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Invertible Graphs of Finite Groups

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Abstract

We investigate some properties of invertible graphs of finite groups, which are newly defined in this paper. The main results have been proved using finite group classification. For each finite group, the size, the girth, the diameter, the clique number and the chromatic number have been studied. These studies show that the invertible graphs are weakly perfect. Specifically, formulas for enumerating a total number of edges in the invertible graph of the Symmetric group and Dihedral group have been derived. Further, the relations between isomorphic, non-isomorphic groups and their invertible graphs are presented.

Keywords: Self-inverse elements, Mutual inverse elements, Weakly-perfect, Isomorphic graphs, Invertible graph.

1 Introduction

Abstract algebra is largely concerned with the study of abstract sets and endowed with one or more binary operations. In this paper, we consider one of the basic algebraic structures known as group. The concept of finite group plays a fundamental role in the theory of group-theoretic graphs. The aim of this paper is to discuss some of the interconnections which exist between graphs and groups. Many authors in graph theory specify so many specific graph-theoretic properties, and results have analogs for algebraic systems such as semi groups, groups, rings, fields, etc. Our main purpose inthis paper is to describe some interactions between finite graphs, and finite groups have been exploited to give new results about group-theoretic graphs. The theory of group-theoretic graphs has provided an interesting and powerful structural abstract approach to the study of the symmetries and non-symmetries of various configurations in the modern design theory and communication science. In recent years, a theory of group-theoretic graphs has found many applications in engineering and applied science, and many articles have been published on group-theoretic graphs such as [1]-[5].

In the present investigation, we write about a group theoretic graph, namely, invertible graph $IG(G)$ of a basic algebraic structure G, a finite group. However, finite group is a core in this paper. Our algebraic approach here is realized on group theoretic graphs with group elements and their corresponding binary operation. Although it is not quite elementary, it is an important aspect in dealing with the inter relation between simple graphs and finite groups.

For a finite group G, we denote by $S(G)$ the set of all self inverse elements, and by $M(G)$ – the set of mutual inverse elements of G. In this paper, we prove that there are some relations between G, $S(G)$, $M(G)$ and $IG(G)$. We classify the finite groups whose invertible graph is one of connected, complete but not bipartite graphs. Also we prove that $IG(G)$ is never Eulerian. For any given finite group G, we estimate the degree, the size, the girth, the diameter, the clique number and the chromatic number. We also discuss isomorphic theorems with some applications and structure of invertible graphs of finite abelian, non-abelian and cyclic groups.

2 Definition and Notations

Now we recall some basic definitions and notations of group theory from [6]. Let G be a finite group with identity e. Then the number of elements in G is the order of G and is denoted by $|G|$. If $a \in G$, then the order of a is $|a| = |<a>|$, where $<a>= \{a^n : n = 0, \pm 1, \pm 2, ...\}$ is a cyclic subgroup of G generated by a. If $G =<a>$, then G must be a cyclic group.

Usually, Z_n is the group of integers over addition modulo n, $U_n = Z_n^*$ is the group of multiplicative inverse elements of modulo n, $Z_p^* = Z_p - \{0\}$ is the multiplicative group of integers modulo p, S_n is the symmetric group of degree n, D_n is the dihedral group of order $2n$, Q_8

is the quaternion group. Further, we have $G_{2p} = \{2, 4, 6, ..., 2(p-1)\}$, a group of order $p-1$ with respect to multiplication modulo $2p$, a prime $p > 2$.

Theorem 1. *(Lagrange's Theorem, [6]) If H is a subgroup of the finite group G, then the order of H divides the order of G.*

Theorem 2. [6] *If $a \in G$ and G is finite, then $|a|$ divides $|G|$.*

We use [7] and [8] for the standard terminology of simple and algebraic graphs, respectively. Let X be a finite simple graph. We denote the vertex set and the edge set of X is $V(X)$ and $E(X)$, respectively. If $a \in V(X)$, then the degree of a denoted by $deg(a)$. If a and b are two adjacent vertices of X then we write $a-b$. A graph X in which any two distinct vertices are adjacent is said to be complete. If any two vertices a and b in X are connected by a path $a = a_0 - a_1 - ... - a_n = b$ then X is called connected graph. A path is a cycle if $a = b$. The length of a path or a cycle is the number of distinct edges in it. A cycle of length n is denoted by C_n.

3 Self and Mutual Inverse Elements of a Group

We introduce in this section the concepts of self and mutual inverse elements of a finite group with a few examples. The results of this section, though simple, are used throughout the paper.

Definition 1. *Let $(G, *)$ be a finite group with the identity e. Then an element $a \in G$ is called a self inverse element of G if $a = a^{-1}$, where a^{-1} is the inverse of a in G. The set of self inverse elements of G is $S(G)$ and its cardinality is $|S(G)|$.*

Next, an element $a \in G$ is called a mutual inverse element of G if there exists $b \in G$ such that $a * b = b * a = e$. The set of mutual inverse elements of G is denoted by $M(G)$. In particular, $M(G) = \{a \in G : a \neq a^{-1}\}$.

In the preceding definition, we have temporarily reverted to the $*$ notation for group operations to remind you that in a specific group, the operation might be addition, multiplication, or something else.

For any finite group G, $S(G)$ is a subgroup of G and $M(G)$ is not a subgroup of G. If $|G| > 2$ and G is a finite cyclic group, then $S(G) \neq G$. We are now ready to state and prove several results about $S(G)$ of G. The proof of the first theorem is implicit in our discussion of a finite cyclic group.

Theorem 3. *Let G be a finite cyclic group. Then*

$$|S(G)| = \begin{cases} 1, & \text{if } |G| \text{ is odd} \\ 2, & \text{if } |G| \text{ is even} \end{cases}.$$

Proof. For each finite cyclic group G, we have $G = S(G) \cup M(G)$ and $S(G) \cap M(G) = \phi$. Now consider two cases on $|G|$.
Case 1. If $|G|$ is odd, then we have to prove that $|S(G)| = 1$. Suppose $|S(G)| \geq 2$. Assume that $|S(G)| = 2$. Therefore, $S(G) = \{e, a : a^2 = e\}$. This implies that $|a| = 2$. By the Theorem 2, $|a|\,\big|\,|G|$, which is a contradiction to the fact that $|G|$ is odd. Hence $|S(G)| = 1$.
Case 2. If $|G|$ is even, then we shall show that $|S(G)| = 2$. Without loss of generality we may assume that $|S(G)| = 3$. This implies that every non-identity of $S(G)$ has order 2. That is, $a \neq b$ in $S(G)$ such that $a = a^{-1}$ and $b = b^{-1} \Rightarrow (ab) = (ab)^{-1}$, since G is abelian. It is clear that $ab \in S(G)$, and $S(G) = \{e, a, b, ab\}$ which is a contradiction to our assumption that $|S(G)| = 3$. Hence $|S(G)| = 2$. □

We next observe one of the most important results of $S(G)$. That is, if G is not a cyclic group of even order, then $|S(G)| \geq 2$. The following examples illustrate this point.

Example 1. *Since $e = e^{-1}$, $a = a^{-1}$, $b = b^{-1}$ and $c = c^{-1}$ in the Klein group $V_4 = \{e, a, b, c\}$, therefore, $|S(V_4)| = 4$.*

Example 2. *$|S(S_3)| = 4$, $|S(D_3)| = 4$, $|S(Q_8)| = 2$, $|S(Z_2 \times Z_2)| = 4$.*

According to the above examples, the following consequences specify the orders of $S(G)$ and $M(G)$ in a given finite group G.

Corollary 1. *Let G be a finite group of even order. Then $|S(G)|$ and $|M(G)|$ are both even.*

Proof. It is obvious, since a finite group G can be written as disjoint union of $S(G)$ and $M(G)$. □

Corollary 2. *Let G be a finite group of odd order. Then $|S(G)| = 1$ and $|M(G)| = |G| - 1$.*

Proof. It is obviously true because $|G|$ is odd if and only if $S(G) = \{e\}$. □

Example 3. $|S(Z_3 \times Z_3)| = 1$, $|S(Z_3 \times Z_5)| = 1$.

Remark 1. [9] *If there is a one-to-one mapping $a \leftrightarrow a'$ of the elements of a group G onto those of a group G', and if $a \leftrightarrow a'$ and $b \leftrightarrow b'$ imply $ab \leftrightarrow a'b'$, then we say that G and G' are isomorphic and write $G \cong G'$. If we put $a' = f(a)$ and $b' = f(b)$ for $a, b \in G$, then $f : G \to G'$ is a bijection satisfying $f(ab) = a'b' = f(a)f(b)$.*

Lemma 1. *Let G and G' be any two finite groups. If $G \cong G'$, then $S(G) \cong S(G')$. But converse is not true.*

Proof. Suppose $G \cong G'$. Then, by the Remark 1, there exists a group isomorphism f from G onto G' with the relation $f(a) = a'$, for every $a \in G$ and $a' \in G'$. Now define a map $\varphi : S(G) \to S(G')$ by the relation $\varphi(s) = s'$ for every s in $S(G)$. Let $s, t \in S(G)$. If $\varphi(s) = \varphi(t)$, then $s' = t' \Rightarrow f(s) = f(t) \Rightarrow s = t$, since f is one-to-one. By the way φ was constructed, we see that φ is onto. The only condition that remains to be checked is that φ is operation preserving. To do this, let s and t belong to $S(G)$. Then obviously $s, t \in G$. Therefore $\varphi(st) = f(st) = s't' = f(s)f(t) = \varphi(s)\varphi(t)$. Hence $S(G) \cong S(G')$. But the converse of this result is not true. For example, $S(U_6) = \{1, 5\}$ and $S(U_{10}) = \{1, 9\}$. It is clear that $S(U_6) \cong S(U_{10})$, but U_6 is not isomorphic to U_{10}. □

4 Properties of Invertible Graphs

This section introduces invertible graph of a finite group and a study of its basic properties such as degree, size, connectedness and completeness. Further, we obtain a formula for finding the clique number, the chromatic number and hence prove that invertible graph is weakly perfect.

We begin with the notion and definition of the invertible graph of a finite group.

Definition 2. *An undirected simple graph $IG(G)$ is called invertible graph of a finite group G whose vertex set is G and two distinct vertices a and b in G are adjacent in $IG(G)$ if and only if either $a \neq b^{-1}$, or, $b \neq a^{-1}$, where a^{-1} is the inverse of the element a in G.*

Before exploring the results and concepts of invertible graphs, instead of $a * b$, we shall write ab. The preceding definition can be visualized as shown in Figure 1. If $f_1(x) = x, f_2(x) = 1 - x, f_3(x) = \frac{1}{x}, f_4(x) = 1 - \frac{1}{x}, f_5(x) = \frac{1}{1-x}, f_6(x) = \frac{x}{1-x}$ are functions from $R - \{0, 1\}$ to $R - \{0, 1\}$, then the set $G = \{f_1, f_2, f_3, f_4, f_5, f_6\}$ is a non-abelian group under composition of functions. Here $S(G) = \{f_1, f_2, f_3, f_6\}$.

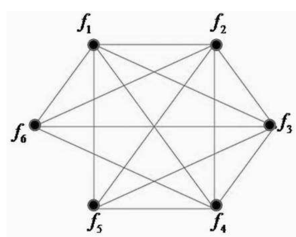

Figure 1. The Graph $IG(G)$.

Theorem 4. *For any finite group G, the invertible graph of G is a connected graph.*

Proof. It is obvious since e is the identity element of G, $ae \neq e$ for every $a \neq e$ in G, so that the vertex e is adjacent with remaining all the vertices of $IG(G)$. Hence $IG(G)$ is a connected graph. □

Theorem 5. *Let a be an element of a finite group G. Then*

$$deg(a) = \begin{cases} |G| - 1, & \text{if } a \in S(G) \\ |G| - 2, & \text{if } a \notin S(G) \end{cases} .$$

Proof. If $a \in S(G)$, then there exists $b \neq a$ in G such that $ab = a^{-1}b \neq e$. This implies that the vertex a is adjacent to all other vertices of $IG(G)$ if and only if $a \in S(G)$, therefore it is easy to derive that the degree of vertex a is $|G| - 1$.

If $a \notin S(G)$, then a has mutual inverse, say $b \neq a^{-1}$ in G such that $ab = e = ba$. It is clear that the vertex a is adjacent to all vertices of $IG(G)$ except b. However, if $a \notin S(G)$, then the vertex a is not adjacent to exactly one vertex of the graph $IG(G)$. Hence $deg(a) = |G| - 2$. □

Theorem 6. [8] *A connected graph is Eulerian if and only if the degree of each vertex is even.*

Corollary 3. *The invertible graph $IG(G)$ is never Eulerian.*

Proof. By the Theorem 5, it is clear that the degree of each vertex in $IG(G)$ is either $|G| - 1$ or $|G| - 2$. If $|G|$ is even, then $|G| - 1$ is odd. On the other hand, if $|G|$ is odd, then $|G| - 2$ is also odd. Hence in both cases, we found that the degree of each vertex in $IG(G)$ cannot be even. Thus, by the Theorem 6, the result follows. □

In view of the Theorem 5, the following Remark is obvious.

Remark 2. *Let $|G| > 3$ and $S(G) \neq G$. Then the graph $IG(G)$ is never a regular, a cycle, a star and a triangle free graph.*

Theorem 7. [8] *The total number of edges of the simple graph of order n is $\binom{n}{2}$.*

By combining Theorems 5 and 7, we can easily count the number of edges (size) in an invertible graph of a given finite group. For convenience, we introduce the following theorem.

Theorem 8. *For any finite group G, the size of invertible graph $IG(G)$ is $\frac{1}{2}(|S(G)|(|G| - 1) + |M(G)|(|G| - 2))$.*

By using Theorem 8, we derive a formula for enumerating the total number of edges in $IG(S_n)$ and $IG(D_n)$, respectively.

Theorem 9. *The size of invertible graph of the symmetric group S_n, $n > 1$, is $\frac{1}{2}((n!)^2 - 2n! + s(n))$, where $s(n)$ is the number of self inverse elements in S_n.*

Proof. Let $s(n)$ be the number of elements in S_n satisfying the relation $a = a^{-1}$, for every $a \in S_n$. Then $s(n)$ satisfies the recurrence relation, see [10], $s(n+2) = s(n+1) + (n+1)s(n)$, where $s(1) = 1$, $s(2) = 2$. In view of the Theorem 8, the size of the graph $IG(S_n)$ is $|E(IG(S_n))| = \frac{1}{2}(s(n)(n!-1) + (n! - s(n))(n!-2)) = \frac{1}{2}((n!)^2 - 2n! + s(n))$. □

Theorem 10. *The size of invertible graph of the Dihedral group of order $2n$ is*

$$|E(IG(D_n))| = \begin{cases} \frac{1}{2}(4n^2 - 3n), & \textit{if } n \textit{ is even} \\ \frac{1}{2}(4n^2 - 3n + 1), & \textit{if } n \textit{ is odd} \end{cases}.$$

Proof. Let $D_n = \{1, a, a^2, ..., a^{n-1}, b, ab, a^2b, ..., a^{n-1}b : a^n = 1, b^2 = 1, bab^{-1} = a^{-1}\}$. In D_n, the n elements $b, ab, a^2b, .., a^{n-1}b$ always have order 2. If n is even, then $a^{n/2}$ also has order 2. Therefore, the total number of elements in D_n, n is even, satisfying the relation $x = x^{-1}$ is $n + 2$. Similarly, if n is odd, there are $n + 1$ self inverse elements in D_n. However,

$$|S(D_n)| = \begin{cases} n+2, & \text{if } n \text{ is even} \\ n+1, & \text{if } n \text{ is odd} \end{cases} and |M(D_n)| = \begin{cases} n-2, & \text{if } n \text{ is even} \\ n-1, & \text{if } n \text{ is odd} \end{cases}.$$

By the Theorem 8, we have

$$|E(IG(D_n))| = \begin{cases} \frac{1}{2}(4n^2 - 3n), & \text{if } n \text{ is even} \\ \frac{1}{2}(4n^2 - 3n + 1), & \text{if } n \text{ is odd} \end{cases}.$$

□

Corollary 4. *The size of an invertible graph of a finite cyclic group G is*

$$|E(IG(G))| = \begin{cases} \frac{1}{2}(|G|-1)^2, & \text{if } n \text{ is even} \\ \frac{1}{2}(|G|^2 - 2|G| + 2), & \text{if } n \text{ is odd} \end{cases}.$$

Proof. Let G be a finite cyclic group. Then there are two cases for $|G|$.
Case 1. Let $|G|$ be odd. Then $|G|-1$ is even. In view of Theorem 8, the total number of non-adjacent edges in $IG(G)$ is $\frac{1}{2}(|G|-1)$. But the maximum number of edges in a simple graph of order $|G|$ is $\binom{|G|}{2}$. So in this case the total number of edges in $IG(G)$ is $\binom{|G|}{2}$ - $\frac{1}{2}(|G|-1) = \frac{1}{2}(|G|-1)^2$.
Case 2. Let $|G|$ be even. Then, in view of Theorem 5, there are exactly $\frac{|G|}{2} - 1$ pairs of distinct vertices that satisfy the relation $ab = e$ in G. Therefore, the total number of non-adjacent pairs in $IG(G)$ is $\frac{|G|}{2} - 1$. So in this case the total number of edges in $IG(G)$ is $\binom{|G|}{2}$ -$(\frac{|G|}{2} - 1) = \frac{1}{2}(|G|^2 - 2|G| + 2)$. □

By using Theorem 8, the following short table illustrates the way we can easily determine the size of an invertible graph of some finite groups.

Group G	Z_p^*	G_{2p}	$U(2^k)$	S_3
Size of $IG(G)$	$\frac{1}{2}(p^2 - 4p + 5)$	$\frac{1}{2}(p^2 - 4p + 5)$	$2^{k-1}(2^k - 2) + 2$	14

Theorem 11. *Let $S(G) \neq G$. Then $IG(G)$ is never a complete graph.*

Proof. Suppose on the contrary that, $IG(G)$ is a complete graph. Then by the Theorem 7, the size of $IG(G)$ is $\binom{|G|}{2}$= $\frac{|G|}{2}(|G|-1)$, but in view of Theorem 8, we arrived at a contradiction to the completeness of $IG(G)$. □

Our next theorem shows how the bi-implication of $S(G) = G$ and completeness of $IG(G)$ are intertwining.

Theorem 12. *The invertible graph $IG(G)$ is complete if and only if $S(G) = G$.*

Proof. **Necessity.** Suppose that $IG(G)$ is a complete graph of a finite group G. Then any two vertices a and b in G are adjacent in $IG(G)$. Consequently $ab \neq e$, for every $a, b \in G$. This implies that $a \neq b^{-1}$ and $b \neq a^{-1}$. Therefore $a = a^{-1}$ and $b = b^{-1}$. That is, $a, b \in S(G)$. This shows that $G \subseteq S(G)$, also since $S(G) \subseteq G$. Hence $S(G) = G$.
Sufficiency. Let $S(G) = G$. Suppose $IG(G)$ is not a complete graph. Then there exist distinct vertices a and b in G such that $ab = e$ and $ba = e$. This implies that $a^{-1} = b$ and $b^{-1} = a$. It is clear that $a, b \notin S(G)$. Therefore, $S(G) \neq G$, which is a contradiction to our hypothesis, and hence $IG(G)$ is complete. □

We are now ready to prove a number of useful consequences of Theorem 12.

Corollary 5. *The graph $IG(G)$ is complete if and only if G is isomorphic to one of the groups, $Z_2 \times Z_2$, U_4, U_6, U_8, U_{12} and V_4.*

Proof. It is true from the fact that the Klein four-group V_4 is isomorphic to $Z_2 \times Z_2$, U_8, U_{12}. Also the group Z_2 is isomorphic to U_4, U_6. □

Corollary 6. *The invertible graph of G is complete if and only if $|G| = 2$.*

Proof. We have, $|G| = 2 \Leftrightarrow G = \{e, a : a^2 = e\} \Leftrightarrow G = S(G)$. □

Before going to further properties of invertible graph, let us consider the following example for the description of the result in the Theorem 12.

Example 4. *The invertible graph of the group $(P(X), \Delta)$ is complete. Let $X = \{a, b, c\}$ and let $A = \{a\}$, $B = \{b\}$, $C = \{c\}$ so that $\overline{A} = \{b, c\}$, $\overline{B} = \{a, c\}$ and $\overline{C} = \{a, b\}$. Then $P(X) = \{\phi, A, B, C, \overline{A}, \overline{B}, \overline{C}, X\}$ is* *an abelian group with respect to the symmetric difference* Δ *of sets and* $S(P(X)) = P(X)$ *but* $P(X)$ *is not a Klein four-group. The Figure 2 shows the complete invertible graph of the group* $(P(X), \Delta)$.

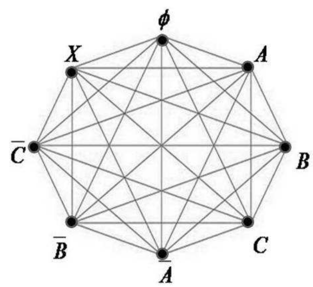

Figure 2. The invertible graph of $(P(X), \Delta)$.

Theorem 13. [8] *A simple graph is bipartite if and only if it does not have any odd cycle.*

Theorem 14. *If* $|G|$ *is composite, then* $IG(G)$ *is not a bipartite graph.*

Proof. Assume that $|G|$ is composite. Suppose $IG(G)$ is a bipartite graph. Then there exists a bipartition $(S(G), M(G))$. Without loss of generality we assume that $e \notin M(G)$. Since $|G|$ is composite, so there exists at least one self-inverse element $s \neq e$ in $S(G)$. If $m \in M(G)$ such that $m^{-1} \neq m$, then clearly $es \neq e$, $sm \neq e$ and $me \neq e$. Therefore, the triads (s, e, m) of the graph $IG(G)$ form a triangle. This violates the condition in the Theorem 13 for a bipartite graph. Hence $IG(G)$ is not a bipartite graph. □

Theorem 15. *Let* $|G| > 3$*. Then the girth of invertible graph* $IG(G)$ *is* 3*.*

Proof. We know that the girth of a simple graph is the length of a smallest cycle. Here there exist two cases.
Case 1. Suppose $S(G) = G$ and $|G| > 3$. Then, by the Theorem 12, $IG(G)$ is complete. Therefore $IG(G)$ has a smallest cycle of length 3. Hence, $gir(IG(G)) = 3$.

Case 2. Suppose $S(G) \neq G$ and $|G| > 3$. The vertex e in $IG(G)$ is adjacent to all other vertices. For this reason we can choose two vertices $a \neq e$ and $b \neq e$ in $IG(G)$ such that $a^{-1} = a$ and $b^{-1} \neq b$. Then $ab \neq ab^{-1} \Rightarrow ab \neq a^{-1}b^{-1} \Rightarrow ab \neq (ba)^{-1} \Rightarrow ab \neq e$, since $S(G) \neq G$. This implies that $ea \neq e, ab \neq e$ and $be \neq e$, so the graph $IG(G)$ always has a three cycle $e - a - b - e$, which is the smallest. Hence, $gir(IG(G)) = 3$. □

For distinct vertices x and y of a simple graph X, the diameter of X is $diam(X) = max\{d(x, y) : x, y \in V(X)\}$, where $d(x, y)$ is the length of the shortest path from x to y in X.

Theorem 16. *If $|G| > 1$, then the diameter of invertible graph is either* 1 *or* 2.

Proof. Let G be a finite group with $|G| > 1$. Then we consider the following two cases on $S(G)$.
Case 1. Suppose $S(G) = G$. In view of Theorem 12, $IG(G)$ is complete, hence $diam(IG(G)) = 1$.
Case 2. Suppose $S(G) \neq G$. Then, $IG(G)$ is never a complete graph. Let us assume that a and b are any two vertices in $IG(G)$. However, if the vertex $a \neq e$ is adjacent to vertex $b \neq e$, then trivially $d(a, b) = 1$. Otherwise, if a is not adjacent to b in $IG(G)$, then clearly $d(a, b) > 1$, where $a \neq e$ and $b \neq e$, but in the graph $IG(G)$ there always exists a path $a-e-b$ of the shortest length 2. It follows that $diam(IG(G)) = 2$.

From Case 1 and Case 2 we conclude that the diameter of invertible graph is either 1 or 2. □

A clique in a simple graph X is a complete subgraph. A clique Y in X is called maximal if no vertex set outside of Y is adjacent to all members of Y. The size of the largest clique in X is called the clique number $\omega(X)$. Simply, $\omega(X)$ is the maximum number of pair wise adjacent vertices. For any simple graph X, $1 \leq \omega(X) \leq |V(X)|$.

Theorem 17. *Let G be a finite group. Then the clique number of $IG(G)$ is $\omega(IG(G)) = \frac{1}{2}(|G| + |S(G)|)$.*

Proof. If $s \in S(G)$, then clearly the vertex s is adjacent to all other vertices of $IG(G)$ since $sa \neq e$ for every a in G. Therefore, the pair of non-adjacent vertices in $IG(G)$ is of degree $\frac{1}{2}|M(G)|$, and hence the total number of mutually adjacent vertices in $IG(G)$ is $|G| - \frac{1}{2}|M(G)| = \frac{1}{2}(|G| + |S(G)|)$, which is $\omega(IG(G))$. □

Theorem 18. *Let G be a finite cyclic group. Then the clique number of $IG(G)$ is* $\omega(IG(G)) = \begin{cases} \frac{1}{2}(|G|+1), & \text{if } |G| \text{ is odd} \\ \frac{1}{2}(|G|+2), & \text{if } |G| \text{ is even} \end{cases}$.

Proof. We know that the order of invertible graph $IG(G)$ of a finite cyclic group G is $|G|$. We need the following two cases on $|G|$.
Case 1. $|G|$ is odd. Trivially any two distinct vertices a and b are non adjacent if and only if $ab = e$. It follows that any vertex a is non-adjacent with exactly one vertex b, and hence total number of such vertices in $IG(G)$ is $|G| - 1$. It is clear that, the pair of non-adjacent vertices in $IG(G)$ is of degree $\frac{1}{2}(|G| - 1)$, and hence the total number of mutually adjacent vertices in $IG(G)$ is $|G| - \frac{1}{2}(|G| - 1) = \frac{1}{2}(|G| + 1)$, which is the size of a maximum clique.
Case 2. $|G|$ is even. So in this case the pair of non-adjacent vertices in $IG(G)$ is of degree $\frac{1}{2}(|G| - 2)$. Hence total number of mutually adjacent vertices in $IG(G)$ is $|G| - \frac{1}{2}(|G| - 2) = \frac{1}{2}(|G| + 2)$, which is the size of a maximum clique. This completes the proof of the theorem. □

Example 5. *The following table shows the values of $\omega(IG(G))$ for some non-cyclic groups G.*

Non-cyclic group	V_4	Q_8	S_3	D_3	D_4
$\omega(IG(G))$	4	5	5	5	7

Definition 3. *A simple graph X is n-colorable if there exists a colouring of X which uses n colours. The minimum number of colors required*

to color a graph X is called the chromatic number and is denoted by $\chi(X)$. Note that $\omega(X) \leq \chi(X) \leq |V(X)|$. If $\chi(X) = \omega(X)$, then the graph X is called weakly-perfect.

Definition 4. *In a simple graph X, the set of pair-wise non-adjacent vertices is called an independent set of vertices.*

Theorem 19. *Let G be a finite group. Then the chromatic number of $IG(G)$ is $\chi(IG(G)) = \frac{1}{2}(|G| + |S(G)|)$.*

Proof. **Case 1.** Suppose that $S(G) \neq G$. In this case $a \in M(G)$ if and only if a is not adjacent with exactly one vertex in $IG(G)$. Therefore, the maximum independent set of $IG(G)$ is of size 2, moreover, total number of such independent sets in $IG(G)$ is $\frac{1}{2}|M(G)|$. For all these vertices we need $\frac{1}{2}|M(G)|$ colors, since each independent set in $IG(G)$ is uniquely colorable. But, vertices in $S(G)$ are adjacent with all the vertices in $M(G)$, and thus we require for $S(G)$ more colors distinct from these colors. Hence, the minimum number of colors required to colour the invertible graph is

$$\chi(IG(G)) = \frac{1}{2}|M(G)| + |S(G)| = \frac{1}{2}(|G| + |S(G)|),$$

since $G = S(G) \cup M(G)$ and $S(G) \cap M(G) = \phi$.
Case 2. Suppose $S(G) = G$. Then, trivially, $|M(G)| = 0$. But, by the Theorem 12 the graph $IG(G)$ is complete, therefore the required result is obviously true. That is, $\chi(IG(G)) = \frac{1}{2}(|G| + |S(G)|)$. □

Theorem 20. *Let G be a finite cyclic group. Then the chromatic number of $IG(G)$ is*

$$\chi(IG(G)) = \begin{cases} \frac{1}{2}(|G| + 1), & \text{if } |G| \text{ is odd} \\ \frac{1}{2}(|G| + 2), & \text{if } |G| \text{ is even} \end{cases}.$$

Proof. **Case 1.** Suppose that $|G| = 2$. Then obviously, $IG(G) \cong K_2$, and hence $\chi(IG(G)) = 2$.

Case 2. Suppose that $|G| > 2$. Then $IG(G)$ is never a complete graph since $S(G) \neq G$ for any finite cyclic group G with $|G| > 2$. Now we shall show the required result with the help of the following two subcases.

Subcase 1. Suppose that $|G|$ is even. Then, $|S(G)| = 2$, since G is cyclic. Therefore the order of the independent set in the graph $IG(G)$ is $\frac{1}{2}(|G| - 2)$. In fact each independent set is uniquely colorable, it means that for all these vertices we need $\frac{1}{2}(|G| - 2)$ colors. However two vertices in $S(G)$ are adjacent with remaining all vertices in $IG(G)$, thus minimum number of colors to color the invertible graph is $\frac{1}{2}(|G| - 2) + 2 = \frac{1}{2}|G| + 1$.

Subcase 2. Suppose that $|G|$ is odd. Then, $|S(G)| = 1$, since G is cyclic. Therefore the order of independent set in $IG(G)$ is $\frac{1}{2}(|G| - 1)$. Here one vertex in $S(G)$ is adjacent with all other vertices of $IG(G)$, thus we require one more color from these colors. Hence,

$$\chi(IG(G)) = \frac{1}{2}(|G| - 1) + 1 = \frac{1}{2}(|G| + 1).$$

□

By combining Theorems 17 and 19, we can easily prove that the invertible graph of any finite group is weakly perfect.

Theorem 21. *For any finite cyclic group G, the graph $IG(G)$ is weakly perfect.*

Proof. It follows directly from Theorems 17 and 19, since $\omega(IG(G)) = \chi(IG(G))$. □

5 Isomorphic properties of IG(G)

In this section, we examine isomorphic properties of invertible graphs of finite groups in detail and determine their important characteristics. We begin with a few examples.

Example 6. *The isomorphic groups and their invertible graphs are traced in Figure 3.*

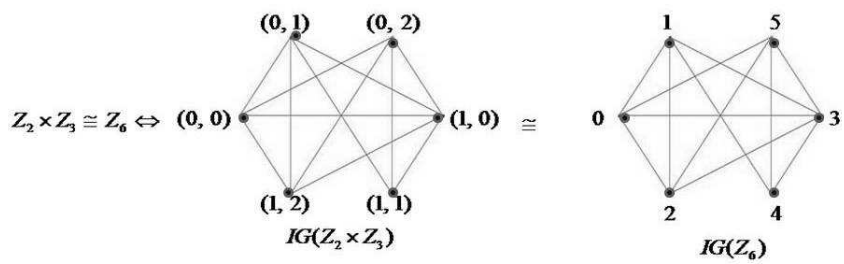

Figure 3. $Z_2 \times Z_3 \cong Z_6 \Leftrightarrow IG(Z_2 \times Z_3) \cong IG(Z_6)$.

Example 7. *Figure 4 shows that groups are not isomorphic and their invertible graphs are also not isomorphic.*

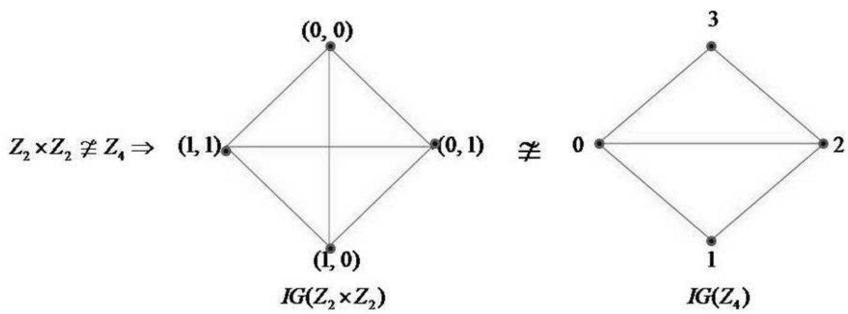

Figure 4. $Z_2 \times Z_2 \ncong Z_4 \Leftrightarrow IG(Z_2 \times Z_2) \ncong IG(Z_4)$.

Example 8. *Figure 5 shows that groups are not isomorphic but their invertible graphs are isomorphic. Consider the cyclic group* $G = \{I, A, B, C, D, E, F, G, H\}$ *with respect to addition modulo* 9 *and* $G' = \{I', A', B', C', D', E', F', G', H'\}$ *is an abelian but not cyclic group with respect to addition modulo 3, where* $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}, C = \begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix}, D = \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}, E = \begin{bmatrix} 5 & -5 \\ 5 & -5 \end{bmatrix}, F = \begin{bmatrix} 6 & -6 \\ 6 & -6 \end{bmatrix},$

$G = \begin{bmatrix} 7 & -7 \\ 7 & -7 \end{bmatrix}, H = \begin{bmatrix} 8 & -8 \\ 8 & -8 \end{bmatrix}, I' = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, A' = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \varphi : G \to G', C' = \begin{bmatrix} x+1 & -(x+1) \\ x+1 & -(x+1) \end{bmatrix}, D' = \begin{bmatrix} 2x+1 & -(2x+1) \\ 2x+1 & -(2x+1) \end{bmatrix}, E' = \begin{bmatrix} x+2 & -(x+2) \\ x+2 & -(x+2) \end{bmatrix}, F' = \begin{bmatrix} 2x+2 & -(2x+2) \\ 2x+2 & -(2x+2) \end{bmatrix}, G' = \begin{bmatrix} 2x & -2x \\ 2x & -2x \end{bmatrix}, H' = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$ *and x is an indeterminate over Z_3. Under the mapping $\varphi : G \to G'$ such that $\varphi(I) = I'$, $\varphi(A) = A'$, $\varphi(B) = B'$, $\varphi(C) = C'$, $\varphi(D) = D'$, $\varphi(E) = E'$, $\varphi(F) = F'$, $\varphi(G) = G'$, $\varphi(H) = H'$. Hence, the fact that G is not isomorphic to G' implies that $IG(G) \cong IG(G')$.*

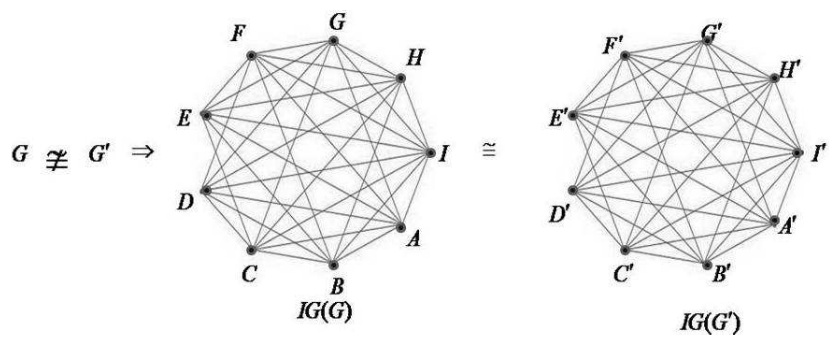

Figure 5. $G \ncong G' \Rightarrow IG(G) \cong IG(G')$.

As the above examples suggest, the invertible graphs of isomorphic groups are isomorphic but converse need not to be true. So, the next theorem completely characterises all isomorphic invariable graphs.

Theorem 22. *Let G and G' be finite groups. If $G \cong G'$, then $IG(G) \cong IG(G')$. But the converse is not true.*

Proof. Suppose that $G \cong G'$. Then there is a group isomorphism f from G onto G' such that $f(a) = a'$, for every element a in G and a' in G'. Now, define a map φ from $IG(G)$ to $IG(G')$ by the relation $\varphi(a) = f(a)$, for every vertex $a \in G$. By Remark 1, φ is a bijection. Now let us prove that φ preserves adjacency. For this let $ab \neq e$, then

$f(ab) \neq f(e)$. That implies $f(a)f(b) \neq f(e)$. That is, $\varphi(a)\varphi(b) \neq e'$. So the vertex $\varphi(a)$ is adjacent to the vertex $\varphi(b)$ in $IG(G')$. Similarly, if a is not adjacent to b in $IG(G)$, then $\varphi(a)$ is also not-adjacent to $\varphi(b)$ in $IG(G')$. This shows that $IG(G) \cong IG(G')$. The converse of this statement is false, as the Example 8 shows. That is, if $IG(G) \cong IG(G')$, it does not necessarily follow that $G \cong G'$. $\square$

Let G be a finite group. Then an isomorphism from G onto G is called a group automorphism and set of all automorphisms of G is denoted by $Auto(G)$. Further, an isomorphism from a simple graph X to itself is called graph automorphism of X, and the set of all graph automorphisms forms a group under the operation of composition. This group is also denoted by $Auto(X)$ and is called automorphism group of a graph X.

The following result is an analogous result between $Auto(G)$ and $Auto(IG(G))$.

Theorem 23. *If G is a finite group, then $Auto(G) \subseteq Auto(IG(G))$. But the converse is not true.*

Proof. Let $\psi \in Auto(G)$. Then $\psi : G \to G'$ is a group isomorphism from G onto itself. We shall now show that $\psi \in Auto(IG(G))$. Suppose vertices a and b in G are adjacent in $IG(G)$. Then, either $ab \neq e$, or, $ba \neq e. \Rightarrow \psi(ab) \neq e$, or, $\psi(ba) \neq e. \Rightarrow \psi(a)\psi(b) \neq e$, or, $\psi(b)\psi(a) \neq e. \Rightarrow$ The vertex $\psi(a)$ is adjacent to the vertex $\psi(b)$ in $IG(G)$.

This shows that ψ is a graph isomorphism from $IG(G)$ onto itself. It is clear that $\psi \in Auto(IG(G))$. Hence, $Auto(G) \subseteq Auto(IG(G))$. But, the converse of this result is not true. For this we consider the group $Z_5 = \{0, 1, 2, 3, 4\}$ with respect to addition modulo 5. Define a map $\psi : Z_5 \to Z_5$ by $\psi(0) = 0$, $\psi(1) = 2$, $\psi(2) = 3$ and $\psi(3) = 4$. It is clear that $Auto(Z_5) \subseteq IG(Z_5)$. But $\psi(1 \oplus_5 2) = \psi(Z_3) = 4$ and $\psi(1) \oplus_5 \psi(2) = 2 \oplus_5 3 = 0$. Therefore, $\psi(1 \oplus_5 2) \neq \psi(1) \oplus_5 \psi(2)$ so that ψ is not a homomorphism of group Z_5. $\square$

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References

[1] Xuan Long Ma, Hua Quan Wei, and Guo Zhong, "The cyclic graph of a finite group," *Algebra*, vol. 2013, pp. 1–7, 2013, Art.ID. 107265. Available: http://dx.doi.org/10.1155/2013/107265.

[2] T. Chalapathi and R.V.M.S.S. Kiran Kumar, "The structural properties of even free graphs," *Int. J. Comput. Applications*, vol. 121, no. 2, pp. 19–21, July 2015. Available: http://doi.org/10.5120/21512-4476.

[3] Peter J. Cameron and Shamik Ghosh, "The power graph of a finite group," *Discrete Mathematics*, vol. 311, no. 6, pp. 1220–1222, July 2011. Available: https://doi.org/10.1016/j.disc.2010.02.011.

[4] M. Sattanathan and R. Kala, "An introduction to order prime graph," *Int. J. Contemp. Math. Sciences*, vol. 4, no. 10, pp. 467–474, Jan. 2009.

[5] Timothy C. Burness and Elisa Covato, "On the prime graph of simple groups," *Bulle. of Aus. Math. Socie.*, vol. 91, no. 2, pp. 227–240, Apr. 2015. Available: https://doi.org/10.1017/S0004972714000707.

[6] Jonathan K. Hodge, Steven Schlicker, and Ted Sundstrom, *Abstract Algebra: An Inquiry Based Approach* (Textbooks in Mathematics), CRC Press, 2014, 595 p. ISBN-13: 978-1466567061, ISBN-10: 1466567066.

[7] Norman Biggs, *Algebraic graph theory* (Cambridge Mathematical Library), 2nd ed., Cambridge, USA: Cambridge Univ. Press, 1993, 216 p. ISBN-10: 0521458978, ISBN-13: 978-0521458979.

[8] Vitaly I. Voloshin, *Introduction to graph theory*, New York: Nova Sci. Publishers. Inc, 2009, 144 p. ISBN-10: 1606923749, ISBN-13: 978-1606923740.

[9] *Encyclopedic Dictionary of Mathematics: The Mathematical Society of Japan*, 2nd ed., vol. 1, Kiyosi Ito, Ed., The MIT Press, 2000, 2168 p. ISBN-10: 0262590204, ISBN-13: 978-0262590204.

[10] Vivek Sahai and Vikas Bist, *Algebra*, 3rd ed., Oxford, United Kingdom: Alpha Science International Ltd, 2008, 347 p. ISBN-10: 1842654063, ISBN-13: 9781842654064.

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