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Coloring hyperplanes of CAT(0) cube complexes

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Abstract

In these notes, we briefly describe the results which we will present in the talk *Coloring hyperplanes of CAT(0) cube complexes.* They are based on the papers [4] and [5].

Keywords: geometry, combinatorics, CAT(0) cube complexes, isometric embedding, hyperplanes.

1 CAT(0) cube complexes

In his seminal paper [7], among many other results, Gromov gave a nice combinatorial characterization of *CAT(0) cube complexes* as simply connected cube complexes in which the links of 0-cubes are *simplicial flag complexes.* Subsequently, Sageev [9] introduced and investigated the concept of *hyperplanes* of CAT(0) cube complexes, showing in particular that each hyperplane is itself a CAT(0) cube complex and divides the complex into two CAT(0) cube complexes. These two results identify CAT(0) cube complexes as the basic objects in geometric group theory. For instance, many well-known classes of groups are known to act nicely on CAT(0) cube complexes. On the other hand, [3] established that the 1-skeleta of CAT(0) cube complexes are exactly the median graphs, i.e. the graphs in which any triplet of vertices admit a unique median vertex. Median graphs and related median structures have been investigated in several contexts by quite a number of authors for more than half a century. They have many nice properties and admit numerous characterizations relating them to other discrete structures. Barthélemy and Constantin [1] showed that pointed median graphs are exactly the domains of event structures with binary conflict (investigated in computer science in concurrency theory [10, 8]).

All CAT(0) cube complexes $\mathbf{X}$ and median graphs – the 1-skeleta $G(\mathbf{X})$ of $\mathbf{X}$ – are intimately related to hypercubes: they are constituted of cubes and themselves embed isometrically into hypercubes. The minimum dimension of a hypercube into which $G(\mathbf{X})$ (or $\mathbf{X}$) isometrically embeds equals the number of hyperplanes of $\mathbf{X}$, or, equivalently, the number of equivalence classes of the transitive closure of the "opposite" relation of edges of $G(\mathbf{X})$ on 2-cubes of $\mathbf{X}$. While the dimension of the smallest hypercube into which the median graph $G(\mathbf{X})$ embeds is easy to determine, the problem of determining the least number $\tau(\mathbf{X}) = \tau(G(\mathbf{X}))$ of tree factors necessary for an isometric embedding of the 1-skeleton of $\mathbf{X}$ into a cartesian product of trees is hard. For arbitrary CAT(0) cube complexes $\mathbf{X}$, the value $\tau(\mathbf{X})$ is closely related to the chromatic number of the so-called *crossing graph* $\Gamma_{\#}(\mathbf{X})$ of $\mathbf{X}$. $\Gamma_{\#}(\mathbf{X})$ can be viewed as the intersection graph of the hyperplanes of $\mathbf{X}$: its vertices are the hyperplanes of $\mathbf{X}$ sensu [9] and two hyperplanes are adjacent in $\Gamma_{\#}(\mathbf{X})$ iff they cross (or, equivalently, they intersect).

Extending the fact that $\tau(\kappa(G)) = \chi(G)$, it can be shown that the equality $\tau(\mathbf{X}) = \chi(\Gamma_{\#}(\mathbf{X}))$ holds for all CAT(0) cube complexes $\mathbf{X}$. Since an arbitrary graph can be realized as the crossing graph of a CAT(0) cube complex $\mathbf{X}$, to better capture the structure of $\mathbf{X}$, the concept of the *contact graph* $\Gamma(\mathbf{X})$ of $\mathbf{X}$ introduced in [6] is useful: the vertices of $\Gamma(\mathbf{X})$ are the hyperplanes of $\mathbf{X}$ and two hyperplanes are adjacent in $\Gamma(\mathbf{X})$ iff they cross or osculate (i.e., their carriers touch each other). $\Gamma(\mathbf{X})$ can be also viewed as the intersection graph of the carriers of the hyperplanes of $\mathbf{X}$. The clique number $\omega(\Gamma(\mathbf{X}))$ of the contact graph of $\mathbf{X}$ is exactly the maximum degree in $G(\mathbf{X})$ of a 0-cube of $\mathbf{X}$, i.e., to the maximum number of 1-cubes incident to a 0-cube of $\mathbf{X}$. The contact graph $\Gamma(\mathbf{X})$ always contains the crossing graph $\Gamma_{\#}(\mathbf{X})$. $\Gamma(\mathbf{X})$ also hosts the *pointed contact graph* $\Gamma_\alpha(\mathbf{X})$ of the 1-skeleton $G_\alpha(\mathbf{X})$ of $\mathbf{X}$ pointed at arbitrary vertex α. The graph $\Gamma_\alpha(\mathbf{X})$ has hyperplanes of $\mathbf{X}$ as vertices and two hyperplanes H, H' are adjacent in $\Gamma_\alpha(\mathbf{X})$ if and only if they are adjacent in $\Gamma(\mathbf{X})$ and two incident 1-cubes, one crossed by H and another crossed by H', are directed away from the common origin.

Pairwise-independently, F. Haglund, G. Niblo, M. Sageev, and the first author of these notes asked the following question:

Question 1. *Is it true that all CAT(0) cube complexes* $\mathbf{X}$ *with uniformly bounded degrees can be isometrically embedded into a finite number of trees?*

2 Event structures

Question 1 is closely related with the conjecture of Rozoy and Thiagarajan [8] (also called the *nice labeling problem*) asserting that:

Question 2. *Any event structure with finite (out)degree admits a labeling with a finite number of labels.*

An *event structure* is a triple $\mathcal{E} = (E, \leq, \smile)$, where E is a set of *events*, $\leq$ is a partial order on E, called *causal dependency*, and $\smile$ is a symmetric, irreflexive binary relation on E called *conflict*. For all e, e', e'', if $e \smile e'$ and $e' \leq e''$, then $e \smile e''$. The events e and e' are *concurrent* if they are incomparable in the partial ordering $\leq$ and $e \not\smile e'$. The events e and e' are *independent* if they are either concurrent or in minimal conflict. An *independent set* is a set of pairwise independent events in E. The *degree* of E is the maximum cardinality of an independent set in E. In [8], Rozoy and Thiagarajan formulated the *nice labeling problem* for event structures (Question 2). A *labeling* is a map $\lambda : E \to \Lambda$, where Λ is some alphabet, and λ is a *nice labeling* if $\lambda(e) \neq \lambda(e')$ whenever e and e' are independent. Solving the nice labeling problem for $\mathcal{E}$ entails constructing a nice labeling λ such that Λ is finite. The *domain* $\mathcal{D}(\mathcal{E})$ of the event structure $\mathcal{E}$ is defined as follows. A *configuration* C is a subset $C \subseteq E$ of the set of events such that no two elements of C are in conflict, and, if $e \leq e' \in C$ are not in conflict, then $e \in C$. The domain $\mathcal{D}(\mathcal{E})$ is the set of all such configurations C, ordered by inclusion. This construction naturally gives rise to a median graph and an accompanying CAT(0) cube complex associated to $\mathcal{E}$. Indeed, let

$G = G(\mathcal{E})$ be the graph whose vertices are the elements of the domain $\mathcal{D}(\mathcal{C})$, with C and C' joined by an edge if and only if $C = C' \cup \{e\}$ for some $e \in E - C$. In this situation, the edge $C'C$ is directed from C' to C. In other words, an event $e \in E$ is viewed as a minimal change from one configuration to another [10].

As noted above, pointed median graphs are exactly the domains of event structures [1]. Then, in view of the bijection between median graphs and 1-skeleta of CAT(0) cube complexes, the nice labeling problem for such event structures can be equivalently viewed as the colouring problem of the pointed contact graph $\Gamma_\alpha(\mathbf{X})$ of the CAT(0) cube complex $\mathbf{X}$ associated to the domain of the event structure. Since $\chi(\Gamma_\alpha(\mathbf{X})) \leq \chi(\Gamma(\mathbf{X}))$ and $\chi(\Gamma_\#(\mathbf{X})) \leq \chi(\Gamma(\mathbf{X}))$, in relation with Questions 1 and 2, the following question is natural:

Question 3. *Is it true that the chromatic number $\chi(\Gamma(\mathbf{X}))$ of the contact graph of a CAT(0) cube complex $\mathbf{X}$ of degree Δ can be bounded by a function ϵ of Δ?*

3 Results

Since $\omega(\Gamma(\mathbf{X})) = \Delta$ and $\Gamma_\#(\mathbf{X}), \Gamma_\alpha(\mathbf{X})$ are subgraphs of $\Gamma(\mathbf{X})$, all three questions can be reformulated, namely: which of the classes of graphs $\Gamma_\#(\mathbf{X}), \Gamma_\alpha(\mathbf{X})$, and $\Gamma(\mathbf{X})$ are χ-bounded? A class $\mathcal{C}$ of graphs is called χ-*bounded* if there exists a function f such that $\chi(G) \leq f(\omega(G))$ for any graph G of $\mathcal{C}$. Via a series of nontrivial examples, Burling [2] showed that the class of intersection graphs of axis-parallel boxes of $\mathbb{R}^3$ is not χ-bounded. Based on Burling's examples, it was recently shown in [4] that for CAT(0) cube complexes the classes of graphs $\Gamma(\mathbf{X})$ and $\Gamma_\alpha(\mathbf{X})$ are not χ-bounded, thus disproving the nice labeling conjecture of [8]:

Theorem 1. *[4] There exists a pointed median graph $\widetilde{G}^*_\alpha$ of maximum out-degree 5 such that the chromatic number of its pointed contact graph $\Gamma(\widetilde{G}^*_\alpha)$ is infinite. In particular, any nice labeling of the event structure $\mathcal{E}_\alpha$ (of degree 5) whose domain is $\widetilde{G}^*_\alpha$, requires an infinite number of labels.*

We adapted this counterexample by using the recubulation technique from [6] to show that the class of crossing graphs $\Gamma_{\#}(\mathbf{X})$ of CAT(0) cube complexes is also not χ-bounded, thus answering in the negative the first open question:

Theorem 2. *[5] For any $n > 0$, there exists a CAT(0) cube complex $\mathbf{X}_n$ with constant maximum degree such that any colouring of the crossing graph of $\mathbf{X}_n$ requires more than n colours, i.e., any isometric embedding of $\mathbf{X}_n$ into a Cartesian product of trees requires $> n$ trees. There exists an infinite CAT(0) cube complex $\mathbf{X}$ with constant maximum degree which cannot be isometrically embedded into a Cartesian product of a finite number of trees, i.e., the chromatic number of its crossing graph is infinite.*

On the other hand, and this is the main contribution of [5], we show that in the case of 2-dimensional CAT(0) cube complexes $\mathbf{X}$ the contact graphs $\Gamma(\mathbf{X})$ (and therefore the crossing and the pointed contact graphs) are χ-bounded by a polynomial function in $\omega(\Gamma(\mathbf{X})) = \Delta$, thus showing that in the 2-dimensional case the three questions have positive answers; this is the content of our main result:

Theorem 3. *[5] Let $\mathbf{X}$ be a 2-dimensional CAT(0) cube complex such that the degrees of all its vertices are bounded by Δ. Then there exists $M < \infty$, independent of $\mathbf{X}$, such that $\chi(\Gamma(\mathbf{X})) \leq \epsilon(\Delta) = M\Delta^{26}$. In particular, $\tau(\mathbf{X}) \leq \epsilon(\Delta)$, i.e. the 1-skeleton of $\mathbf{X}$ isometrically embeds into the Cartesian product of at most $\epsilon(\Delta)$ trees. Finally, all event structures of (out)degree Δ_0, whose domains are 2-dimensional, admit a nice labeling with at most $\epsilon(\Delta_0 + 2)$ labels.*

We actually obtain the following bound: $\chi(\Gamma(\mathbf{X})) \leq \epsilon(\Delta) = 1165226\Delta^{26}$, or, simply $M = 1165226$.

Idea of the proof: To show that the chromatic number $\chi(\Gamma(\mathbf{X}))$ of the contact graph $\Gamma(\mathbf{X})$ is polynomially bounded in Δ, we show that the edges of $\Gamma(\mathbf{X})$ can be distributed over six spanning subgraphs of $\Gamma(\mathbf{X})$, such that the chromatic numbers of each of these subgraphs can be polynomially bounded. As a result, each vertex of $\Gamma(\mathbf{X})$ (hyperplane of $\mathbf{X}$) receives a sextuple of colours, each colour corresponding to the colour received by this vertex in the colouring of the corresponding subgraph. Since each edge of $\Gamma(\mathbf{X})$ is present in at least one spanning subgraph, the sextuple-colouring of the hyperplanes of $\mathbf{X}$ is a correct colouring of the contact graph $\Gamma(\mathbf{X})$. The number of colours is the product of the six numbers of colours used to colour the spanning subgraphs, whence it is polynomial in Δ. In Sections 4-6, one after another, we will define and colour the six spanning subgraphs. For this, we will study the geometrical and the combinatorial properties of contact graphs of 2-dimensional CAT(0) cube complexes.

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