Note on the power graph of finite simple groups

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Abstract. A graph Γ is said to be 2−connected if Γ does not have a cut vertex. The power graph $\mathcal{P}(G)$ of a group G is the graph which has the group elements as vertex set and two elements are adjacent if one is a power of the other. In an earlier paper, it is conjectured that there is no non-abelian finite simple group with a 2−connected power graph. Bubboloni et al. [3] and independently Doostabadi and Farrokhi D. G. [11], presented counterexamples for this conjecture. The aim of this paper is to first modify this conjecture and then prove this modified conjecture for the sporadic groups, Ree groups ${}^2F_4(q)$ and ${}^2G_2(q)$, the Chevalley groups $A_1(q), B_2(q), C_3(q)$ and $F_4(q)$, the unitary group $U_3(q)$, the symplectic group $S_4(q)$ and the projective special linear group $PSL(3, q)$, where q is a prime power.

1. Introduction

The investigation of graphs related to groups is an important topic in algebraic combinatorics. This paper is devoted to the study of power graphs, which were introduced by Kelarev and Quinn in [13]. These authors in [16, 14, 15] studied the same structures on a semigroup. The power graph $\mathcal{P}(G)$ of a finite group G is a simple graph in which $V(\mathcal{P}(G)) = G$ and two vertices are adjacent if and only if one of them is a power of the other. We encourage the interested reader to consult [1] for a survey of all recent results on this topic.

Let us review some facts on power graphs of a finite group. Chakrabarty et al. [6] classified the complete power graphs and obtained a formula for the number of edges in a power graph. Cameron and Ghosh [4] proved that non-isomorphic finite groups may have isomorphic power graphs, but that finite abelian groups with isomorphic power graphs must be isomorphic. It is also conjectured that [4] two finite groups with isomorphic power graphs have the same number of elements of each order. Later Cameron [5] responded affirmatively to this conjecture.

Mirzargar et al. [20] considered some graph invariants of the power graphs into account and conjectured that the power graph of a cyclic group of order n has the maximum number of edges between the power graphs of all groups of order n. This conjecture recently proved by Curtin and Pourgholi [9]. Moghaddamfar et al. [21] defined the proper power graph $\mathcal{P}^{\star}(G)$ as a graph constructed from $\mathcal{P}(G)$ by deleting the identity element of G. They provided necessary and sufficient

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conditions for a proper power graph $\mathcal{P}^\star(G)$ to be a strongly regular graph, a bipartite graph or a planar graph. In a recent paper [22], the authors determined, up to isomorphism, the structure of a finite group G whose power graph has exactly n spanning trees, $n < 5^3$, and obtained a new characterization of the alternating group A_5 by tree-number of its power graph. Finally in [19], the second author of the present paper computed the automorphism group of the power graphs of cyclic groups.

A graph Γ is said to be $2-$*connected* if Γ does not have a cut vertex. It is easy to see that $\mathcal{P}^\star(G)$ is connected if and only if $\mathcal{P}(G)$ is $2-$connected. Pourgholi et al. [23], proved some results about characterization of simple groups by power graphs. They proposed the following open question:

Question. *Does there exist a non-abelian simple group with a $2-$connected power graph?*

Following Bubboloni et al. [3], we assume that P is the set of prime numbers and $b, c \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers. Set

$$bP + c = \{x \in \mathbb{N} \mid x = bp + c, for\ some\ p \in P\}$$

and define

$$A = P \cup (P + 1) \cup (P + 2) \cup (2P) \cup (2P + 1).$$

They proved that $\mathcal{P}(A_n)$ is $2-$connected if and only if $n = 3$ or $n \notin A$. In Theorems 3.6 and 3.7 of [11], the authors proved that the proper power graphs of the projective special linear group $PSL(2, q)$, q is prime power, and the Suzuki group $Sz(2^{2n+1})$ are disconnected. This shows that their power graphs are not $2-$connected. They also reproved [3, Theorem A] with a difference in the case that $\frac{n-2}{2}$ is prime. We conjecture that:

Conjecture. *The power graph of a non-abelian simple group G is $2-$connected if and only if G is isomorphic to the alternating group A_n, where $n = 3$ or $n \notin A$.*

The aim of this paper is to prove this conjecture for some classes of finite simple groups. For a finite group G, we denote by $\pi_e(G)$ a set of all element orders of group G. This set is closed under divisibility and hence is uniquely determined by a set $\mu(G)$ of elements in $\pi_e(G)$ which are maximal under the divisibility relation. The set of all divisors of a natural number n is denoted by $\beta(n)$. Our other notations are standard and taken mainly from [8].

We will prove the following theorem:

Main Theorem. *Let q be a power of a prime number. The proper power graphs of the sporadic groups, Ree groups ${}^2F_4(q)$ and ${}^2G_2(q)$, the Chevalley groups $A_1(q)$, $A_2(q)$, $B_2(q)$, $C_3(q)$ and $F_4(q)$, the projective unitary group $U_3(q)$ and the projective symplectic group $S_4(q)$ are disconnected.*

2. Proof of the main theorem

The aim of this section is to prove our main theorem. We separated our proof into four subsections. In the first subsection, it is proved that the power graph of sporadic groups are not $2-$connected. In the second subsection, the $2-$connectivity of $\mathcal{P}(^2F_4(q))$ and $\mathcal{P}(^2G_2(q))$ are investigated. Our third subsection is devoted to connectedness of the proper power graph of the Chevalley groups $A_1(q)$, $A_2(q)$, $B_2(q)$, $C_2(q)$ and $F_4(2^m)$. In our final subsection, the power graphs of $U_3(q)$ and $S_4(q)$ are taken into account.

2.1. The sporadic groups

Two positive integers r and s are said to be incomparable if r is not divisible by s and s is not divisible by r. Suppose G is a finite group and $G \setminus \{e\}$ can be partitioned into two subsets A and B such that for each element $a \in A$ and $b \in B$, $|a|$ and $|b|$ are coprime. Then the proper power graph $\mathcal{P}^\star(G)$ will be disconnected. We apply this simple fact to prove that the power graphs of the sporadic groups are not $2-$connected.

Define $S(M_{11}) = S(M_{12}) = S(M_{22}) = S(McL) = \{11\}$, $S(M_{23}) = S(M_{24}) = \{23\}$, $S(J_1) = S(J_3) = S(HN) = \{19\}$, $S(J_2) = \{7\}$, $S(He) = \{17\}$, $S(J_4) = \{23, 29, 31, 37, 43\}$, $S(Co_1) = S(Co_3) = \{23\}$, $S(Co_2) = \{11, 23\}$, $S(O'N) = \{31\}$, $S(Ly) = \{67\}$, $S(Ru) = S(Fi_{24}) = \{29\}$, $S(HS) = \{7, 11\}$, $S(Th) = \{13, 19, 31\}$, $S(Suz) = \{11, 13\}$, $S(B) = \{31, 47\}$, $S(M) = \{41, 71\}$, $S(Fi_{22}) = \{13, 17\}$, $S(Fi_{23}) = \{17, 23\}$. For an arbitrary sporadic group G, we assume that $A(G) = \{g \in G \mid |g| \in S(G)\}$ and $B(G) = G \setminus (A(G) \cup \{e\})$. We now apply computer algebra system GAP [12] to prove that for each $x \in A(G)$ and $y \in B(G)$, $|x|$ and $|y|$ are coprime, proving the following result:

Theorem 1. *The power graphs of the sporadic groups are not $2-$connected.*

2.2. The power graph of the Ree groups $^2F_4(q)$ and $^2G_2(q)$

The aim of this section is to prove the power graph of $^2F_4(q)$ and $^2G_2(q)$ are not $2-$connected. Suppose $\mu(G)$ denotes the set of all maximal elements of $\pi_e(G)$ with divisibility order. We first consider the group $^2F_4(q)$, where $q = 2^{2m+1}$ and $m \geqslant 1$. Deng and Shi [10, Lemma 3] proved that

$$\begin{aligned}\pi_e(^2F_4(q)) = {} & \{1, 2, 4, 8, 12, 16\} \cup \beta(2(q+1)) \cup \beta(4(q-1)) \\ & \cup \beta(4(q+\sqrt{2q}+1)) \cup \beta(4(q-\sqrt{2q}+1)) \cup \beta(q^2-1) \cup \beta(q^2+1) \\ & \cup \beta(q^2-q+1) \cup \beta((q-1)(q+\sqrt{2q}+1)) \cup \beta((q-1)(q-\sqrt{2q}+1)) \\ & \cup \beta(q^2+q\sqrt{2q}+q+\sqrt{2q}+1) \cup \beta(q^2-q\sqrt{2q}+q-\sqrt{2q}+1).\end{aligned}$$

Set $\alpha = q^2 + q\sqrt{2q} + q + \sqrt{2q} + 1, X = \beta(\alpha) \setminus \{1\}$, $Y = \pi_e(G) \setminus (X \cup \{1\})$ and $Z = \{q+1, q-1, q^2+1, q+\sqrt{2q}+1, q-\sqrt{2q}+1, q^2-q+1, q^2-q\sqrt{2q}+q-\sqrt{2q}+1\}$.

We claim that for integer $\gamma \in Z$, $(\alpha, \gamma) = 1$. To prove this, it is enough to notice that by simple divisions of appropriate components, we have:

$$\begin{aligned}
\alpha &= (q+\sqrt{2q})(q+1)+1 \\
&= (q+\sqrt{2q}+2)(q-1)+(2\sqrt{2q}+3) \\
&= q(q+\sqrt{2q}+1)+(\sqrt{2q}+1) \\
&= (q^2-q+1)+\sqrt{2q}(q+\sqrt{2q}+1) \\
&= (q^2-q\sqrt{2q}+q-\sqrt{2q}+1)+2\sqrt{2q}(q+1) \\
&= (q+2\sqrt{2q}+4)(q-\sqrt{2q}+1)+(3\sqrt{2q}-3) \\
&= (1+\sqrt{2q})(q^2+1)+(q\sqrt{2q}+q-q^2\sqrt{2q}).
\end{aligned} \tag{1}$$

To explain, we choose the case of $q-\sqrt{2q}+1$. If the prime p divides α and $q-\sqrt{2q}+1$ then by the fourth equation in (1),

$$\alpha = (3\sqrt{2q}-3)+(q+2\sqrt{2q}+4)(q-\sqrt{2q}+1).$$

This shows that $p|3\sqrt{2q}-3 = 3\times 2^{m+1}-3$ and so $p|3\times 2^{2m+1}$. Since p is odd, $p=3$ and $3|q-\sqrt{2q}+1 = 2^{2m+1}-2^{m+1}+1$. Thus, $2^{2m+1}-2^{m+1} \equiv -1 (\mod 3)$. On the other hand, for each positive integer k, $2^{2k+1} \equiv 2 (\mod 3)$ which implies that $3|2^{m+1}$, a contradiction. Using a similar argument, all cases lead to contradiction. Hence, we obtain a partition $\pi_e(G)X \cup Y \cup \{1\}$ such that elements of X and Y are mutually coprime. Therefore, we have proved the following result:

Theorem 2. *The power graph of the Ree group ${}^2F_4(q)$ is not $2-$connected.*

We now consider the groups ${}^2G_2(q)$, where $q = 3^{2m+1}$ and $m \geq 0$. It is well-known that ${}^2G_2(3) \cong Aut(SL(2,8))$ and for $m \geqslant 1$ the groups ${}^2G_2(q)$ are simple. Staroletove [26, Lemma 3.5], proved that

$$\mu({}^2G_2(q)) = \left\{q+\sqrt{3q}+1, q-\sqrt{3q}+1, q-1, \frac{q+1}{2}, 6\right\}.$$

Set $\alpha = q+\sqrt{3q}+1$ and $T = \{q-\sqrt{3q}+1, q-1, \frac{q+1}{2}, 6\}$. We prove that α does not have a common prime factor with an element of T. This is an immediate consequence of the fact that $\alpha = (q-\sqrt{3q}+1)+2\sqrt{3q} = (q-1)+(2+\sqrt{3q}) = 2\frac{q+1}{2}+\sqrt{3q}$. This shows that by removing the identity element, the resulting graph will be disconnected. We have proved the following result:

Theorem 3. *The power graph of the Ree group ${}^2G_2(q)$ is not $2-$connected.*

2.3. The power graphs of $A_1(q), A_2(q), B_2(q), C_2(q)$ and $F_4(2^m)$

In this section, it is proved that the power graphs of the groups $A_1(q)$, $A_2(q)$, $B_2(q), C_2(q)$ and $F_4(2^m)$ are not $2-$connected. We start by the simple group

$G = A_1(q)$, where q is an odd prime power. Staroletove [26, Lemma 3.5], proved that $\mu(A_1(q)) = \left\{\frac{q+1}{2}, \frac{q-1}{2}, p\right\}$. Since $p \not| \frac{q+1}{2}$ and $p \not| \frac{q-1}{2}$, by removing the identity element, the elements of order p will be separate from other elements. Thus, we proved the following:

Theorem 4. *The graph $\mathcal{P}(A_1(q))$ is not 2−connected.*

We now consider the simple group $A_2(q)$, where q is a prime power. Simpson [24] proved that

$$\mu(A_2(q)) = \begin{cases} \left\{q-1, \frac{p(q-1)}{3}, \frac{q^2-1}{3}, \frac{q^2+q+1}{3}\right\} & d=3 \textit{ and } q \textit{ is odd},\\ \left\{p(q-1), q^2-1, q^2+q+1\right\} & d=1 \textit{ and } q \textit{ is odd},\\ \left\{4, q-1, \frac{2(q-1)}{3}, \frac{q^2-1}{3}, \frac{q^2+q+1}{3}\right\} & d=3 \textit{ and } q \textit{ is even},\\ \left\{4, 2(q-1), q^2-1, q^2+q+1\right\} & d=1 \textit{ and } q \textit{ is even}, \end{cases}$$

where $d = (3, q-1)$.

We first assume that q is odd and $d = 3$. Set

$$\alpha = \frac{q^2+q+1}{3} \quad \text{and} \quad X = \left\{q-1, \frac{p(q-1)}{3}, \frac{q^2-1}{3}\right\}.$$

Since $\alpha = (q+2)\frac{q-1}{3} + 1 = q\frac{q-1}{3} + \frac{2q+1}{3}$, by a similar argument as Proposition 2, α and elements of X are coprime. Hence $\mathcal{P}(A_2(q))$ is not 2−connected. Next assume that q is odd, $d = 1$, $\alpha = q^2+q+1$ and $X = \{p(q-1), q^2-1\}$. Again since $\alpha = q(q-1)+(2q+1)$, α is coprime with $p(q-1)$ and q^2-1 which shows that $\mathcal{P}(A_2(q))$ is not 2−connected.

We now assume that $q = 2^m$ and $d = 3$. Define:

$$\alpha = \frac{q^2+q+1}{3} \quad \text{and} \quad X = \left\{4, \frac{q-1}{3}, \frac{q^2-1}{3}\right\}.$$

Since $\alpha = (q+2)\frac{q-1}{3} + 1$, it can easily prove that α is coprime to all elements of X which implies that the proper power graph of $A_2(q)$ is disconnected. Finally, if $q = 2^m$, $d = 1$, $\alpha = q^2+q+1$ and $X = \{4, q-1, q^2-1\}$, then α an elements of X are coprime. Therefore, we have proved the following:

Theorem 5. *The proper power graph of $A_2(q)$ is not connected.*

We now proceed to consider the simple group $G = B_2(q)$, where q is a prime power. Srinivasan [25] proved that:

$$\mu(B_2(q)) = \begin{cases} \frac{q^2+1}{(2,q-1)}, \frac{q^2-1}{(2,q-1)}, p(q+1), p(q-1) & p>3\\ \\ \frac{q^2+1}{(2,q-1)}, \frac{q^2-1}{(2,q-1)}, p(q+1), p(q-1), p^2 & p \in \{2,3\} \end{cases}$$

We consider three cases as follows:

a. *$q = p^m$, where $p > 3$ is prime and m is a natural number.* In this case $(2, q-1) = 2$. Define $\alpha = \frac{q^2+1}{2}$ and $X = \{p(q+1), p(q-1), \frac{q^2-1}{2}\}$. Then α is coprime with all elements of X and by a similar argument as Proposition 2, $\mathcal{P}(B_2(q))$ is not $2-$connected.

b. *$q = 3^m$, where m is a natural number.* In this case we have again $(2, q-1) = 2$ and by choosing $\alpha = \frac{q^2+1}{2}$ and $X = \{3(q+1), 3(q-1), \frac{q^2-1}{2}, 9\}$, we can see that α does not have a common divisor with an element of X. So, $\mathcal{P}(B_2(q))$ is not $2-$connected.

c. *$q = 2^m$, where m is a natural number.* In this case, $(2, q-1) = 1$. Set $\alpha = q^2+1$ and $X = \{p(q+1), p(q-1), q^2-1, 4\}$. Then a similar argument as Cases a and b shows that $\mathcal{P}(B_2(q))$ is not $2-$connected.

Thus, we have proved the following result:

Theorem 6. *The proper power graph of $B_2(q)$ is not connected.*

We now consider the group $C_2(q)$, where q is an odd prime power. Staroletove [26, Lemma 3.5] proved that:

$$\mu(C_2(q)) = \begin{cases} \left\{\frac{q^2+1}{2}, \frac{q^2-1}{2}, p(q+1), p(q-1)\right\} & p \neq 3, \\ \left\{\frac{q^2+1}{2}, \frac{q^2-1}{2}, p(q+1), p(q-1), 9\right\} & p = 3. \end{cases}$$

We consider two separate cases as follows:

a. *$q = p^m$, where $p > 3$ is prime and m is a natural number.* In this case, we define $\alpha = \frac{q^2+1}{2}$ and $X = \{p(q+1), p(q-1), \frac{q^2-1}{2}\}$. Some similar calculations as above show that α is coprime with all elements of X and so $\mathcal{P}(B_2(q))$ is not $2-$connected.

b. *$q = 3^m$, where m is a natural number.* A similar argument as Case b in the proof of Proposition 6 completes this case.

Hence, we have proved the following result:

Theorem 7. *The proper power graph of $C_2(q)$ is not connected.*

We end this subsection by investigation of the power graph of the group $F_4(q)$, $q = 2^m$ and $m \geq 1$. Coa et al. [7, Lemma 1.6] proved that:

$\mu(F_4(q)) = \{16,\ 8(q-1), 8(q+1), 4(q^2-1), 4(q^2+1), 4(q^2-q+1), 4(q^2+q+1), 2(q-1)(q^2+1), 2(q+1)(q^2+1), 2(q^3-1), 2(q^3+1), (q^2-1)(q^2-q+1), q^4-q^2+1, (q^2-1)(q^2+q+1), q^4-1, q^4+1\}$.

Define $\alpha = q^4 - q^2 + 1$ and

$$X = \{q-1, q+1, q^2+1, q^2-q+1, q^2+q+1, q^4-1, q^4+1\}.$$

By using a similar argument as above, one can see that it is possible to partition the group $F_4(2^m)$ into the set of all elements that their orders are divisors of α and its complement. Again by deleting the identity element, the resulting graph will be disconnected. So, we have:

Theorem 8. *The proper power graph of $F_4(2^m)$ is not connected.*

2.4. The power graphs of $U_3(q)$ and $S_4(q)$

The aim of this section is to prove the proper power graph of $U_3(q)$ and $S_4(q)$ are disconnected. We start by the simple groups $U_3(q)$, where q is an odd prime power. This group is defined as $U_3(q) = \frac{SU_3(q)}{Z(SU_3(q))}$, where $SU_3(q)$ is the set of all invertible 3×3 matrices A on $GF(q^2)$ such that $detA = 1$ and $A\overline{A^T} = I$, and $Z(SU_3(q))$ denotes its center. It is well-known that $|U_3(q)| = \frac{q^3(q^3+1)(q^2-1)}{d}$, where $d = (3, q-1)$. Aleeva [2, Lemma 10] proved that if q is odd then the maximal element orders of this group is as follows:

$$\mu(U_3(q)) = \begin{cases} \left\{\frac{q^2-q+1}{3}, \frac{q^2-1}{3}, \frac{p(q+1)}{3}, q+1\right\} & d = 3, \\ \left\{q^2-q+1, q^2-1, p(q+1)\right\} & d = 1. \end{cases}$$

We now consider the following two cases:

1. $d = 3$. Suppose $\alpha = \frac{q^2-q+1}{3}$ and $X = \left\{q+1, \frac{p(q+1)}{3}, \frac{q^2-1}{3}\right\}$. If we partition $U_3(q)$ into the set of all elements such that their orders are divisors of α and its complement, then by removing the identity element the resulting graph will be disconnected.

2. $d = 1$. In this case by choosing $\alpha = q^2+q+1$ and $X = \{p(q+1), q^2-1\}$, one can easily prove that α and elements of X are coprime. Thus, $U_3(q)$ is not $2-$connected.

We have proved the following:

Theorem 9. *The proper power graph of $U_3(q)$ is not connected.*

We end this paper by considering the simple group $S_4(q)$, $q = p^m$ and p is an odd prime. Srinivasan [25] proved that:

$$\pi_e(S_4(q)) = \beta(\frac{q^2+1}{2}) \cup \beta(\frac{q^2-1}{2}) \cup \beta(p(q+1)) \cup \beta(p(q-1));\ p \neq 3,$$
$$\mu(S_4(q)) = \left\{\frac{q^2+1}{2}, \frac{q^2-1}{2}, 3(q+1), 3(q-1), 9\right\};\ p = 3.$$

We consider two separate cases as follows:

1. Set $\alpha = \frac{q^2+1}{2}$ and $X = \left\{\frac{q^2-1}{2}, p(q+1), p(q-1)\right\}$. Then
$$A = \{x \in S_4(q) \mid o(x)|\alpha\} \quad \text{and} \quad B = S_4(q) \setminus A$$
is a partition of $S_4(q)$ such that by removing the identity element, the resulting graph will disconnected. This proves that $\mathcal{P}(S_4(q))$ is $2-$connected.

2. Set $\alpha = \frac{q^2+1}{2}$ and $X = \left\{\frac{q^2-1}{2}, 3(q+1), 3(q-1), 9\right\}$. A similar argument as Case (1), completes our argument.

Therefore, the following result is proved.

Theorem 10. *The proper power graph of $S_4(q)$, $q = p^m$ and p is an odd prime, is not connected.*

The proof of the main theorem follows from Theorems $1 - 10$.

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