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Prime one-sided ideals in ordered semigroups

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Abstract. We prove that the following are equivalent: (1) an ordered semigroup S with zero and identity is right weakly regular; (2) $(AA] = A$ for any right ideal A of S; (3) $A \cap I = (AI]$ for any right ideal A and two-sided ideal I of S; (4) $B \cap I \subseteq (BI]$ for any bi-ideal B and two-sided ideal I of S; (5) $B \cap I \cap A \subseteq (BIA]$ for any bi-ideal B, right ideal A and two-sided ideal I of S; and prove that S is a fully prime right ordered semigroup if and only if S is right weakly regular and the set of all two-sided ideals of S is totally ordered.

1. Introduction

One-sided ideals of a prime type of a ring have been studied by K. Koh in [6]. One-sided prime ideals have been considered by J. Dauns in [3], the author considered prime right ideals of a ring. F. Hansen [4] studied one-sided prime ideals, the paper contained some results on prime right ideals in a weakly regular ring. W.D. Blair and H. Tsutsui studied fully prime rings, it was shown a necessary and sufficient condition for a ring to be fully prime is that every ideal is idempotent and the set of ideals is totally ordered [2]. F. Alarcan and D. Polkawska described fully prime semirings, the authors characterized semirings where every ideal is prime (fully prime semirings) as those having a totally ordered lattice with every ideal idempotents [1]. Recently, prime one-sided ideals in a semiring and a Γ-semiring have been introduced and studied by R. Jagatap and Y. Pawar in [5] and by M. Shabir and M.S. Iqbal in [7]. An ordered semigroup $(S, ., \leqslant)$ is a semigroup $(S, .)$ together with an ordered relation $\leqslant$ on S which is compatible with the

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semigroup operation. In this paper, we consider prime one-sided ideals in an ordered semigroup. Indeed, we mainly consider right weakly regular ordered semigroups and fully prime right ordered semigroups. Let S be an ordered semigroup with zero and identity. It is proved that the following are equivalent: (1) S is right weakly regular; (2) $(AA] = A$ for any right ideal A of S; (3) $A \cap I = (AI]$ for any right ideal A and a two-sided ideal I of S; (4) $B \cap I \subseteq (BI]$ for any bi-ideal B and two-sided ideal I of S; (5) $B \cap I \cap A \subseteq (BIA]$ for any bi-ideal B, right ideal A and two-sided ideal I of S. Moreover, a characterization of fully prime right ordered semigroups will be given in terms of right weakly regularity and the set of all two-sided ideals. Indeed, it is proved that S is a fully prime right ordered semigroup if and only if S is right weakly regular and the set of all two-sided ideals of S is totally ordered (i.e., for any ideals A and B of S, $A \subseteq B$ or $B \subseteq A$).

An *ordered semigroup* $(S, ., \leqslant)$ consists of a semigroup $(S, .)$ together with an ordered relation $\leq$ on S which is compatible with the semigroup operation (i.e., for any $a, b, c \in S$, $a \leqslant b$ implies $ca \leqslant cb$ and $ac \leqslant bc$). For $A, B \subseteq S$, we write AB for $\{ab \in S \mid a \in A, b \in B\}$ and write $(A]$ for $\{x \in S \mid \exists a \in A, x \leqslant a\}$, i.e.

$$AB = \{ab \in S \mid a \in A, b \in B\};$$

$$(A] = \{x \in S \mid \exists a \in A, x \leqslant a\}.$$

It is observed that

(1) $A \subseteq (A]$;

(2) if $A \subseteq B$, then $(A] \subseteq (B]$;

(3) $((A]] = (A]$;

(4) $(A](B] \subseteq (AB]$;

(5) $((A](B]] = (AB]$;

(6) $(A \cup B] = (A] \cup (B]$;

(7) $(A \cap B] \subseteq (A] \cap (B]$.

A nonempty subset A of S is called a *right ideal* (of S) if

(1) $ax \in A$ for any $a \in A$ and $x \in S$ (i.e., $AS \subseteq A$);

(2) $(A] = A$ (i.e., if $a \in A$ and $x \in S$ such that $x \leqslant a$, then $x \in A$).

A left ideal of S can be defined similarly: a nonempty subset A of S is called a *left ideal* (of S) if

(1) $xa \in A$ for any $a \in A$ and $x \in S$ (i.e., $SA \subseteq A$);

(2) $(A] = A$ (i.e., if $a \in A$ and $x \in S$ such that $x \leqslant a$, then $x \in A$).

A nonempty subset A of S is called a *two-sided ideal* (it is abbreviated by *ideal*) of S if it is both a left and a right ideal of S. An element 0 of S is called a *zero* if $0a = a0 = 0$ for all $a \in S$. An element 1 of S is called an *identity* if $a1 = 1a = a$ for all $a \in S$. If S has the identity, then the principal right ideal of S generated by a is of the form $(aS]$; the principal left ideal of S generated by a is of the form $(Sa]$; and the principal ideal of S generated by a is of the form $(SaS]$.

2. Main results

Hereafter, S is an ordered semigroup with zero 0 and identity 1. We begin this section with the definition of prime right ideals of S.

Definition 2.1. Let P be a right ideal of S. Then P is called a *prime right ideal* of S if for any right ideals A and B of S, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

Theorem 2.2. *Let P be a right ideal of S. Then P is a prime right ideal of S if and only if for any $a, b \in S$, $aSb \subseteq P$ implies $a \in P$ or $b \in P$.*

Proof. Assume that P is a prime right ideal of S. Let $a, b \in S$ be such that $aSb \subseteq P$; then

$$(aS](bS] \subseteq ((aS](bS]] = ((aS)(bS)] \subseteq (PS] \subseteq (P] = P.$$

Since $(aS]$ and $(bS]$ are right ideals of S, $(aS] \subseteq P$ or $(bS] \subseteq P$. Hence $a \in P$ or $b \in P$. Conversely, assume that for any $a, b \in S$, $aSb \subseteq P$ implies $a \in P$ or $b \in P$. Let A and B be right ideals of S such that $AB \subseteq P$. Suppose that $A \nsubseteq P$, i.e. there exists $a \in A \setminus P$. Let $b \in B$. Then

$$aSb \subseteq (aSb] \subseteq (ASB] \subseteq (AB] \subseteq (P] = P.$$

By assumption, $a \in P$ or $b \in P$. Thus $b \in P$. Therefore $B \subseteq P$ and hence P is a prime right ideal of S. □

Definition 2.3. Let M be a proper right ideal of S. Then M is said to be *maximal* if there is no any proper right ideal of S containing M properly.

Theorem 2.4. *If M is a maximal right ideal of S, then M is a prime right ideal of S.*

Proof. Let M be a maximal right ideal of S. To show that M is a prime right ideal of S, let $a, b \in S$ be such that $aSb \subseteq M$. Suppose that $a \notin M$. We have $M \cup (aS]$ is a right ideal of S. Since M is a maximal right ideal of S and $M \subset M \cup (aS]$, $M \cup (aS] = S$. Then $1 \in M$ or $1 \in (aS]$. If $1 \in M$, then $b = 1b \in M$. If $1 \in (aS]$, let $1 \leqslant as$ for some $s \in S$. Consider:

$$b = 1b \leqslant asb \in aSb \subseteq M.$$

Therefore $b \in M$ and by Theorem 2.2, M is a prime right ideal of S. □

Theorem 2.5. *Let P be a prime right ideal of S. For $a \in S \setminus P$,*

$$(P : a) = \{x \in S \mid ax \in P\}$$

is a prime right ideal of S.

Proof. Clearly, $0 \in (P : a)$. If $x \in (P : a)$ and $s \in S$, then $ax \in P$; hence $a(xs) = (ax)s \in P$. If $x \in (P : a)$ and $s \in S$ such that $s \leqslant x$, then $as \leqslant ax \in P$; hence $as \in P$ (i.e., $s \in (P : a)$). Therefore $(P : a)$ is a right ideal of S. Let B and C be right ideals of S such that $BC \subseteq (P : a)$; then $a(BC) \subseteq P$. Consider:

$$(aB](aC] \subseteq ((aB](aC]] = ((aB)(aC)] \subseteq (aBC] \subseteq (P] = P.$$

Then $(aB] \subseteq P$ or $(aC] \subseteq P$. Hence $B \subseteq (P : a)$ or $C \subseteq (P : a)$. Hence $(P : a)$ is a prime right ideal of S. □

Similarly, we have the following result:

Theorem 2.6. *Let P be a prime right ideal of S. Then*

$$\{x \in S \mid Sx \subseteq P\}$$

is the largest ideal of S contained in P.

Definition 2.7. Let P be a right ideal of S. Then P is said to be a *semiprime right ideal* of S if for any right ideal A of S, $AA \subseteq P$ implies $A \subseteq P$.

It is observed that every prime right ideal is a semiprime right ideal.

Theorem 2.8. *Let P be a right ideal of S. Then P is a semiprime right ideal of S if and only if for any $a \in S$, $aSa \subseteq P$ implies $a \in P$.*

Proof. Assume that P is semiprime right ideal of S. Let $a \in S$ be such that $aSa \subseteq P$; then

$$(aS](aS] \subseteq ((aS](aS]] = ((aS)(aS)] \subseteq (PS] \subseteq (P] = P.$$

Since $(aS]$ is a right ideal of S, $(aS] \subseteq P$. Hence $a \in P$. Conversely, assume that for any $a \in S$, $aSa \subseteq P$ implies $a \in P$. Let A be a right ideal of S such that $AA \subseteq P$. Let $a \in A$. Then

$$aSa \subseteq (aSa] \subseteq (ASA] \subseteq (AA] \subseteq (P] = P.$$

By assumption, $a \in P$. Therefore $A \subseteq P$. Hence P is a semiprime right ideal of S. □

Definition 2.9. Let A be a right ideal of S. Then A is said to be *irreducible* if for any right ideals B and C of S, $B \cap C = A$ implies $B = A$ or $C = A$.

Definition 2.10. Let A be a right ideal of S. Then A is said to be *strongly irreducible* if for any right ideals B and C of S, $B \cap C \subseteq A$ implies $B \subseteq A$ or $C \subseteq A$.

Theorem 2.11. *Let A be a right ideal of S. If $x \notin A$, then there exists an irreducible right ideal of S containing A and not containing x.*

Proof. Assume that $x \notin A$. Clearly, the set of right ideals of S containing A and not containing x is nonempty. Consider a set $\{A_\alpha \mid \alpha \in \Lambda\}$ of a chain of right ideals of S containing A and not containing x. Then $\cup_{\alpha\in\Lambda} A_\alpha$ is a right ideal of S containing A and not containing x. By Zorn's lemma, the set of right ideals of S containing A and not containing x contains a maximal element, denoted by M. Let B and C be right ideals of S such that $B \cap C = M$. Suppose that $M \subset B$ and $M \subset C$. Then $x \in B$ and $x \in C$. Since $x \notin M$, $x \notin B$ or $x \notin C$. This is a contradiction. Hence $M = B$ or $M = C$. Therefore M is irreducible □

Theorem 2.12. *Any proper right ideal A of S is the intersection of irreducible right ideals of S containing A.*

Proof. Let A be a proper right ideal of S, $\{A_\alpha \mid \alpha \in \Lambda\}$ the set of irreducible right ideals of S containing A. Then $A \subseteq \cap_{\alpha\in\Lambda} A_\alpha$. If $x \notin A$, then there exists an irreducible right ideal A_{α_0} of S such that $A \subseteq A_{\alpha_0}$ and $x \notin A_{\alpha_0}$. Then $x \notin \cap_{\alpha\in\Lambda} A_\alpha$. Hence $\cap_{\alpha\in\Lambda} A_\alpha \subseteq A$. Thus $A = \cap_{\alpha\in\Lambda} A_\alpha$. Therefore A is the intersection of irreducible right ideals of S containing A. □

Theorem 2.13. *Let P be a right ideal of S. If P is strongly irreducible semiprime, then P is prime.*

Proof. Assume that P is strongly irreducible semiprime. To show that P is prime, let A and B be right ideals of S such that $AB \subseteq P$. We have

$$(A \cap B)(A \cap B) \subseteq AB \subseteq P.$$

Since $A \cap B$ is a right ideal of S and P is semiprime, $A \cap B \subseteq P$. From P is strongly irreducible, it follows that $A \subseteq P$ or $B \subseteq P$. Hence P is prime. □

Definition 2.14. An ordered semigroup S is called *right weakly regular* if $a \in (aSaS]$ for all $a \in S$.

Theorem 2.15. *The following conditions are equivalent:*

(1) *S is right weakly regular;*

(2) *$(AA] = A$ for any right ideal A of S;*

(3) *$A \cap I = (AI]$ for any right ideal A and ideal I of S.*

Proof. Assume that S is right weakly regular. Let A be a right ideal of S. Then $(AA] \subseteq A$. If $a \in A$, then

$$a \in (aSaS] \subseteq (ASAS] \subseteq (AA].$$

Then $A \subseteq (AA]$. Hence $A = (AA]$. Therefore $(AA] = A$ for any right ideal A of S. Conversely, assume that $(AA] = A$ for any right ideal A of S. To show that S is right weakly regular, let $a \in S$. Since $(aS]$ is a right ideal of S, $((aS](aS]] = (aS]$. Thus

$$a \in (aS] = ((aS](aS]] = (aSaS].$$

Therefore S is right weakly regular. This proves that (1) is equivalent to (2).

To show that (2) is equivalent to (3) assume that $(AA] = A$ for any right ideal A of S. Let A be a right ideal and I an ideal of S. We have $(AI] \subseteq A \cap I$. From $A \cap I$ is a right ideal of S, it follows that

$$A \cap I = ((A \cap I)(A \cap I)] \subseteq (AI].$$

Then $A \cap I = (AI]$. Hence $A \cap I = (AI]$ for any right ideal A and ideal I of S. Conversely, assume that $A \cap I = (AI]$ for any right ideal A and ideal I of S. Let B be a right ideal of S. We have $(SBS]$ is an ideal of S. Consider:

$$B = B \cap (SBS] = (B(SBS]] \subseteq ((B](SBS]] = (BSBS] \subseteq (BB].$$

Hence $(BB] = B$. Therefore, $(BB] = B$ for any right ideal B of S. □

Theorem 2.16. *S is right weakly regular if and only if every right ideal of S is semiprime.*

Proof. Assume that S is right weakly regular. Let P be a right ideal of S. Let A be a right ideal of S such that $AA \subseteq P$. By assumption and Theorem 2.15, $A = (AA]$. Thus $A \subseteq P$. Hence P is semiprime. Conversely, assume that every right ideal of S is semiprime. To show that S is right weakly regular, let B be a right ideal of S. Since $(BB]$ is a right ideal of S, $(BB]$ is semiprime. From $BB \subseteq (BB]$, it follows that $B \subseteq (BB]$. Since $(BB] \subseteq B \subseteq (BB]$, $(BB] = B$. By Theorem 2.15, S is right weakly regular. □

Theorem 2.17. *Let S be right weakly regular and P an ideal of S. Then P is prime if and only if P is irreducible.*

Proof. It is clear that if P is prime, then P is irreducible. Assume that P is irreducible. Let A and B be ideals of S such that $AB \subseteq P$. By Theorem 2.15, $A \cap B \subseteq P$. Then $(A \cap B) \cup P = P$. This means $(A \cup P) \cap (B \cup P) = P$. By assumption, $A \cup P = P$ or $B \cup P = P$. Hence $A \subseteq P$ or $B \subseteq P$. Therefore P is prime. □

Definition 2.18. We call S a *fully prime right ordered semigroup* if all right ideals of S are prime right ideals. For a *fully semiprime right ordered semigroup* can be defined similarly.

Theorem 2.19. *If S is a fully prime right ordered semigroup, then S is right weakly regular and the set of ideals of S is totally ordered.*

Proof. If S is a fully prime right ordered semigroup, then all right ideals of S are prime right ideals of S. Since every prime right ideal is semiprime and Theorem 2.16, S is right weakly regular. Let A and B be ideals of S. Then $A \cap B$ is a right ideal of S. By assumption, $A \cap B$ is prime. Since $AB \subseteq A \cap B$, $A \subseteq A \cap B$ or $B \subseteq A \cap B$. This means $A = A \cap B$ or $B = A \cap B$. Therefore $A \subseteq B$ or $B \subseteq A$. Hence S is right weakly regular and the set of ideals of S is totally ordered. □

Theorem 2.20. *If S is right weakly regular and the set of ideals of S is totally ordered, then S is a fully prime right ordered semigroup.*

Proof. Assume that S is right weakly regular and the set of ideals of S is totally ordered. It is to show that S is a fully prime right ordered semigroup. Let P be a right ideal of S. To show that P is prime, let A and B be right ideals of S such that $AB \subseteq P$. We have $A \subseteq B$ or $B \subseteq A$; $(AA] = A$, $(BB] = B$. If $A \subseteq B$, then

$$A = (AA] \subseteq (AB] \subseteq (P] = P.$$

Similarly, for $B \subseteq A$, we have $B \subseteq P$. Hence P is prime. Therefore S is a fully prime right ordered semigroup. □

Now we give a characterization of a fully prime right ternary semiring followed by Theorems 2.19 and Theorem 2.20.

Theorem 2.21. *S is a fully prime right ordered semigroup if and only if S is right weakly regular and the set of ideals of S is totally ordered.*

We recalled that a subsemigroup B of S is called a *bi-ideal* of S if $BSB \subseteq B$ and $(B] = B$ (i.e., if $b \in B$ and $x \in S$ such that $x \leqslant b$, then $x \in B$).

Theorem 2.22. *S is right weakly regular if and only if $B \cap I \subseteq (BI]$ for any bi-ideal B and ideal I of S.*

Proof. Assume that S is right weakly regular. Let B be a bi-ideal and I an ideal of S. Let $x \in B \cap I$. By assumption, $x \in (xSxS]$. Then

$$x \in (xSxS] \subseteq (xS(xSxS]S] \subseteq (xSxSxSS] \subseteq (BSBSISS] \subseteq (BI].$$

Hence $B \cap I \subseteq (BI]$. Conversely, assume that $B \cap I \subseteq (BI]$ for any bi-ideal B and an ideal I of S. Let A be a right ideal of S. It is observed that A is a bi-ideal of S. Using assumption, we have

$$A = A \cap (SAS] \subseteq (A(SAS]] = (ASAS] \subseteq (AA] \subseteq A.$$

Thus $A = (AA]$. By Theorem 2.15, S is right weakly regular. □

Theorem 2.23. *S is right weakly regular if and only if $B \cap I \cap A \subseteq (BIA]$ for any bi-ideal B, right ideal A and ideal I of S.*

Proof. Assume that S is right weakly regular. Let B be a bi-ideal, A a right ideal and I an ideal of S. Let $x \in B \cap I \cap A$. By assumption, $x \in (xSxS]$. Then

$$x \in (xSxS] = (xS(xSxS]S] \subseteq (xSxSxSS] \subseteq (B(SIS)(ASS)] \subseteq (BIA].$$

Hence $B \cap I \cap A \subseteq (BIA]$. Conversely, assume that $B \cap I \cap A \subseteq (BIA]$ for any bi-ideal B, right ideal A and ideal I of S. Let A be a right ideal of S. From A is a bi-ideal of S and assumption, we have

$$A = A \cap S \cap A \subseteq (ASA] \subseteq (AA] \subseteq (A] = A.$$

Thus $A = (AA]$. By Theorem 2.15, S is right weakly regular. □

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