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CONSTRUCTION OF TILINGS AND BEHAVIOUR OF GEODESICS

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Abstract: On a special interest are tilings in hyperbolic n-space. In this work the proposed construction could be considered as wel as constructive demonstration related to the theorem of existence of non-face-to-face tilings of hyperbolic n-dimensional space by equal, convex and compacte polytopes. The article also studies the upper bound for the number of faces of a n-dimensional hyperbolic tile.

Keywords:non-face-to-face anisohedral tilings, horocyclic strip, horosphere, hyperbolic n -space

JEL Classification: C60, F60, C22

Introduction

Let's address now to the anisohedral tilings and to the second question of the eighteenth problem of D.Hilbert (see [3], pp.51): is there a tiling of space by equal polytopes which cannot be transformed into the isohedral by permutation of the polytopes? For the case of Euclidean space (D.Hilbert, probably, thought about this case, although it was not mentioned) the answer to this question was given by Reinhard). Below, following [1] and [2], we will give examples of similar tilings (both face-to-face, and non-face-to-face) in *n*–dimensional hyperbolic space.

Construction and a proof of the existence of anisohedral and non- face-to-face tilings in high dimensional hyperbolic spaces Λ^n ($n \geq 2$)

Firstly we prove the assertion:

Theorem 3. *In the hyperbolic n–space Λ^n of dimension $n \geq 2$, there exists anisohedral and non-face-to-face tiling composed of congruent convex polyhedral tiles, which can't be transformed into tile-transitive tiling using any permutation of the polyhedral tiles.*

Proof. In 1975, K. Böröczky published some ingenious constructions of tilings in the hyperbolic plane Λ^2 [1]. Below we show how it is possible to receive a anisohedral not face-to-face tiling from that Böröczky face-to-face decomposition in hyperbolic *n*–space by congruent convex and compact polyhedra. Thus, we find another remarkable property of the Böröczky's tiling and, at the same time, we prove the theorem formulated above.

a) The case $n = 2$ dimension. Here we show the construction for two-dimensional hyperbolic plane (case) Λ^2 (see Figure 5). An analogous construction works for arbitrary dimension. We start from (one of) Böröczky tiling [1]. It is a non-crystallographic tiling of a hyperbolic plane by equal pentagons. We give an explicit construction of Böröczky's prototile and describe shortly the situation in the hyperbolic plane Λ^2. Figure 5 illustrates the construction of the Böröczky tiling in the upper half-plane model. Let *l* be a line in the two-dimensional hyperbolic plane Λ^2. Let $\sum_0^1$ be an horocycle with *l* as an axis; *l* is oriented and directed towards the concave side of the oricycle (Figure 5, see page 17). Let O_0 be the point of orthogonal intersection of the horocycle $\sum_0^1$ with the axis *l*. Draw some horocycle $\sum_0^1$ orthogonal to *l* ($\sum_0^1$ have a ideal point Ω at infinity), for which the selected line *l* is its axis. From the point of intersection O_0 (of axis *l* with horocycle $\sum_0^1$) we set aside segments equal length on the horocycle. For simplicity, the further construction can be carried out in one of the half-planes (the half-plane model) defined by the straight line, and then the received tiling of a half-plane by reflection in the line *l* to move the second half-plane, defined by the straight line *l*. Suppose that $O_0, A_0, B_0, C_0, D_0, \ldots$ - the partition points of horocycle. Draw through them the straight line $l_{A_0}, l_{B_0}, l_{C_0}, l_{D_0}, \ldots,$ parallel *l* (in the chosen direction). Through the same point draw and equidistantes $h_{A_0}, h_{B_0}, h_{C_0}, h_{D_0}, \ldots$ with base *l*. On the line l_{B_0}, denote the point of intersection A_1 of this axis with equidistant h_{A_0}. Through the receved point A_1 pass a horocycle $\sum_1^1$, which has the some *l*-axis at tue initial $\sum_0^1$ horocycle (to say with the same *l*–axis and a common ideal point Ω at infinity). Let O_1 be the point of intersection of the horocycle $\sum_1^1$ with the axis *l*. Obviously, the length of the horocyclic (horocycal) segment $O_1 A_1$ is equal to the length of the horocyclic segment $O_0 A_0$. The set of division points of horocycle $\sum_0^1$ determines the regular (infinite), a broken line ω_0, inscribed in this horocycle. Obviously, the horocyclic translation *t* along line *l* by the vector $\overrightarrow{O_0 O_1}$ moves horocycle $\sum_0^1$ in horocycle $\sum_1^1$, point $A_0 \in \sum_0^1$ to point $A_1 \in \sum_1^1$, the segment $O_0 A_0$ in segment $O_1 A_1$, all partition of horocycle $\sum_0^1$ moving in partition of horocycle $\sum_1^1$ (as inscribed in his regular a broken line ω_0 passes in the regular broken line ω_1, inscribed in $\sum_1^1$). Thus over everyone segment of the regular broken line ω_1, inscribed in "the lower" horocycle $\sum_1^1$, it turns out (according to the construction) two congruent adjacent segments of the regular broken line ω_0, inscribed in the "upper" horocycle $\sum_0^1$ (the bunch of parallels with the centre Ω carries out the corresponding projection).

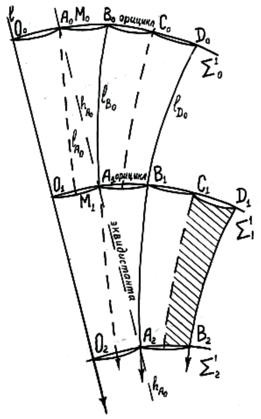

Fig. 5.

Constructing the Böröczky anisohedral not face-to-face tiling in the upper half-plane of the hyperbolic plane Λ^2

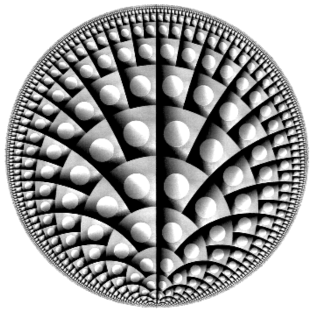

Fig. 6.

Böröczky's tiling in Poincare's interpretation II

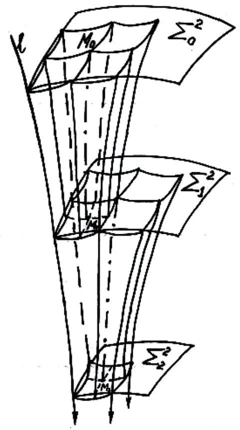

Fig. 7.

Constructing the Böröczky anisohedral

non- face-to-face tiling in the upper half-space model of the hyperbolic 3-space Λ^3

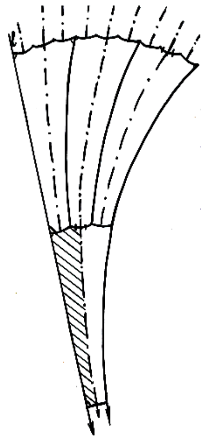

Fig. 8.

Possible variations of tilings

Segments $O_1 A_1$, $O_0 A_0$ and $A_0 B_0$ together with axes l and l_{B_0} limit some convex pentagon, whose axis l_{A_0} is the axis of symmetry. She is straight line, projecting a common vertex A_0 of segments $O_0 A_0$ and $A_0 B_0$ in the middle M_1 of "basis" $O_1 A_1$.

Let a horocyclic turn w (about the ideal point Ω), defined by axes l and l_{B_0}. Then each of the movements by the group $\Gamma_1 =< w >$ will moves "strip region" (a horocyclic strip) between horocycle $\sum_0^1$ and $\sum_1^1$ in itself. The polygon $O_0 A_0 B_0 A_1 O_1$ which maps ("multiplied") by the group Γ_1, will divide this strip into convex equal irregular pentagons. For brevity further we will say, that we have map (multiplied) a considered polygon (further–a polyhedron) of symmetry group of tiling by horocycle $\sum_1^1$ (hereinafter – the group of tiling corresponding horosphere). Repeating so the divided horocyclic strip between horocycles mouvements of the group $\Gamma_2 =< t >$, we obtain the anisohedral and face-to-face tiling of all plane Λ^2 on such (convex, equal) pentagons-Böröczky's tiling of the hyperbolic plane Λ^2**.** Generally speaking, for the purposes of the theory of packing, (for which tiling was under construction) to connect partition points of horocycles by segments (to construct an infinite horocycal regular polygon) absolutely is not necessarily: it is enough to partition plane into the pentagons, limited by horocyclic segments and lengths segments of their axes (thus two segments of the upper base actually are continuations of one another). It is this tiling (conducted in the Poincare's interpretations) which usually also is showed to the reader (see Fig. 6). On Fig. 6 in each (curvilinear) pentagon it is placed by one circle. It would seem that the packing density of circles is possible to be characterised by the ratio of area of a circle to the area of a curvilinear pentagon. But if we replace a curvilinear pentagon with the rectilinear, for the same reasons as density of the same packing of circles, it can take the ratio of the area of a circle to the area of rectilinear pentagon. But the latter is less than the area of a curvilinear pentagon on the area of horocycal segment, as it shows the incorrectness of such definition of packing density.

Now (attention!) each of pentagons of this tiling we will cut its axis of symmetry into two congruent quadrangles (for example a pentagon $O_0 A_0 B_0 A_1 O_1$ by a segment $A_0 M_1$ will be divided into two quadrangles: $O_0 A_0 M_1 O_1$ and $A_0 B_0 A_1 M_1$). If we make such cuts in all pentagons of Böröczky's tiling in hyperbolic plane Λ^2 we will see that on "upper" "base" of (each) quadrangle, two quadrangles of the "upper" layer have been arranged with their "lower" bases (in Figure 5 one such quadrangle is shaded). Thus we have an not face-to-face tiling of hyperbolic plane Λ^2 into equal (convex) quadrangles.

b) The case $n = 3$ **dimension.**

Theorem 4. *In the hyperbolic 3–space* Λ^3*, there exists a anisohedral not face-to-face tiling composed of congruent convex polyhedral tiles, which can't be transformed into isohedral tiling using any permutation of the polyhedral tiles.*

Proof makes use of the so-called Böröczky tiling of the hyperbolic 3-space Λ^3 by congruent polyhedra and, for the reader's convience, we start with a short description of this tiling. Figure 7 illustrates the construction of the Böröczky tiling in the upper half-space model. Suppose the hyperbolic 3-space Λ^3 has curvature -1 and let $\sum_0^2$ be an horosphere. It is well-known that $\sum_0^2$ is isometric to the Euclidean plane. In three-dimensional hyperbolic space Λ^3 the tiling has an analogous construction (see Figure 7). If very briefly the description of the face-to-face Böröczky tiling can be explaned as follows.

In plain words, the construction goes as follows. Let us describe it for hyperbolic 3-space Λ^3. Consider a collection of concentric horospheres, where consecutive horospheres have equal distance. Each horosphere is conformal to the Euclidean plane R^2. So consider a partition of each horosphere into the canonical (refering to the standart unit square - at the heart of the construction) tiling of R^2 by geodesic unit squares. Erect on each geodesic square a prism, such that the top of the prism is made of four geodesic squares of the next layer. This yields a tiling of hyperbolic 3-space Λ^3, where each tile carries four tiles (geodesic squares) on its top. These "polyhedral layers" fit together and produce the Böröczky tiling of the whole hyperbolic three-dimensional space anisohedrally and face-to-face.

A more detailed description looks like. Let l be an oriented line in 3-space Λ^3 through O_0. Choose a horosphere $\sum_0^2$ orthogonal to l, intersecting the axis l at a point O_0. A horosphere is partitioned by an orthogonal net of horocycles into geodesic squares (edge-to-edge tiling of $\sum_0^2$ with pairwise equal geodesic squares), so that the point $O_0 = \sum_0^2 \cap l$ is a vertex of this partition (with an edge $O_0 A_0$) (Figure 7). Let -2-dimensional cube (square) in $\sum_0^2$, centered at $\sum_0^2 \cap l$. So divided horosphere into squares we subject to the translation t along an axis l on a vector, whose length is equal to length (previously constructed) segment $O_0 O_1$. It is easy to see that the lenth of the translation vector $O_0 O_1$ for t is identical to that of the analogous translation in the two dimensional case. Let $\sum_1^2$ be another horosphere, such that $\sum_1^2$ is concentric with $\sum_0^2$. It is obvious enough that in one square of tiling on horosphere $\sum_1^2 = t\left(\sum_0^2\right)$ will project a star of the vertex M_0 of the partition $\sum_0^2$, consisting of four squares (see Figure 7, $O_0 M_0$ – diagonal a horospheric square on $\sum_0^2$). Thus the horosphere axis, passing through the node M_0 of the partition $\sum_0^2$, will pass through the center M_1 of a square of partition in of horosphere $\sum_1^2$. Taking the convex hull of the vertices considered squares, we obtain a a cell of the Böröczky tiling- polyhedron with 13 vertices. In other words, polyhedron with *9* facets (a *9*–face polyhedra), limited by $2^2 = 4$ "upper" facets (by four Euclidean squares in $\sum_1^2$), one "lower" facet (a Euclidean square in $\sum_0^1$), and 4 aside facets (four pentagons, congruents to pentagons of two – dimensional Böröczky's tiling of hyperbolic plane Λ^2 **).** By construction, the Böröczky prototile (polyhedron, a cell) has 2^{n-1} +2* $(n-1)$ +1= $2^2 + 2 \cdot 2 + 1 = 9$ facets. One "lower" facet (a Euclidean square in $\sum_0^1$), $2^2 = 4$ "upper" facets (by four Euclidean squares in $\sum_1^2$), and 4 aside facets. The 4 aside facets are Böröczky prototiles of dimension two. The axis $M_0 M_1$ is the axis of symmetry of this *9*-face polyhedron. Four planes of symmetry of it *9*-face polyhedron pass through this axis, two of which ("coordinate") also pass through the edges of the squares (incident to the node M_0 of edges), inscribed in the "upper" horosphere $\sum_0^2$. These two planes cut *9*-face polyhedron into four equal convex "prismatic" hexahedron. To obtain a Böröczky's tiling for *3*-space Λ^3, it is enough at first to repeat *9*-face polyhedron by symmetry group Γ_1^2 of partitions horosphere $\sum_1^2$ (and by that to obtain a "polyhedral layer" of *9*-face polytopes, laying between horospheres $\sum_0^2$ and $\sum_1^2$), then the received layer to multiply by the group $\Gamma_2 =< t >$. Get a sequence of polyhedra whose 9 "upper" vertices lie on the horosphere $\sum_0^2$ (and belong to the set of vertices of the tiling $\sum_0^2$) and 4 "bottom" vertices lie on the horosphere $\sum_1^2$ (and belong to the set of vertices of the tiling $\sum_1^2$). As a result we get a tiling of a "polyhedral layer" with vertices on $\sum_0^2$ and $\sum_1^2$. These "polyhedral layer" fit together and produce the Böröczky tiling of the whole hyperbolic 3-space. Specifically, for $n \geq 2$, there exist a Böröczky polyhedron $\mathrm{P_n}$ with $(n^2 + 5)$ facets, which tiles hyperbolic 3-space Λ^3.

To get the corresponding non-face-to-face tiling of the hyperbolic *3*-space Λ^3 into convex equal "prismatic" hexahedron it is suffices to cut each *9*-face polyhedron of the Böröczky's tiling into four prismatic polyhedra by the above- mentioned "coordinates" planes of symmetry. The tilings (face-to-face and non-face-to-face) of *n*–dimensional hyperbolic space are under construction almost literally in the same way through partition of corresponding $(n-1)$–horospheres into geodesic $(n-1)$ –cube (cubiliaj).

c) The case *n = 4* dimension.

Let us give the construction of *4*–dimensional prototile. Let us describe it for hyperbolic *4*-space

Λ^4. If in short. Consider a collection of concentric *3*–horospheres, where consecutive *3*–horospheres have equal distance. Each *3*–horosphere is conformal to the Euclidean *3*-space E^3. So consider a partition of each *3*–horosphere into the canonical (refering to the standart unit cube - at the heart of the construction) tiling of *3*-space E^3 by geodesic unit *3*–cubes. Erect on each geodesic *3*–cube a prism, such that the top of the prism is made of eight geodesic *3*–cubes of the next layer. This yields a tiling of hyperbolic *4*-space Λ^4, where each tile carries eight tiles (geodesic *3*–cubes) on its top. These "polyhedral layers" fit together and produce the Böröczky tiling of the whole hyperbolic four-dimensional space anisohedrally and face-to-face.

A more detailed description looks like. In $n = 4$ the similar construction leads to, that the star of cubes in three-dimensional partition of *3*–horosphere $\sum_0^3$ into *3*–cubes, with a common vertex M_0, will be projected in one cube of horosphere $\sum_1^3 = t\left(\sum_0^3\right)$ and thus the axis $M_0 M_1$ will pass through the centre of symmetry M_1 of the cube of the "lower" base. Taking the convex hull of the vertices of the star in "upper" tiling and corresponding to its cube of the "lower" tiling ("base"), we obtain a polyhedron with nine cubic facets (one in the "base" -"lower" facet (a one Euclidean *3*–cube in $\sum_0^3$) and the eight – over it-"upper" facets (by eight Euclidean *3*–cube in $\sum_1^3$), and six aside facets, congruent to facets of three-dimensional Böröczky's tiling of hyperbolic *3*-space Λ^3. By construction, the Böröczky prototile (polyhedron) has $2^{n-1}+2*(n-1)+1=$ 8+6+1=13 facets. One "lower" facet (a Euclidean *3*–cube in $\sum_0^3$, 8 "upper" facets (by eight Euclidean *3*–cubes in $\sum_1^3$), and 6 aside facets.

The hyperplanes of symmetry of this polyhedron, defined by the axis $M_0 M_1$ and incidental of M_0 two–dimensional faces of "upper" cubes ("coordinate" the hyperplanes), cut this four–dimensional polytope into eight "prismatic" polytopes. Acting with symmetry group Γ_1^3 on tiling a horosphere $\sum_1^3$ up into cube, we "fill" a layer between horospheres $\sum_0^3$ and $\sum_1^3$. Acting with the group $\Gamma_2 =< t >$, we "multiply" the constructed layer and we receive a Böröczky's tiling for *4*-space Λ^4. Cutting each polyhedron of this tiling as indicated above ("coordinate" planes of symmetry) into the "prismatic" polyhedra, we receive not face-to-face tiling of hyperbolic *4*-space Λ^4 into equal prismatic convex polyhedra: on the cubic "upper" base of the "lower" horospheric layer there are (are arranged) eight "lower" bases of the "upper" horospheric layer of prisms.

d) In n **dimensions.** The construction in the general (*n*–dimensional) case should now be clear. Throughout this section, we will make strong use of the fact that any $(n-1)$–dimensional horosphere $\sum^{n-1}$ in hyperbolic *n*-space Λ^n is isometric to the Euclidean space E^{n-1}, see for instance . Let us give the construction of *n*–dimensional prototile. Let us describe it for hyperbolic *n*-space Λ^n. If in short. Consider a collection of concentric $(n-1)$–dimensional horospheres, where consecutive $(n-1)$–horospheres have equal distance. Each $(n-1)$–horosphere is conformal to the Euclidean $(n-1)$-dimensional space. So consider a partition of each $(n-1)$–horosphere into the canonical (refering to the standart unit $(n-1)$–cube - at the heart of the construction) tiling of Euclidean $(n-1)$-dimensional space by geodesic unit $(n-1)$–cubes. Erect on each geodesic $(n-1)$–

cube a prism, such that the top of the prism is made of 2^{n-1} geodesic $(n-1)$–cubes of the next layer. This yields a tiling of hyperbolic *n*-space Λ^n, where each tile carries 2^{n-1} tiles (geodesic $(n-1)$–cubes) on its top. These "polyhedral layers" fit together and produce the Böröczky tiling of the whole hyperbolic *n*-dimensional space Λ^n anisohedrally and face-to-face.

A more detailed description looks like. Choose an oriented line l and an $(n-1)$–dimensional horosphere $\sum_0^{n-1}$ orthogonal to it. Partition $\sum_0^{n-1}$ into geodesic $(n-1)$–cubes, corresponding to its translation t (along the axis l of horosphere $\sum_0^{n-1}$) partition of a horosphere $\sum_1^{n-1} = t\left(\sum_0^{n-1}\right)$ into geodesic $(n-1)$–cubes and it is noticed, that the star of node M_0 of cubes on horosphere $\sum_0^{n-1}$ are projected into one cube of the horosphere $\sum_1^{n-1}$. Next, we construct the convex hull of vertices of the "upper" star and the "lower" cube. By construction, the Böröczky prototile (polyhedron) has 2^{n-1} +2*$(n-1)$+1 facets. One "lower" facet (a Euclidean $(n-1)$–cube in $\sum_0^{n-1}$, 2^{n-1} "upper" facets (by 2^{n-1} Euclidean $(n-1)$–cubes in $\sum_1^{n-1}$), and 2* $(n-1)$ aside facets. The received polyhedron "multiplys" by means of symmetry group Γ_1^{n-1} of tiling of horosphere $\sum_1^{n-1}$ into cubes. So it is formed horospherical layer from such polyhedra between horospheres $\sum_0^{n-1}$ and $\sum_1^{n-1}$. This horospherical layer "multiplys" by group $\Gamma_2 =< t >$, that leads to the Böröczky's tiling of hyperbolic *n*-space Λ^n. Finally, each polytope of this tiling is cut by ("coordinate") hyperplanes of symmetry (spanned on axis $M_0 M_1$ and incident to the vertex M_0 $(n-2)$–dimensional faces of the star of the "upper" partition) into the 2^{n-1}-"prismatic" polytopes, which form the desired not face-to-face tiling of hyperbolic n–space Λ^n: on the "roof" of one prismatic polytope of the "lower" horospherical layer there are arranged 2^{n-1}"bases" of " prismatic polytopes of the "upper" horospherical layer. Using the ideas applied in the study to (face-to-face) Böröczky's tiling in [2], it is easy to see that the constructed not face-to-face tilings of *n*–dimensional hyperbolic space by equal, finite and convex polytopes is anisohedral and also cannot be transformed into isohedral (tile-transitive, regular) tilings using polytopes permutation of the tiling as well.

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