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Cooperative properties of multiple quantum scattering: II. Coherentlasing

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Abstract

The description model of the multiple scattering lasers using the superposition states between the generated photons in the ensemble of bi-modes of the resonator field, we introduced a concept of indistinguishable energy portions generated in the resonator following multiple scattering. Each of these quasi-quanta has energy equal to the difference between the pumping and scattering quantum energies at each step of multiple scattering. The conversion of the photons in the external electromagnetic mode of the resonator destroys this cooperative correlation between the bi-modes and established conservation lows during the cooperative emission. The master equation containing two parameters connected with the gain of quasi-energy portions, and their annihilation due to the losses from the resonator, is proposed. The competition between these two processes is numerically studied. An attractive problem is connected with established quantum correlations between the photons belonging to non-adjacent modes of the cavity. The time behavior of the evolution of correlations between such modes is observed. This conception may be used for the teleportation of information in other modes during the multiple Raman conversion.

1. Introduction

Multiple scattering process of particles was at the center of attention in many investigations (see for example [1–6]). The classical aspects of multiple scattering lights are in the potential applications in holography, medical instrumentation, laser spectroscopy [6–8], LIDAR [9, 10], and nonlinear optical mixers [11–15]. Recently, the specific attention is given to the new type of coherent emissions, which occur not only among the same quanta but between the photon groups generated in the nonlinear interaction of the electromagnetic field (EMF) with emitters (atoms, molecules, biomolecules, etc). The quantum aspects of this type of emission were intensively studied in [16–22], but multiple conversions of the photons and their quantum correlations remain today in development studies [22]. A pump laser beam and a weaker probe light are co-propagating in scattered medium consisting of a gas. This type of light generation supports the idea of coherent correlation that appears in the bi-modal field, in which the entangled photons are generated. A physical characteristic of the radiation formed from the blocks of well-correlated bi-modes must be determined by the intensity of the electric field of each mode, and propriety in such superposition.

An attractive aspect of the problem consists in the selective two-quantum excitation of some atoms, or molecules of the system, where it is necessary to minimize the dipole active radiation in comparison with Raman emission. The last idea can be applied in microbiology [18, 23–25], where a selective dis-activation of some molecular structures (for example of viruses) in the tissue may become possible in induced Raman excitation. In such situations appears a necessity for a good description of both the amplitude and phase of this new type of radiation formed from bimodal correlated photons. Another application of multiple scattering coherence may be used in photon recycling inside solar cells. Here the photon absorption is accompanied by the excitation of

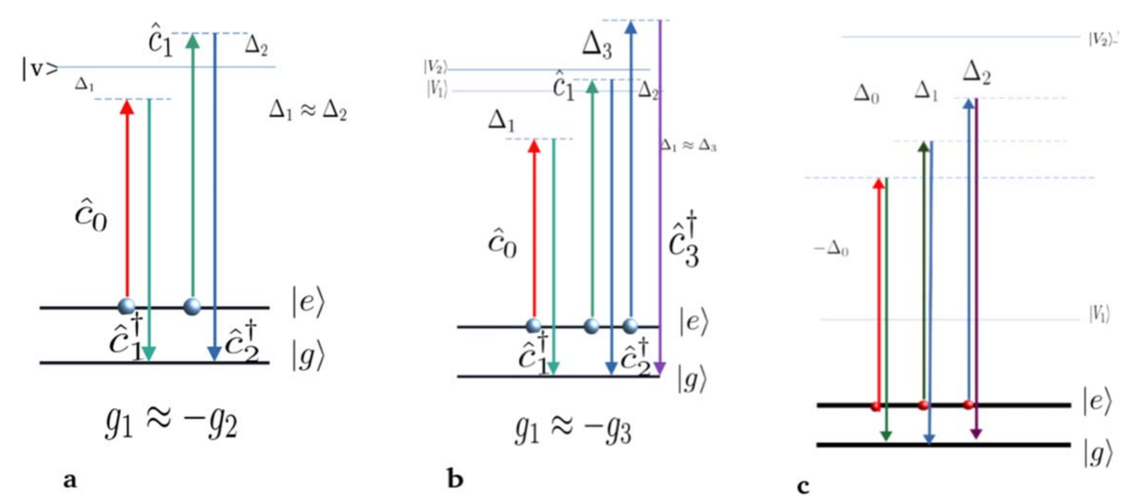

Figure 1. Possible realization of $su(2)$ and $su(1, 1)$ symmetries in the multiple scattering transitions. *a*. Realization of two steps $su(2)$ scattering with the same detuning relative to the virtual state V; *b*. The possible connections between the scattering amplitudes in the three steps process for $su(2)$ symmetry, when the frequency of the second step of such scattering is situated between the two virtual states. *c*. The possible realization of $su(1, 1)$ symmetry in multiple scattering, when the process is situated between the two virtual states of the radiator at a relatively big energetic distance with the same magnitude of transition amplitudes through both scattering processes.

the change carrier and reemitted another photon with lower energy, which in the next step participates in the same cycle of reabsorption and generation of a new carrier in semiconductors like perovskites [26–29].

In [30] we take into consideration the quantum correlations between two-photon lasing and induced scattering emission in a single act cooperative scattering emission in which the pump field is converted into anti-Stokes one. This article is focused on multistep cooperative scattering, in which the emission anti-Stokes photons may be reabsorbed for generation of the next anti-Stokes quanta during the multistep scattering effect . In the 'n' steps of the scattering process, we have the pump mode at frequency, ω_{p_0}, stimulated by the excited ensemble of radiators relative to the transition energy, $\hbar\omega_r$ in the new modes, $\omega_1 = \omega_{p_0} + \omega_r$; $\omega_2 = \omega_{p_0} + 2\omega_r$ ..., $\omega_n = \omega_{p_0} + n\omega_r$ (see figure 1).

Taking into consideration that the emission field contains superposition of these multiple components of EMF discussed in multiple scattering quantum nutation [31], $\mathbf{E}^{(+)}(z,\ t) = \mathbf{E}^{(+)}_{p_0}(z,\ t) + \mathbf{E}^{(+)}_{as1}(z,\ t) + \mathbf{E}^{(+)}_{as2}(z,\ t) + \ldots + \mathbf{E}^{(+)}_{asn}(z,\ t)$, we introduce the new characteristic of the such cavity field: $\hat{\Pi}_1^+(t) \sim \hat{E}^{(+)}_{asj}(t)\hat{E}^{(-)}_{asj-1}(t)$; $\hat{\Pi}_2^+(t) \sim \hat{E}^{(+)}_{asj}(t)\hat{E}^{(-)}_{asj-2}(t)$; ...; $\hat{\Pi}_\alpha^+(t) \sim \hat{E}^{(+)}_{asj}(t)\hat{E}^{(-)}_{asj-\alpha}(t)$; ... . In this paper the main attention is focused to the sum on label, j, of the product components, $\hat{E}^{(+)}_j(t)\hat{E}^{(-)}_{j+\alpha}(t)$, for $\alpha > 1$, which contains the photon correlations from non- adjacent multiple scattering fields described with classical phase, $\phi_{k\alpha} = i\alpha\omega_r t + i\alpha Kz$. Here $\hbar\alpha\omega_r = \hbar\omega_{j+\alpha} - \hbar\omega_j$, and $\alpha K = k_{j+\alpha} - k_j$ corresponds to the excitation of the cavity with $\alpha -$ portions of energy, $\hbar\omega_r$; K is the wave vector of bimodal components in each step of higher-order induced lasing. The detection possibilities of such cooperative phenomena after the propagation of correlated photons through different fibers are discussed.

In order to understand the possibilities of observation of quantum correlations between different mode components of the multiple scattering lasing, the new type of master equation is obtained in section 2 and detailed described in appendix A. The quantum correlation between the field components is done in section 3, using the system of vectors for cooperative representation of multiple scattering processes, described in appendix B. This approach gives us the possibility to describe the photon correlation from non-adjacent steps of multiple scattering processes given in same section 3. Using the projection of the master equation for the quantum lasing in multiple scattering modes, the cooperative correlations between the photons which belong to the bimodal field of the multiple scattered processes were analytically estimated and numerically represented. New cooperative aspects between cavity photons belonging to the bimodal field were established and annualized in discussions and conclusions 5.

2. Correlation functions and lasing process

As follows from [31], the reducing degree of freedom in the multiple scattering pump processes in the closed cavity field is described by collective photon operators, $\hat{\Lambda}_n^- = \sum_{p=0}^{n-1} g_{p+1}\hat{c}_{p+1}\hat{c}_p^\dagger / \chi_n$ and

$\hat{\Lambda}_n^+ = \sum_{p=0}^{n-1} g_{p+1}\hat{c}_p\hat{c}_{p+1}^\dagger / \chi_n$, the symmetry of which depends on the commutation relation between them. Some of these symmetries really follow from positions of virtual stats, through which take place the multiple Raman emissions. The multiple scattering part of Hamiltonian, $\hat{H}_I = \hbar\chi_n\hat{\Lambda}_n^-\hat{D}^+ + H.c.$, depends on the scattering constant, g_p, included in the bimodal operator, $\hat{\Lambda}_n^+$(or $\hat{\Lambda}_n^-$),in the process of de-excitation, $\hat{D}^-$, (excitation, $\hat{D}^+$) of the atomic ensemble relative to the exited and ground states. For example, $su(2)$ symmetry in two steps Raman conversion may be realized when the main virtual state is placed symmetrically relative to pump mode, $\Delta_0 = \omega_{v2} - \omega_{p_0} > 0$ for the first anti-Stokes generation, and with negative detuning, $\Delta_1 = \omega_{v2} - \omega_1 < 0$, for second anti-Stokes one, so that may be realized the situation $g_1 = -g_2$ (see figure 1(a). The three steps of Raman may be possibly relative to two virtual levels, considering that the resonance of the second step is situated between them, like this is represented in figure 1(b). The four steps of Raman scattering may be approximated in the same position of the virtual level, as this is described in figures 1(a) and (b). The $su(1, 1)$ symmetry needs special attention considering that all scattered modes are placed between two virtual levels. Considering both virtual levels are far from resonance with scattering components as this is represented in the figure 1(c), we represent the scattering amplitudes, $\{g_p\}$, as a superposition of the transitions through both virtual labels, $g_p \sim 1/(\omega_{v_1 2} - \omega_{p_0}) + 1/(\omega_{v_2 2} - \omega_{p_0})$. It is possible to obtain the linear dependence of the amplitude, g_p, on its step order after the substitutions $\omega_p = \omega_{p_0} + p\omega_r$, and considering that $\Delta_0 = \omega_{v_1 2} - \omega_{p_0} = \omega_{p_0} - \omega_{v_2 2}$. Here the pump field is at frequency $\omega_{p_0} \equiv \omega_0$. In this situation, the scattering amplitude becomes proportional to the step order, $g_p \sim 2p\omega_r/(\Delta_0^2 - p^2\omega_r^2)$, for large detuning from both virtual levels represented in figure 1 (c).

In this section, we put the injection of excited atoms in the evanescent zone a possibility to find the lasing condition of the photons in the bimodal cavity/fibers modes as this is represented in figure 1. For this, we develop the method of elimination of atomic operators, when the mean number of radiators, $N = 2j$, pumped into the evanescent zone losses a small part of inversion D_z during the flay time. After the elimination of the operators of the scattered photons in the external field the generalized equation (10) of [31] for the atomic and cavity field operators, $\hat{O}(t)$, takes the following form,

$$
\begin{aligned}
\frac{d}{dt}\hat{O}(t) = {} & i\sum_{p=0}^{n}\omega_p[\hat{c}_p^\dagger(t)\hat{c}_p(t), \hat{O}(t)] + i\omega_r D_z(t), \hat{O}(t)] \\
& + i\chi_n\{\{[\hat{\Lambda}_n^-(t)\hat{D}^+(t), \hat{O}(t)]\rangle - \langle[\hat{O}(t), \hat{D}^-(t)\hat{\Lambda}_n^+]\} \\
& + \sum_p \Gamma_p\{[\hat{c}_p^\dagger(t)\hat{D}^+(t), \hat{O}(t)]\hat{c}_p(t)\hat{D}^-(t) \\
& + H.c.\{[\hat{O}(t)]^+ \to \hat{O}(t)\}\} \\
& - \sum_{j=1}^{N_r}\left\{\frac{\gamma_p}{2}\sum_{\alpha\neq e,g}[|\alpha(t)\rangle\langle e(t)|_j, \hat{O}(t)]|e(t)\rangle\langle\alpha(t)|_j\right. \\
& + \gamma[\hat{D}_j^-(t), \hat{O}_\perp(t)]\hat{D}_j^+(t) + H.c.\{[\hat{O}_\perp(t)]^+ \to \hat{O}_\perp(t)\}\}. \qquad (1)
\end{aligned}
$$

The first term of Exp (1) describes the commutator of field operator with the free field which contains the pump modes at frequency, ω_0 and scattered cavity field at frequencies $\omega_p = \omega_0 + p\omega_r$. The second term of this exprssion contains the commutator of operator, $\hat{O}(t)$, with the interaction part of Hamiltonian, $\hat{H}_I$, which converted the photons from the pump mode, ω_0, into other modes during the multiple scattering process. The third term takes into consideration the scattered process into the external field described by Hamiltonian [31], $\hat{H}_{BC}^e = \sum_{k,p}\hbar\chi_{kp}\{\hat{c}_p\hat{b}_k^\dagger\hat{D}^- + H.c.$. According to the method described in [31], the creation, $\hat{b}_k^\dagger$, and annihilation, $\hat{b}_k$, operators in the outside modes of resonators were eliminated. After this elimination we introduced the losses of photons from the cavity, $\Gamma_p = \pi\sum_k|\chi_{p,k}|^2\delta(\omega_k - \omega_p - \omega_r)$, during this scattering process described by matrix elements, $\chi_{p,k}$. The operators, $\hat{\Lambda}_n^-(t)$ and $\hat{\Lambda}_n^+$, describe the creation or annihilation of the portion of energy equal to, $\hbar\omega = \hbar(\omega_{p+1} - \omega_p) = \hbar\omega_r$. We observe that field operator, $\hat{\Lambda}(t)$, commutes with atomic one, $\hat{D}^+(t)$, $\hat{D}^-(t)$ and $D_z(t)$.

Let us firstly consider that the operators, $\hat{O}(t)$ belongs to the cavity filed one, $\hat{\Lambda}(t)$. Representing this generalized operator as a superposition of bimodal operator, $(\hat{\Lambda}_n^+(t))^k(\hat{\Lambda}_n^-(t))^m$, in which k and m are aleatory power integer numbers, we intend to describe the proprieties of the cooperative lasing process, in the language of annihilation, $\hat{\Lambda}_n^-(t)$, and generation, $\hat{\Lambda}_n^+$, operators for portion of energy equal to $\hbar\omega_r$. As follows from the definition of these operators, their commutator is described by the expression,

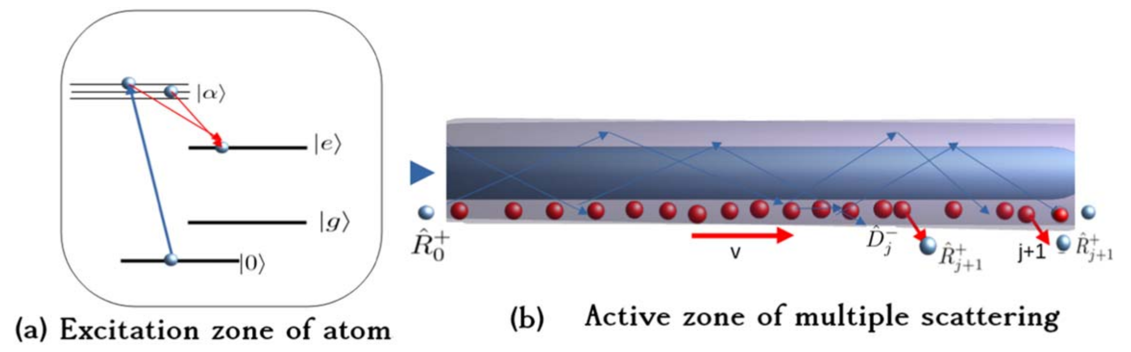

Figure 2. (a) Representation of the possible pump methods of excited atoms in the process of the non coherent excitation from fundamental state $|0\rangle_j$ to working levels $|e\rangle_j$, $|g\rangle_j$ through the $|\alpha\rangle_j$ excited states of the atom. (b) The attenuation of cooperative atomic polarization in the active lazing zone described by fiber evanescent field during the collision process between the $j-$, and $(j+1)-$ atoms.

$$\Delta\hat{\Lambda}_n = [\hat{\Lambda}_n^+, \hat{\Lambda}_n^-] = \frac{1}{\chi_n^2}\sum_{p=0}^{n-1}(g_p^2 - g_{p+1}^2)\hat{c}_p\hat{c}_p^\dagger.$$

According to the analyses proposed in [31] we observe that for $g_p/\chi_n = \sqrt{p(2s-p+1)}$ this commutator is reduced to su(2) algebra, $[\hat{\Lambda}_n^+, \hat{\Lambda}_n^-] = 2\hat{\Lambda}_n^z$. Here, $\hat{\Lambda}_n^z = \sum_{k=0}^{2s}(p-s)\hat{c}_p^\dagger\hat{c}_p$, $s = nl/2$, where, l is the initial number of photons in the pump mode, and n is the number of scattered steps. For big number of scattering steps, $n \gg 1$, and, $g_p/\chi_n = p$ the commutator is reduced to $su(1, 1)$ symmetry, $[\hat{\Lambda}^+, \hat{\Lambda}^-] \simeq -\hat{\Lambda}^z$, where $\hat{\Lambda}_z = \lim_{n\to\infty}\sum_{k=0}^{n-1}(k+1)^2\{\hat{c}_{k+1}\hat{c}_{k+1}^\dagger - \hat{c}_k\hat{c}_k^\dagger\}/2$. Of course as in the superradiance the scattering effect in other modes outside the cavity destroys the intrinsic symmetry of bimodal field operators, $\hat{\Lambda}_n^+$, $\hat{\Lambda}_n^-$ and $\Delta\hat{\Lambda}_n$.

In order to obtain the master equation for bimodal field operators let's consider that the generalized operator, $\hat{O}(t)$ belongs to atomic subsystem, $\hat{D}(t) \equiv \hat{D}_z(t)$, $\hat{D}^+(t)$, $\hat{D}^-(t)$, where the collective excitation, $\hat{D}^+(t) = \sum_{j=1}^{N_r}|e(t)\rangle\langle g(t)|_j \exp[iKz_j]$, and lowering, $\hat{D}^-(t) = \sum_{j=1}^{N_r}|g(t)\rangle\langle e(t)|_j \exp[-iKz_j]$, atomic operators belonging in to $su(2)$ symmetry, $[\hat{D}^+(t), \hat{D}^-] = 2\hat{D}_z(t)$. The atomic cooperative inversion can be represented through the inversion of each atom, $\hat{D}_z(t) = \sum_{j=1}^{N_r}[|e(t)\rangle\langle e(t)|_j - |g(t)\rangle\langle g(t)|_j]/2$. In the above representation z_j is the position of j- atom, and $K = k_{p+1} - k_p$ is the difference between the anti-Stokes and Stokes wave vectors of adjacent modes of cavity in quasi-linear scattering representation (see figure 2). In case when the atoms are situated at distance commensurable with $\lambda \sim 2\pi/K$, the phase argument, $\phi_j = Kz_j$, plays an important role of common polarization of the atomic stream. In the opposite situation, $|z_j| \ll \lambda$, this phase argument can be neglected, $\exp[-iKz_j] \sim 1$.

The pump rate of exited atoms, γ_p, is described by the last terms of Exp. (1), and takes into consideration that the j - atom is incoherently excited in the state $|e\rangle_j$ through other $|\alpha(t)\rangle$ atomic states represented in figure 2, $\sum_{\alpha\neq e,g}[|\alpha(t)\rangle\langle e(t)|_j, \hat{D}(t)]|e(t)\rangle\langle\alpha(t)|_j/\gamma_p$, so that this term gives non-zero contribution only for atomic inversion, $\hat{D}(t) = \hat{D}_z(t)$. According to Exp (1) the equation for atomic inversion becomes,

$$\frac{d}{dt}\hat{D}_z(t) = -\gamma_p(\hat{D}_z(t) - N_r/2) - i\chi_n(\hat{\Lambda}_n^-(t)\hat{D}^+(t) - \hat{D}^-(t)\hat{\Lambda}_n^+) - 2\hat{D}^+(t)\hat{D}^-(t)\sum_k \Gamma_k\hat{c}_k^\dagger(t)\hat{c}_k(t). \quad (2)$$

Here we take into consideration the conservation of excitation number in each atom, $\sum_{\alpha\neq e,g}|\alpha(t)\rangle\langle\alpha(t)|_j + |e(t)\rangle\langle e(t)|_j + |g(t)\rangle\langle g(t)|_j = \hat{N}_j$. Neglecting the number of atoms in 'ground' state at the initial stage of lasing process,$\langle|g\rangle\langle g|_j\rangle = 0$, we can approximate the expression, $\gamma_p\sum_{j=1}^{N_r}\sum_{\alpha\neq e,g}|\alpha(t)\rangle\langle\alpha(t)|_j/2$, with the first term of equation (2) , $\gamma_p[N_r/2 - \hat{D}_z(t)]$. Here $N_r = \sum_j\hat{N}_j$; $\hat{D}_{zj}(t) \sim |e(t)\rangle\langle e(t)|_j/2$. Considering that the polarization of j radiator may be transmitted to polarization of $(j+1)$ radiator from outside lazing zone during the atomic collision, we can describe this process by interaction Hamiltonian, $\hat{H}_{dr} = \hbar\kappa\sum_j\hat{D}_j^-(t)\hat{R}_{j+1}^+(t) + H.c.$, in which the operators. $\hat{D}_j^\pm(t)$ and $\hat{R}_{j+1}^\mp(t)$ describe the atomic polarization inside and outside the active lasing zone, respectively. Here, we assumed that the operator $\hat{R}_l^+(t) = |e(t)\rangle\langle g(t)|_l \exp[iK(z_l + \delta\phi_l(x, y))]$ shifted the plan position (x,y) to a random phase, $K\delta\phi_l(x, y)>1$, so that the average interaction with fiber modes can be neglect $[\hat{H}_l, \hat{R}_l^\pm(t)] = 0$. According to this representation illustrated in figure 2, this phenomenological interaction has many analogies with the transfer of photons from cavity modes to outside one used in traditional approach described in [32–35]. In the mean field approximation, after the elimination of outside polarization of non-interaction atoms, $\hat{R}^{+}(t)_{l+1}$ and $\hat{R}^{-}(t)_{l+1}$, we obtain the attenuation rate of polarization as this is represented in the last term of the generalized equation (1),

$$Los = \sum_{j=1}^{N_r} \gamma_j \{[\hat{D}_j^{-}(t), \hat{D}_{\perp}(t)]\hat{D}_j^{+}(t) + \hat{D}_j^{-}(t)[\hat{D}_{\perp}(t), \hat{D}_j^{+}(t)]\},$$

where $\gamma_j = \kappa^2\epsilon/[(\omega_{rj} - \omega_{rj+1}) + \epsilon^2]$, $\epsilon^{-1} \sim \ell/v$ is proportional to the flying time of the excited atom moving along the cavity with length l, with the mean value of velocity, v. In the resonance case, $\omega_{rj} = \omega_{rj+1}$ this expression coincide with the last term of generalized equation (1). Considering that $(j+1)$ - atom may leave the active zone of resonator, when another outside atom enters in it (See figure 2), we introduced in the generalized equation (1) the operator modification, $\hat{D}_{\perp}(t) \equiv \{\hat{D}^{\alpha}(t)(1 + \delta_{z,\alpha}), \alpha = z, `+', `-'\}$, takes into consideration only polarization modification. According to this description, the operator of atomic polarization satisfy the equation,

$$\frac{d}{dt}\hat{D}^{+}(t) = \{i(\omega_r + i\gamma) + 2[\hat{D}_z(t) - 1]\sum_k \Gamma_k \hat{c}_k^{\dagger}(t)\hat{c}_k(t)\}\hat{D}^{+}(t) - 2i\chi_n \hat{D}_z(t)\hat{\Lambda}_n^{+}(t). \tag{3}$$

If the atoms don't take part in collision process, the κ parameter may be proportional to ϵ, so that polarization loses take place dues to finite flaying time of atoms inside the active emission zone, $\gamma^{-1} \sim \ell/v$. The equation for operator $\hat{D}^{-}(t)$, is Hermit conjugate to the equation (3). In the situation, when the flying time is shorter than the nutation period described in [36, 37].

For elimination of atomic polarization from the generalized equation (1) we represent the solution of the atomic operators as a sum of free and sources parts,

$$\begin{aligned} \hat{D}^{+}(t) &= \hat{D}_f^{+}(t) + \hat{D}_s^{+}(t); \\ \hat{D}^{-}(t) &= \hat{D}_f^{-}(t) + \hat{D}_s^{-}(t); \\ \hat{D}_z(t) &= j|E\rangle\langle E|(1 - \exp(-t/\gamma_p) + \hat{D}_{zs}(t). \end{aligned} \tag{4}$$

Here the solutions of operators $\hat{D}^{+}(t)$, $\hat{D}^{-}(t)$ and $\hat{D}_z(t)$ are represented in the anti-normal form relative the excited states of the mean number of atoms, N_r, in the evanescent zone, $|e\rangle = |e_1, e_2, \ldots, e_{N_r}\rangle$. The free parts of the solutions for, $\hat{D}^{+}(t)$, and $\hat{D}^{-}(t)$,

$$\hat{D}_f^{+}(t) = \mathcal{T}\exp\{i(\omega_r + i\gamma)t + 2\int_0^t d\tau[\hat{D}_z(t-\tau) - 1] \times \sum_p \Gamma_p \hat{c}_p^{\dagger}(t-\tau)\hat{c}_p(t-\tau)\}\hat{D}^{+}(0); \tag{5}$$

$$\hat{D}_f^{-}(t) = \hat{D}^{-}(0)\tilde{\mathcal{T}}\exp\{-i(\omega_r - i\gamma)t + 2\int_0^t d\tau[\hat{D}_z(t-\tau) - 1] \times \sum_p \Gamma_p \hat{c}_p^{\dagger}(t-\tau)\hat{c}_p(t-\tau)\}, \tag{6}$$

are represented so that their action on the initial state, $|E\rangle$, give zero contribution in anti-normal arrangement of operators, $\hat{D}_f^{+}(t)|E\rangle = 0$, and the same result we obtained by acting with operator $\hat{D}_f^{-}(t)$ on the state $\langle E|$, $\langle E|\hat{D}_f^{-}(t) = 0$. As the operators don't commute between them in the different time moments in Exps. (5) and (6) we introduced the chronological time-ordered product, $\mathcal{T}$, and antichronological time product, $\tilde{\mathcal{T}}$, of operators, respectively [48]. After such representation in the bimodal field equation (1) remain only the source part of the solutions for $\hat{D}^{+}(t)$ and $\hat{D}^{-}(t)$:

$$\begin{aligned} \hat{D}_s^{+}(t) &= -2i\chi_n \mathcal{T}\int_0^t d\tau \exp\{i(\omega_r + i\gamma)\tau + 2\int_0^{\tau} d\tau_1[\hat{D}_z(t-\tau_1) - 1] \\ &\times \sum_p \Gamma_p \hat{c}_p^{\dagger}(t-\tau_1)\hat{c}_p(t-\tau_1)\}\hat{\Lambda}_n^{+}(t-\tau)\hat{D}_z(t-\tau); \\ \hat{D}_s^{-}(t) &= 2i\chi_n \tilde{\mathcal{T}}\int_0^t d\tau \hat{\Lambda}_n^{-}(t-\tau)\hat{D}_z(t-\tau)\exp\{-i(\omega_r - i\gamma)\tau \\ &+ 2\int_0^{\tau} d\tau_1[\hat{D}_z(t-\tau_1) - 1]\sum_p \Gamma_p \hat{c}_p^{\dagger}(t-\tau_1)\hat{c}_p(t-\tau_1)\}. \end{aligned}$$

Further action in the generalized equation (1) consists in the introducing of solutions for inversion operator (4) respecting the anti-normal product of atomic operators from sources part,

$$
\begin{aligned}
\hat{D}_{zs}(t) = -i\int_0^t \exp(-\tau/\gamma_p)\sum_n \chi_n[\hat{\Lambda}_n^-(t-\tau)\hat{D}^+(t-\tau) \\
- \hat{D}^-(t-\tau)\hat{\Lambda}_n^+(t-\tau)]d\tau - 2\int_0^t \exp(-\tau/\gamma_p)\hat{D}^+(t-\tau) \\
\times \hat{D}^-(t-\tau)\sum_{p'}\Gamma_{p'}\hat{c}_{p'}^\dagger(t-\tau)\hat{c}_{p'}(t-\tau)d\tau.
\end{aligned}
$$

In the above description, we have two small parameters, $\kappa \sim 4\chi^2/\gamma_p\gamma < 1$, and, $\vartheta_k = 2\Gamma_k/\gamma_p$, during the interaction of the bimodal field with excited atomic ensemble pumped in the evanescent zone of the cavity. The first parameter corresponds to multiple scattering processes in which the number of cavity photons is conserved. Another parameter, $\Gamma_p \equiv \Gamma_p^{(1)}$ described in ([31]) corresponds to the losses of photons in external modes during the multiple scattering processes.

The master equation for bimodal cavity field in multiple conversion was obtained in appendix A. Introducing the super-operators, we can represent this master equation for the bimodal cavity field (A13) in the following compact form,

$$
\begin{aligned}
&\frac{d}{dt}\hat{W}(t) - i\sum_{p=0}^{n}\omega_p[W(t), \hat{c}_p^\dagger\hat{c}_p] \\
&= N_r\left\{\frac{1}{1-(\kappa/4)[\hat{\Lambda}_n^-, \hat{\Lambda}_n^+\hat{V}\ldots]} - 1\right\}\hat{W}(t) \\
&- \sum_p \Gamma_p\{[\hat{c}_p^\dagger, \hat{c}_p\{N_r\hat{W}(t) \\
&+ (N_r^2/4)\left(\frac{1}{1+\kappa\hat{\Lambda}_n^+\overbrace{\ldots}\hat{\Lambda}_n^-} - 1\right)\hat{W}(t)\}] \\
&+ H.c.,
\end{aligned}
\tag{7}
$$

in which we used the compact represents of the serial decomposition (A13). Here, the action of super-operator has similarities to the decomposition rules described by Exps. (A8) and (A11) of appendix A ,

$$
\begin{aligned}
\{1/(1-(\kappa/4)[\hat{\Lambda}_n^-, \hat{\Lambda}_n^+\hat{V}\ldots]) - 1\} = (\kappa/4)W_1(t) + (\kappa/4)^2\hat{W}_2(t) + \ldots \\
+ (\kappa/4)^l\hat{W}_l(t) + \ldots
\end{aligned}
$$

It acts on the density matrix operator according to the below definition:
$W_1(t) = [\hat{\Lambda}_n^-, \hat{\Lambda}_n^+\hat{V}\ldots]\hat{W}(t) = [\hat{\Lambda}_n^-, \hat{\Lambda}_n^+\hat{V}\hat{W}(t)]$; $W_2(t) = [\hat{\Lambda}_n^-, \hat{\Lambda}_n^+\hat{V}\hat{W}_1(t)], \ldots, \hat{W}_l(t) = [\hat{\Lambda}_n^-, \hat{\Lambda}_n^+\hat{V}\hat{W}_{l-1}(t)]$.
The last term describes the dress of the density matrix obtained in the decomposition,

$$
\begin{aligned}
\left(\frac{1}{1+\kappa\hat{\Lambda}_n^+\overbrace{\ldots}\hat{\Lambda}_n^-} - 1\right)\hat{W}(t) = -\kappa\hat{W}_1^d(t) + \kappa^2\hat{W}_2^d(t) - \\
\ldots + (-\kappa)^l\hat{W}_l^d(t) + \ldots,
\end{aligned}
$$

due to the superradiant decay of the atoms. Here $\hat{W}_1^d(t) = \hat{\Lambda}_n^+\overbrace{\ldots}\hat{\Lambda}_n^-\hat{W}(t) = \hat{\Lambda}_n^+\hat{W}(t)\hat{\Lambda}_n^-$;
$\hat{W}_2^d(t) = \hat{\Lambda}_n^+\hat{W}_1^d(t)\hat{\Lambda}_n^-; \ldots; \hat{W}_l^d(t) = \hat{\Lambda}_n^+\hat{W}_{l-1}^d(t)\hat{\Lambda}_n^-$.

Below, we discuss the possibility of super-radiant transfer of energy from the radiator system to scattered photos into cavity modes. Let us take into consideration bimodal excitations in the lowest decomposition of generalized equation (1), the multiple aspects of scattering parameters and the renormalization of loss parameters. For this, considering that the lifetime of the excited radiators is less than the inverse value of Rabi nutation frequency and that the atomic flux contains a big number of atoms on the excited state, we eliminate the atomic operators from the generalized equation (1) using the procedure proposed in [30]

$$
\begin{aligned}
\frac{d}{dt}\langle\hat{\Lambda}(t)\rangle = (\kappa/4)\{\langle[\hat{\Lambda}_n^-(t), \hat{\Lambda}(t)]\hat{\Lambda}_n^+(t)\rangle + H.c.\} \\
+ \sum_p \Lambda_p\{\langle[\hat{c}_p^\dagger(t), \hat{\Lambda}(t)]\hat{c}_p(t)\rangle + H.c.\}.
\end{aligned}
\tag{8}
$$

Here $\kappa = N_r\chi_n^2/\gamma$, and γ is the pump rate of atoms in the cavity. The coefficient, $\Lambda_p = N_r\Gamma_p + \zeta$, contains the sum of losses from the cavity due to the scattering of the photons in the external field, Γ_p, by radiators prepared in the excited states defined by the master equation (7). The direct photon losses from the optical cavity, ζ, are introduced here according to the traditional method used in quantum optics [38]. After substitution of the generalized operator, $\hat{\Lambda}(t)$, with the operator of the number of excitations in the resonator modes, $\hat{W}_f(t) = \sum_{p=0}^n p\hat{c}_p^\dagger(t)\hat{c}_p(t)$, we obtain the following equation,

$$\frac{d}{dt}\langle\hat{W}_f(t)\rangle = 2\kappa\langle\hat{\Lambda}_n^-(t)\hat{\Lambda}_n^+(t)\rangle - 2\sum_p p\Lambda_p\langle\hat{c}_p^\dagger(t)\hat{c}_p(t)\rangle. \quad (9)$$

It is not difficult to observe that in the good cavity limit, $\kappa > \bar{\Lambda}_p$, we may neglect the second term of the equation (9). In this situation, taking into consideration the possibilities to realize the $su(2)$ symmetry between the coefficients of the multiple scattering processes, we may use the conservation of the Casimir vector which helps us to represent the bimodal phase correlation through the inversion one, $\hat{\Lambda}_n^-(t)\hat{\Lambda}_n^+(t) = L(L+1) - \hat{L}_n^z(\hat{L}_n^z + 1)$. Taking into consideration that, $\hat{W}_f(t) = \hat{L}_n^z(t) + L$ in the semi-classical approximation of superradiant process [39, 40] we obtain the following dependence, $\langle\hat{W}_f(t)\rangle = sl - 1/2 - (sl + 1/2)\tanh\{2\kappa(sl + 1/2)(t - t_0)\}$ of bimodal cavity excitations on the number of initial photons in the pump field l and number of steps, $n = 2s$. Here, $t_0 = \ln(2sl)/[4\kappa(L + 1/2)]$ is the delay time of generation rate, $\langle d\hat{W}_f(t)/dt\rangle$ of the cavity excitation. To observe this effect we must open the cavity taking into consideration the second term in the expression (9). Considering that the delay time, t_0, of the maximum of super-radiance process has the magnitude of $1/(N_r\Gamma_p)$ and $1/\zeta$, we may approximate the cavity losses, $\Gamma_p = \Gamma_0(p+1)(2s-p)$, by its expression in the maximum of superradiance $\Gamma_p \sim \Gamma_0 s$, in which $p \approx n/2$. In this situation we substitute, Λ_p with its mean value, $\bar{\Lambda}$, and reduce the second term of equation (9) to $2\bar{\Lambda}\langle\hat{W}_f(t)\rangle$. the (9) maybe by the expression,

$$\frac{d}{dt}\langle\hat{W}_f(t)\rangle = 2k\{(L + 1/2 - \delta)^2 - [\langle\hat{W}_f(t)\rangle - L + \delta + 1/2]\} + 2\bar{\Lambda}, \quad (10)$$

from which follows that the radius of the Bloch sphere, L, is diminished by the value, $\delta = \bar{\Lambda}/(2\kappa)$. The correction to the correlation radius may be larger than one, $\delta > 1$ when the losses from the cavity satisfy the inequality $2\kappa < \bar{\Lambda}$. The solution of this equation is similar to the traditional Dicke superradiance one [39, 40], in which the cooperative number is less than usually used in literature by the value of parameter, δ. In this semi-classical approach, we observe that the principal cooperative number in the cavity is described by expression, $L = sl$, as in the multiple scattering process [31], where l is the number of photons in the pump field, and s is half of the number of steps $s = n/2$.

3. Quantum aspects of correlation functions in multiple scattering of pump field

When the system becomes open, the conservation laws and symmetry are violated. In this situation the returning to the initial field operator becomes a necessity. From the analyses of the behavior of moments of the photon emission from the cavity, we observed that the set of vectors obtained by multiple acting with the operator, $(\hat{\Lambda}_n^+)^k$, on the initial state of the cavity field, $|l\rangle_0|0\rangle_1...|0\rangle_n$, must be represented not only through collective states , $|L, m\rangle_f$, but through the products of the Fock state of each mode component, $|L, m\rangle_f \sim |n_0\rangle_0|n_1\rangle_1...|n_n\rangle_n$ (see appendix B). Here the cooperative number, L=sl, depends on the number of photons of the scattered modes, $l = n_0 + n_1 + ... + n_n$, and the number of steps, $n = 2s$. Sometimes the last representation is the basic one because the symmetries disused in the [31] may help us to understand the cooperative process in the special cases of the level positions of the radiator, expressed through the states of each scattered mode. In appendix B we construct the set of wave vectors obtained from the initial state of the scattered field consisting of l photons in the pump field and zero photons in the scattered one after the consecutive action of the cooperative operator, $\hat{\Lambda}_n^+$. These states are expressed through the linear superposition of the product of Fock states for each converted photon state without the restrictions to transition amplitudes, g_p . In this situation, it is better to use this new set of vectors of the cavity field for the projection of the master equation (7) obtained after the anti-normal representation of photon operators. Following the appendix B we can obtain the recurrence form wave-vectors obtained after the $(\hat{\Lambda}^+)^p$ action on the initial state according to the described above algorithm,

$$|\widetilde{l, p}>_{(\alpha)} = \sum_{k=0}^{\lfloor p/\alpha\rfloor}\left(\frac{g_1 g_2 \cdots g_\alpha}{\alpha!}\right)^k \sqrt{C_l^k}\,|l - \widetilde{k, p} - \alpha k>_{(\alpha-1)}\,|k\rangle_\alpha, \quad (11)$$

which connects the wave fictions of α - steps conversion of multiple scattering processes with the linear combination between the wave function for $\alpha - 1$- scattering steps and new supplementary Fock state, $|k\rangle$. The analytic representation in two, three to fifes steps of multiple scattering processes is carefully described in appendix B.

Using the set of wave functions, (B5), (B10) and (B9), we can calculate the time-dependent correlation functions described in the introduction 1. It is better to define moments of the creation, $\hat{\Lambda}_n^+$, and annihilation, $\hat{\Lambda}_n^-$, operators of the quasi quanta with energy, $\hbar\omega_r$, in the cavity and their actions on the wave functions. For the two-step scattering process of the multiple Raman emission, we can define the correlation functions of the moments of these operators,

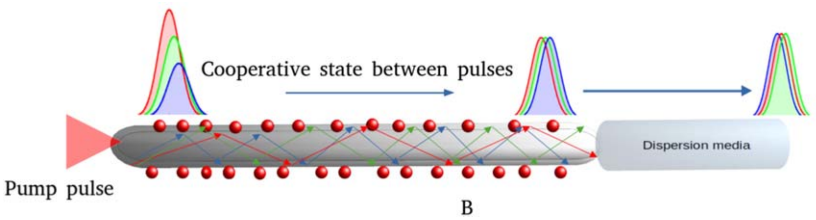

Figure 3. The pump field enters into the fiber and covers the evanescent zone, in which the excited system of atoms is placed. After cooperative two-step conversion, the generated light enters the dispersion zone (maybe the same fiber) which can modify the optical way as a function of the frequency.

$$K_m^a(p, p') = <l, p'|(\hat{\Lambda}_2^-)^m(\hat{\Lambda}_2^+)^m|l, p>_{(2)} . \tag{12}$$

Taking into consideration the proprieties of the wave vectors (B5), we obtain the following expression for these correlations,

$$K_m^a(p, p') = \frac{[(p+m)!]^2}{[p!]^2\chi_2^{2m}} \frac{\sum_{k=\lceil (p+m)/2 \rceil}^{p+m}\left(\frac{\lambda}{2}\right)^{2(p+m-k)} C_k^{2k-p-m} g^{2k} C_l^k}{\sum_{k=\lceil p/2 \rceil}^{p}\left(\frac{g_2}{2}\right)^{2(p-k)} C_k^{2k-p} g_1^{2k} C_l^k}\delta_{p,p'}.$$

The intensity of bimodal modes may be defined as a product of the strength of adjacent modes. Its normal or anti-normal representation of the bimodal field can be written in the forms (B5)

$$\langle p'|\hat{G}_{N(A)}^{(1,n)}|p\rangle = \begin{cases} <l, p'|\hat{\Pi}_{1n}^+(z, t)\hat{\Pi}_{1n}^-(z, t)|l, p>_{(2)} & \textit{for normal} \\ <l, p'|\hat{\Pi}_{1n}^-(z, t)\hat{\Pi}_{1n}^+(z, t)|l, p>_{(2)} & \textit{for antinormal} \end{cases}. \tag{13}$$

Here the bimodal field characteristics in the multiple steps of scattering are represented through the photons of two adjacent modes, which are by the upper indexes, and 'n'. Each term of the field characteristic, $\hat{\Pi}_{1m}^+(z, t)$, oscillates with the transition frequency of the of scattering, $\omega_r = \omega_p - \omega_{p-1}$,

$$\begin{aligned}\hat{\Pi}_{1m}^+(z, t) &= \hat{E}_p^{(+)}(z, t)\hat{E}_1^{(-)}(z, t) + \hat{E}_1^{(+)}(z, t)\hat{E}_2^{(-)}(z, t) \\ &\quad + ... + \hat{E}_{m-1}^{(+)}(z, t)\hat{E}_m^{(-)}(z, t) \\ &= \{q_0 q_1 \hat{c}_0 \hat{c}_1^\dagger + q_1 q_2 \hat{c}_1 \hat{c}_2^\dagger + ... + q_{m-1} q_m \hat{c}_{m-1}\hat{c}_m^\dagger\}\exp[i\omega_r t - iK_1 z],\end{aligned} \tag{14}$$

Here, $E_j^{(-)}(z, t) = q_j \hat{b}_j^\dagger \exp[i\omega_j t - ik_j z]$, is the EMF strength of j- mode. In the multistep scattering, we have the opportunity to describe the possible correlations of bimodal field components.

Following the scheme represented in figure 3 (B), we give below a description of the two-step process introducing the phase advance, φ_a, and phase delay, ψ_r, of the first and second terms of the field characteristic (14) of the two-step scattered photons in accordance with the chirp conception of the pulse propagation through the normal dispersive medium [41]. For simplicity, we consider this phase advance, $\varphi_a = \delta n\omega_r z/c$, and phase delay, $\psi_r = \delta n\omega_r z/c$ relative to the central frequency field component, ω_1, takes the same value, $\phi = \psi_r = \varphi_a$, when the refraction indexes for pump mod, $n_0 = n_1 - \delta n$, and the second scattered field, $n_0 = n_1 + \delta n$, are equally displaced relative to central scattered mode at frequency, ω_1. Here, z and c are respectively the propagation distance and light velocity in a vacuum. In this situation, we can represent the bimodal field characteristic after the propagation through the medium in the following form, $\hat{\Pi}_{12}^+(z, t) = \hat{E}_p^{(+)}(z, t)\hat{E}_1^{(-)}(z, t)\exp(i\phi) + \hat{E}_1^{(+)}(z, t)\hat{E}_2^{(-)}(z, t)\exp(-i\phi)$. In this case, the correlation functions of the two-step scattering on this phase.

$$\begin{aligned}\langle p'|G_A^{(1,2)}(t, \phi)|p\rangle &= <l, p'|\hat{\Pi}_{1m}^-(z, t)\hat{\Pi}_{1m}^+(z, t-\tau)|l, p>_{(2)} \\ &= [q_0^2 q_1^2\langle p'|\hat{K}_{1A}^{(2)}|p\rangle + q_1^2 q_2^2\langle p'|\hat{K}_{2A}^{(2)}|p\rangle \\ &\quad + q_0 q_1^2 q_2\langle p'|\hat{K}_{3A}^{(2)}(\phi)|p\rangle],\end{aligned} \tag{15}$$

where

$$\begin{aligned}\langle p'|\hat{K}_{1A}^{(1,2)}|p\rangle &= <l, p'|\hat{c}_1\hat{c}_1^\dagger\hat{c}_0^\dagger\hat{c}_0|l, p>_{(2)} \\ &= \delta_{p,p'}A_{lp}^{(2)}\sum_{k=\lceil p/2 \rceil}^{p}\left(\frac{g_2}{2}\right)^{2(p-k)} g_1^{2k} C_l^k C_k^{p-k}(2k-p+1)(l-p);\end{aligned} \tag{16}$$

$$\langle p'|\hat{K}_{2A}^{(1,2)}|p\rangle = <l,\, p'|\hat{c}_1^\dagger\hat{c}_1\hat{c}_2\hat{c}_2^\dagger|l,\, p>_{(2)} \\ = A_{lp}^{(2)}\sum_{k=\lceil p/2\rceil}^{p}\left(\frac{g_2}{2}\right)^{2(p-k)} g_1^{2k}C_l^kC_k^{p-k}(2k-p)(p-k+1)\delta_{p,p'}, \tag{17}$$

and

$$\langle p'|\hat{K}_{3A}^{(1,2)}(\delta_r)|p\rangle = <l,\, p'|\hat{c}_0^\dagger\hat{c}_1\hat{c}_1\hat{c}_2^\dagger\exp(ik_i\delta_r) + H.c|l,\, p>_{(2)} \\ = 2\cos(\phi)A_{lp}^{(2)}\sum_{k=\lceil p/2\rceil}^{p}\left(\frac{g_2}{2}\right)^{2(p-k)-1} \\ \times g_1^{2k+1}C_l^kC_k^{2k-p}(p-k)(l-k)\delta_{p,p'}. \tag{18}$$

The similar expression as (15) we obtain for normal form of correlation function, $\langle p'|G_N^{(1,2)}(t,\, \phi)|p\rangle = <l,\, p'|\hat{\Pi}_{1m}^+(z,\, t)\hat{\Pi}_{1m}^-(z,\, t)|l,\, p>_{(2)}$. For simplicity, we represent its analytic form of this expression for equal amplitudes of field components described in Exp. (14), $q_0 = q_1 = q_2$. After the introduction of wave functions (11) and field operators (14) we obtain the following analytic expression,

$$\langle p'|G_N^{(1,2)}(t,\, \phi)|p\rangle = \delta_{p,p'}q_0^4A_{lp}^{(2)}\sum_{k=\lceil p/2\rceil}^{p}\left(\frac{g_2}{2}\right)^{2(p-k)} g_1^{2k}C_k^{2k-p}C_l^k \\ \times \{(l-k+1)(2k-p) + (p-k)(2k-p+1) \\ + 2(p-k)(l-k)\cos(\phi)\frac{2g_1}{g_2}\Big\}. \tag{19}$$

We observe that the third term in the expressions (15) and (19) describes the interference between the first and second terms of the bimodal field characteristic, $\hat{\Pi}_{1m}^+(z,\, t)$. This interference depends on the phase advance and phase delay of the two waves with frequencies, ω_p, and, $\omega_p + 2\omega_r$, relative to the central one, $\omega_p + \omega_r$. The behavior of the quantum fluctuations of this process also depends on the difference of these ways in the interferometer schemes or in the dispersive medium. The square fluctuations are described by the mean value of matrix elements,

$$\langle p||\Delta_{GN}^2|p\rangle = \langle p|:\, (G_N^{(1,2)}(t,\, \phi))^2:\, |p\rangle - (\langle p|G_N^{(1,2)}(t,\, \phi)|p\rangle)^2,$$

where

$$\langle p|:(G_N^{(1,2)}(t,\, \phi))^2:\, |p\rangle = \langle p|\hat{\Pi}_{2m}^-(z,\, t)\hat{\Pi}_{2m}^-(z,\, t)\hat{\Pi}_{2m}^+(z,\, t)\hat{\Pi}_{2m}^+(z,\, t)|p\rangle \\ = A_{lp}^{(2)}\sum_{k=\lceil p/2\rceil}^{p}\left(\frac{g_2}{2}\right)^{2(p-k)}\{g_1^{2k}C_k^{2k-p}C_l^k \\ \times [(l-k+1)(l-k+2)(2k-p-1)(2k-p) \\ + (l-k+1)(p-k)(4k-2p+1)^2 \\ + (p-k)(p-k-1)(2k-p+2)(2k-p+1)] \\ + 2\left(\frac{2g_1}{g_2}\right)\sqrt{C_{k+1}^{2k-p+2}C_l^{k+1}C_k^{2k-p}C_l^k}\,(4k-2p+1)\cos[\phi] \\ \times [\{[(l-k)\sqrt{(p-k)(l-k+1)(2k-p+1)(2k-p+2)} \\ + (p-k)\sqrt{(l-k)(p-k+1)(2k-p)(2k-p-1)}\,] \\ + 2\left(\frac{2g_1}{g_2}\right)^2\cos[2\phi]\sqrt{C_{k+2}^{2k-p+4}C_l^{k+2}C_k^{2k-p}C_l^k} \\ \times (2k-p+2)(2k-p+1)\sqrt{(l-k-1)(l-k)(p-k)(p-k-1)}\,\}.$$

Let us find the correlations between the bimodal field components oscillations with frequency, $2\omega_r$. Here we considered the field strength component product, $\hat{E}_{m-2}^{(+)}(z,\, t)\hat{E}_m^{(-)}(z,\, t) \sim \exp[2i\omega_r t + i(k_m - k_{m-2})z]$... The correlation function of such a process is described by the functions $\langle p'|\hat{G}_A^{(2,m)}(t,\, t-\tau)|p\rangle = \langle p'|\hat{\Pi}_{2m}^-(z,\, t)\hat{\Pi}_{2m}^+(z,\, t-\tau)|p\rangle$. In the multi-step scattering process, the number of this terms are less with one than in the correlation function (13). According to the definition of such field characteristics,

$$\hat{\Pi}_{21}^{-}(z,\,t) = \hat{E}_p^{(-)}(z,\,t)\hat{E}_{a2}^{(+)}(z,\,t) \\ = q_0 q_2 \hat{c}_0^{\dagger}\hat{c}_2 \exp[-2i\omega_r t + i(k_2 - k_p)z],$$

follows that the correlation function contains only one term,

$$\begin{aligned} &\langle p'|\hat{G}_{N(A)}^{(2,1)}(t,\,t-\tau)|p\rangle \\ &= \begin{cases} q_0^2 q_2^2 < l,\,p'|\hat{c}_0\hat{c}_2^{\dagger}\hat{c}_2\hat{c}_0^{\dagger}|l,\,p >_{(2)} \exp[i2i\omega_r\tau], & for \quad N.p.; \\ q_0^2 q_2^2 < l,\,p'|\hat{c}_2\hat{c}_0^{\dagger}\hat{c}_0\hat{c}_2^{\dagger}|l,\,p >_{(2)} \exp[i2i\omega_r\tau], & for \quad A.p. \end{cases} \\ &= \delta_{p,p'} q_0^2 q_2^2 A_{lp}^{(2)} \sum_{k=\lceil p/2 \rceil}^{p} \left(\frac{g_2}{2}\right)^{2(p-k)} g_1^{2k} C_l^k C_k^{2k-p} \\ &\times \begin{cases} (l-k)^*(p-k+1), & for \quad N.p.; \\ (l-k+1)^*(p-k) & for \quad A.p. \end{cases}. \end{aligned} \tag{20}$$

The quantum aspect of this problem can be estimated by projecting the master equation (7) of cavity excitations on the states (11) of the bimodal system of the cavity field described in appendix B. Due to the fact that in the open system, the total number of photons in the cavity/fiber is not conserved, we propose to decompose the density matrix, $\hat{W}(t)$, not only on the converted photon number, p, but also on the total number, l, of the collective photon states,

$$\hat{W}(t) = \sum_{l,p} Q_{lp}(t)|l,\,p >_{\alpha} < l,\,p|_{\alpha}.$$

To obtain the equation for probabilities $P_{lp}(t)$ that the cavity system excited by $l-$ photons can generate p— bimodal excitations in the cavity, we introduce the unity in the orthogonal sets of wave vectors obtained in the appendix B,

$$\sum_{l,p}|l,\,p >_{\alpha} < l,\,p|_{\alpha} = 1,$$

where $<l,\,p|l',\,p' >_{\alpha} = \delta_{l,l'}\delta_{p,p'}$. Using the decomposition of the density matrix on the system of wave vectors (B5) of the two-step scattering process, we obtain the following system of differential equations for the statistical weights, Q_{lp}, of the density matrix,

$$\begin{aligned} \frac{d}{dt}Q_{lp}(t) &= 2\kappa\{Q_{lp-1}(t)C_p^2 - C_{p+1}^2 Q_{lp}(t)\} \\ &+ 2G_0\{l(A_{lp}^{(2)}/A_{l-1p}^{(2)})^2 Q_{l-1p}(t) - (l+1-D_p)Q_{lp}(t)\} \\ &+ 2\Lambda_p\{(l+1)(A_{l+1p}^{(2)}/A_{lp}^{(2)})^2 Q_{l+1,p} - lQ_{lp}\} \\ Q_{l0}(0) &= 1, \quad l = ...,l+2,\,l_0+1,\,l_0,\,l_0-1,\,l_0-2\,...;\,p = 0,\,1,...l, \end{aligned} \tag{21}$$

where the first term describes the conversion process and the second, takes into consideration the cavity losses. The coefficients $\{(A_{l+1p}^{(2)}/A_{lp}^{(2)})^2\}$ and $\{C_p^2\}$ are square amplitudes of the losses and conversion processes in the system described by Exps (B5). The coefficient is expressed through the mean value of excitation numbers of photons, which leave the pump mode,

$$D_p = [A_{lp}^{(2)}]^2 \sum_{k=\lceil p/2 \rceil}^{p} k\left(\frac{g_2}{2}\right)^{2(p-k)} g_1^{2k} C_k^{2k-p} C_l^k. \tag{22}$$

Following the description of the short pulse duration of the pump field, we considered a convenient to use the Hamiltonian description of the multiple scatterings of photons. In this situation, for a quantum description point of view, we propose to use the uncertainty principle between the pulse duration and energy, $\Delta t\Delta E \geqslant \hbar$, where the first term describes the conversion process proportional to the pulse duration, $\delta t \sim \tau_p$, and the second term describes the dispersion of the number of photons, l relative its mean value l_0, $\delta E \sim \hbar\omega_0|(l-l_0)|$. In this situation, the solution of the master equation (21) can be averaged supplementary, taking into consideration this uncertainty principle. It is convenient to propose this supplementary average procedure for the initial set of Fock states, $|l\rangle$, $l = 1, 2,\ldots,l_0, l+1,\ldots$, of the pump, field considering it as a normal distribution relative to its mean value $w(l) = (1/\sqrt{2\pi\sigma^2})\exp[-(l-l_0)^2/2\sigma^2]$, where σ is the dispersion connected with pulse duration and photon statistics in the pump field, l_0 is the mean value of the photons in the pump field.

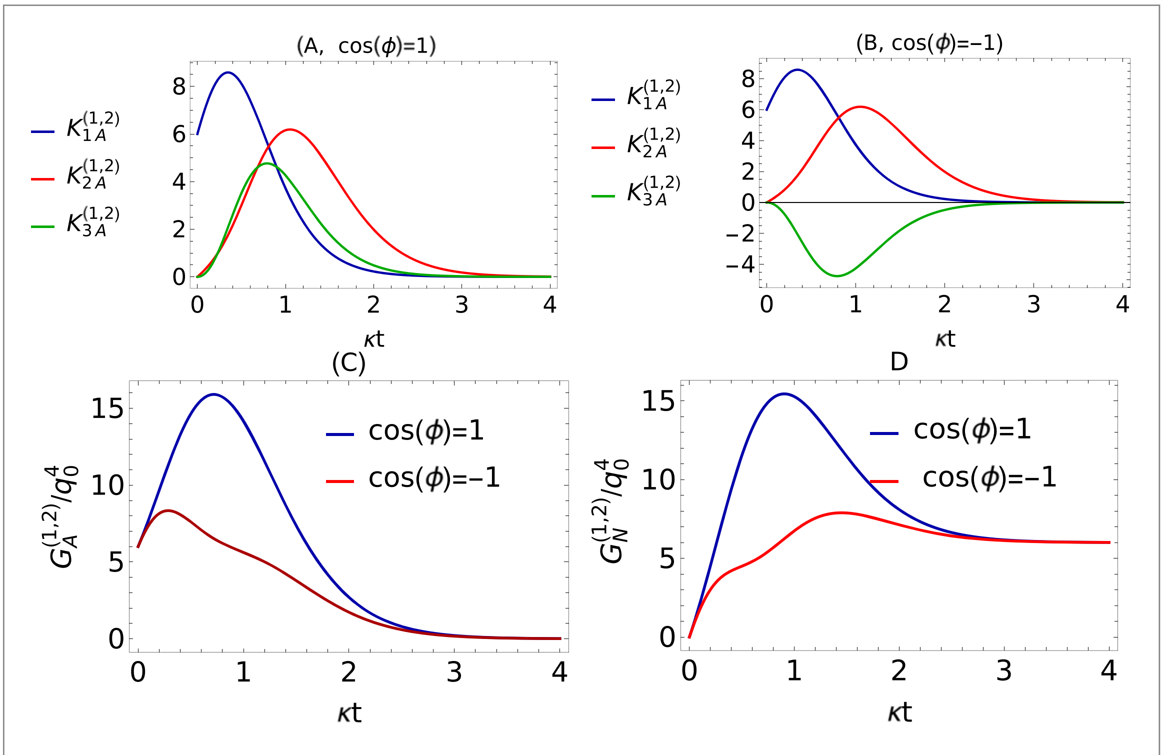

Figure 4. Description of the relationships between different components of multiple Raman scattering components using bimodal correlation functions. In figures (A) and (B) the correlation functions, $K_{1A}^{(2,1)}$ and $K_{2A}^{(2,1)}$ and $K_{3A}^{(2,1)}$ for two value of the phase delay, $\cos(\phi) = 1$ and $\cos(\phi) = -1$, have the opposite signs. This describes the relationships between different components of multiple Raman scattering components using bimodal correlation functions. In figures (A) and (B) the correlation functions, $K_{1A}^{(2,1)}$ and $K_{2A}^{(2,1)}$ and $K_{3A}^{(2,1)}$ for two value of the phase delay, $\cos(\phi) = 1$ and $\cos(\phi) = -1$. have the opposite signs correlates relative to their amplitude superposition. Figures C and D corresponds to the behavior of the total bimodal field characteristics, $G_A^{(1,2)}$ and $G_N^{(1,2)}$ for two values of the $\cos(\phi) = \pm 1$ Here $l_0 = 6$.

4. Numerical results and discussions

The time behavior of the moments averaged by the statistical weights of the system of differential equations (21) is plotted in figures 4–6. According to this conception, the correlations (16), (17) and (18) describe the development of the coherence effect between bimodal components of the pump and first anti-Stokes component $\langle \hat{E}_p^{(+)}(z,\,t)\hat{E}_1^{(-)}(z,\,t)\hat{E}_1^{(+)}(z,\,t)\hat{E}_p^{(-)}(z,\,t)\rangle$; correlation between the first anti-Stokes component second one, $\langle \hat{E}_1^{(+)}(z,\,t)\hat{E}_2^{(-)}(z,\,t)\hat{E}_2^{(-)}(z,\,t)\hat{E}_1^{(-)}(z,\,t)\rangle$, and their cross-correlation, $\langle \hat{E}_p^{(+)}(z,\,t)\hat{E}_1^{(-)}(z,\,t)\hat{E}_2^{(-)}(z,\,t)\hat{E}_1^{(-)}(z,\,t)\rangle + H.c.$ The mean value of these correlations can be obtained after the average of these functions using the statistical weights (21). After this procedure of mediation, we can use normal distribution of the photons in the pump field so that the correlations takes the forms,

$$K_{iA}^{(1,2)} = \sum_{l}\sum_{p=0}^{2l} w(l)Q_{lp}(t)\langle p|\hat{K}_{iA}^{(1,2)}|p\rangle,\quad i = 1,\,2,\,3.$$

As follows from figures 4 (A) and (B) the interference between $K_{1A}^{(1,2)}$ and $K_{2A}^{(1,2)}$ bimodal fields is described by the correlation function, $K_{3A}^{(1,2)}(\phi) = K_{3A}^{(1,2)}\cos\phi$. This is possible not only during the propagation through the dispersive medium, but with the changing optical way of the bimodal field characteristics, $\hat{E}_p^{(+)}(z,\,t)\hat{E}_1^{(-)}(z,\,t)$, and $\hat{E}_p^{(+)}(z,\,t-\tau)\hat{E}_1^{(-)}(z,\,t-\tau)$, in the heterodyne interferometry [42, 43]. In figure 4 (A) we take the same sign of the splitting of the bimodal amplitudes, $q_0 = q_2$, which corresponds to the phase shift, $\cos(\phi) = 1$. For demonstration of the phase sensitivity of these correlations in figure 4 (B) we take the opposite signs, $\cos(\phi) = 1$. The dependence on the intensity of the output signal of the bimodal field detected in the anti-normal, $G_A^{(1,2)}$, and normal , $G_N^{(1,2)}$, products of the bimodal field characteristics,

$$G_{A(N)}^{(1,2)}(t,\,t) = q_0^2 q_1^2 K_{1A(N)}^{(1,2)} + q_1^2 q_2^2 K_{2A(N)}^{(1,2)} + q_0 q_1^2 q_2 K_{3A(N)}^{(1,2)}\cos\phi, \tag{23}$$

are plotted in the figures 6 (C) and (D). The significant modification of output bimodal signal detected at differences of the frequencies, $\omega_r = \omega_{i-1} - \omega_i$ as function of phase delay is observed in these figures in case when $\cos(\phi) = 1$ and $\cos(\phi) = -1$. The plot was made for the same single photon amplitudes, $q_0^2 = q_1^2 = q_2^2$. The behavior is like in the traditional interference effect, but we must emphases here the square dependence on the

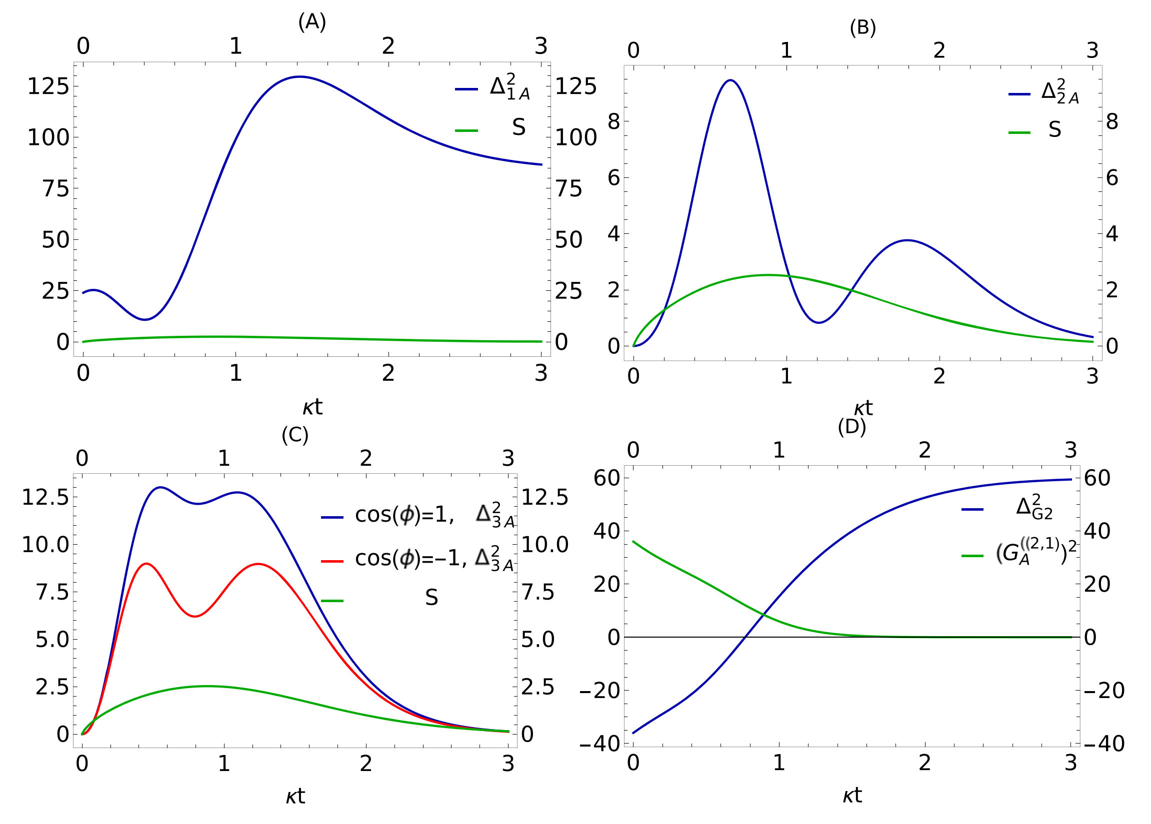

Figure 5. The behavior of quantum fluctuations of each step component (A) and (B) of bimodal field in the cooperative scattering and their correlations (C). The evolution of quantum fluctuations of the correlations between pump and second step scattered photons is plotted in figure (D). All numerical simulations was made for the parameters: $l_0 = 6$, $|g_1|=|g_2|$; $|q_0| = |q_1| = |q_2|$.

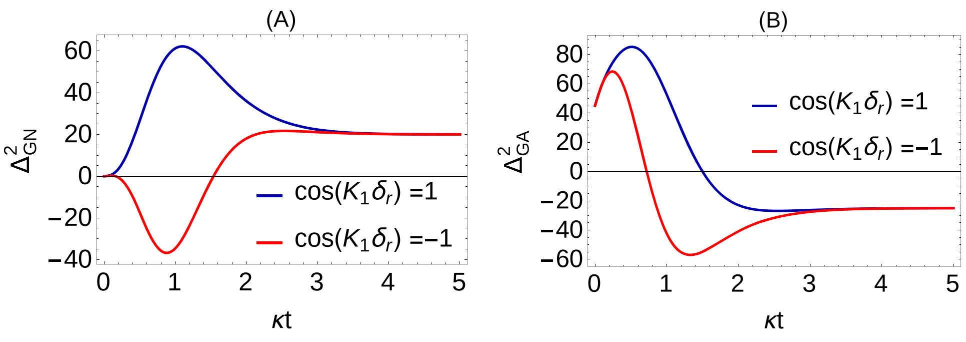

Figure 6. The dependence of the total fluctuation of the intensity of bimodal field for the in-phase delay of one of the components, $\phi = K_1\delta_r = 2n\pi$ (or $\cos(K_1\delta_r) = 1$) and for anti-phase delay, $\phi = K_1\delta_r = \pi + 2n\pi$ (or $\cos(K_1\delta_r) = -1$). Figure *A* and *B* are plotted respectively for normal and anti-normal arrangement of the operators in the correlation functions for $l_0 = 5$.

field strength of the characteristic of the bimodal field. These correlations were established during cooperative phenomena between two steps, Raman emission. During the interference between the amplitudes, $q_0^2 q_1^2 K_{1A}^{(1,2)}$, and, $q_1^2 q_2^2 K_{2A}^{(1,2)}$, is possible the retardation path of two bimodal waves, $\hat{E}_{p_0}^{(+)}(z,\ t)\hat{E}_1^{(-)}(z,\ t)$, and $\hat{E}_1^{(-)}(z-\delta_r,\ t)\hat{E}_2^{(+)}(z-\delta_r,\ t)$, relative to pump one, $\hat{E}_{p_0}^{(+)}(z,\ t)$, and second step anti-Stokes components, $\hat{E}_2^{(+)}(z-\delta_r,\ t)$. In this situation the interference term, $\hat{E}_{p_0}^{(+)}(z,\ t)\hat{E}_1^{(-)}(z,\ t)\hat{E}_1^{(-)}(-\delta_r,\ t)\hat{E}_2^{(+)}(z-\delta_r,\ t)+H.c.$ of the expression (15) contains this delay like in Young interference scheme, $K_{3A}^{(1,2)}(\delta_r) = \cos(K_1\delta_r)K_{11}^{(3)}(0)$. This takes place when the phase achieved the value, $\phi = K_1\delta_r$, where the bimodal wave vector is defined by the relations, $K_1 = k_1 - k_p = k_2 - k_1$. The possible interference between such bimodal field can be realized in the fiber regime of the propagation of radiation.

The square fluctuations and entropy of these correlates may be estimated according to the definition of each correlations plotted in figure 5,

$$\Delta_i^2 = \sum_l \sum_{p=0}^{2l} w(l) Q_{lp}(t) \langle p| \breve{A}_n \{\hat{K}_{11}^{(i)} \hat{K}_{11}^{(i)}\} |p\rangle - (K_{11}^{(i)})^2, \quad i = 1, 2, 3;$$

$$S = -\sum_l \sum_{p=0}^{2l} w(l) Q_{lp}(t) \ln Q_{lp}(t). \tag{24}$$

Here the notation, $\breve{A}_n\{...\}$, represents the anti-normal product of the bimodal operators $\hat{\Pi}_{1m}^{-}(z, t)$ and $\hat{\Pi}_{1m}^{+}(z, t)$. We observe that the maximal values of fluctuations and entropy are achieved in the same time interval. The square fluctuations and entropy of these correlates may be estimated according to the definition of each correlation,

$$\Delta_{G2}^2 = \sum_l \sum_{p=0}^{2l} w(l) Q_{lp}(t) \langle p| :(G_N^{(1,2)}(t, \phi))^2: |p\rangle - \\ - \left(\sum_l \sum_{p=0}^{2l} w(l) Q_{lp}(t) \langle p| G_N^{(1,2)}(t, \phi) |p\rangle \right)^2, \tag{25}$$

which for simplicity we take equal one for vacuum field, $|q_0| = |q_1| = |q_2| = 1$, in the numerical plot of Exp. (25). As follows from the numerical representation described by figure 6, the total fluctuations contains the interference between the each bimodal components described in Exps. (24) and depends on the phase shifts through the $\cos(2\phi)$. Two values of this interference function, $\cos(2\phi) = 1$, and $\cos(2\phi) = -1$, are plotted in this figure. This interference describes the higher entangled between the photons in the multiple scattering process.

For the description of the quantum correlations between the pump and second anti-Stokes mode, $G_{A(N)}^{(2,1)}(t)$, (20) we introduce the characteristics,

$$G_{A(N)}^{(2,1)}(t) = \sum_l \sum_{p=0}^{2l} w(l) Q_{lp}(t) \langle p| \hat{G}_{A(N)}^{(2,2)} |p\rangle, \tag{26}$$

correlation,

$$\Delta_R^2 = \sum_l \sum_{p=0}^{2l} w(l) Q_{lp}(t) \langle p| AN\{\hat{R}_{21} \hat{R}_{21}\} |p\rangle - \{R_{12}\}^2. \tag{27}$$

If we take into consideration hay steps scattering process, described by the set of wave functions (B9), we can obtain the large numbers of interference terms in Exps. (23) and (26). Numerical behavior of cross-correlation between the pump and second anti-Stokes modes is represented in figure 4 (D) for the same parameters of the system $g_0^2 = g_2^2 = 1$. As this correlates is proportional to the number of the photons, we plot in this figure and photon number cross-correlation between the modes, $\Delta R_{A(N)} = G_{A(N)}^{(2,1)}(t) - \langle \hat{c}_2 \hat{c}_2^\dagger \rangle \langle \hat{c}_0^\dagger \hat{c}_0 \rangle$, where $\langle \hat{c}_i \hat{c}_i^\dagger \rangle = \sum_l \sum_{p=0}^{2l} w(l) Q_{lp}(t) \langle p| \hat{c}_i \hat{c}_i^\dagger |p\rangle, i = 0, 1, 2$ mean value of the photon number in each mode.

5. Discussions and conclusions

The transfer of phase correlation from one mode to other modes of quantum generators in multiple scattering processes plays an important role in quantum information. This effect is named teleportation [44, 45] and possibilities to construct such equipment play an important role from classical and quantum points of view.

The possibilities to detect such correlations are represented in figures 3 and 7. The first proposed method uses the interference between the pulses with different frequency propagation ways through the dispersive medium situated after the lasing volume of multiple scattering. The second one is possibly improving the losses from lasing volume using contact fiber with excited atoms situated in evanescent zones of fibers, as this is represented in figure 7(a). The two types of detection represented in figure 7 may be used for the detection of the transfer of photon excitations between non- adjacent modes. The first consists of the detection of a bimodal field using photon current correlation of two detectors (C), and the second one consists in the detection of interference of two fiber branches of bimodal on the two-photon excitation, as in the traditional interferometer using detectors like $D1$ or $D2$ represented in figure 7(b). For the detection of correlations for non-adjacent modes in the multi-step scattering, we propose schematic photodetectors consisting of N - independent atoms prepared in the ground states. The first detector has the excitation energy, $E_e \sim \hbar\omega_r$, equal to the energy of each portion of the quasi-particle in the lasing volume, but the second one has the double excitation energy, $E_e \sim 2\hbar\omega_r$, which can be used for the detection of correlation between the k and $k + 2$ steps of scattering process. Of course, in the particular situation of the two-step Raman process, the excitations of such detectors are possible using two adjacent or non-adjacent mode components.

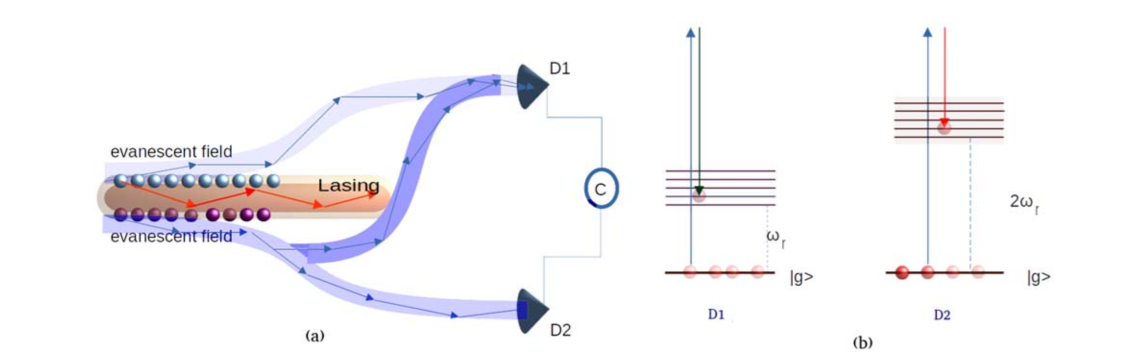

Figure 7. (a) Detection of a bimodal field using photon current correlation of two detectors (C) or correlation or interference of two fiber branches on the scattering excitation, as in the traditional interferometer (D1). (b) Schematic representation of two photodetectors consisting of N-independent atoms prepared in the ground states. One detector has the excitation energy $E_e \sim \hbar\omega_r$ equal to the energy of each portion of the quasi-particle in the lasing volume. The second detector has the double excitation energy, $E_e \sim 2\hbar\omega_r$. The excitations of such detectors are possible using two adjacent or non-adjacent mode components of the two-step Raman process.

The inelastic stimulated Raman scatters in the fibers were observed in [46, 47]. Here the pump is converted by vibrating SiO_2 molecules in a glass lattice, whereas the feedback is provided by elastic Rayleigh scattering of the SRS-induced Stokes wave on sub-micron irregularities of the glass structure, with a small part of scattered light coming back into the fiber. Though the feedback is very weak, it is sufficient for lasing in passive fiber, given that the integral Raman gain is proportional to the fiber length and pump power. The higher-order Stokes generation likely comes from photons that have the longest path lengths, thus have the most gain in the random lasing process [46, 47]. The pump quantum efficiency of converting the pump 1.05 μm into the output radiation is almost independent of the Stokes order, the pump amounting to 79, 83, and 77 for the 1st (1.11 μm), 2nd (1.17 μm), and 3rd (1.23 μm) order, respectively, at the polarization extinction ratio 22 dB for all orders. The laser bandwidth grows with increasing order, but it is almost independent of power in the $(1-10)$ W range, amounting to $\sim$1, $\sim$ 2, and $\sim$ 3 nm for orders $1-3$, respectively.

Acknowledgments

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).Data will be available from 8 September 2022.

Appendix A. Master equation of multiple scatting lasing

We propose the elimination procedure of atomic polarization operators using the anti-normal representation of the solutions (1) in which two small parameters will be taken into consideration. Considering the ratio, $\varsigma/(1+\varsigma)$ of the expression, $\varsigma = \kappa\langle\hat{\Lambda}^-(t)\hat{\Lambda}^\dagger(t)\rangle + \sum_k \vartheta_k \hat{c}_k^\dagger(t)\hat{c}_k(t)$, as a small parameter of the system we propose the method of decomposition of master equation for cavity field on this common parameter. This parameter, $\epsilon = \varsigma/(1+\varsigma)$, depends on gain, κ, and losses, ϑ_k, rates and helps us to avoid the divergences connected with relatively large values of the parameter, ς, in the serial decomposition. Indeed, for large values of the parameter, $\varsigma > 1$ the traditional serial decomposition, $\varsigma - \varsigma^2$, may become negative, although it is positive by definition, $\epsilon > 0$. Considering that in the initial stage, the arrived radiators are in the excited states we obtain the following equation for mean value of the multiple scattering operator, $\hat{\Lambda}$, in anti-normal representation,

$$\begin{aligned}\frac{d}{dt}\langle\hat{\Lambda}(t)\rangle = i\sum_{p=0}^{n}\omega_p\langle[\hat{c}_p^{\dagger}(t)\hat{c}_p(t),\,\hat{\Lambda}(t)]\rangle\\ + i\chi_n\langle[\hat{\Lambda}_n^{-}(t),\,\hat{\Lambda}(t)]\hat{D}_s^{+}(t)\rangle + \langle\hat{D}_s^{-}(t)[\hat{\Lambda}_n^{+},\,\hat{\Lambda}(t)]\rangle\\ + \sum_p\Gamma_p\{2\langle[\hat{c}_p^{\dagger}(t),\,\hat{\Lambda}(t)]\hat{c}_p(t)\hat{D}_z(t)\rangle\\ + \langle\hat{D}_s^{-}(t)[\hat{c}_p^{\dagger}(t),\,\hat{\Lambda}(t)]\hat{c}_p(t)\hat{D}_z(t)\hat{D}_s^{+}(t)\rangle\\ + H.c.\{[\hat{\Lambda}(t)]^{+}\to\hat{\Lambda}(t)\}\}.\end{aligned} \tag{A1}$$

Here the mean value of operators was made on the inital state of atoms and feld operators, $|E\rangle_r|L\rangle_f = |e_1, e_2,\ldots,e_{N_r}\rangle_r \otimes |l, 0,\ldots,n\rangle_f$. The last term of equation (3) represents the losses from the system in which the atomic operator, $\hat{D}^{+}(t)\hat{D}^{-}(t)$, is represented through the anti-normal expression, $\hat{D}^{+}(t)\hat{D}^{-}(t) = 2\hat{D}_z(t) + \hat{D}^{-}(t)\hat{D}^{+}(t)$. Considering that the pumping process takes place in a short time interval in comparison with the scattering process, we represent sources operators, $\hat{D}_s^{-}(t)$ and $\hat{D}_s^{+}(t)$, described by expressions (5) and (6) in the Born-Markov approximation,

$$\begin{aligned}\hat{D}_s^{+}(t) = +\frac{2\chi_n}{\omega_r-\omega+i\tilde{\gamma}}\hat{D}_z(t)\hat{\Lambda}_n^{+}(t);\\ \hat{D}_s^{-}(t) = \frac{2\chi_n}{\omega_r-\omega-i\tilde{\gamma}}\hat{\Lambda}_n^{-}(t)\hat{D}_z(t).\end{aligned} \tag{A2}$$

Here the losses parameter are normalized, $\tilde{\gamma} = \gamma - 2(\hat{D}_z(t) - 1)\sum_k\Gamma_k\hat{c}_k^{\dagger}(t)\hat{c}_k(t)$. In the same approximation the inversion equation (2) can be represented in the following form,

$$\begin{aligned}j = [1 + \sum_p\vartheta_k\hat{c}_p^{\dagger}(t)\hat{c}_p(t)]\hat{D}_z(t) + i\frac{\chi_n}{\gamma_p}[\hat{\Lambda}_n^{-}(t)\hat{D}_s^{+}(t) - \hat{D}_s^{-}(t)\hat{\Lambda}_n^{+}(t)]\\ + i\frac{\chi_n}{\gamma_p}[\hat{\Lambda}_n^{-}(t)\hat{D}_f^{+}(t) - \hat{D}_f^{-}(t)\hat{\Lambda}_n^{+}(t)]\\ + [j(j+1) - \hat{D}_z^{2}]\sum_k\vartheta_k\hat{c}_k^{\dagger}(t)\hat{c}_k(t).\end{aligned} \tag{A3}$$

In the last term of Exp. (A3) we take into consideration the conservation low Bloch vector, $j(j+1) \approx D_z(D_z - 1) + \hat{D}^{+}(t)\hat{D}^{-}(t)$, when the pump rate have the same magnitude as a loses of fling atoms from the evanescent zone: $\gamma_p \simeq \gamma$ the last loss. After the substitution of Born-Markov representation for sources part of atomic operators (A2) in the right-hand side of the equation (A3) we obtain relative the inversion $\hat{D}_z$ a solvable square equation,

$$\begin{aligned}\hat{D}_z(t) = \frac{1}{\hat{\epsilon}(t)}\{1 - \{1 - 2\hat{\epsilon}(t)\{j - j(j+1)\sum_k\vartheta_k\hat{c}_k^{\dagger}(t)\hat{c}_k(t)\\ + i\chi_n[\hat{\Lambda}_n^{-}(t)\hat{D}_f^{+}(t) - \hat{D}_f^{-}(t)\hat{\Lambda}_n^{+}(t)]/\gamma_p\}\}^{0.5}.\end{aligned} \tag{A4}$$

Here we have the small parameter,

$$\hat{\epsilon} = \frac{2\sum_k\vartheta_k\hat{c}_k^{\dagger}(t)\hat{c}_k(t)}{1 + \sum_k\vartheta_k\hat{c}_k^{\dagger}(t)\hat{c}_k(t) + \tilde{\kappa}\hat{\Lambda}_n^{-}(t)\hat{\Lambda}_n^{+}(t)}.$$

Considering that the interaction with external electromagnetic field is smaller than the interaction with the cavity field due to the small parameter, $\hat{\epsilon}(t)$, we represent the root in the serial decomposition, $\sqrt{1-\hat{X}} \approx 1 - \hat{X}/2$, so that expression (A4) take the form known in the literature (see for example [48–52]),

$$\begin{aligned}\hat{D}_z(t) = \frac{j[1-(j+1)\sum_k\vartheta_k\hat{c}_k^{\dagger}(t)\hat{c}_k(t)]}{1 + \sum_p\vartheta_k\hat{c}_k^{\dagger}(t)\hat{c}_k(t) + \tilde{\kappa}\hat{\Lambda}_n^{-}(t)\hat{\Lambda}_n^{+}(t)}\\ + \frac{i\frac{\chi_n}{\gamma_p}[\hat{\Lambda}_n^{-}(t)\hat{D}_f^{+}(t) - \hat{D}_f^{-}(t)\hat{\Lambda}_n^{+}(t)]}{1 + \sum_k\vartheta_k\hat{c}_k^{\dagger}(t)\hat{c}_k(t) + \tilde{\varsigma}\hat{\Lambda}_n^{-}(t)\hat{\Lambda}_n^{+}(t)}.\end{aligned} \tag{A5}$$

48−52We can substitute in source operators (A2) the inversion operator with the right hand part of Exp. (A5). After that we return to generalized equation (1) introducing the solutions (4) in anti-normal forms Taking into consideration that the action of operator $\hat{D}_f^{+}(t)$ and $\hat{D}_f^{-}(t)$f on initially excited radiator state, $|E\rangle_r$ give zero contribution, $\hat{D}_f^{+}(t)|E\rangle_r = 0$ and $\langle E|_r\ \hat{D}_f^{-}(t) = 0$, we continue to eliminate the atomic polarization during the lasing. In the first approximations, the terms proportional to $\tilde{\varsigma}$ and ϑ_n give the following contribution in the generalized equation,

$$\frac{d}{dt}\langle\hat{\Lambda}(t)\rangle = i\sum_{p=0}^{n}\omega_p\langle[\hat{c}_p^{\dagger}(t)\hat{c}_p(t),\ \hat{\Lambda}(t)]\rangle$$
$$+\left\{\langle[\hat{\Lambda}_n^{-}(t),\ \hat{\Lambda}(t)]\hat{\Lambda}_n^{+}(t)\frac{i\chi_n^2\tilde{N}}{(\omega_r-\omega+i\tilde{\gamma})}\hat{V}\right\rangle$$
$$+\sum_k\vartheta_k\{2\langle[\hat{c}_k^{\dagger}(t),\ \hat{\Lambda}(t)]\hat{c}_k(t)\hat{D}_z(t)\rangle$$
$$+\langle\hat{\Lambda}_n^{-}(t)[\hat{c}_k^{\dagger}(t),\ \hat{\Lambda}(t)]\hat{c}_k(t)\frac{4\chi_n^2}{(\omega_r-\omega)^2+\tilde{\gamma}^2}$$
$$\times[j(j+1)-\hat{D}_z(t)-\hat{D}^{-}(t)\hat{D}^{+}(t)]\hat{\Lambda}_n^{+}(t)\rangle\}$$
$$+2i\chi_n^2\frac{i\chi_n}{\gamma_p}\{\langle[\hat{\Lambda}_n^{-}(t),\ \hat{\Lambda}(t)]\hat{\Lambda}_n^{+}(t)\frac{1}{(\omega_r-\omega+i\tilde{\gamma})}\hat{V}\hat{D}_f^{-}(t)\hat{\Lambda}_n^{+}\rangle$$
$$+H.c.(\hat{\Lambda}^{+}(t)\to\hat{\Lambda}(t))\},\qquad\text{(A6)}$$

where $\tilde{N} - N[1-(j+1)\sum_k\vartheta_k\hat{c}_k^{\dagger}(t)\hat{c}_k(t)]$, $\hat{D}_z^2(t) = j(j+1) - \hat{D}_z(t) - \hat{D}^{-}(t)\hat{D}^{+}(t)$, and

$$\hat{V}(t) = \frac{1}{1+\sum_k\vartheta_k\hat{c}_k^{\dagger}(t)\hat{c}_k(t)+\tilde{\kappa}\hat{\Lambda}_n^{-}(t)\hat{\Lambda}_n^{+}(t)}.$$

It is created an impression that if we permute the operator, $\hat{D}_f^{-}(t)$, to the left hand of the correlation function $\langle[\hat{\Lambda}_n^{-}(t),\ \hat{O}(t)]\hat{\Lambda}_n^{+}(t)\hat{V}\hat{D}_f^{-}(t)\hat{\Lambda}_n^{+}\rangle$, we obtain the zero contribution in the last term of the Exp. (A6). Such a permutation is possible if we take into consideration that $\hat{D}_f^{-}(t) = \hat{D}^{-}(t) - \hat{D}_s^{-}(t)$. In this situation only the first term, of this difference can be permuted in the left hand of the correlation function so that after the permutation we can return and eliminate it

$$\langle[\hat{\Lambda}_n^{-}(t),\ \hat{O}(t)]\hat{\Lambda}_n^{+}(t)\hat{V}\hat{D}_f^{-}(t)\hat{\Lambda}_n^{+}\rangle$$
$$=\langle[\hat{D}_s^{-}(t)[\hat{\Lambda}_n^{-}(t),\ \hat{O}(t)]\hat{\Lambda}_n^{+}(t)\hat{V}(t)\hat{D}_s^{-}(t)]\hat{\Lambda}_n^{+}\rangle.$$

This re-normalization of the photon losses, ϑ_k from the cavity is connected to super-radiant effect of the confinement of bimodal field and atomic subsystem. So in the first approximation on the development on the parameter , $\tilde{\varsigma}$, the equation for $\langle\hat{\Lambda}(t)\rangle$ becomes,

$$\frac{d}{dt}\langle\hat{\Lambda}(t)\rangle^{(1)} = i\sum_{p=0}^{n}\omega_p\langle[\hat{c}_p^{\dagger}(t)\hat{c}_p(t),\ \hat{\Lambda}(t)]\rangle$$
$$+\{\langle[\hat{\Lambda}(t),\ \hat{\Lambda}_n^{-}(t)]\hat{\Lambda}_n^{+}(t)\frac{i\chi_n^2\tilde{N}_r}{(\omega_r-\omega+i\tilde{\gamma})}\hat{V}(t)\rangle$$
$$+N\sum_k\vartheta_k\{\langle[\hat{c}_k^{\dagger}(t),\ \hat{\Lambda}(t)]\hat{c}_k(t)\rangle$$
$$+\frac{4\chi_n^2j^2}{(\omega_r-\omega)^2+\tilde{\gamma}^2}\langle\hat{\Lambda}_n^{-}(t)[\hat{c}_k^{\dagger}(t),\ \hat{\Lambda}(t)]\hat{c}_k(t)\hat{\Lambda}_n^{+}(t)\rangle$$
$$+H.c.[(\hat{\Lambda}(t))^{+}\to\hat{\Lambda}(t)].$$

The next contribution can be obtained introducing the free part fluctuations terms in the right hand site of the generalized equation (A6). Let's now find the analytical representations of gain and losses terms of generalized equation (A6). Gain contribution is described by the expression, $G(t) = \langle[\hat{\Lambda}_n^{-}(t),\ \hat{\Lambda}(t)]\hat{D}_s^{+}(t)\rangle$, and its Hermite conjugate form. After consecutive elimination of the introduction of the free part of operator, $\hat{D}^{+}(t)$, and inversion, $\hat{D}_z(t)$, we obtain the following serial decomposition

$$G(t) = -N_r\sum_{k=1}^{\infty}(\tilde{\kappa}/4)^k\langle[\ldots[[\hat{\Lambda}(t),\ \hat{\Lambda}_n^{-}(t)]\hat{\Lambda}_n^{+}\hat{V}(t),\ \hat{\Lambda}_n^{-}(t)]$$
$$\times\hat{\Lambda}_n^{+}\hat{V}(t),\ldots,\hat{\Lambda}_n^{-}]\hat{\Lambda}_n^{+}\hat{V}(t)\rangle.\qquad\text{(A7)}$$

Here we can introduce the super-operator, $1/\{1-\hat{X}\}$, the serial decomposition, $1+\hat{X}+\hat{X}^2+\hat{X}^3+\ldots+$ of which is defined by the following recurrent actions, $\hat{K}_1 = \hat{X}\hat{\Lambda}(t)= (\tilde{\kappa}/4)[\hat{\Lambda}(t),\ \hat{\Lambda}_n^{-}(t)]\hat{\Lambda}_n^{+}\hat{V}(t)$; $K_2 = \hat{X}^2\hat{\Lambda}(t) = \hat{X}\hat{K}_1= (\tilde{\kappa}/4)[\hat{K}_1,\ \hat{\Lambda}_n^{-}(t)]\hat{\Lambda}_n^{+}\hat{V}(t),\ldots,\ \hat{K}_n = \hat{X}^n\hat{\Lambda}(t) = (\tilde{\kappa}/4)[\hat{K}_{n-1},\ \hat{\Lambda}_n^{-}(t)]\hat{\Lambda}_n^{+}\hat{V}(t)\ldots$ In this situation the serial decomposition (A7) can be represented in the following compact form,

$$G(t) = -N_r \frac{1}{1 - (\tilde{\kappa}/4)[..., \hat{\Lambda}_n^-(t)]\hat{\Lambda}_n^+\hat{V}(t)}\hat{\Lambda}(t) + N_r. \tag{A8}$$

After the substitution of this super-operator in the right hand cite of expression (A8), we observe that the first term of the decomposition is reduced with $+N_r$ and the next terms represent the serial decomposition (A7).

Let's give the similar representation for losses term,

$L(t) = \sum_p \Gamma_p\{\langle[\hat{c}_p^\dagger(t),\ \hat{\Lambda}(t)]\hat{c}_p(t)\hat{D}^+(t)\hat{D}^-(t)\rangle$, and its Hermite conjugate. After that the first elimination of the free parts of operators $\hat{D}^-(t)$ and $\hat{D}^+(t)$ in Exps. (A1) and (A6) we must again introduce operators $\hat{D}^-(t)$ and $\hat{D}^+(t)$ through the free and sources parts in the anti-normal representation of the losses term of Exp. (A6). In second elimination this correlation becomes equal to the expression,

$$\begin{aligned} L(t) = \sum_p &\Gamma_p\{2\langle[\hat{c}_p^\dagger(t),\ \hat{\Lambda}(t)(t)]\hat{c}_p(t)\hat{D}_z(t)\rangle \\ &+ \frac{4\chi_n^2 j(j+1)}{(\omega_r - \omega)^2 + \tilde{\gamma}^2}\langle\hat{\Lambda}_n^-(t)[\hat{c}_p^\dagger(t),\ \hat{\Lambda}(t)(t)]\hat{c}_p(t)\hat{\Lambda}_n^+(t)\rangle \\ &- \frac{4\chi_n^2}{(\omega_r - \omega)^2 + \tilde{\gamma}^2}\langle\hat{\Lambda}_n^-(t)[\hat{c}_p^\dagger(t),\ \hat{\Lambda}(t)(t)]\hat{c}_p(t)\hat{\Lambda}_n^+(t)\hat{D}_z(t)\rangle\}. \\ &- \left(\frac{4\chi_n^2}{(\omega_r - \omega)^2 + \tilde{\gamma}^2}\right)^2\langle\hat{\Lambda}_n^-(t)\hat{\Lambda}_n^-(t)\hat{D}_z^2(t)[\hat{c}_p^\dagger(t),\ \hat{\Lambda}(t)(t)] \\ \times\ &\hat{c}_p(t)\hat{\Lambda}_n^+(t)\hat{\Lambda}_n^+(t)\rangle\}. \end{aligned} \tag{A9}$$

The procedure is repeated again as in Exps. (A1) and (A6). We mast introduce again in the right hand side of Exp. (A9) the the expression for inversion represented through the anti-normal form of $\hat{D}^-(t)$ and $\hat{D}^+(t)$ operators, $\hat{D}_z^2(t) = j(j+1) - \hat{D}_z(t) - \hat{D}^-(t)\hat{D}^+(t)$. After that permute, $\hat{D}^-(t)$, in the left hand and $\hat{D}^+(t)$ in the right one of the correlations (A9). We can infinitely repeat this procedure representing the correlate, $L(t)$, in the serial decomposition,

$$\begin{aligned} L(t) = \sum_p &\Gamma_p\{2\langle[\hat{c}_p^\dagger(t),\ \hat{\Lambda}(t)]\hat{c}_p(t)\hat{D}_z(t)\rangle \\ &+ j(j+1)\{\tilde{\kappa}\langle\hat{\Lambda}_n^-(t)[\hat{c}_p^\dagger(t),\ \hat{\Lambda}(t)]\hat{c}_p(t)\hat{\Lambda}_n^+(t)\rangle - ... -(-\tilde{\kappa})^l \\ \times\ &\langle\hat{\Lambda}_n^-(t)\hat{\Lambda}_n^-(t)...\ \hat{\Lambda}_n^-(t)[\hat{c}_p^\dagger(t),\ \hat{\Lambda}(t)]\hat{c}_p(t)\hat{\Lambda}_n^+(t)\hat{\Lambda}_n^+(t)...\ \hat{\Lambda}_n^+(t)\rangle - ...\} \\ &- \tilde{\kappa}\{\langle[\hat{c}_p^\dagger(t),\ \hat{\Lambda}(t)]\hat{c}_p(t)\hat{D}_z(t)\rangle - ... \\ &- (-\tilde{\kappa})^l\langle\hat{\Lambda}_n^-(t)\hat{\Lambda}_n^-(t)...\ \hat{\Lambda}_n^-(t)[\hat{c}_p^\dagger(t),\ \hat{\Lambda}(t)] \\ \times\ &\hat{c}_p(t)\hat{\Lambda}_n^+(t)\hat{\Lambda}_n^+(t)...\ \hat{\Lambda}_n^+(t)\hat{D}_z(t)\rangle - ...\}. \end{aligned} \tag{A10}$$

Here $\tilde{\kappa} = 4\chi_n^2/[(\omega_r - \omega)^2 + \tilde{\gamma}^2]$. For further elimination of the operator of inversion, $\hat{D}_z(t)$, we use the representation (A5) for this operator and follow the methodology of elimination of its free part as in correlation function, $G(t)$. But it is not difficult to observe that for the large number of atoms , $N_r = 2j \gg 1$, we can neglect the last terms proportional to $\hat{D}_z(t)$ (here, $j > \langle\hat{D}_z(t)\rangle$) in the expansion (A10) relative the terms proportional to j^2 . In this case the losses corelation of generalized equation becomes,

$$\begin{aligned} L(t) = N_r(1 + N_r/4)&\langle([\hat{c}_p^\dagger(t),\ \hat{\Lambda}(t)]\hat{c}_p(t)\rangle \\ &- \frac{N_r^2}{4}\left\langle\frac{1}{1 + \kappa\hat{\Lambda}_n^-(t) \risingdotseq \hat{\Lambda}_n^+(t)}[\hat{c}_p^\dagger(t),\ \hat{O}(t)]\hat{c}_p\right\rangle. \end{aligned} \tag{A11}$$

Here the function 1-$\frac{1}{1 + \kappa\hat{\Lambda}_n^-(t) \risingdotseq \hat{\Lambda}_n^+(t)}$ can be represented in serial decomposed, $\hat{Y} - \hat{Y}^2 + \hat{Y}^3 - ... -(-\kappa)^l\hat{Y}^l ...$, relative the superoperator $\hat{Y} \ = \ \kappa\hat{\Lambda}_n^-(t) \risingdotseq \hat{\Lambda}_n^+(t)$, with the definition of action on the commutator, $\hat{Z} = [\hat{c}_p^\dagger(t),\ \hat{O}(t)]\hat{c}_p : \hat{\Xi}_1 = \hat{Y}\hat{Z} = \kappa\hat{\Lambda}_n^-(t)[\hat{c}_p^\dagger(t),\ \hat{O}(t)]\hat{c}_p\hat{\Lambda}_n^+(t)$, $\hat{\Xi}_2 = \hat{Y}\hat{Z}_1 = \kappa\hat{\Lambda}_n^-(t)\hat{Z}_1\hat{\Lambda}_n^+(t)$ and so one, $\hat{\Xi}_l = \hat{Y}\hat{Z}_{l-1} = \hat{Y}^l\hat{Z} = \kappa\hat{\Lambda}_n^-(t)\hat{Z}_{l-1}\hat{\Lambda}_n^+(t)$.

Unifying the the gain in multiple scattering process (A7) and losses (A10) we obtain the serial expansion in the terms of bimodal field operators of generalized equation,

$$\frac{d}{dt}\langle\hat{\Lambda}(t)\rangle \quad - i\sum_{p=0}^{n}\omega_p\langle[\hat{c}_p^\dagger(t)\hat{c}_p(t),\, \hat{\Lambda}(t)]\rangle$$
$$= N_r\sum_{k=1}(\kappa/4)^k\langle[...[[\hat{\Lambda}_n^-(t),\, \hat{\Lambda}(t)]\hat{\Lambda}_n^+\hat{V}(t),\, \hat{\Lambda}_n^-(t)]\hat{\Lambda}_n^+\hat{V}(t),\ldots,\hat{\Lambda}_n^-]\rangle$$
$$+ N_r\sum_p\Gamma_p\langle[\hat{c}_p^\dagger(t),\, \hat{\Lambda}(t)]\hat{c}_p(t)\rangle$$
$$+ (N_r^2/4)\{\langle\kappa(\hat{\Lambda}_n^-(t)[\hat{c}_p^\dagger(t),\, \hat{\Lambda}(t)]\hat{c}_p(t)\hat{\Lambda}_n^+(t)\rangle$$
$$- (-\kappa)^l\langle\hat{\Lambda}_n^-(t)\hat{\Lambda}_n^-(t)...\, \hat{\Lambda}_n^-(t)[\hat{c}_p^\dagger(t),\, \hat{\Lambda}(t)]\hat{c}_p(t)\hat{\Lambda}_n^+(t)\hat{\Lambda}_n^+(t)...\, \hat{\Lambda}_n^+(t)\rangle$$
$$- ...\} + H.c.[(\hat{\Lambda}(t))^+ \to \hat{\Lambda}(t)]. \tag{A12}$$

The return to master equation of density matrix, $\hat{W}(t)$, of the bimodal cavity field can be obtained after transformation from Heisenberg to Schrödinger pictures under the operator traces, $\langle\hat{\Lambda}(t)\rangle = Sp\{\hat{W}(t)\hat{\Lambda}(t)\} = Sp\{\hat{W}(t)\hat{\Lambda}\}$. From equation (A12) we obtain the master equation,

$$\frac{d}{dt}\hat{W}(t) - i\sum_{p=0}^{n}\omega_p[\hat{W}(t),\, \hat{c}_p^\dagger\hat{c}_p]$$
$$= N_r\sum_{k=1}^{\infty}(\kappa/4)^k[\hat{\Lambda}_n^-,\, \hat{\Lambda}_n^+\hat{V}...[\hat{\Lambda}_n^-,\, \hat{\Lambda}_n^+\hat{V}[\hat{\Lambda}_n^-,\, \hat{\Lambda}_n^+\hat{V}\hat{W}(t)]]...]$$
$$- \sum_p\Gamma_p\{[\hat{c}_p^\dagger,\, \hat{c}_p\{N_r\hat{W}(t) - (N_r^2/4)\sum_{l=1}^{\infty}\hat{\Lambda}_n^+\hat{\Lambda}_n^+ ...\, \hat{\Lambda}_n^+W(t)$$
$$\times\, \hat{\Lambda}_n^-\hat{\Lambda}_n^- ...\, \hat{\Lambda}_n^-(-\kappa)^l\}] + H.c. \tag{A13}$$

We can return to the problem of multiple cooperative scattering of photons using this master equation. We observe that the gain term of the master equation contain cooperative phenomena between the photons generated in the bimodal field. In the second term of this equation the losses increases being normalized by super—radiant cooperative process between the atoms lake in the Dicke super-fluorescence.

Appendix B. Wave vectors of cavity EMF modes created from initial states in multiple Raman scattering

1. Let's start with two steps of multiple scattering process and acts with excitation bimodal operator, $\hat{\Lambda}_2^+ = (g_1\hat{c}_0\hat{c}_1^\dagger + g_2\hat{c}_1\hat{c}_2^\dagger)/\chi_2$, on the initial state, $|l, 0>_{(2)} = |l\rangle_0|0\rangle_1|0\rangle_2$, of the three boson modes of the cavity, in which is realized two steps scattering process represented in figure 4(a)

$$\hat{\Lambda}_2^+|l\rangle_0|0\rangle_1|0\rangle_2 = \sqrt{l}|l-1\rangle_0|1\rangle_1|0\rangle_2/\chi_2 = C_1|l,\, 1>_{(2)}\, .$$

For the new state which represents that one photon was converted we ask the normalization procedure, $<l, 1|l, 1>_{(2)} = 1$, which gives us the following expression for the coefficient, $C_1 = \sqrt{l}g_1/\chi_2$. It describes the amplitude of the transition probability, $|<l,\, 1|\hat{\Lambda}_2^+|l,\, 0>_{(2)}\, |^2$, from the state with zero converted photons into new state with one portion of energy, $\hbar\omega_r = \hbar(\omega_1 - \omega_{p_0})$. Let us again act on the new obtained vector, $|l, 1>_{(2)} = |l-1\rangle_s|1\rangle_b|0\rangle_a$, with excitation operator, $\hat{\Lambda}_2^+$. We obtain the superposition,

$$\hat{\Lambda}^+|l,\, 1>_{(2)} = \{g_1^2\sqrt{2(l-1)}\,|l-2\rangle_0|2\rangle_1|0\rangle_2$$
$$+ g_1g_2|l-1\rangle_0|0\rangle_1|1\rangle_2\}\Big/\sqrt{g_1^2 l}\chi_2 = C_2|l,\, 2>_{(2)}\, . \tag{B1}$$

The label, 2, of this new state, $|l, 2>_{(2)}$, indicate us that the cavity received two portion of energies equal $2\times(\hbar\omega_r)$. The obtained energy is realized in two possibilities: first way corresponds to the generation of second anti-Stokes photons and second one describes the reabsorption of the anti-Stokes photon with frequency, ω_1 and generation of another photon with frequency, ω_2. Asking the normalization to unity, $\langle l, -l+2|l, -l+2\rangle = 1$, we easily find the normalization coefficient, $C_2 = 2\sqrt{g_1^4C_l^2 + g_1^2g_2^2/4}\Big/(\chi_2\sqrt{g_1^2 l})$. This new wave vector becomes the superposition of above described states.

$$|l,\, 2>_{(2)} = \frac{g_1^2\sqrt{C_l^2}\,|l-2\rangle_0|2\rangle_1|0\rangle_2 + g_1(g_2/2)|l-1\rangle_0|0\rangle_1|1\rangle_2}{\sqrt{g_1^4C_l^2 + g_1^2(g_2/2)^2C_l^1}}, \tag{B2}$$

represented through binomial coefficients, $\binom{n}{k} = C_n^k = n!/[k!(n-k)!]$. The cavity may obtain the third portion of energy in the bimodal conversion, $\hbar\omega_r$, if we act again with operator, $\hat{\Lambda}_2^+$, on the new state, $|l, 2>$,

$$\hat{\Lambda}_2^+|l,\,2>_{(2)} = \frac{\{g_1^3\sqrt{3^2C_l^3}\,|l-3\rangle_s|3\rangle_b|0\rangle_a + 3g_1^2g_2/2\sqrt{2C_l^2}\,|l-2\rangle_s|1\rangle_b|1\rangle_a\}}{\chi_2\sqrt{g_1^2C_l^2 + g_1^2(g_2/2)^2}} = C_3|l,\,3>_{(2)}\,. \quad \text{(B3)}$$

From this representation we obtain the next expression for normalization coefficient, $C_3 = 3\sqrt{g_1^6C_l^3 + 2g_1^4(g_2^2/4)C_l^2}\Big/\sqrt{g_1^2C_l^2 + g_1^2(g_2/2)^2}$. At this step we observe only the conversion of the photon with energy, $\hbar\omega_1 = \hbar\omega_{p_0} + \hbar\omega_r$, from pump mode to anti-Stokes one. The increasing of superposition terms in this two step conversion may be observed on the fourth excitation of cavity modes with the portion of energy, $\hbar\omega_r$, after the action of the next operator, $\hat{\Lambda}_2^+$ on the state $|l, 3>$: $\hat{\Lambda}_2^+|l,\,3> = C_4|l,\,3>$, from which follows the wave vector,

$$|l,\,4>_{(2)} = [g_1^4\sqrt{C_l^4}\,|l-4\rangle_0|4\rangle_1|0\rangle_1 + g_1^3(g_2/2)\sqrt{3C_l^3}\,|l-3\rangle_0|2\rangle_1|1\rangle_2 + g_1^2(g_2/2)^2\sqrt{C_l^2}\,|l-2\rangle_0|0\rangle_1|2\rangle_2] \Big/ \sqrt{C_l^4g^8 + 3g_1^6(g_2/2)^2C_l^3 + g_1^2(g_2/2)^4C_l^2}\,, \quad \text{(B4)}$$

in which from the normalization wave vector $<l, 3|l, 3> = 1$, follows the next expression for coefficient, $C_4 = 4\sqrt{C_l^4g_1^8 + 3g_1^6(g_2/2)^2C_l^3 + g_1^4(g_2/2)^4C_l^2}/(\chi_2\sqrt{g_1^6C_l^3 + 2g^4(g_2^2/4)C_l^2})$. We made four consecutive actions of operator, $\hat{\Lambda}_2^+$, on the initial state in order to understand the behavior of the superposition coefficients. In the final, we propose that the general low of the p-times actions of the generation operator, $\hat{\Lambda}_2^+$, for the creation $p\times(\hbar\omega_r)$- portions of energies in the cavity is described by the wave vector,

$$|l,\,p>_{(2)} = A_{lp}^{(2)}\sum_{k=\lceil p/2\rceil}^{p}\left(\frac{g_2}{2}\right)^{p-k}g_1^k\sqrt{C_k^{2k-p}C_l^k}\,|l-k\rangle_0|2k-p\rangle_1|p-k\rangle_2, \quad \text{(B5)}$$

where, $A_{lp}^{(2)} = 1\Big/\sqrt{\sum_{k=\lceil p/2\rceil}^{p}\left(\frac{g_2}{2}\right)^{2(p-k)}C_k^{2k-p}g_2^{2k}C_l^k}$, is the normalized coefficient of this state, $\lceil p/2\rceil = p/2$ for even numbers, $\lceil p/2\rceil = (p+1)/2$, for odd numbers. Let us consider that this expression is correct and acts again with operator, $\hat{\Lambda}_2^+$, on the state $|l, p>_{(2)}$

$$\hat{\Lambda}_2^+|l,\,p>_{(2)} = A_{lp}\sum_{k=p/2}^{p}\left(\frac{g_2}{2}\right)^{p-k}g_1^k\sqrt{C_k^{2k-p}C_l^k}\,\hat{\Lambda}_2^+|l-k\rangle_0|2k-p\rangle_1|p-k\rangle_2 = C_{p+1}|l,\,p+1>_{(2)}\,.$$

From which follows the identity,

$$C_{p+1}|l,\,p+1>_{(2)} = A_{lp}\Bigg\{\sum_{k=\lceil(p+1)/2\rceil}^{p+1}\Bigg\{\left(\frac{g_2}{2}\right)^{p+1-k}g_1^k \times\sqrt{C_{k-1}^{2k-2-p}C_l^{k-1}(l+1-k)(2k-1-p)}\,|l-k\rangle_s|2k-1-p\rangle_b|p+1-k\rangle_a + \sum_{k=\lceil p+1/2\rceil}^{p+1}\left(\frac{g_2}{2}\right)^{p+1-k}2g_1^k\sqrt{C_k^{2k-p}C_l^k(2k-p)(p-k+1)} \times|l-k\rangle_s|2k-1-p\rangle_b|p-k+1\rangle_a\Bigg\}.$$

Taking into consideration the identities, $C_{k-1}^{2k-2-p}C_l^{k-1}(l+1-k)(2k-1-p) = C_k^{2k-1-p}C_l^k(2k-1-p)^2$, and, $C_k^{2k-p}(2k-p)(p-k+1) = C_k^{2k-p-1}(p-k+1)^2$, we obtain that the new vector, $|l, p+1>_{(2)}$, by replacing p by $p+1$ with coefficients,

$$C_{p+1} = \frac{(p+1)\sqrt{\sum_{k=\lceil(p+1)/2\rceil}^{p+1}\left(\frac{g_2}{2}\right)^{2(p+1-k)}C_k^{2k-p-1}g_1^{2k}C_l^k}}{\chi_2\sqrt{\sum_{k=\lceil p/2\rceil}^{p}\left(\frac{g_2}{2}\right)^{2(p-k)}C_k^{2k-p}g_1^{2k}C_l^k}},$$

which demonstrate the viability of the representation (B5). $g_2/\chi_2 = g_1/\chi_2 = \sqrt{2}$, we obtain the following identity, $C_p = \sqrt{p(2l-p+1)}$, from which follows that the sum takes the form,

$$\sum_{k=\lceil p/2\rceil}^{p}2^{2k}C_k^{2k-p}C_l^k = 2^pC_{2l}^p.$$

2. Below, we want to generalize the induced multiple scattering vectors for the α- steps. Let us study the Raman lasing in the three steps of conversion of photons. In this situation the excitation and de-excited of the cavity field with the portion of energy, $\hbar\omega_r$, are described by the expressions, $\hat{\Lambda}_3^+ = (g_1\hat{c}_0\hat{c}_1^\dagger + g_2\hat{c}_1\hat{c}_2^\dagger + g_3\hat{c}_2\hat{c}_3^\dagger)/\chi_3$, and, $\hat{\Lambda}_3^- = (g_1\hat{c}_1\hat{c}_0^\dagger + g_2\hat{c}_2\hat{c}_1^\dagger + g_3\hat{c}_3\hat{c}_2^\dagger)/\chi_3$, where the renormalization coefficient may be regarding as, $\chi_3 = \sqrt{g_1^2 + g_2^2 + g_3^2}$. In the three step induced scattered photons, the number of modes in which can be generated the converted photons are equal to three: $\omega_1 = \omega_{p_0} + \omega_r$, $\omega_2 = \omega_{p_0} + 2\omega_r$, $\omega_3 = \omega_{p_0} + 3\omega_r$. The initial state vector is represented by the product, $|l, 0>_{(3)} = |l\rangle_0|0\rangle_1|0\rangle_2|0\rangle_3$. Firsts two actions to this state is accompanied by the generation of portions with energies in the bimodal states of cavity, $\hbar\omega_r$, $2\hbar\omega_r$, and have many similarities with two-steps Raman conversion: $\hat{\Lambda}_3^+|l\rangle_0|0\rangle_1|0\rangle_2|0\rangle_3 = g_1\sqrt{l}|l-1\rangle_0|1\rangle_1|0\rangle_2|0\rangle_3/\chi_3$,

$$\begin{aligned}[\hat{\Lambda}_3^+|l, 2>_{(3)} = &[g_1^3 3!\sqrt{C_l^3}|l-3\rangle_0|3\rangle_1|0\rangle_2|0\rangle_3\\ &+ 3g_1^2 g_2\sqrt{2!C_l^2}|l-2\rangle_0|1\rangle_1|1\rangle_2|0\rangle_3\\ &+ g_1 g_2 g_3\sqrt{1!C_l^1}|l-1\rangle_0|0\rangle_1|0\rangle_2|1\rangle_3]/[\chi_3 g_1\sqrt{g_1^2 2^2 C_l^2 + g_2^2 l}]\\ = &A_3|l, 3>_{(3)}.\end{aligned}$$

Here according to the procedure proposed for two steps Raman emission the coefficient, A_3, must be found from normalization condition, $\langle l, 3|l, 3\rangle_{(3)} = 1$. We observe that the new populated state, $|n\rangle_3$, looks as a superposition function composed of the two-step wave function and new photon converted state $|n\rangle_3$,

$$|l, 3>_{(3)} \sim |\widetilde{l, 3}>_{(2)}|0\rangle_3 + \frac{g_1 g_2 g_3}{3!}\sqrt{C_l^1}|l-1, 0>_{(2)}|1\rangle_3 = |\widetilde{l, 3}>_{(3)},$$

where the first term contains the excitation of the cavity field with three portions of energies in the two steps conversion of the initial pumped photons. The first term is described by non normalized wave function (B2), $|\widetilde{l, 3}>_{(2)} = g^2\sqrt{C_l^2}|l-2\rangle_0|2\rangle_1|0\rangle_2 + g_1(g_2/2)|l-1\rangle_0|0\rangle_1|1\rangle_2$ multiplied to vacuum state in the third scattered mode, $|0\rangle_3$. The second term demonstrates that one photon leaved two-step state, $|l-1, 0>_{(2)} = |l\rangle_0|0\rangle_1|0\rangle_2$, so that in the pump field we have $l-1$ photons and one photon in new scattered field $|1\rangle_3$. In order to avoid the complicated coefficients in the procedure of the construction of wave function for, $p \times (h\omega_r)$, excitation portions of energy, we propose to use weave functions, which are non-normalized to unity,

$$|\widetilde{l, p}>_{(2)} = \sum_{k=[p/2]}^{p}\left(\frac{g_2}{2}\right)^{p-k} g_1^k\sqrt{C_k^{2k-p}C_l^k}|l-k\rangle_0|2k-p\rangle_1|p-k\rangle_2, \tag{B6}$$

described by the representation (B5). The next action of the excitation operator on the state, $|\widetilde{l, 3}>_{(3)}$, don't give the conversion of the next photon in the new conversion step mode, ω_3, and according to the Exp. (B3) the action of operator, $\hat{\Lambda}_3^+$, on the state, $|\widetilde{l, 4}>_{(2)}$, only increases its amplitude,

$$|\widetilde{l, 4}>_{(3)} = \frac{\chi_3\hat{\Lambda}_3^+|\widetilde{l, 3}>_{(3)}}{4} = |\widetilde{l, 4}>_{(2)}|0\rangle_3 + \frac{g_1 g_2 g_3}{6}\sqrt{l}|\widetilde{l-1, 1}>_{(2)}|1\rangle_3. \tag{B7}$$

described by the representation (B5). The next action of the excitation operator on the state, $|\widetilde{l, 3}>_{(3)}$, don't give the conversion of the next photon in the new conversion step mode, $\omega_3 -$, and according to the Exp. (B3) the action of operator, $\hat{\Lambda}_3^+$, on the state, $|\widetilde{l, 4}>_{(2)}$, only increases its amplitude,

$$|\widetilde{l, 5}>_{(3)} = |\widetilde{l, 5}>_{(2)}|0\rangle_3 + \frac{1}{2\times 3}g_1 g_2 g_3\sqrt{l}|\widetilde{l-1, 2}>_{(2)}|1\rangle_3.$$

Beginning with this collective scattering state we observe that in the third mode appear the second photon after multiple scattering. The term, $\hat{\Lambda}_3^+|\widetilde{l-1, 2}>_{(2)}|1\rangle_3$, will generate the second photon in the state, $|2\rangle_3$,

$$\begin{aligned}|\widetilde{l, 5}>_{(3)} = &\frac{\chi_3\hat{\Lambda}_3^+|\widetilde{l, 5}>_{(3)}}{6}\\ = &|\widetilde{l, 6}>_{(2)}|0\rangle_3 + \frac{g_1 g_2 g_3}{2\times 3}\sqrt{l}|\widetilde{l-1, 3}>_{(2)}|1\rangle_3\\ &+ \frac{1}{6\times 6}g_1^2 g_2^2 g_3^2\sqrt{\frac{l(l-1)}{2}}|\widetilde{l-2, 0}>_{(2)}|2\rangle_3,\end{aligned} \tag{B8}$$

Taking this propriety into consideration, we can create and three - steps conversations of the pump field into new anti-Stokes modes. Now can generalize this conversion to α − steps generating the wave function for this process,

$$|\widetilde{l,p}\rangle_{(\alpha)} = \sum_{k=0}^{\lfloor p/\alpha \rfloor} \left(\frac{g_1 g_2 \cdots g_\alpha}{\alpha!}\right)^k \sqrt{C_l^k}\, |\widetilde{l-k,\, p-\alpha k}\rangle_{(\alpha-1)}\, |k\rangle_\alpha. \tag{B9}$$

According to the indication method we may propose the following generalized expression for wave function, $|\widetilde{l,p}\rangle_{(3)}$, for generation of p portions of energy, $\hbar\omega_r$, into the cavity after the p- times conversion action of the operator, , $\hat{\Lambda}_3^+$, on the state, $|l\rangle_0|0\rangle_1|0\rangle_2|0\rangle_3$,

$$|\widetilde{l,p}\rangle_{(3)} = \sum_{k=0}^{\lfloor p/3 \rfloor} \left(\frac{g_1 g_2 g_3}{6}\right)^k \sqrt{C_l^k}\, |\widetilde{l-k,\, p-3k}\rangle_{(2)}\, |k\rangle_3, \tag{B10}$$

where

$$|\widetilde{l-k,\, p-3k}\rangle_{(2)} = \sum_{r=\lceil (p-3k)/2 \rceil}^{p-3k} \left(\frac{g_2}{2}\right)^{p-3k-r} g_1^r \sqrt{C_r^{p-3k-r} C_{l-k}^r} \times |l-k-r\rangle_0 |2r-p+3k\rangle_1 |p-3k-r\rangle_2.$$

To demonstrate this proposal, we take the next action of the generation operator, $\hat{\Lambda}_3^+$, on the state (B10), so that the next wave vector, $|\widetilde{l,p+1}\rangle_{(3)} = \chi_3 \hat{\Lambda}_3^+ |\widetilde{l,p+1}\rangle_{(3)} \big/ (p+1)$, takes the form,

$$\begin{aligned}
|\widetilde{l,p+1}\rangle_{(3)} = {} & \frac{1}{(p+1)} \sum_{k=0}^{p/3} \left(\frac{g_1 g_2 g_3}{6}\right)^k (p-3k+1) \\
& \times \sqrt{C_l^k}\, |\widetilde{l-k,\, p-3k+1}\rangle_{(2)}\, |k\rangle_3 \\
& + \frac{1}{(p+1)} \sum_{k=0}^{p/3} \left(\frac{g_1 g_2 g_3}{6}\right)^k g_3 (p-3k+1) \sqrt{C_l^k} \\
& \times \sum_{r=(p-3k)/2}^{p-3k} \left(\frac{g_2}{2}\right)^{p-3k-r} g_1^r \sqrt{C_r^{p-3k-r} k(p-3k-r) C_{l-k}^r} \\
& \times |l-k-r\rangle_0 |2r-p+3k\rangle_1 |p-3k-r-1\rangle_2 |k+1\rangle_3.
\end{aligned} \tag{B11}$$

Using the identities $C_l^{k-1} k(l-k+1) = k^2 C_l^k$ and $C_{r+1}^{p+1-3k+1-r} k(p+1-3k+1-r) C_{l-k+1}^{r+1} = k(l-k+1) C_r^{p+1-3k-r} C_{l-k}^r$ we can reduce the state (B11) to the same form of Exp. (B10)

$$\begin{aligned}
|\widetilde{l,p+1}\rangle_{(3)} = {} & |\widetilde{l-k,\, p+1}\rangle_{(2)}\, |0\rangle_3 \\
& + \frac{1}{(p+1)} \sum_{k=1}^{(p+1)/3} \left(\frac{g_1 g_2 g_3}{6}\right)^k (p-3k+1) \\
& \times \sqrt{C_l^k}\, |\widetilde{l-k,\, p-3k+1}\rangle_{(2)}\, |k\rangle_3 \\
& + \frac{1}{(p+1)} \sum_{k=1}^{p/3+!} \left(\frac{g_1 g_2 g_3}{6}\right)^k 3k \sqrt{C_l^k}\, |\widetilde{l-k,\, p-3k+1}\rangle_{(2)}\, |k\rangle_3 \\
= {} & \sum_{k=0}^{(p+1)/3} \left(\frac{g_1 g_2 g_3}{6}\right)^k \sqrt{C_l^k}\, |\widetilde{l-k,\, p-3k+1}\rangle_{(2)}\, |k\rangle_3.
\end{aligned}$$

It is not difficult to normalize the wave function (B10). For this we can use the representation of the two steps wave function through the normalized to one wave functions (B5),
$|\widetilde{l-k,\, p-3k}\rangle_{(2)} = A_{l-k,p-3k}^{-1} |l-k,\, p-3k\rangle_{(2)}$. In this situation we may easily find the normalized coefficient of the three steps wave function, $|l,p\rangle_{(3)} = A_{l,p}^{(3)} |\widetilde{l,p}\rangle_{(3)}$, described by (B10),

$$|l,p\rangle_{(3)} = A_{l,p}^{(3)} \sum_{k=0}^{\lfloor p/3 \rfloor} \left(\frac{g_1 g_2 g_3}{3!}\right)^k \sqrt{C_l^k}\, (A_{l-k,p-3k}^{(2)})^{-1} |l-k,\, p-3k\rangle_{(2)}\, |k\rangle_3, \tag{B12}$$

where the expression, $A_{l,p}^{(3)} = 1/\sum_{k=0}^{p/3} \left(\frac{g_1 g_2 g_3}{6}\right)^{2k} C_l^k (A_{l-k,p-3k}^{(2)})^{-2}$; is normalized to unity coefficient. In this approach we can reduce to similar expression and the wave function of the two-steps Raman conversion (B6), Using the identity, $C_l^{p-k} C_{l-p+k}^{2k-p} = C_l^k C_k^{p-k}$, we easily can reduce Exp. (B6) to the similar one

$$|\widetilde{l,p}\rangle_{(2)} = \sum_{r=0}^{p/2} \left(\frac{g_1 g_2}{2}\right)^r \sqrt{C_l^r}\, |\widetilde{l-r,\, p-2r}\rangle_{(1)}\, |r\rangle_2. \tag{B13}$$

This wave functions were created using the excited operators, $\{\hat{\Lambda}_\alpha^\dagger\}$, $\alpha = 2, 3, \ldots$. It is not so difficult to observe that after the action of the lowering operators, $\hat{\Lambda}_\alpha^-$, to the wave function, $|\widetilde{l,p}\rangle_{(\alpha)}$, transform it into, $|\widetilde{l,p-1}\rangle_{(\alpha)}$, only in the special relations between the coefficients, $g_0, g_1, g_2, \ldots, g_{\alpha+1}$, where $\alpha = 1, 2, 3, \ldots$.

These functions are orthogonal and can be used for the decomposition of the bimodal density matrix on them $\hat{W}(t) = \sum_{l,p} P_{lp} |\widetilde{l, p} >_{(\alpha)} < \widetilde{l, p}|_{\alpha}$. Observing the symmetry like $su(2)$ or $su(1, 1)$ help us to reduce the degrees of freedom in the Hilbert space of bimodal field.

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