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On a family of Hamiltonian cubic planar differential systems

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Abstract. A class of planar perturbed Hamiltonian systems are studied in the present work in order to identify the limit cycles. The closed curves of the unperturbed associated Hamiltonian system are described. Using the Abelian integral method we find the detection functions. Numerical explorations are presented to illustrate the distribution of the limit cycles.

Mathematics subject classification: 34C07, 37G15.
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1 Introduction

In [1] the authors study a family of dynamical systems given by

$$\dot{x} = yP_0(x),\ \dot{y} = -x + P_2(x)y^2 + P_3(x)y^3, \tag{1}$$

for some particular polynomials P_0, P_2, P_3 proving the existence of eight limit cycles. Considering first a more general class of dynamical systems of the form

$$\dot{x} = yP_0(x, y),\ \dot{y} = P_1(x, y) + P_2(x, y)y^2 + P_3(x, y)y^3, \tag{2}$$

and particularizing $P_0(x, y), P_1(x, y), P_2(x, y), P_3(x, y)$ we get a system given by

$$\dot{x} = 4by\left(-y^2 + ax^2 + 1\right),\ \dot{y} = 4ax\left(x^2 - by^2 - 1\right). \tag{3}$$

This system is a Hamiltonian system with the Hamilton function given by

$$H(x, y) = -(ax^4 + by^4) + 2abx^2y^2 + 2\left(ax^2 + by^2\right). \tag{4}$$

Indeed, given the Hamilton function (4), the associated Hamiltonian system is

$$\dot{x} = \frac{\partial H}{\partial y},\ \dot{y} = -\frac{\partial H}{\partial x}, \tag{5}$$

which easily leads to (3). A similar Hamiltonian system has been studied in [2–4]. In this work we study the existence of the limit cycles of some perturbations of the system (5). The problem of the existence of limit cycles in a planar differential system of a given degree is known as the Hilbert's 16th problem. This problem is still unsolved even for the quadratic polynomial differential systems. A method used to deal with such problems is the Abelian integral method [5]. Other methods can be found in [6, 7] and [8]. Further reading on limit cycles can be found in [9–15]

This paper is organized as follows. In Section 2, we give some preliminary results. In Section 3, we study the global portrait of the unperturbed system. In the last Section 4, we find the numerical values of the detection functions of the perturbed system. Finally, we present the distribution of the limit cycles in a particular representative case.

2 Preliminary results

It is known that one way to produce limit cycles is by perturbing an Hamiltonian system which has one or more centers, in such a way that limit cycles bifurcate in the perturbed system from some of the periodic orbits in the original system.

The following perturbed Hamiltonian system

$$\begin{cases} \dot{x} = y(1 + x^2 - ay^2) + \varepsilon x(ux^n + vy^n - \lambda), \\ \dot{y} = -x(1 - cx^2 + y^2) + \varepsilon y(ux^n + vy^n - \lambda) \end{cases} \tag{6}$$

where $ac > 1, a > c > 0, 0 < \varepsilon \ll 1$, u, v, λ are the real parameters and $n = 2k, k$ integer positive, has been studied in [16–19].

The following result is reported in [17].

Theorem 1. *Consider the perturbed Hamiltonian system*

$$\dot{x} = \frac{\partial H}{\partial y} + P(x, y, \varepsilon), \quad \dot{y} = -\frac{\partial H}{\partial x} + Q(x, y, \varepsilon). \tag{7}$$

Assume that $P(x, y, \varepsilon)$, $Q(x, y, \varepsilon)$ are polynomials of given degree (perturbations of the Hamiltonian system), $P(x, y, 0) = Q(x, y, 0) = 0$, the closed curve $\Gamma^h : H(x, y) = h$ defined by the Hamiltonian $H(x, y)$ of the system (7) is a periodic orbit that extends outside as h increasing, and $\Gamma^h(D)$ is the area inside Γ^h. If there exists h_0 such that function

$$A(h) = \int\limits_{\Gamma^h(D)} [P''_{x\varepsilon}(x, y, 0) + Q''_{y\varepsilon}(x, y, 0)]dxdy \tag{8}$$

satisfies $A(h_0) = 0, A'(h_0) \neq 0, \varepsilon A'(h_0) < 0 (> 0)$, then system (7) has only one stable (unstable) limit cycle nearby Γ^{h_0} for ε very small. If Γ^h constricts inside as h increasing, the stability of the limit cycle is opposite with above. If $A(h) \neq 0$, then system (7) has no limit cycle.

The integral $A(h)$ is called the *Abelian integral* [5].
If the form of the system (7) is:

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial y} + \varepsilon x(p(x,y) - \lambda), \\ \dot{y} = -\dfrac{\partial H}{\partial x} + \varepsilon y(q(x,y) - \lambda) \end{cases} \tag{9}$$

where $p(0,0) = q(0,0) = 0$, then, using the above Theorem 1, from $A(h) = 0$, we get:

$$\lambda = \lambda(h) = \frac{\int\limits_{\Gamma^h(D)} f(x,y)dxdy}{2\int\limits_{\Gamma^h(D)} dxdy} \tag{10}$$

where $f(x,y) = xp'_x(x,y) + yp'_y(x,y) + p(x,y) + q(x,y)$.

This function $\lambda(h)$ is called *the detection function* of the system (9).

From Theorem 1 and using the detection function $\lambda(h)$ we get the following result:

Proposition 2. *a) If $(h_0, \lambda(h_0))$ is an intersecting point of line $\lambda = \lambda_0$ and the detection curve $\lambda = \lambda(h)$, and $\lambda'(h_0) > 0 (< 0)$, then system (9) has only one stable (unstable) limit cycle nearby Γ^{h_0} when $\lambda = \lambda_0$; b) If line $\lambda = \lambda_0$ and the detection curve $\lambda = \lambda(h)$ have no intersecting point, then the system (9) has no limit cycle when $\lambda = \lambda_0$. If the Γ^h constricts inside as h increasing, the stability of the limit cycle is opposite with above.*

Consider in the following the perturbed Hamiltonian system given by:

$$\begin{cases} \dot{x} = 4by\left(-y^2 + ax^2 + 1\right) + P\left(x, y, \varepsilon\right), \\ \dot{y} = 4ax\left(x^2 - by^2 - 1\right) + Q\left(x, y, \varepsilon\right) \end{cases} \tag{11}$$

where $P(x,y,\varepsilon) = \varepsilon x((n+2)vy^n - c\dfrac{s+1}{r+1}x^r y^s - ux^2y^2 - \lambda)$, $Q(x,y,\varepsilon) = \varepsilon y((n+2)ux^n + cx^r y^s - ux^2y^2 - \lambda)$, $r + s = n$, $0 < a < b$, $ab < 1$, $0 < \varepsilon \ll 1$ u, v, λ, c are the real parameters and $n = 2k, k$ positive integer.

3 The behavior of the unperturbed system

The unperturbed system corresponding to system (11) is the system (11) in the case $\varepsilon = 0$.

System (3) has nine finite singular points and they are:

$$A_1\left(\frac{1}{ab-1}\sqrt{(1-ab)(1+b)}, \quad \frac{1}{ab-1}\sqrt{(1-ab)(a+1)}\right),$$

$$A_2\left(\frac{1}{ab-1}\sqrt{(1-ab)(1+b)}, \quad \frac{1}{ab-1}\sqrt{(1-ab)(a+1)}\right),$$

$$A_3\left(-\frac{1}{ab-1}\sqrt{(1-ab)(1+b)}, \quad \frac{1}{ab-1}\sqrt{(1-ab)(a+1)}\right),$$

$$A_4\left(-\frac{1}{ab-1}\sqrt{(1-ab)(1+b)},\quad \frac{1}{ab-1}\sqrt{(1-ab)(a+1)}\right),$$
$$B_{1,2}(0,\pm1);\quad C_{1,2}(\pm1,0)\quad\text{and}\quad O(0,0).$$

By computing eigenvalues at each singular point we have that O, A_1, A_2, A_3, A_4 are centers while the other singular points B_1, B_2, C_1, C_2 are hyperbolic saddle points.

As we said above, the Hamiltonian of the system (3) is

$$H(x,y) = -(ax^4+by^4)+2abx^2y^2+2\left(ax^2+by^2\right)=h. \tag{12}$$

Hence $H(A_i)=\dfrac{a+b+2ab}{1-ab}$, $i=\overline{1,4}$, $H(C_k)=a$, $H(B_k)=b$, $k=1,2$ and $H(O)=0$.

Because $0<a<b$ we get that: $H(O)<H(B_1)<H(C_1)<H(A_1)$.

In polar coordinates, $x=r\cos\theta, y=r\sin\theta$, the system (3) becomes:

$$r'=-r^3p'(\theta)+rq'(\theta),\quad \theta'=-q(\theta)+r^2p(\theta) \tag{13}$$

and the Hamiltonian (12)

$$H(r,\theta)=-r^4p(\theta)+2r^2q(\theta)=h, \tag{14}$$

where

$$p(\theta)=a\cos^4\theta+b\sin^4\theta-2ab\cos^2\theta\sin^2\theta,\quad q(\theta)=a\cos^2\theta+b\sin^2\theta. \tag{15}$$

Remark. The equilibrium points A_1, A_2, A_3, A_4 lie on the lines $d_\pm : \theta = \pm\arctan\sqrt{\dfrac{a+1}{b+1}}$.

Theorem 3. *([2, 20])*

As h varies on the real line, the level curves defined by Hamiltonian (14) can be divided as follows:

1. $\Gamma_1^h : -\infty<h<0$, this corresponds to an orbit that surrounds all critical points, Fig. 1 a).

2. $\Gamma_2^h\cup\Gamma_1^h : 0<h<a$, this corresponds to an orbit $\left(\Gamma_2^h\right)$ that surrounds only the origin and a curve of type $\left(\Gamma_1^h\right)$, Fig. 1 b)-a).

3. $\Gamma_3^h : a<h<b$, this corresponds to two symmetric orbits that do not intersect the Ox axis but encircle the rest of critical points, Fig. 2 b).

If $h=a$ we get four heteroclinic orbits connecting two critical fixed points C_1 and C_2, Fig.2 a).

4. Γ_4^h : $b<h<\dfrac{a+b+2ab}{1-ba}$, this corresponds to four orbits that surround respectively the $A_i, i=\overline{1,4}$, equilibrium points, Fig.3 b). If $h=b$ we get four homoclinic orbits connecting two critical fixed points B_1 and B_2, Fig.3 a).

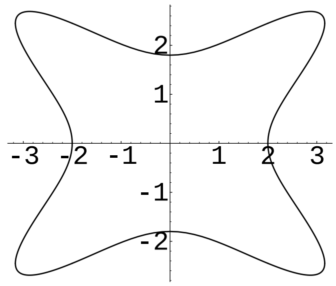

Figure 1. Orbit of type a) L_1 (left) b) L_2 and L_1 (right) .

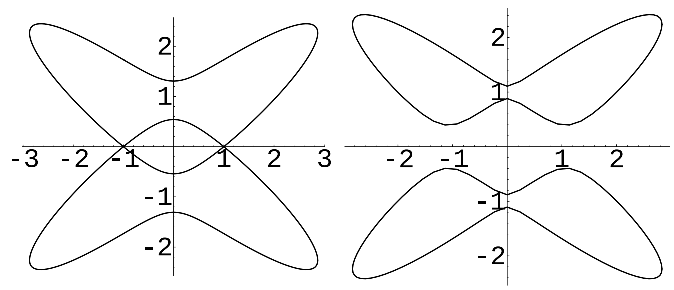

Figure 2. a) Four heteroclinic orbits connecting two critical points C_1 and C_2 (left) b) Two orbits of type L_3 (right) .

4 Numerical explorations

In this section for precisely chosen parameters a and b we numerically compute the detection curves. The four detection curves for a given h depend on the parameters u and v, (see Tables 1-4). Then for two given values of u and v, the detection curves can be plotted on the (h, λ)-plane, as can be seen in Figs.4, 5. By the Proposition 2 and the detection function graphs, we deduce the number and distribution of limit cycles. We consider here the case $n = 8$, that corresponds to perturbation of nine order.

From (14), we get

$$r_{1,2} = r_{\pm}^2(\theta, h) = \frac{1}{p(\theta)} \left(q(\theta) \pm \sqrt{q^2(\theta) - hp(\theta)} \right) \tag{16}$$

and from $\dot{\theta} = -1 + r^2 p(\theta) = 0$ we have:

$$\theta_1(h) = \frac{1}{2} \arccos \left[\left(-B + \sqrt{B^2 - AC} \right) / A \right],$$

$$\theta_2(h) = \frac{1}{2} \arccos \left[\left(-B - \sqrt{B^2 - AC} \right) / A \right]$$

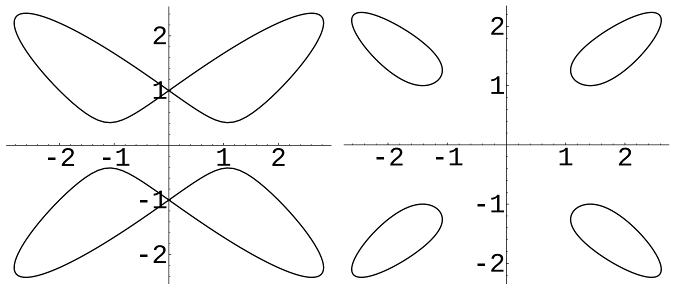

Figure 3. a) Four homoclinic orbits connecting the critical points B_1 and B_2 (left) b) Four orbits of type L_4 (right) .

where $A = a^2 - ah - 2ab\,(h+1) + b^2 - bh$, $B = a^2 - ah - b^2 + bh$, $C = a^2 - ah + 2ab\,(h+1) + b^2 - bh$.

From the form of the perturbation terms

$P\,(x, y, \varepsilon) = \varepsilon x((n+2)vy^n - c\frac{s+1}{r+1}x^r y^s - ux^2y^2 - \lambda)$, $Q\,(x, y, \varepsilon) = \varepsilon y((n+2)ux^n + cx^r y^s - ux^2y^2 - \lambda)$ we have $\frac{\partial^2 P\,(x,y,\varepsilon)}{\partial x \partial \varepsilon} + \frac{\partial^2 Q\,(x,y,\varepsilon)}{\partial y \partial \varepsilon} = u\,(n+2)\,x^n + v\,(n+2)\,y^n - 6ux^2y^2 - 2\lambda$.

Therefore, the four detection functions corresponding to the four closed curves Γ_j^h, $j = \overline{1,4}$, for the above perturbations are :

$$\lambda_j(h) = \frac{\int\limits_{\Gamma_j^h(D)} \left[(n+2)\,(ux^n + vy^n) - 6ux^2y^2\right] dxdy}{2 \int\limits_{\Gamma_j^h(D)} dxdy}, j = \overline{1,4} \tag{17}$$

In polar coordinates and for $a = \frac{1}{3}$, $b = 2$ and $n = 8$, (17) leads to:

$$\lambda_1(h) = \frac{\int\limits_0^{2\pi} \left(r_1^4\,(\theta,h)\,g\,(\theta) - r_1^3\,(\theta,h)\,g_1\,(\theta)\right) d\theta}{\int\limits_0^{2\pi} r_1\,(\theta,h)\,d\theta}, \; -\infty < h < 1/3,$$

$$\lambda_2(h) = \frac{\int\limits_0^{2\pi} \left(r_2^4\,(\theta,h)\,g\,(\theta) - r_2^3\,(\theta,h)\,g_1\,(\theta)\right) d\theta}{\int\limits_0^{2\pi} r_2\,(\theta,h)\,d\theta}, \; 0 < h < 1/3,$$

$$\lambda_3(h) = \frac{\int\limits_{\theta_2(h)}^{\pi-\theta_2(h)} \left[\left(r_1^4(\theta,h) - r_2^4(\theta,h)\right) g(\theta) - \left(r_1^3(\theta,h) - r_2^3(\theta,h)\right) g_1(\theta)\right] d\theta}{\int\limits_{\theta_1(h)}^{\pi-\theta_1(h)} \left(r_1(\theta,h) - r_2(\theta,h)\right) d\theta},$$

$$1/3 < h < 2,$$

$$\lambda_4(h) = \frac{\int\limits_{\theta_1(h)}^{\theta_2(h)} \left[\left(r_1^4(\theta,h) - r_2^4(\theta,h)\right) g(\theta) - \left(r_1^3(\theta,h) - r_2^3(\theta,h)\right) g_1(\theta)\right] d\theta}{\int\limits_{\theta_1(h)}^{\theta_2(h)} \left(r_1(\theta,h) - r_2(\theta,h)\right) d\theta},$$

$$2 < h < 11,$$

where $g(\theta) = u\cos^8\theta + v\sin^8\theta, g_1(\theta) = u\cos^2\theta\sin^2\theta$ and $r_{1,2}(\theta,h) = r_{\pm}^2(\theta,h)$.

Table 1
The values of the detection function $\lambda_1(h)$ when $a = 1/3,\ b = 2,\ n = 8$

h	$\lambda_1(h)$	h	$\lambda_1(h)$	h	$\lambda_1(h)$
-20	6.87u+1800.40v	-19	6.21u+1676.58v	-18	5.58u+1556.51v
-17	4.98u+1440.23v	-16	4.41u+1327.76v	-15	3.87u+1219.13v
-14	3.35u+1114.37v	-13	2.87u+1013.53v	-12	2.43u+916.64v
-11	2.01u+823.75v	-10	1.62u+734.91v	-9	1.27u+650.16v
-8	0.95u+569.56v	-7	0.66u+493.20v	-6	0.41u+421.13v
-5	0.19u+353.45v	-4	0.007u+290.27v	-3	-0.139u+231.71v
-2	-0.247u+177.93v	-1	-0.314u+129.12v	0.	-0.335u+85.66v
0.01	-0.335u+ 85.26v	0.04	-0.335u+ 84.05v	0.07	-0.335u+ 82.85v
0.1	-0.335u+ 81.65v	0.13	-0.334u+ 80.47v	0.16	-0.334u+ 79.29v
0.19	-0.334u+ 78.12v	0.22	-0.334u+ 76.96v	0.25	-0.333u+ 75.81v
0.31	-0.332u+ 73.56v	0.32	-0.332u+ 73.20v	0.33	-0.332u+72.85v

Table 2
The values of the detection function $\lambda_2(h)$ when $a = 1/3,\ b = 2,\ n = 8$

h	$\lambda_2(h)$	h	$\lambda_2(h)$
0.	0	0.033	$-3.32^{-6}u + 2.77^{-8}v$
0.066	$-0.0000129u + 4.49^{-7}v$	0.099	$-0.0000274u + 2.30^{-6}v$
0.132	$-0.0000432u + 7.37^{-6}v$	0.165	-0.0000530u +0.0000182v
0.198	-0.0000935u+ 0.0000305v	0.231	-0.0003127u+ 0.0000558v
0.264	-0.0003479u+ 0.0000938v	0.297	-0.0002976u+0.0001508v
0.33	0.0001670u+0.0002286v		

Table 3
The values of the detection function $\lambda_3(h)$ when $a = 1/3,\ b = 2,\ n = 8$

h	$\lambda_3(h)$	h	$\lambda_3(h)$	h	$\lambda_3(h)$
0.33	-0.348u+ 76.25v	0.43	-0.352u+ 73.63v	0.53	-0.353u+ 70.60v
0.63	-0.353u+ 67.42v	0.73	-0.352u+ 64.15v	0.83	-0.350u+ 60.84v
0.93	-0.347u+ 57.48v	1.03	-0.344u+ 54.10v	1.13	-0.340u+ 50.69v
1.23	-0.335u+ 47.25v	1.33	-0.329u+ 43.78v	1.43	-0.323u+ 40.27v
1.53	-0.315u+ 36.70v	1.63	-0.306u+ 33.03v	1.73	-0.296u+ 29.20v
1.83	-0.283u+ 25.07v	1.93	-0.269u+ 20.24v	1.98	-0.261u+ 17.021v

Table 4
The values of the detection function $\lambda_4(h)$ when $a = 1/3,\ b = 2,\ n = 8$

h	$\lambda_4(h)$	h	$\lambda_4(h)$	h	$\lambda_4(h)$
2	-0.259u+ 15.35v	2.1	-0.223u+ 9.40v	2.12	-0.213u+ 8.71v
2.14	-0.202u+ 8.10v	2.16	-0.191u+ 7.55v	2.2	-0.168u+ 6.61v
2.4	-0.055u+ 3.76v	2.6	0.044u+2.393v	2.8	0.126u+1.641v
3	0.193u+1.189v	3.2	0.246u+0.898v	3.6	0.320u+0.563v
4	0.365u+0.386v	4.4	0.391u+0.281v	4.8	0.403u+0.215v
5	0.405u+0.190v	5.2	0.405u+0.170v	5.4	0.404u+0.152v
5.6	0.401u+0.138v	5.8	0.397u+0.125v	6.	0.392u+0.114v
6.4	0.380u+ 0.096v	6.8	0.364u+0.082v	7.2	0.347u+ 0.071v
7.6	0.327u+0.062v	8.	0.307u+0.054v	8.4	0.285u+0.048v
8.8	0.263u+ 0.042v	9.2	0.240u+0.038v	9.6	0.216u+0.034v
10	0.192u+0.030v	10.4	0.168u+0.027v	10.8	0.143u+0.024v

From Tables 1–4 we have the discrete values of the four detection functions, which can now be plotted as shown in Figs. 4, 5. Using Proposition 2 and from these Figures (4, 5), one gets the following theorem:

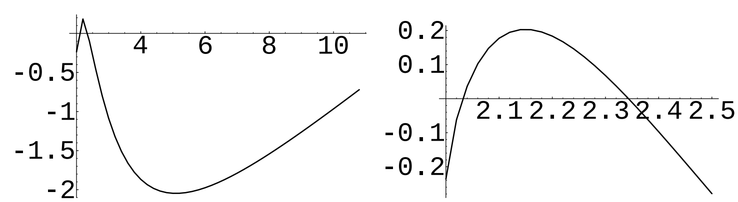

Figure 4. Detection curve λ_4 of system (11) for $a = 1/3,\ b = 2,\ n = 8,\ u = -5$ and $v = -0.1$.

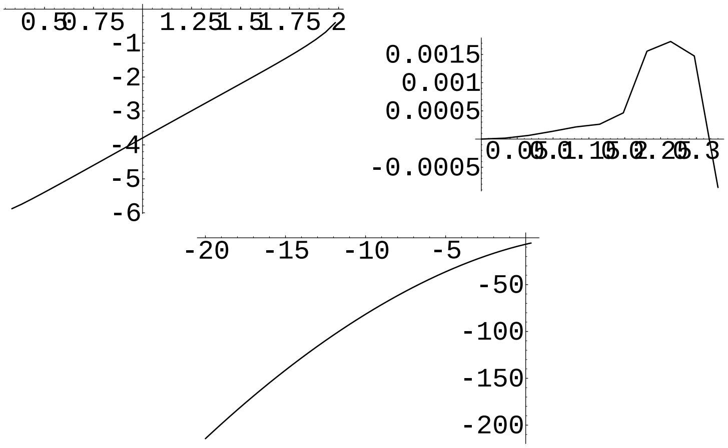

Figure 5. Detection curve λ_3 (left), λ_2 (middle), λ_1 (right) of system (11) for $a = 1/3,\ b = 2,\ n = 8,\ u = -5$ and $v = -0.1$.

Theorem 4. *For $a = 1/3,\ b = 2,\ n = 8,\ u = -5,\ v = -0.1$ and $0 < \varepsilon \ll 1$, we have the following distribution of limit cycles:*

a) If $\lambda < -5.88$, the system (11) has at least one limit cycle in the neighborhood of the orbit of type Γ_1^h, Fig.6a),

b) If $-5.88 < \lambda < -2.045$, the system (11) has at least two limit cycles in the neighborhood of each orbit of type Γ_3^h, Fig.6b),

c) If $-2.045 < \lambda < -0.72$, the system (11) has at least ten limit cycles, two of which in the neighborhood of each orbit of type Γ_4^h and one in the neighborhood of each orbit of type Γ_3^h, Fig.6c),

d) If $-0.72 < \lambda < -0.236$, the system (11) has at least six limit cycles, one of which in the neighborhood of each orbit of type Γ_4^h and Γ_3^h, Fig.7a),

e) If $-0.236 < \lambda < -0.000858$, the system (11) has at least eight limit cycles, two of which in the neighborhood of each orbit of type Γ_4^h, Fig.7b),

f) If $-0.000858 < \lambda < 0$, the system (11) has at least nine limit cycles, two of which in the neighborhood of each orbit of type Γ_4^h and one in the neighborhood of the orbit of type Γ_2^h, Fig.7c),

g) If $0 < \lambda < 0.00173$, the system (11) has at least ten limit cycles, two of which in the neighborhood of each orbit of type Γ_4^h and Γ_2^h, Fig.8, *h) If* $0.00173 < \lambda < 0.202$*, the system (11) has at least eight limit cycles, two of which in the neighborhood of each orbit of type* Γ_4^h*, Fig.7b),*

i) If $\lambda > 0.202$*, the system (11) has no limit cycles,*

where the last point of λ_1 *is* $(0.33, -5.88)$*, the first point of* λ_2 *is* $(0, 0)$*, the maximum of* λ_2 *is* $(0.264, 0.00173)$*, the last point of* λ_2 *is* $(0.33, -0.000858)$*, the first point of* λ_3 *is* $(0.33, -5.88)$*, the last point of* λ_3 *is* $(2, -0.236)$*, the first point of* λ_4 *is* $(2, -0.236)$*, the maximum of* λ_4 *is* $(2.14, 0.202)$*, the minimum of* λ_4 *is* $(5, -2.04)$*, and the last point computed of* λ_4 *is* $(10.8, -0.72)$*.*

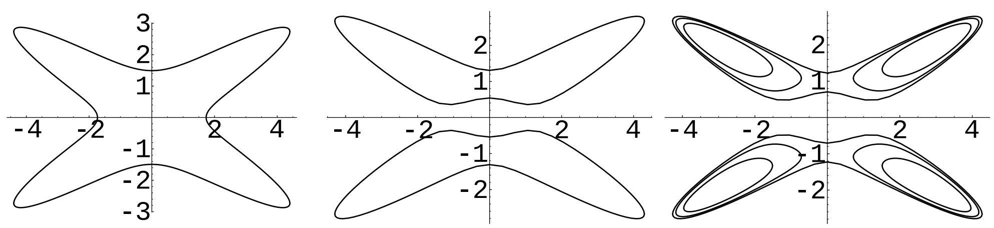

Figure 6. Distribution diagram corresponding to: a) one (left), b) two (middle) c) ten (right) limit cycles of system (11).

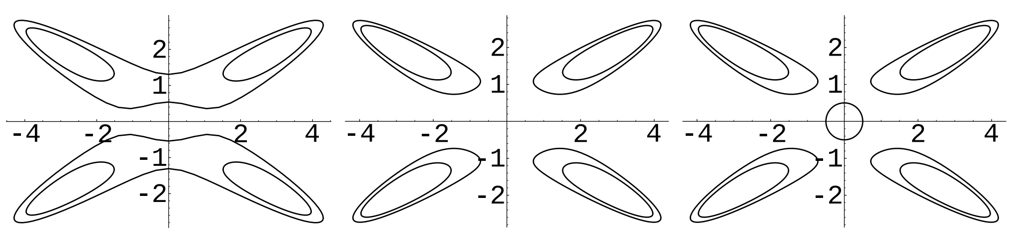

Figure 7. Distribution diagram corresponding to: a) six (left), b) eight (middle) c) nine (right) limit cycles of system (11).

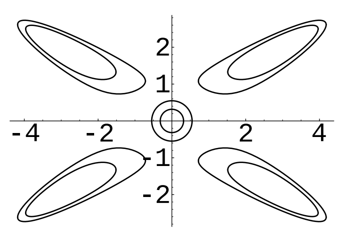

Figure 8. Distribution diagram corresponding to ten limit cycles of system (11).

5 Conclusion

In this paper we investigated a planar perturbed Hamiltonian system. The system possesses nine equilibria, five of type centers and four of type hyperbolic. The unperturbed system displays four different level curves, depending on the values of the parameter h on the real line. The Abelian integral method was employed to study the perturbed Hamiltonian system. By numerical explorations we illustrated the existence, number and distribution of limit cycles. In further papers, naturally we intend to deal with perturbations of higher order as $n = 10$ and $n = 12$.

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