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On entropicity in n-ary semigroups

Sonia Dog

Abstract. We investigate entropicity and the generalized entropic property in n-ary semigroups derived from binary semigroups satisfying for some fixed $k \geqslant 2$ the identity $x^k = x$.

1. We say that an n-ary semigroup (S, f), where $n > 2$, has the *entropic property* or is *entropic* (*medial* in other terminology), if it satisfies the identity

$$f(f(x_{11}, \ldots, x_{1n}), f(x_{21}, \ldots, x_{2n}) \ldots, f(x_{n1}, \ldots, x_{nn})) =$$

$$f(f(x_{11}, \ldots, x_{n1}), f(x_{12}, \ldots, x_{n2}) \ldots, f(x_{1n}, \ldots, x_{nn})).$$

If in (S, f) there exist n-ary terms $t_1, t_2, \ldots, t_n$ such that (S, f) satisfies the identity

$$f(f(x_{11}, \ldots, x_{1n}), f(x_{21}, \ldots, x_{2n}) \ldots, f(x_{n1}, \ldots, x_{nn})) =$$

$$f(t_1(x_{11}, \ldots, x_{n1}), t_2(x_{12}, \ldots, x_{n2}) \ldots, t_n(x_{1n}, \ldots, x_{nn})),$$

then we say that (S, f) has the *generalized entropic property*. These two properties, studied by many authors with various names, are not equivalent in general. The entropicity of n-ary semigroups is a generalization of *mediality*:

$$xy \cdot zu = xz \cdot yu,$$

and *semimediality*:

$$xy \cdot zx = xz \cdot yx$$

of binary algebras (cf. for example [17] or [22]).

The entropicity and the generalized entropicity in idempotent n-ary semigroups were studied in [15]. Below we give very simple (almost trivial) proofs of results given in this paper. We also present some generalizations of these results.

2. We start with some comments on entropic n-ary groups ($n > 2$).

In [11] it is proved that an n-ary group (S, f) is entropic if and only if it is *semiabelian*, i.e., if

$$f(x_1, x_2, \ldots, x_{n-1}, x_n) = f(x_n, x_2, \ldots, x_{n-1}, x_1)$$

for all $x_1, x_2, \ldots, x_n \in S$.

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From [5] (Corollary 15) it follows that an n-ary group (S,f) is entropic if and only if for some $a \in S$ and all $x,y \in S$ we have

$$f(x,a,\ldots,a,y) = f(y,a,\ldots,a,x).$$

Thus, by Gluskin-Hosszú theorem (cf. [12, 13]) any entropic n-ary group (S,f) can be presented in the form

$$f(x_1,x_2,\ldots,x_n) = x_1 \circ \varphi(x_2) \circ \varphi^2(x_3) \circ \cdots \circ \varphi^{n-2}(x_{n-1}) \circ x_n \circ b,$$

where $(S,\circ)$ is an abelian group, φ its automorphism such that $\varphi^{n-1} = id$ and $\varphi(b) = b$ for some fixed element $b \in S$ (cf. [7]). Moreove, as it is proved in [24] (see also [10]), $(A,\circ)$, φ and b are uniquely determined.

3. *Mal'cev n-semigroups*, i.e., n-ary semigroups (S,f) satisfying the identities

$$f(x,y,\ldots,y) = x \quad \text{and} \quad f(y,\ldots,y,x) = x, \tag{1}$$

studied in [15], are in fact n-ary groups. This follows from Proposition 3.1 in [8]. It also can be deduced from results proved in [25]. Hence, Mal'cev n-semigroups (as n-ary groups) are cancellative, i.e.,

$$f(x_1,\ldots,x_{i-1},a,x_{i+1},\ldots,x_n) = f(x_1,\ldots,x_{i-1},b,x_{i+1},\ldots,x_n) \Rightarrow a = b \tag{2}$$

for all $i = 1,\ldots,n$ and $a,b,x_1,\ldots,x_n \in S$.

On the other hand, an n-ary semigroup is cancellative if and only if it satisfies (2) for some $i = 2,3,\ldots,n-1$ or, equivalently, for $i = 1$ and $i = n$ (Lemma 2 in [6]). Hence, in an idempotent i-cancellative n-ary semigroup (S,f) we have

$$f(x,y,\ldots,y) = f(x,y,\ldots,y,f(y,\ldots,y)) = f(f(x,y,\ldots,y),y,\ldots,y),$$

which implies the first identity of (1). Analogously we obtain the second identity. (It is Lemma 3.2 in [15]). Thus an idempotent n-ary semigroup i-cancellative for some $i = 2,\ldots,n-1$ or for $i = 1$ and $i = n$ is an n-ary group satisfying (1). Hence, Proposition 3.3 in [15] is trivial.

As a simple consequence, we obtain Theorem 3.5 from [15]: *a Mal'cev n-semigroup is entropic if and only if it is semiabelian.*

4. By Gluskin-Hosszú theorem, for any ternary Mal'cev semigroup (S,f), as for a ternary group, there exists a group $(S,\cdot)$, its automorphism φ and an element $b \in S$ such that $\varphi(b) = b$ and

$$f(x,y,z) = x \cdot \varphi(y) \cdot \varphi^2(z) \cdot b.$$

Since a ternary Mal'cev semigroup is idempotent, $\varphi(x) \cdot \varphi^2(x) \cdot b = e$. Hence, $\varphi(x) = x^{-1} \cdot b^{-1}$ and $b^{-1} = \varphi(b^{-1}) = e$. Therefore, $b = e$ and $\varphi(x) = x^{-1}$. Thus, for any ternary Mal'cev semigroup (S, f) there is an abelian group $(A, +)$ such that

$$f(x, y, z) = x - y + z.$$

From this, as a simple consequence, we obtain all results proved in Section 4 in [15]. Theorem 4.5 in [15] is a special case of Artamonov's Proposition 6 (cf. [2]). It also can be deduced from the description of free n-ary groups presented by Shchuchkin [14, 18, 19] and Sioson [23].

5. We say that an n-ary semigroup (S, f) is *derived* from a semigroup $(S, \cdot)$ if $f(x_1, x_2, \ldots, x_n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$ for all $x_1, \ldots, x_n \in S$. Obviously, an n-ary semigroup derived from an entropic semigroup is entropic, too. For a *surjective* semigroup, i.e., a semigroup $(S, \cdot)$ with the property $S^2 = S$, we have a stronger result.

Proposition 1. *An n-ary semigroup (S, f) derived from a surjective semigroup $(S, \cdot)$ is entropic if and only if $(S, \cdot)$ is entropic.*

Proof. An entropic n-ary semigroup (S, f) derived from a surjective semigroup $(S, \cdot)$ is semiabelian and each its element can be presented as a multiplication of $n-1$ elements of S. Thus, for any $x, y, a, b \in S$, we have

$$\begin{aligned} xa \cdot by &= x(a_2a_3 \cdots a_n) \cdot (b_2b_3 \cdots b_n)y = x(a_2a_3 \cdots a_nb_2)b_3 \cdots b_ny \\ &= x(b_2a_3 \cdots a_na_2)b_3 \cdots b_ny = xb_2(a_3a_4 \cdots a_na_2b_3)b_4 \cdots b_ny \\ &= xb_2(b_3a_4 \cdots a_na_2a_3)b_4 \cdots b_ny = xb_2b_3(a_4 \cdots a_na_2a_3b_4)b_5 \cdots b_ny \\ &= \ldots = x(b_2b_3 \cdots b_n) \cdot (a_2a_3 \cdots a_n)y = xb \cdot ay, \end{aligned}$$

which completes the proof. □

We say that a semigroup $(S, \cdot)$ is *k-idempotent* ($k > 1$), if for all $x \in S$ we have $x^k = x$. An n-ary semigroup derived from an n-idempotent semigroup is obviously idempotent, but a $(k+1)$-ary semigroup derived from a k-idempotent semigroup is not idempotent. So, results proved for n-ary semigroups derived from k-idempotent semigroups are a significant generalization of results proved in [15].

Lemma 2. *Any k-idempotent semigroup has at least one idempotent.*

Proof. In a 2-idempotent semigroup each element is idempotent. It is clear. For $k > 2$, we have $a^{k-1} = a^ka^{k-2} = a^{k-1}a^{k-1}$, which means that in a k-idempotent semigroup each element a^{k-1} is idempotent. □

Proposition 3. *A k-idempotent semigroup is entropic if and only if it is semimedial.*

Proof. An entropic semigroup is obviously semimedial. To prove the converse statement observe that in a semimedial semigroup

$$(xyzu)^2 = (xy(zu)x)yzu = (x(zu)yx)yzu = x(z(uyx)yz)u = x(zy(uyx)z)u$$
$$= xzy(uy(xz)u) = xzy(u(xz)yu) = (xzyu)^2$$

and

$$(xzyu)(xyzu) = xzy(u(xy)zu) = xz((yu)zx(yu)) = (xzyu)(xzyu) = (xzyu)^2.$$

Thus, in a k-idempotent semigroup

$$xyzu = (xyzu)^k = (xyzu)^2(xyzu)^{k-2} = (xzyu)^2(xyzu)^{k-2} = \ldots = (xzyu)^k$$
$$= xzyu, \qquad \text{if } k \text{ is even.}$$

For $k = 2t+1$ we have

$$xyzu = (xyzu)^{2t+1} = (xyzu)^2(xyzu)^{2t-1} = \ldots = (xzyu)^{2t}(xyzu)$$
$$= (xzyu)^{2t-1}(xzyu)(xyzu) = (xzyu)^{2t-1}(xzyu)^2 = (xzyu)^{2t+1}$$
$$= xzyu,$$

which completes our proof. □

As a consequence, we obtain

Corollary 4. [15, Lemma 6.3] *An n-idempotent semigroup is entropic if and only if it is semimedial.*

Proposition 5. *An n-ary semigroup derived from a k-idempotent semigroup is semiabelian if and only if it is commutative.*

Proof. Let (S, f) be a semiabelian n-ary semigroup derived from a k-idempotent semigroup $(S, \cdot)$. Then for all $x, y \in S$ we have

$$xy = xx^{k-1}y = xx^{k-1}x^{k-1}y = \ldots = xx^{k-1}\ldots x^{k-1}y = f(x, x^{k-1}, \ldots, x^{k-1}y)$$
$$= f(y, x^{k-1}, \ldots, x^{k-1}x) = yx^{k-1}\ldots x^{k-1}x = yx,$$

which means that $(S, \cdot)$ is a commutative semigroup. Consequently, (S, f) is commutative, too.

The converse statement is obvious. □

Corollary 6. [15, Corollary 6.6] *For an n-ary semigroup (S, f) derived from an n-idempotent semigroup $(S, \cdot)$ the following statements are equivalent:*

(a) *(S, f) is semiabelian,*

(b) *$(S, \cdot)$ is commutative,*

(c) *(S, f) is commutative.*

6. Below we present simple proofs of some other results presented in [15]. For this we will use the concept of the covering semigroup.

As is well known (cf. for example [3] or [4]) each n-ary semigroup (S, f) can be isomorphically embedded into some semigroup $(S^*, \cdot)$, called the *covering* or *enveloping semigroup*, in this way that $f(x_1, x_2, \ldots, x_n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$ for all

$x_1, \ldots, x_n \in S \subseteq S^*$. The construction of such semigroup is very similar to the construction of the covering group for an n-ary group (cf. [16]). Unfortunately, as it was observed in [9], two non-isomorphic n-ary semigroups (groups) may have the same covering semigroup (group).

Proposition 7. [15, Proposition 5.1] *An associative and idempotent n-ary operation f satisfying the identities*

$$f(x, \ldots, x, y) = f(y, x, \ldots, x) = f(x, y, x, \ldots, x)$$

or

$$f(x, \ldots, x, y) = f(y, x, \ldots, x) = f(x, \ldots, x, y, x)$$

is commutative.

Proof. Let (S, f) be an idempotent n-ary semigroup satisfying the above identities. Then in its covering semigroup $(S^*, \cdot)$, for every $x, y \in S$, we have $x^n = x$ and $x^{n-1}y = yx^{n-1} = xyx^{n-2}$. So,

$$xy = x^n y = x \cdot x^{n-1}y = x \cdot yx^{n-1} = xyx^{n-2} \cdot x = yx^{n-1} \cdot x = yx^n = yx$$

for every $x, y \in S$. Hence f is a commutative operation. □

An n-ary semigroup (S, f) is called a *left zero n-semigroup* if it satisfies the identity $f(x_1, \ldots, x_n) = x_1$. If it satisfies the identity $f(x_1, \ldots, x_n) = x_n$, then it is called a *right zero n-semigroup*.

Proposition 8. [15, Proposition 5.3] *Let (S, f) be an n-ary semigroup. If (S, f) satisfies the identity $f(x, \ldots, x, y) = x$, then it is a left zero semigroup. If (S, f) satisfies the identity $f(y, x, \ldots, x) = x$, then it is a right zero semigroup.*

Proof. In the covering semigroup of (S, f) for all $x, y \in S$ we have $x^{n-1}y = x$ and $x^n = x$. Thus $xy = x^n y = x \cdot x^{n-1}y = xx$, and consequently,

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &= (x_1x_2)x_3x_4 \cdots x_n = (x_1x_1)x_3x_4 \cdots x_n = x_1(x_1x_3)x_4 \cdots x_n \\ &= x_1x_1x_1x_4 \cdots x_n = \ldots = x_1 \cdots x_1(x_1x_n) = x_1 \end{aligned}$$

for all $x_1, \ldots, x_n \in S$. Hence (S, f) is a left zero semigroup.

The second sentence can be proved analogously. □

7. In the case of an n-ary semigroup (S, f) derived from a binary semigroup $(S, \cdot)$, the generalized entropic property has the form

$$\begin{gathered} (x_{11} \cdot \ldots \cdot x_{1n}) \cdot (x_{21} \cdot \ldots \cdot x_{2n}) \cdot \ldots \cdot (x_{n1} \cdot \ldots \cdot x_{nn}) = \\ t_1(x_{11}, \ldots, x_{n1}) \cdot t_2(x_{12}, \ldots, x_{n2}) \cdot \ldots \cdot t_n(x_{1n}, \ldots, x_{nn}), \end{gathered} \tag{3}$$

where $t_1, t_2, \ldots, t_n$ are some n-ary terms of (S, f).

We start with the following two lemmas which are a generalization of results proved in [15] for idempotent n-ary semigroups derived from binary semigroup containing an idempotent element. We do not assume that considered semigroups are idempotent.

Lemma 9. *If an n-ary semigroup (S, f) derived from a binary semigroup $(S, \cdot)$ with an idempotent e satisfies the generalized entropic property* (3)*, then for every $a \in S$ we have*

$$t_1(e, a, e, \ldots, e) \cdot e = eae = e \cdot t_n(a, e, \cdot, e).$$

Proof. The proof is the same as the proof of Lemma 6.4 in [15]. □

Lemma 10. *If an n-ary semigroup (S, f) derived from a binary semigroup $(S, \cdot)$ with an idempotent e satisfies the generalized entropic property* (3)*, then*

$$eabe = ebeae$$

for all $a, b \in S$.

Proof. The proof is the same as the proof of Lemma 6.5 in [15]. □

Theorem 11. *If an n-ary semgroup (S, f) derived from a k-idempotent semigroup $(S, \cdot)$ has the generalized entropic property, then $(S, \cdot)$ is entropic.*

Proof. For any $a \in S$ consider the set

$$S_a = \{a^{k-1} s a^{k-1} : s \in S\}.$$

It is not difficult to see that $(S_a, \cdot)$ is a semigroup and $e = a^{k-1}$ is its neutral element. So, (S_a, f) is an n-ary subsemigroup of (S, f). Moreover, from the generalized entropic property (3), for all $x_1, \ldots, x_n \in S_a$ we have

$$\begin{aligned} x_1 \cdot x_2 \cdot \ldots \cdot x_n &= (x_1 \cdot e \cdot \ldots \cdot e) \cdot (x_2 \cdot e \cdot \ldots \cdot e) \cdot \ldots \cdot (x_n \cdot e \cdot \ldots \cdot e) \\ &= t_1(x_1, x_2, \ldots, x_n) \cdot t_2(e, e, \ldots, e) \cdot \ldots \cdot t_n(e, e, \ldots, e) \\ &= t_1(x_1, x_2, \ldots, x_n). \end{aligned}$$

Thus $t_1(x_1, x_2, \ldots, x_n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$. Analogously we obtain $t_k(x_1, x_2, \ldots, x_n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$ for other terms $t_2, \ldots, t_n$.

So, in this case, (3) has the form

$$\begin{aligned} &(x_{11} \cdot \ldots \cdot x_{1n}) \cdot (x_{21} \cdot \ldots \cdot x_{2n}) \cdot \ldots \cdot (x_{n1} \cdot \ldots \cdot x_{nn}) = \\ &(x_{11} \cdot \ldots \cdot x_{n1}) \cdot (x_{12} \cdot \ldots \cdot x_{n2}) \cdot \ldots \cdot (x_{1n} \cdot \ldots \cdot x_{nn}), \end{aligned}$$

which, for $x_{1n} \neq e$, $x_{n1} \neq e$ and $x_{ij} = e$ in other cases, gives the commutativity of $(H_a, \cdot)$. Therefore, for all $a, b, c \in S$ we have

$$a^{k-1}b \cdot ca^{k-1} = a^{k-1}ba^{k-1} \cdot a^{k-1}ca^{k-1} = a^{k-1}ca^{k-1} \cdot a^{k-1}ba^{k-1} = a^{k-1}c \cdot ba^{k-1}.$$

So, $a^{k-1}b \cdot ca^{k-1} = a^{k-1}c \cdot ba^{k-1}$. This implies $ab \cdot ca = ac \cdot ba$. Thus $(S, \cdot)$ is semimedial. Proposition 3 completes the proof. □

Corollary 12. [15, Theorem 6.7] *Let (S, f) be an idempotent n-ary semigroup derived from an n-idempotent semigroup $(S, \cdot)$ and assume that (S, f) has the generalized entropic property. Then $(S, \cdot)$ is entropic.*

Theorem 13. *A k-idempotent semigroup has the generalized entropic property if and only if it is entropic.*

Proof. Let $(S, \cdot)$ be a k-idempotent semigroup satisfying the generalized entropic property. Then there are terms t_1 and t_2 such that

$$xy \cdot zu = t_1(x, z)t_2(y, u)$$

for all $x, y, z, u \in S$. By Lemma 2, the set E_S of all idempotents of a semigroup $(S, \cdot)$ is non-empty. Since $ef \cdot ef = t_1(e, e)t_2(f, f) = ef$ for all $e, f \in E_S$, $(E_S, \cdot)$ is a subsemigroup of $(S, \cdot)$. By [1, Proposition 3.11], $(E_S, \cdot)$ is entropic. This completes the proof for $k = 2$ because in this case $S = E_S$.

For $k > 2$, for every $a \in S$ we have $a = axa$ and $ax = xa$, where $x = a^{k-2}$. So, $(S, \cdot)$ is a completely regular semigroup whose idempotents forms a subsemigroup, i.e., a normal band. Hence, in this case, the proof is identical with the second part of the proof of Theorem 6.8 in [15]. Namely, by [20, Theorem 4.1 and Corollary 4.4], a semigroup $(S, \cdot)$ is a normal band of groups. Thus, by [21, Theorem 3.2], it is a subdirect product of a band B and a semilattice L of groups. Since, by the definition of subdirect product, B and L are homomorphic images of a semigroup $(S, \cdot)$, they satisfy all identities satisfied by $(S, \cdot)$. Hence, they satisfy the generalized entropic property. This means that a band B is entropic (see the case $k = 2$). Every group with the generalized entropic property is commutative [1, Proposition 4.7] and a semilattice of commutative groups is commutative. Thus a semilattice L is commutative and hence entropic. Consequently, $(S, \cdot)$, as a subdirect product of entropic B and L, is entropic. □

Corollary 14. [15, Theorem 6.8] *An n-idempotent semigroup $(S, \cdot)$ has the generalized entropic property if and only if it is entropic.*

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E-mail: sonia.dog@sovamua.com