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TWO-DIMENSIONAL GREEN'S FUNCTION FOR THERMAL STRESSES IN A SEMI-LAYER UNDER A POINT HEAT SOURCE

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We present specific new expressions for thermal stresses as Green's functions for a plane boundary value problem of steady-state thermoelasticity for a semi-layer. We also obtain new integration formulas of Green's type, which determine the thermal stresses in the form of integrals of the products of the given distributed internal heat source, boundary temperature, and heat flux and derived kernels. Elementary functions results obtained are formulated in a theorem, which is proved using the harmonic integral representations method to derive thermal stresses Green's functions, which are written in terms of Green's functions for Poisson's equation. A new solution to particular two-dimensional boundary value problem for a semi-layer under a boundary constant temperature gradient is obtained in explicit form. Graphical presentations for thermal stresses Green's functions created by a unit heat source (line load in out-of-plane direction) and by a temperature gradient are also included.

Keywords: Green's functions; Harmonic integral representations; Heat conduction; Temperature gradient; Thermal stresses; Thermoelastic volume dilatation

INTRODUCTION

Green's function method (GFM) gives solutions to boundary value problems (BVPs), including those of thermoelasticity [1–8], in the form of integrals. The major difficulty with this method is the construction of Green's functions (GFs) themselves. There are presently many classic methods of constructing GFs, but only a few are suitable for BVPs of thermoelasticity. It is our opinion that this situation arises because the governing equations of steady-state thermoelasticity have a vector character, but the present practice is to apply the same classical methods as used to solve scalar differential equations. The lead author has developed a new approach [9, 10] for constructing main thermoelastic Green's functions (MTGFs). This approach is based on an effective unified method, called the

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harmonic integral representations method (HIRM) [11]. It involves new integral representations for MTGFs via Green's functions for Poisson's equation (GFPE). In the existing HIRM, classical methods are used for GFPE construction. For MTGFs construction, the same mathematical procedure for both 2D- and 3D-BVPs of thermoelasticity can be followed. We begin this article by discussing HIRM and the GΘ convolution method.

Two Methods for Deriving MTGFs

Two methods are proposed for derivation of displacements MTGFs $U_i(x, \xi)$, $i = 1, 2, 3$ in steady-state isotropic thermoelasticity: GΘCM [12–18] and HIRM [11]. The success of both these methods depends on the ability to derive GFs for Poisson's equation (GFPE) [19–23], and on the ability to derive volume dilatation $\Theta^{(i)}(x, \xi)$ of BVPs for Lamé's equations in the theory of elasticity [24–26]. In particular, GOCM is based on the convolution formula [12–18]:

$$U_i(x, \xi) = \gamma \int_V G_T(x, z)\ \Theta^{(i)}(z, \xi)\, dV(z); \quad x, z, \xi \in V \tag{1}$$

Integration is taken on the volume of the body V; $\gamma = \alpha_t(2\mu + 3\lambda)$ is the thermoelastic constant; λ, μ are Lamé's elastic constants; α_t is the coefficient of the linear thermal expansion; G_T is the GF for a BVP of steady-state heat conduction corresponding to an unit internal point (line load in out-of-plane direction) heat source, and $\Theta^{(i)}$ are influence functions of unit concentrated body forces onto elastic volume dilatation.

Then, HIRM is based on the harmonic integral representations (HIR) for MTGFs [9–11]:

$$\begin{aligned} U_i(x, \xi) = {} & \frac{\gamma \xi_i}{2\mu} G_T(x, \xi) - \frac{\lambda + \mu}{2\mu} \xi_i \Theta(x, \xi) - \frac{\gamma}{2(\lambda + 2\mu)} x_i G_i(x, \xi) \\ & - \int_\Gamma \left[V_i(x, y) \frac{\partial}{\partial n_\Gamma} - \frac{\partial V_i(x, y)}{\partial n_\Gamma} \right] G_i(y, \xi)\, d\Gamma(y) \end{aligned} \tag{2}$$

where Θ is thermoelastic volume dilatation (TVD); $\partial/\partial n_\Gamma$ is the derivative with respect to external normal n_Γ to surface Γ of the body V; G_i and G_T are GFPE, which are linked with boundary conditions for U_i and temperature T respectively; $x \equiv (x_1, x_2, x_3)$ is the point of application of heat source F; $\xi \equiv (\xi_1, \xi_2, \xi_3)$ is the point of finding U_i; $x, \xi \in V$; $y \in \Gamma$. Finally the functions $V_i(x, y)$ are defined as follows:

$$V_i(x, y) = U_i(x, y) + \frac{\xi_i}{2\mu}\left[(\lambda + \mu)\Theta(x, y) - \gamma G_T(x, y)\right], \quad i = 1, 2, 3 \tag{3}$$

In addition the integral representation for TVD Θ is written in the form [10, 11]:

$$\Theta(x, \xi) = \frac{\gamma}{\lambda + 2\mu} G_\Theta(x, \xi) + \int_\Gamma \left[\frac{\partial \Theta(x, y)}{\partial n_\Gamma} - \Theta(x, y) \frac{\partial}{\partial n_\Gamma} \right] G_\Theta(y, \xi)\, d\Gamma(y) \tag{4}$$

where G_Θ is the fundamental solution that is linked with the boundary conditions for TVD. Finally, if MTGFs U_i are known, then thermoelastic displacements $u_i(\xi)$ within a thermoelastic body V are determined by the following integral formula [9–18]:

$$u_i(\xi) = a^{-1}\int_V F(x)\, U_i(x,\xi)\, dV(x) - \int_{\Gamma_D} T(y)\frac{\partial U_i(y,\xi)}{\partial n_y} d\Gamma_D(y) + \int_{\Gamma_N} \frac{\partial T(y)}{\partial n_y} U_i(y,\xi)\, d\Gamma_N(y) + a^{-1}\int_{\Gamma_M}\left[\alpha T(y) + a\frac{\partial T(y)}{\partial n_y}\right] U_i(y,\xi)\, d\Gamma_M(y);\quad i = 1, 2, 3 \tag{5}$$

where a is thermal conductivity; α is the coefficient of heat conductivity; Γ_D, Γ_N and Γ_M are the surfaces on which the boundary conditions of Dirichlet's, Neumann's and mixed types are given, respectively: temperature $T(y)$, heat flux $a\partial T(y)/\partial n_y$ and a heat exchange between exterior medium and surface of the body described by a convection law, $\alpha T(y) + a\partial T(y)/\partial n_y$. Note that use of GΘCM led to some new thermoelastic GFs and Green-type integral formulas for semi-infinite thermoelastic Cartesian bodies in terms of elementary functions: half-plane [18], quadrant [15, 16], half-space [12, 14] and quarter-space [13, 17]. In similar fashion HIRM led to useful formulas for MTGFs in semi-infinite thermoelastic Cartesian bodies, including octants [9–11]. These results indicated that both methods were used for Cartesian bodies that have no straight lines or planes parallel to the Cartesian ones.

RESTRICTIONS OF THE GΘCM

Obtaining solutions for Cartesian bodies that have straight lines or planes parallel to the Cartesian axes or planes by using GΘCM is not possible. This situation is explained by the fact that, for such kind of domains, the thermoelastic potential functions are as yet unknown. This means that the solution of Poisson's equation $\nabla^2 U = f$ with respect to thermoelastic potential function U is yet unknown (f is fundamental solution to Laplace operator for considered domain).

We remember that the function U must be known when we have to compute integral (1) using Green's formula

$$\int_V \left(\varphi\nabla^2 Q - \psi\nabla^2 P dV\right) dV = \int_\Gamma (\varphi\partial Q/\partial n - \psi\partial P/\partial n)\, d\Gamma \tag{6}$$

to transform the volume integral in a surface integral.

As examples, for half-space [12, 14], quarter-space [13, 17], and octant the singular fundamental solution of the Laplace operator ∇^2 and thermoelastic potential function is:

$$f = (4\pi)^{-1} R^{-1} = (1/4\pi)\sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2 + (x_3-\xi_3)^2} \quad U = (8\pi)^{-1} R \tag{7}$$

but for a half-plane [18] and quadrant [15, 16]

$$f = -(2\pi)^{-1}\ln r;\; r = \sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2},\quad U = -(8\pi)^{-1} r^2(\ln r - 1) \tag{8}$$

Finally, for a strip, semi-strip and rectangle

$$
\begin{aligned}
&f = -(4\pi)^{-1} \ln E \quad or \quad f = -(4\pi)^{-1} \ln \tilde{E} \\
&E = 1 - 2e^{(\pi/a_2)(x_1-\xi_1)} \cos(\pi/a_2)(x_2 - \xi_2) + e^{(2\pi/a_2)(x_1-\xi_1)} \quad or \\
&\tilde{E} = 1 - 2e^{(\pi/2a_2)(x_1-\xi_1)} \cos(\pi/2a_2)(x_2 - \xi_2) + e^{(\pi/a_2)(x_1-\xi_1)}
\end{aligned} \tag{9}
$$

but the function U remains unknown.

Objectives

The main objective of this research was to formulate a theorem on the derivation of two-dimensional (2D) MTGFs for displacements and stresses and Green-type integral formula for a specific BVP for a half-strip with different types of mixed homogeneous mechanical and thermal boundary conditions. Therefore, it is necessary to first obtain from Eqs. (2)–(4) the integral representations for MTGFs for a half-strip, and to elaborate a technique for computing of a new special integral which appears on one parallel site of the half-strip [27]. The validity of the derived MTGFs was checked with respect to points $x \equiv (x_1, x_2)$ and $\xi \equiv (\xi_1, \xi_2)$. Another important objective of this research was to solve a particular BVP of thermoelasticity using derived MTGFs. To reach this objective we used the following Green's-type integral formula for thermal stresses that is proposed in [15]:

$$
\begin{aligned}
\sigma(\xi) = a^{-1} \int_V F(x)\, \sigma^*(x, \xi)\, dV(x) - \int_{\Gamma_D} T(y) \frac{\partial \sigma^*(y, \xi)}{\partial n_y} d\Gamma_D(y) \\
+ \int_{\Gamma_N} \frac{\partial T(y)}{\partial n_y} \sigma^*(y, \xi)\, d\Gamma_N(y) + a^{-1} \int_{\Gamma_M} \left[\alpha T(y) + a \frac{\partial T(y)}{\partial n_y} \right] \sigma^*(y, \xi)\, d\Gamma_M(y)
\end{aligned} \tag{10}
$$

where $\sigma(\xi)$ and $\sigma^*(x, \xi)$ are the thermal stress tensors. The components of these tensors: $\sigma_{ij}(\xi)$; $i, j = 1, 2, 3$ and $\sigma^*_{il}(x, \xi)$ are created by the prescribed thermal data and by an internal unit point heat source, respectively. The thermal stresses σ_{ij} and σ^*_{ij} are determined by using the Duhamel–Neumann law [5, 6]:

$$
\sigma_{ij} = \mu\left(u_{i,j} + u_{j,i}\right) + \delta_{ij}(\lambda\theta - \gamma T); \quad \theta = u_{k.k}; \quad i, j, k = 1, 2, 3 \tag{11}
$$

$$
\sigma^*_{ij} = \mu\left(U_{i,j} + U_{j,i}\right) + \delta_{ij}(\lambda\Theta - \gamma G_T); \quad \Theta = U_{k,k}(x, \xi) \tag{12}
$$

It should be noted that the main advantage of the integral formula (5) is that it allows us to determine directly (without pre-determining the thermoelastic displacements and of the temperature field) the thermoelastic stresses in the form of integrals, containing products of the known thermal data and kernels σ^*_{ij}. Finally the last objective is the computer evaluation and graphical presentation of the derived TSGEs σ^*_{ij} and thermal stresses σ_{ij} for one particular BVP of thermoelasticity (see Appendix).

DERIVATION AND CHECK OF THE 2D MTGFS U_i, TSGFS σ^*_{ij} AND GREEN-TYPE INTEGRAL FORMULA FOR σ_{ij}

Here we consider the problem of static equilibrium of the thermoelastic semi-layer, located in the plane strain problem conditions (plane strain problem for a semi-strip). The half-strip is exposed by a unit point internal heat source (in this case the TSGFs σ^*_{ij} have to be constructed), and distributed thermal actions (in this case we derive an appropriate Green-type integral formula for the solution σ_{ij} to a BVP of thermoelasticity). Next we derive the TSGFs σ^*_{ij}. Derivation of the Green-type integral formula for thermal stresses σ_{ij} within a half-strip is presented next.

Theorem. *Let the field of displacements $u_i(\xi)$ in inner points $\xi \equiv (\xi_1, \xi_2)$ of the thermoelastic half-strip $V\,(0 \le x_1 < \infty;\ 0 \le x_2 \le a_2)$ be determined by Lamé equations* [5, 6]

$$\mu\nabla^2 u_i(\xi) + (\lambda+\mu)\,\theta_{,i}(\xi) - \gamma T_{,i}(\xi) = 0; \quad i = 1, 2 \tag{13}$$

but in the points $y \equiv (0, y_2)$, $y \equiv (y_1, 0)$, and $y \equiv (y_1, a_2)$ of boundary lines $\Gamma_{10}\,(y_1 = 0;\ 0 \le y_2 \le a_2)$, $\Gamma_{20}\,(0 \le y_1 < \infty; y_2 = 0)$ and $\Gamma_{21}\,(0 \le y_1 < \infty; y_2 = a_2)$ the following homogeneous locally mixed mechanical boundary conditions are given:

$$\begin{aligned} &u_1 = \sigma_{12} = 0, \quad \xi_1 = 0,\ 0 \le \xi_2 \le a_2; \quad \sigma_{22} = u_1 = 0; \quad \xi_2 = 0, \quad 0 \le \xi_1 \le \infty; \\ &u_2 = \sigma_{21} = 0, \quad \xi_2 = a_2, \quad 0 \le \xi_1 \le \infty \end{aligned} \tag{14}$$

where σ_{22} and $\sigma_{12} = \sigma_{21}$ are the normal and the tangential stresses which are determined by the well-known Duhamel–Neumann law (11) *at $i, j = 1, 2$.*

In addition let the temperature field $T(\xi)$ in Eq. (13), *generated by the inner heat source $F(\xi)$, boundary temperature and heat fluxes satisfy the following BVP of steady-state heat conduction*

$$\nabla^2 T(\xi) = -a^{-1}F(\xi)\,, \ \ \xi \in V, \tag{15a}$$

$$\begin{aligned} &a\partial T/\partial n_{y1} = S_{10}(y)\,, \quad y \equiv (0, y_2) \in \Gamma_{10}; \quad T(y) = T_{20}(y)\,, \quad y \equiv (y_1, 0) \in \Gamma_{20}; \\ &a\partial T/\partial n_{y2} = S_{21}(y)\,, \quad y \equiv (y_1, a_2) \in \Gamma_{21} \end{aligned} \tag{15b}$$

If the inner heat source and boundary thermal data satisfy the conditions:

$$\begin{aligned} &\int_0^{+\infty}\int_0^{a_2} |F(x)|\, dx_1 dx_2 < \infty; \\ &x \equiv (x_1, x_2)\,; \int_0^{a_2} |T_{10}(0, y_2)|\, dy_2 < \infty, \ \int_0^{+\infty} |T_{20}(y_1, 0)|\, dy_1 < \infty, \\ &\int_{-\infty}^{+\infty} |S_{21}(y_1, a_2)|\, dy_1 < \infty \end{aligned} \tag{16}$$

then the solution of BVP in Eqs. (13)–(16) *of thermoelasticity for unknown thermal stresses $\sigma_{ij}(\xi)$ exists and it can be presented by the following Green-type integral* *formula,*

$$\sigma_{ij}(\xi) = \frac{1}{a}\left[\int_0^{+\infty}\int_0^{a_2} F(x)\,\sigma^*_{ij}(x,\xi)\,dx_1dx_2 + \int_0^{a_2} S_{10}(0,y_2)\,K_{ij}(0,y_2;\xi)\,dy_2 + \int_0^{+\infty} S_{21}(y_1,a_2)\,\Pi_{ij}(y_1,a_2;\xi)\,dy_1\right] - \int_0^{+\infty} T_{20}(y_1,0)\,Q_{ij}(y_1,0;\xi)\,dy_1 \tag{17}$$

The kernels (TSGFs $\sigma^*_{ij}(x,\xi)$*) in Eq.* (17) *at* $i,j=1,2$; $K_{ij}(0,y_2;\xi) = \sigma^*_{ij}(y,\xi)$, $y \equiv (0,y_2)$; $\Pi_{ij}(y_1,a_2;\xi) = \sigma^*_{ij}(y,\xi)$, $y \equiv (y_1,a_2)$ *and* $Q_{ij}(y_1,0;\xi) = -(\partial/\partial y_2)\,\sigma^*_{ij}(y,\xi)$, $y \equiv (y_1,0)$ *respectively: of an inner unit heat source* $F = a\delta(x-\xi)$; *of unit heat fluxes* $S_{10} = a\delta(x-\xi)$, $S_{21} = a\delta(x-\xi)$ *on the* Γ_{10} , Γ_{21} *and of a unit temperature* $T_{20} = \delta(y-\xi)$ *on the* Γ_{20} *onto unknown thermal stresses* $\sigma_{ij}(y,\xi)$ *in Eq.*(18) *are determined as follows:*

$$\sigma^*_{11}(x,\xi) = -\frac{\mu\gamma}{4\pi(\lambda+2\mu)}\left[\left(1-\xi_1\frac{\partial}{\partial\xi_1}\right)\ln\frac{\overline{E}\overline{E}_1\tilde{E}_2\tilde{E}_{12}}{\tilde{E}\tilde{E}_1\overline{E}_2\overline{E}_{12}} + x_1\frac{\partial}{\partial\xi_1}\ln\frac{\overline{E}\tilde{E}_1\tilde{E}_2\overline{E}_{12}}{\tilde{E}\overline{E}_1\overline{E}_2\tilde{E}_{12}}\right] \tag{18a}$$

$$\sigma^*_{22}(x,\xi) = -\frac{\mu\gamma}{4\pi(\lambda+2\mu)}\left[\left(1+\xi_1\frac{\partial}{\partial\xi_1}\right)\ln\frac{\overline{E}\overline{E}_1\tilde{E}_2\tilde{E}_{12}}{\tilde{E}\tilde{E}_1\overline{E}_2\overline{E}_{12}} - x_1\frac{\partial}{\partial\xi_1}\ln\frac{\overline{E}\tilde{E}_1\tilde{E}_2\overline{E}_{12}}{\tilde{E}\overline{E}_1\overline{E}_2\tilde{E}_{12}}\right] \tag{18b}$$

$$\sigma^*_{12}(x,\xi) = \frac{\gamma\mu}{4\pi(\lambda+2\mu)}\frac{\partial}{\partial\xi_2}\left[\xi_1\ln\frac{\overline{E}\overline{E}_1\tilde{E}_2\tilde{E}_{12}}{\tilde{E}\tilde{E}_1\overline{E}_2\overline{E}_{12}} - x_1\ln\frac{\overline{E}\tilde{E}_1\tilde{E}_2\overline{E}_{12}}{\tilde{E}\overline{E}_1\overline{E}_2\tilde{E}_{12}}\right] \tag{18c}$$

where the functions $\bar{E}$, $\bar{E}_2$, $\tilde{E}$, $\tilde{E}_2$, $\bar{E}_1$, $\bar{E}_{12}$, $\tilde{E}_1$, $\tilde{E}_{12}$ *are defined by expressions:*

$$\bar{E} = \bar{E}(x,\xi) = 1 + 2e^{(\pi/2a_2)(x_1-\xi_1)}\cos(\pi/2a_2)(x_2-\xi_2) + e^{(\pi/a_2)(x_1-\xi_1)},\quad \bar{E}_2 = \bar{E}_2(x,\xi) = \bar{E}(x;\xi_1,-\xi_2) \tag{19a}$$

$$\tilde{E} = \tilde{E}(x,\xi) = 1 - 2e^{(\pi/2a_2)(x_1-\xi_1)}\cos(\pi/2a_2)(x_2-\xi_2) + e^{(\pi/a_2)(x_1-\xi_1)},\quad \tilde{E}_2 = \tilde{E}_2(x,\xi) = \tilde{E}(x;\xi_1,-\xi_2) \tag{19b}$$

$$\bar{E}_1 = \bar{E}_1(x,\xi) = 1 + 2e^{(\pi/2a_2)(x_1+\xi_1)}\cos(\pi/2a_2)(x_2-\xi_2) + e^{(\pi/2a_2)(x_1+\xi_1)};\quad \bar{E}_{12} = \bar{E}_{12}(x,\xi) = \bar{E}_1(x;-\xi_1,\xi_1) \tag{19c}$$

$$\tilde{E}_1 = \tilde{E}_1(x,\xi) = 1 - 2e^{(\pi/2a_2)(x_1+\xi_1)}\cos(\pi/2a_2)(x_2-\xi_2) + e^{(\pi/2a_2)(x_1+\xi_1)};\quad \tilde{E}_{12} = \tilde{E}_{12}(x,\xi) = \tilde{E}_1(x;-\xi_1,\xi_1) \tag{19d}$$

- for kernels $\sigma^*_{ij}(x,\xi)$;

$$\Pi_{11}(y_1,a_2;\xi) = \sigma^*_{11}(y_1,a_2;\xi) = -\frac{\gamma\mu}{2\pi(\lambda+2\mu)}\left[\left(1-\xi_1\frac{\partial}{\partial\xi_1}\right)\ln\frac{\bar{E}_{a_2}\tilde{E}_{1a_2}}{\tilde{E}_{a_2}\bar{E}_{1a_2}} + y_1\frac{\partial}{\partial\xi_1}\ln\frac{\bar{E}_{a_2}\bar{E}_{1a_2}}{\tilde{E}_{a_2}\tilde{E}_{1a_2}}\right] \tag{20a}$$

$$\Pi_{22}(y_1, a_2; \xi) = \sigma^*_{22}(y_1, a_2; \xi)$$
$$= -\frac{\gamma\mu}{2\pi(\lambda+2\mu)}\left[\left(1+\xi_1\frac{\partial}{\partial\xi_1}\right)\ln\frac{\bar{E}_{a_2}\tilde{E}_{1a_2}}{\tilde{E}_{a_2}\bar{E}_{1a_2}} - y_1\frac{\partial}{\partial\xi_1}\ln\frac{\bar{E}_{a_2}\bar{E}_{1a_2}}{\tilde{E}_{a_2}\tilde{E}_{1a_2}}\right] \quad (20b)$$

$$\Pi_{12}(y_1, a_2; \xi) = \sigma^*_{12}(y_1, a_2; \xi) = \frac{\gamma\mu}{2\pi(\lambda+2\mu)}\frac{\partial}{\partial\xi_2}\left[\xi_1\ln\frac{\bar{E}_{a_2}\tilde{E}_{1a_2}}{\tilde{E}_{a_2}\bar{E}_{1a_2}} - y_1\ln\frac{\bar{E}_{a_2}\bar{E}_{1a_2}}{\tilde{E}_{a_2}\tilde{E}_{1a_2}}\right] \quad (20c)$$

- *for kernels* $\Pi_{ij}(y_1, 0; \xi)$, *where functions* $\bar{E}_{a_2}$, $\bar{E}_{1a_2}$, $\tilde{E}_{a_2}$ *and* $\tilde{E}_{1a_2}$ *are determined from functions* (19a)–(19d) *by changing point* $x \equiv (x_1, x_2) \in V$ *with point* $y \equiv (y_1, y_2 = a_2) \in \Gamma_{21}$;

$$K_{11}(0, y_2; \xi) = \sigma^*_{11}(y, \xi)\big|_{y_1=0} = -\frac{\mu\gamma}{2\pi(\lambda+2\mu)}\left(1-\xi_1\frac{\partial}{\partial\xi_1}\right)\ln\frac{\tilde{E}_{20}\bar{E}_0}{\bar{E}_{20}\tilde{E}_0} \quad (21a)$$

$$K_{22}(0, y_2; \xi) = \sigma^*_{22}(y, \xi)\big|_{y_1=0} = \frac{\mu\gamma}{2\pi(\lambda+2\mu)}\left(1+\xi_1\frac{\partial}{\partial\xi_1}\right)\ln\frac{\tilde{E}_{20}\bar{E}_0}{\bar{E}_{20}\tilde{E}_0} \quad (21b)$$

$$K_{12}(0, y_2; \xi) = \sigma^*_{12}(y, \xi)\big|_{y_1=0} = \frac{\gamma\mu}{2\pi(\lambda+2\mu)}\xi_1\frac{\partial}{\partial\xi_2}\ln\frac{\tilde{E}_{20}\bar{E}_0}{\bar{E}_{20}\tilde{E}_0} \quad (21c)$$

- *for kernels* $K_{ij}(0, y_2; \xi)$, *where functions* $\bar{E}_0$, $\bar{E}_{20}$, $\tilde{E}_0$*and* $\tilde{E}_{20}$ *are determined from functions* (19*a*)–(19*d*) *by changing point* $x \equiv (x_1, x_2) \in V$ *with point* $y \equiv (y_1 = 0, y_2) \bigcup \Gamma_{10}$, *and*

$$Q_{11}(y_1, 0; \xi) = -\frac{\partial}{\partial y_2}\sigma^*_{11}(y, \xi)\big|_{y_2=0}$$
$$= \frac{\gamma\mu}{2\pi(\lambda+2\mu)}\frac{\partial}{\partial\xi_2}\left[\left(1-\xi_1\frac{\partial}{\partial\xi_1}\right)\ln\frac{\tilde{E}_0\tilde{E}_{10}}{\bar{E}_0\bar{E}_{10}} + y_1\frac{\partial}{\partial\xi_1}\ln\frac{\tilde{E}_0\overline{E}_{10}}{\bar{E}_0\tilde{E}_{10}}\right] \quad (22a)$$

$$Q_{22}(y_1, 0; \xi) = -\frac{\partial}{\partial y_2}\sigma^*_{22}(y, \xi)\big|_{y_2=0}$$
$$= \frac{\gamma\mu}{2\pi(\lambda+2\mu)}\frac{\partial}{\partial\xi_2}\left[\left(1+\xi_1\frac{\partial}{\partial\xi_1}\right)\ln\frac{\tilde{E}_0\tilde{E}_{10}}{\bar{E}_0\bar{E}_{10}} - y_1\frac{\partial}{\partial\xi_1}\ln\frac{\tilde{E}_0\overline{E}_{10}}{\bar{E}_0\tilde{E}_{10}}\right] \quad (22b)$$

$$Q_{12}(y_1, 0; \xi) = -\frac{\partial}{\partial y_2}\sigma^*_{12}(y, \xi)\big|_{y_2=0} = -\frac{\gamma\mu}{2\pi(\lambda+2\mu)}\frac{\partial^2}{\partial\xi_2^2}\left[\xi_1\ln\frac{\tilde{E}_0\tilde{E}_{10}}{\bar{E}_0\bar{E}_{10}} - y_1\ln\frac{\tilde{E}_0\overline{E}_{10}}{\bar{E}_0\tilde{E}_{10}}\right] \quad (22c)$$

- *for kernels* $Q_{ij}(y_1, 0; \xi)$, *where functions* $\bar{E}_0$, $\bar{E}_{20}$, $\tilde{E}_0$ *and* $\tilde{E}_{20}$ *are determined from Eqs.* (19a)–(19d) *by changing point* $x \equiv (x_1, x_2) \in V$ *with point* $y \equiv (y_1, y_2 = 0) \in \Gamma_{20}$.

Derivation of the MTGFs U_i

Derivation of MTGFs for a half-strip requires solution of the following Lamé and Poisson-type equations:

$$\mu\nabla_{\xi}^{2}U_i\left(x,\xi\right)+\left(\lambda+\mu\right)\Theta_{,\xi_i}\left(x,\xi\right)-\gamma G_{T,\xi_i}\left(x,\xi\right);\quad i=1,2 \tag{23}$$

$$\nabla^2 G_T\left(x,\xi\right)=-\delta\left(x-\xi\right),\ x,\xi\in V \tag{24}$$

where $\delta\left(x-\xi\right)$ is Dirac's function. Equations (23) and (24) have to be solved under the following homogeneous mechanical and thermal conditions on MTGFs U_i, TSGFs σ_{ij}^* and GFPE G_T:

$$U_1\left(x,y\right)=\sigma_{12}^*\left(x,y\right)=0,\quad \partial G_T\left(x,y\right)/\partial n_1=0;\ x,\xi\in V;\ y\equiv\left(0,y_2\right)\in\Gamma_{10} \tag{25a}$$

$$U_1\left(x,y\right)=\sigma_{22}^*\left(x,y\right)=0,\quad G_T\left(x,y\right)=0;\ x,\xi\in V;\ y\equiv\left(y_1,a_2\right)\in\Gamma_{20} \tag{25b}$$

$$\sigma_{21}^*\left(x,y\right)=U_2\left(x,y\right)=0,\ x,\xi\in V;\ \partial G_T\left(x,y\right)\big/\partial n_{y2}=0;\ y\equiv\left(y_1,0\right)\in\Gamma_{21} \tag{25c}$$

The application of harmonic integral representations (2)–(4) and the technique described in [27], for BVP (23)–(25c) leads to the following structural formulas:

$$U_1\left(x,\xi\right)=\frac{\gamma}{2\left(\lambda+2\mu\right)}\left[\xi_1 G_T\left(x,\xi\right)-x_1 G_1\left(x,\xi\right)\right] \tag{26a}$$

$$U_2\left(x,\xi\right)=-\frac{\gamma}{2\left(\lambda+2\mu\right)}\left[x_1\int\frac{\partial G_2\left(x,\xi\right)}{\partial x_2}dx_1-\xi_1\int\frac{\partial G_T\left(x,\xi\right)}{\partial\xi_2}d\xi_1\right.$$
$$\left.+\iint\frac{\partial G_T\left(x,\xi\right)}{\partial\xi_2}d^2\xi_1\right]. \tag{26b}$$

- for MTGFs; and

$$\Theta\left(x,\xi\right)=\frac{\gamma}{\lambda+2\mu}G_T\left(x,\xi\right) \tag{27}$$

- for TVD.

In Eqs. (26a) and (27), Green's functions $G_1\left(x,\xi\right)$, $G_2\left(x,\xi\right)$ and $G_T\left(x,\xi\right)$ are linked with the boundary conditions (25a)–(25c) as follows:

$$U_1=\sigma_{21}^*=0;\quad G_{T,1}=0\Rightarrow U_1=U_{1,2}=U_{2,1}\Rightarrow\Theta_{,1}=G_1$$
$$=G_{2,1}=G_{\Theta,1}=G_{T,1}=0 \tag{28a}$$

- on the boundary segment of straight line Γ_{10},

$$U_1=\sigma_{22}^*=0;\ G_T=0\Rightarrow U_1=U_{1,1}=U_{2,2}=0\Rightarrow\Theta=G_1=G_{2,2}=G_\Theta=G_T=0 \tag{28b}$$

- on the boundary semi-straight line Γ_{20}, and

$$\sigma^*_{21} = U_2 = 0; \quad G_{T,2} = 0 \Rightarrow U_{1,2} = 0; \quad U_2 = 0; \quad U_{2,1} = 0 \Rightarrow \Theta_{,2} = 0;$$
$$G_{1,2} = 0; \quad G_2 = 0; \quad G_{\Theta,2} = 0; \quad G_{T,2} = 0 \tag{28c}$$

on the boundary semi-straight line Γ_{21}.

Check of the MTGFs U_i with Respect to Point $x \equiv (x_1, x_2)$

Here we check the MTGFs $U_i(x, \xi)$ in Eqs. (26a) and (26b) for the half-strip derived earlier. So, according to results [9–11] the MTGFs $U_i(x, \xi)$ determined by Eqs. (26a) and (26b) must satisfy, over the coordinates of the point $x \equiv (x_1, x_2)$ of application of the heat source, the equation

$$\nabla_x^2 U_i(x, \xi) = -\gamma \Theta^{(i)}(x, \xi) \tag{29}$$

with boundary conditions, similar to those for GFPE G_T in Eqs. (25a)–(25c). In Eq. (29) $\Theta^{(i)}(x, \xi) = U^{(i)}_{j,j}(x, \xi)$ are the influence functions of a unit point body force onto volume dilatation and with the homogeneous boundary conditions (14), rewritten with respect to point $x \equiv (x_1, x_2)$ for displacements $U^{(i)}_j(x, \xi)$ (components of Green's tensor). According to handbook [26] (the problem 12.L.8 and the corresponding answer) the volume dilatation

$$\Theta^{(i)}(x, \xi) = -\frac{1}{\lambda + 2\mu}\frac{\partial G_\Theta}{\partial \xi_i} = -\frac{1}{\lambda + 2\mu}\frac{\partial G_T}{\partial \xi_i} \tag{30}$$

Also, Laplace operator from MTGFs in Eqs. (26a) and (26b) gives us the expressions:

$$\begin{aligned}\nabla_x^2 U_1(x, \xi) &= \nabla_x^2 \frac{\gamma}{2(\lambda + 2\mu)}\left[\xi_1 G_T(x, \xi) - x_1 G_1(x, \xi)\right] \\ &= -\frac{\gamma}{2(\lambda + 2\mu)} 2\frac{\partial G_1}{\partial x_1} = \frac{\gamma}{\lambda + 2\mu}\frac{\partial G_T}{\partial \xi_1}\end{aligned} \tag{31a}$$

$$\begin{aligned}\nabla_x^2 U_2(x, \xi) &= \nabla_x^2 \frac{\gamma}{2(\lambda + 2\mu)}\left[-x_1 \int \frac{\partial G_2(x, \xi)}{\partial x_2} dx_1 + \xi_1 \int \frac{\partial G_T(x, \xi)}{\partial \xi_2} d\xi_1 \right. \\ &\quad \left. - \iint \frac{\partial G_T(x, \xi)}{\partial \xi_2} d^2\xi_1\right] \\ &= -\frac{\gamma}{2(\lambda + 2\mu)} 2\frac{\partial}{\partial x_1}\int \frac{\partial G_2(x, \xi)}{\partial x_2} d\xi_1 \\ &= \frac{\gamma}{2(\lambda + 2\mu)} 2\frac{\partial}{\partial \xi_1}\int \frac{\partial G_T(x, \xi)}{\partial \xi_2} d\xi_1 = \frac{\gamma}{\lambda + 2\mu}\frac{\partial G_T}{\partial \xi_2}\end{aligned} \tag{31b}$$

So, taking into account the last results (30)–(31b) we see that Eq. (29) is satisfied.

Next substituting in the expressions $U_i(x, \xi)$, determined by Eqs. (26a) and (26b), $x \to y \equiv (y_1 = 0, y_2)$, $x \to y \equiv (y_1, y_2 = 0)$ and $x \to y \equiv (y_1, y_2 = a_2)$ we can see that boundary conditions similar to those in Eqs. (25a)–(25c) for GFPE G_T:

$$\partial U_1(y, \xi)/\partial n_{10} = -\frac{\partial}{\partial x_1} U_1(x, \xi)\big|_{x_1=0}$$
$$= -\frac{\gamma}{2(\lambda + 2\mu)}\left[\xi_1 G_{T,x_1}(x, \xi) - x_1 G_{1,x_1}(x, \xi) - G_1(x, \xi)\right]\big|_{x_1=0} = 0 \quad (32a)$$

$$U_1(x, \xi)\big|_{x_2=0} = \frac{\gamma}{2(\lambda + 2\mu)}\left[\xi_1 G_T(x, \xi) - x_1 G_1(x, \xi)\right]\big|_{x_2=0} = 0 \quad (32b)$$

$$U_{1,2}(x, \xi)\big|_{x_2=a_2} = \frac{\gamma}{2(\lambda + 2\mu)}\left[\xi_1 G_{T,2}(x, \xi) - x_1 G_{1,2}(x, \xi)\right]\Big|_{x_2=a_2} = 0 \quad (32c)$$

$$\partial U_2(y, \xi)/\partial n_{10} = -\frac{\partial}{\partial x_1} U_2(x, \xi)\big|_{x_1=0} = -\frac{\gamma}{2(\lambda + 2\mu)}$$
$$\times\left[-x_1 \frac{\partial G_2(x, \xi)}{\partial x_2} - \int \frac{\partial G_2(x, \xi)}{\partial x_2} dx_1 + \xi_1 \int \frac{\partial^2 G_T(x, \xi)}{\partial \xi_2 \partial x_1} d\xi_1 \right.$$
$$\left. - \iint \frac{\partial^2 G_T(x, \xi)}{\partial \xi_2 \partial x_1} d^2\xi_1 \right]\big|_{x_1=0} = 0 \quad (32d)$$

$$U_2(x, \xi)\big|_{x_2=0} = \frac{\gamma}{2(\lambda + 2\mu)}\left[-x_1 \int \frac{\partial G_2(x, \xi)}{\partial x_2} dx_1 + \xi_1 \int \frac{\partial G_T(x, \xi)}{\partial \xi_2} d\xi_1 \right.$$
$$\left. - \iint \frac{\partial G_T(x, \xi)}{\partial \xi_2} d^2\xi_1 \right]\big|_{x_2=0} = 0 \quad (32e)$$

$$U_{2,2}(x, \xi)\big|_{x_2=a_2} = \frac{\gamma}{2(\lambda + 2\mu)}\left[-x_1 \int \frac{\partial^2 G_2(x, \xi)}{(\partial x_2)^2} dx_1 + \xi_1 \int \frac{\partial^2 G_T(x, \xi)}{\partial \xi_2 \partial x_2} d\xi_1 \right.$$
$$\left. - \iint \frac{\partial^2 G_T(x, \xi)}{\partial \xi_2 \partial x_2} (d\xi_1)^2 \right]\big|_{x_2=a_2} = 0 \quad (32f)$$

because in the last equality we have the relation $\int \frac{\partial^2 G_2(x,\xi)}{(\partial x_2)^2} dx_1 \big|_{x_2=a_2} = -\int \frac{\partial^2 G_2(x,\xi)}{(\partial x_1)^2} dx_1 \big|_{x_2=a_2} = 0$.

The boundary conditions (32a)–(32f) are satisfied due to the boundary conditions (28a)–(28c) for Green's functions G_T, G_1 and G_2.

Check of the MTGFs U_i with Respect to Point $\xi \equiv (\xi_1, \xi_2)$

In terms of the coordinates of the point of observation $\xi \equiv (\xi_1, \xi_2)$, the MTGFs $U_i(x, \xi)$, must satisfy Eq. (23) and boundary conditions in Eqs. (26a) and (26b). Indeed, application of the Laplace operator from MTGFs in Eqs. (26a) and (26b) gives the expressions:

$$\mu\nabla_\xi^2 U_1(x,\xi) + (\lambda+\mu)\,\Theta_{,\xi_1}(x,\xi) - \gamma G_{T,\xi_1}(x,\xi)$$

$$= \mu\frac{\gamma}{2(\lambda+2\mu)}\nabla_\xi^2\left[\xi_1 G_T(x,\xi) - x_1 G_1(x,\xi)\right] + (\lambda+\mu)\frac{\gamma}{\lambda+2\mu}G_{T,\,\xi_1}(x,\xi)$$

$$-\gamma G_{T,\xi_1}(x,\xi) = \frac{\mu\gamma}{2(\lambda+2\mu)}2G_{T,\,\xi_1}(x,\xi)$$

$$+\frac{(\lambda+\mu)\gamma}{(\lambda+2\mu)}G_{T,\,\xi_1}(x,\xi) - \gamma G_{T,\xi_1}(x,\xi) = 0. \quad (33a)$$

Also

$$\mu\nabla_\xi^2 U_2(x,\xi) + (\lambda+\mu)\,\Theta_{,\xi_2}(x,\xi) - \gamma G_{T,\xi_2}(x,\xi)$$

$$= \mu\frac{\gamma}{2(\lambda+2\mu)}\nabla_\xi^2\left[\frac{\gamma}{2(\lambda+2\mu)}\left(-x_1\int\frac{\partial G_2(x,\xi)}{\partial x_2}dx_1 + \xi_1\int\frac{\partial G_T(x,\xi)}{\partial\xi_2}d\xi_1\right.\right.$$

$$\left.\left.-\iint\frac{\partial G_T(x,\xi)}{\partial\xi_2}d^2\xi_1\right)\right]$$

$$+(\lambda+\mu)\frac{\gamma}{\lambda+2\mu}G_{T,\,\xi_2}(x,\xi) - \gamma G_{T,\xi_2}(x,\xi)$$

$$= \frac{\mu\gamma}{2(\lambda+2\mu)}2G_{T,\,\xi_2}(x,\xi) + \frac{(\lambda+\mu)\gamma}{(\lambda+2\mu)}G_{T,\,\xi_2}(x,\xi) - \gamma G_{T,\xi_1}(x,\xi) = 0, \quad (33b)$$

where the volume dilatation Θ was calculated by using MTGFs in Eqs. (26a) and (26b) as follows:

$$\Theta(x,\xi) = U_{1,\xi_1}(x,\xi) + U_{2,\xi_2}(x,\xi) = \frac{\gamma}{\lambda+2\mu}G_T(x,\xi) \quad (34)$$

So, from (33a) and (33b), we conclude that Eq. (23) is satisfied. Also we can see that the MTGFs $U_i(x,\xi)$ determined by Eqs. (26a) and (26b), with respect to points $\xi \equiv (\xi_1, \xi_2)$, satisfy the boundary conditions (25a)–(25b). Indeed, from the expressions in Eqs. (26a) and (26b) and boundary conditions (28a)–(28c) for Green's functions G_1, G_2, G_T, it follows that the boundary conditions (25a)–(25b) are satisfied. So, we obtain:

$$U_1(x,\xi)\big|_{\xi_1=0} = \frac{\gamma}{2(\lambda+2\mu)}\left[\xi_1 G_T(x,\xi) - x_1 G_1(x,\xi)\right]\big|_{\xi_1=0} = 0 \quad (35a)$$

$$U_1(x,\xi)\big|_{\xi_2=0} = \frac{\gamma}{2(\lambda+2\mu)}\left[\xi_1 G_T(x,\xi) - x_1 G_1(x,\xi)\right]\big|_{\xi_2=0} = 0 \quad (35b)$$

$$U_{1,2}(x,\xi)\big|_{\xi_2=a_2} = \frac{\gamma}{2(\lambda+2\mu)}\left[\xi_1 G_{T,2}(x,\xi) - x_1 G_{1,2}(x,\xi)\right]\big|_{\xi_2=a_2} = 0 \quad (35c)$$

$$U_{2,1}(x,\xi)\big|_{\xi_1=0} = \frac{\gamma}{2(\lambda+2\mu)} \times \left[-x_1\int\frac{\partial^2 G_2(x,\xi)}{\partial x_2\partial\xi_1}dx_1 + \int\frac{\partial G_T(x,\xi)}{\partial\xi_2}d\xi_1 + \xi_1\int\frac{\partial^2 G_T(x,\xi)}{\partial\xi_2\partial\xi_1}d\xi_1 - \iint\frac{\partial^2 G_T(x,\xi)}{\partial\xi_2\partial\xi_1}d^2\xi_1\right]\Big|_{\xi_1=0} = 0 \tag{35d}$$

$$U_{2,2}(x,\xi)\big|_{\xi_2=0} = \frac{\gamma}{2(\lambda+2\mu)}\left[-x_1\int\frac{\partial^2 G_2(x,\xi)}{\partial x_2\partial\xi_2}dx_1 + \xi_1\int\frac{\partial^2 G_T(x,\xi)}{(\partial\xi_2)^2}d\xi_1 - \iint\frac{\partial^2 G_T(x,\xi)}{(\partial\xi_2)^2}(d\xi_1)^2\right]\Big|_{\xi_2=0} = 0 \tag{35e}$$

$$U_{2}(x,\xi)\big|_{\xi_2=a_2} = \frac{\gamma}{2(\lambda+2\mu)}\left[-x_1\int\frac{\partial G_2(x,\xi)}{\partial x_2}dx_1 + \xi_1\int\frac{\partial G_T(x,\xi)}{\partial\xi_2}d\xi_1 - \iint\frac{\partial G_T(x,\xi)}{\partial\xi_2}d^2\xi_1\right]\Big|_{\xi_2=a_2} = 0 \tag{35f}$$

Thus, we have proved that derived MTGFs $U_i(x,\xi)$ within a half-strip in Eqs. (26a) and (26b) satisfy the respective BVPs of thermoelasticity described by Eq. (23) and boundary conditions (25a)–(25c).

Derivation of the TSGFs σ^*_{ij}

To derive TSGFs σ^*_{ij} first we need to know the expressions for GFPE. So, in Eqs. (26a) and (26b) the expressions for GFPE for half-strip with boundary conditions (28a)–(28c) can be rewritten from references [23, 26]:

$$G_1 = \frac{1}{4\pi}\ln\frac{\overline{E}\tilde{E}_1\tilde{E}_2\overline{E}_{12}}{\tilde{E}\overline{E}_1\overline{E}_2\tilde{E}_{12}};\quad G_2(x,\xi) = \frac{1}{4\pi}\ln\frac{\bar{E}\bar{E}_1\bar{E}_2\overline{E}_{12}}{\tilde{E}\tilde{E}_1\tilde{E}_2\tilde{E}_{12}};$$

$$G_\Theta(x,\xi) = G_T(x,\xi) = \frac{1}{4\pi}\ln\frac{\overline{EE_1}\tilde{E}_2\tilde{E}_{12}}{\tilde{E}\tilde{E}_1\overline{E}_2\overline{E}_{12}} \tag{36}$$

Next, by using the Duhamel–Neumann law (12), constructive formulas (26a) and (26b) for MTGFs U_i and the expressions (36) for $G_1(x,\xi)$ and $G_T(x,\xi)$ we obtain the following expressions for TSGFs σ^*_{ij}:

$$\sigma^*_{11}(x,\xi) = -\frac{\mu\gamma}{(\lambda+2\mu)}\left[\left(1-\xi_1\frac{\partial}{\partial\xi_1}\right)G_T + x_1\frac{\partial}{\partial\xi_1}G_1\right]$$

$$= -\frac{\mu\gamma}{4\pi(\lambda+2\mu)}\left[\left(1-\xi_1\frac{\partial}{\partial\xi_1}\right)\ln\frac{\overline{EE_1}\tilde{E}_2\tilde{E}_{12}}{\tilde{E}\tilde{E}_1\overline{E}_2\overline{E}_{12}} + x_1\frac{\partial}{\partial\xi_1}\ln\frac{\overline{E}\tilde{E}_1\tilde{E}_2\overline{E}_{12}}{\tilde{E}\overline{E}_1\overline{E}_2\tilde{E}_{12}}\right] \tag{37a}$$

$$\sigma_{22}^{*}(x,\xi) = -\frac{\gamma\mu}{(\lambda+2\mu)}\left[\left(1+\xi_1\frac{\partial}{\partial\xi_1}\right)G_T - x_1\frac{\partial}{\partial\xi_1}G_1\right]$$
$$= -\frac{\mu\gamma}{4\pi(\lambda+2\mu)}\left[\left(1+\xi_1\frac{\partial}{\partial\xi_1}\right)\ln\frac{\overline{E}\overline{E}_1\tilde{E}_2\tilde{E}_{12}}{\tilde{E}\tilde{E}_1\overline{E}_2\overline{E}_{12}} - x_1\frac{\partial}{\partial\xi_1}\ln\frac{\overline{E}\tilde{E}_1\tilde{E}_2\overline{E}_{12}}{\tilde{E}\overline{E}_1\overline{E}_2\tilde{E}_{12}}\right] \quad (37b)$$

$$\sigma_{12}^{*}(x,\xi) = \frac{\gamma\mu}{(\lambda+2\mu)}\frac{\partial}{\partial\xi_2}\left[\xi_1 G_T(x,\xi) - x_1 G_1(x,\xi)\right]$$
$$= \frac{\gamma\mu}{4\pi(\lambda+2\mu)}\frac{\partial}{\partial\xi_2}\left[\xi_1\ln\frac{\overline{E}\overline{E}_1\tilde{E}_2\tilde{E}_{12}}{\tilde{E}\tilde{E}_1\overline{E}_2\overline{E}_{12}} - x_1\ln\frac{\overline{E}\tilde{E}_1\tilde{E}_2\overline{E}_{12}}{\tilde{E}\overline{E}_1\overline{E}_2\tilde{E}_{12}}\right] \quad (37c)$$

Finally, omitting the functions $\tilde{E}_1\ \tilde{E}_{12}\ \overline{E}_1\ \overline{E}_{12}$, which contain the inferior index "1," we obtain the expressions for thermal stresses to respective BVP of thermoelasticity for the strip $V(-\infty < x_1 < \infty;\ 0 \le x_2 \le a_2)$.

Note that graphics of the TSGFs σ_{ij}^{*} in Eqs. (37a)–(37c) plotted by using computer software Maple 15 are presented in Figures 1a–3a, found in the Appendix.

Derivation of the Green-type Integral Formula for $\boldsymbol{\sigma_{ij}}$

The Green-type integral formula (17) can be obtained by using, the rewritten for half-strip, the general integral formula (10) taking into account boundary conditions (14), (15b) and expressions (37a)–(37c) for σ_{ij}^{*}. Finally, calculating by using expressions (37a)–(37c) the other influence functions: $K_{ij}(0,y_2;\xi) = \sigma_{11}^{*}(0,y_2;\xi)$ (on marginal segment Γ_{10}), $\Pi_{ij}(y_1,a_2;\xi) = \sigma_{ij}^{*}(y_1,a_2;\xi)$ (on marginal line Γ_{21}), $Q_{ij}(y_1,0;\xi) = \partial\sigma_{ij}^{*}(y_1,0;\xi)/\partial n_{y2}$ (on marginal line Γ_{20}) and substituting them in the rewritten formula (10), we obtain the integral solution (18)–(22c) to respective non-homogeneous BVPs (13)–(15b) for a thermoelastic half-strip.

EXPLICIT THERMAL STRESS TO A PARTICULAR BVP FOR HALF-STRIP

Here we present an example of application of Green's-type integral formula for thermal stresses in Eqs. (17)–(22c) to the solution of particular BVPs of thermoelasticity for the half-strip V.

Example

To determine thermal stresses $\sigma_{ij}(\xi); i,j = 1,2$ in the half-strip $V \equiv (0 \le x_1 < \infty,\ 0 \le x_2 \le a_2)$ caused by the following thermal boundary conditions given on the marginal segment Γ_{10} and on the semi-straight lines Γ_{20}, Γ_{21}:

$$T(y) = \begin{cases} T_{20}(y) = T_0 = const, & y \equiv (y_1,0) \in [a_1 \le y_1 \le b_1] \in \Gamma_{20}; \quad T_0 > 0 \\ T_{20}(y) = 0, & y \equiv (y_1,0) \in [0 \le y_1 < a_1) \bigcup (b_1 < y_1 < \infty) \in \Gamma_{20}; \end{cases}$$
$$\partial T(0,y_2)/\partial n_{y_1} = -\partial T(0,y_2)/\partial n_{y_1} == S_{10}(0,y_2) = 0; \quad y \equiv (0,y_2) \in \Gamma_{10}; \quad (38)$$
$$\partial T(y_1,a_2)/\partial n_{y_2} = \partial T(y_1,a_2)/\partial n_{y_2} == S_{21}(y_1,a_2) = 0; \quad y \equiv (y_1,a_2) \in \Gamma_{21},$$

Thus on the boundaries we define homogeneous mechanical boundary conditions in Eq. (14). According to the above formulated theorem, the solution of the above-mentioned BVP of thermoelasticity can be obtained using the Green-type integral formula in Eq. (17) in the absence of inner heat source, heat fluxes on Γ_{10}, Γ_{21} and at a temperature $T_{20}(y_1)$ on Γ_{20}:

$$\sigma_{ij}(\xi) = -\int_{a_1}^{b_1} T_{20}(y_1, 0)\, Q_{ij}(y_1, 0; \xi)\, dy_1 \tag{39}$$

where kernels $Q_{ij}(y_1, 0; \xi)$ are defined by (22a)–(22c). Thus, substituting (22a)–(22c) and (38) in Eq. (39) we obtain the following integral formula for thermal stresses:

$$\begin{aligned}\sigma_{11}(\xi) &= -\int_{a_1}^{b_1} T_{20}(y_1, 0)\, Q_{11}(y_1, 0; \xi)\, dy_1 \\ &= -\frac{\gamma\mu T_0}{2\pi(\lambda+2\mu)}\frac{\partial}{\partial\xi_2}\int_{a_1}^{b_1}\left[\left(1-\xi_1\frac{\partial}{\partial\xi_1}\right)\ln\frac{\tilde{E}_0\tilde{E}_{10}}{\bar{E}_0\bar{E}_{10}} + y_1\frac{\partial}{\partial\xi_1}\ln\frac{\tilde{E}_0\bar{E}_{10}}{\bar{E}_0\tilde{E}_{10}}\right]dy_1\end{aligned} \tag{40a}$$

$$\begin{aligned}\sigma_{22}(\xi) &= -\int_{a_1}^{b_1} T_{20}(y_1, 0)\, Q_{22}(y_1, 0; \xi)\, dy_1 \\ &= -\frac{\gamma\mu T_0}{2\pi(\lambda+2\mu)}\frac{\partial}{\partial\xi_2}\int_{a_1}^{b_1}\left[\left(1+\xi_1\frac{\partial}{\partial\xi_1}\right)\ln\frac{\tilde{E}_0\tilde{E}_{10}}{\bar{E}_0\bar{E}_{10}} - y_1\frac{\partial}{\partial\xi_1}\ln\frac{\tilde{E}_0\bar{E}_{10}}{\bar{E}_0\tilde{E}_{10}}\right]dy_1\end{aligned} \tag{40b}$$

$$\begin{aligned}\sigma_{12}(\xi) &= -\int_{a_1}^{b_1} T_{20}(y_1, 0)\, Q_{12}(y_1, 0; \xi)\, dy_1 \\ &= \frac{\gamma\mu T_0}{2\pi(\lambda+2\mu)}\int_{a_1}^{b_1}\frac{\partial^2}{\partial\xi_2^2}\left[\xi_1\ln\frac{\tilde{E}_0\tilde{E}_{10}}{\bar{E}_0\bar{E}_{10}} - y_1\ln\frac{\tilde{E}_0\bar{E}_{10}}{\bar{E}_0\tilde{E}_{10}}\right]dy_1\end{aligned} \tag{40c}$$

where

$$\bar{E}_0 = 1 + 2e^{(\pi/2a_2)(y_1-\xi_1)}\cos(\pi/2a_2)(\xi_2) + e^{(\pi/a_2)(y_1-\xi_1)} \tag{41a}$$

$$\tilde{E}_0 = \tilde{E}(x, \xi) = 1 - 2e^{(\pi/2a_2)(y_1-\xi_1)}\cos(\pi/2a_2)(\xi_2) + e^{(\pi/a_2)(y_1-\xi_1)} \tag{41b}$$

$$\bar{E}_{10} = 1 + 2e^{-(\pi/2a_2)(y_1+\xi_1)}\cos(\pi/2a_2)(\xi_2) + e^{-(\pi/2a_2)(y_1+\xi_1)} \tag{41c}$$

$$\tilde{E}_{10} = 1 - 2e^{-(\pi/2a_2)(y_1+\xi_1)}\cos(\pi/2a_2)(\xi_2) + e^{-(\pi/2a_2)(y_1+\xi_1)} \tag{41d}$$

After computing the integrals in (40a)–(40c) we obtain the following final analytical expressions for thermoelastic stresses:

$$\begin{aligned}\sigma_{11}(\xi) = -\frac{\gamma\mu T_0}{2\pi(\lambda+2\mu)}\Bigg[4\left(\bar{f}_1+\tilde{f}_1-\bar{f}-\tilde{f}\right) \\ - \frac{\partial}{\partial\xi_2}\left(y_1\ln\frac{\tilde{E}_0\tilde{E}_{10}}{\bar{E}_0\bar{E}_{10}} + \xi_1\ln\frac{\bar{E}_0\tilde{E}_{10}}{\tilde{E}_0\bar{E}_{10}}\right)\Bigg]_{y_1=a_1}^{y_1=b_1}\end{aligned} \tag{42a}$$

$$\sigma_{22}(\xi) = -\frac{\gamma\mu T_0}{2\pi(\lambda+2\mu)}\left[\frac{\partial}{\partial\xi_2}\left(y_1 \ln\frac{\tilde{E}_0\tilde{E}_{10}}{\bar{E}_0\bar{E}_{10}} + \xi_1 \ln\frac{\bar{E}_0\tilde{E}_{10}}{\tilde{E}_0\bar{E}_{10}}\right)\right]_{y_1=a_1}^{y_1=b_1} \tag{42b}$$

$$\sigma_{12}(\xi) = -\frac{\gamma\mu T_0}{2\pi(\lambda+2\mu)}\left[\left(\xi_1\frac{\partial}{\partial\xi_1} - 1\right)\ln\frac{\bar{E}_0\tilde{E}_{10}}{\tilde{E}_0\bar{E}_{10}} - y_1\frac{\partial}{\partial\xi_1}\ln\frac{\bar{E}_0\bar{E}_{10}}{\tilde{E}_0\tilde{E}_{10}}\right]_{y_1=a_1}^{y_1=b_1} \tag{42c}$$

where

$$\bar{f} = arctg\frac{e^{(\pi/2a_2)(y_1-\xi_1)} + \cos(\pi/2a_2)(\xi_2)}{\sin(\pi/2a_2)(\xi_2)}$$
$$\tilde{f} = arctg\frac{e^{(\pi/2a_2)(y_1-\xi_1)} - \cos(\pi/2a_2)(\xi_2)}{\sin(\pi/2a_2)(\xi_2)} \tag{43a}$$

$$\bar{f}_1 = arctg\frac{e^{-(\pi/2a_2)(y_1+\xi_1)} + \cos(\pi/2a_2)(\xi_2)}{\sin(\pi/2a_2)(\xi_2)}$$
$$\tilde{f}_1 = arctg\frac{e^{-(\pi/2a_2)(y_1+\xi_1)} - \cos(\pi/2a_2)(\xi_2)}{\sin(\pi/2a_2)(\xi_2)} \tag{43b}$$

Note that graphics of thermal stresses in Eqs. (42a)–(43b) are plotted (using computer software Maple 15) and presented in Figures 1b, 2b and 3b.

CONCLUSIONS

An extension of the HIRM for derivation of MTGFs $U_i(x, \xi)$ [9–11] on Cartesian domains, which contains parallel lines or its parts (planes or its parts) to the coordinate axes (coordinate planes) is proposed. A theorem about the constructing and checking MTGFs $U_i(x, \xi)$, TSGFs $\sigma^*_{ij}(x, \xi)$ and new Green-type integral formulas (17)–(22c) is formulated to a specific BVP for half-strip in terms of GFPE. All results are obtained in terms of elementary functions. The explicit solution for a particular BVP of thermoelasticity for half-strip is included. Both, the derived TSGFs $\sigma^*_{ij}(x, \xi)$ and thermal stresses $\sigma_{ij}(\xi)$ to this particular BVP for thermoelastic half-strip were evaluated numerically and graphically by using Maple 15 computer software. The proposed technique of constructing TSGFs for a half-strip could be applied to many other 2D- and 3D-canonical domains, that have straight lines or its parts (planes or its parts) parallel to coordinate axes (coordinate planes) of Cartesian system of coordinates.

NOMENCLATURE

TSGFs	Thermal stresses Green's functions
2D	Two-dimensional
3D	Three-dimensional
BVP	Boundary values problem
GFM	Green's function method
GΘCM	GO convolution method

TVD	Thermoelastic volume dilatation
HIR	Harmonic integral representations
HIRM	Harmonic integral representations method
MTGFs	Main thermoelastic Green's functions
GFPE	Green's functions for Poisson equation
GFs	Green's functions
MPa	Mega Pascal
K	Degrees Kelvin

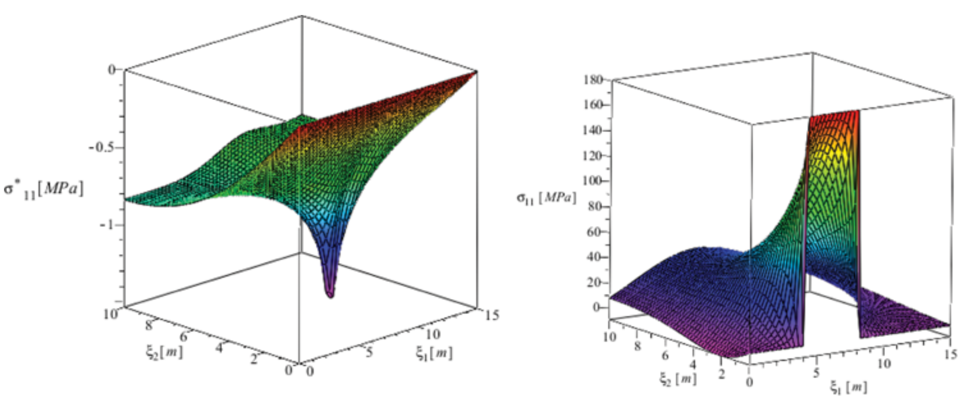
APPENDIX

Graphs of Normal and Tangential Thermal Stresses within Half-Strip caused by the Unit Point Heat Source and by a Constant Boundary Temperature Gradient

Graphs of thermal stresses σ^*_{11}, σ^*_{22}, σ^*_{12} , caused by the unit point heat source and of thermal stresses σ_{11}, σ_{22}, σ_{12}, caused by a constant boundary temperature gradient are constructed at the following values of elastic and thermal constants: the Poisson ratio $\nu = 0.3$, modulus of elasticity $E = 2.1 \cdot 10^5$ MPa, and coefficient of linear thermal expansion $\alpha = 1.2 \cdot 10^{-5}\,(\mathrm{K})^{-1}$. The behavior of the normal thermal stresses σ^*_{11} caused by the inner unit point heat source and of the thermal stresses σ_{11} caused by the boundary temperature gradient, calculated by the formulas in Eqs. (37a) and (42a) are shown in Figure 1a and Figure 1b, respectively.

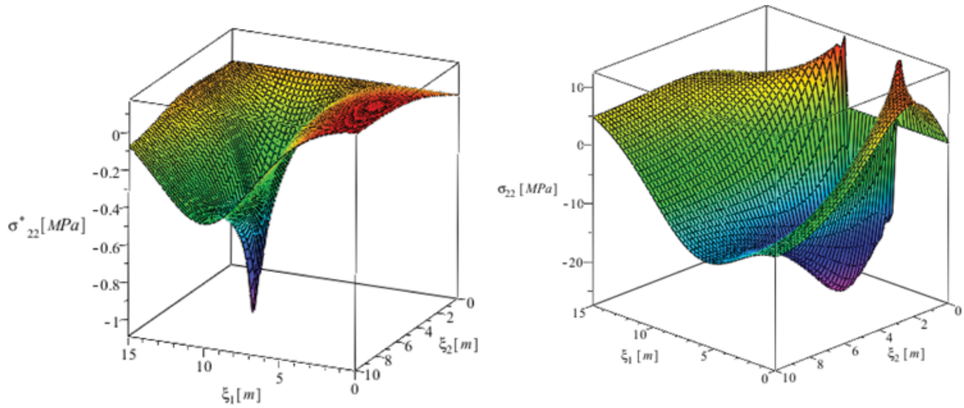
The behaviors of the normal thermal stresses σ^*_{22} caused by the inner unit point heat source and of thermal stresses σ_{22} caused by the boundary temperature gradient, calculated by the formulas (37b) and (42b) are shown in Figure 2a and in Figure 2b, respectively.

The behavior of the tangential thermal stresses σ^*_{12} caused by the inner unit point heat source and of the tangential thermal stresses σ_{12}, caused by the boundary



(a) (b)

Figure 1 Graphs of normal thermal stresses σ^*_{11} and σ_{11} in the half-strip V at $0 \le \xi_1 \le 15m$, $0 \le \xi_2 \le 10m$, caused by a unit heat source applied at inner point $x_1 = 10m\,, x_2 = 5m$ - a; and by the constant temperature gradient $T_0 = 50\,\mathrm{K}$, acting on the segment $4 \le y_1 \le 8m$ of the boundary line Γ_{20} - b.

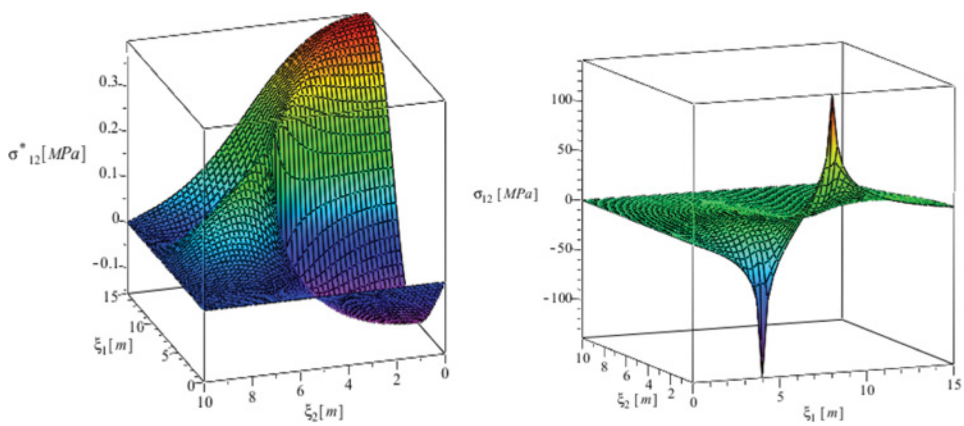


(a) (b)

Figure 2 Graphs of normal thermal stresses σ_{22}^{*} and σ_{22} in the half-strip V for $0 \le \xi_1 \le 15m$, $0 \le \xi_2 \le 10m$, caused by a unit heat source applied at inner point $x_1 = 10m$, $x_2 = 5m$ - a; and by the constant temperature gradient $T_0 = 50\,\mathrm{K}$, acting on the segment $4 \le y_1 \le 8m$ of the boundary semi-straight line Γ_{20} - b.

temperature gradient, calculated by the formulas in (37c) and (42c) are shown in Figure 3a and in Figure 3b, respectively.

One of the major features that can be seen in Figures 1–3 is the boundary conditions for thermal stresses σ_{11} σ_{12}, σ_{22}, created by constant temperature gradient $T_0 = 350\,\mathrm{K}$; and for thermal stresses σ_{11}^{*}, σ_{12}^{*}, σ_{22}^{*}, created by a unit source applied at $x_1 = 2m$, $x_2 = 5m$ are satisfied.



(a) (b)

Figure 3 Graphs of tangential stresses σ_{12}^{*} and σ_{12} in the half-strip V for $0 \le \xi_1 \le 15m$, $0 \le \xi_2 \le 10m$, caused by a unit heat source applied at inner point $x_1 = 10m$, $x_2 = 5m$ - a; and by the constant temperature gradient $T_0 = 50\,\mathrm{K}$, acting on the segment $4 \le y_1 \le 8m$ of the boundary semi-straight line Γ_{20} - b.

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