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Minimal ideals of Abel-Grassmann's groupoids

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Abstract. We study minimal (0-minimal) ideals, simple (0-simple) Abel-Grassmann's groupoids and zeroids of an Abel-Grassmann's groupoid S. We consider S containing a minimal ideal which is the union of all minimal left ideals of S. The completely simple Abel-Grassmann's groupoid which is equal to the union of all its nonzero minimal left ideals is investigated. In addition, we discuss a universally minimal left ideal of S which is a right ideal and is the kernel of S. Finally, we prove that S contains a left zeroid if and only if it contains a universally minimal left ideal.

1. Introduction and preliminaries

The concept of an *Abel-Grassmann's groupoid* (abbreviated as *AG-groupoid*) was first introduced by M. A. Kazim and M. Naseeruddin in 1972 which they called a *left almost semigroup* [7]. P. Holgate [6] called the same structure a *left invertive groupoid*. P. V. Protić and N. Stevanović later called such a groupoid an *Abel-Grassmann's groupoid* [12]. An *AG-groupoid* is in fact a groupoid S satisfying the *left invertive law* $(ab)c = (cb)a$. The left invertive law can be stated by introducing the braces on the left of ternary commutative law $abc = cba$. An AG-groupoid satisfies the *medial law* $(ab)(cd) = (ac)(bd)$. Since AG-groupoids satisfy the medial law, they belong to the class of entropic groupoids. If an AG-groupoid S contains a left identity, then it satisfies the *paramedial law* $(ab)(cd) = (dc)(ba)$ and the identity $a(bc) = b(ac)$ [11]. An AG-groupoid is an algebraic structure which is midway between a groupoid and a commutative semigroup. Consequently, an AG-groupoid has many properties similar to to the properties of semigroups (cf. for example [3], [4] and [5]), but AG-groupoids (also AG-groupoids with a left identity) are non-associative and non-commutative in general.

The minimal ideals are interesting not only in itself but it also influences the other properties of semigroups. In the literature, some interesting articles on minimal ideals and their properties can be found, for instance, see [1, 2, 8] and [9].

In this paper, we investigate minimal ideals in a non-associative and non-commutative AG-groupoid. We also discuss zeroids and divisibility in an AG-groupoid and relate them with minimal ideals.

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By a *unitary AG-groupoid*, we mean a AG-groupoid S with a left identity e. It is worth noticing that if S is a unitary AG-groupoid then $Se = eS = S$ and $S = S^2$. A groupoid with the property $S = S^2$ is called *surjective*.

If $I \subseteq S$ and $SI \subseteq I$ $(IS \subseteq I)$, then I is called a *left* (*right*) *ideal* of S. If I is both a left and right ideal of S, then I is called a *two-sided ideal* or simply an *ideal* of S. A left ideal L of an AG-groupoid S is *minimal* if every left ideal M of S included in L coincides with L. A similar statement holds for the right ideal. Let S^* be an AG-groupoid and $S^* \supseteq S \supseteq A$ such that A is a left ideal of S and S is a left ideal of S^* with the assumption that A is idempotent. Then A is a left ideal of S^*. In fact, the following equalities always hold.

$$S^*A = S^* \cdot AA \subseteq S^* \cdot AS = A \cdot S^*S \subseteq AS = AA \cdot SS = SA \cdot SA \subseteq A.$$

Notice that the property of being left ideal is transitive only if we impose an extra condition on a left ideal A. In general, being a left ideal is not transitive. If S is an AG-groupoid and A and B are ideals of the same type, then $A \cap B$ is either empty or an ideal of the same type as A and B. Also if S is an AG-groupoid, then the union of any collection of ideals of the same type is an ideal of the same type.

If there is an element 0 of an AG-groupoid $(S, \cdot)$ such that $x0 = 0x = 0$ for all $x \in S$, then 0 is the *zero element* of S.

2. Minimal and 0-minimal ideals

In [8], the authors studied minimal ideals of an AG-groupoid. They have shown that if L is a minimal left ideal of a unitary AG-groupoid, then Lc forms a minimal left ideal of S for all $c \in S$ which is a consequence of the following lemma.

Lemma 2.1. *Let L be a left ideal of a unitary AG-groupoid S. Then the following conditions are equivalent:*

(*i*) *L is a minimal left ideal of S;*
(*ii*) *$Lx = L$ for every $x \in L$;*
(*iii*) *$Sx = L$ holds for every $x \in L$.*

Proof. $(i) \Rightarrow (ii)$. Let L be a minimal left ideal of S and $x \in L$. Then $Lx \subseteq L$. Moreover, $S \cdot Lx = Se \cdot Lx = SL \cdot ex \subseteq Lx$. Thus, Lx is a left ideal of S and, by the minimality of L, we have $Lx = L$ for every $x \in L$.

$(ii) \Rightarrow (iii)$ is simple.

$(iii) \Rightarrow (i)$. Let L be a left ideal of S such that $Sx = L$ holds for every $x \in L$. Assume that M is a left ideal of S which is contained in L and let $x \in M$. Then $x \in L$ and therefore, $L = Sx \subseteq SM \subseteq M$. Hence $L = M$. □

Lemma 2.2. *A left ideal L (a right ideal R of a unitary AG-groupoid S is a minimal left (right) ideal of S if and only if $L = Sa$ for all $a \in L$ (respectively, $R = Sa^2$ for every $a \in R$).* □

Theorem 2.3. *If a unitary AG-groupoid S contains a minimal left ideal L such that L is idempotent, then S contains a minimal ideal which is the union of all the minimal left ideals of S.*

Proof. Assume that L is a minimal left ideal of S. Then, as a consequence of Lemma 2.1, LS is the union of all the minimal left ideals of S, that is $LS = \bigcup_{s\in S} Ls$. Now we have the following equalities:

$$LS \cdot S = SS \cdot L = SL = SS \cdot LL = LL \cdot SS = LS,$$

and

$$S \cdot LS = SS \cdot LS = SL \cdot SS \subseteq LS.$$

Hence, we can easily show that LS is an ideal of S. Further, we may suppose that I is an ideal of S such that $I \subseteq LS$. Then $S(I \cdot LS) \subseteq LS$. Therefore by the minimality of L, we have $I \cdot LS = L$. Thus $LS = (I \cdot LS)S \subseteq IS \cdot S \subseteq I$. Hence, we can see that S contains a minimal two sided ideal which is a union of all the minimal left ideals of S. □

Corollary 2.4. *A unitary AG-groupoid S will have no proper ideals if and only if S is the union of all its minimal left ideals.* □

Corollary 2.5. *If a unitary AG-groupoid S contains a minimal left ideal L and an ideal I such that L is idempotent then $L \subseteq I$.* □

Theorem 2.6. *Let L, R and I be the minimal left, minimal right and minimal ideal of a unitary AG-groupoid S respectively such that L is idempotent and $R \subseteq I$. Then $I = LR = LS \cdot R = LS = SR = LI = IR$.*

Proof. Since $L^2 = L$, and $R \subseteq I$ we have $S{\cdot}LR = L{\cdot}SR = L(SS{\cdot}R) = L(RS{\cdot}S) \subseteq LR$ and $LR \cdot S = SR \cdot L = SR \cdot LL = SL \cdot RL \subseteq LR$. So, LR is an ideal of S and therefore by minimality of I again, we have $I \subseteq LR$. Also it is easy to see that $LR \subseteq I$, which shows hat $I = LR$. Thus,

$$\begin{aligned} S(LS \cdot R) &= (SS)(LS \cdot R) = (S \cdot LS)(SR) \\ &= (SS \cdot LS)(SR) = (SL \cdot SS)(SR) \\ &\subseteq (L \cdot SS)(SR) \subseteq LS \cdot R, \end{aligned}$$

and

$$\begin{aligned} (LS \cdot R)S &= (LS \cdot R)(SS) = (LS \cdot S)(RS) \subseteq SL \cdot R \\ &= (SS \cdot LL)R = (SL \cdot SL)R \subseteq LS \cdot R. \end{aligned}$$

Hence, $LS \cdot R$ is an ideal of S and, by the minimality of I, we obtain $I \subseteq LS \cdot R$. Also it is easy to see that $LS{\cdot}R \subseteq I$, which implies that $I = LS{\cdot}R$. The remaining results can be proved in the similar manner. □

Corollary 2.7. *If L, L', R, R' are minimal left and minimal right ideals of a unitary AG-groupoid S respectively, then $LR = L'R'$.* □

Lemma 2.8. *If L is a minimal left ideal of a unitary AG-groupoid S, then L is an AG-groupoid without proper left ideal.*

Proof. Let L' be a left ideal of L, then $LL' \subseteq L$. As L is a left ideal of S, we have $S \cdot LL' = Se \cdot LL' = SL \cdot eL' \subseteq LL'$, The above result shows that $LL^{'}$ is a left ideal of S contained in L and therefore by the minimality of L, we have $L = LL^{'} \subseteq L^{'}$. This equality shows that $L = L^{'}$ and thus L contains no proper left ideal. □

Definition 2.9. A left (right) ideal M of an AG-groupoid S with zero is called 0-*minimal* if $M \neq \{0\}$ and $\{0\}$ is the only left (right) ideal of S properly contained in M.

Theorem 2.10. *Let M be a 0-minimal ideal of a unitary AG-groupoid S with zero such that $M^2 \neq \{0\}$ and $S \neq \{0\}$. If $R \neq \{0\}$ is a right ideal of S contained in M, then $R^2 \neq \{0\}$.*

Proof. Let R be right ideal of S, then it is easy to show that RS is an ideal of S. Therefore by the 0-minimality of M, either $RS = \{0\}$ or $RS = M$. Let $RS = \{0\}$. Since R is nonzero and would appear as an ideal of S contained in M, therefore $R = M$. Thus, $M^2 \subseteq MS = RS = \{0\}$. This contradicts the hypothesis of M. Thus $RS = M$ and therefore $M^2 = RS \cdot RS = R^2S$, which shows that $R^2 \neq \{0\}$. □

Lemma 2.11. *Let S be a unitary AG-groupoid with zero and $S \neq \{0\}$. Then $Sa \cdot S = S$ for every $0 \neq a \in S$ if $\{0\}$ is the only left ideal of S.*

Proof. Assume that $S^2 \neq \{0\}$ and $\{0\}$ is the only left ideal of S. Further, suppose that $C = \{c \in S : Sc \cdot S = \{0\}\} \neq \emptyset$. If $x \in C$ and $y \in S$, then

$$(S \cdot yx)S = (y \cdot Sx)(SS) = (yS)(Sx \cdot S) = (Sx)(yS \cdot S) \subseteq Sx \cdot S = \{0\} .$$

The above equality implies $yx \in C$. Thus $yx \in SC \subseteq C$ which means that C is a left ideal of S. Therefore, either $C = \{0\}$ or $C = S$. For the last case, we have

$$SC \cdot S = S^2S = SS = S = \{0\},$$

which contradicts our assumption. Hence, we have $C = \{0\}$ and $Sa \cdot S \neq \{0\}$ for all $0 \neq a \in S$. Since $Sa \cdot S$ is a left ideal of S, we have $Sa \cdot S = S$. □

Theorem 2.12. *If a 0-minimal ideal A of a unitary AG-groupoid S with zero contains at least one 0-minimal left ideal of S and $A^2 \neq \{0\}$, then every left ideal of A is also a left ideal of S.*

Proof. Assume that $L \neq \{0\}$ is a left ideal of A and $a \in L \setminus \{0\}$. By Lemmas 2.2, 2.11 and the fact that $A^2 \neq \{0\}$, we obtain $Aa \cdot A = A$ and $Aa \neq \{0\}$. By Lemma 6.8 [8], S contains a left ideal L_1 such that $a \in L_1 \subseteq A$. Since Aa is a nonzero left ideal of S contained in L_1, we have $Aa = L_1$. Thus, $a \in Aa$. Therefore $L \subseteq \cup\{Aa : a \in L\}$. To show the converse statement let $x \in \cup\{Aa : a \in L\}$. Then there exist elements $b \in A$ and $c \in L$ such that $x = bc$. Since $AL \in L$, it is evident that $x \in L$. Thus $L = \cup\{Aa : a \in L\}$. By the union of a set of ideals, $\cup\{Aa : a \in L\}$ is a left ideal of S. □

3. Simple and completely 0-simple AG-groupoids

In this section, we consider an AG-groupoid which contains a zero but contains no proper ideal except zero. If zero is the only element of an AG-groupoid, then it would be a proper ideal. The fact that the intersection of two nonzero minimal ideals might contain a zero element of an AG-groupoid differentiates it from the class of non-zero ideals.

Theorem 3.1. *If an AG-groupoid S without zero has at least one minimal left ideal, then the sum of all its minimal left ideals is a two-sided ideal of S.*

Proof. Let A_α be the minimal left ideals of S and $B = \sum_\alpha A_\alpha$. Then B is a left ideal. In fact: $SB = S\sum_\alpha A_\alpha = \sum_\alpha SA_\alpha \subseteq \sum_\alpha A_\alpha = B$. Also let $a \in S$, then $Ba = \sum_\alpha A_\alpha a$. But since $A_\alpha a$ is a minimal left ideal of S is contained in the sum of all minimal left ideals, *i.e* $A_\alpha a \subseteq B$ holds for all $a \in S$. It shows that $Ba \in BS \subseteq B$. Hence B is a two sided ideal of S. □

Theorem 3.2. *An AG-groupoid without zero having at least one minimal left ideal is the sum of all its minimal left ideals if and only if it is simple.*

Proof. Let S be simple and has at least one minimal left ideal L. By Theorem 3.1 the sum B of all the minimal left ideals is a two sided ideal of S. Thus $B = S$. As $B \subset S$ is contrary to the definition of simplicity of S.

Conversely, suppose that $S = \sum_\alpha L_\alpha$. Suppose that S has a two-sided subideal A distinct from S, i.e., $AS \subseteq A \subset S$ and $SA \subseteq A \subset S$. Then AL_α is a left ideal of S contained in L_α. In fact: $S(AL_\alpha) = A(SL_\alpha) \subseteq AL_\alpha \subset L_\alpha$. Since every L_α is a left ideal of S, according to the minimality, $AL_\alpha = L_\alpha$. Therefore, $AS = A\sum_\alpha L_\alpha = \sum_\alpha AL_\alpha = \sum L_\alpha = S$, which contradicts our supposition. Thus S has no proper two sided ideal and hence is simple. □

In a unitary AG-groupoid S the situation $Sa \neq S$ ($Sa^2 \neq S$) for every $a \in S$ is possible. Indeed, such situation take place in a unitary AG-groupoid S with the following multiplication table:

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	c	d	e
c	a	e	b	c	d
d	a	d	e	b	c
e	a	c	d	e	b

Definition 3.3. An AG-groupoid S is called *left* (*right*) *simple* if S is the only left (right) ideal of S. It is called *simple* if it contains no proper ideal.

Theorem 3.4. *A unitary AG-groupoid S is left* (*right*) *simple if and only if $Sa = S$* ($Sa^2 = S$) *for every $a \in S$.*

Proof. Suppose that S is a left simple AG-groupoid. Let $a \in S$, Then

$$S \cdot Sa = SS \cdot Sa = aS \cdot SS = aS \cdot S = SS \cdot a = Sa.$$

Thus Sa is a proper left ideal of S, but this contradicts our assumption. So, $Sa = S$.

Conversely, suppose that $Sa = S$ for all $a \in S$. Let L be a left ideal and $b \in L$. Then $S = Sb \subseteq SL \subseteq L$ and hence $S = L$.

Let S be right simple and $a \in S$. Then

$$Sa^2 \cdot S = SS \cdot a^2 S = S \cdot a^2 S = a^2 \cdot SS = SS \cdot aa = Sa^2.$$

The above shows that Sa^2 is a proper right ideal of S, which is a contradiction to the fact that S is right simple and therefore $Sa^2 = S$.

The converse statement is obvious. □

Definition 3.5. An AG-groupoid S with zero is called 0-*simple* (*left* 0-*simple*, *right* 0-*simple*) if $S^2 \neq \{0\}$ and $\{0\}$ is the only ideal (left ideal, right ideal) of S.

Theorem 3.6. *Let S be a unitary AG-groupoid with zero and $S \neq \{0\}$. Then S is left* (*right*) 0-*simple if and only if $Sa \cdot S = S$* ($Sa^2 \cdot S = S$) *for every $0 \neq a \in S$.*

Proof. The first part of the proof is a consequence of Lemma 2.11. To prove the second part assume that $Sa \cdot S = S$. Then $S^2 \neq \{0\}$ because $S = Sa \cdot S \subseteq S^2$. Let $A \neq \{0\}$ be a left ideal of S and $a \in A$, then $S = Sa \cdot S \subseteq SA \cdot S \subseteq A$. Hence S is left 0-simple. Similarly it can be proved for a right 0-simple AG-groupoid. □

Corollary 3.7. *A unitary AG-groupoid S without zero is left* (*right*) *simple if and only if $Sa \cdot S = S$* ($Sa^2 \cdot S = S$) *for all $a \in S$.* □

Lemma 3.8. *Let $\{0\}$ be the only ideal properly contained in a unitary AG-groupoid S with 0. Then S is 0-simple.*

Proof. Since S^2 is an ideal of S. We have either $S^2 = \{0\}$ or $S^2 = S$. If $S^2 = \{0\}$, then $S = \{0\}$ and $\{0\}$ is not the proper ideal of S, a clear contradiction. Now if $S^2 \neq \{0\}$, then by definition, S is 0-simple. □

Lemma 3.9. *Let L (R) be a 0-minimal left (right) ideal of a 0-simple unitary AG-groupoid S with zero. Then $Sa = L$ ($Sa^2 = R$) for $a \in L \backslash 0$ ($a \in R \backslash 0$).*

Proof. Let L be a 0-minimal left ideal of S and $a \in L\backslash 0$. Then Sa is a left ideal of S contained in L. By minimality of L, either $Sa = \{0\}$ or $Sa = L$. The case $Sa = 0$ is impossible because $a \neq \{0\}$ and therefore $Sa = L$. Similarly in the case for a right ideal. □

Definition 3.10. If S is an AG-groupoid with zero such that $S^2 \neq \{0\}$ and has no proper nonzero ideal and has minimal left and minimal right nonzero ideals, then S is said to *completely simple AG-groupoid with zero.*

Theorem 3.11. *Let L be a minimal left ideal of a completely simple unitary AG-groupoid S with zero such that L is idempotent. Then $LS = S = A$, where A is a nonzero left ideal of S contained in LS.*

Proof. Let S be an AG-groupoid and L be a nonzero minimal left ideal such that $L^2 = L$. Since we have $LS \cdot S = (LL \cdot SS)S = (LS \cdot LS)S \subseteq LL \cdot S \subseteq LS$, and $S \cdot LS = SS \cdot LS = SL \cdot SS \subseteq LS$, we see that LS is an ideal of S. If $LS = \{0\}$, then there exists only one minimal left ideal L, i.e., the zero ideal and S reduces to L. Therefore $LS = SS = S^2 = \{0\}$, which the contradicts the argument of S. Hence our assumption is false and hence $LS = S$. Let A be a nonzero left ideal of S contained in LS. Let $a \in LS$. Then there exists $b \in L$ and $y \in S$ such that $a = by$. Since $A \subseteq LS$, therefore $0 \neq f \in A$ has the form $f = ty$ for $t \in L$ and $y \in S$. According to Lemma 2.1, every $b \in L$ has the form $b = st$ where $s \in S$. Therefore, $a = by = st \cdot y = se \cdot ty = se \cdot f \in SA \subseteq A$. It follows that $LS \subseteq A$ and hence $LS = A$. □

Corollary 3.12. *Let L be an idempotent minimal left ideal of a completely simple unitary AG-groupoid S with zero. Then LS is a minimal left ideal of S.* □

Theorem 3.13. *If S is a completely simple unitary AG-groupoid with a zero and L and R are nonzero minimal left and right ideals of S respectively such that L and R are idempotents. Then $RL \neq \{0\}$. If $LR \neq \{0\}$, then $LR = S$.*

Proof. Similarly as in the proof of Theorem 3.11 we can prove that $LS = S$ and $SR = S$. Hence,

$$\begin{aligned} S &= SS = SR \cdot LS = (SS \cdot RR)(LS) = (RS \cdot L)S = (LS \cdot RR)S \\ &= (LR \cdot SR)S = S(LR \cdot R) \cdot S = (S \cdot RL)S. \end{aligned}$$

The above equality implies that $RL \neq \{0\}$. If $LR \neq \{0\}$, then

$$S \cdot LR = SS \cdot LR = L(SS \cdot R) = L(RS \cdot S) \subseteq LR$$

and $LR \cdot S = (LL \cdot R)S \subseteq LR$, which shows that LR is a two sided ideal of S and therefore $LR = S$. □

Corollary 3.14. *If L is a nonzero minimal left ideal of a completely simple unitary AG-groupoid S, then $LR = S$ for some nonzero minimal right ideal R of S.* □

4. Zeroids and divisibility in AG-groupoids

The concept of zeroids in an AG-groupoid was given by Q. Mushtaq in [10], where it is shown that every AG-groupoid has a left zeroid and characterized an AG-groupoid in terms of zeroids.

Definition 4.1. An element u of an AG-groupoid S is said to be a *left* (*right*) *zeroid* of S if for every element $a \in S$, there exists $x \in S$ such that $u = xa$ ($u = ax$), that is $u \in Sa$ ($u \in aS$). An element is called *zeroid* if it is both a left and a right zeroid.

Definition 4.2. A left (right) ideal of an AG-groupoid S is called an *universally minimal left ideal* of S if it is contained in every left (right) ideal of S. If an AG-groupoid S has a minimal ideal K, then K is called the *kernel* of S.

Lemma 4.3. *A unitary AG-groupoid S contains a left zeroid if and only if it contains a universally minimal left ideal L and L contains all the left zeroids of S.*

Proof. Assume that S contains a left zeroid and L consist of all left zeroids of S. Then for $a \in SL$ there exists $x \in S$ and $y \in L$ such that $a = xy$. Since L is the set of all left zeroids, $y = bc$ for some $b \in S$. Thus

$$a = xy = x \cdot bc = ex \cdot bc = cb \cdot xe = (xe \cdot b)c.$$

So, a is a left zeroid belonging to L. Hence $SL \subseteq L$ and L is a left ideal of S. Let L_1 be a left ideal of S. Then for $b \in L_1$, $Sb \subseteq SL_1 \subseteq L_1$. Let $z \in L$, then since z is a left zeroid, $z \in Sb \subseteq L_1$ and therefore $L \subseteq L_1$.

Conversely, if S contains a universally minimal left ideal L, then for any $x \in S$, Sx is a left ideal of S and $L \subseteq Sx$. Hence for every $a \in L$ we have $a = yx$ for some $y \in S$. Thus we a is a left zeroid of S. □

Lemma 4.4. *An universally minimal left ideal of a unitary AG-groupoid S is a right ideal of S and is the kernel of S.*

Proof. Assume that L is an universally minimal left ideal of S. Let $p \in LS$. Then $p = xy$ for $x \in L$ and $y \in S$. By Theorem 2.3, Ly is a minimal left ideal of S and by definition of L, $L \subseteq Ly$ and hence $L = Ly$. Thus $p \in Ly = L$ and therefore $LS \subseteq L$, which shows that L is a right ideal of S. By definition, L contains no proper ideal and hence is the kernel of S. □

Theorem 4.5. *In a unitary AG-groupoid S with zeroids, every left zeroid is a right zeroid and vice versa. The set of all zeroids of S is the kernel of S.*

Proof. The proof follows from Lemmas 4.7 and 4.4. □

An element $a \in S$ is said to be *divisible on the left* (*right*) *by* $b \in S$ if there exist $x, y \in S$ such that $a = ax$ ($a = yb$).

Theorem 4.6. *Let a and b be two distinct elements of a unitary AG-groupoid S. Then a is divisible by b on the right if and only if the left ideal of a is contained in the left ideal of b.*

Proof. Suppose that a is divisible by b on the right. Then for some $x \in S$, $a = xb$. Thus

$$\begin{aligned} Sa \cup a &= S \cdot xb \cup xb \subseteq S \cdot Sb \cup Sb = SS \cdot Sb \cup Sb \\ &= bS \cdot SS \cup Sb = Sb \cup Sb = Sb \subseteq Sb \cup b. \end{aligned}$$

Conversely, let $Sa \cup a \subseteq Sb \cup b$. Since a and b are distinct elements, therefore we have $a \in Sb$, this means that there exists some $y \in S$ such that $a = yb$. □

Corollary 4.7. *If some elements of a unitary AG-groupoid S are divisible by all the elements of S, then the collection of such elements is a universally minimal left ideal of S.*

Proof. Let B be a non-empty collection of all such elements which are divisible by all the elements of S on the right, then B is a left ideal of S. Indeed, for $a_1, a_2 \in S$, there exists $x \in S$ such that $b = xa_1$ for $b \in S$. Thus

$$a_2b = a_2 \cdot xa_1 = ea_2 \cdot xa_1 = a_1x \cdot a_2e = (a_2e \cdot x)a_1.$$

So, a_2b is divisible on the right by $a_1 \in S$ and hence $a_2b \in B$.

Let L be any arbitrary left ideal of S. Then for $l \in L$ and $b \in B$, there exists $x \in S$ such that $b = xl \in B$. Hence, $B \subseteq L$ and it is an universally minimal left ideal of S. □

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