work is to provide us with specific data to be used along with similar material for higher degree curves, for the purpose of dealing with questions regarding Darboux and algebraic integrability. We collect here in a systematic way information on quadratic systems having invariant lines of exactly four total multiplicity.

This material may also be used in studying quadratic systems which are small perturbations of integrable ones. In fact, as we have already indicated in the introduction, the maximum number of limit cycles of some subclasses of the quadratic class can be obtained by perturbing integrable systems having a rational first integral and invariant lines.

This article forms the basis for the study of some moduli spaces of quadratic systems, under the group action. One such moduli space which we intend to study in a following article is the moduli space of the closure within the quadratic class of the class $\mathbf{QSL_4}$.

3 Divisors associated to configurations of invariant lines

Consider real differential systems of the form:

$$(S)\qquad \begin{cases} \dfrac{dx}{dt} = p_0 + p_1(x,y) + \ p_2(x,y) \equiv p(x,y), \\ \dfrac{dx}{dt} = q_0 + q_1(x,y) + \ q_2(x,y) \equiv q(x,y) \end{cases} \tag{8}$$

with

$$\begin{aligned} &p_0 = a_{00}, \quad p_1(x,y) = a_{10}x + a_{01}y, \quad p_2(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\ &q_0 = b_{00}, \quad q_1(x,y) = b_{10}x + b_{01}y, \quad q_2(x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2. \end{aligned}$$

Let $a = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02})$ be the 12-tuple of the coefficients of system (8) and denote $\mathbb{R}[a,x,y] = \mathbb{R}[a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02},\ b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02}, x, y]$.

Notation 1. *Whenever we refer to some specific point in* $\mathbb{R}^{12}$ *rather than a 12-tuple parameter we shall denote such a point in* $\mathbb{R}^{12}$ *by* $\boldsymbol{a} = (\boldsymbol{a}_{00}, \boldsymbol{a}_{10} \ldots, \boldsymbol{b}_{02})$*. Each particular system* (8) *yields an ordered 12-tuple* $\boldsymbol{a}$ *of its coefficients.*

Notation 2. *Let*

$$\begin{aligned} P(X,Y,Z) =& p_0(\boldsymbol{a})Z^2 + p_1(\boldsymbol{a},X,Y)Z + p_2(\boldsymbol{a},X,Y) = 0,\\ Q(X,Y,Z) =& q_0(\boldsymbol{a})Z^2 + q_1(\boldsymbol{a},X,Y)Z + q_2(\boldsymbol{a},X,Y) = 0. \end{aligned}$$

We denote $\quad \sigma(P,Q) = \{w \in \mathbb{P}_2(\mathbb{C}) \mid P(w) = Q(w) = 0\}.$

Definition 7. We consider formal expressions $\mathbf{D} = \sum n(w)w$ where $n(w)$ is an integer and only a finite number of $n(w)$ in $\mathbf{D}$ are nonzero. Such an expression is called: *i)* a zero-cycle of $\mathbb{P}_2(\mathbb{C})$ if all w appearing in $\mathbf{D}$ are points of $\mathbb{P}_2(\mathbb{C})$; *ii)* a divisor of $\mathbb{P}_2(\mathbb{C})$ if all w appearing in $\mathbf{D}$ are irreducible algebraic curves of $\mathbb{P}_2(\mathbb{C})$; *iii)* a divisor of an irreducible algebraic curve $\mathfrak{C}$ in $\mathbb{P}_2(\mathbb{C})$ if all w in $\mathbf{D}$ are points of the curve $\mathfrak{C}$. We call degree of the expression $\mathbf{D}$ the integer $\deg(\mathbf{D}) = \sum n(w)$. We call support of $\mathbf{D}$ the set $\operatorname{Supp}(\mathbf{D})$ of all w appearing in $\mathbf{D}$ such that $n(w) \neq 0$.

Definition 8. We say that an invariant affine straight line $\mathcal{L}(x,y) = ux+vy+w = 0$ (respectively the line at infinity $Z = 0$) for a quadratic vector field $\tilde{D}$ has multiplicity m if there exists a sequence of real quadratic vector fields $\tilde{D}_k$ converging to $\tilde{D}$, such that each $\tilde{D}_k$ has m (respectively $m-1$) distinct invariant affine straight lines $\mathcal{L}_i^j = u_i^j x + v_i^j y + w_i^j = 0$, $(u_i^j, v_i^j) \neq (0,0)$, $(u_i^j, v_i^j, w_i^j) \in \mathbb{C}^3$, converging to $\mathcal{L} = 0$ as $k \to \infty$ (with the topology of their coefficients), and this does not occur for $m+1$ (respectively m).

Notation 3. *Let us denote by*

$$\begin{aligned} \mathbf{QS} \;&=\; \left\{ (S) \;\middle|\; \begin{array}{c} (S) \textit{ is a real system } (1) \textit{ such that } \gcd(p(x,y),q(x,y)) = 1 \\ \textit{and} \quad \max\big(\deg(p(x,y)), \deg(q(x,y))\big) = 2 \end{array} \right\};\\ \mathbf{QSL} \;&=\; \left\{ (S) \in \mathbf{QS} \;\middle|\; \begin{array}{c} (S) \textit{ possesses at least one invariant affine line or} \\ \textit{the line at infinity has multiplicity at least two} \end{array} \right\}. \end{aligned}$$

In this section we shall assume that systems (8) belong to **QS**.

We define below the geometrical objects (divisors or zero-cycles) which play an important role in constructing the invariants of the systems.

Definition 9.

$$\begin{aligned} \mathbf{D}_{S}(P,Q) \;&=\; \sum_{w\in\sigma(P,Q)} I_w(P,Q)w;\\ \mathbf{D}_{S}(P,Q;Z) \;&=\; \sum_{w\in\{Z=0\}} I_w(P,Q)w;\\ \widehat{\mathbf{D}}_{S}(P,Q,Z) \;&=\; \sum_{w\in\{Z=0\}} \Big(I_w(C,Z),\, I_w(P,Q)\Big)w \quad \text{if} \quad Z \nmid C(X,Y,Z);\\ \mathbf{D}_{S}(C,Z) \;&=\; \sum_{w\in\{Z=0\}} I_w(C,Z)w \quad \text{if} \quad Z \nmid C(X,Y,Z), \end{aligned}$$

where $C(X,Y,Z) = YP(X,Y,Z) - XQ(X,Y,Z)$, $I_w(F,G)$ is the intersection number (see [11]) of the curves defined by homogeneous polynomials $F,\ G \in \mathbb{C}[X,Y,Z]$, $\deg(F), \deg(G) \geq 1$ and $\{Z=0\} = \{[X:Y:0] \,|\, (X,Y) \in \mathbb{C}^2 \setminus (0,0)\}$.

We denote by $\#A$ the number of points of a finite set A.

Notation 4.

$$n_{\mathbb{R}}^{\infty} = \#\{w \in Supp\, \mathbf{D}_S(C,Z) \,\big|\, w \in \mathbb{P}_2(\mathbb{R})\}.$$

A complex projective line $uX + vY + wZ = 0$ is invariant for the system (S) if either it coincides with $Z = 0$ or it is the projective completion of an invariant affine line $ux + vy + w = 0$.

Notation 5. *Let* $(S) \in \mathbf{QSL}$. *Let us denote*

$$\mathbf{IL}(S) = \left\{ \ l \ \left| \ \begin{array}{c} l \textit{ is a line in } \mathbb{P}_2(\mathbb{C}) \textit{ such} \\ \textit{that } l \textit{ is invariant for } (S) \end{array} \right. \right\};$$
$$M(l) = \textit{the multiplicity of the invariant line } l \textit{ of } (S).$$

Remark 2. We note that the line $l_\infty : Z = 0$ is included in $\mathbf{IL}(S)$ for any $(S) \in \mathbf{QS}$.

Let $l_i\ :\ f_i(x,y) = 0,\ i = 1, \ldots, k$, be all the distinct invariant affine lines (real or complex) of a system $(S) \in \mathbf{QSL}$, in case they exist. Let $l_i'\ :\ \mathcal{F}_i(X,Y,Z) = 0$ be the complex projective completion of l_i.

Notation 6. *We denote*

$$\mathcal{G}\ :\quad \prod_i \mathcal{F}_i(X,Y,Z)\, Z = 0; \quad Sing\, \mathcal{G} = \{w \in \mathcal{G} |\ w \textit{ is a singular point of } \mathcal{G}\};$$
$$\nu(w) = \textit{ the multiplicity of the point } w, \textit{ as a point of } \mathcal{G}.$$

Definition 10.

$$\begin{aligned} \mathbf{D}_{\mathbf{IL}}(S) &= \sum_{l \in \mathbf{IL}(S)} M(l) l, \quad (S) \in \mathbf{QSL}; \\ \operatorname{Supp} \mathbf{D}_{\mathbf{IL}}(S) &= \{\, l \mid l \in \mathbf{IL}(S)\}. \end{aligned}$$

Notation 7.

$$\begin{aligned} M_{\mathbf{IL}}(S) &= \deg \mathbf{D}_{\mathbf{IL}}(S); \\ N_{\mathbb{C}}(S) &= \#\operatorname{Supp} \mathbf{D}_{\mathbf{IL}}; \\ N_{\mathbb{R}}(S) &= \#\{l \in \operatorname{Supp} \mathbf{D}_{\mathbf{IL}} \,\big|\, l \in \mathbb{P}_2(\mathbb{R})\}; \\ n^{\mathbb{R}}_{\mathcal{G},\sigma}(S) &= \#\{\omega \in \operatorname{Supp} \mathbf{D}_S(P,Q) \mid \omega \in \mathcal{G}\big|_{\mathbb{R}^2}\}; \\ d^{\mathbb{R}}_{\mathcal{G},\sigma}(S) &= \sum_{\omega \in \mathcal{G}|_{\mathbb{R}^2}} I_\omega(P,Q); \\ m_{\mathcal{G}}(S) &= \max\{\nu(\omega) \mid \omega \in Sing\, \mathcal{G}|_{\mathbb{C}^2}\}; \\ m^{\mathbb{R}}_{\mathcal{G}}(S) &= \max\{\nu(\omega) \mid \omega \in Sing\, \mathcal{G}|_{\mathbb{R}^2}\}. \end{aligned} \tag{9}$$

For brevity we sometimes just write $M_{\mathbf{IL}}$, $N_{\mathbb{C}}$, ..., $m_{\mathcal{G}}^{\mathbb{R}}$.

In the following sections we shall prove the integrability of the quadratic differential systems having invariant lines of total multiplicity four, including the line at infinity and including possible multiplicities of the lines. Their possible configurations as well as invariant conditions with respect to the group action distinguishing these configurations were given in [27]. All possible such configurations for this class are found in Diagram 1 of Theorem 4.1 in [27]. This Theorem will be needed in the following sections so we reproduce it below. It also helps in illustrating how the concepts introduced in this section are used.

Notation 8. *We denote by* $\mathbf{QSL_4}$ *the class of all real quadratic differential systems* (8) *with* p, q *relatively prime* $((p,q)=1)$, $Z \nmid C$, *and possessing a configuration of invariant straight lines of total multiplicity* $M_{\mathbf{IL}}=4$ *including the line at infinity and including possible multiplicities of the lines.*

Theorem 5. *(Schlomiuk and Vulpe [27]) The class* $\mathbf{QSL_4}$ *splits into 46 distinct subclasses indicated in Diagram 1 with the corresponding Configurations 4.1–4.46, where the complex invariant straight lines are indicated by dashed lines. If an invariant straight line has multiplicity* $k>1$, *then the number* k *appears near the corresponding straight line and this line is in bold face. We indicate next to the real singular points of the systems, situated on the invariant lines, their multiplicities as follows:* $(I_\omega(p,q))$ *if* ω *is a finite singularity,* $(I_\omega(C,Z),\ I_\omega(P,Q))$ *if* ω *is an infinite singularity with* $I_w(P,Q)\neq 0$ *and* $(I_\omega(C,Z))$ *if* ω *is an infinite singularity with* $I_\omega(P,Q)=0$.

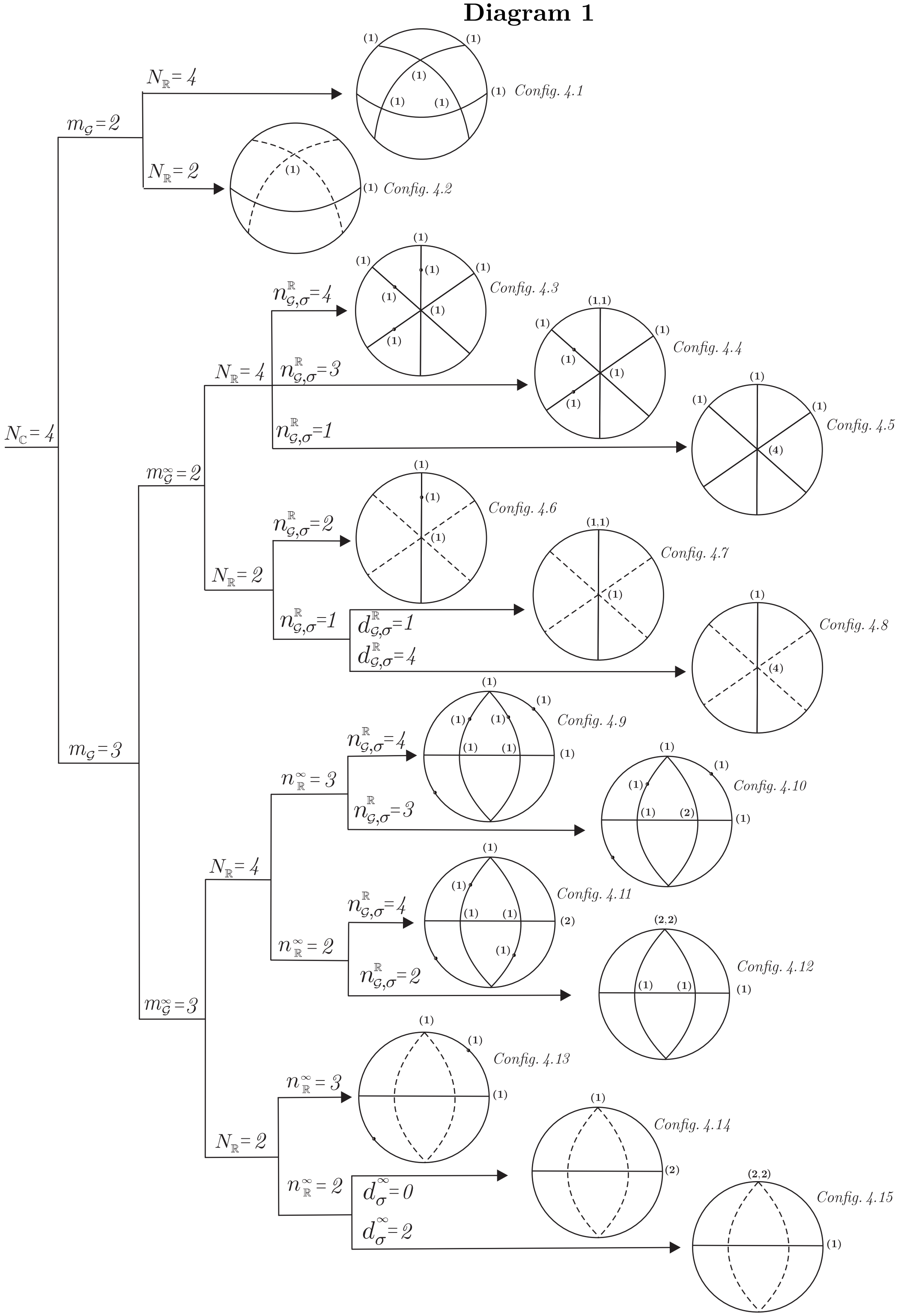
4 Integrability and phase portraits of the systems in the class of quadratic systems with total multiplicity four

4.1 Darboux integrating factors and first integrals

Theorem 6. *Consider a quadratic system* (8) *in* $\mathbf{QSL_4}$. *Then this system has either a polynomial inverse integrating factor which splits into linear factors over* $\mathbb{C}$ *or an integrating factor which is Darboux generating in the usual way a Liouvillian first integral. Out of 46 cases, 26 lead to Darboux integrals which produce, depending on the values of the parameters, 30 Darboux integrals. In the remaining cases the first integral involves special functions such as for example Hypergeometric functions, or Appell or Beta functions, etc. Furthermore the quotient set of* $\mathbf{QSL_4}$ *under the action of the affine group and time rescaling is formed by: (i) a set of 20 orbits; (ii) a set of twenty-three one-parameter families of orbits and (iii) a set of ten two-parameter families of orbits. A system of representatives of the quotient space is given in Table 1. This table also lists the corresponding cofactors of the lines as well as the inverse integrating factors and first integrals of the systems.*

Proof of Theorem 6. In [27] we obtained a total of 46 canonical forms for all the systems in the class $\mathbf{QSL_4}$. They correspond to the 46 possible configurations

Diagram 1



$N_{\mathbb{C}}=4$
$m_{\mathcal{G}}=2$
$N_{\mathbb{R}}=4$
Config. 4.1
$N_{\mathbb{R}}=2$
Config. 4.2
$m_{\mathcal{G}}^{\infty}=2$
$N_{\mathbb{R}}=4$
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=4$
Config. 4.3
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=3$
Config. 4.4
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=1$
Config. 4.5
$N_{\mathbb{R}}=2$
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=2$
Config. 4.6
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=1$
$d_{\mathcal{G},\sigma}^{\mathbb{R}}=1$
Config. 4.7
$d_{\mathcal{G},\sigma}^{\mathbb{R}}=4$
Config. 4.8
$m_{\mathcal{G}}=3$
$m_{\mathcal{G}}^{\infty}=3$
$N_{\mathbb{R}}=4$
$n_{\mathbb{R}}^{\infty}=3$
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=4$
Config. 4.9
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=3$
Config. 4.10
$n_{\mathbb{R}}^{\infty}=2$
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=4$
Config. 4.11
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=2$
Config. 4.12
$N_{\mathbb{R}}=2$
$n_{\mathbb{R}}^{\infty}=3$
Config. 4.13
$n_{\mathbb{R}}^{\infty}=2$
$d_{\sigma}^{\infty}=0$
Config. 4.14
$d_{\sigma}^{\infty}=2$
Config. 4.15

Diagram 1 *(continued)*

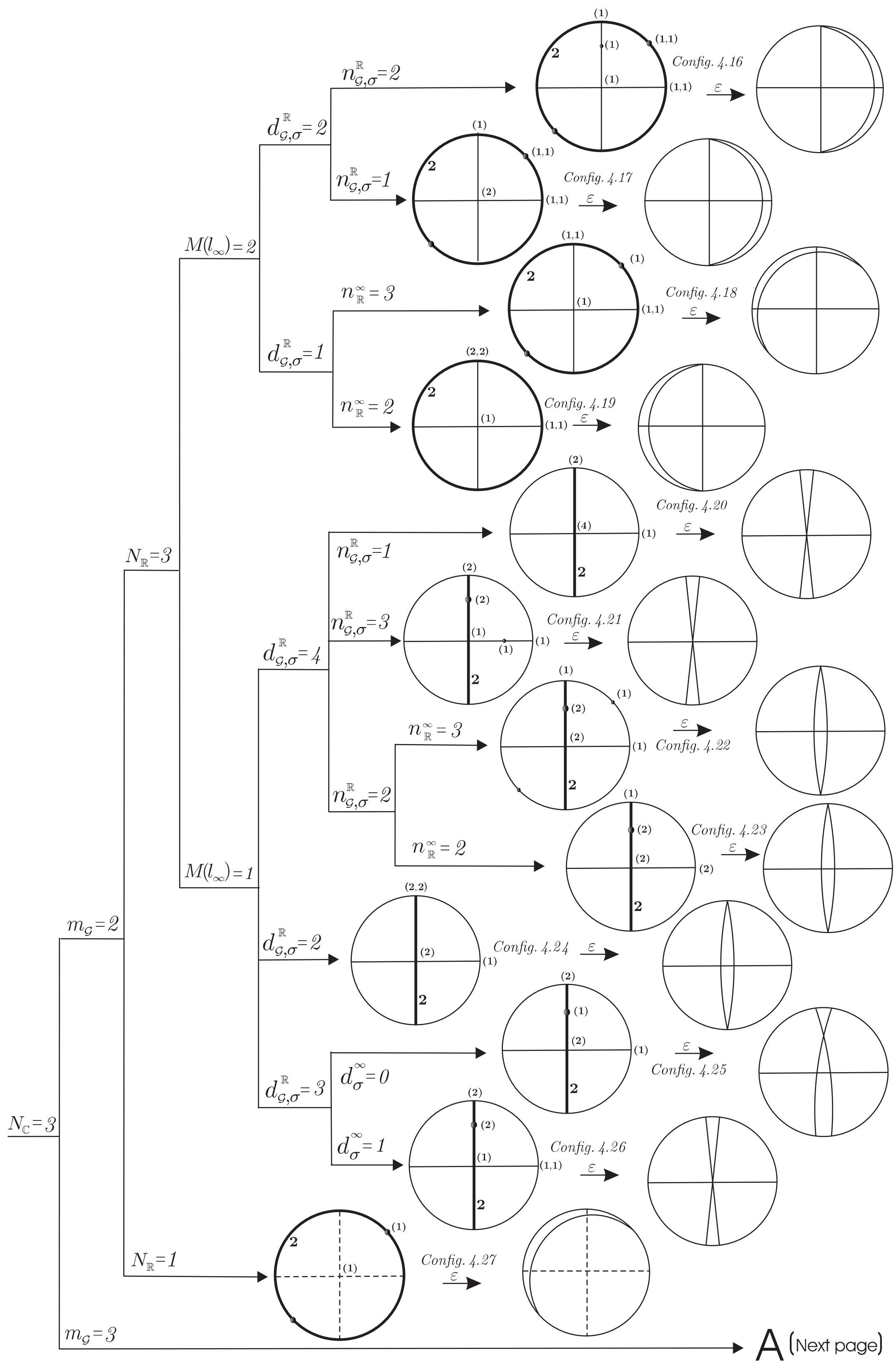


Diagram 1 *(continued)*

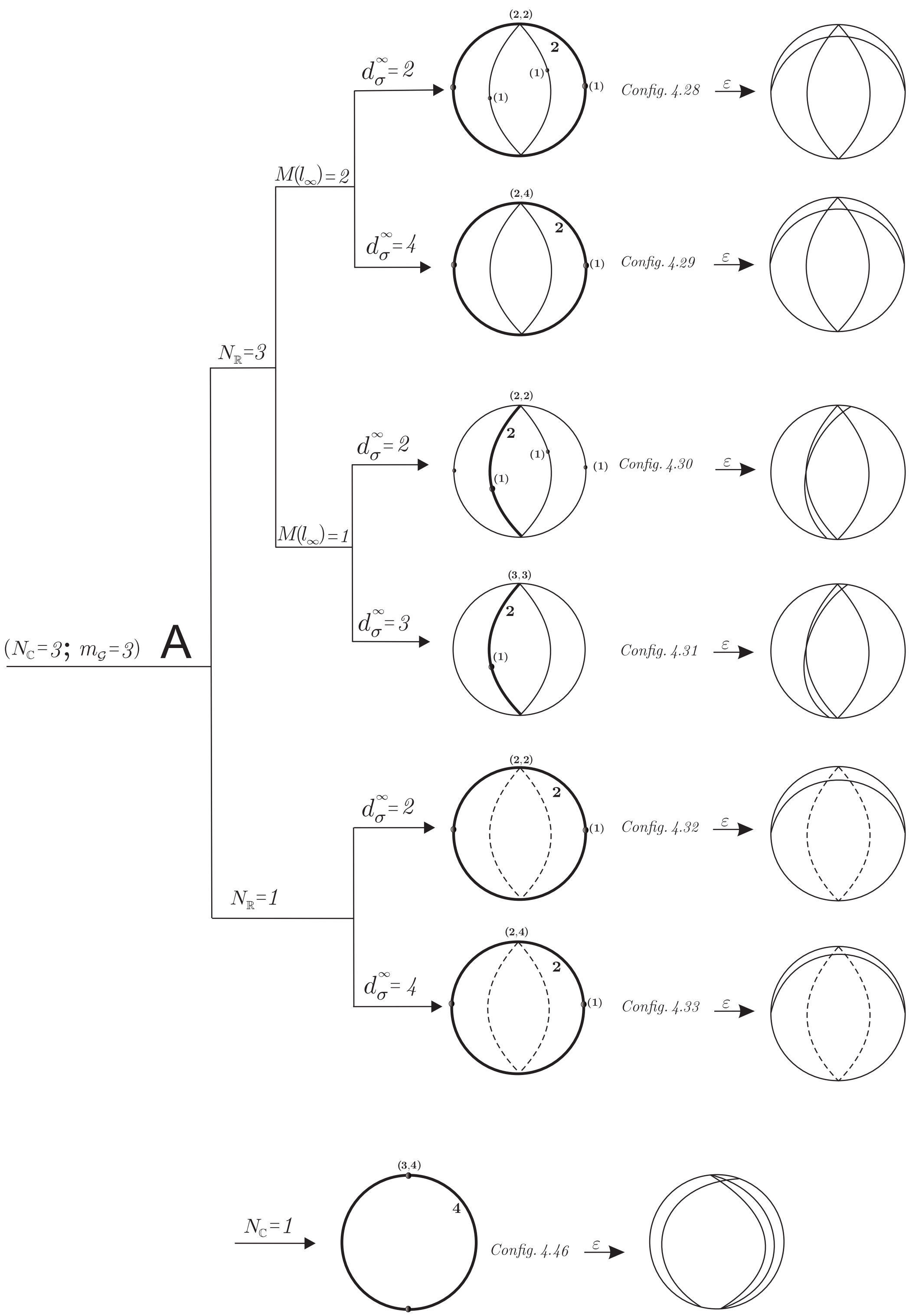
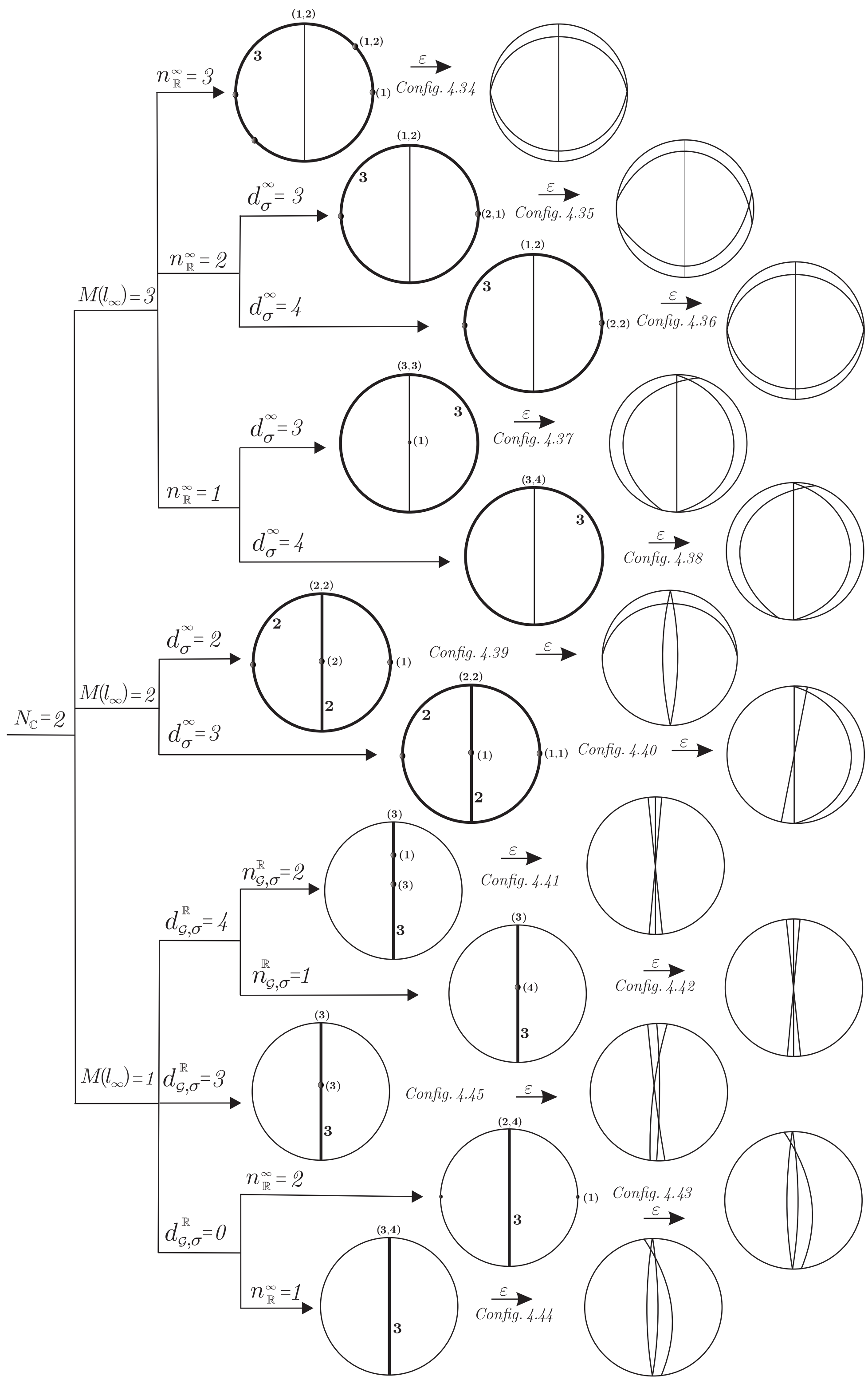


Diagram 1 *(continued)*



of invariant lines listed in Diagram 1. We take each one of these canonical forms, check their invariant lines with their respective multiplicities and determine their cofactors. As Darboux' work showed, these are instrumental in determining the integrating factors by showing linear dependence over $\mathbb{C}$ of the cofactors (of the invariant lines or of the exponential factors) together with the divergence of the vector field. Once the integrating factor is found one proceeds in the usual way to integrate the resulting differential equation (see Section 2). This integration can be done using MAPLE or MATHEMATICA. The calculations for the 46 cases considered yield the results given in **Table 1**.

$$\mathcal{F}_1 = x^h y^g (1+x-y)^{1-g-h};$$

$$\mathcal{F}_2 = x^{-2h}\Big[(x-h-1)+i(y+g)\Big]^{h+1-ig}\Big[(x-h-1)-i(y+g)\Big]^{h+1+ig};$$

$$\widetilde{\mathcal{F}}_2 = x^{-2h}\Big[(x-h-1)^2+(y+g)^2)\Big]^{h+1}\exp\Big[2g\,\mathrm{ArcTan}\,\frac{y+g}{x-h-1}\Big];$$

$$\mathcal{F}_3 = -x^h y^g (x-y)^{-(g+h)}\Big(\frac{y}{x}\Big)^{-g}\Big(1-\frac{y}{x}\Big)^{g+h}\Big[(1+gx)\,\mathrm{Beta}\Big[\frac{y}{x},\,g,\,1-g-h\Big]+$$

$$+(h-1)x\,\mathrm{Beta}\Big[\frac{y}{x},\,g+1,\,1-g-h\Big]+\int_{\omega_0}^{x}\Psi_3(\omega)d\omega,$$

$$\text{where}\quad \Psi_3(x) = x^{h-1}y^g(x-y)^{-(g+h)}\Bigg[y-x+x\Big(\frac{y}{x}\Big)^{-g}\Big(1-\frac{y}{x}\Big)^{g+h}$$

$$\Big[g\,\mathrm{Beta}\Big[\frac{y}{x},\,g,\,1-g-h\Big]+(h-1)\,\mathrm{Beta}\Big[\frac{y}{x},\,g+1,\,1-g-h\Big]\Big]\Bigg];\quad \frac{\partial}{\partial y}\Psi_3 = 0;$$

$$\mathcal{F}_4 = \Big(\frac{y}{x-y}\Big)^g\Big[g(x-y)+\Big(\frac{x-y}{x}\Big)^g\mathrm{Hypergeometric2F1}\Big[g,\,g,\,g+1,\,\frac{y}{x}\Big]\Big]\ (g\neq -1);$$

$$\widetilde{\mathcal{F}}_4 = xy^{-1}\exp\Big[\frac{(y-x)(y-x+1)}{y}\Big]\quad\text{for } g=-1;$$

$$\mathcal{F}_5 = x^h y^g (y-x)^{1-g-h};$$

$$\mathcal{F}_6 = -x^{-h}\int\mathcal{E}(x,y)\mathcal{H}(x,y)^{(h-1)/2}\big[gx+(h+1)y\big]dy+\int_{\omega_0}^{x}\Psi_6(\omega)d\omega,\ \text{where}$$

$$\Psi_6(x) = \mathcal{E}(x,y)\mathcal{H}(x,y)^{(h-1)/2}x^{-1-h}\big[(gx+hy-1)(y+1)-x^2\big]+$$

$$+x^{-h}\int\mathcal{E}(x,y)\mathcal{H}(x,y)^{(h-3)/2}\big[ghx^2+(h^2-1)xy-g(y+1)(gx+hy-1)\big]dy-$$

$$-hx^{-1-h}\int\mathcal{E}(x,y)\mathcal{H}(x,y)^{(h-1)/2}\big[gx+(h+1)y\big]dy,\qquad \frac{\partial}{\partial y}\Psi_6 = 0;$$

$$\mathcal{E}(x,y) = e^{-g\mathrm{ArcTan}[x/(1+y)]},\quad \mathcal{H}(x,y) = x^2+(y+1)^2;$$

$$\mathcal{F}_7 = \mathcal{F}_6\Big|_{h=0};$$

$$\mathcal{F}_8 = x^{-2h}(x+iy)^{h+1-ig}(x-iy)^{h+1+ig};$$

Table 1

Orbit representative	Invariant lines and their multiplicities / Respective cofactors	Inverse integrating factor $\mathcal{R}_i$ / First integral $\mathcal{F}_i$
1) $\begin{cases}\dot x = gx+gx^2+(h-1)xy,\\ \dot y=-hy+(g-1)xy+hy^2,\end{cases}$ $(g,h)\in\mathbb{R}^2,\ gh(g+h-1)\neq 0,$ $(g-1)(h-1)(g+h)\neq 0$	$x\,(1),\quad y\,(1),$ $x-y+1\,(1)$	$\mathcal{R}_1=xy(x-y+1)$
	$g(x+1)+y(h-1),$ $x(g-1)+h(y-1),$ $gx+hy$	$\mathcal{F}_1$
2) $\begin{cases}\dot x = gx^2+(h+1)xy,\\ \dot y=h[g^2+(h+1)^2]+2ghy-x^2\\ \quad +(g^2+1-h^2)x+gxy+hy^2,\end{cases}$ $(g,h)\in\mathbb{R}^2,\ h(h+1)\neq 0,$ $g^2+(h-1)^2\neq 0$	$x\,(1),\ \pm i(y+g)+$ $x-h-1\,(1)$	$\mathcal{R}_2=x\big[(y+g)^2+$ $(x-h-1)^2\big]$
	$gx+(h+1)y,$ $\mp i(x+h+h^2)+$ $g(x+h)+hy$	$\mathcal{F}_2,\ \widetilde{\mathcal{F}}_2$
3) $\begin{cases}\dot x = x+gx^2+(h-1)xy,\\ \dot y=y+(g-1)xy+hy^2,\end{cases}$ $(g,h)\in\mathbb{R}^2,\ gh(g+h-1)\neq 0,$ $(g-1)(h-1)(g+h)\neq 0$	$x\,(1),\quad y\,(1),$ $x-y\,(1)$	$\mathcal{R}_3=x^{1-h}y^{1-g}\times$ $(x-y)^{g+h}$
	$gx+1+y(h-1),$ $x(g-1)+hy+1,$ $gx+hy+1$	$\mathcal{F}_3$
4) $\begin{cases}\dot x = x+gx^2-xy,\\ \dot y=y+(g-1)xy,\end{cases}$ $g\in\mathbb{R},\ g(g-1)\neq 0$	$x\,(1),\ x-y\,(1),\ y\,(1)$	$\mathcal{R}_4=xy^{1-g}(x-y)^g$
	$gx+1-y,\ gx+1$ $x(g-1)+1$	$\mathcal{F}_4,\ \widetilde{\mathcal{F}}_4$
5) $\begin{cases}\dot x = gx^2+(h-1)xy,\\ \dot y=(g-1)xy+hy^2,\end{cases}$ $(g,h)\in\mathbb{R}^2,\ gh(g+h-1)\neq 0,$ $(g-1)(h-1)(g+h)\neq 0$	$x\,(1),\ x-y\,(1),\ y\,(1)$	$\mathcal{R}_5=xy(x-y)$
	$gx+y(h-1), gx+hy,$ $x(g-1)+hy$	$\mathcal{F}_5$
6) $\begin{cases}\dot x = gx^2+(h+1)xy,\\ \dot y=-1+gx+(h-1)y\\ \quad -x^2+gxy+hy^2,\ (g,h)\in\mathbb{R}^2,\end{cases}$ $h(h+1)\big[g^2+(h-1)^2\big]\neq 0$	$x\,(1),\ \mathcal{I}_\pm=x\pm$ $i(y+1)\,(1)$	$\mathcal{R}_6=\mathcal{I}_+^{(1-h-ig)/2}\times$ $\mathcal{I}_-^{(1-h+ig)/2}x^{h+1}$
	$gx+(h+1)y,$ $\pm ix+1+gx+hy$	$\mathcal{F}_6$
7) $\begin{cases}\dot x = gx^2+xy,\quad g\in\mathbb{R},\\ \dot y=-1+gx-y-x^2+gxy\end{cases}$	$x\,(1),\ \mathcal{I}'_\pm=x\pm$ $i(y+1)\,(1)$	$\mathcal{R}_7={\mathcal{I}'_+}^{(1-ig)/2}\times$ ${\mathcal{I}'_-}^{(1+ig)/2}\ x$
	$gx+y,$ $\pm ix+1+gx$	$\mathcal{F}_7$
8) $\begin{cases}\dot x = gx^2+(h+1)xy,\ (g,h)\in\mathbb{R}^2,\\ \dot y=-x^2+gxy+hy^2,\end{cases}$ $h(h+1)\big[g^2+(h-1)^2\big]\neq 0$	$x\,(1),\ x\pm iy\,(1)$	$\mathcal{R}_8=x(x^2+y^2)$
	$gx+(h+1)y,$ $\pm ix+gx+hy$	$\mathcal{F}_8,\ \widetilde{\mathcal{F}}_8$

Table 1 (*continued*)

Orbit representative	Invariant lines and their multiplicities Respective cofactors	Inverse integrating factor $\mathcal{R}_i$ First integral $\mathcal{F}_i$
9) $\begin{cases}\dot{x}=x^2-1,\ (g,h)\in\mathbb{R}^2,\\ \dot{y}=(y+h)[y+(1-g)x-h],\end{cases}$ $g(g-1)[(g\pm1)^2-4h^2]\neq0$	$y+h\,(1)$, $\mathcal{I}''_{\pm}=x\pm1\,(1)$	$\mathcal{R}_9=(y+h)^2\times \mathcal{I}_+''^{\,(g+1-2h)/2}\times \mathcal{I}_-''^{\,(g+1+2h)/2}$
	$x(1-g)+y-h$, $x\mp1$	$\mathcal{F}_9,\ \widetilde{\mathcal{F}}_9,\ \widehat{\mathcal{F}}_9,\ \mathcal{F}_9^*$
10) $\begin{cases}\dot{x}=x^2-1,\quad g\in\mathbb{R},\\ \dot{y}=(y+g)(y+2gx-g),\end{cases}$ $g(2g-1)\neq0$	$y+g\,(1)$, $x\pm1\,(1)$	$\mathcal{R}_{10}=(x+1)^{1-2g}\times (y+h)^2(x-1)$
	$2gx+y-g$, $x\mp1$	$\mathcal{F}_{10},\ \widetilde{\mathcal{F}}_{10}$
11) $\begin{cases}\dot{x}=(x+g)^2-1,\quad g\in\mathbb{R},\\ \dot{y}=y(x+y),\qquad g\neq\pm1\end{cases}$	$y\,(1)$, $\mathcal{I}'''_{\pm}=x+ +g\pm1\,(1)$	$\mathcal{R}_{11}=\mathcal{I}_+'''^{\,(1-g)/2}\times \mathcal{I}_-'''^{\,(1+g)/2}y^2$
	$x+y$, $x+g\mp1$	$\mathcal{F}_{11},\ \widetilde{\mathcal{F}}_{11}$
12) $\begin{cases}\dot{x}=(x+h)^2-1,\ (g,h)\in\mathbb{R}^2,\\ \dot{y}=(1-g)xy,\ g(g-1)\neq0,\end{cases}$ $(h^2-1)\big[\,h^2(g-1)^2-(g+1)^2\big]\neq0$	$y\,(1)$, $x+h\pm1\,(1)$	$\mathcal{R}_{12}=(x+h+1)\times (x+h-1)y$
	$x+y$, $x+h\mp1$	$\mathcal{F}_{12}$
13) $\begin{cases}\dot{x}=x^2+1,\quad (g,h)\in\mathbb{R}^2,\\ \dot{y}=(y+h)[y+(1-g)x-h],\end{cases}$ $g(g-1)\big[(g+1)^2+h^2\big]\neq0$	$y+h\,(1)$, $x\pm i\,(1)$	$\mathcal{R}_{13}=(y+h)^2\times (x+i)^{(1+g+2ih)/2}\times (x-i)^{(1+g-2ih)/2}$
	$x(1-g)+y-h$, $x\mp i$	$\mathcal{F}_{13},\ \widetilde{\mathcal{F}}_{13}$
14) $\begin{cases}\dot{x}=(x+g)^2+1,\\ \dot{y}=y(x+y),\qquad g\in\mathbb{R}\end{cases}$	$y\,(1)$, $x+g\pm i\,(1)$	$\mathcal{R}_{14}=y^2\times (x+g+i)^{(1+ig)/2}\times (x+g-i)^{(1-ig)/2}$
	$x+y$, $x+g\mp i$	$\mathcal{F}_{14},\ \widetilde{\mathcal{F}}_{14}$
15) $\begin{cases}\dot{x}=(x+h)^2+1,\\ \dot{y}=(1-g)xy,\quad (g,h)\in\mathbb{R}^2,\end{cases}$ $g(g-1)\big[(g+1)^2+h^2\big]\neq0$	$y\,(1)$, $x+h\pm i$, (1)	$\mathcal{R}_{15}=y\times \big[(x+h)^2+1\big]$
	x, $x+h\mp i$	$\mathcal{F}_{15},\ \widetilde{\mathcal{F}}_{15}$
16) $\begin{cases}\dot{x}=g+x,\qquad g\in\mathbb{R},\\ \dot{y}=y(y-x),\quad g(g-1)\neq0\end{cases}$	$x+g\,(1)$, $y\,(1)$	$\mathcal{R}_{16}=e^xy^2\times (x+g)^{1-g}$
	1, $y-x$	$\mathcal{F}_{16}$
17) $\begin{cases}\dot{x}=x,\\ \dot{y}=y(y-x)\end{cases}$	$x\,(1)$, $y\,(1)$	$\mathcal{R}_{17}=x\,e^xy^2$
	1, $y-x$	$\mathcal{F}_{17}$
18) $\begin{cases}\dot{x}=g(g+1)+gx+y,\ g\in\mathbb{R},\\ \dot{y}=y(y-x),\ g(g+1)\neq0\end{cases}$	$y\,(1)$, $x-y+g+1\,(1)$	$\mathcal{R}_{18}=y\times (x-y+g+1)$
	$y-x$, $y+g$	$\mathcal{F}_{18}$

Table 1 (*continued*)

Orbit representative	Invariant lines and their multiplicities / Respective cofactors	Inverse integrating factor $\mathcal{R}_i$ / First integral $\mathcal{F}_i$
19) $\begin{cases}\dot x = g + x, \quad g \in \mathbb{R},\\ \dot y = -xy, \quad g(g-1) \neq 0\end{cases}$	$x+g\,(1),\quad y\,(1)$	$\mathcal{R}_{19} = y(x+g)$
	$1,\quad -x$	$\mathcal{F}_{19}$
20) $\begin{cases}\dot x = x(gx+y), \quad g \in \mathbb{R},\\ \dot y = (g-1)xy+y^2,\ g(g-1) \neq 0\end{cases}$	$x\,(2),\quad y\,(1)$	$\mathcal{R}_{20} = x^2 y$
	$gx+y,\ x(g-1)+y$	$\mathcal{F}_{20}$
21) $\begin{cases}\dot x = x(gx+y),\ g(g-1) \neq 0,\\ \dot y = (y+1)(gx-x+y),\ g \in \mathbb{R}\end{cases}$	$x\,(2),\quad y+1\,(1)$	$\mathcal{R}_{21} = x^{g+1} \times e^{-(gx+y+1)/x} \times (y+1)^{1-g}$
	$gx+y,\ x(g-1)+y$	$\mathcal{F}_{21},\ \widetilde{\mathcal{F}}_{21}$
22) $\begin{cases}\dot x = gx^2,\ g \in \mathbb{R},\ g(g-1) \neq 0,\\ \dot y = (y+1)[y+(g-1)x-1]\end{cases}$	$x\,(2),\quad y+1\,(1)$	$\mathcal{R}_{22} = x^{(g+1)/g} \times (y+1)^2 e^{-2/(gx)}$
	$x,\quad x(g-1)+y-1$	$\mathcal{F}_{22}$
23) $\begin{cases}\dot x = x^2+xy,\\ \dot y = (y+1)^2\end{cases}$	$x\,(1),\quad y+1\,(2)$	$\mathcal{R}_{23} = x^2(y+1) \times e^{-1/(y+1)}$
	$x+y,\ y+1$	$\mathcal{F}_{23}$
24) $\begin{cases}\dot x = (x+1)^2, \quad g \in \mathbb{R},\\ \dot y = (1-g)xy,\ g(g-1) \neq 0\end{cases}$	$x+1\,(2),\quad y\,(1)$	$\mathcal{R}_{24} = (x+1)^2\, y$
	$x+1,\quad x$	$\mathcal{F}_{24}$
25) $\begin{cases}\dot x = gx^2+xy, \quad g(g-1) \neq 0,\\ \dot y = y+(g-1)xy+y^2,\ g \in \mathbb{R}\end{cases}$	$x\,(2),\quad y\,(1)$	$\mathcal{R}_{25} = x^2\, y$
	$gx+y,\ x(g-1)+y+1$	$\mathcal{F}_{25}$
26) $\begin{cases}\dot x = xy,\\ \dot y = (y+1)(y-x)\end{cases}$	$x\,(2),\quad y\,(1)$	$\mathcal{R}_{26} = x(y+1) \times e^{-(y+1)/x}$
	$y,\quad y-x$	$\mathcal{F}_{26}$
27) $\begin{cases}\dot x = 2gx+2y, \quad g \in \mathbb{R},\\ \dot y = g^2+1-x^2-y^2\end{cases}$	$y+g \pm i(x-1)\,(1)$	$\mathcal{R}_{27} = (x-1)^2 + (y+g)^2$
	$g-y \pm i(x+1)$	$\mathcal{F}_{27},\ \widetilde{\mathcal{F}}_{27}$
28) $\begin{cases}\dot x = x^2-1, \quad g \in \mathbb{R},\\ \dot y = x+gy, \quad g(g^2-4) \neq 0\end{cases}$	$x+1\,(1),\ x-1\,(1)$	$\mathcal{R}_{28} = (x-1)^{1+g/2} \times (x+1)^{1-g/2}$
	$x-1,\ x+1$	$\mathcal{F}_{28}$
29) $\begin{cases}\dot x = x^2-1, \quad g \in \mathbb{R},\\ \dot y = g+x, \quad g \neq \pm 1\end{cases}$	$x+1\,(1),\ x-1\,(1)$	$\mathcal{R}_{29} = x^2-1$
	$x-1,\ x+1$	$\mathcal{F}_{29}$
30) $\begin{cases}\dot x = (x+1)(gx+1),\ g \in \mathbb{R},\\ \dot y = 1+(g-1)xy,\ g(g^2-1) \neq 0\end{cases}$	$x+1\,(2),\ gx+1\,(1)$	$\mathcal{R}_{30} = (x+1)^2 \times (gx+1)^{(g-1)/g}$
	$gx+1,\ x+1$	$\mathcal{F}_{30},\ \widetilde{\mathcal{F}}_{30},\ \widehat{\mathcal{F}}_{30}$

Table 1 (*continued*)

Orbit representative	Invariant lines and their multiplicities	Inverse integrating factor $\mathcal{R}_i$
	Respective cofactors	First integral $\mathcal{F}_i$
31) $\begin{cases}\dot{x}=x(x+1), & g\in\mathbb{R},\\ \dot{y}=g-x^2+xy, & g(g+1)\neq 0\end{cases}$	$x+1\,(2),\quad x\,(1)$	$\mathcal{R}_{31}=x(x+1)^2$
	$x,\quad x+1$	$\mathcal{F}_{31}$
32) $\begin{cases}\dot{x}=x^2+1, & g\in\mathbb{R},\\ \dot{y}=x+gy, & g\neq 0\end{cases}$	$x\pm i\,(1)$	$\mathcal{R}_{32}=(x+i)^{1+ig/2}\times (x-i)^{1-ig/2}$
	$x\mp i$	$\mathcal{F}_{32}$
33) $\begin{cases}\dot{x}=x^2+1, & g\in\mathbb{R},\\ \dot{y}=g+x\end{cases}$	$x\pm i\,(1)$	$\mathcal{R}_{33}=x^2+1$
	$x\mp i$	$\mathcal{F}_{33},\ \widetilde{\mathcal{F}}_{33}$
34) $\begin{cases}\dot{x}=g, & g\in\{-1,1\},\\ \dot{y}=y(y-x)\end{cases}$	$y\,(1)$	$\mathcal{R}_{34}=y^2\,e^{x^2/(2g)}$
	$y-x$	$\mathcal{F}_{34}$
35) $\begin{array}{l}\dot{x}=g+y,\\ \dot{y}=xy,\quad g\in\{-1,1\}\end{array}$	$y\,(1)$	$\mathcal{R}_{35}=y$
	x	$\mathcal{F}_{35}$
36) $\begin{array}{l}\dot{x}=g,\\ \dot{y}=xy,\quad g\in\{-1,1\}\end{array}$	$y\,(1)$	$\mathcal{R}_{36}=y$
	x	$\mathcal{F}_{36}$
37) $\begin{cases}\dot{x}=x, & g(g^2-1)\neq 0\\ \dot{y}=gy-x^2, & g\in\mathbb{R}\end{cases}$	$x\,(1)$	$\mathcal{R}_{37}=x^{g+1}$
	1	$\mathcal{F}_{37},\ \widetilde{\mathcal{F}}_{37}$
38) $\begin{array}{l}\dot{x}=x,\\ \dot{y}=g-x^2,\quad 0\neq g\in\mathbb{R}\end{array}$	$x\,(1)$	$\mathcal{R}_{38}=x$
	1	$\mathcal{F}_{38}$
39) $\dot{x}=x^2,\quad \dot{y}=x+y$	$x\,(2)$	$\mathcal{R}_{39}=x^2e^{-1/x}$
	x	$\mathcal{F}_{39}$
40) $\dot{x}=1+x,\quad \dot{y}=1-xy$	$x+1\,(2)$	$\mathcal{R}_{40}=(x+1)^2e^{-x}$
	1	$\mathcal{F}_{40}$
41) $\begin{cases}\dot{x}=gxy, & g\in\{-1,1\}\\ \dot{y}=y-x^2+gy^2\end{cases}$	$x\,(3)$	$\mathcal{R}_{41}=x^2e^{-g(y+g)^2/(2x^2)}$
	y	$\mathcal{F}_{41}$
42) $\begin{cases}\dot{x}=gxy, & g\in\{-1,1\}\\ \dot{y}=-x^2+gy^2\end{cases}$	$x\,(3)$	$\mathcal{R}_{42}=x^3$
	y	$\mathcal{F}_{42}$
43) $\begin{cases}\dot{x}=gx^2, & g(g^2-1)\neq 0\\ \dot{y}=1+(g-1)xy, & g\in\mathbb{R}\end{cases}$	$x\,(3)$	$g\neq\frac{1}{2}:\ \mathcal{R}_{43}=x^{2g}\times\big[1+(2g-1)xy\big]^{1-g};$ $g=\frac{1}{2}:\ \widetilde{\mathcal{R}}_{43}=x^3e^{-xy}$
	x	$\mathcal{F}_{43},\ \widetilde{\mathcal{F}}_{43}$

Table 1 (*continued*)

Orbit representative	Invariant lines and their multiplicities / Respective cofactors	Inverse integrating factor $\mathcal{R}_i$ / First integral $\mathcal{F}_i$
44) $\begin{cases}\dot{x} = x^2, \quad g \in \{-1, 1\} \\ \dot{y} = g - x^2 + xy\end{cases}$	$x\,(3)$	$\mathcal{R}_{44} = x^3$
	x	$\mathcal{F}_{44}$
45) $\begin{cases}\dot{x} = gxy, \quad g \in \{-1, 1\} \\ \dot{y} = x - x^2 + gy^2\end{cases}$	$x\,(3)$	$\mathcal{R}_{45} = x^3$
	y	$\mathcal{F}_{45}$
46) $\dot{x} = 1, \quad \dot{y} = y - x^2$	$-$	$\mathcal{R}_{46} = e^x$
	$-$	$\mathcal{F}_{46}$

$$\widetilde{\mathcal{F}}_8 = x^{-2h}(x^2+y^2)^{h+1}\exp\Big[2g\,\mathrm{ArcTan}\,\frac{y}{x}\Big];$$

$$\mathcal{F}_9 = (y+h)^{-1}(x^2-1)^{(1-g)/2}e^{2h\,\mathrm{ArcTanh}[x]} + \int_{\omega_0}^{x} e^{2h\,\mathrm{ArcTanh}[\omega]}(\omega^2-1)^{-(g+1)/2}d\omega,$$

$$\text{if} \quad h(g+1)\neq 0;$$

$$\widetilde{\mathcal{F}}_9 = \frac{(x+1)^h(x^2-1)}{(x-1)^h(y+h)} + 2\frac{(1-x)^h}{(x-1)^h}\,\mathrm{Beta}\Big[\frac{x+1}{2},\, h+1,\, 1-h\Big], \quad \text{for } \begin{cases} g=-1, \\ h\neq -1 \end{cases};$$

$$\widehat{\mathcal{F}}_9 = (x+1)^{-2}\,\exp\Big[x+\frac{(x-1)^2}{y-1}\Big], \quad \text{for } g=h=-1;$$

$$\mathcal{F}_9^* = \frac{(x^2-1)^{(1-g)/2}}{y} + \frac{x(1-x^2)^{(1+g)/2}}{(x^2-1)^{(1+g)/2}}\,\mathrm{Hypergeometric2F1}\Big[\frac{1}{2},\, \frac{g+1}{2},\, \frac{3}{2},\, x^2\Big],$$

$$\text{for } h=0;$$

$$\mathcal{F}_{10} = (y+g)^{-1}(x^2-1)^g e^{2g\,\mathrm{ArcTanh}[x]} + \int_{\omega_0}^{x} e^{2g\,\mathrm{ArcTanh}[\omega]}(\omega^2-1)^{g-1}d\omega, \quad (g\neq 1);$$

$$\widetilde{\mathcal{F}}_{10} = (x-1)^2\,\exp\Big[x+\frac{(x+1)^2}{y+1}\Big], \quad \text{for } g=1;$$

$$\mathcal{F}_{11} = y^{-1}(x+g+1)^{1/2}(x+g-1)^{-1/2}e^{g\,\mathrm{ArcTanh}[x+g]}\Big[(g+1)(x+g-1)$$

$$+\,2y\,\mathrm{Hypergeometric2F1}\Big[1,\, \frac{g+1}{2},\, \frac{g+3}{2},\, \frac{x+g+1}{x+g-1}\Big]\Big], \quad (g\neq -3);$$

$$\widetilde{\mathcal{F}}_{11} = (x-2)\exp\Big[\frac{(x-4)^2+2y}{y(x-2)}\Big], \quad \text{for } g=-3;$$

$$\mathcal{F}_{12} = y^{2/(g-1)}(x+h+1)^{1+h}(x+h-1)^{1-h};$$

$$\mathcal{F}_{13} = (x^2+1)^{(1-g)/2}(y+h)^{-1}\exp\big[-2h\mathrm{ArcTan}\,x\big]+$$
$$\int_{\omega_0}^{x}(\omega^2{+}1)^{-(1+g)/2}\exp\big[-2h\,\mathrm{ArcTan}\,\omega\big]d\omega,\ \ (h{\neq}0);$$
$$\widetilde{\mathcal{F}}_{13} = y^{-1}\,(x^2+1)^{(1-g)/2}+x\ \mathrm{Hypergeometric2F1}\Big[\frac{1}{2},\ \frac{g+1}{2},\ \frac{3}{2},-x^2\Big],\quad \text{for } h=0;$$
$$\mathcal{F}_{14} = y^{-1}\,\big[1+(x+g)^2\big]^{1/2}\ \exp\big[-g\,\mathrm{ArcTan}\,[g+x]\big]$$
$$+\int_{\omega_0}^{x}\big[1+(\omega+g)^2\big]^{-1/2}\ \exp\big[-g\,\mathrm{ArcTan}\,[g+\omega]\big]\,d\omega,\quad (g\neq 0);$$
$$\widetilde{\mathcal{F}}_{14} = y^{-1}\,(x^2+1)^{1/2}+\mathrm{ArcSinh}[x],\quad \text{for } g=0;$$
$$\mathcal{F}_{15} = y^{2/(g-1)}(x+h+i)^{1-ih}(x+h-i)^{1+ih};$$
$$\widetilde{\mathcal{F}}_{15} = y\big[(h+x)^2+1\big]^{(g-1)/2}\ \exp\big[h(1-g)\,\mathrm{ArcTan}[x+h]\big];$$
$$\mathcal{F}_{16} = -(g+x)^g y^{-1}e^{-x}+e^g\mathrm{Gamma}[g,g+x];$$
$$\mathcal{F}_{17} = y^{-1}e^{-x}+\mathrm{ExpIntegralEi}[-x];$$
$$\mathcal{F}_{18} = e^x y^g(x-y+g+1)^{-g-1};$$
$$\mathcal{F}_{19} = e^x y(x+g)^{-g};$$
$$\mathcal{F}_{20} = x^{1-g}\,y^g\,e^{y/x};$$
$$\mathcal{F}_{21} = xe^{(gx+y+1)/x}\Big(\frac{y+1}{-x}\Big)^g+\int_{\omega_0}^{-x/(gx+y+1)}e^{-1/\omega}\omega^{-(1+g)}(g\omega+1)^{g-1}\,d\omega\ \ (g\neq -1);$$
$$\widetilde{\mathcal{F}}_{21} = \frac{x(x+1)}{y+1}\,e^{(y-x+1)/x}+e^{-1}\mathrm{ExpIntegralEi}\Big[1,\ -\frac{y+1}{x}\Big]\ \ \text{for } g=-1;$$
$$\mathcal{F}_{22} = g(y+1)^{-1}x^{(g-1)/g}e^{2/(gx)}+x^{-1/g}\Big(-\frac{2}{gx}\Big)^{-1/g}\mathrm{Gamma}\Big[\frac{1}{g},-\frac{2}{gx}\Big];$$
$$\mathcal{F}_{23} = (y+1)x^{-1}e^{1/(y+1)}-\mathrm{ExpIntegralEi}\Big[\frac{1}{y+1}\Big];$$
$$\mathcal{F}_{24} = (x+1)^{g-1}\,y\,e^{(g-1)/(x+1)};$$
$$\mathcal{F}_{25} = x^{1-g}\,y^g\,e^{(y+1)/x};$$
$$\mathcal{F}_{26} = e^{(y+1)/x}\,x+\mathrm{ExpIntegralE}\Big[1,-\frac{y+1}{x}\Big];$$
$$\mathcal{F}_{27} = e^x\,\big[y+g-i(x-1)\big]^{1-ig}\big[y+g+i(x-1)\big]^{1+ig};$$
$$\widetilde{\mathcal{F}}_{27} = \Big[(x-1)^2+(y+g)^2\Big]\exp\Big[x+2g\ \mathrm{arctg}\ \Big(\frac{y+g}{x-1}\Big)\Big];$$
$$\mathcal{F}_{28} = y\left(\frac{x-1}{x+1}\right)^{-g/2}+\frac{x^2}{4}(1-x)^{g/2}(x-1)^{-g/2}\times$$
$$\Big\{\mathrm{AppellF1}\big[2,\ 1+\frac{g}{2},\ -\frac{g}{2},\ 3,\ x,\ -x\big]+\mathrm{AppellF1}\big[2,\ \frac{g}{2},\ 1-\frac{g}{2},\ 3,\ x,\ -x\big]\Big\};$$

$$\mathcal{F}_{29} = e^{2y}\,(x-1)^{-1-g}\,(x+1)^{-1+g};$$

$$\mathcal{F}_{30} = y(x+1)^{-1}(gx+1)^{1/g} + \frac{1}{2g-1}(x+1)^{-2}(gx+g)^{1/g}\times$$
$$\text{Hypergeometric2F1}\Big[\frac{2g-1}{g},\frac{g-1}{g},\frac{3g-1}{g},\frac{g-1}{g(x+1)}\Big], \quad (g\neq 1/2, 1/3);$$

$$\widetilde{\mathcal{F}}_{30} = (x+1)^{-2}\exp\Big[\frac{(x+2)^2y+2}{x+1}\Big], \quad \text{for} \quad g=1/2;$$

$$\widehat{\mathcal{F}}_{30} = (x+1)^{-4}\exp\Big[-x+\frac{(x+3)^3y+12}{3(x+1)}\Big], \quad \text{for} \quad g=1/3;$$

$$\mathcal{F}_{31} = x^{-g}(x+1)^{1+g}\ \exp\Big[\frac{y-g+1}{x+1}\Big];$$

$$\mathcal{F}_{32} = y\,e^{-g\,\text{ArcTan}[x]} - \int_{\omega_0}^{x} e^{-g\,\text{ArcTan}[\omega]}\frac{\omega}{\omega^2+1}d\omega;$$

$$\mathcal{F}_{33} = e^{-2y}(x-i)^{1-ig}(x+i)^{1+ig};$$

$$\widetilde{\mathcal{F}}_{33} = (x^2+1)^{-1}\exp\Big[2(y+g\ \text{ArcTan}\frac{1}{x})\Big];$$

$$\mathcal{F}_{34} = y^{-1}\exp\big[-\frac{x^2}{2g}\big] + \frac{\sqrt{\pi}}{\sqrt{2g}}\ \text{Erf}\big[\frac{x}{\sqrt{2g}}\big];$$

$$\mathcal{F}_{35} = y^{-2g}\exp\big[x^2-2y\big];$$

$$\mathcal{F}_{36} = y\ \exp\big[-x^2/2g\big];$$

$$\mathcal{F}_{37} = x^{-g}\big[x^2+(2-g)y\big] \quad (g\neq 2);$$

$$\widetilde{\mathcal{F}}_{37} = y\ \exp\big[y/x^2\big] \quad \text{for} \quad g=2;$$

$$\mathcal{F}_{38} = x^{-2g}\exp\big[x^2+2y\big];$$

$$\mathcal{F}_{39} = e^{1/x}\,y + \text{ExpIntegralEi}\Big[\frac{1}{x}\Big];$$

$$\mathcal{F}_{40} = e^{x+1}(x+1)^{-1}\,(y+1) - \text{ExpIntegralEi}\big[1+x\big];$$

$$\mathcal{F}_{41} = x\sqrt{-g}\ \exp\Big[\frac{(gy+1)^2}{2gx^2}\Big] + \sqrt{\pi/2}\ \text{Erf}\Big[\frac{gy+1}{\sqrt{-2g}\,x}\Big];$$

$$\mathcal{F}_{42} = x^{2/g}\ e^{y^2/x^2};$$

$$\mathcal{F}_{43} = x^{1-2g}\big[1+(2g-1)xy\big]^g, \quad (g\neq 1/2);$$

$$\widetilde{\mathcal{F}}_{43} = x^{-2}\,e^{xy}, \quad \text{for } g=1/2;$$

$$\mathcal{F}_{44} = x^2\,e^{(g+2xy)/x^2};$$

$$\mathcal{F}_{45} = x^2\,e^{(2x+gy^2)/x^2};$$

$$\mathcal{F}_{46} = e^{-x}\big[(x+1)^2-y+1\big].$$

5 Phase portraits

In order to construct the phase portraits corresponding to quadratic systems given by Table 1 we use the configurations of invariant straight lines already established in [27] as well as the CT-comitants constructed in [25] and [27] as follows.

Consider the polynomial $\Phi_{\alpha,\beta} = \alpha P + \beta Q \in \mathbb{R}[a, X, Y, Z, \alpha, \beta]$ where $P = Z^2 p(X/Z, Y/Z)$, $Q = Z^2 q(X/Z, Y/Z)$, $p,\ q \in \mathbb{R}[a, x, y]$ and $\max(\deg_{(x,y)} p, \deg_{(x,y)} q) = 2$. Then

$$\begin{aligned}\Phi_{\alpha,\beta} =\ & c_{11}(\alpha,\beta)X^2 + 2c_{12}(\alpha,\beta)XY + c_{22}(\alpha,\beta)Y^2 + 2c_{13}(\alpha,\beta)XZ\\ &+ 2c_{23}(\alpha,\beta)YZ + c_{33}(\alpha,\beta)Z^2,\\ \Delta_3(a,\alpha,\beta) =\ & \det ||c_{ij}(\alpha,\beta)||_{i,j\in\{1,2,3\}}, \quad \Delta_2(a,\alpha,\beta) = \det ||c_{ij}(\alpha,\beta)||_{i,j\in\{1,2\}}.\end{aligned}$$

Using the differential operator $(f,g)^{(k)} = \sum_{h=0}^{k}(-1)^h \binom{k}{h} \dfrac{\partial^k f}{\partial x^{k-h}\partial y^h}\ \dfrac{\partial^k g}{\partial x^h \partial y^{k-h}}$ which is called *transvectant of index k of (f,g)*, $f,\ g \in \mathbb{R}[a,x,y]$ (cf.[13],[18]) we shall construct the following needed invariant polynomials:

$$\begin{aligned}
C_i(a,x,y) &= yp_i(a,x,y) - xq_i(a,x,y),\ \ i = 0,1,2;\\
D_i(a,x,y) &= \frac{\partial}{\partial x}p_i(a,x,y) + \frac{\partial}{\partial y}q_i(a,x,y),\ \ i = 1,2;\\
D(a,x,y) &= 4\Delta_3(a,-y,x);\\
B_3(a,x,y) &= (C_2,D)^{(1)} = Jacob\,(C_2,D)\,,\\
B_2(a,x,y) &= (B_3,B_3)^{(2)} - 6B_3(C_2,D)^{(3)},\\
B_1(a) &= \mathrm{Res}_{\,x}\,(C_2,D)\,/y^9 = -2^{-9}3^{-8}\,(B_2,B_3)^{(4)}\,,\\
M(a,x,y) &= (C_2,C_2)^{(2)} \ = \ 2\,\mathrm{Hess}\,\big(C_2(a,x,y)\big);\\
\eta(a) &= \mathrm{Discriminant}\,\big(C_2(a,x,y)\big);\\
K(a,x,y) &= \big(p_2,q_2\big)^{(1)} \ = \ \mathrm{Jacob}\,\big(p_2,q_2\big);\\
\mu(a) &= \mathrm{Res}_{\,x}(p_2,q_2)/y^4 \ = \ \mathrm{Discriminant}\,\big(K(a,x,y)\big)/16;\\
H(a,x,y) &= 4\Delta_2(a,-y,x);\\
N(a,x,y) &= K(a,x,y) + H(a,x,y);\\
\theta(a) &= \mathrm{Discriminant}\,\big(N(a,x,y)\big);\\
H_1(a) &= -\big(((C_2,C_2)^{(2)},C_2)^{(1)},D\big)^{(3)};\\
H_2(a,x,y) &= (C_1,\ 2H-N)^{(1)} - 2D_1N;\\
H_3(a,x,y) &= (C_2,D)^{(2)};\\
H_4(a) &= \big((C_2,D)^{(2)},(C_2,D_2)^{(1)}\big)^{(2)};\\
H_5(a) &= \big((C_2,C_2)^{(2)},(D,D)^{(2)}\big)^{(2)} + 8\big((C_2,D)^{(2)},(D,D_2)^{(1)}\big)^{(2)};\\
H_6(a,x,y) &= 16N^2(C_2,D)^{(2)} + H_2^2(C_2,C_2)^{(2)};
\end{aligned}$$

$$
\begin{aligned}
H_7(a) &= (N, C_1)^{(2)};\\
H_8(a) &= 9\big((C_2, D)^{(2)}, (D, D_2)^{(1)}\big)^{(2)} + 2\Big[(C_2, D)^{(3)}\Big]^2;\\
H_9(a) &= -\Big(\big((D, D)^{(2)}, D,\big)^{(1)} D\Big)^{(3)};\\
H_{10}(a) &= \big((N, D)^{(2)},\ D_2\big)^{(1)};\\
H_{11}(a, x, y) &= 8H\big[(C_2, D)^{(2)} + 8(D, D_2)^{(1)}\big] + 3H_2^2;\\
N_1(a, x, y) &= C_1(C_2, C_2)^{(2)} - 2C_2(C_1, C_2)^{(2)},\\
N_2(a, x, y) &= D_1(C_1, C_2)^{(2)} - \big((C_2, C_2)^{(2)}, C_0\big)^{(1)},\\
N_3(a, x, y) &= (C_2, C_1)^{(1)},\\
N_4(a, x, y) &= 4\,(C_2, C_0)^{(1)} - 3C_1 D_1,\\
N_5(a, x, y) &= \big[(D_2, C_1)^{(1)} + D_1 D_2\big]^2 - 4\big(C_2, C_2\big)^{(2)}\big(C_0, D_2\big)^{(1)},\\
N_6(a, x, y) &= 8D + C_2\Big[8(C_0, D_2)^{(1)} - 3(C_1, C_1)^{(2)} + 2D_1^2\Big].
\end{aligned}
$$

Remark 3. We note that by Discriminant (C_2) of the cubic form $C_2(a, x, y)$ we mean the expression given in Maple via the function "discrim$(C_2, x)/y^6$".

The CT-comitants indicated below (for detailed definitions of the notions involved see [26]) were constructed in [26] for the purpose of classifying the phase portraits in the vicinity of infinity of quadratic differential systems.

We consider the differential operator $\mathcal{L} = x \cdot \mathbf{L}_2 - y \cdot \mathbf{L}_1$ acting on $\mathbb{R}[a, x, y]$ constructed in [4], where

$$
\begin{aligned}
\mathbf{L}_1 &= 2a_{00}\frac{\partial}{\partial a_{10}} + a_{10}\frac{\partial}{\partial a_{20}} + \frac{1}{2}a_{01}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{10}} + b_{10}\frac{\partial}{\partial b_{20}} + \frac{1}{2}b_{01}\frac{\partial}{\partial b_{11}},\\
\mathbf{L}_2 &= 2a_{00}\frac{\partial}{\partial a_{01}} + a_{01}\frac{\partial}{\partial a_{02}} + \frac{1}{2}a_{10}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{01}} + b_{01}\frac{\partial}{\partial b_{02}} + \frac{1}{2}b_{10}\frac{\partial}{\partial b_{11}}.
\end{aligned}
$$

Then setting $\mu_0(a) = \mu(a) = \operatorname{Res}_x(p_2, q_2)/y^4$ we construct the following polynomials:

$$
\begin{aligned}
\mu_i(a, x, y) &= \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0),\ i = 1, .., 4;\\
\kappa(a) &= (M, K)^{(2)}/4;\\
\kappa_1(a) &= (M, C_1)^{(2)};\\
L(a, x, y) &= 4K(a, x, y) + 8H(a, x, y) - M(a, x, y);\\
R(a, x, y) &= L(a, x, y) + 8K(a, x, y);\\
K_1(a, x, y) &= p_1(x, y)q_2(x, y) - p_2(x, y)q_1(x, y);\\
K_2(a, x, y) &= 4\,Jacob(J_2, \xi) + 3\,Jacob(C_1, \xi)D_1 - \xi(16J_1 + 3J_3 + 3D_1^2);\\
K_3(a, x, y) &= 2C_2^2(2J_1 - 3J_3) + C_2(3C_0K - 2C_1J_4) + 2K_1(3K_1 - C_1D_2),
\end{aligned}
$$

where $\mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$ and $J_1 = Jacob(C_0, D_2)$, $J_2 = Jacob(C_0, C_2)$, $J_3 = Discrim(C_1)$, $J_4 = Jacob(C_1, D_2)$, $\xi = M - 2K$.

The local behavior of the trajectories in the neighborhood of a hyperbolic singular point (i.e. whose eigenvalues have non-zero real parts) is determined by the linearization of the system at this point (see for instance [14]). The simplest kind of singularities are: saddles, nodes, foci, centers and saddle–nodes. Their description can be found in most textbooks (see for example [1, Chapter IV]). We will call *anti–saddle* a singular point at which the linearization of the system has a matrix with positive determinant. In this case the singular point is either a node, or a focus or a center.

We shall use the following notations for a singular point $M_i(x_i, y_i)$:

$$\Delta_i = \left| \begin{array}{cc} p'_x(x,y) & p'_y(x,y) \\ q'_x(x,y) & q'_y(x,y) \end{array} \right|_{(x_i,y_i)}; \quad \rho_i = \big(p'_x(x,y) + q'_y(x,y)\big)\Big|_{(x_i,y_i)}; \quad \delta_i = \rho_i^2 - 4\Delta_i.$$

The following lemma is very useful for checking, in invariant form, conditions for existence of a center in terms of the coefficients of the systems (8) with $a_{00} = b_{00} = 0$, presented in the tensorial form:

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta, \quad (j, \alpha, \beta = 1, 2). \tag{10}$$

Here the notations $x^1 = x,\ x^2 = y,\ a^1_1 = a_{10}, \dots, a^2_{22} = b_{02}$ are used.

Lemma 7. [30] *The singular point* $(0,0)$ *of a quadratic system* (10) *is a center if and only if* $I_2 < 0$, $I_1 = I_6 = 0$ *and one of the following sets of conditions holds:*

$$1)\ I_3 = 0; \quad 2)\ I_{13} = 0; \quad 3)\ 5I_3 - 2I_4 = 13I_3 - 10I_5 = 0,$$

where

$$I_1 = a^\alpha_\alpha, \quad I_2 = a^\alpha_\beta a^\beta_\alpha, \quad I_3 = a^\alpha_p a^\beta_{\alpha q} a^\gamma_{\beta\gamma} \varepsilon^{pq}, \quad I_4 = a^\alpha_p a^\beta_{\beta q} a^\gamma_{\alpha\gamma} \varepsilon^{pq},$$
$$I_5 = a^\alpha_p a^\beta_{\gamma q} a^\gamma_{\alpha\beta} \varepsilon^{pq}, \quad I_6 = a^\alpha_p a^\beta_\gamma a^\gamma_{\alpha q} a^\delta_{\beta\delta} \varepsilon^{pq}, \quad I_{13} = a^\alpha_p a^\beta_{qr} a^\gamma_{\gamma s} a^\delta_{\alpha\beta} a^\mu_{\delta\mu} \varepsilon^{pq} \varepsilon^{rs}.$$

and the tensor ε *has the coordinates:* $\varepsilon^{12} = -\varepsilon^{21} = 1, \quad \varepsilon^{11} = \varepsilon^{22} = 0.$

To construct the phase portraits of quadratic systems possessing invariant lines of total multiplicity four we examine all the families, following step by step the canonical forms from **Table 1**. For the canonical systems corresponding to *Config. 4.i* we shall use the notation $(S_{4.i})$. To obtain the phase portraits we use the behavior of the vector fields on their invariant lines which can easily be established, as well as the behavior in the vicinity of infinity given by [26]. In general this information turns out to be sufficient. Whenever necessary we add extra arguments.

Theorem 8 (**Main Theorem**). *i) The total number of topologically distinct phase portraits in the class of quadratic differential systems with invariant lines of total multiplicity four is 69.*

ii) In Table 2 we give necessary and sufficient conditions, invariant with respect to the action of the affine group and time rescaling, for the realization of each one *of the phase portraits corresponding to the given configuration of invariant lines. More precisely the first column of Table 2 contains the list of all 46 configurations of invariant lines of total multiplicity four. In the second column we list the necessary and sufficient invariant conditions (obtained in [27]) for the realization of each configuration. The last column contains the names of the phase portraits. Whenever for a configuration Config. 4.i we have several phase portraits, we split the corresponding place in the last column into smaller boxes containing the names of these portraits. In the third column are listed the additional conditions needed for the realization of the corresponding phase portrait in the last column.*

Remark 4. Eleven of the 46 configurations from Diagram 1 produce each a unique phase portrait. Each one of the remaining 35 configurations produces several topologically distinct phase portraits. The total number of phase portraits thus obtained is 93 (see Tables 3(a)-3(d)). However only 69 of these phase portraits are topologically distinct. For example in the subclass with two real singularities at infinity (two pairs of opposite singularities on the Poincaré disk), the 38 cases of possible configurations of invariant lines lead to only 26 topologically distinct phase portraits.

Remark 5. a) In the subclass with one real and two complex singularities at infinity (two opposite singularities on the Poincaré disk), the 11 cases of possible configurations of invariant lines lead to 9 topologically distinct phase portraits.
b) In the subclass with only one singularity at infinity (real) (two opposite singularities on the Poincaré disk), the 16 cases of possible configurations of invariant lines lead to 15 topologically distinct phase portraits.
c) Some phase portraits in a) are topologically equivalent to portraits found in the case b) leading to a total of 18 topologically distinct phase portraits for the union of the two cases a) and b) (See Confrontation Table).

Proof of the Main Theorem. The first step in the proof is to construct the phase portrait *Picture 4.i* (or phase portraits *Picture 4.i(j)*, $j \in \{a, b, c, d, e\}$), $i \leq 46$, associated to a configuration *Config. 4.i.* This leads to 93 distinct such possibilities, with not all phase portraits topologically distinct. At the same time we also give necessary and sufficient conditions, invariant with respect to the action of the group for having each one of the 93 situations obtained. Here by situation we mean an ordered couple formed by a configuration and by one of the possible phase portraits associated to it. In the second part of the proof (see page 77) we look for topologically equivalent phase portraits appearing in the 93 cases and form the list of phase portraits which appear to be topologically distinct. Finally we show that the phase portraits in this list are indeed distinct.

We now proceed to the first step mentioned above.

$$\textit{Config. 4.1}: \quad \begin{cases} \dot{x} = gx + gx^2 + (h-1)xy, & (g-1)(h-1)(g+h) \neq 0, \\ \dot{y} = -hy + (g-1)xy + hy^2, & gh(g+h-1) \neq 0. \end{cases} \qquad (S_{4.1})$$

Finite singularities: $M_1(0,0)\big[\Delta_1 = -gh,\ \delta_1 = (g+h)^2\big]$; $M_2(0,1)\big[\Delta_2 = h(g+h-1),\ \delta_2 = (g-1)^2\big]$; $M_3(-1,0)\big[\Delta_3 = g(g+h-1),\ \delta_3 = (h-1)^2\big]$; $M_4(-h,g)\big[\Delta_4 = -gh(g+h-1),\ \delta_4 = 4gh(g+h-1)$.

Table 2

Configuration	*Necessary and sufficient conditions*	*Additional conditions for phase portraits*	*Phase portrait*
Config. 4.1	$\eta > 0,\ B_3 = 0,\ \theta \neq 0,\ H_7 \neq 0$	$\mu_0 > 0$	*Portrait 4.1(a)*
		$\mu_0 < 0,\ K < 0$	*Portrait 4.1(b)*
		$\mu_0 < 0,\ K > 0$	*Portrait 4.1(c)*
Config. 4.2	$\eta < 0,\ B_3 = 0,\ \theta \neq 0,\ H_7 \neq 0$	$\mu_0 > 0,\ \mathcal{G}_1 \neq 0$	*Portrait 4.2(a)*
		$\mu_0 > 0,\ \mathcal{G}_1 = 0$	*Portrait 4.2(b)*
		$\mu_0 < 0,\ \mathcal{G}_1 \neq 0$	*Portrait 4.2(c)*
		$\mu_0 < 0,\ \mathcal{G}_1 = 0$	*Portrait 4.2(d)*
Config. 4.3	$\eta > 0,\ B_3 = 0,\ \theta \neq 0,$ $H_7 = 0,\ H_1 \neq 0,\ \mu_0 \neq 0$	$\mu_0 > 0$	*Portrait 4.3(a)*
		$\mu_0 < 0,\ K < 0$	*Portrait 4.3(b)*
		$\mu_0 < 0,\ K > 0$	*Portrait 4.3(c)*
Config. 4.4	$\eta > 0,\ B_3 = 0,\ \theta \neq 0,$ $H_7 = 0,\ H_1 \neq 0,\ \mu_0 = 0$	$K < 0$	*Portrait 4.4(a)*
		$K > 0$	*Portrait 4.4(b)*
Config. 4.5	$\eta > 0,\ B_3 = 0,\ \theta \neq 0,$ $H_7 = 0,\ H_1 = 0$	$\mu_0 > 0$	*Portrait 4.5(a)*
		$\mu_0 < 0,\ K < 0$	*Portrait 4.5(b)*
		$\mu_0 < 0,\ K > 0$	*Portrait 4.5(c)*
Config. 4.6	$\eta < 0,\ B_3 = 0,\ \theta \neq 0,$ $H_7 = 0,\ \mu_0 \neq 0,\ H_9 \neq 0$	$\mu_0 > 0$	*Portrait 4.6(a)*
		$\mu_0 < 0$	*Portrait 4.6(b)*
Config. 4.7	$\eta < 0, B_3 = 0, \theta \neq 0,$ $H_7 = \mu_0 = 0$	–	*Portrait 4.7*
Config. 4.8	$\eta < 0,\ B_3 = 0,\ \theta \neq 0,$ $H_7 = 0,\ \mu_0 \neq 0,\ H_9 = 0$	$\mu_0 > 0$	*Portrait 4.8(a)*
		$\mu_0 < 0$	*Portrait 4.8(b)*
Config. 4.9	$\eta > 0,\ B_2 = \theta = H_7 = 0,$ $\mu_0 B_3 H_4 H_9 \neq 0$ *and either* $H_{10} N > 0$ *or* $N = 0,\ H_8 > 0$	$\mathcal{G}_2 > 0, H_4 > 0, \mathcal{G}_3 < 0$	*Portrait 4.9(a)*
		$\mathcal{G}_2 < 0$	*Portrait 4.9(b)*
		$\mathcal{G}_2 > 0, H_4 < 0$	
		$\mathcal{G}_2 > 0, H_4 > 0, \mathcal{G}_3 > 0$	*Portrait 4.9(c)*
Config. 4.10	$\eta > 0,\ B_3 \neq 0,\ B_2 = \theta = 0,$ $\mu_0 \neq 0, H_7 = H_9 = 0, H_{10} N > 0$	$H_4 > 0, \mathcal{G}_3 > 0$	*Portrait 4.10(a)*
		$H_4 < 0$	*Portrait 4.10(b)*
		$H_4 > 0, \mathcal{G}_3 < 0$	*Portrait 4.10(c)*
	$\eta > 0, B_3 H_4 \neq 0,$ $B_2 = N = H_9 = 0, H_8 > 0$	–	
Config. 4.11	$\eta = 0,\ M B_3 \neq 0,\ B_2 = \theta = 0,$ $H_7 = 0,\ \mu_0 \neq 0,\ H_{10} > 0$	$H_4 > 0$	*Portrait 4.11(a)*
		$H_4 < 0$	*Portrait 4.11(b)*
Config. 4.12	$\eta = 0,\ M \neq 0,\ B_3 = \theta = 0,$ $K H_6 \neq 0, H_7 = \mu_0 = 0, H_{11} > 0$	$\mu_2 > 0,\ L > 0$	*Portrait 4.12(a)*
		$\mu_2 > 0,\ L < 0$	*Portrait 4.12(b)*
		$\mu_2 < 0,\ K < 0$	*Portrait 4.12(c)*
		$\mu_2 < 0, K > 0, L > 0$	*Portrait 4.12(d)*
		$\mu_2 < 0, K > 0, L < 0$	*Portrait 4.12(e)*

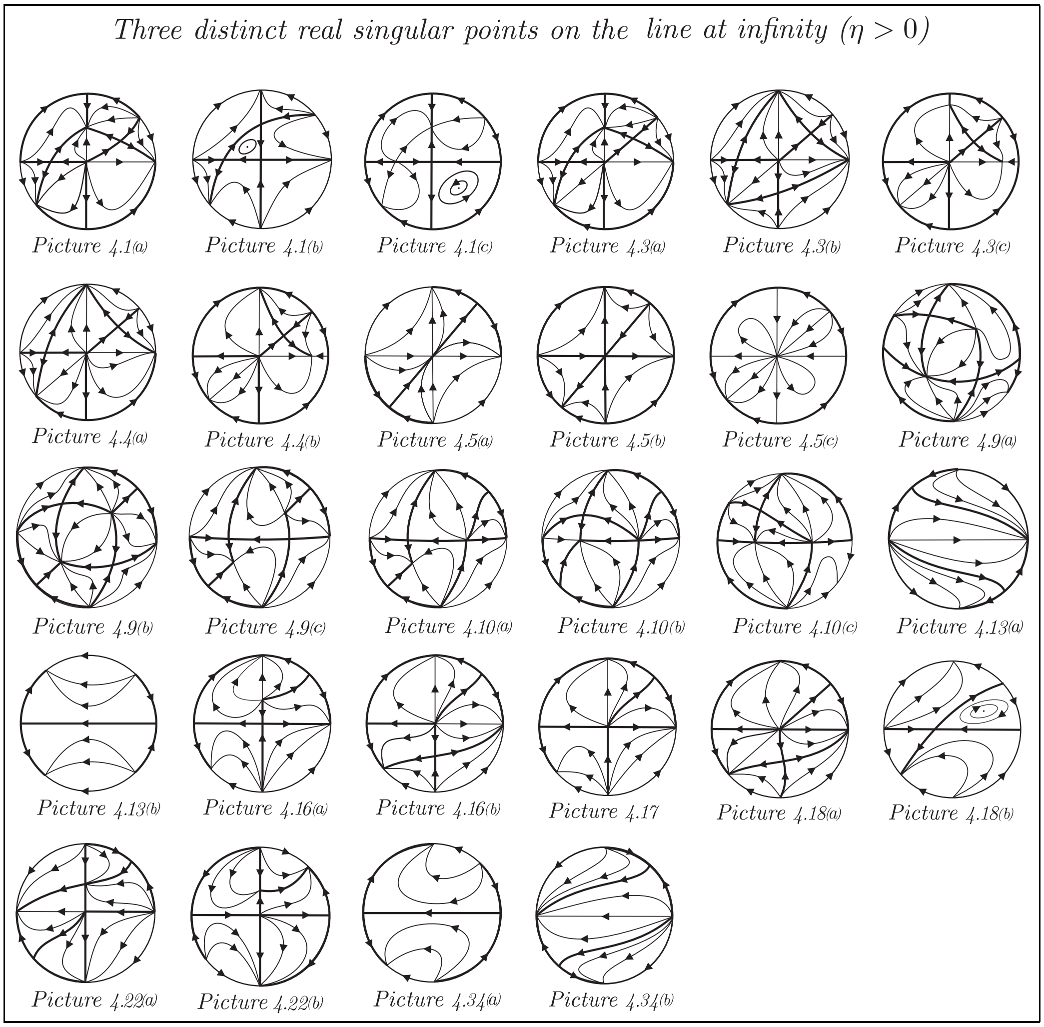
Table 2(*continued*)

Configuration	*Necessary and sufficient conditions*	*Additional conditions for phase portraits*	*Phase portrait*
Config. 4.13	$\eta>0, B_3\neq 0, B_2=\theta=0,$ $\mu_0\neq 0, H_7=0,$ $H_9\neq 0, NH_{10}<0$	$\mathcal{G}_2<0$	*Portrait 4.13(a)*
		$\mathcal{G}_2>0$	*Portrait 4.13(b)*
	$\eta>0, B_3H_4\neq 0,$ $B_2=N=0, H_8<0$	–	
Config. 4.14	$\eta=0,\ MB_3\neq 0,\ B_2=\theta=0,$ $H_7=0,\ \mu_0\neq 0,\ H_{10}<0$	–	*Portrait 4.14*
Config. 4.15	$\eta=0, M\neq 0, B_3=\theta=0,$ $KH_6\neq 0,\ \mu_0=H_7=0,\ H_{11}<0$	$L>0$	*Portrait 4.15(a)*
		$L<0$	*Portrait 4.15(b)*
Config. 4.16	$\eta>0,\ B_3\neq 0,\ B_2=\theta=0,$ $\mu_0=H_7=0,\ H_9\neq 0$	$\mathcal{G}_2>0$	*Portrait 4.16(a)*
		$\mathcal{G}_2<0$	*Portrait 4.16(b)*
Config. 4.17	$\eta>0,\ B_3\neq 0,\ B_2=\theta=0,$ $\mu_0=H_7=H_9=0,\ H_{10}\neq 0$	–	*Portrait 4.17*
Config. 4.18	$\eta>0,\ B_3=\theta=0,$ $\mu_0=0,\ H_7\neq 0$	$\mu_2L>0$	*Portrait 4.18(a)*
		$\mu_2L<0$	*Portrait 4.18(b)*
Config. 4.19	$\eta=0, M\neq 0, B_3=\theta=K=0,$ $NH_6\neq 0,\ \mu_0=H_7=0,\ H_{11}\neq 0$	$\mu_3K_1<0$	*Portrait 4.19(a)*
		$\mu_3K_1>0$	*Portrait 4.19(b)*
Config. 4.20	$\eta=0,\ M\neq 0,\ B_3=0,\ \theta\neq 0,$ $H_7=0,\ D=0$	$\mu_0>0$	*Portrait 4.20(a)*
		$\mu_0<0$	*Portrait 4.20(b)*
Config. 4.21	$\eta=0,\ M\neq 0,\ B_3=0,\ \theta\neq 0,$ $H_7=0,\ D\neq 0,\ \mu_0\neq 0$	$\mu_0>0$	*Portrait 4.21(a)*
		$\mu_0<0$	*Portrait 4.21(b)*
Config. 4.22	$\eta>0,\ B_3\neq 0,\ B_2=\theta=0,$ $\mu_0\neq 0,\ N\neq 0,\ H_7=H_{10}=0$	$H_1>0$	*Portrait 4.22(a)*
		$H_1<0$	*Portrait 4.22(b)*
	$\eta>0, B_3H_4\neq 0, B_2=\theta=N=H_8=0$	–	
Config. 4.23	$\eta=0,\ MB_3\neq 0,\ B_2=\theta=0,$ $\mu_0\neq 0,\ H_7=H_{10}=0$	–	*Portrait 4.23*
Config. 4.24	$\eta=0, M\neq 0, B_3=\theta=0,$ $KH_6\neq 0,\ \mu_0=H_7=H_{11}=0$	$L>0$	*Portrait 4.24(a)*
		$L<0$	*Portrait 4.24(b)*
Config. 4.25	$\eta=0,\ M\neq 0,\ B_3=0,\ \theta\neq 0,$ $H_7\neq 0$	$\mu_0>0$	*Portrait 4.25(a)*
		$\mu_0<0$	*Portrait 4.25(b)*
Config. 4.26	$\eta=0,\ M\neq 0,\ B_3=0,\ \theta\neq 0,$ $H_7=0,\ D\neq 0,\ \mu_0=0$	–	*Portrait 4.26*
Config. 4.27	$\eta<0,\ B_3=\theta=0,$ $N\neq 0,\ H_7\neq 0$	$\mathcal{G}_1\neq 0$	*Portrait 4.27(a)*
		$\mathcal{G}_1=0$	*Portrait 4.27(b)*
Config. 4.28	$\eta=0,\ M\neq 0,\ B_3=\theta=0,$ $\mu_0=N=K=0,\ N_1N_2\neq 0,$ $N_5>0,\ D\neq 0$	–	*Portrait 4.28*

Table 2(*continued*)

Configuration	*Necessary and sufficient conditions*	*Additional conditions for phase portraits*	*Phase portrait*
Config. 4.29	$\eta=0,\ M\neq 0,\ B_3=\theta=\mu_0=0,$ $N=K=0,\ N_1N_2\neq 0,\ N_5>0,\ D=0$	$\mu_4>0$	*Portrait 4.29(a)*
		$\mu_4<0$	*Portrait 4.29(b)*
Config. 4.30	$\eta=0,\ MB_3\neq 0,\ B_2=\theta=\mu_0=0,$ $N\neq 0,\ H_7=H_6=0,\ K\neq 0,\ H_{11}\neq 0$	$\mu_2>0$	*Portrait 4.30(a)*
		$\mu_2<0$	*Portrait 4.30(b)*
Config. 4.31	$\eta=M=0,\ B_3=\theta=0,$ $N\neq 0,\ N_6\neq 0,\ H_{11}\neq 0$	$K_3>0$	*Portrait 4.31(a)*
		$K_3<0$	*Portrait 4.31(b)*
Config. 4.32	$\eta=0,\ M\neq 0,\ B_3=\theta=\mu_0=0,$ $N=K=0,\ N_1N_2\neq 0,\ N_5<0,\ D\neq 0$	–	*Portrait 4.32*
Config. 4.33	$\eta=0,\ M\neq 0,\ B_3=\theta=\mu_0=0,$ $N=K=0,\ N_1N_2\neq 0,\ N_5<0,\ D=0$	–	*Portrait 4.33*
Config. 4.34	$\eta>0,\ B_3\neq 0,\ B_2=\theta=0,$ $\mu_0=H_7=H_9=H_{10}=0$	$H_4<0$	*Portrait 4.34(a)*
		$H_4>0$	*Portrait 4.34(b)*
Config. 4.35	$\eta=0,\ M\neq 0,\ B_3=\theta=0,$ $N\neq 0,\ \mu_0=0,\ H_7\neq 0$	$\mu_3K_1>0$	*Portrait 4.35(a)*
		$\mu_3K_1<0$	*Portrait 4.35(b)*
Config. 4.36	$\eta=0, M\neq 0, B_3=\theta=K=0,$ $NH_6\neq 0,\ \mu_0=H_7=0,\ H_{11}=0$	$\kappa_2<0$	*Portrait 4.36(a)*
		$\kappa_2>0$	*Portrait 4.36(b)*
Config. 4.37	$\eta=M=0,\ B_3=\theta=N=0,$ $N_3D_1\neq 0,\ N_6\neq 0,\ D\neq 0$	$\mu_3K_1>0, K_3\geq 0$	*Portrait 4.37(a)*
		$\mu_3K_1>0, K_3<0$	*Portrait 4.37(b)*
		$\mu_3K_1<0$	*Portrait 4.37(c)*
Config. 4.38	$\eta=M=0,\ B_3=\theta=N=0,$ $N_3D_1\neq 0,\ N_6\neq 0,\ D=0$	$\mu_4>0$	*Portrait 4.38(a)*
		$\mu_4<0$	*Portrait 4.38(b)*
Config. 4.39	$\eta=0,\ M\neq 0,\ B_3=\theta=\mu_0=0,$ $N=K=0,\ N_1N_2\neq 0,\ N_5=0$	–	*Portrait 4.39*
Config. 4.40	$\eta=0,\ MB_3\neq 0,\ B_2=\theta=\mu_0=0,$ $N\neq 0,\ H_7=H_6=0,\ K=0$	–	*Portrait 4.40*
Config. 4.41	$\eta=M=0,\ B_3=0,\ \theta\neq 0,$ $H_7=0,\ D\neq 0$	$\mu_0>0$	*Portrait 4.41(a)*
		$\mu_0<0$	*Portrait 4.41(b)*
Config. 4.42	$\eta=M=0,\ B_3=0,\ \theta\neq 0,$ $H_7=0,\ D=0$	$\mu_0>0$	*Portrait 4.42(a)*
		$\mu_0<0$	*Portrait 4.42(b)*
Config. 4.43	$\eta=0,\ MB_3\neq 0,\ B_2=\theta=\mu_0=0,$ $N\neq 0,\ H_7=H_6=0,\ K\neq 0,\ H_{11}=0$	$L<0$	*Portrait 4.43(a)*
		$L>0, R\geq 0$	*Portrait 4.43(b)*
		$L>0, R<0$	*Portrait 4.43(c)*
Config. 4.44	$\eta=M=0,\ B_3=\theta=0,$ $N\neq 0,\ N_6\neq 0,\ H_{11}=0$	$K_3>0$	*Portrait 4.44(a)*
		$K_3<0$	*Portrait 4.44(b)*
Config. 4.45	$\eta=M=0,\ B_3=0,\ \theta\neq 0,$ $H_7\neq 0$	$\mu_0>0$	*Portrait 4.45(a)*
		$\mu_0<0$	*Portrait 4.45(b)*
Config. 4.46	$\eta=M=0,\ B_3=\theta=N=0,$ $N_3D_1\neq 0,\ N_6=0$	–	*Portrait 4.46*

Table 3(a)



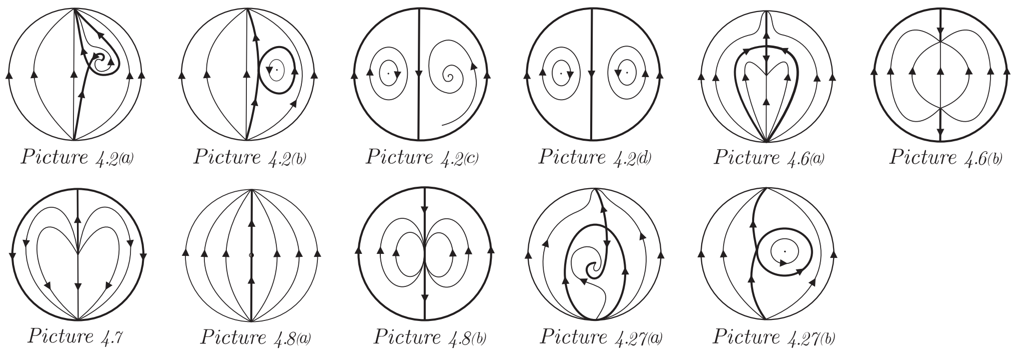
For systems $(S_{4.1})$ calculations yield: $K = 2\big[g(g-1)x^2 + 2ghxy + h(h-1)y^2\big]$, $\mu_0 = gh(g+h-1)$, $\quad \mathrm{sign}\,(\Delta_1\Delta_2\Delta_3\Delta_4) = \mathrm{sign}\,(\mu_0)$.

According to [5] a quadratic system cannot possess four anti–saddles, and neither could it possess four saddles. For this reason we obtain two saddles and two anti–saddles for $\mu_0 > 0$ and either (α) one saddle and three anti–saddles or (β) three saddles and one anti–saddle for $\mu_0 < 0$.

Assume $\mu_0 > 0$. As the singular points M_1, M_2 and M_3 are located on invariant lines and for M_4 we have $\mathrm{sign}\,(\delta_4) = \mathrm{sign}\,(\mu_0)$, we conclude that in this case a system $(S_{4.1})$ possesses two saddles and two nodes. Considering the existence of the invariant lines $x = 0$, $y = 0$ and $y = x + 1$ and the fact that the sum of Poincaré

Table 3(b)

One real and two imaginary singular points on the line at infinity ($\eta < 0$)



Picture 4.2(a) *Picture 4.2(b)* *Picture 4.2(c)* *Picture 4.2(d)* *Picture 4.6(a)* *Picture 4.6(b)*

Picture 4.7 *Picture 4.8(a)* *Picture 4.8(b)* *Picture 4.27(a)* *Picture 4.27(b)*

indices for finite singularities is zero, and at infinity we have 6 simple singularities (on the Poincaré disk), these must be: one couple of opposite saddles and two couples of opposite nodes and we get the phase portrait given by *Picture 4.1(a)*.

For $\mu_0 < 0$ we have $gh(g+h-1) < 0$ and then $\delta_4 < 0$, i.e. the singular point M_4 is either a focus or a center. We claim that it is a center. Indeed, via the translation of the origin of coordinates to this point we get the family of systems

$$\dot{x} = -ghx + h(1-h)y + gx^2 + (h-1)xy, \quad \dot{y} = g(g-1)x + ghy + (g-1)xy + hy^2.$$

Applying Lemma 7 to these systems we calculate: $I_1 = I_6 = I_3 = 0, \quad I_2 = 2gh(g+h-1)$. Thus, sign $(I_2) =$ sign (μ_0) and since $\mu_0 < 0$ (i.e. $I_2 < 0$) according to Lemma 7 we obtain that the singular point M_4 is a center, so our claim is proved.

On the other hand, for $\mu_0 < 0$ the T-comitant K becomes a binary form with well determined sign as $\text{Discrim}(K) = \mu_0/16$.

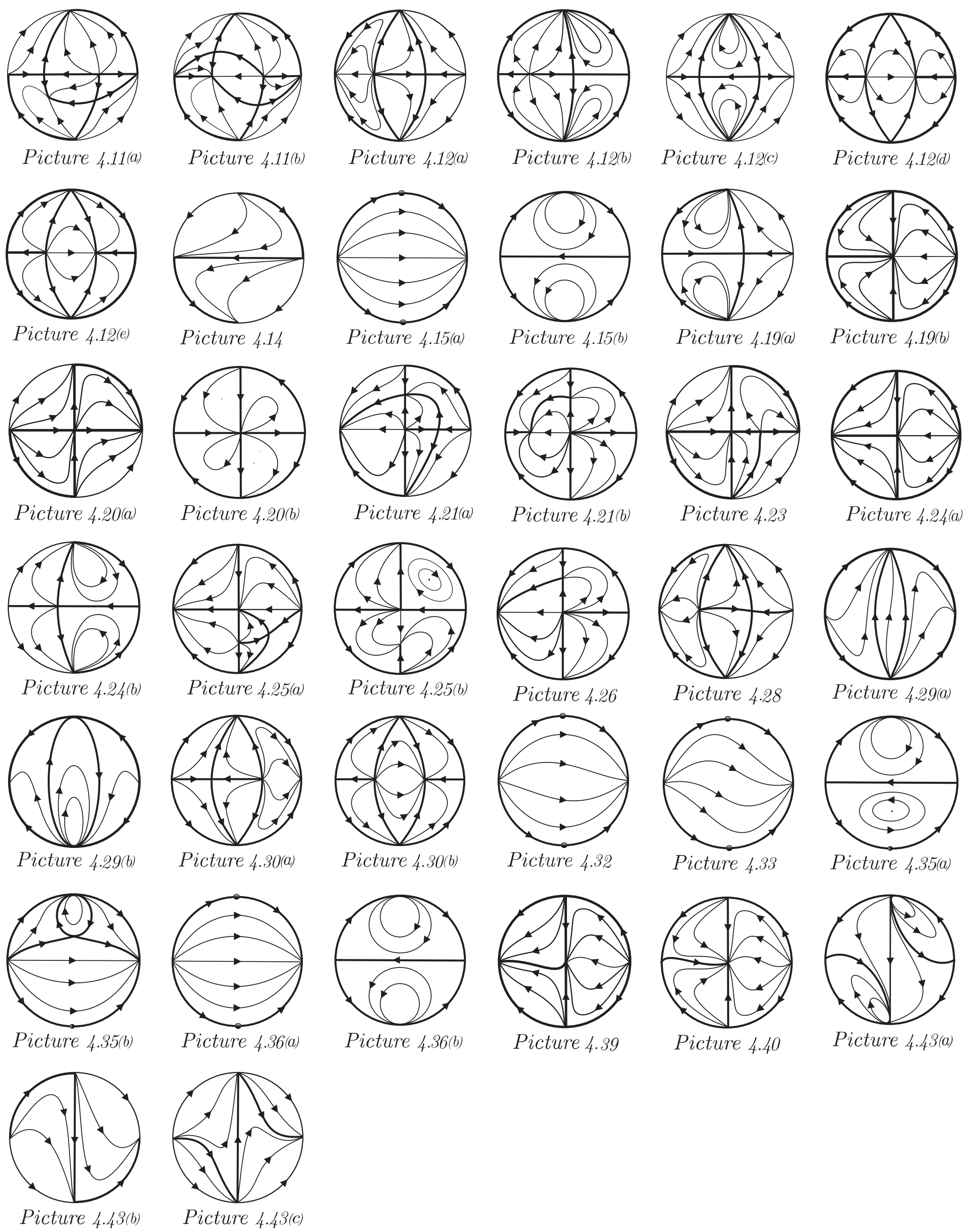
Assume $K < 0$. Then $0 < g < 1$, $0 < h < 1$ and from $\mu_0 < 0$ we obtain $g+h-1 < 0$. In this case we have $\Delta_i < 0$ for all $i \in \{1,2,3\}$ and hence, besides a center systems $(S_{4.1})$ possess three saddles. Moreover, for these values of the parameters g and h the singular point $M_4(-h,g)$ is placed inside of the triangle $\triangle\ M_1M_2M_3$. So, considering the existence of the invariant lines $x=0$, $y=0$ and $y = x+1$ and the fact that the sum of Poincaré indices for finite singularities is -2, we must have 6 nodes at infinity (3 in the projective plane) and we get the *Picture 4.1(b)*.

Suppose now that $K > 0$. Then $g(g-1) > 0$, $h(h-1) > 0$ and we claim that in this case besides the center M_4 systems $(S_{4.1})$ possess two nodes and one saddle. Indeed, supposing the contrary we obtain that all three M_i must be saddles, as $\Delta_1\Delta_2\Delta_3 = -(g^2h^2(g+h-1)^2 < 0$. Hence, $\Delta_i < 0$ for $i = 1,2,3$. From $\Delta_1 < 0$ we get $gh > 0$ and then the condition $\mu_0 < 0$ implies $g+h-1 < 0$. Then the condition $\Delta_2 < 0$ yields $h > 0$ (and hence $g > 0$). Due to $K < 0$ we get the contradiction: $g > 1$, $h > 1$ and $g+h-1 < 0$. This proves our claim.

So, systems $(S_{4.1})$ possess one saddle, two nodes and one center, and the last point is outside the triangle $\triangle\ M_1M_2M_3$. Clearly in this case at infinity we have

Table 3(c)

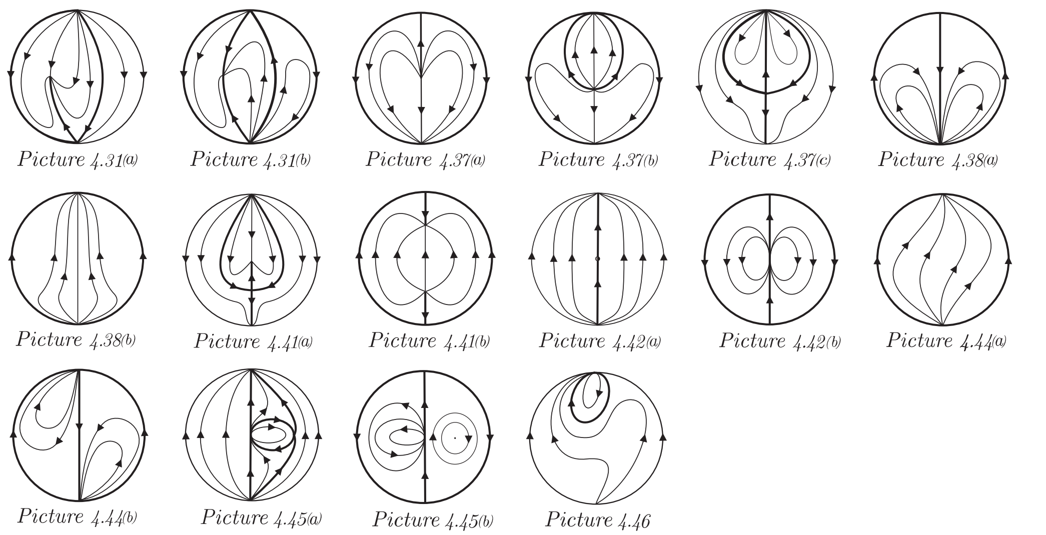
Two distinct real singular points on the line at infinity ($\eta = 0,\ M \neq 0$)



two saddles and one node (as the sum of Poincaré indeces for infinite singularities has to be -1). Considering the existence of the above indicated invariant lines we arrive at the phase portrait given by *Picture 4.1(c)*.

Table 3(d)

Only one singular point on the line at infinity, which is real ($\eta = 0 = M,\ C_2 \neq 0$)



Picture 4.31(a), *Picture 4.31(b)*, *Picture 4.37(a)*, *Picture 4.37(b)*, *Picture 4.37(c)*, *Picture 4.38(a)*, *Picture 4.38(b)*, *Picture 4.41(a)*, *Picture 4.41(b)*, *Picture 4.42(a)*, *Picture 4.42(b)*, *Picture 4.44(a)*, *Picture 4.44(b)*, *Picture 4.45(a)*, *Picture 4.45(b)*, *Picture 4.46*

So, systems ($S_{4.1}$) possess one saddle, two nodes and one center, and the last point is outside the triangle $\triangle\ M_1M_2M_3$. Clearly in this case at infinity we have two saddles and one node (as the sum of Poincaré indeces for infinite singularities has to be -1). Considering the existence of the above indicated invariant lines we arrive at the phase portrait given by *Picture 4.1(c)*.

$$\textit{Config. 4.2}: \begin{cases} \dot{x} = gx^2 + (h+1)xy, \qquad h(h+1)\Big[g^2 + (h-1)^2\Big] \neq 0, \\ \dot{y} = h\big[g^2+(h+1)^2\big]+(g^2+1-h^2)x+2ghy- x^2+gxy+hy^2. \end{cases} \qquad (S_{4.2})$$

Finite singularities: $M_1(0,0)\big[\Delta_1 = [g^2+(h+1)^2](h+1)^2 > 0,\ \delta_1 = -4(h+1)^4 < 0,\ \rho_1 = 2g(h+1)\big]$; $M_2(-h(h+1), gh)\big[\Delta_2 = h\big[g^2 + (h+1)^2\big](h+1)^2,\ \delta_2 = -4h\big[g^2 + (h+1)^2\big](h+1)^2,\ \rho_2 = 0\big]$. Thus the singular point M_1 is either a focus or a center. To determine the conditions for M_1 to be a center, we make a translation and move this point to the origin of coordinates. We get the systems

$$\dot{x} = (1+h+x)(gx+y+hy), \quad \dot{y} = -(h+1)^2x + g(h+1)y - x^2 + gxy + hy^2,$$

for which calculations yield: $I_1 = 2g(h+1)$, $\quad I_6 = g(h+1)^3(5+6h-3g^2-3h^2)/2$,

$$I_2 = 2g^2(h+1)^2 - 2(h+1)^4, \quad I_{13} = g(h+1)\Big[g^2(9h+8) + h(3h+1)^2\Big]/4.$$

Using Lemma 7 we see that M_1 is a center if and only if $g = 0$. If $g \neq 0$ this point is a strong focus. To distinguish between a focus and a center we define a new affine invariant as follows: $\mathcal{G}_1 = ((C_2, \tilde{E})^{(2)}, D_2)^{(1)}$, where $\tilde{E}(a,x,y) = \Big[D_1(2\omega_1 - \omega_2) - 3(C_1,\omega_1)^{(1)} - D_2(3\omega_3 + D_1D_2)\Big]/72$ and $\omega_1(a,x,y) = (C_2, D_2)^{(1)}$, $\omega_2(a,x,y) = (C_2, C_2)^{(2)}$, $\omega_3(a,x,y) = (C_1, D_2)^{(1)}$.

Since for the systems $(S_{4.2})$ calculation yields $\mathcal{G}_1 = 2g(h+1)\big[g^2+(3h+1)^2\big]$, it is clear that the condition $g=0$ is equivalent to $\mathcal{G}_1=0$.

Let us examine the point M_2. For systems $(S_{4.2})$ calculations yield: $\mu_0 = -h\big[g^2+(h+1)^2\big]$. Hence $\text{sign}\,(\delta_2) = \text{sign}\,(\mu_0) = -\text{sign}\,(\Delta_2)$. Therefore the point M_2 is a saddle if $\mu_0>0$ and it is either a focus or a center if $\mu_0<0$. Translating this point at the origin of coordinates we get the systems

$$\dot{x} = (x-h^2-h)[gx+(h+1)y], \quad \dot{y} = (h+1)(g^2+h+1)x+gh(h+1)y-x^2+gxy+hy^2,$$

for which $I_1 = I_6 = I_3 = 0$, $I_2 = -2h(h+1)^2\big[g^2+(h+1)^2\big]$. Consequently, by Lemma 7 the point M_2 is a center if $\mu_0<0$.

We note, that the product of the abscissas of finite singularities equals $-h(h+1)^2$. This means that both points are on the same side (respectively on different sides) of the invariant line $x=0$ if $\mu_0>0$ (respectively $\mu_0<0$).

It remains to observe that at infinity there are only two real simple singular points. When M_2 is a saddle, since M_1 is an anti-saddle (index +1), then the two infinite points must be nodes. When M_2 is a center, since M_1 is an anti-saddle, the two infinite points are saddles. In the last case the invariant line $x=0$ is a separatrix of the saddle at infinity.

Thus, we obtain: *Picture 4.2(a)* if $\mu_0>0$ and $\mathcal{G}_1\neq 0$; *Picture 4.2(b)* if $\mu_0>0$ and $\mathcal{G}_1=0$; *Picture 4.2(c)* if $\mu_0<0$ and $\mathcal{G}_1\neq 0$; *Picture 4.2(d)* if $\mu_0<0$ and $\mathcal{G}_1=0$.

$$\textit{Config. 4.3}: \begin{cases} \dot{x} = \dot{x} = x+gx^2+(h-1)xy, & gh(g+h-1)\neq 0, \\ \dot{y} = y+(g-1)xy+hy^2, & (g-1)(h-1)(g+h)\neq 0. \end{cases} \qquad (S_{4.3})$$

Finite singularities: $M_1(0,0)\big[\Delta_1=1,\ \delta_1=0\big]$; $M_2\Big(0,-\frac{1}{h}\Big)\Big[\Delta_2=-\frac{1}{h}, \delta_2 = \frac{(h+1)^2}{h^2}\Big]$; $M_3\Big(-\frac{1}{g},0\Big)\Big[\Delta_3=-\frac{1}{g}, \delta_3=\frac{(g+1)^2}{g^2}\Big]$; $M_4\Big(-\frac{1}{g+h-1},-\frac{1}{g+h-1}\Big)\Big[\Delta_4 = \frac{1}{g+h-1}, \delta_4 = \frac{(g+h-2)^2}{(g+h-1)^2}\Big]$. For systems $(S_{4.3})$ calculations yield: $\mu_0 = gh(g+h-1)$,

$$K = 2\big[g(g-1)x^2+2ghxy+h(h-1)y^2\big]; \quad \text{sign}\,(\Delta_1\Delta_2\Delta_3\Delta_4) = \text{sign}\,(\mu_0).$$

Since $\delta_i\geq 0$ for all points M_i we conclude that systems $(S_{4.3})$ possess two saddles and two nodes if $\mu_0>0$ and they possess either (α) one saddle and three nodes or (β) three saddles and one node if $\mu_0<0$.

Assume $\mu_0>0$. Then we have two nodes (one of them being the point M_1) and two saddles. Considering the existence of the invariant lines $x=0$, $y=0$ and $y=x$ and the fact that the sum of Poincaré indices for finite singularities is zero, then at infinity we have six simple singularities: two saddle and four nodes and we get the phase portrait given by *Picture 4.3(a)*.

If $\mu_0<0$ the T-comitant K becomes a sign defined binary form as $\text{Discrim}(K) = \mu_0/16$.

Assume $K<0$. Then $0<g<1$, $0<h<1$ and from $\mu_0<0$ we obtain $g+h-1<0$. In this case $\Delta_i<0$ for all $i\in\{2,3,4\}$ and hence, besides the star node M_1 systems $(S_{4.1})$ possess three saddles. Moreover, for these values of the parameters g and h the singular point $M_1(0,0)$ is placed inside of the triangle $\triangle\ M_2M_3M_4$. So, considering the existence of the invariant lines $x=0$, $y=0$ and $y=x$ and the fact that the sum of Poincaré indices for finite singularities is -2, we have six nodes at infinity and we get the *Picture 4.3(b)*.

Suppose now $K>0$. We claim that in this case besides the star node M_1 systems $(S_{4.3})$ possess two nodes and one saddle. Indeed, supposing the contrary, we obtain that all three M_i must be saddles, as $\operatorname{sign}(\Delta_2\Delta_3\Delta_4)=\operatorname{sign}(\mu_0)=-1$. Therefore, from $\Delta_2<0$ and $\Delta_3<0$ we get $h>0$ and $g>0$ respectively, and then the condition $\mu_0<0$ implies $g+h-1<0$. On the other the condition $K>0$ yields $g(g-1)>0$ and $h(h-1)>0$ and we get $g>1$ and $h>1$. This contradicts $g+h-1<0$ and hence proves our claim.

So, systems $(S_{4.3})$ possess one saddle and three nodes. Clearly in this case at infinity we have four saddles and two nodes (as the sum of Poincaré indices for infinite singularities has to be -2). Considering the presence of the above mentioned invariant lines we obtain the phase portrait given by *Picture 4.3(c)*.

$$\textit{Config. 4.4}:\quad \begin{cases} \dot{x}=x+gx^2-xy, & g\in\mathbb{R},\\ \dot{y}=y+(g-1)xy, & g(g-1)\neq 0. \end{cases} \qquad (S_{4.4})$$

Finite singularities: $M_1(0,0)\big[\Delta_1=1,\ \delta_1=0\big]$; $M_2\big(-\frac{1}{g},0\big)\big[\Delta_2=-\frac{1}{g},\ \delta_2=\frac{(g+1)^2}{g^2}\big]$; $M_3\big(\frac{1}{1-g},\frac{1}{1-g}\big)\big[\Delta_3=\frac{1}{g-1},\ \delta_3=\frac{(g-2)^2}{(g-1)^2}\big]$. For systems $(S_{4.3})$ calculations yield: $\mu_0=0$, $K=2g(g-1)x^2$, $\operatorname{sign}(\Delta_2\Delta_3)=-\operatorname{sign}(K)$. We observe, that the family of systems $(S_{4.4})$ is a subset of the family $(S_{4.3})$ defined by the condition $h=0$. So, since the singular point $M_2\big(0,-1/h\big)$ tends to infinity when $h\to 0$ we conclude that the infinite point $N(0,1,0)$ of systems $(S_{4.4})$ is a double point (a saddle-node).

On the other hand it is easy to determine that besides the star node M_1, systems $(S_{4.4})$ possess two saddles if $g(g-1)<0$ (i.e. $K<0$) and they possess one saddle and one node if $g(g-1)>0$ (i.e. $K>0$). Therefore, taking into consideration the invariant lines $x=0$, $y=0$ and $y=x$ and the sum of Poincaré indices we get the *Picture 4.4(a)* if $K<0$ and the *Picture 4.4(b)* when $K>0$.

$$\textit{Config. 4.5}:\quad \begin{cases} \dot{x}=gx^2+(h-1)xy, & (g-1)(h-1)(g+h)\neq 0,\\ \dot{y}=(g-1)xy+hy^2, & gh(g+h-1)\neq 0. \end{cases} \qquad (S_{4.5})$$

We observe that $(S_{4.5})$ is a family of homogenous systems, each having only the origin as finite singular point. These systems possess three invariant lines: $x=0$, $y=0$ and $y=x$. Hence $\eta>0$. We also have $\mu_0\neq 0$. Hence according to Table 4 in [26] we have the following possibilities for singular points at infinity: i) If $\mu_0>0$ we have two saddles and four nodes; ii) If $\mu_0<0$ and $\kappa<0$ we have four saddles and two nodes; If $\mu_0<0$ and $\kappa>0$ we have six nodes.

For systems $(S_{4.5})$ calculations yield:

$$\mu_0=gh(g+h-1),\quad \kappa=-16\big[g(g-1)+h(h-1)+gh\big],$$
$$K=2\big[g(g-1)x^2+2ghxy+h(h-1)y^2\big].$$

Mapping the sign of μ_0 in the plane h, g we determine that $\mu_0 < 0$ in the shaded areas of *Figure 1*. Hence in the shaded areas, K has well determined sign as indicated. On the same figure it is also easy to observe that for $\mu_0 < 0$ the following relation holds: $\operatorname{sign}(\kappa) = -\operatorname{sign}(K)$.

Thus, we obtain: *Picture 4.5(a)* if $\mu_0 > 0$; *Picture 4.5(b)* if $\mu_0 < 0$ and $K < 0$; *Picture 4.5(c)* if $\mu_0 < 0$ and $K > 0$.

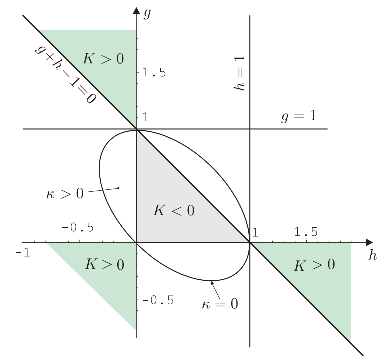


Figure 1

$$\text{Config. 4.6}: \quad \begin{cases} \dot{x} = gx^2 + (h+1)xy, & h(h+1)\Big[g^2 + (h-1)^2\Big] \neq 0, \\ \dot{y} = -1 + gx + (h-1)y - x^2 + gxy + hy^2. \end{cases} \tag{$S_{4.6}$}$$

Finite singularities: $M_1(0,-1)\big[\Delta_1 = (h+1)^2,\ \delta_1 = 0\big]$ – a node; $M_2(0, 1/h)\big[\Delta_2 = (h+1)^2/h,\ \delta_2 = (h^2-1)^2/h^2\big]$ – a node if $h > 0$ and a saddle if $h < 0$. For systems $(S_{4.6})$ we calculate $\mu_0 = -h\big[g^2 + (h+1)^2\big]$, i.e. $\operatorname{sign}(\mu_0) = -\operatorname{sign}(h)$.

It remains to observe that at infinity there exist only one real singular point which is simple. Since M_1 is an anti-saddle (index $+1$), the infinite point is a node (index $+1$) when M_2 is a saddle and it is a saddle (index -1) when M_2 is a node. In the last case the invariant line $x = 0$ is a separatrix of the saddle at infinite. Hence we obtain *Picture 4.6(a)* if $\mu_0 > 0$ and *Picture 4.6(b)* if $\mu_0 < 0$.

$$\text{Config. 4.7}: \quad \dot{x} = gx^2 + xy, \quad \dot{y} = -1 + gx - y - x^2 + gxy, \quad g \in \mathbb{R}. \tag{$S_{4.7}$}$$

Finite singularities: $M_1(0,-1)\big[\Delta_1 = 1,\ \delta_1 = 0\big]$ – a node. We observe that the family of systems $(S_{4.7})$ is a subset of the family $(S_{4.6})$ defined by the condition $h = 0$. So, since the singular point $M_2\big(0, -1/h\big)$ tends to infinite when $h \to 0$ we conclude that the infinite point $N(0,1,0)$ of systems $(S_{4.7})$ is a double point (a saddle-node). This leads to the *Picture 4.7*

$$\text{Config. 4.8}: \quad \begin{cases} \dot{x} = gx^2 + (h+1)xy, \\ \dot{y} = -x^2 + gxy + hy^2, \quad h(h+1)\Big[g^2 + (h-1)^2\Big] \neq 0. \end{cases} \tag{$S_{4.8}$}$$

For systems $(S_{4.8})$ we calculate $\mu_0 = -h\big[g^2 + (h+1)^2\big]$, $\eta = -4 < 0$. According to [26] at infinity there exist two opposite nodes if $\mu_0 > 0$ and two opposite saddles if $\mu_0 < 0$.

Thus, taking into consideration the real invariant line $x = 0$ of systems $(S_{4.8})$ we obtain *Picture 4.8(a)* if $\mu_0 > 0$ and *Picture 4.8(b)* if $\mu_0 < 0$.

$$\text{Config. 4.9}: \quad \begin{cases} \dot{x} = x^2 - 1, \quad g(g-1)[(g \pm 1)^2 - 4h^2] \neq 0, \\ \dot{y} = (y+h)[y + (1-g)x - h]. \end{cases} \tag{$S_{4.9}$}$$

Finite singularities: $M_1(-1,-h)\big[\Delta_1 = 2(2h - g + 1),\ \delta_1 = (2h - g - 1)^2\big]$;
$M_2(1,-h)\big[\Delta_2 = -2(2h + g - 1),\ \delta_2 = (2h + g + 1)^2\big]$;

$M_3(-1,h-g+1)\big[\Delta_3=-2(2h-g+1),\ \delta_3=(2h-g+3)^2\big];\ M_4(1,h+g-1)\big[\Delta_4=2(2h+g-1),\ \delta_4=(2h+g-3)^2\big]$. For systems $(S_{4.9})$ calculations yield: $\mu_0=1>0$, $\eta=g^2>0$, $\operatorname{sign}(\Delta_1\Delta_2\Delta_3\Delta_4)=1$.

Since $\delta_i\geq 0$ for all points M_i we conclude that systems $(S_{4.9})$ possess two saddles and two nodes in the finite part of its phase plane. From the behavior of trajectories at infinity, according to [26] we have four nodes and two opposite saddles. More concretely, we have the node $N_1(0,1,0)$ and the singular points $N_2(1,0,0)$ and $N_3(1,g,0)$ as well as their opposites. It is not hard to find out that the point $N_2(1,0,0)$ (respectively, $N_3(1,g,0)$) ia a saddle (respectively, a node) if $g<0$ and it is a node (respectively, a saddle) if $g>0$.

We note that the first equation depends only on x. $\dot{x}>0$ for x outside $[-1,1]$ and $\dot{x}<0$ for $x\in(-1,1)$. This yields the orientation of the vector field on the invariant line $x=-h$. The phase portrait on the invariant lines $x=\pm 1$ is easily obtained by replacing these values in the second equation which becomes $\dot{y}=(y+h)(y-(h+g-1))$ for $x=1$ and $\dot{y}=(y+h)(y-(h-g+1))$ for $x=-1$. Hence $\dot{y}>0$ for y outside the interval determined by the roots of the polynomials on the right hand sides and $\dot{y}<0$ for y inside this interval. The sign of $\dot{y}$ thus depends on whether or not $-h$ is smaller or greater than $h+g-1$ (respectively $h-g+1$) which amounts to checking the sign of $2h+g-1$ (respectively $2h-g+1$). As the phase portrait around infinity depends on the sign of g the full discussion, which is elementary, depends on the sign of $g(2h-g+1)(2h+g-1)$.

Case 1) We first assume that $g(2h-g+1)(2h+g-1)>0$. This could occur if either *i)* all three factors are positive or *ii)* two of the factors are negative and the third one is positive.

In the case *i)* we have that $-h<h-g+1$ and $-h<h+g-1$ so the points M_3 and M_4 lie above the line $y=-h$, M_3 being a saddle and M_4 being a node while M_1 is a node and M_2 is a saddle. This yields phase portrait *Picture 4.9(b)*.

In the case *ii)* we observe that the case when the first two factors are negative and the third one is positive cannot occur. Indeed, in this case we would necessarily have $-(g-1)<2h<g-1$ which yields a contradiction as $g>0$. So we only need to consider the cases when only the first and last factors are negative or when only the second and last one are negative. In the first situation we have that $h+g-1<-h<h-g+1$, so M_3 and M_4 are respectively above and below the invariant line $y=-h$. M_1 and M_2 are nodes and M_3 and M_4 are saddles. Considering the behavior at infinity we have that N_2 is a saddle and N_3 is a node located on the negative side of the u-axis and the phase portrait is *Picture 4.9(b)*. If only $2h-g+1$ and $2h+g-1$ are negative and $g>0$, the points M_3 and M_4 are both below the line $y=-h$, M_3 is a node and M_4 is a saddle while M_1 is a saddle and M_2 is a node. N_2 is a node and N_3 is on the positive side of the u-axis and it is a saddle. This yields again the phase portrait *Picture 4.9(b)*.

Thus we conclude that in the case $g(2h-g+1)(2h+g-1)>0$ the phase portrait of systems $(S_{4.9})$ corresponds to *Picture 4.9(b)*.

Case 2) Suppose now that $g(2h-g+1)(2h+g-1)<0$. This could occur if all three factors are negative or if only one is negative and the other two are positive.

In the first case M_3 and M_4 are both below the line $y = -h$ and M_1 and M_3 are nodes while M_2 and M_4 are saddles. N_2 is a saddle and N_3 lies on the negative side of the u-axis and it is a node. This yields picture *Picture 4.9(a)*.

It remains to consider the cases when only one of the three factors is negative. If $g < 0$ then M_3 and M_4 are both above the line $y = -h$ and M_3 and M_4 are both below the line $y = -h$ and M_1 and M_4 are nodes while M_3 and M_2 are saddles. N_2 is a saddle and N_3 is on the negative side of the u-axis and it is a node. So in this case we get *Picture 4.9(a)*. If only the second factor is negative, i.e. $2h - g + 1 < 0$ we have $h-g+1 < -h < h+g-1$ and hence M_3 and M_4 are nodes situated on the opposite sides of the line $y = -h$ and M_1 and M_2 are saddles. In this case N_2 is a node and N_3 is a saddle situated in the positive side of u. Hence the phase portrait is *Picture 4.9(c)*. If only the third factor is negative, i.e. $2h + g - 1 < 0$ then M_1 and M_2 are nodes and M_3 and M_4 are saddles located on the opposite sides of the line $y = -h$. In this case N_2 is a node and N_3 is located on the positive side of u and it is a saddle. The phase portrait is therefore *Picture 4.9(b)*.

For each phase portrait assembling together the above conditions we get:

- *Picture 4.9(b)* $\Leftrightarrow$ either $g\big[4h^2-(g-1)^2\big]>0$ or $g\big[4h^2-(g-1)^2\big]<0$ and $0<g<1$;
- *Picture 4.9(a)* $\Leftrightarrow$ $g\big[4h^2-(g-1)^2\big]<0$ and $g<0$;
- *Picture 4.9(c)* $\Leftrightarrow$ $g\big[4h^2-(g-1)^2\big]<0$ and $g>1$.

In order to determine the corresponding invariant conditions we construct the following affine invariants:

$$\mathcal{G}_2 = 8H_8 - 9H_5, \quad \mathcal{G}_3 = (\mu_0 - \eta)H_1 - 6\eta(H_4 + 12H_{10}).$$

Since for the systems $(S_{4.9})$ we have $G_2 = -2^9 3^3 g\big[4h^2 - (g-1)^2\big]$, $H_4 = 48(1-g)\big[4h^2-(g+1)^2\big]$, $G_3 = 6gH_4$, we conclude that these three invariant polynomials distinguish the phase portraits of systems $(S_{4.9})$ for this configuration as it is indicated in the Table 2.

$$\textit{Config. 4.10}: \quad \dot{x} = x^2 - 1, \quad \dot{y} = (y+g)(y+2gx-g), \quad g(2g-1) \neq 0. \qquad (S_{4.10})$$

Finite singularities: $M_1(-1,-g)\big[\Delta_1 = 8g,\ \delta_1 = 4(2g-1)^2\big]$ – a node if $g>0$ and a saddle if $g<0$; $M_2(1,-g)\big[\Delta_2 = 0,\ \rho_2 = 2\big]$ – a saddle-node [1]; $M_3(-1,3g)\big[\Delta_3 = -8g,\ \delta_3 = 4(2g+1)^2\big]$ – a node if $g<0$ and a saddle if $g>0$. For systems $(S_{4.10})$ calculations yield: $\mu_0 = 1 > 0$, $\eta = (2g-1)^2 > 0$. Hence according to [26] at infinity we have six singularities: the node $N_1(0,1,0)$ with its opposite and the singular points $N_2(1,0,0)$ and $N_3(1,1-2g,0)$ with there opposites. It is not hard to find out that the point $N_2(1,0,0)$ (respectively, $N_3(1,1-2g,0)$) is a saddle (respectively, a node) if $1-2g<0$ and it is a node (respectively, a saddle) if $1-2g>0$.

a) Assume first $g<0$, i.e. $M_1(-1,-g)$ is a saddle and $M_3(-1,3g)$ is a node. Since in this case $1-2g>0$ we obtain that $N_2(1,0,0)$ is a node and $N_3(1,1-2g,0)$

is a saddle. Taking into consideration the location of these singularities we get the *Picture 4.10(a)*.

b) Assume now $g > 0$. Then $M_1(-1, -g)$ is a node and $M_3(-1, 3g)$ is a saddle. Since the type of infinite singularities depends on sign $(1 - 2g)$ we shall consider two subcases: $1 - 2g > 0$ and $1 - 2g < 0$.

b_1) If $1 - 2g > 0$ then as in the previous case $N_2(1, 0, 0)$ is a node and $N_3(1, 1 - 2g, 0)$ is a saddle. Taking into consideration the relative location of the singularities of systems $(S_{4.10})$ for $0 < g < 1/2$, we obtain in this case the *Picture 4.10(b)*.

b_2) Supposing $1 - 2g < 0$ we have at infinity the saddle $N_2(1, 0, 0)$ and the node $N_3(1, 1 - 2g, 0)$. It is easy to observe that in this case we have a separatrix connection, between finite saddle-node M_2 and infinite saddle $N_2(1, 0, 0)$. So, in the same manner above we get the *Picture 4.10(c)*.

It remains to construct the respective affine invariant conditions. For systems $(S_{4.10})$ we have $H_4 = 384g(2g - 1)$. Therefore, if $H_4 < 0$ (i.e. $0 < g < 1/2$) we obtain *Picture 4.10(b)*, whereas for $H_4 > 0$ we have either *Picture 4.10(a)* or *Picture 4.10(c)*. We observe that for systems $(S_{4.10})$ calculations yield: $\mathcal{G}_3 = -2304g(2g-1)^2$ and hence, for $H_4 > 0$ we get *Picture 4.10(a)* if $\mathcal{G}_3 > 0$ and *Picture 4.10(c)* if $\mathcal{G}_3 < 0$.

$$\textit{Config. 4.11}: \quad \dot{x} = (x + g)^2 - 1, \quad \dot{y} = y(x + y), \quad g^2 - 1 \neq 0. \tag{$S_{4.11}$}$$

Finite singularities: $M_1(-1 - g,\, 0)\big[\Delta_1 = 2(g + 1),\ \delta_1 = (g - 1)^2\big]$; $M_2(1 - g,\, 0)\big[\Delta_2 = -2(g - 1),\ \delta_2 = (g + 1)^2\big]$; $M_3(-1 - g,\, g + 1)\big[\Delta_3 = -2(g + 1),\ \delta_3 = (g + 3)^2\big]$; $M_4(1 - g,\, g - 1)\big[\Delta_4 = 2(g - 1),\ \delta_4 = (g - 3)^2\big]$.

Evidently, that we have two nodes and two saddles, and which singularities are nodes and which ones are saddles depends on sign $(g^2 - 1)$.

For systems $(S_{4.11})$ calculations yield: $\mu_0 = 1 > 0$, $\eta = 0$, $M = -8y^2 \neq 0$, $C_2 = -xy^2$. So, according to [26] at infinity besides the node $N_1(0, 1, 0)$ systems $(S_{4.11})$ possess a double point $N_1(1, 0, 0)$, which is a saddle-node.

We shall examine three cases: $g < -1$, $-1 < g < 1$ and $g > 1$.

a) Case $g < -1$. Then the singular points M_1 and M_4 are saddles, whereas M_2 and M_3 are nodes. Moreover, M_3 and M_4 are on the same part of the invariant line $y = 0$. Thus we get the phase portrait given by *Picture 4.11(a)*.

b) Case $-1 < g < 1$. In this case the singular points $M_{1,2}$ are nodes and $M_{3,4}$ are saddles. And clearly M_3 and M_4 are on different sides of the invariant line $y = 0$. So we obtain *Picture 4.11(b)*.

c) Case $g > 1$. Then the singular points M_1 and M_4 are nodes, whereas M_2 and M_3 are saddles. In this case M_3 and M_4 are on the same part of the invariant line $y = 0$. Therefore, we get the phase portrait with is topologically equivalent to *Picture 4.11(a)*.

It remains to note that for the systems $(S_{4.11})$ we have $H_4 = 48(g^2 - 1)$ and evidently this invariant polynomial distinguishes *Picture 4.11(a)* $(H_4 > 0)$ from *Picture 4.11(b)* $(H_4 < 0)$.

$$\textit{Config. 4.12}: \begin{cases} \dot{x} = (x + h)^2 - 1, & g(g - 1)(h^2 - 1) \neq 0, \\ \dot{y} = (1 - g)xy, & h^2(g - 1)^2 - (g + 1)^2 \neq 0. \end{cases} \tag{$S_{4.12}$}$$

Finite singularities: $M_1(-1-h,\,0)\big[\Delta_1 = -2(h+1)(g-1),\ \delta_1 = [h(g-1)+(g+1)]^2\big]$; $M_2(1-h,\,0)\big[\Delta_2 = 2(h-1)(g-1),\ \delta_2 = [h(g-1)-(g+1)]^2\big]$. Since $\Delta_1\Delta_2 = -4(g-1)^2(h^2-1)$ and $\Delta_1+\Delta_2 = -4(g-1)$ we conclude that systems $(S_{4.12})$ possess a saddle and a node if $h^2-1>0$. For $h^2-1<0$ these systems possess two saddles if $g>1$ and they possess two nodes if $g<1$.

To determine the behavior of the trajectories at the infinity according to [26] for systems $(S_{4.12})$ we calculate:

$$\eta = 0,\ M = -8g^2x^2 \neq 0,\ C_2 = gx^2y,\ \mu_0 = \mu_1 = \kappa = \kappa_1 = 0,\ L = 8gx^2,$$
$$\mu_2 = (g-1)^2(h^2-1)x^2,\ K = 2(1-g)x^2,\ K_2 = 192(2g^2-g+1)x^2.$$

We observe that by [26] the point $N_1(0,1,0)$ is of the multiplicity 4 (consisting of two finite and two infinite points which have coalesced). We also note that $K_2>0$ for any value of parameter g.

a) Case $\mu_2>0$. Then $h^2-1>0$ and systems $(S_{4.12})$ possess one saddle and one node. As $\operatorname{sign}(L) = \operatorname{sign}(g)$, following [26] we shall consider two subcases: $L>0$ and $L<0$.

a_1) *Assume* $L>0$. Since $K_2>0$ according to [26, Table 4] the behavior of the trajectories in the vicinity of infinity is given by *Figure 19*. Taking into consideration the invariant lines we get *Picture 4.12(a)*.

a_2) *Suppose* $L<0$. Then from [26, Table 4] we get *Figure 17*. So, in the same manner as above we obtain the phase portrait given by *Picture 4.12(b)*.

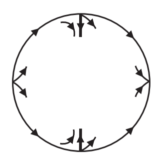


Figure 19

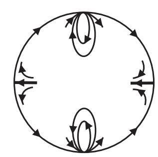


Figure 17

b) Case $\mu_2<0$. In this case we have $h^2-1<0$ and as it is determined above systems $(S_{4.12})$ possess two saddles if $g>1$ and they possess two nodes if $g<1$. As for these systems $K = 2(1-g)x^2$ we have $\operatorname{sign}(K) = -\operatorname{sign}(g-1)$ and we shall consider two subcases: $K<0$ and $K>0$.

b_1) *Assume first* $K<0$, i.e. $g>1$ and the finite singular points are both saddles. On the other hand for the infinite points the relation $L = 8gx^2>0$ holds and according to [26, Table 4] this leads to the *Figure 10*. So, taking into consideration the invariant lines of systems $(S_{4.12})$ we obtain the phase portrait given by *Picture 4.12(c)*.

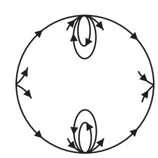


Figure 10

b_2) *Assume now* $K>0$. Then $g<1$ and the finite singular points are both nodes. According to [26, Table 4] the behavior of the trajectories in the vicinity of infinity in this case depends on the sign of the invariant polynomials $L = 8gx^2$.

If $L>0$ (i.e. $g>0$) we obtain *Figure 27*, whereas for $L<0$ (then $K<0$) we get *Figure 29*. Therefore, considering the existence of the invariant lines of systems $(S_{4.12})$ we obtain the *Picture 4.12(d)* if $L>0$ and the *Picture 4.12(e)* if $L<0$.

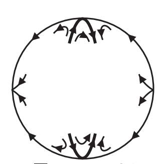


Figure 27

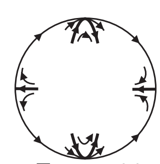


Figure 29

$$\textit{Config. 4.13}: \begin{cases} \dot{x} = x^2+1, \quad g(g-1)\big[(g+1)^2+h^2\big] \neq 0, \\ \dot{y} = (y+h)[y+(1-g)x-h]. \end{cases} \qquad (S_{4.13})$$

No finite singularities. These systems have invariant lines $y = -h$ and $x = \pm i$. Calculations yield $\eta = g^2 > 0$, $\mu_0 = 1 > 0$ and according to [26] on the line at infinity there exist two nodes and one saddle. Due to the existence of the real invariant line $y = -h$ we have to distinguish when the point $N_1(1, 0, 0)$ is a saddle (having a saddle connection) and when it is a node. Constructing the respective to $(S_{4.13})$ family of systems at infinity we get

$$\dot{u} = gu + h(g-1)z - u^2 + h^2 z^2 + uz^2, \quad \dot{z} = z + z^3.$$

Since the singularity $(0, 0)$ of these systems corresponds to $N_1(1, 0, 0)$ we conclude that the singular point N_1 is a saddle if $g < 0$ and it is a node if $g > 0$. To distinguish these two possibilities we shall use the affine invariant G_2. For systems $(S_{4.13})$ we calculate $\mathcal{G}_2 = 13824g\big[4h^2 + (g-1)^2\big]$. Thus, $\mathcal{G}_2 \neq 0$ and $\operatorname{sign}(\mathcal{G}_2) = \operatorname{sign}(g)$. Hence we get *Picture 4.13(a)* if $\mathcal{G}_2 > 0$ and *Picture 4.13(b)* if $\mathcal{G}_2 < 0$.

$$\textit{Config. 4.14}: \quad \dot{x} = (x+g)^2 + 1, \quad \dot{y} = y(x+y), \quad g \in \mathbb{R}. \qquad (S_{4.14})$$

No finite singularities. Calculations yield $\eta = 0$, $M = -8y^2 \neq 0$, $C_2 = -xy^2$, $\mu_0 = 1 > 0$. Thus the singular point $N_1(1, 0, 0)$ is a double point and according to [26] on the line at infinity at infinity there exist one node and one saddle-node (double). Hence, taking into account the real invariant line $y = 0$ we get *Picture 4.14* for any value of the parameter g.

$$\textit{Config. 4.15}: \quad \begin{cases} \dot{x} = (x+h)^2 + 1, & g(g-1) \neq 0, \\ \dot{y} = (1-g)xy, & (g+1)^2 + h^2 \neq 0. \end{cases} \qquad (S_{4.15})$$

No finite singularities. Calculations yield

$$\eta = 0,\ M = -8g^2x^2 \neq 0,\ C_2 = gx^2y,\ \mu_0 = \mu_1 = \kappa = \kappa_1 = 0,\ L = 8gx^2,$$
$$\mu_2 = (g-1)^2(h^2+1)x^2,\ K = 2(1-g)x^2,\ K_2 = -192(2g^2 - g + 1)x^2.$$

We observe [26] that the point $N_1(0, 1, 0)$ is of the multiplicity 4 (two finite and two infinite points have coalesced at this point). We also note that $\mu_2 > 0$ and $K_2 < 0$ for any value of parameters $(g, h) \in \mathbb{R}^2$.

Thus according to [26, Table 4] this leads to the *Figure 8* if $L > 0$ and to the *Figure 17* (see above) if $L < 0$. Taking into consideration the existence of the real invariant line $y = 0$ we obtain *Picture 4.15(a)* if $L > 0$ and *Picture 4.15(b)* if $L < 0$.

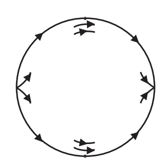


Figure 8

$$\textit{Config. 4.16}: \quad \dot{x} = g + x, \quad \dot{y} = y(y - x), \quad g(g-1) \neq 0. \qquad (S_{4.16})$$

Finite singularities: $M_1(-g, 0)\big[\Delta_1 = g,\ \delta_1 = (g-1)^2\big]$; $M_2(-g, -g)\big[\Delta_2 = -g,\ \delta_2 = (g+1)^2\big]$. We observe that systems $(S_{4.16})$ possess a node and a saddle. To determine the behavior of the trajectories at the infinity according to [26] for these systems we calculate:

$$\eta = 1 > 0,\ C_2 = xy(x-y),\ \mu_0 = \mu_1 = \kappa = 0, \quad L = 8y(y-x),\ \mu_2 = y(y-x).$$

Hence $\mu_2 L = 8y^2(y-x)^2 > 0$ and according to [26, Table 4] on the line at infinity there exist three real singular points, two of which are double and one simple. More precisely, the double points $N_1(1,0,0)$ and $N_2(1,1,0)$ are saddle-nodes, whereas the point $N_2(0,1,0)$ is a node and the geometric configuration corresponds to *Figure 4*.

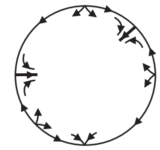


Figure 4

We observe that if the point $M_1(-g,\,0)$, located on the invariant line $y=0$ (as well as on the line $x=-g$) is a saddle (i.e. $g<0$), then we get a saddle connection with the saddle-node $N_1(1,0,0)$.

On the other hand for systems $(S_{4.16})$ we have $\mathcal{G}_2 = -3456g$. So, taking into consideration the invariant lines $x=-g$ and $y=0$ of systems $(S_{4.16})$ we obtain *Picture 4.16(a)* if $\mathcal{G}_2>0$ and *Picture 4.16(b)* if $\mathcal{G}_2<0$.

$$\textit{Config. 4.17}: \qquad \dot x = x, \quad \dot y = y(y-x). \qquad (S_{4.17})$$

We observe that this system can be obtained from the family $(S_{4.16})$ allowing the parameter g to vanish. In this case the points $M_1(-g,\,0)$ and $M_2(-g,\,-g)$ coalesced (at the origin of coordinates), yielding a saddle-node. So, as it can easily be determined, we get *Picture 4.17*.

$$\textit{Config. 4.18}: \qquad \dot x = g(g+1)+gx+y, \quad \dot y = y(y-x), \quad g(g+1)\neq 0. \qquad (S_{4.18})$$

Finite singularities: $M_1(-1-g,\,0)\big[\Delta_1=g(g+1),\ \delta_1=1\big]$; $M_2(-g,\,-g)\big[\Delta_2 = -g(g+1),\ \delta_2 = 4g(g+1),\ \rho_2=0\big]$. We observe that systems $(S_{4.18})$ possess a saddle and a node if $g(g+1)>0$ and they possess a saddle and either a focus or a center if $g(g+1)<0$. We claim that in the second case the point M_2 is a center. Indeed, moving this point to the origin of coordinates we get the systems $\dot x = gx+y,\ \ \dot y = (g-y)(x-y)$, for which considering Lemma 7 we calculate: $I_1=I_6=I_3=0,\ I_2=2g(g+1)$. Since $g(g+1)<0$ by Lemma 7 the point M_2 is a center and our claim is proved.

For systems $(S_{4.18})$ calculations yield:

$$\eta=1,\ C_2=xy(x-y),\ \mu_0=\mu_1=\kappa=0,\ L=8y(y-x),\ \mu_2=g(g+1)y(y-x).$$

Hence $\mu_2 L = 8g(g+1)y^2(y-x)^2 \neq 0$ and then $\operatorname{sign}(\mu_2 L) = \operatorname{sign}\big(g(g+1)\big)$. According to [26, Table 4] on the line at infinity there exist three real singular points, two of which are double and one simple. More precisely, the double points $N_1(1,0,0)$ and $N_2(1,1,0)$ are saddle-nodes, whereas the point $N_2(0,1,0)$ is a node.

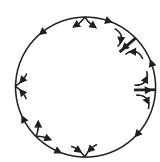


Figure 3

However, depending on the location of the saddle sectors of the saddle-nodes, at infinity there are two distinct configurations. As it was proved in [26] we have the *Figure 4* (see above) if $\mu_2 L>0$ and the *Figure 3* if $\mu_2 L<0$.

a) Case $\mu_2 L>0$. Then $g(g+1)>0$ and systems $(S_{4.18})$ possess one saddle and one node. Taking into consideration the existence of the invariant lines $y=0$ and $x-y+g+1=0$ as well as *Figure 4* we get the *Picture 4.18(a)*.

a) Case $\mu_2 L < 0$. Then $g(g+1) < 0$ and systems $(S_{4.18})$ possess one saddle and one center. Moreover, the behavior of the trajectories at infinity corresponds to *Figure 3.* In this case we obtain the *Picture 4.18(b).*

$$\textit{Config. 4.19}: \qquad \dot{x} = g + x, \quad \dot{y} = -xy, \quad g(g-1) \neq 0. \qquad (S_{4.19})$$

We observe that these systems possess one finite singular point $M_1(-g, 0)$ which is a saddle for $g < 0$ and it is a node if $g > 0$. We shall examine the infinite singularities. Considering [26] for systems $(S_{4.19})$ we calculate: $M = -8x^2 \neq 0,\ C_2 = x^2 y,\ \eta = \mu_0 = \mu_1 = \mu_2 = \kappa = \kappa_1 = L = 0,\ \mu_3 = -gx^2 y,\ K_1 = -x^2 y.$

According to [26] the point $N_1(0,1,0)$ is of the multiplicity 4 (consisting from two finite and two infinite points which have coalesced), while the singular point $N_2(1,0,0)$ is a double point which is a saddle-node (a finite and an infinite singular point being coalesced) Moreover, by [26, Table 4] the behavior of the trajectories at infinity corresponds to *Figure 12* if $\mu_3 K_1 < 0$ and to *Figure 21* if $\mu_3 K_1 > 0$.

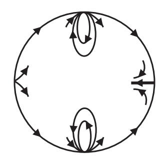


Figure 12

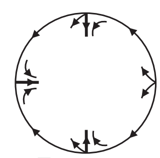


Figure 21

Since $\mu_3 K_1 = gx^4 y^2$ it follows $\operatorname{sign}(\mu_3 K_1) = \operatorname{sign}(g)$. Therefore, taking into account the existence of the invariant lines $y = 0$ and $x = -g$ and *Figures 12* and *21* we obtain *Picture 4.19(a)* if $\mu_3 K_1 < 0$ and *Picture 4.19(b)* if $\mu_3 K_1 > 0$.

$$\textit{Config. 4.20}: \qquad \dot{x} = x(gx+y), \quad \dot{y} = (g-1)xy + y^2, \quad g(g-1) \neq 0. \qquad (S_{4.20})$$

For systems $(S_{4.20})$ calculations yield: $\eta = 0,\ M = -8x^2,\ C_2 = x^2 y,\ \mu_0 = g \neq 0$. We observe that $(S_{4.20})$ is a family of homogenous systems, which possess two invariant lines: $x = 0$ (double) and $y = 0$. According to [26] on the line at infinity, besides the saddle-node $N_1(0,1,0)$ (corresponding to the double line), systems $(S_{4.20})$ have a node if $\mu_0 > 0$ (*Picture 4.20(a)*) and they have a saddle if $\mu_0 < 0$ (*Picture 4.20(b)*).

$$\textit{Config. 4.21}: \qquad \begin{cases} \dot{x} = x(gx+y), \quad g(g-1) \neq 0, \\ \dot{y} = (y+1)(gx - x + y). \end{cases} \qquad (S_{4.21})$$

Finite singularities: $M_1(0,0)\big[\Delta_1 = 0,\ \rho_1 = 1\big]$ – a saddle-node [1]; $M_2(0,-1)$ $\big[\Delta_2 = 1,\ \delta_2 = 0\big]$ – a node; $M_3(1/g,-1)\big[\Delta_3 = -1/g,\ \delta_3 = (g+1)^2/g^2\big]$ – a node if $g < 0$ and a saddle if $g > 0$. For systems $(S_{4.21})$ calculations yield: $\eta = 0,\ M = -8x^2,\ C_2 = x^2 y,\ \mu_0 = g$. Hence according to [26] on the line at infinity we have two singularities: the saddle-node $N_1(0,1,0)$ and the singular point $N_2(1,0,0)$, which is a node if $\mu_0 > 0$ and it is a saddle if $\mu_0 < 0$.

Thus, taking into account the invariant lines $x = 0$ (double) and $y = -1$ we get *Picture 4.21(a)* if $\mu_0 > 0$ and *Picture 4.21(b)* if $\mu_0 < 0$.

$$\textit{Config. 4.22}: \qquad \dot{x} = gx^2, \quad \dot{y} = (y+1)\big[y + (g-1)x - 1\big], \quad g(g-1) \neq 0. \qquad (S_{4.22})$$

Finite singularities: $M_1(0,0)\big[\Delta_1 = 0,\ \rho_1 = 2\big],\ M_2(0,-1)\big[\Delta_2 = 0,\ \rho_2 = -2\big]$ – saddle-nodes [1].

To determine the behavior of the trajectories at the infinity for systems ($S_{4.22}$) we calculate: $\eta = 1 > 0$, $C_2 = xy(x-y)$, $\mu_0 = g^2 > 0$. Thus according to [26, Table 4] on the line at infinity there exists three real singular points: $N_1(1,0,0)$, and $N_2(1,1,0)$ and $N_3(0,1,0)$. More precisely, there are two nodes and one saddle. Using the transformation $x = 1/z$, $y = u/z$) we get the systems

$$\dot{u} = u + (1-g)z - u^2 + z^2, \quad \dot{z} = gz. \tag{11}$$

For the singular point $(0,0)$ (respectively $(1,0)$) of systems (11) corresponding to the point $N_1(1,0,0)$ (respectively $N_2(1,1,0)$) of systems ($S_{4.22}$) we have $\tilde{\Delta}_1 = g$ (respectively $\tilde{\Delta}_2 = -g$). Hence we conclude that besides the node $N_3(0,1,0)$ systems ($S_{4.22}$) possess at infinity the node $N_1(1,0,0)$ and the saddle $N_2(1,1,0)$ if $g > 0$ and they possess the saddle $N_1(1,0,0)$ and the node $N_2(1,1,0)$ if $g < 0$.

On the other hand for systems ($S_{4.22}$) we have $H_1 = 1152g$. Hence, taking into consideration the invariant lines $x = 0$ (double) and $y = -1$ of systems ($S_{4.22}$) we get *Picture 4.22(a)* if $H_1 > 0$ and *Picture 4.22(b)* if $H_1 < 0$.

$$\textit{Config. 4.23}: \qquad \dot{x} = x(x+y), \quad \dot{y} = (y+1)^2. \tag{$S_{4.23}$}$$

Finite singularities: $M_1(0,-1)\big[\Delta_1 = 0,\ \rho_1 = -1\big]$, $M_2(1,-1)\big[\Delta_2 = 0,\ \rho_2 = 1\big]$ – saddle-nodes [1]. For these systems calculations yield: $\eta = 0$, $M = -8x^2$, $C_2 = x^2y$, $\mu_0 = 1 > 0$. We observe [26] that the point $N_1(0,1,0)$ is a double point and it is a saddle-node, whereas the second simple point $N_1(1,0,0)$ is a node. Thus, taking into account the invariant lines $x = 0$ and $y = -1$ (double) we get *Picture 4.23*.

$$\textit{Config. 4.24}: \qquad \dot{x} = (x+1)^2, \quad \dot{y} = (1-g)xy,\ g(g-1) \neq 0. \tag{$S_{4.24}$}$$

Finite singularities: $M_1(-1,0)\big[\Delta_1 = 0,\ \rho_1 = g-1\big]$ – saddle-node [1]. We calculate: $M = -8g^2x^2 \neq 0$, $C_2 = gx^2y$, $\eta = \mu_0 = \mu_1 = \kappa = \kappa_1 = K_2 = 0$, $\mu_2 = (g-1)^2x^2$, $L = 8gx^2$. Since $\mu_2 > 0$ and $K_2 = 0$ by [26, Table 4] the behavior of the trajectories at infinity corresponds to *Figure 19* if $L > 0$ and to *Figure 17* if $L < 0$ (see p. 67). Taking into consideration the existence of the real invariant lines $y = 0$ and $x = -1$ (double) we obtain *Picture 4.24(a)* if $L > 0$ and *Picture 4.24(b)* if $L < 0$.

$$\textit{Config. 4.25}: \qquad \dot{x} = gx^2 + xy, \quad \dot{y} = y + (g-1)xy + y^2, \quad g(g-1) \neq 0. \tag{$S_{4.25}$}$$

Finite singularities: $M_1(0,0)\big[\Delta_1 = 0,\ \rho_1 = 1\big]$ – a saddle-node [1]; $M_2(0,-1)$ $\big[\Delta_2 = 1,\ \delta_2 = 0\big]$ – a node; $M_3(1,-g)\big[\Delta_3 = -g,\ \delta_3 = 4g\big]$ – a saddle if $g > 0$ and either a focus or a center if $g < 0$.

We claim that the point M_3 is a center if $g < 0$. Indeed, translating this point to the origin of coordinates we get the systems $\dot{x} = (1+x)(gx+y)$, $\dot{y} = (g-y)(x-gx-y)$, for which considering Lemma 7 we calculate: $I_1 = I_6 = I_3 = 0$, $I_2 = 2g$. Since $g < 0$ by Lemma 7 the point M_3 is a center and our claim is proved.

On the other hand for systems ($S_{4.25}$) calculations yield: $\eta = 0$, $M = -8x^2$, $C_2 = x^2y$, $\mu_0 = g$. Hence according to [26] at infinity we have two singularities: the saddle-node $N_1(0,1,0)$ and the singular point $N_2(1,0,0)$, which is a node if $\mu_0>0$ and it is a saddle if $\mu_0<0$. Hence, taking into account the invariant lines $x=0$ (double) and $y=-1$ we get *Picture 4.25(a)* if $\mu_0>0$ and *Picture 4.25(b)* if $\mu_0<0$.

$$\textit{Config. 4.26}: \quad \dot{x}=xy, \quad \dot{y}=(y+1)(y-x). \qquad (S_{4.26})$$

Finite singularities: $M_1(0,0)\big[\Delta_1=0,\ \rho_1=1\big]$ – a saddle-node [1]; $M_2(0,-1)\big[\Delta_2=1,\ \delta_2=0\big]$ – a node. For systems $(S_{4.26})$ we calculate: $M=-8x^2\neq 0,\ C_2=x^2y,\ \eta=\mu_0=0,\ \mu_1=y,\ K=2y^2$.

According to [26, Table 4] in this case the behavior of the trajectories at infinity corresponds to *Figure 20*. So, taking into consideration the invariant lines $x=0$ (double) and $y=0$ of systems $(S_{4.26})$ we obtain *Picture 4.26*

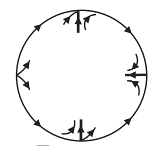


Figure 20

$$\textit{Config. 4.27}: \quad \dot{x}=2gx+2y, \quad \dot{y}=g^2+1-x^2-y^2, \quad g\in\mathbb{R}. \qquad (S_{4.27})$$

Finite singularities: $M_1(-1,\ g)\big[\Delta_1=-4(g^2+1)\big]$ – a saddle; $M_2(1,\ -g)\big[\Delta_2=4(g^2+1),\ \delta_2=-16\big]$. We observe that the singular point M_2 is a strong focus if $g\neq 0$ and it is a center if $g=0$. Indeed, translating this point to the origin of coordinates we get the systems $\dot{x}=2(gx+y)$, $\dot{y}=-2x+2gy-x^2-y^2$, for which we calculate: $I_1=4g,\ I_6=-8g,\ I_{13}=-2g,\ I_2=8(g^2-1)$ So, by Lemma 7 the point M_2 is a center if and only if $g=0$. To determine the behavior of the trajectories at the infinity for systems $(S_{4.27})$ calculations yield: $\eta=-4,\ C_2=x(x^2+y^2),\ \mu_0=\mu_1=\kappa=0,\ \mu_2=4(g^2+1)(x^2+y^2)$. So, according to [26] the unique real infinite singular point $N_1(0,1,0)$ of $(S_{4.27})$ is a node. Therefore, since for these systems we have $\mathcal{G}_1=16g$, we obtain *Picture 4.27(a)* if $\mathcal{G}_1\neq 0$ and *Picture 4.27(b)* if $\mathcal{G}_1=0$.

$$\textit{Config. 4.28}: \quad \dot{x}=x^2-1, \quad \dot{y}=x+gy, \quad g(g^2-4)\neq 0. \qquad (S_{4.28})$$

Finite singularities: $M_1(1,\ -1/g)\big[\Delta_1=2g,\ \delta_1=(g-2)^2\big]$; $M_2(-1,\ 1/g)\big[\Delta_2=-2g,\ \delta_2=(g+2)^2\big]$. We observe that systems $(S_{4.28})$ possess a node and a saddle. For these systems we calculate: $\eta=0,\ M=-8x^2,\ C_2=x^2y,\ \mu_0=\mu_1=\kappa=\kappa_1=0,\ L=8x^2,\ \mu_2=g^2x^2,\ K_2=384x^2$, and according to [26, Table 4] the behavior of the trajectories in the vicinity of infinity corresponds to *Figure 19* (see page 67). Taking into consideration the existence of the real invariant lines $x=\pm 1$ we obtain in both cases (i.e. either $g>0$ or $g<0$) the phase portraits topologically equivalent to *Picture 4.28*.

$$\textit{Config. 4.29}: \quad \dot{x}=x^2-1, \quad \dot{y}=g+x, \quad g^2-1\neq 0. \qquad (S_{4.29})$$

No finite singularities. For these systems calculations yield: $M=-8x^2,\ C_2=x^2y$,

$$\eta=\mu_0=\mu_1=\mu_2=\mu_3=\kappa=\kappa_1=0,\ L=8x^2,\ \mu_4=(g^2-1)x^4,\ K_2=384x^2.$$

We observe that $L > 0$, $K = 0$ and $K_2 > 0$ for any value of parameters $\pm 1 \neq g \in \mathbb{R}$. According to [26, Table 4] the behavior of the trajectories in the vicinity of infinity corresponds to *Figure 18* if $\mu_4 > 0$ and to *Figure 24* if $\mu_4 < 0$. Thus, taking into account the existence of the invariant lines $x = \pm 1$ we get *Picture 4.29(a)* if $\mu_4 > 0$ and *Picture 4.29(b)* if $\mu_4 < 0$.

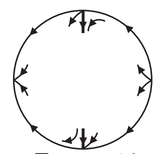


Figure 18

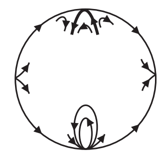


Figure 24

$$\textit{Config. 4.30}: \quad \dot{x} = (x+1)(gx+1), \quad \dot{y} = 1 + (g-1)xy, \quad g(g^2-1) \neq 0. \quad (S_{4.30})$$

Finite singularities: $M_1(-1,\, 1/(g-1))\big[\Delta_1 = (g-1)^2,\ \delta_1 = 0\big]$ – a node; $M_2(-1/g,\, g/(g-1))\big[\Delta_2 = -(g-1)^2/g,\ \delta_2 = (g^2-1)^2/g^2\big]$ – a node if $g < 0$ and a saddle if $g > 0$. For systems $(S_{4.30})$ calculations yield: $\eta = 0,\ M = -8x^2,\ C_2 = x^2y,\ \mu_0 = \mu_1 = \kappa = \kappa_1 = 0$ and

$$L = 8gx^2,\ \mu_2 = g(g-1)^2x^2,\ K_2 = 48(g-1)^2(g^2-g+2)x^2.$$

Since $\operatorname{sign}(\mu_2) = \operatorname{sign}(L) = \operatorname{sign}(g)$ and $K_2 > 0$, according to [26, Table 4] the behavior of the trajectories around the infinity corresponds to *Figure 19* (see p. 67) if $g > 0$ and to *Figure 29* (see p. 67) if $g < 0$. Taking into consideration the real invariant lines and $x+1 = 0$ (double) and $(gx+1) = 0$ we obtain the phase portrait *Picture 4.30(a)* if $\mu_2 > 0$ and *Picture 4.30(b)* if $\mu_2 < 0$.

$$\textit{Config. 4.31}: \quad \dot{x} = x(x+1), \quad \dot{y} = g - x^2 + xy, \quad g(g+1) \neq 0. \qquad (S_{4.31})$$

Finite singularities: $M_1(-1, g-1)\big[\Delta_1 = 1,\ \delta_1 = 0\big]$ – a node. We calculate:

$$\eta = M = 0,\ C_2 = x^3,\ \mu_0 = \mu_1 = \mu_2 = 0,\ \mu_3 = -gx^3, \quad K = 2x^2, \quad K_3 = -6gx^6.$$

Since $\mu_3 K \neq 0$ by [26, Table 4] the behavior of the trajectories in the neighborhood of infinity corresponds to *Figure 37* if $K_3 > 0$ (i.e. $g < 0$) and to *Figure 39* if $K_3 < 0$ (i.e. $g > 0$). Thus, taking into account the invariant lines $x = 0$ and $x = -1$ (double) of systems $(S_{4.31})$ we get *Picture 4.31(a)* if $K_3 > 0$ and *Picture 4.31(b)* if $K_3 < 0$.

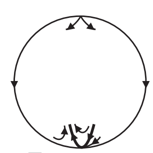


Figure 37

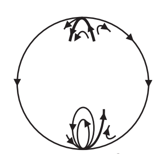


Figure 39

$$\textit{Config. 4.32}: \quad \dot{x} = x^2 + 1, \quad \dot{y} = x + gy, \quad g \neq 0. \qquad (S_{4.32})$$

No finite singularities. For these systems calculations yield: $M = -x^2,\ C_2 = x^2y$,

$$\eta = \mu_0 = \mu_1 = \kappa = \kappa_1 = K = 0\ L = 8x^2,\ \mu_2 = g^2x^2,\ K_2 = -384x^2.$$

We note that $\mu_2 > 0$, $L > 0$ and $K_2 < 0$ for any value of parameter $0 \neq g \in \mathbb{R}$. According to [26, Table 4] the behavior of the trajectories at infinity corresponds to *Figure 8* (see p. 68). This leads to *Picture 4.32*.

$$\textit{Config. 4.33}: \quad \dot{x} = x^2+1, \quad \dot{y} = g+x, \quad g \in \mathbb{R}. \qquad (S_{4.33})$$

This family of systems does not possess real finite singularities and calculations yield:

$$M = -8x^2,\ C_2 = x^2y,\ \eta = \mu_0 = \mu_1 = \mu_2 = \mu_3 = \kappa = \kappa_1 = K = 0,\ L = 8x^2$$

and $\mu_4 = (g^2+1)x^4$, $K_2 = -384x^2$. We observe that $\mu_4 > 0$, $L > 0$ and $K_2 < 0$ for any value of the parameter $g \in \mathbb{R}$. According to [26, Table 4] the behavior of the trajectories around of infinity corresponds to *Figure 8* (see p. 68). Thus we obtain *Picture 4.33*.

$$\textit{Config. 4.34}: \quad \dot{x} = g, \quad \dot{y} = y(y-x), \quad g \in \{-1,1\}. \qquad (S_{4.34})$$

No finite singularities. For these systems we calculate:

$$\eta = 1,\ C_2 = xy(x-y),\ \mu_0 = \mu_1 = \mu_2 = \mu_3 = \kappa = K_1 = 0,\ \mu_4 = g^2y^2(x-y)^2.$$

We note that $\mu_4 \neq 0$ and $K_1 = 0$ for any value of parameter $g \in \{-1,1\}$. According to [26, Table 4] one of the triple points is a node and the other one is a saddle. However we need to distinguish when the point $N_1(1,0,0)$ is a saddle, as in this case the invariant line $y = 0$ will be a separatrix and this leads to a different phase portrait. So, we consider the corresponding systems (obtained via the transformation $x = 1/z$, $y = u/z$):

$$(S): \qquad \dot{u} = -u + u^2 - guz^2, \quad \dot{z} = -gz^3.$$

We observe that systems (S) has two invariant lines: $z = 0$ and $u = 0$. We consider the restrictions on (S) on these lines: $(S)|_{z=0}: \quad \dot{u} = u(u-1)$ and $(S)|_{u=0}: \dot{z} = -gz^3$. On $z = 0$ and for $0 < u < 1$ we have $\dot{u} < 0$ while for $u < 0$, we have $\dot{u} > 0$. Hence on $z = 0$ the point $u = 0$ is an attractive singular point.

Now consider the restriction $(S)|_{u=0}$. We observe, that for $z > 0$, $\operatorname{sign}(\dot{z}) = -\operatorname{sign}(z)$. Hence on $u = 0$ the point $z = 0$ is an attractive singular point if $g > 0$ and it is a repulsing singular point if $g < 0$.

Thus we conclude that the triple singular point $N_1(1,0,0)$ of systems $(S_{4.34})$ is a node if $g > 0$ and it is a saddle if $g < 0$. On the other hand for systems $(S_{4.34})$) we have $H_4 = -48g$. So, considering invariant line $y = 0$ we get *Picture 4.34(a)* if $H_4 < 0$ and *Picture 4.34(b)* if $H_4 > 0$.

$$\textit{Config. 4.35}: \quad \dot{x} = g+y, \quad \dot{y} = xy, \quad g \in \{-1,1\}. \qquad (S_{4.35})$$

Finite singularities: $M_1(0,-g)\big[\Delta_1 = g,\ \delta_1 = -4g\big]$. So, the point M_1 is a saddle if $g < 0$ and we claim that it is a center if $g > 0$. Indeed, translating this point to the origin of coordinates we get the systems $\dot{x} = y, \quad \dot{y} = x(y-g)$, for which calculations yield: $I_1 = I_6 = I_3 = 0,\ I_2 = -2g$. So, by Lemma 7 the point M_1 is a center if and only if $g > 0$ and this proves our claim.

To determine the behavior of the trajectories around the infinity for systems $(S_{4.35})$ we calculate: $\eta = 0,\ M = -8x^2,\ C_2 = x^2y,\ \mu_0 = \mu_1 = \mu_2 = \kappa = L = 0,$ $\kappa_1 = -32,\ \mu_3 = gxy^2,\ K_1 = xy^2$.

Since $\kappa = L = 0,\ \kappa_1 \neq 0$ and $\mu_3 K_1 = gx^2y^4$ (i.e. $sign(g) = \text{sign}\,(\mu_3 K_1)$), according to [26, Table 4] the behavior of the trajectories in the vicinity of infinity corresponds to *Figure 16* if $g < 0$ and to *Figure 9* if $g > 0$. So, considering invariant line $y = 0$ and the type of the singular point $M_1(0, -g)$ we get *Picture 4.35(a)* if $\mu_3 K_1 > 0$ and *Picture 4.35(b)* if $\mu_3 K_1 < 0$.

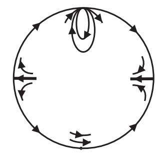


Figure 16

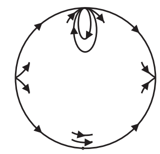


Figure 9

$$\textit{Config. 4.36}: \qquad \dot{x} = g, \quad \dot{y} = xy, g \in \{-1, 1\}. \qquad (S_{4.36})$$

No finite singularities. For these systems we calculate: $M = -8x^2,\ C_2 = -x^2y,$

$$\eta = \mu_0 = \mu_1 = \mu_2 = \mu_3 = \kappa = \kappa_1 = L = K_1 = 0,\ \kappa_2 = g,\ \mu_4 = g^2x^2y^2.$$

We note that $\mu_4 \neq 0$, $L = K_1 = 0$ and $\text{sign}\,(g) = \text{sign}\,(\kappa_2)$. According to [26, Table 4] the behavior of the trajectories around infinity corresponds to *Figure 8* (see p. 68) if $g < 0$ and it corresponds to *Figure 17* (see p. 67) if $g > 0$. Therefore, taking into account the invariant line $y = 0$ we obtain *Picture 4.36(a)* if $\kappa_2 < 0$ and *Picture 4.36(b)* if $\kappa_2 > 0$.

$$\textit{Config. 4.37}: \qquad \dot{x} = x, \quad \dot{y} = gy - x^2, \quad g(g^2 - 1) \neq 0. \qquad (S_{4.37})$$

Finite singularities: $M_1(0,0)\big[\Delta_1 = g,\ \delta_1 = (g-1)^2\big]$ – a saddle if $g < 0$ and a node if $g > 0$. For systems $(S_{4.37})$ calculations yield: $\eta = M = 0,\ C_2 = x^3,$ $\mu_0 = \mu_1 = \mu_2 = 0,\ \mu_3 = -gx^3,\ K = 0,\ K_1 = -x^3,\ K_3 = 6g(2-g)x^6$.

Since $K = 0$ and $\mu_3 K_1 \neq 0$ by [26, Table 4] if $\mu_3 K_1 > 0$ and $K_3 \geq 0$ then the singular point $N_1(0,1,0)$ is a saddle-node (with saddle sectors located on the same part of the line $Z = 0$). Otherwise the behavior of the trajectories around infinity corresponds to *Figure 38* if $\mu_3 K_1 > 0$ and $K_3 < 0$ and it corresponds to *Figure 33* if $\mu_3 K_1 < 0$.

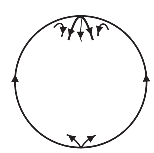


Figure 38

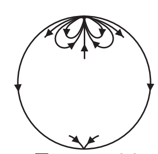


Figure 33

Thus considering the invariant line $x = 0$ and the type of the singularity $M_1(0,0)$ of $(S_{4.37})$ we obtain: *Picture 4.37(a)* when $\mu_3 K_1 > 0$ and $K_3 \geq 0$; *Picture 4.37(b)* when $\mu_3 K_1 > 0$ and $K_3 < 0$; *Picture 4.37(c)* when $\mu_3 K_1 < 0$.

$$\textit{Config. 4.38}: \qquad \dot{x} = x, \quad \dot{y} = g - x^2, \quad 0 \neq g \in \mathbb{R}. \qquad (S_{4.38})$$

No finite singularities. For these systems we calculate:

$$\eta = M = 0,\ C_2 = x^3,\ \mu_0 = \mu_1 = \mu_2 = \mu_3 = 0,\ \mu_4 = -gx^4,\ K = K_3 = 0.$$

Since $K = K_3 = 0$ by [26, Table 4] if $\mu_4 > 0$ then the point $N_1(0,1,0)$ is a node. In the case $\mu_4 < 0$ the behavior of the trajectories at infinity corresponds to *Figure 35*.

Thus taking into account the invariant line $x = 0$ we obtain *Picture 4.38(a)* if $\mu_4 > 0$ and *Picture 4.38(b)* if $\mu_4 < 0$.

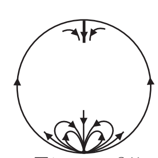


Figure 35

$$\textit{Config. 4.39}: \qquad \dot{x}=x^2, \quad \dot{y}=x+y. \qquad (S_{4.39})$$

Finite singularities: $M_1(0,0)\big[\Delta_1=0,\ \rho_1=1\big]$ – a saddle-node [1]. Calculations yield: $\eta=0,\ M=-8x^2,\ C_2=x^2y,\ \mu_0=\mu_1=\kappa=\kappa_1=0,\ \mu_2=x^2,\ L=8x^2,$ $K_2=0$. As $\mu_2>0$, $L>0$ and $K_2=0$, according to [26, Table 4] the behavior of the trajectories in the neighborhood of infinity corresponds to *Figure 19* (see p. 67).

Thus, taking into account the invariant line $x=0$ we obtain *Picture 4.39.*

$$\textit{Config. 4.40}: \qquad \dot{x}=x+1, \quad \dot{y}=1-xy. \qquad (S_{4.40})$$

Finite singularities: $M_1(-1,-1)\big[\Delta_1=1,\ \delta_1=0\big]$ – a node. For systems $(S_{4.40})$ we calculate: $\eta=0,\ M=-8x^2,\ C_2=x^2y,\ \mu_0=\mu_1=\mu_2=\kappa=\kappa_1=L=0,$ $\mu_3=-x^2y,\ K_1=-x^2y$. We observe that $L=0$ and $\mu_3K_1=x^4y^2>0$. So, according to [26, Table 4] the behavior of the trajectories at infinity corresponds to *Figure 21* (see p. 70). Considering the invariant line $x=0$ we obtain *Picture 4.40.*

$$\textit{Config. 4.41}: \qquad \dot{x}=gxy, \quad \dot{y}=y-x^2+gy^2, \quad g\in\{-1,1\}. \qquad (S_{4.41})$$

Finite singularities: $M_1(0,0)\big[\Delta_1=0,\rho_1=1\big]$; $M_2(0,-1/g)\big[\Delta_2=1,\ \delta_2=0\big]$ – a node. We observe that M_1 is triple, as according to [1, §22] in its vicinity we obtain $\varphi(x)=\tilde{\Delta}_3\,x^3+\ldots=gx^3+\ldots,\ g\in\{-1,1\}$. Moreover, since $\operatorname{sign}(\tilde{\Delta}_3)=\operatorname{sign}(g)$ by [1, §22] we conclude that the triple singular point $M_1(0,0)$ is a (topological) node if $g>0$, and it is a (topological) saddle if $g<0$.

We shall examine the infinite singularities. For systems $(S_{4.41})$ calculations yield: $\eta=0=M$, $C_2=x^3$, $\mu_0=-g^3\neq0$. Hence according to [26, Table 4] the triple singular point $N_1(0,1,0)$ is a node if $\mu_0>0$ (i.e. $g=-1$) and it is a saddle if $\mu_0<0$ (i.e. $g=1$). So, in the first case we get *Picture 4.41(a)*, while in the second one we get *Picture 4.41(b).*

$$\textit{Config. 4.42}: \qquad \dot{x}=gxy, \quad \dot{y}=-x^2+gy^2, \quad g\in\{-1,1\}. \qquad (S_{4.42})$$

We observe that $(S_{4.42})$ are homogenous systems, which possess the triple invariant line $x=0$. As for these systems $\eta=0=M,\ C_2=x^3,\ \mu_0=-g^3\neq0$, then according to [26] the infinite point $N_1(0,1,0)$ is a node if $\mu_0>0$ (*Picture 4.42(a)*) and it is a saddle if $\mu_0<0$ (*Picture 4.42(b)*).

$$\textit{Config. 4.43}: \qquad \dot{x}=gx^2, \quad \dot{y}=1+(g-1)xy, \quad g(g^2-1)\neq0. \qquad (S_{4.43})$$

No finite singularities. For these systems we calculate: $\eta=\mu_0=\mu_1=\mu_2=\mu_3=\kappa=\kappa_1=0,\ M=-8x^2, C_2=x^2y, \mu_4=g^2x^4, L=8gx^2, K=2g(g-1)x^2, R=8g(2g-1)x^2$.

As $\mu_4>0$ according to [26, Table 4] the behavior of the trajectories around infinity corresponds to *Figure 17* (see p. 67) if $L<0$. And since $K\neq0$, in the case $L>0$ we have *Figure 18* (see p. 73) if $R\geq0$ and *Figure 28* if $R<0$. Thus, taking into account the triple invariant line $x=0$ we obtain: *Picture 4.43(a)* if $L<0$; *Picture 4.43(b)* if $L>0$ and $R\geq0$; *Picture 4.43(c)* if $L>0$ and $R<0$.

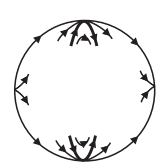


Figure 28

$$\textit{Config. 4.44}: \quad \dot{x} = x^2, \quad \dot{y} = g - x^2 + xy, \quad g \in \{-1, 1\}. \qquad (S_{4.44})$$

No finite singularities. For these systems we calculate: $\eta = M = 0,\ C_2 = x^3,$

$$\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0,\ \mu_4 = g^2x^4,\ K = 2x^2,\ K_3 = -6gx^6.$$

Hence, by [26] the point $N_1(0,1,0)$ is of multiplicity seven (all finite and infinite singularities have coalesced at this point). As $\mu_4 > 0$ and $K \neq 0$, according to [26, Table 4] this point is a node if $K_3 > 0$ (i.e. $g = -1$, we get *Picture 4.44(a)*) and the behavior of the trajectories around infinity is as in *Figure 36* if $K_3 < 0$ (i.e. $g = 1$, we get *Picture 4.44(b)*).

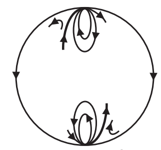


Figure 36

$$\textit{Config. 4.45}: \quad \dot{x} = gxy, \quad \dot{y} = x - x^2 + gy^2, \quad g \in \{-1, 1\}. \qquad (S_{4.45})$$

Finite singularities: $M_1(0,0)\big[\Delta_1 = 0, \rho_1 = 0\big]$; $M_2(1,0)\big[\Delta_2 = g,\ \delta_2 = -4g,\ \rho_2 = 0\big]$. The point M_2 is a saddle if $g < 0$ and we claim that it is a center if $g > 0$. Indeed, translating this point to the origin of coordinates we get the systems $\dot{x} = g(x+1)y,\ \dot{y} = -x - x^2 + gy^2$, for which calculations yield: $I_2 = -2g$, $I_1 = I_6 = I_3 = 0$. By Lemma 7 the point M_2 is a center if and only if $g > 0$ and this has proved our claim.

Let us examine the multiple point $M_1(0,0)$. We observe that M_1 is a nilpotent singular point. According to [1, §22] in its vicinity we calculate $\psi(x) = \tilde{a}_3x^3 + \ldots = -g^2x^3 + \ldots,\quad \sigma(x) = \tilde{b}_1\,x + \ldots = 3gx$. Hence we obtain $a_3 = -g^2 < 0$ and for the quantity γ (see [1, §22]) in this case we obtain: $\gamma = \tilde{b}_1^2 + 8\tilde{a}_3 = g^2 > 0$. Therefore, the triple point is an "elliptic saddle" (i.e. a non–elementary singular point having one elliptic and one hyperbolic sectors [1]).

To determine the behavior of the trajectories at the infinity for systems $(S_{4.45})$ we calculate: $\eta = 0 = M$, $C_2 = x^3$, $\mu_0 = -g^3 \neq 0$. Hence according to [26, Table 4] the triple singular point $N_1(0,1,0)$ is a node if $\mu_0 > 0$ (*Picture 4.45(a)*) and it is a saddle if $\mu_0 < 0$ (*Picture 4.45(b)*).

$$\textit{Config. 4.46}: \quad \dot{x} = 1, \quad \dot{y} = y = y - x^2. \qquad (S_{4.46})$$

No finite singularities. For these systems calculations yield:

$$\eta = M = 0,\ C_2 = x^3,\ \mu_0 = \mu_1 = \mu_2 = \mu_3 = 0,\ \mu_4 = x^4,\ K = 0,\ K_3 = -6x^6.$$

Hence, by [26] the point $N_1(0,1,0)$ is of multiplicity seven seven (all finite and infinite singularities have coalesced at this point). As $\mu_4 > 0$, $K = 0$ and $K_3 < 0$, according to [26, Table 4] the behavior of the trajectories around infinity is as indicated in *Figure 32*. Thus, we obtain *Picture 4.46*.

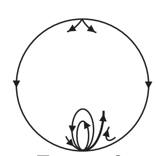


Figure 32

It remains to retain out of the 93 phase portraits *Picture 4.i(j)* in Tables 3(u), $u \in \{a, b, c, d, e\}$ only portraits which are topologically distinct. This is what we now do.

Three real singular points at infinity ($\eta > 0$)

Type of infinite singularities	Number and type of finite singularities; number of canonical regions and of separatrices					Total # of phase portraits
	0	1	2	3	4	
(N,N,N)		4.5(b)			(S,S,S,C): 4.1(b); (S,S,S,N): 4.3(b)	3
(N,N,S)	$0SC$: 4.13(a)$\simeq$ 4.34(b); $1SC_{\infty}^{\infty}$: 4.13(b)$\simeq$ 4.34(a)	4.5(a)	$0SC_f^{\infty}$: 4.22(a); $1SC_f^{\infty}$: 4.22(b)	$0SC_f^{\infty}$,$1SC_f^f$: 4.10(a); $0SC$: 4.10(b) $1SC_f^{\infty}$,$0SC_f^f$: 4.10(c);	$0SC$: 4.1(a) $\simeq$ 4.3(a) $\simeq$ 4.9(b); $1SC_f^{\infty}$, $0SC_f^f$: 4.9(a); $0SC_f^{\infty}$, $1SC_f^f$: 4.9(c);	11
(N,S,S)		4.5(c)			(N,N,N,S): 4.3(c); (N,N,C,S): 4.1(c)	3
(N,S,S-N)				4.4(b)		1
(N,S-N,S-N)		4.17	(N,S): $0SC$: 4.18(a) $\simeq$ 4.16(b); $1SC_f^{\infty}$: 4.16(a); (S,C): 4.18(b)			4
(N,N,S-N)				4.4(a)		1
Total number of topologically distinct phase portraits						23

In order to distinguish topologically the phase portraits of the systems we obtained, we use the following invariants:

- The topological types of the infinite singularities. Whenever we have several sectors on the Poincaré disk we indicate the types of sectors, e.g. PEH means that we have three sectors (on the Poincaré disk): parabolic, elliptic, hyperbolic. In the case $\eta = 0 \neq M$ we place two opposite singularities at infinity at the north and south poles. Then for example in *Picture 4.29(b)* HHH-PEP means that the north pole has three hyperbolic sectors and the south pole has a parabolic sector followed by an elliptic sector and a parabolic one.
- Number and type of distinct finite real singular points.
- The total number SC (respectively the numbers SC_f^f, SC_f^{∞}, SC_{∞}^{∞}) of separatrix connections, i.e. of phase curves connecting two singularities which are

local separatrices of the two singular points (respectively of separatrix connections connecting two infinite singularities, a finite with an infinite singularity, two finite singularities).

One real and two complex singular points at infinity ($\eta < 0$)

Type of the infinite singularity	Number and type of finite singularities		Total number of phase portraits
	1	2	
(N)	4.8(a)	4.2(a)$\simeq$ 4.27(a)(S;F); 4.6(a)(S;N)	3
(S)	4.8(b)	4.2(c)(C,F); 4.2(d)(C,C); 4.6(b)(N;N)	4
(S-N)	4.7		1
(PHP-PHP)		4.2(b)$\simeq$ 4.27(b)	1
Total number of topologically distinct phase portraits			9

Only one singular point (real) at infinity ($\eta = 0 = M$, $C_2 \neq 0$)

Type of infinite singularity	Number and type of finite singularities			Total number of phase portraits
	0	1	2	
(N)	4.38(b)$\simeq$ 4.44(a)	4.31(a)(N) 4.42(a)(HH)	4.41(a)(S,N); 4.45(a) (S,HPEP)	5
(S)		4.42(b)	4.41(b)(N,N); 4.45(b)(EH,C)	3
(S-N)		4.37(a)		1
Existence of an elliptic sector	4.38(a)(HH-EE); 4.44(b)(EH-HE); 4.46(PEH-P)	4.31(b)(N); 4.37(c)(S)		5
(HPH-P)		4.37(b)		1
Total number of topologically distinct phase portraits				15

Two real singular points at infinity ($\eta = 0$, $M \neq 0$)

Type of infinite singularities	Number and type of finite singularities; number of canonical regions and of separatrices					Total # of phase port-raits
	0	1	2	3	4	
(N,N)	4.43(c)				4.11(b)	2
(N,S-N)	$1SC_{\infty}^{\infty}$: 4.14 $\simeq$ 4.43(b); $0SC$: 4.29(a)	4.20(a)	4.23 ((N,S-N)	4.21(a)$\simeq$ 4.25(a)	4.11(a)	6
(N,PEP-PEP)			4.12(c)			1
(N,H-H)	4.15(a)$\simeq$ 4.32$\simeq$ 4.33$\simeq$ 4.36(a)					1
(N, PH-PH)		4.24(a)$\simeq$ 4.39	4.12(a)$\simeq$ 4.30(a)$\simeq$ 4.28			2
(N,HHH-HHH)			4.12(d)			1
(N,PEP-H)		4.35(b)				1
(N,HHH-PEP)	4.29(b)					1
(S,S-N)		4.20(b)		4.21(b) (N,N,S-N); 4.25(b) (N,C,S-N)		3
(S,EP-EP)		4.24(b)	4.12(b)			2
(S,PE-EP)	4.43(a)					1
(S,PHP-PHP)			4.30(b)$\simeq$ 4.12(e)			1
(S,E-H)		4.35(a)				1
(S,E-E)	4.15(b)$\simeq$ 4.36(b)					1
(S-N,S-N)			4.26			1
(S-N,PH-PH)		4.19(b)$\simeq$ 4.40				1
(S-N,EP-EP)		4.19(a)				1
Total number of topologically distinct phase portraits						27

Confrontation of phase portraits with $\eta < 0$ with those with $\eta = 0 = M$, $C_2 \neq 0$

<table>
<tr><td rowspan="2">Type of the infinite singularity in the two cases</td><td colspan="4">Number and type of finite singularities</td><td rowspan="2">Total # of phase port-raits</td></tr>
<tr><td colspan="2">1</td><td colspan="2">2</td></tr>
<tr><td rowspan="2">(N)</td><td>4.8(a)(HH)</td><td>4.31(a)(N);
4.42(a)((HH)</td><td>4.27(a)(S,F);
4.6(a)(S,N)</td><td>4.41(a)(S,N);
4.45(a)(S,HPEP)</td><td rowspan="2">5</td></tr>
<tr><td colspan="2">4.8(a)$\simeq$ 4.42(a)</td><td colspan="2">4.6(a)$\simeq$ 4.41(a)</td></tr>
<tr><td rowspan="2">(S)</td><td>4.8(b)(EE)</td><td>4.42(b)(EE)</td><td>4.2(c)(C,F);
4.2(d)(C,C);
4.6(b)(N,N)</td><td>4.41(b)(N,N);
4.45(b)(EH,C)</td><td rowspan="2">5</td></tr>
<tr><td colspan="2">4.8(b)$\simeq$ 4.42(b)</td><td colspan="2">4.6(b)$\simeq$ 4.41(b)</td></tr>
<tr><td rowspan="2">(S-N)</td><td>4.7(N)</td><td>4.37(a)(N)</td><td colspan="2" rowspan="2"></td><td rowspan="2">1</td></tr>
<tr><td colspan="2">4.7 $\simeq$ 4.37(a)</td></tr>
<tr><td colspan="5">Total number of topologically distinct phase portraits for
$\eta < 0$ or ($\eta = 0 = M$ and $C_2 \neq 0$)</td><td>24 - 5
=19</td></tr>
</table>

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