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On groups with the same type as large Ree groups

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Abstract. Let G be a finite group and $\mathsf{nse}(G)$ be the set of the number of elements with the same order in G. In this article, we prove that the large Ree groups ${}^2\mathrm{F}_4(q)$ with an odd order component prime are uniquely determined by $\mathsf{nse}({}^2\mathrm{F}_4(q))$ and their order. As an immediate consequence, we verify Thompson's problem (1987) for the large Ree groups ${}^2\mathrm{F}_4(q)$ with an odd order component prime.

1. Introduction

In 1987, J. G. Thompson possed a problem which is related to algebraic number fields [15, Problem 12.37]:

For a finite group G and natural number n, set $G(n) = \{x \in G \mid x^n = 1\}$ and define the type of G to be the function whose value at n is the order of $G(n)$. Is it true that a group is solvable if its type is the same as that of a solvable one?

This problem links to the set $\mathsf{nse}(G)$ of *the number of elements of the same order* in G. Indeed, it turns out that if two groups G and H are of the same type, then $\mathsf{nse}(G) = \mathsf{nse}(H)$ and $|G| = |H|$. Therefore, if a group G has been uniquely determined by its order and $\mathsf{nse}(G)$, then Thompson's problem is true for G. One may ask this problem for non-solvable groups, in particular, finite simple groups. In this direction, Shao et al [17] studied finite simple groups whose order is divisible by at most four primes. Following this investigation, such problem has been studied for some other families of simple groups including Suzuki groups $\mathrm{Sz}(q)$, small Ree groups ${}^2\mathrm{G}_2(q)$ and Chevalley groups $\mathrm{F}_4(q)$ with $q = 2^{4n} + 1$ prime [2, 3, 6], see also [4, 7, 8, 10, 12, 16]. In this paper, we study this problem for the large Ree groups ${}^2\mathrm{F}_4(q)$, and prove that

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Theorem 1.1. *Let G be a group with $\mathsf{nse}(G) = \mathsf{nse}({}^2\mathrm{F}_4(q))$ and $|G| = |{}^2\mathrm{F}_4(q)|$, where $q = 2^{2m+1}$ and $q^2 + \sqrt{2q^3} + q + \sqrt{2q} + 1$ or $q^2 - \sqrt{2q^3} + q - \sqrt{2q} + 1$ is prime. Then $G \cong {}^2\mathrm{F}_4(q)$.*

As noted above, as an immediate consequence of Theorem 1.1, we have

Corollary 1.2. *If G is a group with the same type as ${}^2\mathrm{F}_4(q)$, where $q = 2^{2m+1}$ and $q^2 + \sqrt{2q^3} + q + \sqrt{2q} + 1$ or $q^2 - \sqrt{2q^3} + q - \sqrt{2q} + 1$ is prime, then G is isomorphic to ${}^2\mathrm{F}_2(q)$.*

Finally, we give some brief comments on the notation used in this paper. Throughout this article all groups are finite. We denote a Sylow p-subgroup of G by G_p. We also use $\mathsf{n}_p(G)$ to denote the number of Sylow p-subgroups of G. For a positive integer n, the set of prime divisors of n is denoted by $\pi(n)$, and we set $\pi(G) := \pi(|G|)$, where $|G|$ is the order of G. We denote the set of element orders of G by $\omega(G)$ known as *spectrum* of G. For $i \in \omega(G)$, we denote the number of elements of order i in G by $\mathsf{m}_i(G)$ and the set of the number of elements with the same order in G by $\mathsf{nse}(G)$. In other words, $\mathsf{nse}(G) = \{\mathsf{m}_i(G) \mid i \in \omega(G)\}$. The *prime graph* $\Gamma(G)$ of a finite group G is a graph whose vertex set is $\pi(G)$, and two distinct vertices u and v are adjacent if and only if $uv \in \omega(G)$. Assume further that $\Gamma(G)$ has $t(G)$ connected components $\pi_i(G)$, for $i = 1, 2, \ldots, t(G)$. The positive integers n_i with $\pi(\mathsf{n}_i) = \pi_i(G)$ are called order components of G. In the case where G is of even order, we always assume that $2 \in \pi_1$, and π_1 is said to be the even component of G. In this way, π_i and n_i are called odd components and odd order components of G, respectively. Recall that $\mathsf{nse}(G)$ is the set of the number of elements in G with the same order. In other word, $\mathsf{nse}(G)$ consists of the numbers $\mathsf{m}_i(G)$ of elements of order i in G, for $i \in \omega(G)$. Here, φ is the *Euler totient* function.

2. Preliminaries

In this section, we state some useful lemmas which will be used in the proof of the main theorem.

Lemma 2.1. [14, Main Theorem] *The maximal subgroups of ${}^2\mathrm{F}_4(q)$ with $q = 2^{2m+1} \geqslant 8$ are conjugate to one of the subgroups listed in* Table 1.

Lemma 2.2. [5, Theorem 1] and [9, Theorem 2.7.6] *Let G be a Frobenius group of even order with kernel K and complement H. Then $t(G) = 2$,* *and $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$. Moreover, K is nilpotent and $|H|$ divides $|K|-1$.*

Table 1: The maximal subgroups of ${}^2\mathrm{F}_4(q)$ with $q=2^{2m+1}\geqslant 8$.

Maximal subgroup	Conditions
$[q^{11}]:\mathrm{GL}_2(q)$	
$[q^{10}]:(\mathrm{Sz}(q)\times\mathbb{Z}_{q-1})$	
$\mathrm{SU}_3(q(),no.2,$	
$(\mathbb{Z}_{q+1}\times\mathbb{Z}_{q+1}):\mathrm{GL}_2(3)$	
$(\mathbb{Z}_{q-\sqrt{2q}+1}\times\mathbb{Z}_{q-\sqrt{2q}+1}):4\mathrm{S}_4$	$q>8$
$(\mathbb{Z}_{q+\sqrt{2q}+1}\times\mathbb{Z}_{q+\sqrt{2q}+1}):4\mathrm{S}_4$	
$\mathbb{Z}_{q^2-\sqrt{2q^3}+q-\sqrt{2q}+1}:12$	
$\mathbb{Z}_{q^2+\sqrt{2q^3}+q+\sqrt{2q}+1}:12$	
$\mathrm{PGU}_3(q):2$	
$\mathrm{Sz}(q)\wr 2$	
$\mathrm{Sp}_4(q):2$	
${}^2\mathrm{F}_4(q_0)$	$q=q_0^r$ with r prime

A group G is a 2-Frobenius group if there exists a normal series $1\trianglelefteq H\trianglelefteq K\trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H, respectively.

Lemma 2.3. [5, Theorem 2] *Let G be a 2-Frobenius group of even order. Then $t(G)=2$, $\pi_1(G)=\pi(G/K)\cup\pi(H)$ and $\pi_2(G)=\pi(K/H)$. Moreover, G/K and K/H are cyclic groups, and $|G/K|$ divides $|Aut(K/H)|$.*

Lemma 2.4. [11, Theorem 9.1.2] *Let G be a finite group, and let n be a positive integer dividing $|G|$. If $G(n)=\{g\in G\mid g^n=1\}$, then $n\mid|G(n)|$.*

Lemma 2.5. *Let G be a finite group, and let $i\in\omega(G)$. Then $\mathsf{m}_i(G)=k\varphi(i)$, where k is the number of cyclic subgroups of order i in G. Moreover, $\varphi(i)$ divides $\mathsf{m}_i(G)$, and i divides $\sum_{j|i}\mathsf{m}_j(G)$. In particular, if $i>2$, then $\mathsf{m}_i(G)$ is even.*

Proof. The proof is straightforward by Lemma 2.4. □

Lemma 2.6. [1, Lemma 3.1] *The order of ${}^2\mathrm{F}_4(q)$ with $q=2^{2m+1}\geqslant 8$ is coprime to* 17.

3. Proof of the main result

Let $q = 2^{2m+1} \geqslant 8$, and let p be a prime number. Suppose that p is $q^2+\sqrt{2q^3}+q+\sqrt{2q}+1$ or $q^2-\sqrt{2q^3}+q-\sqrt{2q}+1$, and set $F := {}^2\mathrm{F}_4(q)$. Let G be a finite group with $\mathsf{nse}(G) = \mathsf{nse}(F)$ and $|G| = |F|$. We note that ${}^2\mathrm{F}_4(q)$ with $q = 2^{2m+1} \geqslant 8$ is of order $q^{12}(q-1)(q^2+1)(q^3+1)(q^4-1)\cdot f_+(q)\cdot f_-(q)$, where

$$f_\epsilon(q) = q^2 + \epsilon\sqrt{2q^3} + q + \epsilon\sqrt{2q} + 1, \tag{3.1}$$

with $\epsilon = \pm$. We observe by [18] that the simple group ${}^2\mathrm{F}_4(q)$ with $q = 2^{2m+1} \geqslant 8$ has two odd order components, namely, $f_+(q)$ and $f_-(q)$.

Lemma 3.1. *Let $F := {}^2\mathrm{F}_4(q)$ with $q = 2^{2m+1} \geqslant 8$, and let $f_\epsilon(q)$ be as in* (3.1)*. If $p = f_\epsilon(q)$ is prime, then*

(a) $\mathsf{m}_p(F) = (p-1)|F|/(12p)$;

(b) $p \mid \mathsf{m}_i(F)$ *for every* $i \in \omega(F) \setminus \{1, p\}$.

Proof. By Lemma 2.1, F_p is a cyclic group of order p, and so $\mathsf{m}_p(F) = \varphi(p)\mathsf{n}_p(F) = (p-1)\mathsf{n}_p(F)$. According to Lemma 2.1, $|\mathbf{N}_F(\mathrm{F}_p)| = 12p$, and so $\mathsf{n}_p(F) = |F|/12p$. If $i \in \omega(F) \setminus \{1, p\}$, then [13] implies that p is an isolated vertex of $\Gamma(F)$, and so $p \nmid i$ and $pi \notin \omega(F)$. Thus F_p acts fixed point freely on the set of elements of order i in G by conjugation, and hence $|\mathrm{F}_p| \mid \mathsf{m}_i(F)$. Therefore, $p \mid \mathsf{m}_i(F)$. □

Lemma 3.2. *Let $F := {}^2\mathrm{F}_4(q)$, and let G be a group such that $|G| = |F|$ and $\mathsf{nse}(G) = \mathsf{nse}(F)$. Let also p be $f_\epsilon(q)$ defined as in* (3.1)*. If p is prime, then*

(a) $\mathsf{m}_2(G) = \mathsf{m}_2(F)$;

(b) $\mathsf{m}_p(G) = \mathsf{m}_p(F)$;

(c) $\mathsf{n}_p(G) = \mathsf{n}_p(F)$;

(d) *p is an isolated vertex of* $\Gamma(G)$;

(e) $p \mid \mathsf{m}_i(G)$ *for every* $i \in \omega(G) \setminus \{1, p\}$.

Proof. According to Lemma 2.5, for any $i \in \omega(G)$, $i > 2$ if and only if $\mathsf{m}_i(G)$ is even. So $\mathsf{m}_2(G) = \mathsf{m}_2(F)$. By Lemma 2.5, $(\mathsf{m}_p(G), p) = 1$, and so $p \nmid \mathsf{m}_p(G)$. Then by Lemma 3.1, $\mathsf{m}_p(G) \in \{\mathsf{m}_1(F), \mathsf{m}_p(F)\}$, and since

$\mathsf{m}_p(G)$ is even, we deduce that $\mathsf{m}_p(G) = \mathsf{m}_p(F)$. Since G_p and F_p are cyclic groups of order p, it follows that $\mathsf{m}_p(G) = \varphi(p)\mathsf{n}_p(G) = \varphi(p)\mathsf{n}_p(F) = \mathsf{m}_p(F)$. So $\mathsf{n}_p(G) = \mathsf{n}_p(F)$. If p is not an isolated vertex of $\Gamma(G)$, then there exists $r \in \pi(G) - \{p\}$ such that $pr \in \omega(G)$. Thus $\mathsf{m}_{pr}(G) = \varphi(pr)\mathsf{n}_p(G)k$, where k is the number of the cyclic subgroups of order r in $\mathbf{C}_G(G_p)$. Since $\mathsf{n}_p(G) = \mathsf{n}_p(F) = |F|/(12p)$ and $|F| = |G|$, we conclude that $\mathsf{n}_p(G) = |G|/(12p)$. Thus $(p-1)(r-1)|G|/(12p)$ divides $\mathsf{m}_{pr}(G)$. On the other hands, by Lemma 3.1, p is a divisor of $\mathsf{m}_{pr}(G)$. Then $p(p-1)(r-1)|G|/12p$ divides $\mathsf{m}_{pr}(G) < |G|$, and this implies that $r = 2$ and $p < 13$, which is a contradiction. Hence p is an isolated vertex of $\Gamma(G)$. $\square$

Proof of Theorem 1.1. We first prove that the group G is neither a Frobenius group, nor a 2-Frobenius group. Assume to the contrary that G is a Frobenius group or a 2-Frobenius group. If G is a Frobenius group with kernel K and complement H. Then Lemma 2.2 implies that $t(G) = 2, \pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$. Since p is an isolated vertex of $\Gamma(G)$, we deduce that $|K| = p$ and $|H| = |F|/p$, or $|H| = p$ and $|K| = |F|/p$. By Lemma 2.2, $|F|/f_\epsilon(q)$ divides $f_\epsilon(q)-1$ or $f_\epsilon(q)$ divides $[|F|/f_\epsilon(q)]-1$. This implies that $p \mid 11$, which is a contradiction. If G is a 2-Frobenius group, then Lemma 2.3 implies that $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H, respectively, $\pi_1(G) = \pi(G/K) \cup \pi(H)$, $\pi_2(G) = \pi(K/H)$ and $|G/K|$ divides $|\mathrm{Aut}(K/H)|$. Since p is an isolated vertex of $\Gamma(G)$, we deduce that $|K/H| = f_\epsilon(q)$ and $|H| = q^{12}(q-1)(q^2+1)(q^4-1)(q^3+1)\mathrm{F}_{-\epsilon}(q)/|G/K|$. Since $|G/K|$ divides $|\mathrm{Aut}(K/H)|$, we deduce that $|G/K|$ divides $p-1$. On the other hand, since K is a Frobenius group with kernel H, Lemma 2.2 implies that p divides $[q^{12}(q-1)(q^2+1)(q^4-1)(q^3+1)\mathrm{F}_{-\epsilon}(q)/|G/K|] - 1$, and hence p divides $12 - |G/K|$, which is a contradiction.

Therefore, G is neither a Frobenius group, nor a 2-Frobenius group, and hence by [18, Theorem A], G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|\mathrm{Out}(K/H)|$. Moreover, any odd component of G is also an odd component of K/H. Since p is an isolated vertex of $\Gamma(G)$, we deduce that $p \mid |K/H|$ and $t(K/H) \geq 2$. The connected components of the simple group K/H can be read off from [13, 18], and in what follows we discuss all these possibilities. For convenience, we use Lie notation for the finite simple groups of Lie type.

Let K/H be a sporadic simple group or one of the simple groups $\mathrm{A}_2(2)$,

$\mathrm{A}_2(4)$, ${}^2\mathrm{A}_3(2)$, ${}^2\mathrm{A}_5(2)$, $\mathrm{E}_7(2)$, $\mathrm{E}_7(3)$, ${}^2\mathrm{E}_6(2)$ and ${}^2\mathrm{F}_4(2)'$. Then $f_\epsilon(q)$ is equal to one of the prime numbers 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 59, 67, 71, 73, 127, 757 and 1093. This is possible only for $q = 8$ when $\mathrm{F}_-(q) = 37$ and K/H is isomorphic to J_4 or Ly in which case $|K/H|$ does not divide $|G|$.

Let now K/H be an alternating group of degree n. Then since by Lemma 2.6, $17 \notin \pi(G)$, it follows that $n < 17$, and this violates the choice of p which is at least 37.

Let K/H be a finite simple classical group over a finite field of size q'. Then we easily observe by Lemma 2.6 that $17 \nmid q'$. Moreover, if $q'^{16} - 1$ is a divisor of $|K/H|$, then by the Fermat's little theorem, $q'^{16} - 1 \equiv 0 \pmod{17}$, and so $17 \mid |K/H|$ which violates Lemma 2.6. Therefore, we have one of the following possibilities:

K/H	Condition
$\mathrm{A}_n(q')$	$1 \leqslant n \leqslant 16$
${}^2\mathrm{A}_n(q')$	$1 \leqslant n \leqslant 16$
$\mathrm{C}_n(q')$	$2 \leqslant n \leqslant 7$
$\mathrm{B}_n(q')$	$2 \leqslant n \leqslant 7$, q' odd
$\mathrm{D}_n(q')$	$3 \leqslant n \leqslant 8$
${}^2\mathrm{D}_n(q')$	$2 \leqslant n \leqslant 8$

Suppose that K/H is isomorphic to $\mathrm{A}_n(q')$. If $n = 1$, then p is q', $(q' \pm 1)$ or $(q' \pm 1)/2$, and so $p \mp 1$ or $2p \mp 1$ divides $|K/H|$, so does $|G|$, which is a contradiction. If $2 \leqslant n \leqslant 16$ and $(n, q') \neq (2, 2), (2, 4)$, then $n = p'$ or $p' - 1$, and so p is $(q'^{p'} - 1)/[(q' - 1)(p', q' - 1)]$ or $(q'^{p'} - 1)/(q' - 1)$. Therefore, $(p', q' - 1)p - 1$ or $p - 1$ divides $|K/H|$, respectively. But none of these is a divisor of $|G|$, which is a contradiction.

Suppose that K/H is isomorphic to ${}^2\mathrm{A}_n(q')$ for $n = p', p' - 1$ with $(n, q') \neq (3, 2), (5, 2)$. Then p is $(q'^{p'} + 1)/[(q' + 1)(p', q' + 1)]$ or $(q'^{p'} + 1)/(q' + 1)$, which is impossible as neither $(p', q' - 1)p - 1$, nor $p - 1$ divides $|G|$.

Suppose that K/H is isomorphic to $\mathrm{B}_n(q')$ or $\mathrm{C}_n(q')$. Then p is $(q'^n \pm 1)/(2, q' - 1)$, and so $(2, q' - 1)p \mp 1$ has to divide $|G|$, which is a contradiction.

Suppose that K/H is isomorphic to $\mathrm{D}_n(q')$ with $n = p', p' + 1$ and $q' = 2, 3, 4$. Then p is $(q'^{p'} - 1)/(4, q' - 1)$, and so $(4, q' - 1)p + 1$ has to divide $|G|$, which is a contradiction.

Suppose that K/H is isomorphic to ${}^2\mathrm{D}_n(q')$. Then p is $(q'^{p'}+1)/(2,q'-1)$, $2^{n'-1}+1$, $2^{n'}+1$, $(3^n+1)/4$ or $(3^{n-1}+1)/2$. $(2,q'-1)p-1$, $p-1$, $p+1$, $2p-1$, $4p-1$ has to divide $|G|$, which is a contradiction.

If K/H is isomorphic to $G_2(q')$, $\mathrm{F}_4(q')$, $\mathrm{E}_6(q')$, ${}^2\mathrm{E}_6(q')$ or ${}^3\mathrm{D}_4(q')$, then p is $q'^2\pm q'+1$, $q'^4-q'^2+1$ or q'^4+1, $q'^6+q'^3+1$ or $(q'^6+q'^3+1)/3$, $(q'^6\pm q'^3+1)/(3,q\mp 1)$ or $q'^4-q'^2+1$. So $p-1$ or $3p-1$ is a divisor of $|G|$, which is a contradiction.

Suppose that K/H is isomorphic to $\mathrm{E}_8(q')$. Then p is ${q'}^8\pm {q'}^7\mp {q'}^5-{q'}^4\mp {q'}^3\pm q'+1$, ${q'}^8-{q'}^4+1$ or ${q'}^8-{q'}^6+{q'}^4-{q'}^2+1$. If p is ${q'}^8-{q'}^4+1$ or ${q'}^8-{q'}^6+{q'}^4-{q'}^2+1$, then $p-1$ divides $|G|$, which is impossible. If $p={q'}^8\pm {q'}^7\mp {q'}^5-{q'}^4\mp {q'}^3\pm q'+1$, then $2^{m+1}(2^m\pm 1)(2^{2m+1}+1)=q'({q'}^7\pm {q'}^6\mp {q'}^4-{q'}^3\mp {q'}^2\pm 1)$, so we have three possibilities:

(1) $(q',2^{m+1})\neq 1$. Since $(2^{m+1},2^m\pm 1)=(2^{m+1},2^{2m+1}+1)=1$, we have $q'=2^{m+1}$. This implies that ${q'}^{120}\mid |K/H|$ so does $|G|$, which is a contradiction.

(2) $(q',2^{2m+1}+1)\neq 1$. If $3\nmid q'$, then $q'\mid 2^{2m+1}+1=q+1$ and ${q'}^2\nmid q+1$ because $(2^{2m+1}+1,2^m\pm 1)=1$ or 3. This also requires ${q'}^{120}\mid |G|$, which is a contradiction. If $3\mid q'$, then $q'=3^{m'}$ for some positive integer m'. If $(q',2^{2m+1}+1)>3$, then $3^{m'-1}\mid 2^{2m+1}+1=q+1$ but $3^{m'+1}\nmid q+1$. Hence ${q'}^{120}\mid |G|$, which is impossible. We note that the case where $q'=3$ and $(q',2^{2m+1}+1)=3$ cannot occur as $p=q^2\pm\sqrt{2q^3}+q\pm\sqrt{2q}+1$ is a prime number and $q=2^{2m+1}>2$. If $q'=3^{m'}>3$ and $(q',2^{2m+1}+1)=3$, then $3^{m'-1}\mid 2^m\pm 1$ but $3^{m'+1}\nmid 2^m\pm 1$. Since $|K/H|\mid |G|$ we have $q'^{120}\mid |G|$, which is a contradiction.

(3) $(q',2^m\pm 1)\neq 1$. This case can be ruled out by the same manner as in case (2).

Suppose that K/H is isomorphic to ${}^2\mathrm{B}_2(q')$ with $q'=2^{2m'+1}$. Then $p=q'-1$ or $q'\pm\sqrt{2q'}+1$. If $p=q'-1$, then $2^{2m'+1}-2=2^{m+1}(2^m\pm 1)(2^{2m+1}+1)$, and so $m=0$, which is a contradiction. If $p=q'\pm\sqrt{2q'}+1$, then $2^{m'+1}(2^{m'}\pm 1)=2^{m+1}(2^m\pm 1)(2^{2m+1}+1)$ implying that $m=m'$, which is a contradiction.

Suppose that K/H is isomorphic to ${}^2G_2(q')$ with $q'=3^{2m'+1}$. Then $p=q'\pm\sqrt{3q'}+1$, and so $3^{m'+1}(3^{m'}\pm 1)=2^{m+1}(2^m\mp 1)(2^{2m+1}+1)$. Therefore $2^{m+1}\mid 3^{m'}\pm 1$. Note that $(2^m\mp 1,2^{2m+1}+1)=1$ or 3. If $3^{m'}\mid 2^m\mp 1$, then $m=m'=1$, which is impossible. If $3^{m'}\mid 2^{2m+1}+1$, then $q'\mid (q+1)^2$ but $q'^2\nmid (q+1)^2$. Since $q'^3\mid |K/H|$, we have $q'^3\mid |G|$, which is a contradiction.

Therefore, K/H is isomorphic to ${}^2\mathrm{F}_4(q')$, and hence $q' = q$. This forces $H = 1$, and hence $G = K \cong {}^2\mathrm{F}_4(q)$, as claimed. □

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