A note on left loops with WA-property

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Abstract. We study properties of WA-quasigroups with a left identity element, i.e., quasigroups satisfying two identities: $xx \cdot yz = xy \cdot xz$ and $xy \cdot zz = xz \cdot yz$.

1. Introduction

We start from some definitions and examples. Other basic facts about quasigroups and loops can be found in [2] and [13].

Definition 1.1. (cf. [5, 8]) A groupoid $(Q, \cdot)$ is called a *quasigroup* if, on the set Q, there exist operations "\" and "/" such that in the algebra $(Q, \cdot, \backslash, /)$ identities

$$x \cdot (x \backslash y) = y, \tag{1}$$

$$(y/x) \cdot x = y, \tag{2}$$

$$x \backslash (x \cdot y) = y, \tag{3}$$

$$(y \cdot x)/x = y, \tag{4}$$

are fulfilled.

Definition 1.2. (cf. [11, 12]) A quasigroup $(Q, \cdot)$ with the identities

$$xx \cdot yz = xy \cdot xz \quad \text{and} \quad xy \cdot zz = xz \cdot yz \tag{5}$$

is called a *WA-quasigroup* or a *semi-medial quasigroup* (shortly: *SM-quasigroup*) (cf. [14, 15]).

Identities (5) are not equivalent.

Example 1.3. This quasigroup satisfies only the first of these identities.

$*$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	3	2	5	4
2	2	4	0	5	1	3
3	4	2	5	0	3	1
4	3	5	1	4	0	2
5	5	3	4	1	2	0

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Lemma 1.4. *In any WA-quasigroup* $(Q,\cdot)$ *the following identities are true:*

$$x^2\backslash(yz) = (x\backslash y)(x\backslash z), \tag{6}$$

$$(yz)/x^2 = (y/x)(z/x), \tag{7}$$

$$x(y\backslash z) = (xy)\backslash(x^2 z), \tag{8}$$

$$(y/z)x = (yx^2)/(zx), \tag{9}$$

where $xy = z \Leftrightarrow x\backslash z = y \Leftrightarrow z/y = x$.

Proof. (6). It is clear that there exists an element z' such that $x^2\backslash yz' = (x\backslash y)(x\backslash z)$. We must prove that $z' = z$. From the definition of the operation $\backslash$ we have

$$yz' = x^2((x\backslash y)(x\backslash z)) \overset{(5)}{=} x(x\backslash y)\cdot x(x\backslash z) \overset{(1)}{=} yz.$$

Therefore, $z' = z$.

(7). It is clear that there exists an element z' such that $yz'/x^2 = (y/x)(z/x)$. We must prove that $z' = z$. From the definition of the operation $/$ we have

$$yz' = (y/x)(z/x)\cdot x^2 \overset{(5)}{=} (y/x)x\cdot(z/x)x \overset{(2)}{=} yz.$$

Therefore, $z' = z$.

(8). It is clear that there exists an element z' such that $x(y\backslash z) = (xy)\backslash(x^2 z')$. We must prove that $z' = z$. We have

$$x^2 z' = (xy)\cdot x(y\backslash z) \overset{(5)}{=} x^2\cdot y(y\backslash z) \overset{(1)}{=} x^2 z.$$

Therefore, $z' = z$.

(9). It is clear that there exists an element y' such that $(y/z)x = (y'x^2)/(zx)$. We must prove that $y' = y$. As in previous cases

$$y'x^2 = (y/z)x\cdot(zx) \overset{(5)}{=} (y/z)z\cdot x^2 \overset{(2)}{=} yx^2.$$

Therefore, $y' = y$. □

Definition 1.5. (cf. [2]) Let λ and ρ be two maps $Q\to Q$. A quasigroup $(Q,\cdot)$ is called an *LIP-quasigroup* if it satisfies the identity

$$\lambda x\cdot(x\cdot y) = y,$$

and an *RIP-quasigroup* if it satisfies the identity

$$(x\cdot y)\cdot\rho y = x.$$

A quasigroup which is simultaneously an LIP- and RIP-quasigroup is called an *IP-quasigroup*.

Definition 1.6. (cf. [9]) A quasigroup $(Q,\cdot)$ is called a *left Bol quasigroup*, if it satisfies the identity
$$x(y\cdot xz) = R_{e_x}^{-1}(x\cdot yx)\cdot z.$$
It is called a *right Bol quasigroup*, if it satisfies the identity
$$(yx\cdot z)x = yL_{f_x}^{-1}(xz\cdot x),$$
where $xe_x = x = f_x x$.

Definition 1.7. (cf. [2]) A quasigroup $(Q,\cdot)$ is called a *Moufang quasigroup*, if in $(Q,\cdot)$ the following identities are true

$$(xy\cdot z)y = x(y(e_y z\cdot y)), \tag{10}$$

$$y(x\cdot yz) = ((y\cdot xf_y)y)z \tag{11}$$

where $ye_y = y = f_y y$.

In his PhD thesis (see also [4]) I.A. Florja proved that in quasigroups the identities (10) and (11) are equivalent, so Moufang quasigroups can be defined as quasigroups satisfying one of these identities.

We will need the following two lemmas. The first was proved by I.A. Florja in his PhD thesis, the second is proved in the Belousov's book [2].

Lemma 1.8. *A left and right Bol quasigroup is a Moufang quasigroup.*

Lemma 1.9. *A loop isotopic to a Moufang quasigroup is an IP-loop.*

Definition 1.10. A commutative loop $(Q,\cdot)$ with the identity $xx\cdot yz = xy\cdot xz$ is called a *commutative Moufang loop*.

From Definitions 1.2 and 1.10 it follows that any commutative Moufang loop is a WA-quasigroup.

Theorem 1.11. (cf. [7, 12, 15]) *Each loop isotopic to a WA-quasigroup is a commutative Moufang loop.*

2. Properties of left WA-loops

Lemma 2.1. *Any WA-quasigroup with a left identity element is a left Bol quasigroup.*

Proof. If f is a left identity element of a quasigroup $(Q,\cdot)$, then $ff = f$, $L_f x = x$ for all $x\in Q$ and $L_f = \varepsilon$. From Theorem 1.11 it follows that an isotope of the form
$$x\circ y = R_f^{-1}x\cdot L_f^{-1}y = R_f^{-1}x\cdot y \tag{12}$$

of a quasigroup $(Q,\cdot)$ is a commutative Moufang loop. Any commutative Moufang loop $(Q,\circ)$ is an IP-loop, i.e., there exists a permutation I such that

$$Ix\circ(x\circ y)=(y\circ x)\circ Ix=y \tag{13}$$

for all $x,y\in Q$.

Going in the equation (13) to the operation $\cdot$ we have $R_f^{-1}Ix\cdot(R_f^{-1}x\cdot y)=y$, $R_f^{-1}IR_fx\cdot(x\cdot y)=y$. Hence, $I_lx\cdot(x\cdot y)=y$ for

$$I_l=R_f^{-1}IR_f. \tag{14}$$

Thus $(Q,\cdot)$ is an LIP-quasigroup. This, by results of [9], shows that $(Q,\cdot)$ is a left Bol quasigroup. $\square$

Corollary 2.2. *If f is a left identity element of a WA-quasigroup $(Q,\cdot)$, then the translation R_f is an automorphism of $(Q,\cdot)$, and an automorphism of the commutative Moufang loop $(Q,\circ)$ defined by* (12).

Proof. The fact that $R_f\in Aut(Q,\cdot)$ follows from (5) and the equality $ff=f$. Further, using the formula (12), we have: $R_f(x\circ y)=R_fx\circ R_fy$, $R_f(R_f^{-1}x\cdot y)=R_f^{-1}R_fx\cdot R_fy$, $x\cdot R_fy=x\cdot R_fy$. Therefore $R_fI=IR_f$, and (14) takes the form $I_l=I$. $\square$

Lemma 2.3. *Any WA-quasigroup with a right identity element is a right Bol quasigroup.*

Proof. Consider the isotope $(Q,\circ)$ of a WA-quasigroup $(Q,\cdot)$ given by:

$$x\circ y=x\cdot L_e^{-1}y,$$

where e is a right identity of $(Q,\cdot)$. By Theorem 1.11, $(Q,\circ)$ is a commutative Moufang loop. Let 1 be the identity element of $(Q,\circ)$ and I be a permutation of Q such that $x\circ Ix=1$ for all $x\in Q$. Since $(y\circ x)\circ Ix=y$ for all $x,y\in Q$, we have $y=(y\circ x)\circ Ix=(y\cdot L_e^{-1}x)\cdot L_e^{-1}Ix$. Therefore, $(y\cdot x)\cdot L_e^{-1}IL_ex=y$, hence $(y\cdot x)\cdot\rho x=y$ for $\rho=L_e^{-1}IL_e$. So, $(Q,\cdot)$ is an RIP-quasigroup. This, by results of [9] means that $(Q,\cdot)$ is a right Bol quasigroup. $\square$

Lemma 2.4. *Any WA-quasigroup $(Q,\cdot)$ with the left* (*right*) *inverse property is a left* (*right*) *Bol quasigroup.*

Proof. Since a WA-quasigroup with the left inverse property is an LIP-quasigroup, the proof of this lemma is very similar to the proof of Lemma 2.1.

For WA-quasigroups with the right inverse property the proof is analogous. $\square$

Corollary 2.5. *Any WA-quasigroup that is an IP-quasigroup, is a Moufang quasigroup.*

Proof. The proof follows from Lemma 2.4 and Lemma 1.8. $\square$

Definition 2.6. (cf. [2]) The isotope of the form

$$x \circ y = L_a^{-1}(L_a x \cdot y) \tag{15}$$

is called a *right derivative operation of* $(Q, \cdot)$ *generated by* a.

The isotope of the form

$$x \circ y = R_a^{-1}(x \cdot R_a y) \tag{16}$$

is called a *left derivative operation of* $(Q, \cdot)$ *generated by* a.

Theorem 2.7. *Let* $(Q, \cdot)$ *be a WA-quasigroup. Then*

(i) *the right derivative operation* $(Q, \cdot)$ *is a left Bol quasigroup,*

(ii) *the left derivative operation of* $(Q, \cdot)$ *is a right Bol quasigroup.*

Proof. (i). From (15) it follows that a quasigroup $(Q, \circ)$ has a left identity element, namely, $f = e_a$, where $ae_a = a$. Indeed, $e_a \circ y = L_a^{-1}(L_a e_a \cdot y) = L_a^{-1} L_a y = y$. In particular $f \circ f = f$.

We consider the following isotope of a quasigroup $(Q, \circ)$:

$$x + y = (R_f^\circ)^{-1} x \circ y, \tag{17}$$

where $R_f^\circ x = x \circ f$. Then $(Q, +)$ is a loop with the identity element f.

Indeed, $f + y = (R_f^\circ)^{-1} f \circ y = f \circ y = y$, since, if $(R_f^\circ)^{-1} f = z$, then $f = (R_f^\circ) z$, $f = z \circ f$. But, as was mentioned, $f \circ f = f$, therefore, $z = f$. Further we have $x + f = (R_f^\circ)^{-1} x \circ f = R_f^\circ (R_f^\circ)^{-1} x = x$.

Using (15) we can re-write (17) as follows:

$$x + y = L_a^{-1}(L_a (R_f^\circ)^{-1} x \cdot y).$$

Thus the loop $(Q, +)$ is an isotope of a WA-quasigroup $(Q, \cdot)$. By Theorem 1.11 among loop isotopes of a WA-quasigroup $(Q, \cdot)$ there exists a commutative Moufang loop. We recall that any loop isotopic to a Moufang loop is a Mofang loop (cf. [2]). Therefore $(Q, +)$ is a Moufang loop.

Our proof will be complete, if we prove that a quasigroup $(Q, \circ)$ is an LIP-quasigroup.

From $x^{-1} + (x + y) = y$, using (17), we obtain $(R_f^\circ)^{-1} x^{-1} \circ ((R_f^\circ)^{-1} x \circ y) = y$. Now, denoting $(R_f^\circ)^{-1} x^{-1}$ by αx and $(R_f^\circ)^{-1} x$ by βx, we obtain two permutations α, β of the set Q, and the possibility to rewrite the last equation in more useful form $\alpha x \circ (\beta x \circ y) = y$, which is equivalent to $\alpha \beta^{-1} x \circ (x \circ y) = y$.

The last means that $(Q, \circ)$ is an LIP-quasigroup. This completes the proof of (i).

(ii). From (16) it follows that a quasigroup $(Q, \circ)$ has a right identity element $e = f_a$, where $f_a a = a$. Indeed, $x \circ f_a = R_a^{-1}(x \cdot R_a f_a) = R_a^{-1}(x \cdot a) = x$.

We consider the following isotope of the quasigroup (right loop) $(Q,\circ)$:

$$x + y = x \circ (L_e^\circ)^{-1} y, \tag{18}$$

where $L_e^\circ x = e \circ x$. Then $(Q,+)$ is a loop with the identity element e. The proof is similar to the proof in the case (i) and we omit it.

From (18), using (16), we have

$$x + y = R_a^{-1}(x \cdot R_a (L_e^\circ)^{-1} y).$$

Analogously as in (i) we can prove that $(Q,+)$ is a Moufang loop. Next, from $(y + x) + x^{-1} = y$, using (18), we deduce $(y \circ (L_e^\circ)^{-1} x) \circ (L_e^\circ)^{-1} x^{-1} = y$. This shows that $(Q,\circ)$ is an RIP-quasigroup. □

3. Automorphisms of left WA-loops

We start with the following lemma which is a quasigroup folklore.

Lemma 3.1. *In a quasigroup autotopy any two components uniquely define the third.*

Elements of the group $I_h(Q,\cdot) = \{\alpha \in M(Q,\cdot) \,|\, \alpha h = h\}$, where $M(Q,\cdot)$ is the group generated by all left and right translations of a quasigroup $(Q,\cdot)$, are called *inner mappings of* $(Q,\cdot)$ *relative to the element* $h \in Q$ (cf. [2]). Belousov proved (cf. [2]) that the group $I_h(Q,\cdot)$ is generated by all permutations of the form:

$$\begin{aligned} L_{x,y} &= L^{-1}_{x\circ y} L_x L_y\,, &&\text{where} \quad (x \circ y)h = x \cdot yh,\\ R_{x,y} &= R^{-1}_{x\bullet y} R_y R_x\,, &&\text{where} \quad h(x \bullet y) = hx \cdot y,\\ T_x &= L^{-1}_{\sigma x} R_x\,, &&\text{where} \quad \sigma = R_h^{-1} L_h. \end{aligned}$$

Lemma 3.2. *In a WA-quasigroup* $(Q,\cdot)$ *with the left identity element* f *inner permutations* $L_{x,y}$*,* $R_{x,y}$*, and* T_x *relative to the element* f *are automorphisms of* $(Q,\cdot)$*.*

Proof. In our case $L_{x,y} = L^{-1}_{x\circ y} L_x L_y$, where $x \circ y = R_f^{-1}(x \cdot R_f y) = R_f^{-1} x \cdot y$, by Corollary 2.2. Therefore

$$L_{x,y} = L^{-1}_{R_f^{-1} x \cdot y} L_x L_y. \tag{19}$$

Moreover, $x \cdot y = fx \cdot y = f(x \bullet y) = x \bullet y$ implies

$$R_{x,y} = R^{-1}_{x\cdot y} R_y R_x. \tag{20}$$

Since $\sigma x = R_f^{-1} L_f x = R_f^{-1} x$, we also have $T_x = L^{-1}_{R_f^{-1} x} R_x$. Thus

$$L_{x,y} f = R_{x,y} f = T_x f = f. \tag{21}$$

From (5) it follows that for any fixed $a \in Q$ the following triplets (L_a, L_a, L_{a^2}) and (R_a, R_a, R_{a^2}), their inverse and various component-vise products are autotopies of $(Q, \cdot)$. Therefore

$$\begin{aligned}&(L^{-1}_{R_f^{-1}x\cdot y}, L^{-1}_{R_f^{-1}x\cdot y}, L^{-1}_{(R_f^{-1}x\cdot y)^2})(L_x, L_x, L_{x^2})(L_y, L_y, L_{y^2}) = \\ &(L_{x,y}, L_{x,y}, L^{-1}_{(R_f^{-1}x\cdot y)^2}L_{x^2}L_{y^2}).\end{aligned} \tag{22}$$

This means that

$$L_{x,y}f \cdot L_{x,y}z = L^{-1}_{(R_f^{-1}x\cdot y)^2}L_{x^2}L_{y^2}(f\cdot z), \tag{23}$$

whence, applying (21), we obtain

$$L_{x,y}z = L^{-1}_{(R_f^{-1}x\cdot y)^2}L_{x^2}L_{y^2}z \tag{24}$$

for all $x, y, z \in Q$. This, together with (22), shows that $L_{x,y}$ is an automorphism of the quasigroup $(Q, \cdot)$.

Similarly from

$$(R^{-1}_{x\cdot y}, R^{-1}_{x\cdot y}, R^{-1}_{(x\cdot y)^2})(R_y, R_y, R_{y^2})(R_x, R_x, R_{x^2}) = (R_{x,y}, R_{x,y}, R^{-1}_{(x\cdot y)^2}R_{y^2}R_{x^2}) \tag{25}$$

and

$$(L^{-1}_{R_f^{-1}x}, L^{-1}_{R_f^{-1}x}, L^{-1}_{(R_f^{-1}x)^2})(R_x, R_x, R_{x^2}) = (T_x, T_x, L^{-1}_{(R_f^{-1}x)^2}R_{x^2}) \tag{26}$$

it follows that $R_{x,y}$ and T_x are automorphisms of $(Q, \cdot)$. □

Corollary 3.3. *In any WA-quasigroup with a left identity element we have*

$$L_{x,y} = L_{x^2,y^2}, \quad R_{x,y} = R_{x^2,y^2}, \quad T_x = T_{x^2}.$$

Proof. Putting $y = z$ in (5), we obtain

$$(xy)^2 = x^2 \cdot y^2. \tag{27}$$

From (19) and (24) it follows that the identity $L_{x,y} = L_{x^2,y^2}$ will be proved, if we prove that $(R_f^{-1}x \cdot y)^2 = R_f^{-1}x^2 \cdot y^2$. This follows from (27) and the fact that R_f (and its inverse) are automorphisms of $(Q, \cdot)$ (see Corollary 2.2). Indeed, $(R_f^{-1}x)^2 = R_f^{-1}x \cdot R_f^{-1}x = R_f^{-1}(x \cdot x) = R_f^{-1}x^2$. Since $R_{x,y} = R^{-1}_{(x\cdot y)^2}R_{y^2}R_{x^2}$, by (25), from (20) and (27) we obtain $R_{x,y} = R_{x^2,y^2}$.

The third identity can be proved in a similar way. □

Definition 3.4. (cf. [10, 13]) Let $(Q, \cdot)$ be a groupoid. The element $a \in Q$ is called a *left nuclear element* in $(Q, \cdot)$ if $L_{ax} = L_aL_x$ for all $x \in Q$. The set of all left nuclear elements in $(Q, \cdot)$ is called the *left nucleus* of $(Q, \cdot)$ and is denoted by N_l.

It is well known (cf. [2, 3]) that in a quasigroup the set N_l forms a subgroup.

Theorem 3.5. *In a WA-quasigroup $(Q,\cdot)$ with the left identity element f the inner permutations $L_{x,y}$, $R_{x,y}$, and T_x relative to $a\in Q$ are automorphisms of $(Q,\cdot)$ if and only if $a\in N_l$ and the following identity $xy\cdot a = xf\cdot ya$ is satisfied.*

Proof. In this case $L_{x,y} = L^{-1}_{x\circ y}L_xL_y$, where $x\circ y = R_a^{-1}(x\cdot R_ay)$, $R_{x,y} = R^{-1}_{x\bullet y}R_yR_x$, where $ax\cdot y = a(x\bullet y)$ and $T_x = L^{-1}_{\sigma x}R_x$, where $\sigma x = R_a^{-1}L_fa$.

In a similar way as in the proof of Lemma 3.2 (identities (22), (23), and (24)), we can prove that $L_{x,y}$, $R_{x,y}$, and T_x are automorphisms of $(Q,\cdot)$. Then $L^{-1}_{x\circ y}L_xL_yf = f$, i.e., $L_xL_yf = L_{x\circ y}f$. So, $x\cdot yf = L_{x\circ y}f = R_a^{-1}(x\cdot R_ay)f$, which gives $R_f^{-1}(x\cdot yf) = R_a^{-1}(x\cdot R_ay)$. Since R_f^{-1} is an automorphism of $(Q,\cdot)$ (Corollary 2.2), from the last identity we obtain $R_f^{-1}x\cdot y = R_a^{-1}(x\cdot R_ay)$, and consequently, $xy = R_a^{-1}(R_fx\cdot R_ay)$. Thus, $xy\cdot a = xf\cdot ya$ for all $x,y\in Q$.

Moreover, $R^{-1}_{x\bullet y}R_yR_xf = f$ implies $R_yR_xf = R_{x\bullet y}f$. Hence $xy = x\bullet y$ and $ax\cdot y = a(x\bullet y) = a\cdot xy$ for all $x,y\in Q$. Therefore $a\in N_l$.

The converse statement is obvious. □

Lemma 3.6. *The permutation L_aR_a is an automorphism of a WA-quasigroup $(Q,\cdot)$ with the left identity element element f if and only if $a^2 = f$.*

Proof. Since

$$(L_a, L_a, L_{a^2})(R_a, R_a, R_{a^2}) = (L_aR_a, L_aR_a, L_{a^2}R_{a^2}) \tag{28}$$

is an autotopy of $(Q,\cdot)$, we have

$$L_aR_af\cdot L_aR_ay = L_{a^2}R_{a^2}y \tag{29}$$

for all $y\in Q$. This autotopy is an automorphism if and only if $L_{a^2}R_{a^2} = L_aR_a$. The last equality holds if and only if $L_aR_af = f$, i.e., if and only if $a^2 = f$. □

Corollary 3.7. *Let $(Q,\cdot)$ be a WA-quasigroup with the left identity element f. If $a^2 = f$, then $L_aR_a = L_{a^2}R_{a^2} = R_f$.*

Proof. From (28) and the fact that L_aR_a is an automorphism of $(Q,\cdot)$ it follows $L_aR_a = L_{a^2}R_{a^2}$. From (29), $L_aR_af = f$ and $a^2 = f$ we obtain $L_aR_a = R_f$. □

Lemma 3.8. *The permutation $L_{a^2}R_a$ is an automorphism of a WA-quasigroup $(Q,\cdot)$ with the left identity element f if and only if $a^2\cdot a = f$.*

Proof. It is clear that

$$(L_{a^2}, L_{a^2}, L_{(a^2)^2})(R_a, R_a, R_{a^2}) = (L_{a^2}R_a, L_{a^2}R_a, L_{(a^2)^2}R_{a^2})$$

is an autotopy of $(Q,\cdot)$. Therefore $L_{a^2}R_af\cdot L_{a^2}R_ay = L_{(a^2)^2}R_{a^2}y$ is true for all $y\in Q$. This autotopy is an automorphism if and only if $L_{(a^2)^2}R_{a^2} = L_{a^2}R_a$, i.e., if and only if $L_{a^2}R_af = f$. The last condition is equivalent to $a^2\cdot a = f$. □

4. Pseudoautomorphisms and subloops

A bijection θ of a set Q is called a *right pseudoautomorphism* of a quasigroup $(Q,\cdot)$ if there exists at least one element $c\in Q$ such that $(c\cdot\theta x)\cdot\theta y=c\cdot\theta(x\cdot y)$ for all $x,y\in Q$, i.e., if $(L_c\theta,\theta,L_c\theta)$ is an autotopy of a quasigroup $(Q,\cdot)$. The element c is called a *companion* of θ (cf. [2]).

A quasigroup with a right pseudoautomorphism has also a left identity element (cf. [13]).

Lemma 4.1. *In a WA-quasigroup with a left identity element f the translation R_a is a right pseudoautmorphism if and only if the translation L_a is a right pseudoautmorphism and $a^2=f$.*

Proof. Suppose that R_a is a right pseudoautomorphism with the companion k, i.e., a quasigroup $(Q,\cdot)$ has an autotopy (L_kR_a,R_a,L_kR_a). By Lemma 3.6 , L_aR_a, where $a^2=f$, is an automorphism of $(Q,\cdot)$.

Therefore

$$(L_kR_a,R_a,L_kR_a)(L_aR_a,L_aR_a,L_aR_a)^{-1}=(L_kL_a^{-1},L_a^{-1},L_kL_a^{-1})$$

also is an autotopy of $(Q,\cdot)$. The last means that L_a^{-1} is a right pseudoautomorphism of $(Q,\cdot)$. Since the set of all right pseudoatomorphisms of $(Q,\cdot)$ forms a group (cf. [2]), also L_a is a right pseudoautomorphism of $(Q,\cdot)$.

The converse statement is obvious. □

Lemma 4.2. *Let $(H,\cdot)$ be a subquasigroup of a WA-quasigroup $(Q,\cdot)$. Then $(aH,\cdot)$ is a subquasigroup of $(Q,\cdot)$ for any $a=a^2$.*

Proof. By (5), we have $ah_1\cdot ah_2=a^2\cdot h_1h_2=a\cdot h_1h_2\in aH$. Thus the set aH is closed with respect to the quasigroup operation.

The equation $ah_1\cdot x=ah_2$, where $h_1,h_2\in H$, has a unique solution $x\in Q$. Obviously, $x=ax'$ for some $x'\in Q$. Thus, $ah_1\cdot ax'=a\cdot h_1x'=ah_2$. Hence $h_1x'=h_2$, and consequently, $x'\in H$. Therefore $x=ax'\in aH$.

Analogously we prove that the equation $y\cdot ah_1=ah_2$ has a solution in aH. □

Lemma 4.3. *Let $(Q,\cdot)$ be a WA-quasigroup with the left identity element f and $(Q,\circ)$ be a loop defined by* (12). *If $(Q,\cdot)$ satisfies the inverse property, then:*

(*i*) $R_f^2=\varepsilon$, *where ε is the identity permutation,*

(*ii*) $I_l=I,\ \ I_r=IR_f$ *for* $I_lx\cdot xy=y,\ \ xy\cdot I_ry=x,\ \ Ix\circ(x\circ y)=y,$

(*iii*) I_l *and* I_r *are automorphisms of a quasigroup $(Q,\cdot)$ and a loop $(Q,\circ)$,*

(*iv*) $I_lI_r=R_f,\ \ I_lI_r=I_rI_l.$

Proof. (*i*). Indeed, $I_lf=I_rf=f$. Thus $xf\cdot f=x$ for any $x\in Q$. So, $R_f^2=\varepsilon$.

(*ii*). Since $x\circ y\overset{(12)}{=}R_f^{-1}x\cdot y=R_fx\cdot y$, by (*i*), R_f is an automorphism of $(Q,\cdot)$ and the corresponding commutative Moufang loop $(Q,\circ)$ (Theorem 1.11 and Corollary 2.2).

Going now in the equation $I_l x(x \cdot y) = y$ to the loop operation $\circ$ we obtain $R_f I_l x \circ (R_f x \circ y) = y$. Thus $R_f I_l R_f^{-1} x \circ (x \circ y) = y$. Consequently, $I = R_f I_l R_f^{-1}$ and $I_l = R_f^{-1} I R_f = I$, since in $(Q, \circ)$ automorphisms R_f and I commute.

Similarly, going in the equation $(x \cdot y) I_r y = x$ to the loop operation $\circ$ we obtain $R_f(R_f x \circ y) \circ I_r y = x$. Thus, $(x \circ R_f y) \circ I_r y = x$ and $(x \circ y) \circ I_r R_f^{-1} y = x$. Therefore, $I = I_r R_f^{-1}$ and $I_r = I R_f$.

(iii). It is a consequence of (ii) and (iii).

(iv). Since $I = I_l$ and $I_r = I R_f$, we have $I_l I_r = I^2 R_f = R_f$. Analogously, $I_r I_l = I R_f I = R_f$. $\square$

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