QUADRATIC DIFFERENTIAL SYSTEMS WITH THE LINE AT INFINITY OF MAXIMAL MULTIPLICITY

Alexandru Şubă, *prof. univ., dr. hab.,*
IMI „Vladimir Andrunachievici",
Universitatea Pedagogică de Stat „Ion Creangă" din Chişinău

SISTEME DIFERENŢIALE PĂTRATICE CE AU LINIA DE LA INFINIT DE MULTIPLICITATE MAXIMALĂ

Alexandru Şubă, *Dr. Hab., Prof.,*
IMI „Vladimir Andrunachievici",
"Ion Creanga" State Pedagogical University of Chisinau
ORCID:0000-0003-3489-9619
alexandru.suba@math.md

CZU: 517.925
DOI: 10.46727/c.v3.24-25-03-2023.p306-310

Rezumat. În lucrarea de faţă se arată că în clasa sistemelor diferenţiale pătratice multiplicitatea maximală a liniei de la infinit este egală cu cinci, iar în cazurile când aceste sisteme au puncte critice cu rădăcinile ecuaţiei caracteristice pur imaginare multiplicitatea liniei de la infinit este egală cu trei şi punctele considerate sunt de tip centru.
Cuvinte-cheie: sistem diferenţial pătratic, problema centrului, dreaptă invariantă, multiplicitate.

Abstract. In this paper we show that in the class of quadratic differential systems the maximal multiplicity of the line at infinity is five. In the cases when these systems have critical points with purely imaginary eigenvalues the maximal multiplicity of the line at infinity is three and the considered critical points are of the center type.
Keywords: quadratic differential system, center problem, invariant straight line, multiplicity.

1. Introduction

We consider the real polynomial differential system

$$\dot{x} = p(x,y), \quad \dot{y} = q(x,y), \qquad (1)$$

where $\dot{x} = \frac{dx}{dt}, \dot{y} = \frac{dy}{dt}$, and $\mathcal{X}$ is the vector field $\mathcal{X} = p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y}$ associated to systems (1).

Denote $n = \max\{\deg(p), \deg(q)\}$. If $n = 2$ (respectively, $n = 3, n = 4, n = 5$), then the system (1) is called *quadratic* (respectively, *cubic, quartic, quintic*).

Definition 1. *An algebraic curve $f(x,y) = 0, f \in \boldsymbol{C}[x,y]$, is said to be an invariant algebraic curve of (1) if there exists a polynomial $K_f \in \boldsymbol{C}[x,y]$, such that the identity $\mathcal{X}(f) = f(x,y)K_f(x,y)$ holds.*

In particular, a straight line $L \equiv \alpha x + \beta y + \gamma = 0, \alpha, \beta, \gamma \in \boldsymbol{C}$ is called *invariant* for system (1) if there exists a polynomial $K_L \in \boldsymbol{C}[x,y]$ such that the identity holds

$$\alpha p(x,y) + \beta q(x,y) \equiv (\alpha x + \beta y + \gamma)K_L(x,y), (x,y) \in \boldsymbol{R}^2.$$

Definition 2 [1]. *An invariant straight line L has (algebraic) multiplicity $m(L)$ if $m(L)$ is the greatest positive integer such that $L^{m(L)}$ divides $E(\mathcal{X})$, where*

$$E(\mathcal{X}) = p \cdot \mathcal{X}(q) - q \cdot \mathcal{X}(p).$$

Let $P(x,y,Z), Q(x,y,Z)$ be the homogenized polynomials of $p(x,y), q(xy)$, respectively, and denote $\mathcal{X}_\infty = P(x,y,Z)\frac{\partial}{\partial x} + Q(x,y,Z)\frac{\partial}{\partial y}$.

Definition 3. *We say that the line at infinity $Z = 0$ has (algebraic) multiplicity $\nu + 1$ if ν is the greatest positive integer such that Z^νdivides $E_\infty(\mathcal{X}_\infty)$, where*

$$E_\infty(\mathcal{X}_\infty) = P \cdot \mathcal{X}_\infty(Q) - Q \cdot \mathcal{X}_\infty(P).$$

Denote by $L_\infty \equiv Z = 0$ the line at infinity and by $m(L_\infty)$ the multiplicity of L_∞.

The cubic, quartic and quintic differential systems with the multiple invariant straight lines (including the line at infinity) was investigated in [2-13]. In this paper the quadratic differential systems with the line at infinity are classified and for these systems the problem of the center is solved.

2. The maximal multiplicity of the line at infinity in the class of quadratic systems

Consider the quadratic differential system of the general form

$$\begin{cases} \dot{x} = a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 \equiv p(x,y), \\ \dot{y} = b_0 + b_1x + b_2y + b_3x^2 + b_4xy + b_5y^2 \equiv q(x,y). \end{cases} \quad (2)$$

Suppose that

$$\gcd(p,q) = 1 \text{ and } -b_3x^3 + (a_3 - b_4)x^2y + (a_4 - b_5)xy^2 + a_5y^3 \not\equiv 0. \quad (3)$$

For system (2) the polynomial $E_\infty(\mathcal{X}_\infty)$ look as

$$E_\infty(\mathcal{X}_\infty) = C_2 + C_3Z + C_4Z^2 + C_5Z^3 + C_6Z^4 + C_7Z^5,$$

where

$$C_2 = \big((a_4b_3 - a_3b_4)x^2 + (a_5b_3 - a_3b_5)x\,y + (a_5b_4 - a_4b_5)y^2\big)(-b_3x^3 + (a_3 - b_4)x^2y + (a_4 - b_5)xy^2 + a_5y^3),$$

$$\begin{aligned} C_3 = {} & (a_1a_3b_3 - a_3^2b_1 - 2a_4b_1b_3 + a_3b_2b_3 - a_2b_3^2 + a_3b_1b_4 + a_1b_3\,b_4)x^4 + \\ & +(2a_2a_3b_3 - a_3a_4b_1 - 2a_3^2b_2 + 2a_1a_4b_3 - 4a_5b_1b_3 - a_4b_2b_3 - a_1a_3b_4 - a_4b_1b_4 + \\ & a_3b_2b_4 - a_2b_3b_4 + a_1b_4^2 + 2a_3b_1b_5 + 2a_1b_3b_5)x^3y - \\ & 3(a_3a_4b_2 - a_2a_4b_3 - a_1a_5b_3 + a_5b_2b_3 + a_5b_1b_4 + a_1a_3b_5 - a_3b_2b_5 - a_1b_4b_5)x^2y^2 \\ & + (a_4a_5b_1 - a_4^2b_2 - 2a_3a_5b_2 + 4a_2a_5b_3 + a_2a_4b_4 - \\ & 2a_5b_2b_4 - 2a_2a_3b_5 - 2a_1a_4b_5 - 2a_5b_1b_5 + a_4b_2b_5 + a_2b_4b_5 + 2a_1b_5^2)xy^3 + \\ & (a_5^2b_1 - a_4a_5b_2 + 2a_2a_5b_4 - a_2a_4b_5 - a_1a_5b_5 - a_5b_2b_5 + a_2b_5^2)y^4, \end{aligned}$$

$$\begin{aligned} C_4 = {} & (-2a_3^2b_0 - a_1a_3b_1 - a_4b_1^2 + a_3b_1b_2 + a_1^2b_3 + 2a_0a_3b_3 - 2a_4b_0b_3 - 2a_2b_1b_3 \\ & + a_1b_2b_3 + a_3b_0b_4 + a_1b_1b_4 + a_0b_3b_4)x^3 + \\ & (-3a_3a_4b_0 - 2a_5b_1^2 - 3a_1a_3b_2 - a_4b_1b_2 + a_3b_2^2 + 3a_1a_2b_3 + 3a_0a_4b_3 - 4a_5b_0b_3 \\ & - a_2b_2b_3 - a_4b_0b_4 - a_2b_1b_4 + 2a_1b_2b_4 + a_0b_4^2 + 2a_3b_0b_5 + 2a_1b_1b_5 \\ & + 2a_0b_3b_5)x^2y + \end{aligned}$$

$$(-a_4^2b_0 - 2a_3a_5b_0 + a_2a_4b_1 + a_1a_5b_1 - 2a_2a_3b_2 - 2a_1a_4b_2 - 3a_5b_1b_2 + 2a_2^2b_3 + 4a_0a_5b_3 + a_1a_2b_4 + a_0a_4b_4 - 3a_5b_0b_4 - a_1^2b_5 - 2a_0a_3b_5 + 3a_1b_2b_5 + 3a_0b_4b_5)xy^2 + (-a_4a_5b_0 + 2a_2a_5b_1 - a_2a_4b_2 - a_1a_5b_2 - -a_5b_2^2 + a_2^2b_4 + 2a_0a_5b_4 - a_1a_2b_5 - a_0a_4b_5 - 2a_5b_0b_5 + a_2b_2b_5 + 2a_0b_5^2)y^3,$$

$$C_5 = (-3a_1a_3b_0 - 2a_4b_0b_1 - a_2b_1^2 + a_3b_0b_2 + a_1b_1b_2 + 3a_0a_1b_3 - 2a_2b_0b_3 + a_0b_2b_3 + a_1b_0b_4 + a_0b_1b_4)x^2 + (-2a_2a_3b_0 - 2a_1a_4b_0 + a_1a_2b_1 + a_0a_4b_1 - 4a_5b_0b_1 - a_1^2b_2 - 2a_0a_3b_2 - a_4b_0b_2 - a_2b_1b_2 + a_1b_2^2 + 4a_0a_2b_3 + a_0a_1b_4 - a_2b_0b_4 + 2a_0b_2b_4 + 2a_1b_0b_5 + 2a_0b_1b_5)xy + (-a_2a_4b_0 - a_1a_5b_0 + a_2^2b_1 + 2a_0a_5b_1 - a_1a_2b_2 - a_0a_4b_2 - 3a_5b_0b_2 + 2a_0a_2b_4 - a_0a_1b_5 + 3a_0b_2b_5)y^2,$$

$$C_6 = (-a_1^2b_0 - 2a_0a_3b_0 - a_4b_0^2 + a_0a_1b_1 - 2a_2b_0b_1 + a_1b_0b_2 + a_0b_1b_2 + 2a_0^2b_3 + a_0b_0b_4) + (-a_1a_2b_0 - a_0a_4b_0 - 2a_5b_0^2 + 2a_0a_2b_1 - a_0a_1b_2 - a_2b_0b_2 + a_0b_2^2 + a_0^2b_4 + 2a_0b_0b_5)y,$$

$$C_7 = -a_0a_1b_0 - a_2b_0^2 + a_0^2b_1 + a_0b_0b_2.$$

Taking into account (3) and solving the system of identities $\{C_2 \equiv 0,\ C_3 \equiv 0,\ C_4 \equiv 0, C_5 \equiv 0\}$, we obtain the following two solutions

$$a_1 = a_2 = a_3 = a_4 = a_5 = b_2 = b_4 = b_5 = 0,\ \ a_0b_3 \neq 0; \qquad (4)$$
$$a_1 = a_3 = a_4 = b_1 = b_2 = b_3 = b_4 = b_5 = 0,\ \ a_5b_0 \neq 0. \qquad (5)$$

Remark 1. *The transformation of coordinates $x \to y, y \to x$ reduces the system $\{(2),(5)\}$ to the system $\{(2),(4)\}$.*

Under the conditions (4) the polynomial C_6 yields $C_6 = 2a_0^2b_3x \neq 0$. After the substitution $x \to a_0x, y \to a_0^2b_3y,\ \frac{b_0}{a_0^2b_3} = a, \frac{b_1}{a_0b_3} = b$, the system $\{(2),(4)\}$ can by written in the form

$$\dot{x} = 1,\ \dot{y} = a + bx + x^2.\dot{x} = 1,\ \dot{y} = a + bx + x^2. \quad (6)$$

In this way we prove the following theorem.

Theorem 1. *In the class of quadratic systems $\{(2),(3)\}$ the maximal multiplicity of the line at infinity is five. Modulo the affine transformation of coordinates and rescaling the coefficient each quadratic system $\{(2),(3)\}$ with the line at infinity of multiplicity five can be written in the form* (6).

3. The problem of the center for quadratic systems with the line at infinity of maximal multiplicity

We consider the quadratic system of the form

$$\dot{x} = y + ax^2 + cxy + fy^2 \equiv p(x,y),\ \dot{y} = -(x + gx^2 + dxy + by^2) \equiv q(x,y). \quad (7)$$

The critical point $(0,0)$ of system (7) is either a focus or a center. The problem of distinguishing between a center and a focus is called *the problem of the center*. It is well known that $(0,0)$ is a center for system (7) if the system has an axis of symmetry or an analytical integrating factor of the form $\mu(x,y)$ in a neighborhood of $(0,0)$.

In the case of system (7) the inequalities (3) look as

$$\gcd(p,q)=1,\ gx^3+(a+d)x^2y+(b+c)xy^2+fy^3\not\equiv 0 \quad (8)$$

and $E_\infty(\mathcal{X}_\infty)$ is a polynomial of degree three in Z:

$$E_\infty(\mathcal{X}_\infty)=C_2+C_3Z+C_4Z^2+C_5Z^3,$$

where

$$C_2=((ad-cg)x^2+2(ab-fg)xy+(bc-df)y^2)$$
$$(gx^3+(a+d)x^2y+(b+c)xy^2+fy^3),$$
$$C_3=(a^2+ad-2cg-g^2)x^4+(2ab+ac-cd-2ag-dg-4fg)x^3y$$
$$-3(df+cg)x^2y^2+(2ab+bd-cd-2bf-cf-4fg)xy^3$$
$$+(b^2+bc-2df-f^2)y^4,$$
$$C_4=-((c+2g)x+(d+2f)y)(x^2+y^2),$$
$$C_5=-x^2-y^2.$$

Solving in conditions (8) the system of identities $\{C_2\equiv 0, C_3\equiv 0\}$, we obtain the following three solutions:

$$a=b=d=f=g=0, c\neq 0; \qquad (9)$$
$$a=b=c=f=g=0, d\neq 0; \qquad (10)$$
$$c=b-\frac{a^2}{b}, d=a-\frac{b^2}{a}, f=-a, g=-b. \qquad (11)$$

In each set of conditions (9), (10) and (11) the system (7) obtain the form, respectively

$$\dot{x}=y(1+cx),\ \dot{y}=-x, c\neq 0; \qquad (12)$$
$$\dot{x}=y,\ \dot{y}=-x(1+dy), d\neq 0; \qquad (13)$$
$$\dot{x}=y+(bx-ay)(ax+by)/b,$$
$$\dot{y}=-(x+(bx-ay)(ax+by)/a). \qquad (14)$$

Remark 2. *The transformation* $x\to y, y\to x, d\to c$ *reduces the system* (13) *to the system* (12).

Under the conditions (9) (respectively, (10), (11)) the polynomial C_4 look as $C_4=cx\not\equiv 0$ (respectively, $C_4=dx\not\equiv 0, C_4=-(bx-ay)^2(ax+by)/(ab)\not\equiv 0$).

In this way we prove the following theorem.

Theorem 2. *In the class of quadratic systems* $\{(7),(8)\}$ *the maximal multiplicity of the line at infinity is three. Each quadratic system* $\{(2),(3)\}$ *with the line at infinity of multiplicity three has one of the forms* (12), (13), (14).

Remark 3. *For system* (12) (*respectively,* (13), (14))

- the straight line $cx+1=0$ (*respectively,* $dx+1=0, (a^2+b^2)(ax+by)-ab=0$) *is invariant;*
- $\mu=1/(cx+1)$ (*respectively,* $\mu=1/(dx+1),\ \mu=1/((a^2+b^2)(ax+by)-ab)$) *is an integrating factor;*
- *the critical point* $(0,0)$ *is of center type.*

Theorem 3. *Let for quadratic system (2) the eigenvalues of critical point* M_0 *are purely imaginary and the line at infinity is of the maximal multiplicity* $m(L_\infty) = 3$*, Then, this system has: 1) an invariant straight line* $L = 0$*; 2) an integrating factor of the form* $\mu = 1/L$*; 3) an axis of symmetry; 4) a center at* M_0*.*

BIBLIOGRAPHIE

1. CHRISTOPHER, C.; LLIBRE, J.; PEREIRA, J.V. Multiplicity of invariant algebraic curves in polynomial vector fields. *Pacific J. of Math,* 2007, vol. 229, no. 1, 63-117.
2. REPEȘCO, V. Qualitative study of the quartic system with maximal multiplicity of the line of the infinity. *Acta et Commentationes*. *Exact and Natural Sciences*, 2020, no. 2(10), pp.89-96.
3. REPEȘCO, V. Phase portraits of some polynomial differential systems with maximal multiplicity of the line at the infinity. *Acta et Commentationes*. *Exact and Natural Sciences*, 2022, no. 2(14), pp.68-80.
4. ȘUBĂ, A. Center problem for cubic differential systems with the line at infinity of multiplicity four. *Carpathian J. Math*., 2022, vol. 1, 217-222.
5. ȘUBĂ A. Centers of cubic differential systems with the line at infinity of maximal multiplicity. *Acta et Commentationes*. *Exact and Natural Sciences*, 2022, no. 2(14), pp.38-46.
6. ȘUBĂ, A.; TURUTA, S. Solution of the problem of the center for cubic differential systems with the line at infinity and an affine real invariant straight line of total algebraic multiplicity five. *Bulletin of ASM. Mathematics*. 2019, vol. 90, nr. 2, pp. 13-40.
7. ȘUBĂ, A.; TURUTA, S. Cubic differential systems with a weak focus and a real invariant straight line of maximal algebraic multiplicity. *Acta et Commentationes*. *Exact and Natural Sciences*, 2017, no. 2(4), pp.119-130.
8. ȘUBĂ, A.; TURUTA, S. Solution of the center problem for cubic differential systems with one or two affine invariant straight lines of total algebraic multiplicity four. *ROMAI J*., 2020, vol. 15, nr. 2, pp. 101-116.
9. ȘUBĂ, A.; VACARAȘ, O. Cubic differential systems with an invariant straight line of maximal multiplicity. *Annals of the University of Craiova*, Mathematics and Computer Science Series, 2015, v. 42(2), pp. 427--449.
10. ȘUBĂ, A.; VACARAȘ, O. Quartic differential systems with an invariant straight line of maximal multiplicity. *Bulletin of Academy of Sciences of the Republic of Moldova.* Mathematics. 2018, no.1(86), pp. 76-91.
11. ȘUBĂ, A.; VACARAȘ, O. Center problem for cubic differential systems with the line at infinity and an affine real invariant straight line of total multiplicity four. *Bukovinian Math. Journal*, 2021, vol. 9, no. 2, pp. 1-17.
12. TURUTA, S. Solution of the problem of the center for cubic differential systems with three affine invariant straight lines of total algebraic multiplicity four. In: *Bulletin of ASM. Mathematics*. 2020, vol. 92, nr. 1, pp. 89-105.
13. VACARAȘ, O. Maximal multiplicity of the line at infinity for quartic differential systems. *Acta et Commentationes*, *Exact and Natural Sciences*. 2018, no. 2(6), pp. 70-77.