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Normal edge-transitive Cayley graphs on certain groups of orders 4n and 8n

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Abstract. Normal edge-transitive Cayley graph $Cay(G,S)$ where G is the generalized quaternion group Q_{4n} of order $4n$ or a certain group V_{8n} of order $8n$ is investigated. It is shown that up to isomorphism there is only one tetravalent normal edge-transitive Cayley graph when $G \cong Q_{4n}$ is the generalized quaternion group and its automorphism group is found. In the case of V_{8n} we show that there is no normal edge-transitive Cayley graph on V_{8n}.

1. Introduction

We will be concerned with simple graphs, which mean graphs with no multiple edges and loops. Let $\Gamma = (V,E)$ be a graph with vertex set V and edge set E. The edge joining the vertices u and v is denoted by $e = \{u,v\}$. The group of the automorphisms of the graph is denoted by $A = \mathbb{A}ut\,(\Gamma)$, and Γ is called *vertex* or *edge transitive* if A acts transitively on V or E respectively. Let G be a finite group and S be a subset of G such that $S = S^{-1}$ and $1 \notin S$. The Cayley graph of G on S is denoted by $\Gamma = Cay(G,S)$ and has its vertex set G and edge set $e = \{x, sx\}$ where $x \in G$ and $s \in S$. Therefore Γ is a regular graph of valency $|S|$, and it is connected if and only if S generates G. For $g \in G$ the mapping defined by $\rho_g : G \to G$, $\rho_g(x) = xg$, $x \in G$ is a permutation of G preserving the edges of Γ, hence it is an automorphism of Γ. It can be verified that $R(G) = \{\rho_g \,|\, g \in G\}$ is a subgroup of $Aut(\Gamma)$ isomorphic to G which acts regularly on the vertices of Γ, hence Γ is a vertex transitive graph.

For the Cayley graph $\Gamma = Cay(G,S)$ we define the group $Aut(G,S)$ by putting $Aut(G,S) = \{\sigma \in Aut(G) \,|\, \sigma(S) = S\}$. It can be verified that $Aut(G,S)$ is a subgroup of $A = Aut(\Gamma)$ which acts on $R(G)$ by $\rho_g^\sigma := \rho_{\sigma^{-1}(g)}$, where $\sigma \in Aut(G,S)$ and $\rho_g \in R(G)$. Therefore the semi-direct product $R(G) \rtimes Aut(G,S)$ is a subgroup of A.

It is proved in [3] that $N_A(R(G)) = R(G) \rtimes Aut(G,S)$, where $N_A(R(G))$ denotes the normalizer of $R(G)$ in A. In [7] the graph Γ is called *normal* if $R(G)$ is a normal subgroup of A and obviously in this case we have $A = R(G) \rtimes Aut(G,S)$.

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The normality of Cayley graphs has been studied by various authors from different point of views. If one is interested to study the normality of the Cayley graphs it suffices to consider the connected normal Cayley graphs, because in [5] all the disconnected normal Cayley graphs are determined. The research on edge-transitive Cayley graphs of small valency is of interest to many authors. In [6] the authors determined all the tetravalent edge-transitive Cayley graphs on the group $PSL_2(p)$ and Brian P. Corr et al. in [1] determined normal edge-transitive Cayley graphs of Frobenius group of order a product of two different primes. In [8] tetravalent non-normal Cayley graphs of order $4p$, p a prime number, are determined. In [2] the authors studied normal edge-transitive Cayley graphs on group of order $4p$ where p is an odd prime. Motivated by [2] we are interested to investigate normal edge-transitive Cayley graphs on the generalized quaternion group of order 4n and a certain group of order $8n$, where n is an arbitrary natural number. In particular we obtain:

Main result 1. *Let* $Q_{4n} = \langle a, b \mid a^{2n} = b^4 = 1,\ a^n = b^2,\ b^{-1}ab = a^{-1} \rangle$ *be the generalized quaternion group of order* $4n$*. Then up to isomorphism there is only one normal edge-transitive tetravalent Cayley graph of* G *and its automorphism group is isomorphic to* $G \rtimes D_8$ *if* n *is even and isomorphic to* $G \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ *if* n *is odd.*

Main result 2. *Let* $V_{8n} = \langle a, b \mid a^{2n} = b^4 = 1,\ (ab)^2 = (a^{-1}b)^2 = 1 \rangle$ *be a group of order* $8n$*. Then there is no normal edge-transitive Cayley graph on* V_{8n}*.*

2. Preliminary results

Let G be a group and S be a subset of G such that $1 \notin S$. The *Cayley di-graph* (*directed graph*) $Cay(G, S)$ of G relative to S has G as its vertex set and (x, sx) as its edge set, where $x \in G$ and $s \in S$. If S is an inverse closed subset of G, i.e., $S = S^{-1}$, then $Cay(G, S)$ is an undirected graph that is simply called a Cayley graph. The following result can be found for example in [4].

Lemma 2.1. *Let* $\Gamma = Cay(G, S)$ *be the Cayley graph of* G *with respect to* S*. Then the following hold:*

(*i*) $N_A(R(G)) = R(G) \rtimes \mathbb{A}ut(G, S)$.

(*ii*) $R(G) \trianglelefteq A$ *if and only if* $A = R(G) \rtimes \mathbb{A}ut(G, S)$.

(*iii*) Γ *is normal iff* $A_1 = \mathbb{A}ut(G, S)$*, where* A_1 *denotes the stabilizer of the vertex* 1 *under* A.

We set $N = N_A(R(G)) = R(G) \rtimes \mathbb{A}ut(G, S)$ and we remark that for the normal edge-transitivity of $Cay(G, S)$ the group N need only be transitive on undirected edges, and may or may not be transitive on ordered pairs of adjacent vertices. From [4] we have the following result which is useful in our investigation.

Lemma 2.2. *Let $\Gamma = Cay(G, S)$ be an undirected Cayley graph of the group G on S and let $N = N_A(R(G)) = R(G) \rtimes \mathbb{A}ut(G, S)$. Then the following are equivalent:*

(*i*) *Γ is normal edge-transitive.*

(*ii*) *$S = T \cup T^{-1}$ where T is an orbit of $\mathbb{A}ut(G, S)$ on S.*

(*iii*) *There exist a subgroup H of $\mathbb{A}ut(G)$ and $g \in G$ such that $S = g^H \cup (g^{-1})^H$, where $g^H = \{gh \mid h \in H\}$.*

3. Cayley graphs on a certain group of order 4n

First we consider the generalized quaternion group. The generalized quaternion group of order $4n$ has the following presentation:

$$Q_{4n} = \langle a, b \mid a^{2n} = b^4 = 1,\ a^n = b^2,\ b^{-1}ab = a^{-1} \rangle.$$

It is easy to verify that the center Z of Q_{4n} has order 2 generated by $a^n = b^2$ and $\frac{Q_{4n}}{Z} \cong D_{2n}$. The elements of Q_{4n} are of the form $a^i b^j$, $0 \leqslant i \leqslant 2n-1$, $j = 0, 1$. Element orders of Q_{4n} is as follows:

$$O(a^k) = \tfrac{2n}{(k,2n)},\ \ 0 \leqslant k \leqslant 2n-1,\ \ (0, 2n) = 2n,$$

$$O(a^k b) = 4,\ \ 0 \leqslant k \leqslant 2n-1.$$

Proposition 3.1. *The automorphism group of Q_{4n} is of order $2n\varphi(2n)$ and is isomorphic to the semi-direct product $\mathbb{Z}_{2n} \rtimes \Phi_{2n}$, where Φ_{2n} is the group of units of $\mathbb{Z}_{2n}$.*

Proof. Let $\varphi \in \mathbb{A}ut(Q_{4n})$. Then φ is completely determined by defining $\varphi(a)$ and $\varphi(b)$. Since φ preserves order of elements we have $O(\varphi(a)) = 2n$ and $O(\varphi(b)) = 4$. Therefore $\varphi(a) = a^k$, where $1 \leqslant k < 2n$, $(k, 2n) = 1$. If $\varphi(b) = a^l$has order 4, then $\varphi(\langle a, b \rangle) \subseteq \langle a \rangle$ or $G \subseteq \langle a \rangle$ which is a contradiction. Therefore $\varphi(b) = a^l b$, $0 \leqslant l < 2n$. It can be verified that φ in fact defines an automorphism of Q_{4n} and if we set $\varphi_{k,l}(a) = a^k$, $\varphi_{k,l}(b) = a^l b$ with k, l satisfying the above conditions, then $\varphi_{k,l}\varphi_{k',l'} = \varphi_{kk',l+kl'}$, hence:

$$\begin{aligned} \mathbb{A}ut(Q_{4n}) &= \{\varphi_{k,l} \mid k \in \Phi_{2n},\ l \in \mathbb{Z}_{2n}\} \\ &\cong \left\{ \begin{bmatrix} k & l \\ 0 & 1 \end{bmatrix} : k \in \Phi_{2n}, l \in \mathbb{Z}_{2n} \right\} \end{aligned}$$

But if we set

$$N = \left\{ \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix} : l \in \mathbb{Z}_{2n} \right\}$$

and

$$H = \left\{ \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} : k \in \Phi_{2n} \right\},$$

then $\mathbb{A}ut(Q_{4n}) = N \rtimes H \cong \mathbb{Z}_{2n} \rtimes \Phi_{2n}$, where the group Φ_{2n} has order $\varphi(2n)$. The proof is completed now. $\square$

Now let S be a subset of Q_{4n} such that $1 \notin S$, $S = S^{-1}$ and $\langle S \rangle = Q_{4n}$. Our aim is to consider normal edge-transitive Cayley graphs Q_{4n} on S. By Lemma 2.2, elements of S have the same order d and $S = T \cup T^{-1}$ where T is an orbit of $\mathbb{A}ut(G, S)$. If S contains an element of order 2 this element must be b^2 which is a central element and invariant under $\mathbb{A}ut(G, S)$ and S can not break as $S = T \cup T^{-1}$. This implies that $|S|$ should be even. Since $\langle a \rangle$is a cyclic group of order $2n$, for each divisor d of $2n$ there is a unique subgroup of $\langle a \rangle$with order d and elements of order d of $\langle a \rangle$ lie in this subgroup. If $d \neq 4$, elements of order d of Q_{4n} lie in $\langle a \rangle$ and obviously can not generate Q_{4n}.

Next we assume elements of S are of order $d = 4$. Keeping fixed the above notations we state the following:

Proposition 3.2. *S can not contain elements of order 4 contained in $\langle a \rangle$.*

Proof. On the contrary suppose $a^k \in \langle a \rangle \cap S$ has order 4. Then$\frac{2n}{(k,2n)} = 4$ implying $n = 2(k, 2n)$. Hence n must be even and we set $n = 2t$ which implies k is an odd multiple of t, i.e., $k = (2l + 1)t = \frac{(2l+1)n}{2}$. Then from $0 \leqslant k < 2n$ we obtain $l = 0$ or 1, hence $k = \frac{n}{2}$ or$\frac{3n}{2}$. This implies that the only elements of order4 in $\langle a \rangle$ are $a^{\frac{n}{2}}$ and $a^{\frac{3n}{2}}$.

But in this case if we apply the automorphisms φ of Q_{4n} obtained in Proposition 3.1 we see that $\{a^{\frac{n}{2}}, a^{\frac{3n}{2}}\}$ is invariant under $\mathbb{A}ut(Q_{4n})$. Again S can not break as $S = T \cup T^{-1}$ with T as an $\mathbb{A}ut(G, S)$ orbit and this completes the proof. $\square$

By the above proposition if $Cay(G, S)$ is normal edge-transitive, then we will have $S \subseteq \{a^i b \,|\, 0 \leqslant i < 2n\}$.

Proposition 3.3. *Let $0 \leqslant i \neq j < 2n$. Then $\langle a^i b, a^j b \rangle = Q_{4n}$ if and only if $(i - j, 2n) = 1$.*

Proof. Suppose $j < i$, $(i - j, 2n) = d$ and $H = \langle a^i b, a^j b \rangle$. Then using the defining relations for Q_{4n} we deduce $(a^i b)^2 = b^2 \in H$. Therefore $a^{i-j} \in H$. Since $(i - j, 2n) = d$ we obtain $a^d \in H$ and d is the least power of a belonging to H. Now elements of H can be organized as a^{id}, $a^{id}b^2$, $0 \leqslant i < \dfrac{2n}{d}$. Hence$|H| = \frac{4n}{d}$ and $H = Q_{4n}$ if and only if $d = 1$ and the proof is complete. $\square$

Next we turn on tetravalent Cayley graphs of Q_{4n}. By what we proved earlier we have $S = \{a^i b, a^j b, a^i b^{-1}, a^j b^{-1}\}$, where $(i - j, 2n) = 1$. We define the following concept which is needed in the next result.

If G is a group with two subsets S and T such that $1 \notin S$, $1 \notin T$, and if there is an automorphism φ of G such that $\varphi(S) = T$, then $Cay(G, S)$ is isomorphic to $Cay(G, T)$. In this case S and T are called equivalent.

Proposition 3.4. *If* $(i - j, 2n) = 1$, *then* $\{b, ab, b^{-1}, ab^{-1}\}$ *is equivalent to* $\{a^i b, a^j b, a^i b^{-1}, a^j b^{-1}\}$.

Proof. It is enough to apply the automorphism $\varphi_{j-i,i}$ of Q_{4n} to one of the above sets. □

Theorem 3.5. *There is only one tetravalent normal edge-transitive Cayley graph of* Q_{4n} *and the automorphism group of this graph is isomorphic to* $Q_{4n} \rtimes D_8$ *if* n *is even and isomorphic to* $Q_{4n} \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ *if* n *is odd.*

Proof. By Proposition 3.3 we have $S \subseteq \{a^i b \,|\, 0 \leqslant i < 2n\}$ and $|S| = 4$, $S = S^{-1}$, $\langle S \rangle = Q_{4n}$ forces $S = \{a^i b, a^j b, a^i b^{-1}, a^j b^{-1}\}$ for some i, j where $(i - j, 2n) = 1$. Now by Proposition 3.4 we may take $S = \{b, ab, b^{-1}, ab^{-1}\}$. This proves that up to equivalence there is a unique tetravalent normal edge-transitive Cayley graph of Q_{4n}. Next we determine $\mathbb{A}ut(Q_{4n}, S)$.

Since $\langle S \rangle = Q_{4n}$ the group $\mathbb{A}ut(Q_{4n}, S)$ acts on S faithfully, from which we deduce $\mathbb{A}ut(Q_{4n}, S) \leqslant \mathbb{S}_4$. If $\mathbb{A}ut(Q_{4n}, S)$ contains an element σ of order 3, then σ would fix an element say $\alpha \in S$, but in this case $\sigma(\alpha^{-1}) = \alpha^{-1}$ and σ can not be a 3-cycle. Therefore $|\mathbb{A}ut(Q_{4n}, S)|$ is a divisor of 8. It is easy to verify that the elements $\varphi_{1,n}$ and $\varphi_{2n-1,1}$ belong to $\mathbb{A}ut(Q_{4n}, S)$ and $\langle \varphi_{1,n}, \varphi_{2n-1,1} \rangle \cong V_4$ the Klein's four group. We distinguish two cases:

Case (i). n is even. In this case $\varphi_{n-1,1}$ is also an element of $\mathbb{A}ut(G, S)$ of order 4 and $\langle \varphi_{n-1,1}, \varphi_{2n-1,1}, \varphi_{1,n} \rangle \cong D_8$ is a subgroup of $\mathbb{A}ut(Q_{4n}, S)$, hence $\mathbb{A}ut(Q_{4n}, S) \cong D_8$ therefore the automorphism group of $Cay(Q_{4n}, S)$ is isomorphic to $Q_{4n} \rtimes D_8$.

Case (ii). n is odd. In this case we will prove that $\mathbb{A}ut(Q_{4n}, S)$ does not contain an element of order 4. On the contrary suppose $\varphi_{k,l} \in \mathbb{A}ut(Q_{4n}, S)$ is an element of order 4. Therefore we have one of the cases $\varphi_{k,l}(b) = ab$, $\varphi_{k,l}(ab) = b^{-1}$ or $\varphi_{k,l}(b) = ab^{-1}$, $\varphi_{k,l}(ab) = b$. In the first case we obtain $a^l b = ab$ and $a^{k+l} b = b^{-1}$, hence $a^{l-1} = 1$, $a^{k+l+n} = 1$. Since a is of order $2n$ we obtain $k = n - 1$, and because n is odd, $2 | (n - 1, 2n) = (k, 2n) = 1$, a contradiction. In the second case we obtain $a^l b = ab^{-1}$, $a^{k+l} b = b$, hence $a^{l+n-1} = 1$ and $a^{l+k} = 1$. Again from these relations we obtain $k = n - 1$, a contradiction.

Since $\mathbb{A}ut(Q_{4n}, S)$ does not contain elements of order 4 we obtain $\mathbb{A}ut(Q_{4n}, S) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, hence the automorphism group of $Cay(Q_{4n}, S)$ is isomorphic to $Q_{4n} \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ and the proof is complete. □

4. Cayley graph of a group of order 8n

Next we are going to study the normal edge-transitive Cayley graphs of a certain group of order $8n$ whose presentation is given as follows:

$$V_{8n} = \langle a, b \mid a^{2n} = b^4 = 1, (ab)^2 = (a^{-1}b)^2 = 1 \rangle$$

where n is a natural number. Using similar techniques as used in the previous section in finding the automorphism group of Q_{4n} one can prove the following:

Lemma 4.1. *$\mathbb{A}ut(V_{8n})$ is a group of order $4n\varphi(2n)$ if $n > 1$ and it is a group of order 8 if $n = 1$.*

Proof. In fact if $n = 1$, the group $V_1 = D_8$ is the dihedral group of order 8. To define an automorphism f of V_{8n} it is enough to define $f(a)$ and $f(b)$ which can be verified they are of the form:

$$f_{i,r,s,t}(a) = a^i b^r$$
$$f_{i,r,s,t}(b) = a^{2t} b^s,$$

where $(i, 2n) = 1$, $r = 0, 2$, $s = \pm 1$, $1 \leqslant t \leqslant n$. □

Lemma 4.2. *For V_{8n} we have*

$$\langle a^2, b^2, ab \rangle = \{a^{2k}, a^{2k+1}b^{\pm 1}, a^{2k}b^2 \mid 1 \leqslant k \leqslant n\}$$

Proof. If we set $X = \{a^{2k}, a^{2k+1}b^{\pm 1}, a^{2k}b^2 \mid 1 \leqslant k \leqslant n\}$ since $\{a^2, b^2, ab\} \subseteq X$ it is sufficient to show that X is a subgroup of V_{8n} and it is obviously true. □

Theorem 4.3. *There is no normal edge-transitive Cayley graph $Cay(G, S)$ for $G = V_{8n}$ if S has an element of order 2.*

Proof. Suppose $Cay(G, S)$ is a normal edge-transitive Cayley graph and S has an element of order 2.

Elements of order 2 in V_{8n} are $Y = \{a^n, b^2, a^n b^2, a^{2k+1}b^{\pm 1} \mid 1 \leqslant k \leqslant n\}$. Since all elements of S have the same order we have $S \subseteq Y$. If n is even then $\langle S \rangle \subseteq \langle Y \rangle \subseteq \langle a^2, b^2, ab \rangle \neq V_{8n}$, a contradiction. Hence n is odd.

If $S \cap \{a^n, a^n b^2\} = \emptyset$ then $\langle S \rangle \subseteq \langle a^2, b^2, ab \rangle \neq V_{8n}$ a contradiction, hence $S \cap \{a^n, a^n b^2\} \neq \emptyset$. For all $f \in \mathbb{A}ut(G, S)$ we have $f(\{a^n, a^n b^2\}) = \{a^n, a^n b^2\}$, therefore $S \cap \{a^n, a^n b^2\}$ is an orbit of $f \in \mathbb{A}ut(G, S)$ on S and it is a contradiction by Lemma 2.2. □

Theorem 4.4. *There is no normal edge-transitive Cayley graph $Cay(G, S)$ for $G = V_{8n}$ if S has an element of order 4.*

Proof. Suppose $Cay(G, S)$ is a normal edge-transitive Cayley graph and S has an element of order 4. Elements of order 4 in V_{8n} are $a^{2t}b^{\pm 1}$ for odd n and are $\{a^{\frac{n}{2}}, a^{\frac{n}{2}}b^2, a^{2t}b^{\pm 1} \mid 1 \leqslant t \leqslant n\}$.

Since $Cay(G, S)$ is a normal edge transitive Cayley graph all elements of S have order 4. If $(n, 4) = 1$ or $(n, 4) = 4$ then $\langle S \rangle \subseteq \langle a^2, b \rangle \neq V_{8n}$, a contradiction. Hence $(n, 4) = 2$ or equivalently $\frac{n}{2}$ is odd.

If $S \cap \{a^{\frac{n}{2}}, a^{\frac{n}{2}}b^2\} = \emptyset$ then $\langle S \rangle \subseteq \langle a^2, b \rangle \neq V_{8n}$ a contradiction, hence $S \cap \{a^{\frac{n}{2}}, a^{\frac{n}{2}}b^2\} \neq \emptyset$. For all $f \in \mathbb{A}ut(G, S)$ we have $f(\{a^{\frac{n}{2}}, a^{\frac{n}{2}}b^2\}) = \{a^{\frac{n}{2}}, a^{\frac{n}{2}}b^2\}$

therefore $S \cap \{a^{\frac{n}{2}}, a^{\frac{n}{2}}b^2\}$ is an orbit of $\mathbb{A}ut(G,S)$ on S and it is a contradiction by Lemma 2.2. unless $|S| = 4$ and $S = \{a^{\frac{n}{2}}, a^{\frac{n}{2}}b^2, a^{-\frac{n}{2}}, a^{-\frac{n}{2}}b^2\}$ and in these case we also have $\langle S\rangle \neq V_{8n}$. □

Theorem 4.5. *There is no normal edge-transitive Cayley graph on V_{8n}.*

Proof. Suppose $Cay(G,S)$ is a normal edge-transitive Cayley graph. By Theorems 3.3 and 3.4 we know that S can not have elements of order 2 or 4, Hence we have $S \subseteq \{a^i, a^ib^2 \mid 1 \leqslant i \leqslant 2n\}$ consequently $\langle S\rangle \subseteq \langle a, b^2\rangle \neq V_{8n}$, a contradiction. □

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