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# On Baer filters of bounded distributive lattices

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**Abstract.** Following the concept of Baer ideals, we define Baer filters and we will make an intensive investigate the basic properties and possible structures of these filters.

## 1. Introduction

All lattices considered in this paper are assumed to have a least element denoted by 0 and a greatest element denoted by 1, in other words they are *bounded*.

The notion of an order plays an important role not only throughout mathematics but also in adjacent such as logic, computer science and engineering and, hence, ought to be in the literature. Filters of lattices play a central role in the structure theory and are useful for many purposes. The main aim of this article is that of extending some results obtained for ring theory to the theory of lattices. The main difficulty is figuring out what additional hypotheses the lattice or filter must satisfy to get similar results. Nevertheless, growing interest in developing the algebraic theory of lattices can be found in several papers and books (see for example [2, 3, 4, 7, 8, 9, 10]).

An ideal $I$ of a commutative ring $R$ is called a $d$-ideal provided that for each $a \in I$ and $x \in R$, $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(x)$ implies that $x \in I$. The concept of $d$-ideals has been studied by several authors in different forms and by different names. The notion of $d$-ideals in a commutative ring was introduced by Speed [17] who called them Baer ideals. These ideals were also put to good use in 1972 by Evans [5] when characterizing commutative rings that are finite direct sums of integral domains. In [11], Jayaram introduced $fd$-ideals (as strongly Baer ideals) and 0-ideals in reduced rings and characterize quasi regular and von Neumann regular rings. In [13], Khabazian, Safaeeyan and

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Vedadi extended the concept of d-ideals to the category of modules and investigated the modules for which their submodules are $d$-submodules. In [16], Safaeeyan and Taherifar studied $d$-ideals and $fd$-ideals in general rings, and not just the reduced ones. In [1], Anebri, Kim and Mahdou investigated the concepts of $d$-submodules, $fd$-submodules and 0-submodules of a module over a commutative ring. In [15], Mason investigated the concepts of $z$-ideals of a commutative ring.

Let $\pounds$ be a *bounded distributive lattice*. We say that a subset $S \subseteq \pounds$ is *join closed* if $0 \in S$ and $s_1 \vee s_2 \in S$ for all $s_1, s_2 \in S$ (clearly, if $\mathbf{p}$ is a prime filter of $\pounds$, then $\pounds \setminus \mathbf{p}$ is a join closed subset of $\pounds$). If $F, G$ are filters of $\pounds$ and $y \in \pounds$, then we define the *filter quotients* $(G :_{\pounds} F) = \{x \in \pounds : x \vee F \subseteq G\}$ and $(\{1\} :_{\pounds} y) = (1 :_{\pounds} y) = \{z \in \pounds : z \vee y = 1\}$; clearly these are another filters of $\pounds$ and $G \subseteq (G :_{\pounds} F)$. A filter $F$ is said to be a *Baer filter* (resp. *strongly Baer* filter) if $(1 :_{\pounds} f) \subseteq (1 :_{\pounds} x)$ for some $f \in F$ and $x \in \pounds$ implies that $x \in F$ (resp. $(1 :_{\pounds} G) \subseteq (1 :_{\pounds} x)$ for some finite subset $G$ of $F$ and $x \in \pounds$ implies that $x \in F$). $F$ is said to be a 1-*filter* if $F = \{1\}_S(\pounds) = \{x \in \pounds : x \vee s = 1$ for some $s \in S\}$ for some join closed subset $S$ of $\pounds$. For each element $x$ in a lattice $\pounds$, the intersection of all minimal prime filters in $\pounds$ containing $x$ is denoted by $P_x$, and a filter $F$ in $\pounds$ is called a $z^0$-*filter* if $P_x \subseteq F$, for all $x \in F$. A filter $F$ of $\pounds$ is a *strongly* $z^0$-*filter* if $P_A \subseteq F$ for each finite subset $A$ of $F$. For each element $x$ in a lattice $\pounds$, the intersection of all maximal filters in $\pounds$ containing $x$ is denoted by $M_x$, and a filter $F$ in $\pounds$ is called a $z$-*filter* if $M_x \subseteq F$, for all $x \in F$. In the present paper, we are interested in investigating Baer filters to use other notions of Baer, and associate which exist in the literature as laid forth in [1, 11, 15, 16].

Our objective in this paper is to extend the notion of Baer property in commutative rings to Baer property in the lattices, and to investigate the relations between Baer filters, Strongly Baer filters, $z^0$-filters, strongly $z^0$-filters and $z$-filters. Among many results in this paper, the first, introductional section contains elementatary observations needed later on.

In Section 2, we give basic properties of Baer filters. In particular, we show that the class of lattices for which their Baer filters, strongly Baer filters, $z^0$-filters and strongly $z^0$-filters are the same (see Proposition 2.4, Prposition 2.19 and Theorem 2.20). Also, we investigate Baer filters and specify some distinguished classes of Baer filters in a lattice. For example, 1-filters, the filter $(F :_{\pounds} G)$ where $F$ is a Baer filter and $G$ is a filter of $\pounds$ (so $(1 :_{\pounds} H)$ for every filter $H$ of $\pounds$), direct meets and all minimal prime filters are Baer filters (see Lemma 2.5, Lemma 2.6, Proposition 2.7, Proposition 2.9 and Proposition 2.12). In this section we observe that in a lattice $\pounds$, If $\mathbf{p}$ is a prime filter of a lattice $\pounds$, then either $\mathbf{p}$ is a Baer filter or the maximal Baer filters contained in $\mathbf{p}$ are prime Baer filters (see Theorem 2.13).

Section 3 is dedicated to the study of $z$-filters. We show that every minimal prime filter in a semisimple lattice $\pounds$ is a $z$-filter (see Theorem 3.4). We also prove in Theorem 3.5 that if $F$ is a $z$-filter of $\pounds$, then every $\mathbf{p} \in \min(F)$ is a $z$-filter. Here, we observe that in a lattice $\pounds$, Baer filters and $z$-filters are not coincide generally (see Example 3.7). The remaining part of this section is mainly devoted to investigation of lattices $\pounds$ such that when the class of Baer filters is contained in the class of $z$-filters (see Theorem 3.8).

Section 4 concentrates to the relation between Baer filters and prime filters. We prove in Theorem 4.4 that every prime filter of $\pounds$ is a Baer filter if and only if every filter of $\pounds$ is a Baer filter. We also show that $\pounds$ is a classical lattice such that for every finitely generated filter $F \subseteq \mathrm{I}(\pounds)$, $(1 :_{\pounds} F) \neq \{1\}$ if and only if every maximal filter of $\pounds$ is a Baer filter (see Theorem 4.5). Moreover, we prove that in a lattice $\pounds$, every prime Baer filter of $\pounds$ is either a minimal prime or a maximal filter if and only if for each maximal filter $\mathbf{m}$ of $\pounds$ and each $m, n \in \mathbf{m}$, there exists a finite subset $A \subseteq (1 :_{\pounds} m)$ and $d \notin \mathbf{m}$ such that $(1 :_{\pounds} T(A \cup \{m\})) \subseteq (1 :_{\pounds} d \vee n)$ (see Theorem 4.6). Finally, we will show that every prime Baer filter of $\pounds$ is a minimal prime filter if and only if for each $a \in \pounds$, there exists a finitely generated filter $F$ such that $F \subseteq (1 :_{\pounds} a)$ and $(1 :_{\pounds} T(F \cup \{a\})) = \{1\}$ (see Theorem 4.7).

Let us recall some notions and notations. By a lattice we mean a poset $(\pounds, \leqslant)$ in which every couple elements $x, y$ has a g.l.b. (called the meet of $x$ and $y$, and written $x \wedge y$) and a l.u.b. (called the join of $x$ and $y$, and written $x \vee y$). A lattice $\pounds$ is complete when each of its subsets $X$ has a l.u.b. and a g.l.b. in $\pounds$. Setting $X = \pounds$, we see that any nonvoid complete lattice contains a least element 0 and greatest element 1 (in this case, we say that $\pounds$ is a lattice with 0 and 1). A lattice $\pounds$ is called a distributive lattice if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all $a, b, c$ in $\pounds$ (equivalently, $\pounds$ is distributive if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all $a, b, c$ in $\pounds$). A non-empty subset $F$ of a lattice $\pounds$ is called a filter, if for $a \in F$, $b \in \pounds$, $a \leqslant b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if $\pounds$ is a lattice with 1, then $1 \in F$ and $\{1\}$ is a filter of $\pounds$). A proper filter $F$ of $\pounds$ is called prime if $x \vee y \in F$, then $x \in F$ or $y \in F$. A proper filter $F$ of $\pounds$ is said to be maximal if $G$ is a filter in $\pounds$ with $F \subsetneqq G$, then $G = \pounds$. The radical of $\pounds$, denoted by $\mathrm{Rad}(\pounds)$, is the intersection of all maximal filters of $\pounds$.

Let $A$ be subset of a lattice $\pounds$. Then the filter generated by $A$, denoted by $T(A)$, is the intersection of all filters that is containing $A$. A filter $F$ is called finitely generated if there is a finite subset $A$ of $F$ such that $F = T(A)$. A lattice $\pounds$ with 1 is called $\pounds$-domain if $a \vee b = 1$ $(a, b \in \pounds)$, then $a = 1$ or $b = 1$. First we need the following lemma proved in [2, 4, 6, 8, 9].

**Lemma 1.1.** *Let $\pounds$ be a lattice.*

(1) *A non-empty subset $F$ of $\pounds$ is a filter of $\pounds$ if and only if $x \vee z \in F$ and $x \wedge y \in F$ for all $x, y \in F$, $z \in \pounds$. Moreover, since $x = x \vee (x \wedge y)$, $y = y \vee (x \wedge y)$ and $F$ is a filter, $x \wedge y \in F$ gives $x, y \in F$ for all $x, y \in \pounds$.*

(2) *If $F_1, \ldots, F_n$ are filters of $\pounds$ and $a \in \pounds$, then $\bigvee_{i=1}^n F_i = \{\bigvee_{i=1}^n a_i : a_i \in F_i\}$ and $a \vee F_i = \{a \vee a_i : a_i \in F_i\}$ are filters of $\pounds$ and $\bigvee_{i=1}^n F_i = \bigcap_{i=1}^n F_i$.*

(3) *Let $A$ be an arbitrary non-empty subset of $\pounds$. Then $T(A) = \{x \in \pounds : a_1 \wedge a_2 \wedge \cdots \wedge a_n \leqslant x \text{ for some } a_i \in A \ (1 \leqslant i \leqslant n)\}$. Moreover, if $F$ is a filter and $A$ is a subset of $\pounds$ with $A \subseteq F$, then $T(A) \subseteq F$, $T(F) = F$ and $T(T(A)) = T(A)$.*

(4) *If $\pounds$ is distributive, $F, G$ are filters of $\pounds$, and $y \in \pounds$, then $(G :_\pounds F) = \{x \in \pounds : x \vee F \subseteq G\}$, $(F :_\pounds T(\{y\})) = (F :_\pounds y) = \{a \in \pounds : a \vee y \in F\}$ and $(\{1\} :_\pounds y) = (1 :_\pounds y) = \{z \in \pounds : z \vee y = 1\}$ are filters of $\pounds$.*

(5) *If $\{F_i\}_{i \in \Delta}$ is a chain of filters of $\pounds$, then $\bigcup_{i \in \Delta} F_i$ is a filter of $\pounds$.*

(6) *If $\pounds$ is distributive, $G, F_1, \cdots, F_n$ are filters of $\pounds$, then $G \vee (\bigwedge_{i=1}^n F_i) = \bigwedge_{i=1}^n (G \vee F_i)$.*

(7) *If $\pounds$ is distributive and $F_1, \ldots, F_n$ are filters of $\pounds$, then for each $i$ $\bigwedge_{i=1}^n F_i = \{\wedge_{i=1}^n a_i : a_i \in F_i\}$ is a filter of $\pounds$ and $F_i \subseteq \bigwedge_{i=1}^n F_i$.*

## 2. Basic properties of Baer filters

In this section, we collect some basic properties concerning Baer filters and strongly Baer filters and then investigate the relationship among these filters. Throughout this paper we shall assume, unless otherwise stated, that $\pounds$ is a *bounded distributive lattice*. The proof of the following lemma can be found in [6] (with some different proof and notions), but we give the details for convenience.

**Lemma 2.1.** *For the lattice $\pounds$ the following statements hold:*

(1) *If $F$ is a proper filter of $\pounds$ with $F \neq \{1\}$, then $F$ contained in a maximal filter of $\pounds$;*
(2) *Every Maximal filter of $\pounds$ is a prime filter.*

*Proof.* (1). Since the filter $F$ is proper, $\Omega = \{G : G$ is a filter of £ with $F \subseteq G, G \neq \pounds\} \neq \emptyset$. Moreover, $(\Omega, \subseteq)$ is a partial order. Clearly, $\Omega$ is closed under taking unions of chains and so the result follows by Zorn's Lemma.

(2). Assume that $\mathbf{m}$ is a maximal filter of $\pounds$ and let $a \vee b \in \mathbf{m}$ with $a, b \notin \mathbf{m}$. Then $\pounds = \mathbf{m} \wedge T(\{a\})$ which implies that $0 = m \wedge (a \vee s)$ for some $m \in \mathbf{m}$ and $s \in \pounds$. Then $\mathbf{m}$ is a filter gives $b = b \vee (m \wedge (a \vee s)) = (b \vee m) \wedge (b \vee a \vee s) \in \mathbf{m}$ which is impossible. Thus $\mathbf{m}$ is prime. □

**Lemma 2.2.** *Assume that $F$ is a filter of $\pounds$ and let $S$ be a join closed subset of $\pounds$. Then $F_S(\pounds) = \{x \in \pounds : x \vee s \in F$ for some $s \in S\}$ is a filter of $\pounds$ with $F \subseteq F_S(\pounds)$.*

*Proof.* If $f \in F$, then $f \vee s \in F$ $(s \in S)$ gives $F \subseteq F_S(\pounds)$. Let $x_1, x_2 \in F_S(L)$ and $t \in \pounds$. Then $x_1 \vee s_1, x_2 \vee s_2 \in F$ for some $s_1, s_2 \in S$ (so $s_1 \vee s_2 \in S$) gives $(x_1 \wedge x_2) \vee (s_1 \vee s_2), (x_1 \vee t) \vee s_1 \in F$; hence $x_1 \wedge x_2, x_1 \vee t \in F_S(\pounds)$, as needed. □

We remind the reader with the following definition.

**Definition 2.3.** Let $F$ be a filter of $\pounds$.

(1) $F$ is said to be a *Baer filter* if $(1 :_{\pounds} f) \subseteq (1 :_{\pounds} x)$ for some $f \in F$ and $x \in \pounds$ implies that $x \in F$.
(2) $F$ is said to be a *strongly Baer filter* if $(1 :_{\pounds} G) \subseteq (1 :_{\pounds} x)$ for some finite subset $G$ of $F$ and $x \in \pounds$ implies that $x \in F$.
(3) $F$ is said to be a 1-*filter* if $F = \{1\}_S(\pounds)$ for some join closed subset $S$ of $\pounds$.

It can be easily seen that every strongly Baer filter is a Baer filter. It can also be verified that arbitrary intersection of Baer fiters is again a Baer filter. The next result determines the class of lattices for which their Baer filters and strongly Baer filters are the same.

**Proposition 2.4.** *A filter $F$ of a lattice $\pounds$ is a Baer filter if and only if $F$ is a strongly Baer filter.*

*Proof.* It is enough to show that if $F$ is Baer filter, then $F$ is a strongly Baer filter. Let $(1:_\pounds A) \subseteq (1:_\pounds x)$ for some finite subset $A = \{a_1, a_2, \cdots, a_k\}$ of $F$ (so $\bigwedge_{i=1}^k a_i \in F$, as $F$ is a filter) and $x \in \pounds$. Then $(1:_\pounds \bigwedge_{i=1}^k a_i) = \bigcap_{i=1}^k (1:_\pounds a_i) = (1:_\pounds A) \subseteq (1:_\pounds x)$ gives $x \in F$, as $F$ is a Baer filter. This completes the proof. □

**Lemma 2.5.** *For a lattice $\pounds$ the following statements hold:*

(1) *If $S$ is a join closed subset of $\pounds$, then $\{1\}_S(\pounds)$ is a Baer filter.*

(2) *A filter $F$ of $\pounds$ is a Baer filter if and only if for each $f_1, f_2 \in F$ with $(1:_\pounds f_1) \cap (1:_\pounds f_2) \subseteq (1:_\pounds x)$ implies $x \in F$.*

*Proof.* (1). Let $(1:_\pounds a) \subseteq (1:_\pounds x)$ for some $a \in \{1\}_S(\pounds)$ and $x \in \pounds$. Then there exists $s \in S$ such that $a \vee s = 1$ which implies that $s \in (1:_\pounds a) \subseteq (1:_\pounds x)$; hence $x \in \{1\}_S(\pounds)$.

(2). If $F$ is a Baer filter, then $(1:_\pounds f_1 \wedge f_2) = (1:_\pounds f_1) \cap (1:_\pounds f_2) \subseteq (1:_\pounds x)$ gives $x \in F$. Conversely, let $(1:_\pounds A) \subseteq (1:_\pounds x)$ for some finite subset $A = \{a_, a_2, \ldots, a_k\}$ of $F$ and $x \in \pounds$. Then $(1:_\pounds A) = \bigcap_{i=1}^k (1:_\pounds a_i)$ $= (1:_\pounds a_1) \cap \bigcap_{i=2}^k (1:_\pounds a_i) = (1:_\pounds a_1) \cap (1:_\pounds \bigwedge_{i=2}^k a_i) \subseteq (1:_\pounds x)$ gives $x \in F$. □

**Lemma 2.6.** *Let $F$ and $G$ be filters of $\pounds$. If $F$ is a Baer filter, then $(F:_\pounds G)$ is a Baer filter. In particular, $(1:_\pounds H)$ is a Baer filter for every filter $H$ of $\pounds$.*

*Proof.* Let $(1:_\pounds f) \subseteq (1:_\pounds x)$ for some $f \in (F:_\pounds G)$ (so $f \vee G \subseteq F$) and $x \in \pounds$. Then for each $g \in G$, $(1:_\pounds f \vee g) \subseteq (1:_\pounds x \vee g)$ and $f \vee g \in F$ gives $x \vee G \subseteq F$, as $F$ is a Baer filter; hence $x \in (F:_\pounds G)$. The in particular statement is clear. □

A proper filter $F$ of $\pounds$ is said to be a direct meet of $\pounds$ if $\pounds = F \wedge G$ and $F \cap G = \{1\}$ for some filter $G$ of $\pounds$. Compare the next Proposition with Propostion 2.10 (4) in [16].

**Proposition 2.7.** *Every direct meet of a lattice $\pounds$ is a Baer filter.*

*Proof.* Let $F$ be a direct meet of $\pounds$. Then $\pounds = F \wedge G$ and $G \cap F = F \vee G = \{1\}$ for some filter $G$ of $\pounds$. Clearly, $G \cap (1 :_{\pounds} G) = \{1\}$ and $F \subseteq (1 :_{\pounds} G)$. If $x \in (1 :_{\pounds} G)$, then $x = x \wedge 1 \in \pounds = F \wedge G$ gives $x = a \wedge b$ for some $a \in F$ and $b \in G$. This implies that $a, b \in (1 :_{\pounds} G)$ by Lemma 1.1; so $b = 1$ which gives $x = a \in F$. Thus $F = (1 :_{\pounds} G)$. Now the assertion follows from Lemma 2.6. □

Compare the next Proposition with Lemma 3.9 in [12].

**Proposition 2.8.** *Assume that $\pounds$ be a lattice and let $F$ be a filter of $\pounds$. The following statements are equivalent:*

(1) *$F$ is a Baer filter of $\pounds$;*

(2) *$(1 :_{\pounds} (1 :_{\pounds} f)) \subseteq F$ for each $f \in F$;*

(3) *$F = \bigcup_{f \in F}(1 :_{\pounds} (1 :_{\pounds} f))$.*

*Proof.* (1) $\Rightarrow$ (2). Let $x \in (1 :_{\pounds} (1 :_{\pounds} f))$ for some $f \in F$. Then $x \vee (1 :_{\pounds} f) = \{1\}$ gives $(1 :_{\pounds} f) \subseteq (1 :_{\pounds} x)$; so $x \in F$, as $F$ is a Baer filter.

(2) $\Rightarrow$ (3). By (2), $H = \bigcup_{f \in F}(1 :_{\pounds} (1 :_{\pounds} f)) \subseteq F$. If $e \in F$, then $e \in (1 :_{\pounds} (1 :_{\pounds} e)) \subseteq H$ and so we have equality.

(3) $\Rightarrow$ (1). Let $(1 :_{\pounds} f) \subseteq (1 :_{\pounds} x)$ for some $f \in F$ and $x \in \pounds$. This implies that $x \in (1 :_{\pounds} (1 :_{\pounds} x)) \subseteq (1 :_{\pounds} (1 :_{\pounds} f)) \subseteq F$ by (3), as required. □

**Proposition 2.9.** *Let $\pounds$ be a lattice. The following hold:*

(1) *If $\mathbf{p}$ is a prime filter of $\pounds$, then $1_{\mathbf{p}} = \{x \in \pounds : (1 :_{\pounds} x) \cap (\pounds \setminus \mathbf{p}) \neq \{1\}\}$ is a Baer filter;*

(2) *If $x \in \pounds$, then $F = (1 :_{\pounds} (1 :_{\pounds} x))$ is a Baer filter;*

(3) *If $x \in \pounds$, then $(1 :_{\pounds} x)$ is a Baer filter.*

*Proof.* (1). Let $x_1, x_2 \in 1_{\mathbf{p}}$ and $t \in \pounds$. Then there exist $1 \neq a \notin \mathbf{p}$ and $1 \neq b \notin \mathbf{p}$ (so $1 \neq a \vee b \notin \mathbf{p}$) such that $a \vee x_1 = 1 = b \vee x_2$ which implies that $a \vee b \in (1 :_{\pounds} (x_1 \wedge x_2)) \cap (\pounds \setminus \mathbf{p})$; hence $x_1 \wedge x_2 \in 1_{\mathbf{p}}$. Similarly, $x_1 \vee t \in 1_{\mathbf{p}}$. Thus $1_{\mathbf{p}}$ is a filter of $\pounds$. To see that $1_{\mathbf{p}}$ is a Baer filter, at first we show that $1_{\mathbf{p}} = \bigcup_{x \in \pounds \setminus \mathbf{p}}(1 :_{\pounds} x) = H$. If $x \in 1_{\mathbf{p}}$, then $x \vee z = 1$ for some $1 \neq z \in \pounds \setminus \mathbf{p}$. This implies that $1 \neq x \in (1 :_{\pounds} z) \subseteq H$; so $1_{\mathbf{p}} \subseteq H$. Similarly, $H \subseteq 1_{\mathbf{p}}$, and so we have equality. Let $(1 :_{\pounds} a) \subseteq (1 :_{\pounds} x)$ for some $a \in 1_{\mathbf{p}}$ and $x \in \pounds$. Then there exists $1 \neq t \in \pounds \setminus \mathbf{p}$ such that $a \vee t = 1$. Then $t \in (1 :_{\pounds} a) \subseteq (1 :_{\pounds} x)$ which gives $1 \neq t \in (1 :_{\pounds} x) \cap (\pounds \setminus \mathbf{p})$; thus $x \in 1_{\mathbf{p}}$.

(2). It suffices to show that for each $y \in F$, $(1 :_{\pounds} (1 :_{\pounds} y)) \subseteq F$ by Proposition 2.8. Let $z \in (1 :_{\pounds} (1 :_{\pounds} y))$. Then $z \vee (1 :_{\pounds} y) = \{1\} = y \vee (1 :_{\pounds} x)$ gives $(1 :_{\pounds} x) \subseteq (1 :_{\pounds} y) \subseteq (1 :_{\pounds} z)$ which implies that $z \vee (1 :_{\pounds} x) = \{1\}$; so $z \in F$.

(3). Since $(1 :_{\pounds} x) = (1 :_{\pounds} (1 :_{\pounds} (1 :_{\pounds} x)))$, $(1 :_{\pounds} x)$ is a Baer filter by (2) and Lemma 2.6. □

**Proposition 2.10.** *A lattice $\pounds$ is a $\pounds$-domain if and only if it has no nontrivial Baer filter.*

*Proof.* Assume that $F \neq \{1\}$ is a Baer filter of $\pounds$ and let $1 \neq x \in F$. Then $(1 :_{\pounds} x) = \{1\}$, as $\pounds$ is a $\pounds$-domain. Thus for each $y \in \pounds$, $(1 :_{\pounds} x) \subseteq (1 :_{\pounds} y)$ which implies that $y \in F$ since $F$ is a Baer filter. Hence $F = \pounds$. Conversely, for each $x \in \pounds$, $(1 :_{\pounds} x)$ is a Baer filter by Proposition 2.9 (3); so either $(1 :_{\pounds} x) = \{1\}$ or $(1 :_{\pounds} x) = \pounds$. Thus for each $1 \neq x \in \pounds$, $(1 :_{\pounds} x) = \{1\}$. Hence $\pounds$ is a $\pounds$-domain. □

Let $\pounds$ be a lattice. We denote by Spec($\pounds$) the set of all prime filters of $\pounds$. If $F$ is a filter in $\pounds$, the set of all minimal prime filters over $F$ (or belonging to $F$) will be denoted by min($F$). We need the following proposition proved in [6, Proposition 2.7].

**Proposition 2.11.** *For a lattice $\pounds$ the following statements hold:*

(1) *If $F$ is a filter and $\mathbf{p}$ is a prime filter of $\pounds$, then $\mathbf{p} \in \min(F)$ if and only if for each $x \in \mathbf{p}$, there is a $y \notin \mathbf{p}$ such $y \vee x \in F$;*

(2) *If $\mathbf{p}$ is a prime filter of $\pounds$, then $\mathbf{p} \in \min(\pounds)$ if and only if for each $x \in \mathbf{p}$, there is a $y \notin \mathbf{p}$ such that $y \vee x = 1$.*

The next result shows that every minimal prime filter of a lattice $\pounds$ is a Baer filter. Compare the next Proposition with Propostion 2.13 (1) in [16].

**Proposition 2.12.** *If $F$ is a Baer filter of a lattice $\pounds$, then every minimal prime filter over $F$ is a Baer filter.*

*Proof.* Suppose that $\mathbf{p} \in \min(F)$ and let $(1 :_{\pounds} p) \subseteq (1 :_{\pounds} x)$ for some $p \in \mathbf{p}$ and $x \in \pounds$. Then $p \vee p' \in F$ for some $p' \notin \mathbf{p}$ by Proposition 2.11 (1). Clearly, $(1 :_{\pounds} p \vee p') \subseteq (1 :_{\pounds} p' \vee x)$. This implies that $x \vee p' \in F \subseteq \mathbf{p}$, as $F$ is a Baer filter, and therefore $x \in \mathbf{p}$. □

**Theorem 2.13.** *If $\mathbf{p}$ is a prime filter of a lattice $\pounds$, then either $\mathbf{p}$ is a Baer filter or the maximal Baer filters contained in $\mathbf{p}$ are prime Baer filters.*

*Proof.* Set $\Omega = \{F : F \text{ is a Baer filter of } \pounds \text{ and } F \subseteq \mathbf{p}\}$. Then $\{1\} \in \Omega$ and $(\Omega, \subseteq)$ is a partial order. Clearly, $\Omega$ is closed under taking unions of chains and so by Zorn's Lemma, $\Omega$ has a maximal element, say $\mathbf{m}$. It is clear that $\mathbf{p} = \mathbf{m}$ if and only if $\mathbf{p}$ is a prime Baer filter. If $\mathbf{m} \subsetneqq \mathbf{p}$, then there exists a prime filter $\mathbf{m}'$ minimal with respect to $\mathbf{m} \subseteq \mathbf{m}'$ and $\mathbf{m}' \subsetneqq \mathbf{p}$ since $\mathbf{m}'$ will be a Baer filter by Proposition 2.12. So, either $\mathbf{m}' = \mathbf{m}$ which gives $\mathbf{m}$ is prime, or $\mathbf{m} \subsetneqq \mathbf{m}'$ which contradicts the maximality of $\mathbf{m}$. $\square$

**Theorem 2.14.** If $F$ is a 1-filter of $\pounds$, then every $\mathbf{p} \in \min(F)$ is a minimal prime filter of $\pounds$.

*Proof.* By assumption, $F = \{1\}_S(\pounds) = \{x \in \pounds : x \vee s = 1 \text{ for some } s \in S\}$ for some join closed subset $S$ of $\pounds$. By Proposition 2.11 (2), it suffices to show that for each $x \in \mathbf{p}$ there exists $y \notin \mathbf{p}$ such that $y \vee x = 1$. Let $x \in \mathbf{p}$. Then by Proposition 2.11 (1), there is $y \notin \mathbf{p}$ such that $x \vee y \in F$ and $\mathbf{p} \cap S = \emptyset$. So $x \vee y \vee s = 1$ for some $s \in S \setminus \mathbf{p}$. Thus $x \vee y \vee s = 1$ and $y \vee s \notin \mathbf{p}$ and hence $\mathbf{p}$ is a minimal prime filter of $\pounds$. $\square$

Compare the next Theorem with Lemma 2.5 in [16].

**Theorem 2.15.** *Let $F$ be a filter of a lattice $\pounds$. Then $F$ contained in a proper Baer filter if and only if for each finite subset $K$ of $F$, $(1 :_\pounds K) \neq \{1\}$.*

*Proof.* Assume to the contrary, that $F$ is contained in a proper Bear filter $G$ and $K$ a finite subset of $F$ such that $(1 :_\pounds K) = \{1\}$. Let $y \in \pounds$. Then $(1 :_\pounds K) \subseteq (1 :_\pounds y)$ gives $y \in G$; so $G = \pounds$ which is a contradiction. Conversely, suppose that $F$ has the stated property and put

$$H = \{x \in \pounds : (1 :_\pounds K) \subseteq (1 :_\pounds x) \text{ for some finite subset } K \text{ of } F\}.$$

Let $x_1, x_2 \in H$ and $t \in \pounds$. Then there exist finite subsets $H_1, H_2$ of $F$ such that $(1 :_\pounds H_1) \subseteq (1 :_\pounds x_1)$ and $(1 :_\pounds H_2) \subseteq (1 :_\pounds x_2)$. It follows that $(1 :_\pounds H_1 \wedge H_2) \subseteq (1 :_\pounds H_1) \cap (1 :_\pounds H_2) \subseteq (1 :_\pounds x_1) \cap (1 :_\pounds x_2) \subseteq (1 :_\pounds x_1 \wedge x_2)$ and $(1 :_\pounds H_1) \subseteq (1 :_\pounds x_1) \subseteq (1 :_\pounds x_1 \vee t)$; hence $x_1 \wedge x_2, x_1 \vee t \in H$. Therefore $H$ is a filter of $\pounds$. Let $(1 :_\pounds K) \subseteq (1 :_\pounds y)$ for some finite subset $K = \{k_1, \cdots, k_m\}$ of $H$ and $y \in \pounds$. There are finite subsets $K_1, \cdots, K_m$ of $F$ such that $(1 :_\pounds K_i) \subseteq (1 :_\pounds k_i)$ for each $1 \leqslant i \leqslant m$. Set $K' = \bigvee_{i=1}^m K_i \subseteq F$. If $z \in (1 :_\pounds K')$, then $z \vee K' = \{1\}$ gives $z \vee K_i = \{1\}$ (so $z \vee k_i = 1$) for each $1 \leqslant i \leqslant m$ which implies that $z \in (1 :_\pounds K) \subseteq (1 :_\pounds x)$; hence $(1 :_\pounds K') \subseteq (1 :_\pounds x)$ and so $x \in H$. Thus $H$ is a Baer filter. Moreover, if $f \in F$, then $(1 :_\pounds \{f\}) \subseteq (1 :_\pounds f)$ gives $F \subseteq H$. $\square$

Compare the next Theorem with Proposition 2.14 in [16].

**Theorem 2.16.** *If $F_1, F_2, \cdots, F_m$ are filters of $\pounds$ such that for each $i \neq j$, $F_i \wedge F_j = \pounds$, then $\bigcap_{i=1}^m F_i$ is a Baer filter if and only if each $F_i$ $(1 \leqslant i \leqslant m)$ is a Baer filter.*

*Proof.* (1). One side is clear. To see the other side, suppose that $\bigcap_{i=1}^m F_i$ is a Baer filter, $f \in F_j$ for some $1 \leqslant j \leqslant m$ and $b \in \pounds$ such that $(1 :_\pounds f) \subseteq (1 :_\pounds b)$. Set $F = \bigcap_{i=1, i \neq j}^m F_i$. We claim that $F \wedge F_j = \pounds$. On the contrary, assume that $F \wedge F_j \neq \pounds$. Then there is a maximal filer $\mathbf{m}$ of $\pounds$ such that $F \wedge F_j \subseteq \mathbf{m}$ by Lemma 2.1 (1) (so $F_j \subseteq \mathbf{m}$ and $F \subseteq \mathbf{m}$). Then there is a $1 \leqslant s \leqslant m$ with $s \neq j$ such that $F_s \subseteq \mathbf{m}$. Otherwise, for each $1 \leqslant i \leqslant m$ with $i \neq j$, there exists $f_i \in F_i \setminus \mathbf{m}$, but then $\bigvee_{i=1, i \neq j}^m f_i \in F \setminus \mathbf{m}$ By Lemma 2.1 (2), and this contradicts the statement of $F \subseteq \mathbf{m}$. So $\pounds = F_j \wedge F_s \subseteq \mathbf{m}$, a contradiction. Therefore $F_j \wedge F = \pounds$ and hence $0 = f_j \wedge y$ for some $f_j \in F_j$ and $y \in F$. So $b = (b \vee f_j) \wedge (b \vee y)$ and $(1 :_\pounds f \vee y) \subseteq (1 :_\pounds b \vee y)$. Since $f \vee y \in \bigcap_{i=1}^m F_i$ and it is a Baer filter, $b \vee y \in \bigcap_{i=1}^m F_i$. Thus $b \vee y \in F_j$. Since $b \vee f_j \in F_j$ and $b \vee y \in F_j$, $b \in F_j$. Therefore $F_j$ is a Baer filter. $\square$

For each element $x$ in a lattice $\pounds$, the intersection of all minimal prime filters in $\pounds$ containing $x$ is denoted by $P_x$, and a filter $F$ in $\pounds$ is called a $z^0$-filter if $P_x \subseteq F$, for all $x \in F$. Clearly, $P_1 = \bigcap_{1 \in \mathbf{p} \in \min(\pounds)} \mathbf{p} = \bigcap_{\mathbf{p} \in \min(\pounds)} \mathbf{p} = \{1\}$ by [6, Lemma 2.6], $x \in P_x$ and if $a \in P_x$, then $P_a \subseteq P_x$. A filter $F$ of $\pounds$ is a strongly $z^0$-filter if $P_A \subseteq F$ for each finite subset $A$ of $F$. It can be easily seen that every strongly $z^0$-filter is a $z^0$-filter. For each $a \in \pounds$, set $V(a) = \{\mathbf{p} \in \min(\pounds) : a \in \mathbf{p}\}$.

**Proposition 2.17.** *For a lattice $\pounds$ the following statements hold:*

(1) *For every $x \in \pounds$ and a finite subset $A$ of $\pounds$, $(1 :_\pounds A) \subseteq (1 :_\pounds x)$ if and only if $V(A) \subseteq V(x)$, i.e. $P_x \subseteq P_A$;*

(2) *For $a, b \in \pounds$, $(1 :_\pounds a) \subseteq (1 :_\pounds b)$ if and only if $P_b \subseteq P_a$, i.e. $V(a) \subseteq V(b)$.*

*Proof.* (1). Let $\mathbf{p} \in P_A$ and $x \in \pounds$ such that $(1 :_\pounds A) =$

$$(1 :_\pounds A) = \bigcap_{i=1}^{k} (1 :_\pounds a_i) = (1 :_\pounds \bigwedge_{i=1}^{k} a_i) \subseteq (1 :_\pounds x),$$

where $A = \{a_1, a_2, \cdots, a_k\} \subseteq \mathbf{p}$. By Proposition 2.11, there exist the sequence $\{b_1, b_2, \cdots, b_k\} \subseteq \pounds \setminus \mathbf{p}$ such that for each $1 \leqslant i \leqslant k$, $a_i \vee b_i = 1$.

Set $b = \bigvee_{i=1}^{k} b_i$. Then $1 \neq b \notin \mathbf{p}$. By assumption, $b \vee (\bigwedge_{i=1}^{k} a_i) = 1$ gives $b \in (1 :_{\pounds} x)$ and hence $b \vee x = 1 \in \mathbf{p}$. This implies that $x \in \mathbf{p}$, as $\mathbf{p}$ is prime. Thus $V(A) \subseteq V(x)$. Conversely, let $x \in \pounds$ and $A = \{a_1, a_2, \cdots, a_k\}$ be a finite subset of $\pounds$ and $y \in (1 :_{\pounds} A) = \bigcap_{i=1}^{k}(1 :_{\pounds} a_i)$. This implies that $y \vee a_i = 1$ for each $1 \leqslant i \leqslant k$. Then $P_x \subseteq P_A \subseteq \bigcap_{i=1}^{k} P_{a_i}$ gives $x \vee y \in P_{x\vee y} \subseteq \bigcap_{i=1}^{k} P_{a_i \vee y} = P_1 = \{1\}$ and hence $x \vee y = 1$, as needed.

(2). This is clear by (1). □

**Lemma 2.18.** *Let $F$ be a filter of a lattice $\pounds$. The following hold:*

(1) *$F$ is a $z^0$-filter if and only if for each $a \in F$ and $b \in \pounds$, $P_b \subseteq P_a$ implies $b \in F$.*

(2) *$F$ is a strongly $z^0$-filter if and only if for each $a \in \pounds$ and a finite subset $A$ of $\pounds$, $P_a \subseteq P_A$ implies $a \in F$.*

*Proof.* (1). Assume that $F$ is a $z^0$-filter and let $P_b \subseteq P_a$, where $a \in F$ and $b \in \pounds$ which gives $b \in P_b \subseteq P_a \subseteq F$. Conversely, let $x \in F$ and $y \in Px$. Then by assumption, $P_y \subseteq P_x$ and $x \in F$ gives $y \in F$; so $P_x \subseteq F$.

(2). Suppose that $F$ is a strongly $z^0$-filter and let $P_a \subseteq P_A$ for some $a \in \pounds$ and a finite subset $A$ of $\pounds$. By assumption, $a \in P_a \subseteq P_A \subseteq F$. Conversely, let $B$ be a finite subset of $\pounds$ and $z \in P_B$. Then by assumption, $P_z \subseteq P_B$ gives $z \in F$. Thus $P_B \subseteq F$. This completes the proof. □

**Proposition 2.19.** *A filter $F$ of a lattice $\pounds$ is a $z^0$-filter if and only if $F$ is a strongly $z^0$-filter.*

*Proof.* It is enough to show that if $F$ is $z^0$-filter, then $F$ is a strongly $z^0$-filter. Let $P_a \subseteq P_A$ for some $a \in \pounds$ and a finite set $A = \{a_1, a_2, \cdots, a_k\}$ of $\pounds$. Then $(1 :_{\pounds} \bigwedge_{i=1}^{k} a_i) = \bigcap_{i=1}^{k}(1 :_{\pounds} a_i) = (1 :_{\pounds} A) \subseteq (1 :_{\pounds} a)$ by Proposition 2.17; so again by Proposition 2.17, $P_a \subseteq P_{\bigwedge_{i=1}^{k} a_i}$ gives $a \in F$, as $F$ is a $z^0$-filter. Thus $F$ is a strongly $z^0$-filter by Lemma 2.18. □

The following result determines the class of lattices for which their Baer filters and $z^0$-filters are the same.

**Theorem 2.20.** *A filter $F$ of a lattice $\pounds$ is a Baer filter if and only if $F$ is a $z^0$-filter.*

*Proof.* Assume that $F$ is a Baer filter and let $a \in F$ and $b \in \pounds$ such that $P_b \subseteq P_a$. By Proposition 2.17, $(1 :_{\pounds} a) \subseteq (1 :_{\pounds} b)$. Now $F$ is a Baer filter gives $b \in F$. Thus $F$ is a $z^0$-filter. Conversely, suppose that $F$ is a $z^0$-filter and let $(1 :_{\pounds} a) \subseteq (1 :_{\pounds} b)$ for some $a \in F$ and $b \in \pounds$. Then by Proposition 2.17, we have $b \in P_b \subseteq P_a \subseteq F$, i.e. the result holds. □

## 3. Some properties of $z$-filters

For each element $x$ in a lattice $\pounds$, the intersection of all maximal filters in $\pounds$ containing $x$ is denoted by $M_x$, and a filter $F$ in $\pounds$ is called a $z$-filter if $M_x \subseteq F$, for all $x \in F$. Clearly, $M_1 = \text{Rad}(\pounds)$, $x \in M_x$ and if $a \in M_x$, then $M_a \subseteq M_x$. A lattice $\pounds$ is called semisimple provided that $\text{Rad}(\pounds) = \{1\}$.

**Lemma 3.1.** *Let $F$ be a filter of $\pounds$. Then $F$ is a $z$-filter if and only if for each $a \in F$ and $b \in \pounds$, $M_b \subseteq M_a$ implies $b \in F$.*

*Proof.* Assume that $F$ is a $z$-filter and let $M_b \subseteq M_a$, where $a \in F$ and $b \in \pounds$. It follows that $b \in M_b \subseteq M_a \subseteq F$. Conversely, let $x \in F$ and $y \in M_x$. Then by assumption, $M_y \subseteq M_x$ and $x \in F$ gives $y \in F$; so $M_x \subseteq F$. □

**Remark 3.2.** 1. If $\mathbf{m}$ is a maximal filter of $\pounds$, then $M_a \subseteq \mathbf{m}$ for all $a \in \mathbf{m}$. Thus the family of $z$-filters contains the set of maximal filters of $\pounds$.

2. It can be easily seen that any intersection of $z$-filters is a $z$-filter.

3. By (1) and (2), $\text{Rad}(\pounds)$ is a $z$-filter. Moreover, if $x \in \text{Rad}(\pounds)$, $F$ is any $z$-filter and $y \in F$, then $M_x \subseteq M_y$ gives $x \in F$. Therefore $\text{Rad}(\pounds)$ is contained in every $z$-filter.

4. The intersections of maximal filters are the most obvious $z$-filters and they will be called strong z-filters.

5. Suppose that $T(\{x\})$ is a $z$-filter; we show that

$$T(\{x\}) = \bigcap\{\mathbf{m} \in \text{Max}(\pounds) : T(\{x\}) \subseteq \mathbf{m}\}.$$

If $y \in \bigcap\{\mathbf{m} \in \text{Max}(\pounds) : T(\{x\}) \subseteq \mathbf{m}\}$, then $M_y \subseteq M_x$ gives $y \in T(\{x\})$, and so we have equality. Thus any cyclic $z$-filter is a strong $z$-filter.

**Proposition 3.3.** *If $F$ is a $z$-filter, then $(F :_{\pounds} G)$ is a $z$-filter for any $G$.*

*Proof.* Let $M_b \subseteq M_a$ for some $a \in (F :_{\pounds} G)$ and $b \in \pounds$. Then $M_{b\vee g} \subseteq M_{a\vee g}$ for all $g \in G$. Since $a \vee g \in F$, $b \vee g \in F$ for all $g \in G$, i.e. $b \in (F :_{\pounds} G)$. □

**Theorem 3.4.** *Every minimal prime filter in a semisimple lattice $\pounds$ is a $z$-filter.*

*Proof.* Assume that $\mathbf{p}$ is a minimal prime filter of $\pounds$ and let $M_q \subseteq M_p$ for some $p \in \mathbf{p}$ and $q \in \pounds$. Since $\mathbf{p}$ is minimal prime, there exists a $y \notin \mathbf{p}$ such that $p \vee y = 1$ by Proposition 2.11. We claim that $q \vee y = 1$. Assume to the contrary, that $y \vee q \neq 1$. By Lemma 2.1, there exists a maximal filter $\mathbf{m}$ such that $y \vee q \notin \mathbf{m}$, since an element which belongs to every maximal filter is 1, as $\pounds$ is semisimple. Then $\mathbf{m} \wedge T(\{y \vee q\}) = \pounds$, as $\mathbf{m}$ is a maximal filter and so there would be elements $s \in \pounds$ and $m \in \mathbf{m}$ such that $0 = m \wedge (y \vee q \vee s)$, which then implies $p = (p \vee m) \wedge (p \vee y \vee q \vee s) = p \vee m$, and hence $p \in \mathbf{m}$. But $q \in M_q \subseteq M_p \subseteq \mathbf{m}$, so we would have $q \in \mathbf{m}$, and hence $q \vee y \in \mathbf{m}$, leading to a contradiction. Therefore $y \vee q = 1 \in \mathbf{p}$, and since $\mathbf{p}$ is prime with $y \notin \mathbf{p}$, we deduce that $q \in \mathbf{p}$. Thus, $\mathbf{p}$ is a $z$-filter. □

Compare the next theorem with Theorem 1.1 in [15].

**Theorem 3.5.** *If $F$ is a $z$-filter of $\pounds$, then every $\mathbf{p} \in \min(F)$ is a $z$-filter.*

*Proof.* It suffices to show that if $\mathbf{p}$ is a prime filter containing $F$ which is not a $z$-filter, it is not minimal. If $\mathbf{p}$ is not a $z$-filter, then there are elements $q \notin \mathbf{p}$ and $p \in \mathbf{p}$ such that $M_q \subseteq M_p$ by Lemma 3.1. Set $D = (\pounds \setminus \mathbf{p}) \cup H$, where $H = \{p \vee s : s \notin \mathbf{p}\}$. Clearly, $0 \in D$. Let $x, y \in D$. If $x, y \notin \mathbf{p}$, then $x \vee y \notin \mathbf{p}$ gives $x \vee y \in D$. If $x \notin \mathbf{p}$ and $y \in H$, then there exists $u \notin \mathbf{p}$ such that $y = u \vee p$ which implies that $x \vee y = (x \vee u) \vee p \in H \subseteq D$. Similarly, if $x \in H$ and $y \notin \mathbf{p}$, we have $x \vee y \in D$. If $x, y \in H$, then $x = p \vee u$ and $y = p \vee u'$ for some $u, u' \notin \mathbf{p}$. Then $x \vee y = p \vee (u \vee u') \in H \subseteq D$. Thus $D$ is a join closed subset of $\pounds$. If $x \in F \cap D$, then $x \in H$; so $x = p \vee s$ for some $s \notin \mathbf{p}$. By assumption, $M_{q \vee s} \subseteq M_{p \vee s}$ and $p \vee s \in F$ gives $q \vee s \in F \subseteq \mathbf{p}$. But $q, s \notin \mathbf{p}$ and $\mathbf{p}$ is prime. Thus $D \cap F = \emptyset$. By [6, Lemma 2.6 (i)], There is a prime filter $F \subseteq \mathbf{p}'$ which is maximal with respect to the property $\mathbf{p}' \cap F = \emptyset$ and it is clear that $\mathbf{p}' \subsetneqq \mathbf{p}$. Thus $\mathbf{p}$ is not minimal. □

Compare the next corollary with Theorem 1.5 in [15].

**Corollary 3.6.** *If $\mathbf{p}$ is a prime filter of a semisimple lattice $\pounds$, then either $\mathbf{p}$ is a $z$-filter or the maximal $z$-filters contained in $\mathbf{p}$ are prime $z$-filters.*

*Proof.* Set $\Delta = \{G : G$ is a z-filter of $\pounds$ and $G \subseteq \mathbf{p}\}$. Then $\{1\} \in \Delta$ and $\Delta$ is inductive so by Zorn's lemma, $\Delta$ has a maximal element, say $\mathbf{q}$. It is clear that $\mathbf{p} = \mathbf{q}$ if and only if $\mathbf{p}$ is a prime $z$-filter. If $\mathbf{q} \subsetneqq \mathbf{p}$, then there exists a prime filter $\mathbf{q}'$ minimal with respect to $\mathbf{q} \subseteq \mathbf{q}'$ and $\mathbf{q}' \subsetneqq \mathbf{p}$ since $\mathbf{q}'$ will be a $z$-filter by Theorem 3.5. So, either $\mathbf{q}' = \mathbf{q}$ which gives $\mathbf{q}$ is prime, or $\mathbf{q} \subsetneqq \mathbf{q}'$ which contradicts the maximality of $\mathbf{q}$. □

The following example shows that $z$-filters are not necessarily Baer filters.

**Example 3.7.** Let $D = \{a, b, c\}$. Then $\pounds = \{X : X \subseteq D\}$ forms a distributive lattice under set inclusion greatest element $D$ and least element $\emptyset$ (note that if $x, y \in \pounds$, then $x \vee y = x \cup y$ and $x \wedge y = x \cap y$). It can be easily seen that proper filters of $\pounds$ are $\{D\}$, $F_1 = \{D, \{a, b\}\}$, $F_2 = \{D, \{a, c\}\}$, $F_3 = \{D, \{b, c\}\}$, $F_4 = \{D, \{a, c\}, \{a, b\}\{a\}\}$, $F_5 = \{D, \{b, c\}, \{a, b\}\{b\}\}$ and $F_6 = \{D, \{a, c\}, \{c, b\}\{c\}\}$. Then

$$F_3 = (1 :_{\pounds} \{a\}) \subseteq (1 :_{\pounds} \{a, b\}) = F_6, \{a, b\} \in F_5$$

and $\{a\} \notin F_5$. This shows that $F_5$ is not a Baer filter, but $F_5$ is a $z$-filter since it is maximal. So Baer filters and $z$-filters are not coincide generally.

The following theorem shows when the class of Baer filters is contained in the class of $z$-filters. Compare the next Theorem with Propostion 2.9 in [16].

**Theorem 3.8.** *For a lattice $\pounds$ the following statements are equivalent:*

(1) *$\pounds$ is semisimple;*

(2) *Every Baer filter of $\pounds$ is a $z$-filter.*

*Proof.* (1) $\Rightarrow$ (2). Assume that $F$ is a Baer filter of $\pounds$ and let $M_b \subseteq M_a$, where $a \in F$ and $b \in \pounds$. Let $x \in (1 :_{\pounds} a)$. Then $M_b \subseteq M_a$ gives $M_{b \vee x} \subseteq M_{x \vee a} \subseteq M_1 = \mathrm{Rad}(\pounds) = \{1\}$. Hence $b \vee x \in M_{b \vee x} = \{1\}$ which implies that $(1 :_{\pounds} a) \subseteq (1 :_{\pounds} b)$; thus $b \in F$, as $F$ is Baer Filter. Therefore $F$ is a $z$-filter.

(2) $\Rightarrow$ (1). Suppose that every Baer filter is a $z$-filter; so $\{1\}$ is a $z$-filter which gives $\mathrm{Rad}(\pounds) = M_1 \subseteq \{1\}$ and hence $\mathrm{Rad}(\pounds) = \{1\}$. Thus $\pounds$ is semisimple. □

## 4. Further results

This section is devoted to the relation between Baer filters and prime filters. Let us begin the following proposition.

**Proposition 4.1.** *For a lattice $\pounds$ the following statements hold:*

(1) *If $F$ is a filter, $\mathbf{p}$ is a prime filter of $\pounds$ and $F \cap \mathbf{p}$ is a Baer filter, then either $F$ or $\mathbf{p}$ is a Baer filter;*

(2) *If $\mathbf{p}$ and $\mathbf{q}$ are prime filters of $\pounds$ which do not belong to a chain,*

*then* $\mathbf{p}$ *and* $\mathbf{q}$ *are both Baer filters if and only if* $\mathbf{p}\cap\mathbf{q}$ *is a Baer filter;*

(3) *If* $F$ *is a filter,* $\mathbf{m}$ *is a maximal filter of* $\pounds$ *such that* $F \nsubseteq \mathbf{m}$*, then* $F$ *and* $\mathbf{m}$ *are both Baer filters if and only if* $F \cap \mathbf{m}$ *is a Baer filter.*

*Proof.* (1). If $F \subseteq \mathbf{p}$, then $\mathbf{p} \cap F = F$ is a Baer filter. So we may assume that there exists $x \in F$ with $x \notin \mathbf{p}$. Let $(1 :_\pounds p) \subseteq (1 :_\pounds y)$ for some $p \in \mathbf{p}$ and $y \in \pounds$. Then $(1 :_\pounds x \vee p) \subseteq (1 :_\pounds x \vee y)$ and $p \vee x \in \mathbf{p} \cap F$ gives $x \vee y \in \mathbf{p} \cap F$, as $\mathbf{p} \cap F$ is a Baer filter which implies that $y \in \mathbf{p}$. Thus $\mathbf{p}$ is a Baer filter.

(2). We need only prove the converse. Assume that $\mathbf{q} \nsubseteq \mathbf{p}$ (so there exists $x \in \mathbf{q}$ with $x \notin \mathbf{p}$) and let $(1 :_\pounds p) \subseteq (1 :_\pounds y)$ for some $p \in \mathbf{p}$ and $y \in \pounds$. Then $(1 :_\pounds x \vee p) \subseteq (1 :_\pounds x \vee y)$ and $p \vee x \in \mathbf{p} \cap \mathbf{q}$ gives $x \vee y \in \mathbf{q} \cap \mathbf{p} \subseteq \mathbf{p}$, as $\mathbf{q} \cap \mathbf{p}$ is a Baer filter; hence $y \in \mathbf{p}$. Consequently, $\mathbf{p}$ is a Baer filter and so is $\mathbf{q}$ via similar argument.

(3). Since $\mathbf{m} \subsetneqq \mathbf{m} \wedge F \subseteq \pounds$, we have $F \wedge \mathbf{m} = \pounds$. Now the assertion follows from Theorem 2.16. □

An element $x$ of $\pounds$ is called *identity join* of a lattice $\pounds$, if there exists $1 \neq y \in \pounds$ such that $x \vee y = 1$. An element $x$ of $\pounds$ is called *zero-divisor* of a lattice $\pounds$, if there exists $0 \neq y \in \pounds$ such that $x \wedge y = 0$. The set of all identity joins of a lattice $\pounds$ is denoted $\mathrm{I}(\pounds)$ and the set of all zero-divisors of $\pounds$ is denoted $Z(\pounds)$.

**Lemma 4.2.** *If* $\{1\} \neq \mathbf{p}$ *is a prime filter of* $\pounds$ *with* $(1 :_\pounds \mathbf{p}) \neq \{1\}$*, then* $\mathbf{p} \subseteq \mathrm{Id}(\pounds)$*.*

*Proof.* By [7, Proposition 2.2 (iv)], $\mathbf{p} = (1 :_\pounds (1 :_\pounds \mathbf{p}))$. This implies that $\mathbf{p} \subseteq \mathrm{Id}(\pounds)$. □

Following the concept of classical rings (see [13, 3]), we define classical lattices as follows:

**Definition 4.3.** A lattice $\pounds$ is called [*classical* if $\pounds = \mathrm{I}(\pounds) \cup Z(\pounds)$.

The following theorem shows that: when is every prime filter of $\pounds$ a Baer filter? (Compare the next theorem with Proposition 3.2 in [16]).

**Theorem 4.4.** *For a lattice* $\pounds$ *the following statements are equivalent:*

(1) *Every prime filter of* $\pounds$ *is a Baer filter;*

(2) *Every filter of* $\pounds$ *is a Baer filter;*

(3) *For each* $x \in \pounds$*,* $T(\{x\})$ *is a Bear filter;*

(4) *$\pounds$ is a classical lattice and for each $x, y \in \pounds$, $(1 :_\pounds x) \subseteq (1 :_\pounds y)$ implies $y \in T(\{x\})$.*

*Proof.* (1) $\Rightarrow$ (2). Let $F$ be a filter of $\pounds$. Then $F = \bigcap_{F\subseteq \mathbf{p}} \mathbf{p}$ by [6, Lemma 2.6 (ii)]; hence $F$ is a Baer filter of $\pounds$ by (1).

The implication (2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (4). Let $x$ be an arbitrary element of $\pounds$ such that $x \neq 0, 1$. If $x \notin Z(\pounds)$, then there exists a non-zero element $y$ of $\pounds$ such that $x \wedge y \neq 0$; so $x \wedge y \neq 1$. If $T(\{x \wedge y\}) = \pounds$, then $0 = (x \wedge y) \vee s$ for some $s \in \pounds$ gives $x \wedge y = 0$, a contradiction. Thus $T(\{x \wedge y\}) \neq \pounds$. If $(1 :_\pounds x \wedge y) = \{1\}$, then for each $z \in \pounds$, we have $(1 :_\pounds x \wedge y) \subseteq (1 :_\pounds z)$ and hence $z \in T(\{x \wedge y\})$. Therefore, $T(\{x \wedge y\}) = \pounds$, a contradiction. Thus $(1 :_\pounds x \wedge y) \neq \{1\}$. Let $1 \neq a \in (1 :_\pounds x \wedge y)$. Then $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y) = 1$ gives $a \vee x = 1$ which implies that $x \in \mathrm{I}(\pounds)$. Thus $\pounds$ is a classical lattice. Let $x, y \in \pounds$ such that $(1 :_\pounds x) \subseteq (1 :_\pounds y)$. By assumption, $T(\{x\})$ is a Baer filter; hence $y \in T(\{x\})$.

(4) $\Rightarrow$ (1). Suppose that $\mathbf{p}$ is a prime filter of $\pounds$ and let $p \in \mathbf{p}$. We claim that $(1 :_\pounds p) \neq \{1\}$. Otherwise, for each $z \in \pounds$, we have $(1 :_\pounds p) \subseteq (1 :_\pounds z)$ and hence $z \in T(\{p\})$. Therefore, $T(\{p\}) = \pounds \subseteq \mathbf{p}$, a contradiction. Thus $p \in \mathrm{I}(\pounds)$ and so $\mathbf{p} \subseteq \mathrm{I}(\pounds)$. Let $(1 :_\pounds p) \subseteq (1 :_\pounds x)$ for some $p \in \mathbf{p}$ and $x \in \pounds$. By assumption, $x \in T(\{p\}) \subseteq \mathbf{p}$, as needed. □

The following theorem is a lattice counterpart of Theorem 3.1 in [16] describing the structure of maximal ideals of a classical ring.

**Theorem 4.5.** *For a lattice $\pounds$ the following statements are equivalent:*

(1) *$\pounds$ is a classical lattice such that for every finitely generated filter $F \subseteq \mathrm{I}(\pounds)$, $(1 :_\pounds F) \neq \{1\}$;*

(2) *Every maximal filter of $\pounds$ is a Baer filter.*

*Proof.* (1) $\Rightarrow$ (2). Suppose that $\mathbf{m}$ is a maximal filter of $\pounds$. We claim that $\mathbf{m} \subseteq \mathrm{I}(\pounds)$. Assume to the contrary, that there is a $x \in \mathbf{m}$ such that $x \notin \mathrm{I}(\pounds)$. By assumption, there exists a non-zero element $y \notin \mathbf{m}$ such that $x \wedge y = 0$. Then $T(\{y\}) \wedge \mathbf{m} = \pounds$ gives $x = m \wedge (y \vee s)$ for some $m \in \mathbf{m}$ and $s \in \pounds$ which implies that $y \vee s \in \mathbf{m}$ by Lemma 1.1. Then $0 = x \wedge y = m \wedge y \wedge (y \vee s) = m \wedge (y \vee s) = x$, a contradiction. Thus $\mathbf{m} \subseteq \mathrm{I}(\pounds)$. Set

$$G = \{x \in \pounds : (1 :_\pounds A) \subseteq (1 :_\pounds x) \text{ for some finite subset } A \text{ of } \mathbf{m}\}.$$

If $x \in \mathbf{m}$, then $(1 :_{\pounds} x) \subseteq (1 :_{\pounds} x)$ gives $\mathbf{m} \subseteq G$. We claim that $G$ is a proper Baer filter. Let $x, y \in G$ and $a \in \pounds$. Then there are two finite subsets $A$ and $B$ of $\mathbf{m}$ such that $(1 :_{\pounds} A) \subseteq (1 :_{\pounds} x)$ and $(1 :_{\pounds} B) \subseteq (1 :_{\pounds} y)$. Hence,

$$(1 :_{\pounds} A \wedge B) \subseteq (1 :_{\pounds} A) \cap (1 :_{\pounds} B) \subseteq (1 :_{\pounds} x) \cap (1 :_{\pounds} y) \subseteq (1 :_{\pounds} x \wedge y)$$

and $(1 :_{\pounds} A) \subseteq (1 :_{\pounds} x) \subseteq (1 :_{\pounds} x \vee a)$ gives $x \wedge y, x \vee a \in G$. Thus $G$ is a filter of $\pounds$. Let $(1 :_{\pounds} g) \subseteq (1 :_{\pounds} z)$ for some $g \in G$ and $z \in \pounds$. By assumption, there exists a finite subset $H$ of $\mathbf{m}$ such that $(1 :_{\pounds} H) \subseteq (1 :_{\pounds} g)$. Therefore $(1 :_{\pounds} H) \subseteq (1 :_{\pounds} g) \subseteq (1 :_{\pounds} z)$ and hence $z \in G$. So $G$ is a Baer filter. If $y \in G$, then $\{1\} \neq (1 :_{\pounds} T(A)) \subseteq (1 :_{\pounds} A) \subseteq (1 :_{\pounds} y)$ for some finite subset $A$ of $\mathbf{m}$ which implies that $y \in \mathrm{I}(\pounds)$ and so $G \subseteq \mathrm{I}(\pounds)$. Thus $G$ is a proper filter and so by maximality of $\mathbf{m}$ we have $G = \mathbf{m}$ is a Baer filter.

(2) $\Rightarrow$ (1). Let $c \notin Z(\pounds)$. Then there exists a maximal filter $\mathbf{m}'$ of $\pounds$ such that $c \in T(\{c\}) \subseteq \mathbf{m}'$ by Lemma 2.1. If $m \in \mathbf{m}'$, then $(1 :_{\pounds} m) \neq \{1\}$ (otherwise, $T(\{m\}) = \pounds \subseteq \mathbf{m}'$, a contradiction since $\mathbf{m}'$ is a Baer filter) gives $\mathbf{m}' \subseteq \mathrm{I}(\pounds)$ by Lemma 4.2 and so $c \in \mathrm{I}(\pounds)$. Thus $\pounds$ is a classical lattice. Let $H$ be a finitely generated filter of $\pounds$ such that $H \subseteq \mathrm{I}(\pounds)$. Then there is a maximal filter $\mathbf{Q}$ of $\pounds$ such that $H \subseteq \mathbf{Q}$. It follows that $(1 :_{\pounds} H) \neq \{1\}$, as $\mathbf{Q}$ is a Baer filter. This completes the proof. $\square$

Compare the next theorem with Theorem 3.2 in [16]).

**Theorem 4.6.** *For a lattice $\pounds$ the following statements are equivalent:*

(1) *Every prime Baer filter of $\pounds$ is either a minimal prime or a maximal filter;*

(2) *For each maximal filter $\mathbf{m}$ of $\pounds$ and each $m, n \in \mathbf{m}$, there exists a finite subset $A \subseteq (1 :_{\pounds} m)$ and $d \notin \mathbf{m}$ such that $(1 :_{\pounds} T(A \cup \{m\})) \subseteq (1 :_{\pounds} d \vee n)$.*

*Proof.* (1) $\Rightarrow$ (2) Assume to the contrary, that there exists a maximal filter $\mathbf{m}$ of $\pounds$ and $m, n \in \mathbf{m}$ such that $(1 :_{\pounds} T(A \cup \{m\})) \nsubseteq (1 :_{\pounds} n \vee d)$ for each $d \notin \mathbf{m}$ and each finite subset $A \subseteq (1 :_{\pounds} m)$. Set $S = \{n \vee c : c \notin \mathbf{m}\} \cup \{0\}$,

$$G = \{x \in \pounds : (1 :_{\pounds} T(A \cup \{m\})) \subseteq (1 :_{\pounds} x), \text{ where } A \subseteq (1 :_{\pounds} m) \text{ is finite}\}.$$

Let $x, y \in G$ and $a \in \pounds$. Then there are two finite subsets $A$ and $B$ of $(1 :_{\pounds} \mathbf{m})$ such that $(1 :_{\pounds} T(A \cup \{m\}) \subseteq (1 :_{\pounds} x)$ and $(1 :_{\pounds} T(B \cup \{m\}) \subseteq (1 :_{\pounds} y)$. Hence,

$$(1 :_{\pounds} T(A \cup B \cup \{m\}) \subseteq (1 :_{\pounds} T(A \cup \{m\}) \cap (1 :_{\pounds} T(B \cup \{m\})$$

$$\subseteq (1 :_£ x) \cap (1 :_£ y) \subseteq (1 :_£ x \wedge y)$$

and $(1 :_£ T(A \cup \{m\}) \subseteq (1 :_£ x) \subseteq (1 :_£ x \vee a)$ gives $x \wedge y, x \vee a \in G$. Thus $G$ is a filter of $£$. Let $(1 :_£ g) \subseteq (1 :_£ z)$ for some $g \in G$ and $z \in £$. By assumption, there exists a finite subset $C$ of $(1 :_£ m)$ such that $(1 :_£ T(C \cup \{m\}) \subseteq (1 :_£ g) \subseteq (1 :_£ z)$; so $z \in G$ which implies that $G$ is a Baer filter. Clearly, $S$ is a join closed subset of $£$. If $s \in S \cap G$, then $s = n \vee t$ for some $t \notin \mathbf{m}$ and there exists a finite subset $D$ of $(1 :_£ m)$ such that $(1 :_£ T(C \cup \{m\}) \subseteq (1 :_£ n \vee t)$ which is a contradiction. Thus $G \cap S = \emptyset$. Then there exists a $\mathbf{p} \in \min(G)$ such that $\mathbf{p} \cap S = \emptyset$ by [6, Lemma 2.6 (i)]. Moreover, by Proposition 2.12, $\mathbf{p}$ is a Baer filter. Since $(1 :_£ T(A \cup \{m\})) \subseteq (1 :_£ m)$, $m \in G \subseteq \mathbf{p}$. Then by Proposition 2.11, there exists $d \notin \mathbf{p}$ such that $m \vee d = 1$ which implies that $\{d\} \subseteq (1 :_£ m)$. On the other hand $(1 :_£ T(\{d, m\})) \subseteq (1 :_£ d)$. Thus $d \in G \subseteq \mathbf{p}$ which is a contradiction, i.e. (2) holds.

(2) $\Rightarrow$ (1). Let $\mathbf{p}$ be a prime Baer filter of $£$. By Lemma 2.1, there exists a maximal filter $\mathbf{q}$ of $£$ such that $\mathbf{p} \subseteq \mathbf{q}$. If $\mathbf{p} = \mathbf{q}$, then we are done. So we may assume that $\mathbf{p} \neq \mathbf{q}$. Suppose that $\mathbf{p}$ is neither maximal nor minimal prime. By Proposition 2.11, there exists $p \in \mathbf{p}$ such that $p \vee c \neq 1$ for each $c \in £ \setminus \mathbf{p}$. Suppose that $q \in \mathbf{q}$ such that $q \notin \mathbf{p}$. Thus $(1 :_£ p) \cap (£ \setminus \mathbf{p}) = \emptyset$ which implies that $(1 :_£ p) \subseteq \mathbf{p}$. Now by assumption, there exists a finite subset $A$ of $(1 :_£ p)$ and $d \in £ \setminus \mathbf{q}$ such that $(1 :_£ T(A \cup \{p\})) \subseteq (1 :_£ q \vee d)$. Then $T(A \cup \{p\}) \subseteq \mathbf{p}$ and $\mathbf{p}$ is a Baer filter gives $q \vee d \in \mathbf{p}$; hence either $d \in \mathbf{p}$ or $q \in \mathbf{p}$, a contradiction, i.e. (1) holds. □

Compare the next theorem with Theorem 3.3 in [16].

**Theorem 4.7.** *For a lattice $£$ the following statements are equivalent:*

(1) *Every prime Baer filter of $£$ is a minimal prime filter;*

(2) *For each $a \in £$, there exists a finitely generated filter $F$ such that $F \subseteq (1 :_£ a)$ and $(1 :_£ T(F \cup \{a\})) = \{1\}$.*

*Proof.* (1) $\Rightarrow$ (2). Let $a \in £$. If $(1 :_£ a) = \{1\}$, then $(1 :_£ T(\{1\} \cup \{a\})) = \{1\}$. So we may assume that $(1 :_£ a) \neq \{1\}$. Set $G = T(\{a\} \cup (1 :_£ a))$. We claim that there exists a finite subset $A$ of $G$ such that $(1 :_£ A) = \{1\}$. To the contrary assume that for each finite subset $A$ of $G$, $(1 :_£ A) \neq \{1\}$. Set $H = \{x \in £ : (1 :_£ A) \subseteq (1 :_£ x)$ for some finite subset $A \subseteq G\}$.

Let $x, y \in H$ and $u \in £$. So there exist two finite subsets $A, B$ of $G$ such that $(1 :_£ A) \subseteq (1 :_£ x)$ and $(1 :_£ B) \subseteq (1 :_£ y)$. Then

$$(1 :_£ A \wedge B) \subseteq (1 :_£ A) \cap (1 :_£ B) \subseteq (1 :_£ x) \cap (1 :_£ y) \subseteq (1 :_£ x \wedge y)$$

and $(1:_{\pounds} A) \subseteq (1:_{\pounds} x) \subseteq (1:_{\pounds} x \vee u)$; hence $x \wedge y, x \vee u \in H$. Let $(1:_{\pounds} h) \subseteq (1:_{\pounds} z)$ for some $h \in H$ and $z \in \pounds$. Then there exists a finite subset $C$ of $G$ such that $(1:_{\pounds} C) \subseteq (1:_{\pounds} c) \subseteq (1:_{\pounds} z)$; hence $z \in H$. Thus $H$ is a Baer filter. Let $\mathbf{p}$ be a minimal prime filter over $H$. By Proposition 2.12, $\mathbf{p}$ is a Baer filter; so $\mathbf{p}$ is a minimal prime filter of $\pounds$ by (1). Since $\{a\} \subseteq G$ and $(1:_{\pounds} a) \subseteq (1:_{\pounds} a)$, $a \in H \subseteq \mathbf{p}$. Moreover, if $b \in (1:_{\pounds} a)$, then $\{b\} \subseteq (1:_{\pounds} a) \subseteq G$ and $(1:_{\pounds} b) \subseteq (1:_{\pounds} b)$ gives $(1:_{\pounds} a) \subseteq \mathbf{p}$. Now by Proposition 2.11, there exists $c \in \pounds \setminus \mathbf{p}$ such that $c \vee a = 1$ which implies that $c \in (1:_{\pounds} a) \subseteq \mathbf{p}$, a contradiction. Hence there is a finite subset $A = \{a_1, a_2, \cdots, a_k\}$ of $G$ such that $(1:_{\pounds} A) = \{1\}$. Assume that for each $1 \leqslant i \leqslant k$, $a \wedge b_i \leqslant a_i$ (so $a_i = (a_i \vee a) \wedge (a_i \vee b_i)$, where $b_i \in (1:_{\pounds} a)$. Set $F = T(\{b_1, b_2, \cdots, b_k\}) \subseteq (1:_{\pounds} a)$. It remains to show that $(1:_{\pounds} T(F \cup \{a\})) = \{1\}$. Then for each $1 \leqslant i \leqslant k$,

$$\begin{aligned}(1:_{\pounds} b_i) \cap (1:_{\pounds} a) &\subseteq (1:_{\pounds} a \vee a_i) \cap (1:_{\pounds} a_i \vee b_i) = (1:_{\pounds} (a_i \vee a) \wedge (a_i \vee b_i)) \\ &= (1:_{\pounds} a_i).\end{aligned}$$

This implies that

$$\begin{aligned}(1:_{\pounds} T(F \cup \{a\})) &\subseteq (1:_{\pounds} F \cup \{a\}) = \cap_{i=1}^{k} (1:_{\pounds} b_i) \cap (1:_{\pounds} a) \subseteq \cap_{i=1}^{k} a_i \\ &= (1:_{\pounds} A) = \{1\}.\end{aligned}$$

(2) $\Rightarrow$ (1). Let $\mathbf{p}$ be a prime Baer filter and $a \in \mathbf{p}$. By (2), there exits a finitely generated filter $F = T(A)$ of $\pounds$ such that $F \subseteq (1:_{\pounds} a)$ and $(1:_{\pounds} T(F \cup \{a\})) = \{1\}$, where $A$ is a finite set. We claim that $A \cup \{a\} \nsubseteq \mathbf{p}$. Otherewise, for each $y \in \pounds$, $\{1\} = (1:_{\pounds} A \cup \{a\}) \subseteq (1:_{\pounds} y)$ gives $y \in \mathbf{p}$, as $\mathbf{p}$ is a Baer filter of $\pounds$, a contradiction. Hence there exists $z \in A \subseteq (1:_{\pounds} a)$ such that $z \notin \mathbf{p}$ and $z \vee a = 1$. Therefor by Proposition 2.11, $\mathbf{p}$ is a minimal prime filter. □

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