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Criteria for the nonexistence of periodic orbits in planar differential systems

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Abstract. In this work we summarize some well-known criteria for the nonexistence of periodic orbits in planar differential systems. Additionally we present two new criteria and illustrate with examples these criteria.

Mathematics subject classification: 34C05, 34D30.
Keywords and phrases: periodic orbits, Bendixson, Dulac, Liénard equation, angular velocity, planar differential systems.

1 Introduction and statement of the main results

We consider a *planar differential system* that we write as

$$\frac{dx}{dt} = \dot{x} = P(x, y), \qquad \frac{dy}{dt} = \dot{y} = Q(x, y), \tag{1}$$

where $P(x, y)$ and $Q(x, y)$ are C^1 real functions in the variables x and y, and t is the independent variable.

The objective of this note is double, first we recall the more well-known results for the nonexistence of periodic orbits of a differential system (1). Second we provide two new criteria for the nonexistence of periodic orbits of system (1).

As far as we know one of the first criterium of nonexistence is the following one due to Poincaré.

Theorem 1 (Poincaré Method of Tangential Curves)**.** *Consider a family of curves $F(x, y) = C$, where $F(x, y)$ is continuously differentiable. If in a region R the quantity*

$$\frac{dF}{dt} = P\frac{\partial F}{\partial x} + Q\frac{\partial F}{\partial y}$$

has constant sign, and the curve

$$P\frac{\partial F}{\partial x} + Q\frac{\partial F}{\partial y} = 0$$

(which represents the locus of points of contact between curves in the family and the trajectories of (1)*, and is called a tangential curve) does not contain a whole trajectory of* (1) *or any closed branch, then system* (1) *does not possess a periodic orbit which is entirely contained in R.*

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For a proof of Theorem 1 see either Theorem 1.9 of [8], or Proposition 7.9 of [3].

Theorem 2 (Bendixson's Theorem). *Assume that the divergence function $\partial P/\partial x + \partial Q/\partial y$ of system* (1) *has constant sign in a simply connected region R, and is not identically zero on any subregion of R. Then system* (1) *does not have a periodic orbit which lies entirely in R.*

For a proof of Theorem 2 see either Theorem 1.10 of [8], or Section 3.9 of [6], or Proposition 1.133 of [2], or Theorem 7.10 of [3].

Theorem 3 (Dulac's Theorem). *If for system* (1) *there exists a C^1 function $B(x,y)$ in a simply connected region R such that $\partial(BP)/\partial x + \partial(BQ)/\partial y$ has constant sign and is not identically zero in any subregion, then this system* (1) *does not have a periodic orbit lying entirely in R.*

For a proof of Theorem 3 see either Theorem 1.12 of [8], or Theorem 4.8 of [9], or Section 3.9 of [6], or Exercise 1.136 of [2], or Theorem 7.12 of [3].

The well-know Liénard differential equation [4]

$$\ddot{x} + f(x)\dot{x} + g(x) = 0,$$

where $f(x)$ and $g(x)$ are C^1 functions in the open subset R of $\mathbb{R}^2$, can be written as the following first order differential system

$$\dot{x} = y, \qquad \dot{y} = -g(x) - f(x)y. \tag{2}$$

Theorem 4 (Chen–Yang–Zhang–Zhang's Theorem). *Assume that the differential system* (2) *satisfies the following conditions:*

(i) $g(x) = -g(-x)$ *and* $xg(x) > 0$ *if* $x \neq 0$;

(ii) $f(x) = f_1(x) + f_2(x)$ *with* $f_1(x) = f_1(-x)$, $f_2(x) = -f_2(-x)$ *and* $f_1(x) \neq 0$.

Then this system (2) *has no periodic orbits in R.*

Theorem 4 is a particular case of Theorem 1 of [1].

As far as we know the next two criteria for the nonexistence of periodic orbits are new.

Let $f(x,y) = 0$ be a curve, then a point (x_0, y_0) of this curve is a contact point with system (1) if it satisfies $(P\partial f/\partial x + Q\partial f/\partial y)(x_0, y_0) = 0$.

Theorem 5 (Transversal divergence criterium). *Let $D(x,y) = \partial P/\partial x + \partial Q/\partial y$ be the divergence of system* (1). *If the curve $D(x,y) = 0$ has no contact points of even multiplicity with the system* (1), *then this system has no periodic orbits.*

The proof of Theorem 5 is given in Section 2.

Working in polar coordinates (r, θ) where $x = r\cos\theta$ and $y = r\sin\theta$ system (1) writes

$$\dot{r} = \left.\frac{xP + yQ}{\sqrt{x^2+y^2}}\right|_{(x,y)=(r\cos\theta, r\sin\theta)}, \quad \dot{\theta} = \left.\frac{xQ - yP}{x^2+y^2}\right|_{(x,y)=(r\cos\theta, r\sin\theta)}.$$

Theorem 6 (Angular velocity criterium)**.** *Assume that the origin of coordinates is an equilibrium point of a system* (1)*, and that the component γ of the curve $xQ - yP = 0$ passes through the origin of coordinates and locally on one side of this curve we have $xQ - yP > 0$ and on the other side $xQ - yP < 0$. Then system* (1) *has no periodic orbits surrounding the origin crossing the component γ at a point with odd mutiplicity.*

Theorem 6 is proved in Section 3.

2 Proof of Theorem 5

By the Bendixson Theorem any periodic orbit of system (1) must intersect the curve $D(x, y) = 0$. But under the assumptions of Theorem 5 the flow of this system is transversal at all the point of the curve except at its possible contact points of odd multiplicity, but also at these points the flow crosses the curve $D(x, y) = 0$. Hence clearly a periodic orbit cannot intersect the divergence curve $D(x, y) = 0$ and consequently it does not exists. This completes the proof of Theorem 5.

Now we present an application of Theorem 5. We consider the Selkov-Higgins system which is relevant in the study of the glycolysis. This system when one of its parameters is equal to 2 writes

$$\dot{x} = 1 - xy^2 = P(x, y), \qquad \dot{y} = ay(xy - 1) = Q(x, y). \tag{3}$$

The divergence of this system is $D(x, y) = -a + 2axy - y^2$. Now we study the transversality of the flow of system (1) on the curve $D(x, y) = 0$, that is

$$p(y) := \frac{\partial D}{\partial x}P + \frac{\partial D}{\partial x}Q\Big|_{D=0} = \frac{1}{2}(-a^2 + 4ay - 3y^4).$$

Using the formulas of Lu Yang [7] for this quartic polynomial we have

$$D_2 = 0, \quad D_3 = -2592a^2, \quad D_4 = 6912a^4(a^2 - 9).$$

When $|a| > 3$ then $D_4 > 0$ and $D_3 \le 0$ or $D_2 \le 0$, and the polynomial $p(y)$ has no real roots. Consequently by Theorem 5 system (3) has no periodic orbits.

If $a = \pm 3$ then $D_4 = 0$ and $D_3 < 0$, and the polynomial $p(y)$ has one double real root. So system (3) again by Theorem 5 has no periodic orbits.

If $a = 0$ then $D_4 = D_3 = D_2 = 0$ and the polynomial $p(y)$ has one quadruple real root. Hence by Theorem 5 system (3) has no periodic orbits.

Finally if $a \in (-3, 0) \cup (0, 3)$ then $D_4 < 0$, and the polynomial $p(y)$ has two real simple roots. In this case we cannot apply Theorem 5 and system (3) could have periodic orbits for some of these values of a. Indeed in the work [5] values of $a \in (-3, 0) \cup (0, 3)$ are given for which system (3) has periodic orbits.

3 Proof of Theorem 6

Assume that there exists a periodic orbit Γ surrounding the origin which crosses the component γ at a point p with odd multiplicity. Then on this periodic orbit Γ in a neighborhood of p and on one side of γ we have $\dot{\theta} > 0$ and on the other side $\dot{\theta} < 0$, this provides a contradiction because in a small neighborhood of p the periodic orbit must be either $\dot{\theta} \geq 0$, or $\dot{\theta} \leq 0$. This completes the proof of Theorem 6.

Now we present one application of Theorem 6. Consider the differential system

$$\dot{x} = -x(2 + f(x, y)) + y = P(x, y), \qquad \dot{y} = -y(2 + f(x, y)) + x = Q(x, y), \tag{4}$$

where the C^1 function f is such that it, its first and second derivatives vanish at the origin of coordinates. Then the origin of coordinates is a stable node with eigenvalues -1 and -3, and $xQ - yP = x^2 - y^2$, and consequently $\dot{\theta} = 0$ is formed by the two straight lines $y = \pm x$. So by Theorem 6 system (4) cannot have periodic orbits surrounding the origin.

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Some families of quadratic systems with at most one limit cycle

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Abstract. The work of Chicone and Shafer published in 1982 together with the work of Bamon published in 1986 proved that any polynomial differential system of degree two has finitely many limit cycles. But the problem remains open of providing a uniform upper bound for the maximum number of limit cycles that a polynomial differential system of degree two can have, i.e. the second part of the 16th Hilbert problem restricted to the polynomial differential systems of degree two remains open. Here we present six subclasses of polynomial differential systems of degree two for which we can prove that an upper bound for their maximum number of limit cycles is one.

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Keywords and phrases: quadratic systems, 16th Hilbert problem, limit cycles.

1 Introduction and statement of the main results

We deal with polynomial differential systems in $\mathbb{R}^2$ of the form

$$\frac{dx}{dt} = \dot{x} = P(x,y), \qquad \frac{dy}{dt} = \dot{y} = Q(x,y).$$

The degree of such a polynomial system is the maximum of the degrees of the polynomials P and Q. In what follows the polynomial differential systems of degree 2 are simply called *quadratic systems*.

We recall that a *limit cycle* of a differential system is a periodic orbit of this system isolated in the set of all periodic orbits of the system. As far as we know the notion of limit cycle appeared in the work of Poincaré [14] in the year 1885.

At the Second International Congress of Mathematicians, held in Paris in 1900, Hilbert [8] proposed his famous 16th problem, whose second part essentially says: *Find an upper bound for the maximum number of limit cycles that the polynomial differential systems in $\mathbb{R}^2$ of a given degree can have.*

The works of Chicone and Shafer [5] and of Bamon [1] proved that any polynomial differential system of degree 2 has finitely many limit cycles. This result uses previous results of Ilyashenko [9]. Up to now the second part of the 16th Hilbert problem remains unsolved, also for the quadratic systems.

In 1957 Petrovskii and Landis [12] claimed that the polynomial differential systems of degree $n = 2$ have at most 3 limit cycles. Soon (in 1959) a gap was found

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in the arguments of Petrovskii and Landis, see [13]. Later, Lan Sun Chen and Ming Shu Wang [3] in 1979, and Songling Shi [16] in 1982, provided the first quadratic systems having 4 limit cycles, and up to now 4 is the maximum number of limit cycles known for a quadratic system.

We recall the following three well known properties of quadratic systems.

(a) In the region limited by a periodic orbit of a quadratic system there is a unique equilibrium point, see Theorem 2 of Coppel [6], or Theorem 2.8 of Chicone and Jinghuang [4].

(b) A periodic orbit of a quadratic system surrounds a focus or a center, proved by Vorob'ev [17], see also Theorem 6 of Coppel [6].

(d) Quadratic systems having a center have no limit cycles, see Vulpe [18] and Schlomiuk [15].

From these three properties if follows that if a quadratic system has a limit cycle this must surround a focus.

Let O be a focus or a center of a quadratic system, without loss of generality we can assume that O is localized at the origin of coordinates, otherwise we do a translation sending O to the origin of coordinates. Kaptein [10, 11] proved that any quadratic system having a focus or a center at the origin of coordinates can be written as (see also Bautin [2])

$$\begin{aligned} \dot{x} &= \lambda_1 x - y - \lambda_3 x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6 y^2, \\ \dot{y} &= x + \lambda_1 y + \lambda_2 x^2 + (2\lambda_3 + \lambda_4)xy - \lambda_2 y^2. \end{aligned} \tag{1}$$

In order to avoid subindexes we denote

$$\lambda_1 = \lambda, \quad \lambda_2 = a, \quad \lambda_3 = b, \quad \lambda_4 = c, \quad \lambda_5 = d, \quad \lambda_6 = e.$$

Then system (1) becomes

$$\begin{aligned} \dot{x} &= \lambda x - y - bx^2 + (2a + d)xy + ey^2, \\ \dot{y} &= x + \lambda y + ax^2 + (2b + c)xy - ay^2. \end{aligned} \tag{2}$$

The goal of this paper is to give conditions on the parameters of system (2) for the presence of a maximum of one limit cycle for the system surrounding the origin. For this we rely on the paper [7] where a theorem is stated giving conditions for having at most three limit cycles in an Abel differential equation.

A good tool for studying the possible limit cycles surrounding the origin O of the quadratic system (2) is to write this quadratic system in polar coordinates (r, θ), where $x = r\cos\theta$, $y = r\sin\theta$. Then system (2) becomes

$$\begin{aligned} \dot{r} &= \lambda r + f(\theta) r^2, \\ \dot{\theta} &= 1 + g(\theta) r, \end{aligned} \tag{3}$$

where

$$\begin{aligned} f(\theta) &= -a\sin^3\theta + (2b+c+3)\sin^2\theta\cos\theta + (3a+d)\sin\theta\cos^2\theta - b\cos^3\theta, \\ g(\theta) &= -3\sin^3\theta - (3a+d)\sin^2\theta\cos\theta + (3b+c)\sin\theta\cos^2\theta + a\cos^3\theta. \end{aligned} \tag{4}$$

Note that $f(\theta)$ and $g(\theta)$ are homogeneous trigonometric polynomials of degree three.

We define the polynomials

$$\begin{aligned} F(z) &= -az^3 + (2b+c+3)z^2 + (3a+d)z - b, \\ G(z) &= -3z^3 - (3a+d)z^2 + (3b+c)z + a, \end{aligned}$$

note that $f(\theta) = \cos^3\theta F(\tan\theta)$ and $g(\theta) = \cos^3\theta G(\tan\theta)$.

Here first we classify all quadratic systems whose polynomial $G(z)(\lambda G(z) - F(z))$ satisfies the following two properties:
(P1) it has degree six, and
(P2) for all $z \in \mathbb{R}$ the value of $G(z)(\lambda G(z) - F(z))$ is either ≥ 0, or $= 0$, or ≤ 0.

Theorem 1. *Every quadratic system* (2) *satisfying properties* ($P1$) *and* ($P2$) *must be one of the following six forms of quadratic systems*

$$\begin{aligned} \dot{x} &= \lambda x - y - bx^2 - 2axy - a^2y^2/b, \\ \dot{y} &= x + \lambda y + ax^2 + (a^2 - b^2)xy/b - ay^2 \end{aligned} \tag{5}$$

(i.e. $d = 0$, $c = (a^2 - 3b^2)/b$ *and* $e = -a^2/b$ *in* (2)*), with* $b \neq 0$*;*

$$\begin{aligned} \dot{x} &= \lambda x - y - cy^2, \\ \dot{y} &= x + \lambda y + cxy \end{aligned} \tag{6}$$

(i.e. $a = b = d = 0$ *and* $e = c$ *in* (2)*), with* $c \neq 0$*;*

$$\begin{aligned} \dot{x} &= \lambda x - y + dxy + ey^2, \\ \dot{y} &= x + \lambda y + cxy \end{aligned} \tag{7}$$

(i.e. $a = b = 0$ *in* (2)*), with* $c^2 + d^2 + e^2 \neq 0$ *and* $\Delta_i(\lambda, 0, 0, c, d, e) > 0$ *for* $i = 1, 2$*;*

$$\begin{aligned} \dot{x} &= \lambda x - y + (2a+d)xy + ey^2, \\ \dot{y} &= x + \lambda y + ax^2 + cxy - ay^2 \end{aligned} \tag{8}$$

(i.e. $b = 0$ *in* (2)*), where* $\Delta_i(\lambda, a, 0, c, d, e) > 0$ *for* $i = 1, 2$*,*
$c = -a(2a+d-e)(2a+d+e)/((2a+d)e)$ *and* $(2a+d)e \neq 0$*;*

$$\begin{aligned} \dot{x} &= \lambda x - y - bx^2 + (2a+d)xy + ey^2, \\ \dot{y} &= x + \lambda y + ax^2 + cxy - ay^2 \end{aligned} \tag{9}$$

(i.e. $2b + c = c$ *in* (2)*), where* $\Delta_i(\lambda, a, b, c, d, e) > 0$ *for* $i = 1, 2$*, and* $c = -a\left(2ab + 2ae + bd + de + (b-e)\sqrt{(2a+d)^2 + 4be}\right)/(2be)$ *and* $be \neq 0$*;*

$$\begin{aligned} \dot{x} &= \lambda x - y - bx^2 + (2a+d)xy + ey^2, \\ \dot{y} &= x + \lambda y + ax^2 + cxy - ay^2 \end{aligned} \tag{10}$$

(i.e. $2b + c = c$ *in* (2)*), where* $\Delta_i(\lambda, a, b, c, d, e) > 0$ *for* $i = 1, 2$*, and* $c = -a\left(2ab + 2ae + bd + de - (b-e)\sqrt{(2a+d)^2 + 4be}\right)/(2be)$ *and* $be \neq 0$*.*
The functions Δ_i *for* $i = 1, 2$ *are defined in Section* 2*.*

Theorem 2. *The six quadratic families of systems of Theorem* 1 *have only one equilibrium point, the origin of coordinates. Moreover all these quadratic systems have at most one limit cycle, and when it exists it surrounds the origen of coordinates.*

Theorems 1 and 2 are proved in Section 2.

2 Proof of Theorems 1 and 2

Statement (a) of the next proposition is proved in statement (a) of Proposition 8 of Gasull and Llibre [7], and statement (b) of the next proposition is proved in statement (b) of Theorem C also in [7].

Proposition 1. *Let* $A(\theta) = g(\lambda g - f)$*, where the functions* $f(\theta)$ *and* $g(\theta)$ *are defined in* (4)*. Then the following statements hold.*

(a) If $A(\theta) \neq 0$ *and either* $A(\theta) \geq 0$ *or* $A(\theta) \leq 0$*, then system* (2) *has at most one limit cycle surrounding the origin. Furthermore, it can exist only if* $\lambda sign(A(\theta)) < 0$*.*

(b) If $A(\theta) = 0$*, then system* (2) *has at most one limit cycle surrounding the origin.*

From Proposition 1 the next result follows immediately .

Corollary 1. *If for all values of* $z \in \mathbb{R}$ *the polynomial* $G(z)(\lambda G(z) - F(z))$ *is either* ≥ 0*, or* $= 0$*, or* ≤ 0*, then the differential system* (2) *has at most one limit cycle surrounding the origin of coordinates.*

If the polynomial $G(z)(\lambda G(z) - F(z))$ is the zero polynomial, then by Corollary 1 there is at most one limit cycle of system (2) surrounding the origin. Later on we will show that the six quadratic families of systems of Theorem 1 have only a unique equilibrium point, the origin. So Theorems 1 and 2 will be proved when the polynomial $G(z)(\lambda G(z) - F(z))$ is the zero polynomial. So in what follows we assume that this polynomial is distinct from zero.

By assumption (P1) the polynomial $G(z)(\lambda G(z) - F(z))$ has degree six, therefore both polynomials $G(z)$ and $\lambda G(z) - F(z)$ are of degree three, so they have at least one real root. Then such a real root must be common to the polynomials $G(z)$ and $\lambda G(z) - F(z)$, otherwise the assumption (P2) would not hold. Hence the resultant of the polynomials $G(z)$ and $\lambda G(z) - F(z)$ must be zero, i.e.

$$\begin{aligned} R(G, \lambda G - F) = \quad & \left((4a+d)^2 + (3b+c+e)^2\right)\Big(ad(2b+c)(b+e) + be(2b+c)^2 + \\ & 4a^4 + 4a^3 d + a^2\left(3b^2 + 2b(c+3e) + 2ce + d^2 - e^2\right)\Big). \end{aligned}$$

Now we consider two cases.
Case 1: $(4a+d)^2 + (3b+c+e)^2 = 0$. Then $d = -4a$, $e = -3b - c$. Therefore the roots of the polynomial $G(z)$ are $\pm i$ and $-a/(3b+c)$, note that $3b + c \neq 0$ because

the polynomial $G(z)$ has degree 3 and consequently it must have three roots. The roots of the polynomial $\lambda G(z) - F(z)$ are $\pm i$ and $-(b+\lambda a)/(a+\lambda(3b+c))$, and $a+\lambda(3b+c) \neq 0$ because the polynomial $\lambda G(z) - F(z)$ has degree 3.

In order that the polynomial $G(z)(\lambda G(z) - F(z))$ verify that $g \geq 0$ or $g \leq 0$ for all $z \in \mathbb{R}$, we need that the real root of the polynomials $G(z)$ and $\lambda G(z) - F(z)$ coincide, i.e.

$$-a/(3b+c) = -(b+\lambda a)/(a+\lambda(3b+c)). \tag{11}$$

Then if $b \neq 0$ we have that $c = (a^2 - 3b^2)/b$ and

$$G(z)(\lambda G(z) - F(z)) = \frac{a(\lambda a + b)(z^2+1)^2(az+b)^2}{b^2}.$$

Since the function $G(z)(\lambda G(z) - F(z))$ satisfies the assumptions of Corollary 1, so system (2) satisfying $c = (a^2 - 3b^2)/b$ reduces to system (5) and has at most one limit cycle, this limic cycle surrounds the origin. Furthermore also this system has a unique equilibrium point, the origin, as it is easy to check.

If $b = 0$ then from (11) we get that $a = 0$, and consequently

$$G(z)(\lambda G(z) - F(z)) = \lambda c^2 z^2 (1+z^2)^2.$$

Again the function $G(z)(\lambda G(z) - F(z))$ satisfies the assumptions of Corollary 1, so system (2) satisfying $b = a = 0$ reduces to system (6) and has at most one limit cycle, and this limic cycle surrounds the origin. Furthermore also this system has a unique equilibrium point, the origin, as it is easy to verify.

We remark that if we impose that the polynomials $G(z)$ and $\lambda G(z) - F(z)$ be one a multiple of the other, or equivalently that they have exactly the same three roots, then we get exactly the previous two quadratic systems (5) and (6). Hence in what follows we can assume that the polynomials $G(z)$ and $\lambda G(z) - F(z)$ have different roots. Then in order that the polynomial $G(z)(\lambda G(z) - F(z))$ can satisfy the assumption (P2) and since the polynomials $G(z)$ and $\lambda G(z) - F(z)$ are cubic polynomials by the assumption (P1), they must have in common a real root, and the other roots cannot be the same for both polynomials, otherwise we will obtain the quadratic systems (5) and (6).

In summary, we can restrict our attention to the polynomials $G(z)$ and $\lambda G(z) - F(z)$ having a common real root and the other two roots non-real because if one of these two polynomials has the three real roots, then it is not possible that the polynomial $G(z)(\lambda G(z) - F(z))$ satisfies the assumption (P2), i.e. the polynomial $G(z)(\lambda G(z) - F(z))$ would change the sign because not all the real roots of the polynomials $G(z)$ and $(\lambda G(z) - F(z))$ would coincide. This implies that the discriminats of the polynomials $G(z)$ and $\lambda G(z) - F(z)$ must be positive, see an easy proof of this fact in the cubic equation of Wikipedia.

The discriminants $\Delta_i = \Delta_i(\lambda, a, b, c, d, e)$ for $i = 1, 2$ of the polynomials $G(z)$ and $\lambda G(z) - F(z)$ are respectively
$\Delta_1 = 108a^4 + 81a^2b^2 + 54a^2bc + 9a^2c^2 + 108a^3d + 54ab^2d + 36abcd + 6ac^2d + 36a^2d^2 + 9b^2d^2 + 6bcd^2 + c^2d^2 + 4ad^3 + 162a^2be + 108b^3e + 54a^2ce + 108b^2ce + 36bc^2e + 4c^3e +$

$54abde + 18acde - 27a^2e^2$,
and
$\Delta_2 = 108a^4 + 117a^2b^2 + 32b^4 + 90a^2bc + 48b^3c + 9a^2c^2 + 24b^2c^2 + 4bc^3 + 108a^3d + 60ab^2d + 42abcd + 6ac^2d + 36a^2d^2 + 4b^2d^2 + 4bcd^2 + c^2d^2 + 4ad^3 + 90a^2be + 48b^3e + 18a^2ce + 48b^2ce + 12bc^2e + 42abde + 12acde + 4bd^2e + 2cd^2e + 9a^2e^2 + 24b^2e^2 + 12bce^2 + 6ade^2 + d^2e^2 + 4be^3 - 4ab^3\lambda + 6ab^2c\lambda - 2ac^3\lambda + 36a^2bd\lambda + 24b^3d\lambda + 16b^2cd\lambda - 2bc^2d\lambda - 2c^3d\lambda + 18abd^2\lambda + 6acd^2\lambda + 4bd^3\lambda + 2cd^3\lambda + 12ab^2e\lambda - 12abce\lambda - 36a^2de\lambda - 12b^2de\lambda - 14bcde\lambda - 4c^2de\lambda - 18ad^2e\lambda - 2d^3e\lambda - 12abe^2\lambda + 6ace^2\lambda - 12bde^2\lambda - 2cde^2\lambda + 4ae^3\lambda + 216a^4\lambda^2 + 198a^2b^2\lambda^2 + 36b^4\lambda^2 + 144a^2bc\lambda^2 + 60b^3c\lambda^2 + 18a^2c^2\lambda^2 + 37b^2c^2\lambda^2 + 10bc^3\lambda^2 + c^4\lambda^2 + 216a^3d\lambda^2 + 102ab^2d\lambda^2 + 54abcd\lambda^2 + 72a^2d^2\lambda^2 - 8bcd^2\lambda^2 - 4c^2d^2\lambda^2 + 12ad^3\lambda^2 + d^4\lambda^2 + 252a^2be\lambda^2 + 144b^3e\lambda^2 + 72a^2ce\lambda^2 + 132b^2ce\lambda^2 + 34bc^2e\lambda^2 + 2c^3e\lambda^2 + 120abde\lambda^2 + 54acde\lambda^2 + 18bd^2e\lambda^2 + 8cd^2e\lambda^2 - 18a^2e^2\lambda^2 + 36b^2e^2\lambda^2 + 24bce^2\lambda^2 + c^2e^2\lambda^2 - 6ade^2\lambda^2 + 18ab^2c\lambda^3 + 12abc^2\lambda^3 + 2ac^3\lambda^3 + 36a^2bd\lambda^3 + 36b^3d\lambda^3 + 42b^2cd\lambda^3 + 16bc^2d\lambda^3 + 2c^3d\lambda^3 + 6abd^2\lambda^3 - 6acd^2\lambda^3 - 2bd^3\lambda^3 - 2cd^3\lambda^3 - 36abce\lambda^3 - 12ac^2e\lambda^3 - 36a^2de\lambda^3 - 36b^2de\lambda^3 - 42bcde\lambda^3 - 10c^2de\lambda^3 - 6ad^2e\lambda^3 + 18ace^2\lambda^3 + 108a^4\lambda^4 + 81a^2b^2\lambda^4 + 54a^2bc\lambda^4 + 9a^2c^2\lambda^4 + 108a^3d\lambda^4 + 54ab^2d\lambda^4 + 36abcd\lambda^4 + 6ac^2d\lambda^4 + 36a^2d^2\lambda^4 + 9b^2d^2\lambda^4 + 6bcd^2\lambda^4 + c^2d^2\lambda^4 + 4ad^3\lambda^4 + 162a^2be\lambda^4 + 108b^3e\lambda^4 + 54a^2ce\lambda^4 + 108b^2ce\lambda^4 + 36bc^2e\lambda^4 + 4c^3e\lambda^4 + 54abde\lambda^4 + 18acde\lambda^4 - 27a^2e^2\lambda^4$.
We recall that when the discriminant of a cubic polynomial is positive, then such a polynomial has a unique real root.
Case 2:
$ad(2b+c)(b+e) + be(2b+c)^2 + 4a^3(a+d) + a^2\left(3b^2 + 2b(c+3e) + 2ce + d^2 - e^2\right) = 0$.
This equation has the following seven sets of solutions

(s1) $a = e = 0$;

(s2) $b = e = 0$ and $d = -2a$;

(s3) $c = -((2b^2d + 4a^2(a+d) + a(3b^2 + d^2))/(b(2a+d)))$ and $e = 0$;

(s4) $a = b = 0$;

(s5) $c = -((a(2a+d-e)(2a+d+e))/((2a+d)e))$ and $b = 0$;

(s6) $c = -((2(a^2+2b^2)e + 2a^2(b+e) + ad(b+e) + a(b-e)\sqrt{(2a+d)^2 + 4be})/(2be))$;

(s7) $c = -((2(a^2+2b^2)e + 2a^2(b+e) + ad(b+e) - a(b-e)\sqrt{(2a+d)^2 + 4be})/(2be))$.

The polynomial $G(z)(\lambda G(z) - F(z))$ has degree less than 6 for the solutions (s1), (s2) and (s3), so we do not consider these three solutions. While for the solutions from (s4) to (s7) this polynomial has degree 6.

Every one of the solutions from (s4) to (s7) implies that the polynomials $G(z)$ and $\lambda G(z) - F(z)$ have at least one root in common, if additionally we impose that the discriminants of these two polynomials are positive, then these polynomials have one real root in common and two distinct conjugate complex roots. Additionally we shall prove that the quadratic systems satisfying some solution (sk) for $k = 4, \dots, 7$ have a unique equilibrium, the focus localized at the origin of coordinates, therefore by Corollary 1 we obtain that these four families of quadratic systems satisfying some solution (sk) for $k = 4, \dots, 7$ with $\Delta_1 > 0$, $\Delta_2 > 0$, cannot have more than one limit cycle surrounding the origin. Hence Theorems 1 and 2 will be proved.

Now we prove that the quadratic systems satisfying (sk) for $k = 4, \dots, 7$ have a unique equilibrium. Indeed, since the polynomial $\lambda G(z) - F(z)$ has a unique real root and two complex ones, and this real root also is the unique real root of the polynomial $G(z)$, it follows from systems (3) that systems (2) has only one finite equilibrium point, the origin of coordinates. Indeed, the equilibrium points (r^*, θ^*) of system (3) with $r^* \neq 0$ must satisfy that $\lambda g(\theta^*) - f(\theta^*) = 0$ and $r^* = -1/g(\theta^*)$, but if $\lambda g(\theta^*) - f(\theta^*) = 0$ then $1/g(\theta^*) = \infty$. Hence the unique equilibrium point of system (3) is the one with $r = 0$, i.e. the origin of coordinates.

We note that the solutions (s4), (s5), (s6) and (s7) provide the quadratic systems (7), (8), (9) and (10), respectively.

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Time-Reversibility and Ivariants of Some 3-dim Systems

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Abstract. We study time-reversibility and invariants of the group of transformations $x \to x,\ y \to \alpha y,\ z \to \alpha^{-1} z$ for three-dimensional polynomial systems with $0:1:-1$ resonant singular point at the origin. An algorithm to find the Zariski closure of the set of time-reversible systems in the space of parameters is proposed. The interconnection of time-reversibility and invariants of the group mentioned above is discussed.

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Dedicated to the memory of
Professor K. S. Sibirsky

1 Introduction

Let k be a field, let G be a multiplicative group of invertible $n \times n$ matrices with elements in k and, for $A \in G$ and $\mathbf{x} = (x_1, x_2, \dots, x_n) \in k^n$, let $A \cdot \mathbf{x}$ denote the usual action of G on k^n. A polynomial $f \in k[x_1, \dots, x_n]$ is *invariant under* G if $f(\mathbf{x}) = f(A \cdot \mathbf{x})$ for every $\mathbf{x} \in k^n$ and every $A \in G$. The polynomial f is also called an *invariant* of G.

Consider two-dimensional systems of the form

$$\begin{aligned} \dot{x} &= x - \sum_{(p,q)\in S} a_{pq} x^{p+1} y^q, \\ \dot{y} &= -y + \sum_{(p,q)\in S} b_{qp} x^q y^{p+1}, \end{aligned} \tag{1}$$

where the variables x and y and the coefficients of (1) are complex, and $S \subset (\{-1\} \cup \mathbb{N}_0) \times \mathbb{N}_0$ is a finite set, of which every element (p,q) satisfies $p + q \geq 1$. Let ℓ be the cardinality of the set S. Then, $\mathbb{C}^{2\ell}$ is the parameter space of (1), which we denote by $E(a,b)$. The set of polynomials in ordered variables $a_{p_1,q_1}, \dots, a_{p_\ell,q_\ell}, b_{q_\ell,p_\ell}, \dots, b_{q_1,p_1}$ with coefficients in the field k will be denoted by $k[a,b]$.

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After the transformation

$$x' = e^{-i\varphi}x, \quad y' = e^{i\varphi}y \tag{2}$$

(such transformations form a one-parametric group of the parameter φ), we obtain the system

$$\dot{x}' = x' - \sum_{(p,q)\in S} a(\varphi)_{pq} x'^{p+1} y'^{q}, \quad \dot{y}' = -y' + \sum_{(p,q)\in S} b(\varphi)_{qp} x'^{q} y'^{p+1},$$

where the coefficients of the transformed system are

$$a(\varphi)_{pq} = a_{pq} e^{i(p_j - q_j)\varphi}, \quad b(\varphi)_{qp} = b_{qp} e^{-i(p_j - q_j)\varphi}, \tag{3}$$

for $(p,q) \in S$. For any fixed φ the equations in (3) determine an invertible linear mapping U_φ of the space $E(a,b)$ of parameters of (1) onto itself.

The group U_φ of family (1) acts on $E(a,b) = \mathbb{C}^{2\ell}$. The set of polynomial invariants of this group action has been for the first time studied by Sibirsky [12, 13]. Actually, Sibirsky considered the case of the "real" system (1), that is, the case where both equations on the right-hand side of (1) are multiplied by i and the first equation of (1) is the complex conjugate of the second one (such systems are complexifications of real systems, see e.g. [9, Chapter 3]). However, as it is shown in [8] and [9, Chapter 5], the theory for general systems (1) is similar to the theory developed by Sibirsky.

Before we proceed, we fix some notations. For any n-tuple $s = (s_1, s_2, \ldots, s_n)$, $n \geq 1$, let $\hat{s}$ be the permutation $\hat{s} = (s_n, s_{n-1}, \ldots, s_1)$. For two n-tuples $r = (r_1, r_2, \ldots, r_n)$, $s = (s_1, s_2, \ldots, s_n)$ we define the "dot"-product as $r \cdot s = r_1 s_1 + r_2 s_2 + \cdots + r_n s_n$. Given n-tuples r, s, let the ordered pair (r,s) denote the $2n$-tuple generated in the obvious way. Furthermore, we will use a short form of monomial writing as $(a_1, a_2, \ldots, a_n)^{(\nu_1, \nu_2, \ldots, \nu_n)} := a_1^{\nu_1} a_2^{\nu_2} \ldots a_n^{\nu_n} = a^\nu$, where $a = (a_1, \ldots, a_n)$ and $\nu = (\nu_1, \ldots, \nu_n)$.

Let $L_1, L_2 : \mathbb{N}_0^{2\ell} \to \mathbb{Z}$ be homomorphisms of the additive monoid $\mathbb{N}_0^{2\ell}$ defined with respect to the ordered set S by

$$\begin{aligned} L_1(\nu) &= p_1\nu_1 + \cdots + p_\ell \nu_\ell + q_\ell \nu_{\ell+1} + \cdots + q_1 \nu_{2\ell} \\ &= (p, \hat{q}) \cdot \nu, \\ L_2(\nu) &= q_1\nu_1 + \cdots + q_\ell \nu_\ell + p_\ell \nu_{\ell+1} + \cdots + p_1 \nu_{2\ell}) \\ &= (q, \hat{p}) \cdot \nu, \end{aligned} \tag{4}$$

where $p := (p_1, \ldots, p_\ell)$, $q := (q_1, \ldots, q_\ell)$ and $\nu := (\nu_1, \ldots, \nu_{2\ell})$. Furthermore, the map

$$L := L_1 - L_2 : \mathbb{N}_0^{2\ell} \to \mathbb{Z} \tag{5}$$

is a monoid-homomorphism as well, hence the kernel,

$$\widetilde{\mathcal{M}} := \ker L = \{\nu : \ L(\nu) = 0\} \tag{6}$$

is also a monoid. Since U_φ changes only the coefficients of polynomials, a polynomial $f \in \mathbb{C}[a,b]$ is an invariant of the group U_φ if and only if each of its terms is an invariant (see Lemma 3.4 of [12]). Therefore, for the description of polynomial invariants of U_φ, it suffices to find the invariant monomials. By (3), for $\nu \in \mathbb{N}_0^{2\ell}$, $a = (a_{p_1,q_1} \dots a_{p_\ell,q_\ell})$, $b = (b_{q_1,p_1} \dots a_{q_\ell,p_\ell})$, we denote by $[\nu] \in \mathbb{C}[a,b]$ the monomial

$$[\nu] := a_{p_1q_1}^{\nu_1} \cdots a_{p_\ell q_\ell}^{\nu_\ell} b_{q_\ell p_\ell}^{\nu_{\ell+1}} \cdots b_{q_1p_1}^{\nu_{2\ell}} = (a, \hat{b})^\nu. \tag{7}$$

The image of ν under the group action U_φ is the monomial

$$\begin{aligned} U_\varphi([\nu]) &= (a(\varphi), \widehat{b(\varphi)})^\nu \\ &= a(\varphi)_{p_1q_1}^{\nu_1} \cdots a(\varphi)_{p_\ell q_\ell}^{\nu_\ell} b(\varphi)_{q_\ell p_\ell}^{\nu_{\ell+1}} \cdots b(\varphi)_{q_1p_1}^{\nu_{2\ell}} \\ &= a_{p_1q_1}^{\nu_1} e^{i\varphi\nu_1(p_1-q_1)} \cdots a_{p_\ell q_\ell}^{\nu_\ell} e^{i\varphi\nu_\ell(p_\ell-q_\ell)} b_{q_\ell p_\ell}^{\nu_{\ell+1}} e^{i\varphi\nu_{\ell+1}(q_\ell-p_\ell)} \cdots b_{q_1p_1}^{\nu_{2\ell}} e^{i\varphi\nu_{2\ell}(q_1-p_1)} \\ &= e^{i\varphi[\nu_1(p_1-q_1)+\dots+\nu_\ell(p_\ell-q_\ell)+\nu_{\ell+1}(q_\ell-p_\ell)+\dots+\nu_{2\ell}(q_1-p_1)]} a_{p_1q_1}^{\nu_1} \cdots a_{p_\ell q_\ell}^{\nu_\ell} b_{q_\ell p_\ell}^{\nu_{\ell+1}} \cdots b_{q_1p_1}^{\nu_{2\ell}} \\ &= e^{i\varphi(L_1-L_2)(\nu)}\,[\nu] \\ &= e^{i\varphi L(\nu)}\,[\nu]\,. \end{aligned} \tag{8}$$

From (8) we see that the monomial $[\nu]$ defined by (7) is invariant under the group action U_φ, for system (1) if and only if $L(\nu) = 0$, that is, if and only if $\nu \in \widetilde{\mathcal{M}}$. Since, for any $\nu \in \mathbb{N}_0^{2\ell}$,

$$\begin{aligned} L(\nu) &= (p - q, \hat{q} - \hat{p}) \cdot \nu \\ &= (q - p, \hat{p} - \hat{q}) \cdot \hat{\nu} \\ &= -L(\hat{\nu}), \end{aligned} \tag{9}$$

we have $\nu \in \widetilde{\mathcal{M}}$ if and only if $\hat{\nu} \in \widetilde{\mathcal{M}}$, hence the monomial $[\nu]$ is invariant under the group action U_φ if and only if its so-called conjugate

$$\begin{aligned} [\hat{\nu}] &= a_{p_1q_1}^{\nu_{2\ell}} \cdots a_{p_\ell q_\ell}^{\nu_\ell+1} b_{q_\ell p_\ell}^{\nu_\ell} \cdots b_{q_1p_1}^{\nu_1} \\ &= (a, b)^{\hat{\nu}} \end{aligned} \tag{10}$$

is also invariant.

Sibirsky found some important properties of the monoid $\widetilde{\mathcal{M}}$. One of them is the fact that the set $\{[\nu] : \nu \in \widetilde{\mathcal{M}}\}$ is closed under multiplication. From his results one can see that a basis of the monoid $\widetilde{\mathcal{M}}$ (a basis of the invariants of the group U_φ) can be found by sorting, since Sibirsky got a bound for the degree of basis invariants. A simple algorithm to compute generators of $\widetilde{\mathcal{M}}$ based on the Gröbner bases theory was proposed in [4].

With system (1) and the monoind $\widetilde{\mathcal{M}}$ we associate the ideal

$$\widetilde{I}_S = \langle [\nu] - [\hat{\nu}] : \nu \in \widetilde{\mathcal{M}} \rangle.$$

This ideal was called in [4] the Sibirsky ideal of system (1). It was shown by Sibirsky [12, Chapter 3] that in the "real" case if the parameters of the system belong to the variety $\mathbf{V}(I_S)$, then the vector field of the system is symmetric with respect to a line passing through the origin (after reversion of time), that is, it is time-reversible, and, therefore, admits an analytic local first integral in a neighborhood of the origin. Later on the result was generalized to general systems (1) in [7,8], where it was shown that for family (1) not all systems from $\mathbf{V}(I_S)$ are time-reversible, but $\mathbf{V}(I_S)$ is the Zariski closure of the set of time-reversible systems and, therefore, all systems from $\mathbf{V}(I_S)$ admit an analytic first integral in a neighborhood of the origin.

We recall (see e.g. [5]) that in the higher-dimensional case a system of ordinary differential equations

$$\dot{\mathbf{x}} = \mathcal{X}(\mathbf{x}), \tag{11}$$

where $\mathcal{X}(\mathbf{x})$ is a vector function defined on some domain D of $\mathbb{R}^n$ or $\mathbb{C}^n$, is *time-reversible* on D if there exists an involution $\psi : D \to D$ (the involution means that ψ is smooth and $\psi \circ \psi = id_D$) such that

$$D_\psi^{-1} \mathcal{X} \circ \psi = -\mathcal{X}.$$

It is said that a system (11) is *completely integrable* on D if it admits $n-1$ functionally independent analytic first integrals on D. The problem of complete integrability can be also considered as a natural generalization of the center problem for two-dimensional systems to higher dimensions, see e.g. [6, 11, 14].

In this paper we study three-dimensional systems of the form

$$\begin{aligned} \dot{x} &= P_1(x,y,z), \\ \dot{y} &= y + P_2(x,y,z), \\ \dot{z} &= -z + P_3(x,y,z), \end{aligned} \tag{12}$$

where P_j, $j \in \{1,2,3\}$, are polynomial functions on $\mathbb{C}^3$ which vanish together with its first partial derivatives at the origin and present some generalizations of the above mentioned results of Sibirsky and those of [7,8] to the case of system (12).

2 Time-reversibility

The following statement is easily derived from a general result of [6] (see also [10]).

Theorem 1. *If under the interchange of the last two variables a system* (12) *is transformed to a system of the same form but with the right-hand side multiplied by* -1, *then it admits two analytic local first integrals of the form*

$$\Psi_1(x,y,z) = x + \cdots$$

and

$$\Psi_2(x,y,z) = yz + \cdots .$$

In the other words, the statement means that if a system (12) is time-reversible with respect to the linear involution defined on $\mathbb{C}^3$

$$x \mapsto x, \ y \mapsto z, \ z \mapsto y, \tag{13}$$

then it is completely integrable in a neighborhood of the origin.

Without loss of generality we can write a polynomial system (12) in the form

$$\begin{aligned} \dot{x} &= \sum_{(P,Q,R)\in T} a_{PQR} x^P y^Q z^R, \\ \dot{y} &= y - \sum_{(p,q,r)\in S} b_{pqr} x^p y^{q+1} z^r, \\ \dot{z} &= -z + \sum_{(p,q,r)\in S} c_{prq} x^p y^r z^{q+1}, \end{aligned} \tag{14}$$

where $S \subset \mathbb{N}_0 \times (\mathbb{N}_0 \cup \{-1\}) \times \mathbb{N}_0$ is a set of ℓ triplets, all satisfying $1 \le p+q+r \le N$, and $T \subset \mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{N}_0$ is a set of triplets, all satisfying $2 \le P+Q+R \le N$, where N is the degree of (14). Note that the indexing set T is symmetric with respect to the second and third coordinates, i.e. $(P,Q,R) \in T$ if and only if $(P,R,Q) \in T$.

The correctness of the following statement can be verified by straightforward computations, see also (20).

Lemma 2. *Let $\alpha \neq 0$. If a system* (14) *is time-reversible with respect to the involution*

$$\psi(x,y,z) = (x, \alpha z, \alpha^{-1} y), \tag{15}$$

then $a_{PQQ} = 0$ for every $(P,Q,Q) \in T$.

Due to the above lemma, we a priori assume that in (14)

$$a_{PQQ} = 0 \ \text{ for all } (P,Q,Q) \in T$$

or, equivalently, we exclude these parameters from the parameter space. By enumeration we fix an arbitrary order in the indexing set S

$$S = \{(p_1,q_1,r_1), \dots, (p_\ell,q_\ell,r_\ell)\}. \tag{16}$$

Further we split the indexing set T in a disjoint union $T = T_1 \cup T_2$ with $T_1 = \{(P,Q,R) : Q > R\}$ and $T_2 = \{(P,Q,R) : Q < R\}$. Note that T_1 and T_2 have the property that for every $(P,Q,R) \in T_1$ we have $(P,R,Q) \in T_2$, thus both T_1 and T_2 have the same number of elements, say m elements. Then we fix an arbitrary order in T_1:

$$T_1 = \{(P_1,Q_1,R_1), \dots, (P_m,Q_m,R_m)\}. \tag{17}$$

In a natural way, this order induces the order in the set T_2

$$T_2 = \{(P_1,R_1,Q_1), \dots, (P_m,R_m,Q_m)\}.$$

The ring of polynomials with ordered coefficients

$$a_{P_1Q_1R_1}, \cdots a_{P_mQ_mR_m}, a_{P_mR_mQ_m}, \cdots a_{P_1R_1Q_1}, b_{p_1q_1r_1}, \cdots b_{p_\ell q_\ell r_\ell}, c_{p_\ell r_\ell q_\ell}, \cdots c_{p_1r_1q_1} \tag{18}$$

as indeterminates and coefficients in a field k (typically $\mathbb{C}$ or $\mathbb{Q}$) will be denoted by $k[a,b,c]$. Along with the latter ring we will work also with its extension $k[a,b,c,\alpha,w]$ where α and w are variables.

Proposition 3. *1) The Zariski closure of the set of systems in family* (14) *which are time-reversible with respect to involution* (15) *is the variety* $\mathbf{V}(\mathcal{I}_R)$ *of the ideal*

$$\mathcal{I}_R = H \cap \mathbb{C}[a,b,c],$$

where H *is the following ideal in* $\mathbb{C}[a,b,c,\alpha,w]$

$$H = \langle a_{PQR}\alpha^Q + a_{PRQ}\alpha^R, b_{pqr}\alpha^{q+1} - c_{prq}\alpha^{r+1}, \alpha w - 1 : (P,Q,R) \in T,\ (p,q,r) \in S\rangle. \tag{19}$$

2) If the parameters of a system (14) *belong to the variety* $\mathbf{V}(\mathcal{I}_R)$*, then the system is completely integrable.*

Remark 4. *Notice that the above ideal* H *remains the same if we replace the indexing set* T *by only* T_1 *or by* T_2*.*

Proof of Prop. 3. Let $\mathcal{X}$ be the vector field (14). Equating to zero the coefficients of the monomials of the polynomial $D_\psi \cdot \mathcal{X} + \mathcal{X} \circ \psi$ we obtain the system

$$a_{PQR} = -\alpha^{R-Q}a_{PRQ},\ b_{pqr} = \alpha^{r-q}c_{prq}, \quad (P,Q,R) \in T,\ (p,q,r) \in S.$$

That means, system (14) is time-reversible with respect to involution (15) if and only if there is a nonzero α such that

$$a_{PQR}\alpha^Q + \alpha^R a_{PRQ} = 0, \quad b_{pqr}\alpha^q - \alpha^r c_{prq} = 0, \quad (P,Q,R) \in T,\ (p,q,r) \in S \tag{20}$$

or, equivalently, avoiding the possibly negative exponent $q \geq -1$

$$a_{PQR}\alpha^Q + \alpha^R a_{PRQ} = 0, \quad b_{pqr}\alpha^{q+1} - \alpha^{r+1} c_{prq} = 0, \quad (P,Q,R) \in T,\ (p,q,r) \in S.$$

By the Elimination theorem (see e.g. [2,9]) this is the case when the coefficients of (14) belong to the variety of the ideal $\mathcal{I}_R$ defined by (3).

2) By the construction $\mathbf{V}(\mathcal{I}_R)$ is the Zariski closure of systems which are time-reversible with respect to (15). We observe that if a system (14) is time-reversible with respect to (15) then, after the change of coordinates $x_1 = x$, $x_2 = \alpha^{-1}y$, $x_3 = \alpha z$, we obtain the system which is time-reversible with respect to involution (13). By Theorem 1 the obtained system is completely integrable. Thus, $\mathbf{V}(\mathcal{I}_R)$ is the Zariski closure of a set of completely integrable systems. By the results of [11] the set of completely integrable systems is an algebraic set. Therefore systems from $\mathbf{V}(\mathcal{I}_R)$ are completely integrable. $\square$

3 Invariants

Recalling the fixed order (18) in our polynomial indeterminates, we write each monomial in the polynomial ring with these coefficients as indeterminates in the form

$$a_{P_1Q_1R_1}^{\mu_1}\cdots a_{P_mQ_mR_m}^{\mu_n}{a_{P_mR_mQ_m}}^{\mu_{n+1}}\cdots a_{P_1R_1Q_1}^{\mu_{2m}}b_{p_1q_1r_1}^{\nu_1}\cdots b_{p_\ell q_\ell r_\ell}^{\nu_\ell}c_{p_\ell r_\ell q_\ell}^{\nu_{\ell+1}}\cdots c_{p_1r_1q_1}^{\nu_{2\ell}}. \tag{21}$$

Introducing the notations

$$a = (a_{P_1Q_1R_1},\dots,a_{P_mQ_mR_m}), \qquad b = (b_{p_1q_1r_1},\dots,b_{p_\ell q_\ell r_\ell}),$$
$$a' = (a_{P_1R_1Q_1},\dots,a_{P_mR_mQ_m}), \qquad c = (c_{p_1r_1q_1},\dots,c_{p_\ell r_\ell q_\ell}),$$

we set up the monomial (21)

$$\begin{aligned}[\mu;\nu] =&[\mu_1,\dots,\mu_{2m};\nu_1,\dots,\nu_{2\ell}]\\ =&(a,\widehat{a'})^{\mu}(b,\hat{c})^{\nu}.\end{aligned} \tag{22}$$

In particular,

$$[\mu;0] = (a,\widehat{a'})^{\mu} \tag{23}$$

and

$$[0;\nu] = (b,\hat{c})^{\nu}. \tag{24}$$

With systems (14) and the fixed enumeration (17), (16) of indices $(P,Q,R)\in T_1$ and $(p,q,r)\in S$ we associate vectors

$$\begin{aligned}K =&(Q_1-R_1,\dots,Q_m-R_m) = (K_1,\dots,K_m),\\ \kappa =&(q_1-r_1,\dots,q_\ell-r_\ell) = (\kappa_1,\dots,\kappa_\ell)\end{aligned}$$

and the map $L:\mathbb{N}_0^{2m}\times\mathbb{N}_0^{2\ell}\to\mathbb{Z}$, defined by

$$L(\mu,\nu) = (K,-\widehat{K})\cdot\mu + (\kappa,-\hat{\kappa})\cdot\nu, \qquad \mu\in\mathbb{N}_0^{2m},\ \nu\in\mathbb{N}_0^{2\ell}.$$

It is easy to see that L is a homomorphism of the Abelian monoid $\mathbb{N}_0^{2m}\times\mathbb{N}_0^{2\ell}$ into the Abelian monoid $\mathbb{Z}$ and consequently, the kernel of L, denoted by $\mathcal{M} := \{(\mu,\nu):\ L(\mu,\nu)=0\}$ is a submonoid in $\mathbb{N}_0^{2m}\times\mathbb{N}_0^{2\ell}$.

A simple computation gives that for every $\mu\in\mathbb{N}_0^{2m},\ \nu\in\mathbb{N}_0^{2\ell}$

$$L(\mu,\nu) = -L(\hat{\mu},\hat{\nu}),$$

easily providing the following statement.

Lemma 5. $(\mu,\nu)\in\mathcal{M}$ *if and only if* $(\hat{\mu},\hat{\nu})\in\mathcal{M}$.

Let

$$x\to x,\quad y\to\alpha y,\quad z\to\alpha^{-1}z \tag{25}$$

be the one-parametric group U_α of invertible linear transformations of the phase space of systems (14). Similarly to the two-dimensional case in Section 1, we denote the coefficients of the new systems as $a_{PQR}(\alpha), b_{pqr}(\alpha), c_{prq}(\alpha)$. The straightforward computation gives

$$\begin{aligned} a_{PQR}(\alpha) &= \alpha^{R-Q} a_{PQR}, \\ b_{pqr}(\alpha) &= \alpha^{r-q} b_{pqr}, \\ c_{prq}(\alpha) &= \alpha^{q-r} c_{prq}, \end{aligned} \tag{26}$$

for all $(P,Q,R) \in T$, $(p,q,r) \in S$.

Proposition 6. *The monomial* $[\mu;\nu]$ *is invariant under the action of group* (25) *if and only if* $(\mu,\nu) \in \mathcal{M}$.

Proof. The action of the group (25) induces the change of coefficients of (14) according to (26). Recalling (23) and (24) and performing this substitution in $[\mu,\nu]$ we obtain

$$\begin{aligned} U_\alpha([\mu;\nu]) =& [\mu;\nu]\alpha^{(Q-R,\hat{R}-\hat{Q})\cdot\mu+(q-r,\hat{r}-\hat{q})\cdot\nu} \\ =& [\mu,\nu]\alpha^{(K,-\hat{K})\cdot\mu+(\kappa,-\hat{\kappa})\cdot\nu} \\ =& [\mu,\nu]\alpha^{L(\mu,\nu)} \end{aligned}$$

wherefrom the claim easily follows. $\square$

We now define a generalized version of the Sibirsky ideal. For any $\mu \in \mathbb{N}_0^{2m}$ denote $|\mu| = \sum_{j=1}^{2m} \mu_j$.

Definition 7. *The ideal*

$$\mathcal{I}_S = \langle (-1)^{|\mu|}[\mu;\nu] - [\hat{\mu};\hat{\nu}] : (\mu,\nu) \in \widetilde{\mathcal{M}} \rangle$$

is called the Sibirsky ideal of systems (14).

For the proof of our main theorem, we will apply the following theorem ([1], Theorem 2.4.10).

Theorem 8. *Let* J *be an ideal of* $k[y_1,\dots,y_m]$, I *be an ideal of* $k[x_1,\dots x_n]$ *and let* $K = \langle I, y_1 - f_1, \dots, y_m - f_m \rangle \subseteq k[y_1,\dots,y_m,x_1,\dots x_n]$.
Let $\phi : k[y_1,\dots,y_m]/_J \to k[x_1,\dots x_n]/_I$ *be the homomorphism defined by*

$$y_i + J \mapsto f_i + I.$$

Then $\ker\phi = K \cap k[y_1,\dots,y_m](\mathrm{mod} J)$. *That is, if* $\ker\phi = \langle g_1 + J, \dots, g_p + J \rangle$, *then* $K \cap k[y_1,\dots,y_m] = \langle g_1,\dots,g_p \rangle$.

The statement below is our main result and it generalizes a result obtained in [7] for the case of systems (1) to the case of systems (14).

Theorem 9. $\mathcal{I}_R = \mathcal{I}_S$.

Proof. Recall that the ideal H is defined by (19) and the ideal, which we are interested in, is $\mathcal{I}_R = H \cap \mathbb{C}(a,b,c)$. Let $\mathcal{I} = \langle \alpha w - 1\rangle$, $s = (s_1, \ldots, s_m)$, $t = (t_1, \ldots, t_\ell)$. We define a homomorphism $\phi : \mathbb{C}[a,b,c] \to \mathbb{C}[s,t,\alpha,w]_{/\mathcal{I}}$ by

$$\begin{aligned}
a_{P_nQ_nR_n} &\mapsto s_n + \mathcal{I},\\
a_{P_nR_nQ_n} &\mapsto -\alpha^{Q_n-R_n} s_n + \mathcal{I},\\
b_{p_jq_jr_j} &\mapsto t_j + \mathcal{I},\\
c_{p_jr_jq_j} &\mapsto \alpha^{q_j-r_j} t_j + \mathcal{I}, \quad \text{if } q_j \geq r_j,\\
c_{p_jr_jq_j} &\mapsto w^{r_j-q_j} t_j + \mathcal{I}, \quad \text{if } r_j > q_j,\\
& n = 1,2,\ldots,m,\ j = 1,2,\ldots,\ell.
\end{aligned}$$

Recalling the shorthand notation $K_n = Q_n - R_n > 0$, $n = 1,2,\ldots,m$, and $\kappa_j = q_j - r_j$, $j = 1,2,\ldots,\ell$, let

$$\begin{aligned}
\widetilde{H} = \langle \mathcal{I},\ & a_{P_nQ_nR_n} - s_n,\ a_{P_nR_nQ_n} - (-\alpha^{K_n} s_n),\ b_{p_jq_jr_j} - t_j,\ c_{p_{k_j}r_{k_j}q_{k_j}} - t_{k_j}\alpha^{\kappa_{k_j}},\\
& c_{p_{k_i}r_{k_i}q_{k_i}} - w^{-\kappa_{k_i}} t_{k_i} : 1 \leq n \leq m, 1 \leq j \leq \ell, \kappa_{k_j} \geq 0, \kappa_{k_i} < 0\rangle.
\end{aligned}$$

By Theorem 8 (J is taken to be trivial), we have

$$\ker \phi = \widetilde{H} \cap \mathbb{C}[a,b,c]$$

and by Proposition 3, $\mathcal{I}_R = H \cap \mathbb{C}[a,b,c]$.

We next show that $\widetilde{H} \cap \mathbb{C}[a,b,c] = H \cap \mathbb{C}[a,b,c]$. By elimination of $s_1, \ldots, s_m, t_1, \ldots, t_\ell$ from $\widetilde{H}$ we get exactly H. Hence $H = \widetilde{H} \cap \mathbb{C}[a,b,c,\alpha,w]$ and

$$\begin{aligned}
\mathcal{I}_R &= H \cap \mathbb{C}[a,b,c]\\
&= \widetilde{H} \cap \mathbb{C}[a,b,c,\alpha,w] \cap \mathbb{C}[a,b,c]\\
&= \widetilde{H} \cap \mathbb{C}[a,b,c]\\
&= \ker \phi.
\end{aligned}$$

Next we check that $\mathcal{I}_S \subset \ker \phi$, i.e. that

$$\phi([\hat{\mu};\hat{\nu}]) = (-1)^{|\mu|}\phi([\mu;\nu]), \qquad (\mu;\nu) \in \mathcal{M}.$$

Writing in a short way, with $\mu = (\xi, \eta) \in \mathbb{N}_0^m \times \mathbb{N}_0^m$, $\nu = (\zeta, \theta) \in \mathbb{N}_0^\ell \times \mathbb{N}_0^\ell$, we have

$$[\mu; 0] = [\xi, \eta; 0] = a^\xi (\hat{a}')^\eta = \prod_{j=1}^m a_{P_jQ_jR_j}{}^{\xi_j} \prod_{n=1}^m a_{P_nR_nQ_n}{}^{\hat{\eta}_n}$$

and

$$[0;\nu] = [0;\zeta,\theta] = b^\zeta \hat{c}^\theta = \prod_{j=1}^\ell b_{p_jq_jr_j}{}^{\zeta_j} \Pi_{n=1}^\ell c_{p_nr_nq_n}{}^{\hat{\theta}_n}.$$

Now, acting by ϕ on $[\mu;0]$, noting that $\hat{s}^\eta = s^{\hat{\eta}}$ gives us

$$\begin{aligned}\phi([\mu;0]) = \phi([\xi,\eta;0]) &= (-1)^{|\eta|}\alpha^{(\widehat{Q}-\widehat{R})\cdot\eta}s^\xi\hat{s}^\eta + \mathcal{I}\\ &= (-1)^{|\eta|}\alpha^{\widehat{K}\cdot\eta}s^{\xi+\hat{\eta}} + \mathcal{I}\end{aligned} \tag{27}$$

and

$$\begin{aligned}\phi([\hat{\mu};0]) = \phi([\hat{\eta},\hat{\xi};0]) &= (-1)^{|\xi|}\alpha^{(\widehat{Q}-\widehat{R})\cdot\hat{\xi}}s^{\hat{\eta}}\hat{s}^{\hat{\xi}} + \mathcal{I}\\ &= (-1)^{|\xi|}\alpha^{(Q-R)\cdot\xi}s^{\xi+\hat{\eta}} + \mathcal{I}\\ &= (-1)^{|\xi|}\alpha^{K\cdot\xi}s^{\xi+\hat{\eta}} + \mathcal{I}.\end{aligned}$$

By choosing $[\mu;0] = [\xi,\eta;0] \in \mathcal{M}$ we know that $K\cdot\xi = \widehat{K}\cdot\eta$. Moreover, it is easy to check that

$$\phi((-1)^{|\mu|}[\mu;0]) = \phi([\hat{\mu};0]))$$

since $(-1)^{|\xi|+2|\eta|} = (-1)^{|\xi|}$. We have to be a bit careful when computing $\phi(c^\theta)$. Namely, $\phi(c_n^{\theta_n}) = t_n^{\theta_n}\alpha^{\kappa_n\theta_n}$ if $\kappa_n = q_n - r_n \geq 0$ and $\phi(c_j^{\theta_j}) = t^j w^{-\kappa_j\theta_j}$ if $\kappa_j < 0$. Denote by κ_+ the non-negative part of κ, and by κ_- the negative part such that $\kappa = \kappa_+ + \kappa_-$ and $\operatorname{supp}\kappa_+ \cap \operatorname{supp}\kappa_- = \{\}$. Now,

$$\phi([0;\nu]) = \phi([0;\zeta,\theta]) = t^\zeta\hat{t}^\theta\alpha^{(\kappa_+)\cdot\hat{\theta}}w^{-(\kappa_-)\cdot\hat{\theta}} + \mathcal{I} = t^{\zeta+\hat{\theta}}\alpha^{(\kappa_+)\cdot\hat{\theta}}w^{-(\kappa_-)\cdot\hat{\theta}} + \mathcal{I} \tag{28}$$

and

$$\phi([0;\hat{\nu}]) = \phi([0;\hat{\theta},\hat{\zeta}]) = t^{\hat{\theta}+\zeta}\alpha^{(\kappa_+)\cdot\zeta}w^{-(\kappa_-)\cdot\zeta} + \mathcal{I}.$$

We next show that $\alpha^{(\kappa_+)\cdot\hat{\theta}}w^{-(\kappa_-)\cdot\hat{\theta}} - \alpha^{(\kappa_+)\cdot\zeta}w^{-(\kappa_-)\cdot\zeta} \in \mathcal{I}$ as soon as $(0;\zeta,\theta) \in \mathcal{M}$. Denote $u_1 = \kappa_+\cdot\hat{\theta}$, $u_2 = -\kappa_-\cdot\hat{\theta}$, $v_1 = \kappa_+\cdot\zeta$, $v_2 = -\kappa_-\cdot\zeta$. The requirement $(0;\zeta,\theta) \in \mathcal{M}$ tells us that $v_1 - u_1 = v_2 - u_2 =: d$. Assuming that $d \geq 0$ we obtain

$$\begin{aligned}\alpha^{(\kappa_+)\cdot\hat{\theta}}w^{-(\kappa_-)\cdot\hat{\theta}} - \alpha^{(\kappa_+)\cdot\zeta}w^{-(\kappa_-)\cdot\zeta} =&\alpha^{u_1}w^{u_2} - \alpha^{v_1}w^{v_2}\\ =&\alpha^{u_1}w^{u_2}(1-\alpha^d w^d)\\ =&\alpha^{u_1}w^{u_2}f(\alpha,w)(1-\alpha w)\end{aligned}$$

where $f(\alpha,w)$ is a polynomial. We proceed very similarly when $d < 0$. Therefore, $\phi([0;\nu]) = \phi([0;\hat{\nu}])$.

To complete this step of the proof, i.e. to show that all generating binomials of $\mathcal{I}_S$ are in the kernel of ϕ, let $(\mu,\nu) \in \mathcal{M}$. Then, as ϕ is a ring homomorphism,

$$\begin{aligned}\phi((-1)^{|\mu|}[\mu;\nu]) =&\phi((-1)^{|\mu|}[\mu;0])\phi([0;\nu])\\ =&\phi([\hat{\mu};0])\phi([0;\hat{\nu}])\\ =&\phi([\hat{\mu};\hat{\nu}]).\end{aligned}$$

It remains to check that $\ker\phi \subset I_S$. A reduced Gröbner basis G of $\mathbb{C}[a,b,c] \cap \widetilde{H}$ can be found by computing a reduced Gröbner basis of $\widetilde{H}$ using an elimination ordering with $\{a,b,c\} < \{w,\alpha,s,t\}$, and then intersecting it with $\mathbb{C}[a,b,c]$. Since $\widetilde{H}$

is binomial, any reduced Gröbner basis G of $\widetilde{H}$ also consists of binomials. This means that $\mathcal{I}_R = \widetilde{H} \cap \mathbb{Q}[a,b,c] = \ker\phi$ is a binomial ideal. Assume that for some $(\xi,\eta;\zeta,\theta)$, $(\gamma,\delta;\varepsilon,\varphi) \in \mathbb{N}^{2m} \times \mathbb{N}^{2\ell}$, $u \in \mathbb{C}$, the equality $\phi(u[\xi,\eta;\zeta,\theta] - [\gamma,\delta;\varepsilon,\varphi]) = 0$ holds. Without loosing any generality, we assume that $[\xi,\eta;\zeta,\theta]$ and $[\gamma,\delta;\varepsilon,\varphi]$ do not have nontrivial common factors. This implies that $\xi_j\gamma_j = \eta_j\delta_j = 0$, $j = 1,2,\dots,m$, and $\zeta_i\varepsilon_i = \theta_i\delta_i = 0$, $i = 1,2,\dots,\ell$. Suppose

$$\phi(u[\xi,\eta;\zeta,\theta]) = \phi([\gamma,\delta;\varepsilon,\varphi]).$$

We will show that $[\gamma,\delta;\varepsilon,\varphi] = [\hat{\eta},\hat{\xi};\hat{\theta},\hat{\zeta}]$ and $u = (-1)^{|\xi|+|\eta|}$. From (27) and (28) one derives that

$$f := u(-1)^{|\eta|+|\delta|} s^{\xi+\hat{\eta}} t^{\zeta+\hat{\theta}} \alpha^{\widehat{K}\cdot\eta+(\kappa_+)\cdot\hat{\theta}} w^{-(\kappa_-)\cdot\hat{\theta}} - s^{\gamma+\hat{\delta}} t^{\varepsilon+\hat{\varphi}} \alpha^{\widehat{K}\cdot\delta+(\kappa_+)\cdot\hat{\varphi}} w^{-(\kappa_-)\cdot\hat{\varphi}} \in \langle \alpha w - 1\rangle.$$

Computing the value of f at $w = \alpha^{-1}$ we must have 0. But this implies the equality of (possibly rational) monomials

$$s^{\xi+\hat{\eta}} t^{\zeta+\hat{\theta}} \alpha^{\widehat{K}\cdot\eta+(\kappa_+ + \kappa_-)\cdot\hat{\theta}} = s^{\gamma+\hat{\delta}} t^{\varepsilon+\hat{\varphi}} \alpha^{\widehat{K}\cdot\delta+(\kappa_+ + \kappa_-)\cdot\hat{\varphi}} \tag{29}$$

and additionally,

$$u(-1)^{|\eta|+|\delta|} = 1. \tag{30}$$

Comparing the powers at s, t, α in (29) gives

$$\xi + \hat{\eta} = \gamma + \hat{\delta} \tag{31}$$

$$\zeta + \hat{\theta} = \varepsilon + \hat{\varphi} \tag{32}$$

$$\widehat{K}\cdot\eta + \kappa\cdot\hat{\theta} = \widehat{K}\cdot\delta + \kappa\cdot\hat{\varphi}. \tag{33}$$

We will firstly prove and then immediately apply the following technical lemma.

Lemma 10. *Let $\xi,\eta,\gamma,\delta \in \mathbb{N}_0$ be non-negative integers. Assume that*

$$\xi + \eta = \gamma + \delta \tag{34}$$

$$\xi\gamma = 0 \tag{35}$$

$$\eta\delta = 0. \tag{36}$$

Then $(\gamma,\delta) = (\eta,\xi)$.

Proof. Let us firstly assume that $\gamma > \eta$. Then $\gamma \neq 0$ and by (35), $\xi = 0$. Apply (34) to get a contradiction, since δ is not negative. Similarly, if $\eta > \gamma$ we have $\eta \neq 0$ and thus by (36) one obtains $\delta = 0$. This contradicts the non-negativity of ξ. It follows that $\gamma = \eta$ and consequently from (34), $\delta = \xi$ as claimed. □

Let us continue with the proof of Theorem 9. From (31) we observe that $\xi_j + \hat{\eta}_j = \gamma_j + \hat{\delta}_j$ and by our assumption on coprimeness, $\xi_j\gamma_j = \hat{\eta}_j\hat{\delta}_j = 0$ for all $j = 1,2,\dots,m$. Applying Lemma 10 we obtain $\gamma_j = \hat{\eta}_j$ and $\hat{\delta}_j = \xi_j$, $j = 1,\dots,m$

and in turn, $\gamma = \hat{\eta}$ and $\delta = \hat{\xi}$, i.e. $(\gamma, \delta) = (\hat{\eta}, \hat{\xi})$. In a very similar manner we get $(\varepsilon, \varphi) = (\hat{\theta}, \hat{\zeta})$ from (32).

It remains to see that both $[\xi, \eta; \zeta, \theta]$ and $[\gamma, \delta; \varepsilon, \varphi]$ must be in $\mathcal{M}$. By Lemma 5 and inserting $(\gamma, \delta, \varepsilon, \varphi) = (\hat{\eta}, \hat{\xi}, \hat{\theta}, \hat{\zeta})$ into (33) we confirm the claim.

Finally we easily get $u = (-1)^{|\xi|+|\eta|}$ from (30) since $|\delta| = |\hat{\xi}| = |\xi|$. $\square$

A generating set or basis $\mathcal{N}$ of $\mathcal{M}$ is *minimal* if, for each $\nu \in \mathcal{N}$, $\mathcal{N} \setminus \{\nu\}$ is not a generating set. A minimal generating set is called a *Hilbert basis* of $\mathcal{M}$.

Theorem 11. *Let G be the reduced Gröbner basis of $\mathcal{I}_S$ with respect to a chosen term order. Then the following holds.*

1. *Every element of G has the form $(-1)^{|\mu|}[\mu; \nu] - [\hat{\mu}; \hat{\nu}]$, where $(\mu, \nu) \in \mathcal{M}$ and $[\mu; \nu]$ and $[\hat{\mu}; \hat{\nu}]$ have no common factors.*

2. *The set*

$$\begin{aligned}\mathcal{N} = \{&(\mu, \nu), (\hat{\mu}, \hat{\nu}) : (-1)^{|\mu|}[\mu; \nu] - [\hat{\mu}; \hat{\nu}] \in G\} \\ &\cup \{(0, e_j) + (0,\ e_{2\ell-j+1}) : j = 1, \ldots, \ell \text{ and } \pm([0; e_j] - [0; e_{2\ell-j+1}]) \notin G\},\end{aligned}$$

where $e_j = (0, \ldots, 0, \overset{j}{1}, 0, \ldots, 0) \in \mathbb{Q}^{2\ell}$, is a Hilbert basis of $\mathcal{M}$.

The proof of the theorem is similar to the proof of Theorem 5.2.5 in [9].

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A survey on local integrability and its regularity

Yantao Yang and Xiang Zhang

Abstract. In this survey paper, we summarize our results and also some related ones on local integrability of analytic autonomous differential systems near an equilibrium. The results are on necessary conditions related to existence of local analytic or meromorphic first integrals, on existence of analytic normalization of local analytically integrable system, and also on some sufficient conditions for existence of local analytic first integrals. Among which the results are also on regularity of the local first integrals, including analytic and Gevrey smoothness. We also present some open questions for further investigation.

Mathematics subject classification: 34C05, 34A34, 34G05.
Keywords and phrases: Analytic differential systems, integrability, regularity, Gevrey smoothness, normalization.

1. Introduction

For analytic autonomous differential system

$$\dot{y} = \frac{dy}{dx} = F(y), \quad y \in \Omega \subset \mathbb{R}^n, \tag{1}$$

where Ω is an open domain, the problem on existence of local or global first integrals is classical. This problem can be traced back to Poincaré [17] and Darboux [3, 4]. A smooth function $H(y)$ is a first integral of system (1) on Ω if $\langle \nabla H, F(y)\rangle \equiv 0$ except perhaps a zero Lebesgue measure subset. Hereafter $\langle \cdot, \cdot \rangle$ represents the inner product of two vectors in $\mathbb{R}^n$. System (1) is analytic (smooth) integrable if it has $n-1$ functionally independent analytic (smooth) first integrals. Here our system is autonomous, we consider the first integrals only depending on the dependent variables, because our aim is to apply these first integrals to describe the dynamics of the system in the phase space. Of course, we can consider first integrals including also the independent variable. Since we want to study orbits in the phase space, we consider here only the first integrals in the phase variables y.

For a given analytic system (1), as it is well known that if $y = y_0$ is a regular point, system (1) is analytically integrable around y_0, i.e. it has $n-1$ functionally independent local analytic first integrals around y_0. This argument can be verified using the flow-box theorem or proved directly using the solution of the initial value problem with the fixed initial time x_0 and the initial values near y_0. When $y = y_0$ is a singular point of system (1), the problem on existence of functionally independent

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analytic or smooth first integrals becomes very difficult. In this case we can write system (1), after possibly a translation, in the form

$$\dot{y} = Ay + f(y), \quad f(y) = O(|y|^2), \tag{2}$$

where without loss of generality we can assume that A is in Jordan normal form. Let $\mathcal{Y}$ be the vector field associated to this differential system, and set $\mathcal{Y} = \sum_{j=1}^{\infty} \mathcal{Y}_j$, where $\mathcal{Y}_j$ is the jth homogeneous part of $\mathcal{Y}$.

As we will show that the existence and number of functionally independent first integrals of system (2) in a neighborhood of the origin are strongly related to the eigenvalues and their relations. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be the n-tuple of eigenvalues of A. We say that λ is resonant if

$$\mathcal{R} := \{m \in \mathbb{Z}_+^n | \ \langle m, \lambda \rangle = 0, \ |m| \geq 2\} \neq \emptyset,$$

where $\mathbb{Z}_+$ is the set of nonnegative integers, and $|m| = m_1 + \ldots + m_n$ for $m = (m_1, \ldots, m_n) \in \mathbb{Z}_+^n$. The classical Poincaré theorem on integrability [17] states that the analytic differential system (2) has no analytic or formal first integrals in a neighborhood of the origin provided that the eigenvalues of A are not resonant. See also [8, 21].

According to the classical result mentioned above by Poincaré, in order that system (2) has an analytic first integral or a formal first integral near the origin, the eigenvalues of A must be resonant.

1 Necessary conditions on existences of first integrals

In 2008, Chen, Yi and Zhang [2] obtained some necessary conditions on existence and number of functionally independent analytic or formal first integrals.

Theorem 1. *Assume that the number of $\mathbb{Q}_+$–linearly independent elements of $\mathcal{R}$ is m. Then the analytic differential system* (2) *has at most m functionally independent analytic or formal first integrals in a neighborhood of the origin.*

The proof depends on the inductive calculations on analytic or formal first integral of the form

$$H(y) = \sum_{j=q}^{\infty} H_j(y)$$

with H_j homogeneous polynomials of degree j, via

$$\begin{aligned} \mathcal{L}(H_q)(y) &:= \langle \nabla H_q, Ay \rangle = 0, \\ \mathcal{L}(H_j)(y) &= -\sum_{s=2}^{j-q+1} \mathcal{Y}_s(H_{j-s+1}), \quad j = q+1, \ldots \end{aligned}$$

by inductive calculations together with the invertibility of the linear operator $\mathcal{L}$ on each linear space formed by homogeneous polynomials of any given degree. For all functionally independent analytic or formal first integrals $H_1, \ldots, H_m$ of system (2), one can assume without loss of generality [34, 35] that their lowest parts $H_1^0(y), \ldots, H_m^0(y)$ are functionally independent. For a proof, see e.g.[9]. Then the problem is turned to the maximum number of functionally independent monomial solutions of $\mathcal{L}(H_q)(y) = 0$. The solutions of this problem is equivalent to the spectrum of the linear operator

$$\mathcal{L}(H_\ell)(y) = \nabla H_\ell(y) A y$$

on the linear space $\mathcal{H}_\ell(y)$, formed by the homogeneous polynomial of degree ℓ in the n variables y. By [12,31] the spectrum of $\mathcal{L}$ on $\mathcal{H}_\ell(y)$ is $\mathcal{R}_\ell := \{m \in \mathbb{Z}_+^n | \ \langle m, \lambda\rangle, \ |m| = \ell\}$.

In 2007 Shi [20] extended the Poincaré's result to the existence of meromorphic first integrals of the analytic differential system (2). Cong, Llibre and Zhang [6] further developed Shi's result to the version of Theorem 1 on the number of functionally independent meromorphic first integrals.

Theorem 2. *Assume that*

$$\mathcal{R}_Q := \{m \in \mathbb{Z}^n | \ \langle m, \lambda\rangle = 0, \ |m| = |m_1| + \ldots + |m_n| \geq 2\}$$

contains r number of $\mathbb{Q}$-linearly independent elements. Then system (2) *has at most r functionally independent meromorphic first integrals.*

The proof adopts the ideas from those of Theorem 1. For functionally independent meromorphic first integrals

$$H_1(y) = \frac{P_1(y)}{Q_1(y)}, \quad \ldots, \quad H_r(y) = \frac{P_r(y)}{Q_r(y)}$$

one can assume without loss of generality that the lowest order terms

$$H_1^0(y) = \frac{P_1^0(y)}{Q_1^0(y)}, \quad \ldots, \quad H_r^0(y) = \frac{P_r^0(y)}{Q_r^0(y)}$$

are functionally independent. For a proof see e.g.[6, Lemma 6], otherwise one can take polynomials $W_j(z_1, \ldots, z_j)$ such that

$$H_1(y), \ W_2(H_1, H_2), \ \ldots, \ W_r(H_1, \ldots, H_r)$$

have their lowest order rational homogeneous parts being functionally independent. Here the lowest order rational homogeneous part of a meromorphic function $H(y) = P(y)/Q(y)$ with P, Q analytic and having the expansions $P(y) = P^0(y)$+h.o.t and $Q(y) = Q^0(y)$+h.o.t is $P^0(y)/Q^0(y)$, because after expansion

$$H(y) = \frac{P(y)}{Q(y)} = \frac{P^0(y)}{Q^0(y)} + \sum_{j=0}^{\infty} \frac{A^j(y)}{B^j(y)}$$

one has

$$\deg P^0(y) - \deg Q^0(y) < \deg A^j(y) - \deg B^j(y), \qquad \text{for all} \ \ j \geq 1$$

where A^j, B^j's are homogeneous polynomials of degree j. Each $\frac{A^j(y)}{B^j(y)}$ is a rational homogeneous function. The next proofs can be down in a similar way as those in the proof of Theorem 1, by considering the linear operator defined by the linear part of system (2) acting on the set of the lowest order rational homogeneous parts of the meromorphic first integrals.

Associated to Theorems 1 and 2, Llibre, Walcher and Zhang [16] provided a version on local Darboux first integrals of analytic differential systems via Poincaré-Dulac normal form.

Theorems 1 and 2 establish only the necessary conditions on the existence of analytic first integrals. As we know, it is really difficult to provide a sufficient condition on existence of analytic first integrals. The typical one is the center-focus problem in the general case. This is to characterize planar analytic differential systems which have an equilibrium with a pair of pure imaginary eigenvalues. Of course, in this case the two eigenvalues are $\mathbb{Z}_+$-resonant. But the problem whether it admits an analytic first integral was solved only for quadratic differential systems. See example [18, 19, 31].

2 Analytic normalization of local analytically integrable differential systems

As it is well known, the sufficient condition for existence of functionally independent first integrals is hard to be found for general planar analytic even polynomial differential systems. Sometimes the equivalent characterization to analytic integrability of analytic differential systems is helpful to determine local properties of the system near an equilibrium. See for instance the next result by Poincaré.

To state the next result, we recall the definitions on Poincaré-Dulac normal form and resonant terms. For system (2) in $\mathbb{R}^n$ with A in Jordan normal form, if the n-tuple of eigenvalues λ of A have complex conjugate ones, saying for example λ_j and $\lambda_{j+1} = \overline{\lambda_j}$, whose associated variables are y_j and y_{j+1}, we set $z_j = y_j + \sqrt{-1}y_{j+1}$ and $z_{j+1} = y_j - \sqrt{-1}y_{j+1}$. For each real eigenvalue λ_s, whose associated variable y_s is replaced by z_s. In these new coordinates, system (2) becomes

$$\dot{z} = Bz + g(z),$$

with B in lower triangular matrix. Its Poincaré-Dulac normal form system is the one

$$\dot{z} = Bz + h(z),$$

with $h(z)$ consisting of the resonant monomials. A monomial $z^k e_l$ in $h(z)$, $k \in \mathbb{Z}_+^n$, $l \in \{1, \ldots, n\}$, is resonant, if $\langle k, \lambda \rangle = \lambda_l$, where e_l is the unique vector whose lth component is 1 and all others are 0.

In the next results, the Poincaré-Dulac normal form system is of the special form

$$\dot{z} = Bz(1 + \rho(z)), \tag{3}$$

where $\rho(z)$ consists of the monomials of the form z^k satisfying $\langle k, \lambda \rangle = 0$. We also call the monomials in $\rho(z)$ resonant ones. In the two dimensional case with complex eigenvalues, if the eigenvalues are resonant, they must be conjugate pure imaginary ones. Then each resonant monomial in $\rho(z)$ must be of the form $z^k = z_1^{k_1} z_2^{k_2} = (y_1^2 + y_2^2)^{k_1}$, which is real. So when we write system (3) again in real coordinates $y = (y_1, y_2)$, one gets

$$\dot{y} = Ay(1 + \rho(y))$$

with $\rho(y)$ consisting of resonant monomials, which are powers of $y_1^2 + y_2^2$.

Now we can state the next result.

Theorem 3. *Assume that system* (2) *is two dimensional and A has a pair of pure imaginary eigenvalues. Then the following statements hold.*

(a) *The origin is a center if and only if the system is analytically equivalent to its Poincaré-Dulac normal form*

$$\dot{y} = Ay(1 + \rho(y)),$$

where $\rho(y)$ is an analytic function in $y_1^2 + y_2^2$. That is, $\rho(y)$ consists of resonant monomials.

(b) *The origin is a center if and only if the system has an analytic first integral of the form $y_1^2 + y_2^2$+h.o.t.*

Statement (b) is useful in studying the center-focus problem. For a two dimensional polynomial differential system of form (2) with the origin having a pair of pure imaginary eigenvalues, one tries to find its Lyapunov quantities of the system at the origin. According to the Hilbert's basis theorem, among the Lyapunov quantities there are only finitely many ones being independent, all the others are functions of these finite ones. If these finite number of Lyapunov quantities vanish, then all Lyapunov quantities vanish, and so the origin is a center. But the Hilbert's basis theorem does not provide a technique to compute this number. And using mathematical softs one can compute only a small number of Lyapunov quantities. Setting these Lyapunov quantities to be zero provides some conditions on the coefficients of the system. Under these coefficient conditions, if we can find an analytic first integral defined in a neighborhood of the origin, then the origin must be a center.

This classical result by Poincaré was extended to higher dimensional analytic differential systems. We summarize the results on this kind of generalization. For analytically integrable Hamiltonian systems in the Liouvillian sense, Ito [9,10] proved the convergence of a symplectic normalization which sends the Hamiltonian system to its Birkhoff normal form under a so-called strong one resonant condition on the eigenvalues of the linearized system at the equilibrium. Zung [36] in 2005 proved in general that any analytically integrable differential system in Liouvillian sense is analytically equivalent to its Birkhooff normal form via torus action. Ito [11] extended further these results to supperintegrable Hamiltonian systems. On the normal form theory, we refer the readers to Bibikov [1], Chow et al [5], Li [12] and Zung [36, 37] for more information on the general definitions and results.

Beside integrable Hamiltonian systems, Zung [37] in 2002 characterized the convergence of the normalization of analytically integrable differential system to its Poincaré-Dulac normal form via torus action. He did not present the concrete representations of the normal form systems. In 2008, Zhang [28] proved the existence of analytic normalization of an analytically integrable differential system to its Poincaré-Dulac normal form under the assumptions that the origin is nondegenerate, and that the matrix A is diagonal. In 2013 Zhang [29] further released these restrictions and obtained the next results.

Theorem 4. *Assume that the n-tuple of eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_n)$ is not zero, i.e. A has at least one eigenvalue not equal to zero. Then system* (2) *is analytically integrable at the origin, i.e. it has $n-1$ functionally independent analytic first integrals in a neighborhood of the origin if and only if the resonant set $\mathcal{R}$ has $n-1$ $\mathbb{Q}_+$ linearly independent elements and system* (2) *is analytically equivalent to its distinguished normal form*

$$\dot{y} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) y (1 + g(y))$$

by a near identity analytic normalization, where $g(y)$ has no a constant term and is an analytic function in all resonant monomials y^m with $m \in \mathcal{R}$ being $\mathbb{Q}_+$ linearly independent elements.

Hereafter $\mathbb{Q}_+$ and $\mathbb{Z}_+$ are respectively the sets of nonnegative rational numbers and of nonnegative integers.

The proof of sufficiency is very easy, but the proof of necessity is relatively complicated. The proofs are separated in several steps. The first step is to get the Poincaré-Dulac normal form, the second step is to prove that the analytic or formal first integrals of the Poincaré-Dulac normal form system consist of resonant monomials. By analytic integrability of the original system, one gets in the third step the special form of the normal form system as mentioned in the theorem, and that the eigenvalues of A does not satisfy the small divisor condition. The fourth step is to present the concrete expressions of the coefficients of the normalization and $g(y)$ in terms of the coefficients of $f(y)$ in (2), and use them to prove the convergence of the normalization and $g(y)$ by the majorant series and the implicit function theorem.

We remark that Zhang [29] in 2013 also established a version of Theorem 4 for analytically integrable diffeomorphims, where the normal form system has a much involved structure. Some related results to Theorem 2 can also be founded in [15] by Llibre, Pantazi and Walcher. In [16], Llibre et al studied also the effect of local Darboux integrability on existence of analytic normalizations.

On the relation between analytic normalization and analytic integrability of analytic differential systems near an equilibrium, Wu and Zhang [24] extended these results near an equilibrium to a periodic orbit of analytic autonomous differential systems. Du, Romanovski and Zhang [7] further developed the above results to partly integrable analytic differential systems.

The known results on characterization of local integrability near an equilibrium were obtained by using the resonance of the eigenvalues of the linearized matrix of the analytic differential system at the equilibrium. It is possible to obtain some necessary conditions on analytic or meromorphic integrability by using the higher order terms of the systems.

Open problem 1. How to apply the higher order terms of analytic differential systems, beside their linear parts, to obtain more necessary conditions on existence of sufficient number of functionally independent analytic or meromorphic first integrals?

Because any smooth function of first integrals is also a first integral, this causes difficulty in finding first integrals of a given analytic differential systems. Du, Romanovski and Zhang [7] in 2016 provided the next result on the structure of first integrals, which is very useful in finding the first integrals of a given analytic differential system.

Theorem 5. *Let $\mathcal{Y}$ be the analytic vector field associated to system* (2).

(*a*) *There exists a series $\Psi(y)$ such that*

$$\mathcal{Y}(\Psi)(y) = \sum_{m\in\mathcal{R}} v_m y^m$$

where the sum takes over all possible resonant elements in $\mathcal{R}$. v_m's are polynomials in the coefficients of $\mathcal{Y}$, and are called integrable varieties.

(*b*) *If the vector field $\mathcal{Y}$ has ℓ functionally independent analytic or formal first integrals, then $\mathcal{Y}$ has ℓ functionally independent first integrals of the form*

$$H_1(y) = \alpha_1 y^{m_1} + h_1(y), \ \ldots, \ H_\ell(y) = \alpha_\ell y^{m_\ell} + h_\ell(y),$$

where $\alpha_1, \ldots, \alpha_\ell \in \mathbb{R}$, $m_1, \ldots, m_\ell \in \mathbb{Z}_+^n$ are $\mathbb{Q}_+$-linearly independent elements of $\mathcal{R}$, and each h_j is composed of nonresonant monomials in y of degree larger than m_j.

We remark that in statement (*a*), the series $\Psi(y)$ could consist of both resonant monomials and nonresonant monomials, but its resonant monomials could be arbitrarily chosen. Here a resonant monomial y^m is the one with $m \in \mathcal{R}$.

According to this result, we want to know whether there is a corresponding version to Darboux polynomials of a polynomial differential system. If yes, it will bring great simplification on the searching of Darboux polynomials, because any product of Darboux polynomials is also a Darboux polynomial. Recall that for a polynomial vector field $\mathcal{P}(y)$, a polynomial $f(y)$ is a Darboux polynomial of $\mathcal{P}$ if there exists a polynomial $k(y)$ such that

$$\mathcal{P}(f)(y) = k(y)f(y),$$

with $k(y)$ a cofactor of $f(y)$.

Open problem 2. For polynomial differential systems, is there a result similar to Theorem $5(b)$ on Darboux polynomials?

Recall that Darboux polynomials played an important role in characterizing local and global dynamics of polynomial differential systems. For instance, if a polynomial differential system has a sufficient number of Darboux polynomials, say $f_1, \ldots, f_p$, with the corresponding cofactors $k_1, \ldots, k_p$ satisfying

$$c_1 k_1(y) + \ldots + c_p k_p(y) = 0$$

with $c_1, \ldots, c_p \in \mathbb{C}$, then the polynomial differential system has a Darboux first integral

$$H(y) = f_1^{c_1}(y) \cdot \ldots \cdot f_p^{c_p}(y).$$

As we know [31], if a quadratic differential system has an equilibrium as a center, then the system has a Darboux first integral defined in a neighborhood of the equilibrium.

The first two sections provide some known results on necessary conditions for existence of functionally independent analytic or formal or meromorphic first integrals of analytic differential systems, or on the equivalent characterization for existence of analytic normalization via analytic integrability of analytic differential systems. Next we recall some results on sufficient conditions for existence of a given type of first integrals of analytic differential systems near an equilibrium.

3 Sufficient condition on existence of local first integrals

In this direction, there are lots of results, especially on center-focus problem for concrete planar polynomial differential systems. But there are seldom systematic results on general analytic differential systems. As we mentioned previously, in order that system (2) have an analytic or a formal first integral around the origin, the eigenvalues of A must be resonant. The simplest resonances are the cases that there are two eigenvalues, say λ_1, λ_2, satisfying $\lambda_1/\lambda_2 = -1$ and the others are nonresonant, and that one of the eigenvalues is zero, and the others are nonresonant. The former is on the center-focus problem. As we know, there are lots of published papers related to the center-focus problem. And also there have appeared many published books summarizing these results. See e.g. Liu, Li and Huang [14], Romanovski and Shafer [18], Ye [26, 27] and Zhang et al [33]. But as we knew, the center-focus problem is far from being solved, even for cubic differential systems.

Here we recall the results on the latter, which has one zero eigenvalue and the others are nonresonant. This work was initiated from 2003 by Li, Llibre and Zhang

[13]. As before, let $\lambda = (\lambda_1, \mu)$, with $\mu = (\lambda_2, \ldots, \lambda_n)$, be the n-tuple of eigenvalues of A. Set

$$\mathcal{R}_0 := \left\{ m \in \mathbb{Z}_+^{n-1} \mid \lambda_1 = 0, \langle \mu, m \rangle = 0, \ |m| \geq 2 \right\}.$$

The main results in [13] are the following.

Theorem 6. *Assume that a system of form* (2) *is analytic and that* $\mathcal{R}_0$ *is empty, i.e.* μ *does not satisfy any* $\mathbb{Q}_+$*-resonant condition. The following statements hold.*

- (a) *When* $n = 1, 2$*, system* (2) *has an analytic first integral in a neighborhood of the origin if and only if the equilibrium* $y = 0$ *is not isolated. That is, system* (2) *has a curve passing the origin, which is full of equilibria.*
- (b) *When* $n > 2$*, system* (2) *has a formal first integral around the origin if and only if the equilibrium* $y = 0$ *is not isolated.*

In one-dimensional case, there exists a unique eigenvalue, which is zero. So the existence of analytic first integral implies that the system is trivial.

In two-dimensional case, the existence of analytic first integral forces by Theorem 5(b) the existence of analytic first integral of the form

$$H(y) = y_1 + h_1(y)$$

with h_1 consisting of higher order terms. Then after the near identity transformation $z = (z_1, z_2) = \beta(y) := (y_1 + h_1(y), y_2)$, system (2) is equivalently changed to

$$\dot{z}_1 = 0, \quad \dot{z}_2 = \dot{y}_2 = \lambda_2 z_2 + f_2(\beta^{-1} z)$$

Clearly, this last system has the analytic curve $\lambda_2 z_2 + f_2(\beta^{-1} z) = 0$ being full of equilibria, and consequently system (2) has a curve filled up with equilibria. Conversely, $\lambda_2 y_2 + f_2(y) = 0$ has an analytic solution, saying $y_2 = \zeta_2(y_1)$, such that $f_1(y_1, \zeta_2(y_1) \equiv 0$ in a neighborhood of the origin. Then the original system can be written in

$$\dot{y}_1 = (y_2 - \zeta_2(y_1)) g_1(y), \quad \dot{y}_2 = \lambda_2 (y_2 - \zeta_2(y_1)) g_2(y).$$

Comparing with the original system gives that $g_2(y) = 1+$h.o.t. and this last system has the same first integral as that of

$$\dot{y}_1 = g_1(y), \quad \dot{y}_2 = \lambda_2 g_2(y).$$

Its origin is regular, and so it has an analytic first integral near the origin. Hence the original system has an analytic first integral in a neighborhood of the origin.

For higher dimensional system, the proof of (b) follows from the Poincaré-Dulac normal form via a sufficiently higher order cut of the formal transformation, and the order of an isolated equilibrium is independent of the choice of a near identity analytic transformation.

After this result was published in 2003, a long time has passed in before we could determine whether the formal first integral in statement (b) of Theorem 6(b) could be replaced by some first integrals with suitable regularity. Zhang [30] in 2017 answered this problem under suitable conditions on the nonresonant eigenvalues of A.

Theorem 7. *Assume that the elements of μ either all have positive real parts or all have negative real parts. Then system* (2) *has an analytic first integral at the origin if and only if the equilibrium $y = 0$ is not isolated.*

The proof of the necessity follows from statement (b) of Theorem 6, or can be proved independently using the result in Theorem 5(b). The sufficiency could be proved using the existence of an analytic invariant manifold along the curve filled up with the equilibria and the normal form system.

Theorem 7 has a C^∞ version, see Theorem 2 of Zhang [30]. But when the eigenvalues μ have both positive real parts and negative real part, as shown in Theorem 3(b_2) of [30], there exist systems of form (2) which have no analytic first integrals defined in a neighborhood of the origin. Then one has to ask: under this last condition, does system (2) have C^∞ first integrals in a neighborhood of the origin. Zhang [32] had worked on this problem.

Recently, this kind of study has been developed to Gevrey systems of form (2) with μ nonresonance, see [25]. For more information on Gevrey smoothness, see e.g. Stolovitch [22] and Wu et al [23]. As usual, the classes C^∞ or $\mathscr{G}_s$ for $s \geq 1$ denote the sets of functions C^∞ or Gevrey-s smooth. Particularly $\mathscr{G}_1$ is the analytic functional class, and $\mathscr{G}_1 \subseteq \mathscr{G}_s(s \geq 1) \subseteq C^\infty$ and $C^\infty \subseteq \mathbb{F}^n[[x]]$ the set of n dimensional formal series. Now denote the resonant set by

$$\Lambda_r = \left\{(j,k,l) \mid \langle k,\mu\rangle = \mu_l, \quad j+|k| \geq 2, \quad k \in \mathbb{Z}_+^{n-1}, \quad j \in \mathbb{Z}_+, \quad l \in \{2,\ldots,n\}\right\}.$$

Under the condition that the equilibrium $y = 0$ is not isolated, system (2) can be written in

$$\frac{dy_1}{dx} = \hat{f}_1(y_1,z), \quad \frac{dz}{dx} = A_0 z + \hat{f}_2(y_1,z), \tag{4}$$

with $z = (y_2,\ldots,y_n)$, A_0 having the eigenvalues μ, and $\hat{f}_1(y_1,0) \equiv 0$ and $\hat{f}_2(y_1,0) \equiv 0$. Moreover, its Poincaré-Dulac formal normal form system is

$$\frac{dy_1}{dx} = 0, \quad \frac{dz}{dx} = A_0 z + g(y_1,z), \tag{5}$$

where $g(y_1,z) = \sum_{j,k,l\in\Lambda_r} g_{(j,k),l} y_1^j z^k e_l$ with e_l the l-th unit vector. Set

$$q = \min\{|k| \mid (j,k,l) \in \Lambda_r, \quad g_{(j,k),l} \neq 0, \quad \exists j,l\}, \tag{6}$$

and

$$q^* = \min\{j+|k| \mid (j,k,l) \in \Lambda_r, \quad g_{(j,k),l} \neq 0, \quad \exists l\}. \tag{7}$$

Formulating a function Φ as

$$c^{-1}\Phi(t) = \max\{|k\cdot\lambda|^{-1} \mid |k| \leq t, \ k \in \mathbb{Z}_+^d\}, \tag{8}$$

with c a normalized parameter such that $\Phi(1) = 1$. Of course, when $\Phi(t) = t^\mu$, it is of the diophantine type. Next results from [25] characterize the (formal) Gevrey integrability via the diophantine type small divisor condition.

Theorem 8. *Assume that system* (2) *is of Gevrey-s smooth with the origin as a nonisolated equilibrium, and* μ *nonresonant. Then the following statements hold.*

(a) *If* A_0 *has its eigenvalues all with positive real parts or all with negative real parts, then system* (2) *has a local Gevrey-s smooth first integral whose formal series is not trivial.*

(b) *Assume that* A *is diagonal , and* $\Phi(t) = t^{\mu}$ *with* $\mu > 0$ *a constant.*

(b_1) *If* $\partial_z \hat{f}_2(y_1, 0) \equiv 0$ *in* (4) *and* $q < \infty$ *defined in* (6), *there exists a local Gevrey-*s^* *smooth first integrals with nontrivial formal series, where* $s^* = \max\{s, (\mu + q)/(q - 1)\}$.

(b_2) *If* $q^* < \infty$ *defined in* (7), *there exists a nontrivial formal Gevrey-*s^* *first integral, where* $s^* = \max\{s - 1, (\mu + 1)/(q^* - 1)\}$.

The proof will be done by using the homological equation, KAM theory, the Gevrey norm of functions and majorant Gevrey series together with lots of estimations.

After Theorem 8 we naturally have the next questions.

Open problem 3. Among analytic differential systems of form (1) satisfying that $\mathcal{R}_0$ is empty, and the origin is a nonisolated equilibrium and that μ have both elements with positive and negative real parts,

(a) what is the subset in which all systems have analytic first integrals in a neighborhood of the origin?

(b) what is the subset in which all systems have Gevrey first integrals in a neighborhood of the origin?

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The bifurcation diagram of the configurations of invariant lines of total multiplicity exactly three in quadratic vector fields

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Abstract. We denote by $\boldsymbol{QSL}_3$ the family of quadratic differential systems possessing invariant straight lines, finite and infinite, of total multiplicity exactly three. In a sequence of papers the complete study of quadratic systems with invariant lines of total multiplicity at least four was achieved. In addition three more families of quadratic systems possessing invariant lines of total multiplicity at least three were also studied, among them the Lotka-Volterra family. However there were still systems in $\boldsymbol{QSL}_3$ missing from all these studies. The goals of this article are: to complete the study of the geometric configurations of invariant lines of $\boldsymbol{QSL}_3$ by studying all the remaining cases and to give the full classification of this family modulo their configurations of invariant lines together with their bifurcation diagram. The family $\boldsymbol{QSL}_3$ has a total of 81 distinct configurations of invariant lines. This classification is done in affine invariant terms and we also present the bifurcation diagram of these configurations in the 12-parameter space of coefficients of the systems. This diagram provides an algorithm for deciding for any given system whether it belongs to $\boldsymbol{QSL}_3$ and in case it does, by producing its configuration of invariant straight lines.

Mathematics subject classification: 34C23, 34A34.
Keywords and phrases: quadratic differential system, invariant line, singularity, configuration of invariant lines, group action, polynomial invariant.

1 Introduction and the statement of the Main Theorem

We consider here real planar differential systems of the form

$$(S) \qquad \frac{dx}{dt} = p(x,y), \quad \frac{dy}{dt} = q(x,y), \tag{1}$$

where $p, q \in \mathbb{R}[x, y]$, i.e. p, q are polynomials in x, y over $\mathbb{R}$. We call *degree* of a system (S) the integer $\deg(S) = \max(\deg(P), \deg(Q))$. We call quadratic (respectively cubic) differential system such a polynomial system of degree two (respectively three). We shall sometimes use quadratic system instead of quadratic differential system. Each such system generates a complex differential vector field when the dependent variables range over $\mathbb{C}$.

Of the three classical problems on these systems, Hilbert's 16th problem, the problem of Poincaré and the problem of the center, only the last one was solved for the family **QS** of quadratic differential systems. Although it is the simplest

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non-linear class of polynomial systems we are still far from understanding this family. To gain insight into this family, in recent years subfamilies of **QS** began to be studied from a global viewpoint using a variety of methods, among them algebraic and geometric or analytical, also numerical or involving substantial symbolic calculations. In particular families of quadratic systems possessing invariant algebraic curves began to be studied, the simplest ones being those possessing invariant lines.

Every system in **QS** possesses an invariant line, the line at infinity. This line could be simple or multiple, in which case producing several distinct lines in perturbations.

The notion of multiplicity of an invariant line of a system (1) has been introduced in [9]. This concept was extended to the notion of multiplicity of an invariant algebraic curve of a differential system. In the fundamental article [6] several notions of multiplicity of an invariant algebraic curve of a polynomial system were introduced and they were proven to be equivalent in the case of *algebraic solutions* which are algebraic invariant curves defined by polynomials that are irreducible over $\mathbb{C}$. If a system has a finite number of invariant lines $f_i(x, y) = 0$, $i = 1, ..., k$, of respective multiplicities $m_1, ..., m_k$, we call *total multiplicity of the invariant lines* of (S), the number $M = \sum_i m_i + m_\infty$ where m_∞ is the multiplicity of the line at infinity. Since in any system (1) the line at infinity is invariant we always have $m_\infty \geq 1$ and in particular we have this for any system in **QS**.

At the beginning of this century a systematic study of non-degenerate quadratic systems possessing invariant algebraic curves was initiated by Schlomiuk and Vulpe. In the series of articles [9,11,13,14] the authors studied the class $\mathbf{QSL}_{\geq 4}$ of quadratic systems having invariant lines, including the line at infinity, of total multiplicity at least four. We see in [9] that the maximum number of invariant lines, including the line at infinity, of non-degenerate quadratic systems is six.

This study was based on the notion of *configuration of invariant lines* of a real polynomial differential system defined in [14]. We recall here this definition.

Definition 1. Consider a real polynomial differential system (S) endowed with a finite number of invariant algebraic curves $f_i(x, y) = 0$, $i = 1, \ldots, k$, over $\mathbb{C}$. We call *configuration of invariant curves of* (S) the set of curves $f_1 = 0, \ldots, f_k = 0$ and the line at infinity, each endowed with its own multiplicity, together with all the real singular points of (S) situated on these curves, each one of them endowed with its own multiplicity.

The notion of configuration is an affine invariant which is a powerful classification tool. This was clearly seen in the way the topological classification was obtained for the Lotka-Volterra systems which have a total of 112 phase portraits. The geometry of configurations acts like a guiding light to fray our way through this maze of phase portraits. Thus we first obtained the geometric classification by splitting the class according to their 65 distinct configurations of invariant lines that the systems possess. Then we classified topologically each one of these 65 families.

In order to classify all the configurations of the family $\mathbf{QSL}_3$ we first need to say when two configurations $\mathcal{C}_1$, $\mathcal{C}_2$ of invariant lines of two quadratic systems (S_1) and

(S_2) are to be considered as distinct, respectively when two such configurations are to be considered equivalent.

Consider two polynomial differential systems (S_1) and (S_2) such that each has a finite set of singular points and a finite set of invariant lines, including the line at infinity. Let $\mathcal{C}_1$, $\mathcal{C}_2$ be the two configurations of invariant lines of (S_1) and (S_2).

Definition 2. We say that *two configurations $\mathcal{C}_1$, $\mathcal{C}_2$, of (S_1) and (S_2) formed by invariant lines (including the line at infinity) are equivalent* if and only if there is a bijection ϕ between the two sets of invariant lines sending the line at infinity of $\mathcal{C}_1$ to the line at infinity of $\mathcal{C}_2$, sending a line with coefficients in $\mathbb{R}$ of (S_1) to a line with coefficients in $\mathbb{R}$ of (S_2). In addition the map preserves the multiplicities of the invariant lines, and for each invariant line L of $\mathcal{C}_1$ there is a one-to-one correspondence ϕ_L between the set of real singular points of (S_1) situated on the line L and the set of real singular points of the system (S_2) situated on the line $\phi(L)$ which preserves the multiplicities of the singular points and sends a real singular point at infinity to a real singular point at infinity. In addition we have the following:

(i) When we list in a counterclockwise sense the real singular points at infinity on (S_1) starting from a point p on the Poincaré disk, $p_1 = p, ..., p_l$, this correspondence preserves the multiplicities of the singular points and preserves or reverses the orientation.

(ii) We consider the total curves

$$\mathcal{F} : \prod F_j(X;Y;Z)^{m_i} Z^m = 0; \quad \mathcal{F}' : \prod F_j'(X;Y;Z)^{m_i'} Z^{m'} = 0$$

where $F_i(X;Y;Z) = 0$ (respectively $F_i'(X;Y;Z) = 0$) are the projective completions of the lines $\mathcal{L}_i$ (respectively $\mathcal{L}_i'$) and m_i, m_i' are the multiplicities of the curves $F_i = 0$, $F_i' = 0$ and m, m' are respectively the multiplicities of $Z = 0$ in the first and in the second system. Then, there is a one-to-one correspondence between the real singularities of the curves $\mathcal{F}$ and $\mathcal{F}'$ conserving their multiplicities as singular points of the total curves.

After the study of the family $\mathbf{QSL}_{\geq 4}$ mentioned above, the next step is the study of the subfamily $\mathbf{QSL}_3$ of $\mathbf{QS}$ which is the family of all non-degenerate quadratic differential systems with invariant lines of total multiplicity three. The study of this class began with work on the Lotka-Volterra systems (shortly L-V systems), a family important for applications. (Previous literature on L-V systems systems is also mentioned in [16,17].) This is the class of all quadratic differential systems that have two real invariant lines intersecting at a finite point. In [16,17] the authors completed the study of this class by giving its bifurcation diagram in the 12-dimensional space of the coefficients of quadratic systems (1).

The family $\mathbf{QSL}_3$ splits into several subfamilies of $\mathbf{QS}$ according to the geometry of the systems, one of them being the L-V systems. Another subfamily of $\mathbf{QSL}_3$ is the family of non-degenerate real quadratic systems possessing two complex invariant lines intersecting at a (real) finite point. The topological classification for this family was done in [19] but without using the configurations of invariant lines.

The bifurcation diagram in terms of invariant polynomials was done in [3]. But the configurations of invariant lines for systems in this family and occurring in $\mathbf{QSL}_3$ are presented here for the first time.

In [5] one more subfamily of $\mathbf{QSL}_3$ was studied. More exactly, in [5] the authors made the study of the family $\mathbf{QSL}^{2p}$ of quadratic systems possessing one of the following defining properties: two parallel invariant lines or a unique affine line that is double, or an affine invariant line and the double line at infinity or the triple line at infinity.

However, we still have quadratic systems in $\mathbf{QSL}_3$ that were not mentioned so far. These are quadratic differential systems in $\mathbf{QSL}_3$ that are limit points of the L-V systems.

Indeed such systems could be obtained from a generic L-V system using one of the following three possibilities:

(*i*) Two simple invariant lines of a L-V-system from the subfamily $\mathbf{QSL}_3$ coalesced giving a double invariant line and a multiple real singular point at infinity.

(*ii*) One simple invariant line of a L-V system from the subfamily $\mathbf{QSL}_3$ coalesced with infinite line $Z = 0$ giving a double infinite invariant line with the second invariant line remaining in the finite part of the phase plane.

(*iii*) Both simple invariant lines of a L-V system from the subfamily $\mathbf{QSL}_3$ coalesced with infinite line $Z = 0$ producing a triple line at infinity.

The goal of this paper is to complete the study of the configurations of invariant lines of family $\boldsymbol{QSL}_3$ and to present all possible configurations of invariant lines which a non-degenerate quadratic system from the class $\mathbf{QSL_3}$ could have. Our main results are summed up in the following theorem:

Main Theorem. *The following statements hold:*

- *(i) The family* $\mathbf{QSL}_3$ *possesses a total of 81 distinct configurations of invariant lines given in Figure 1.*
- *(ii) The classification of the family* $\mathbf{QSL}_3$ *is done using algebraic invariants and hence it is independent of the normal forms in which the systems may be presented.*
- *(iii) The "bifurcation" diagram of the configurations of invariant lines for systems in the family* $\mathbf{QSL}_3$ *is done in the twelve-dimensional parameter space* $\mathbb{R}^{12}$ *and it is presented in Diagrams 1 and 2. These diagrams give us an algorithm by determining for any given system if it belongs or not to the family* $\mathbf{QSL}_3$ *and in case it belongs to this family, it gives us the specific configuration of invariant lines.*

2 The main invariant polynomials associated to the class $\boldsymbol{QSL}_3$

We consider the class of real quadratic polynomial differential systems

$$\begin{aligned} \dot{x} &= p_0 + p_1(x,y) + p_2(x,y) \equiv P(\tilde{a},x,y), \\ \dot{y} &= q_0 + q_1(x,y) + q_2(x,y) \equiv Q(\tilde{a},x,y) \end{aligned} \tag{2}$$

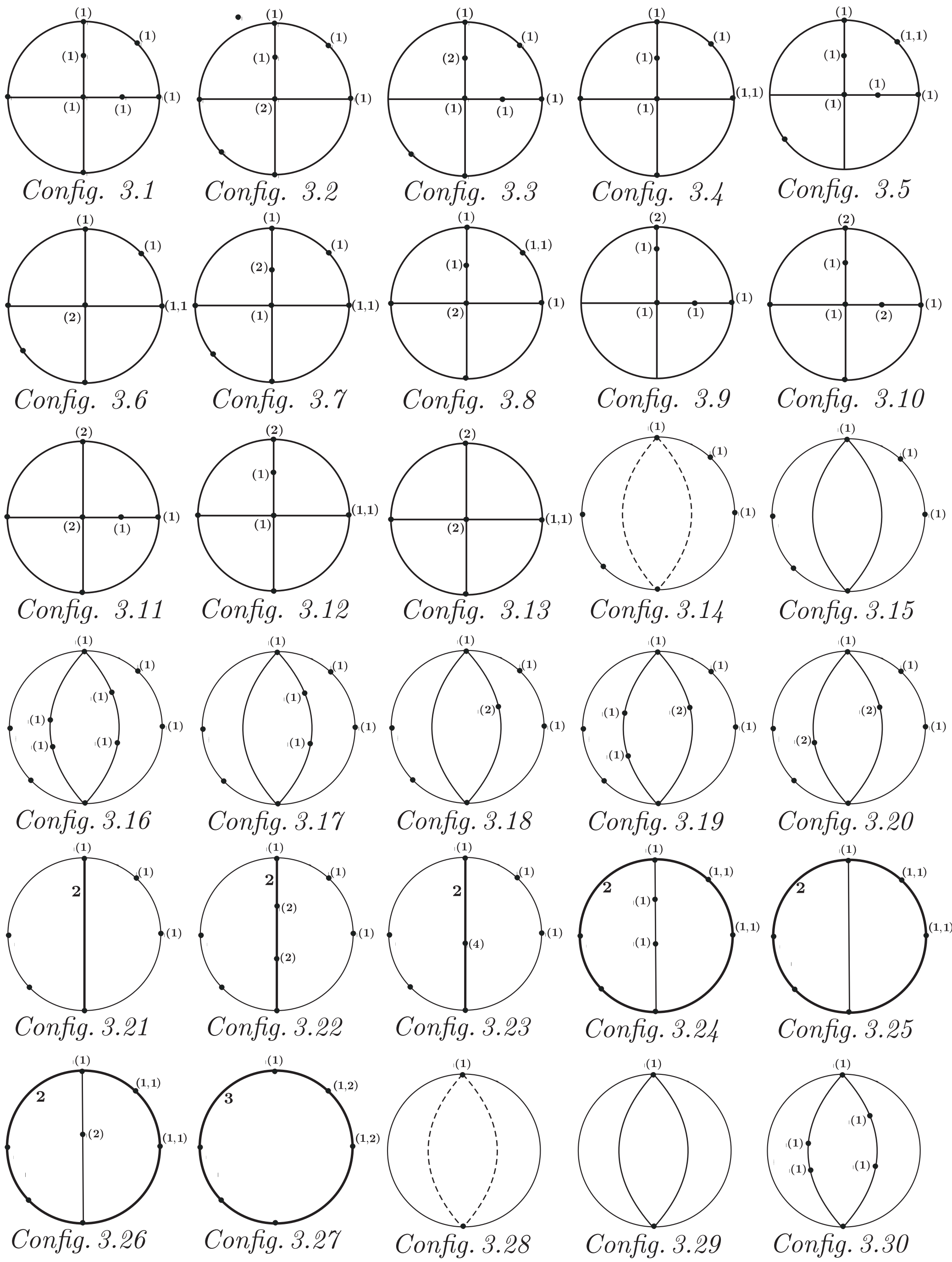

Figure 1. The configurations of quadratic systems in $\mathbf{QSL}_3$

where

$$p_0 = a, \quad p_1(x, y) = cx + dy, \quad p_2(x, y) = gx^2 + 2hxy + ky^2,$$
$$q_0 = b, \quad q_1(x, y) = ex + fy, \quad q_2(x, y) = lx^2 + 2mxy + ny^2$$

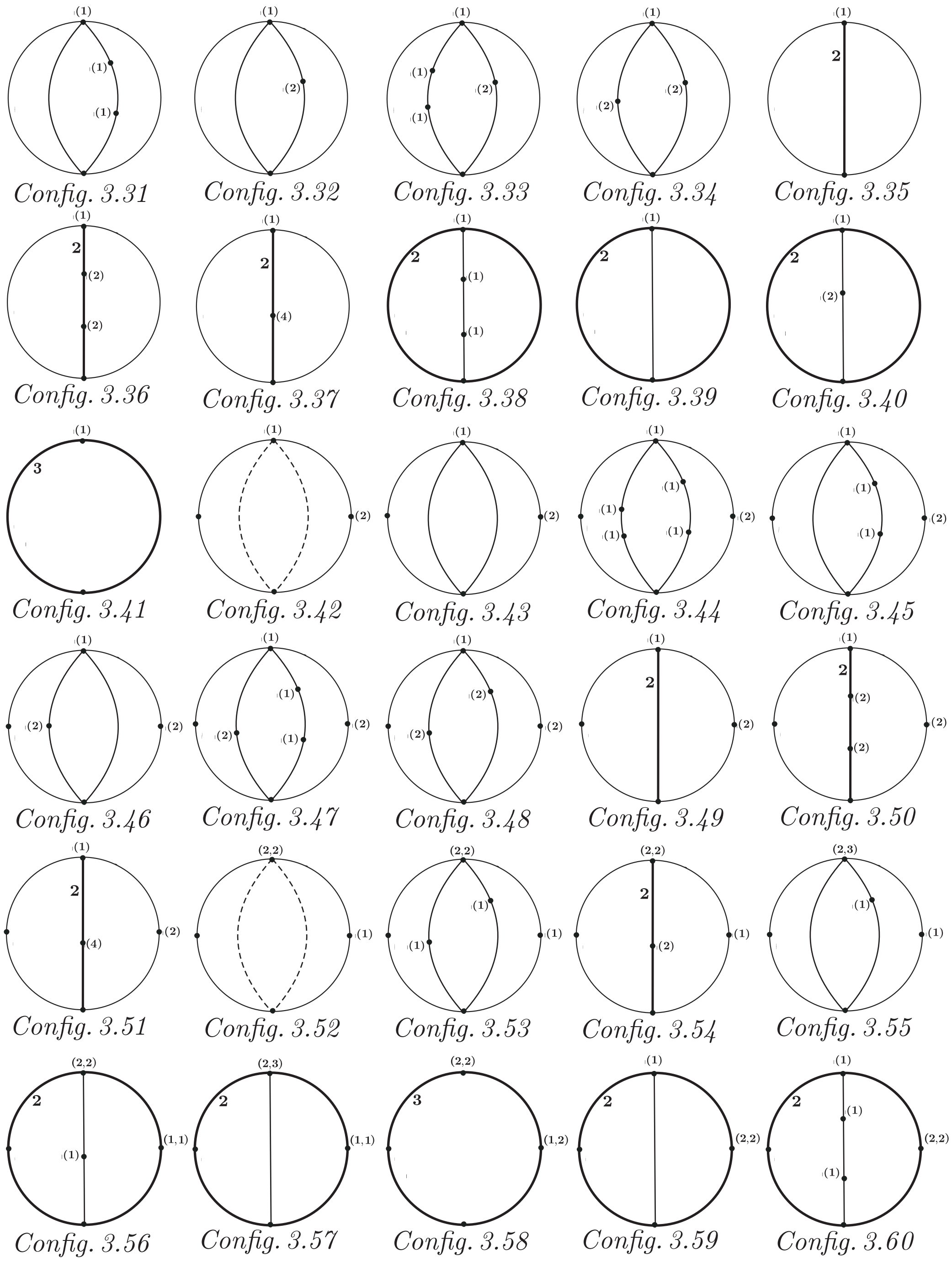

Figure 1 *(continuation).* The configurations of quadratic systems in $\mathbf{QSL}_3$

and with $\max(\deg(p), \deg(q)) = 2$. It is known that on the set $\boldsymbol{QS}$ the group $Aff(2,\mathbb{R})$ of affine transformations on the plane acts (cf. [10]). For every subgroup $G \subseteq Aff(2,\mathbb{R})$ we have an induced action of G on $\boldsymbol{QS}$. We can identify the set $\boldsymbol{QS}$

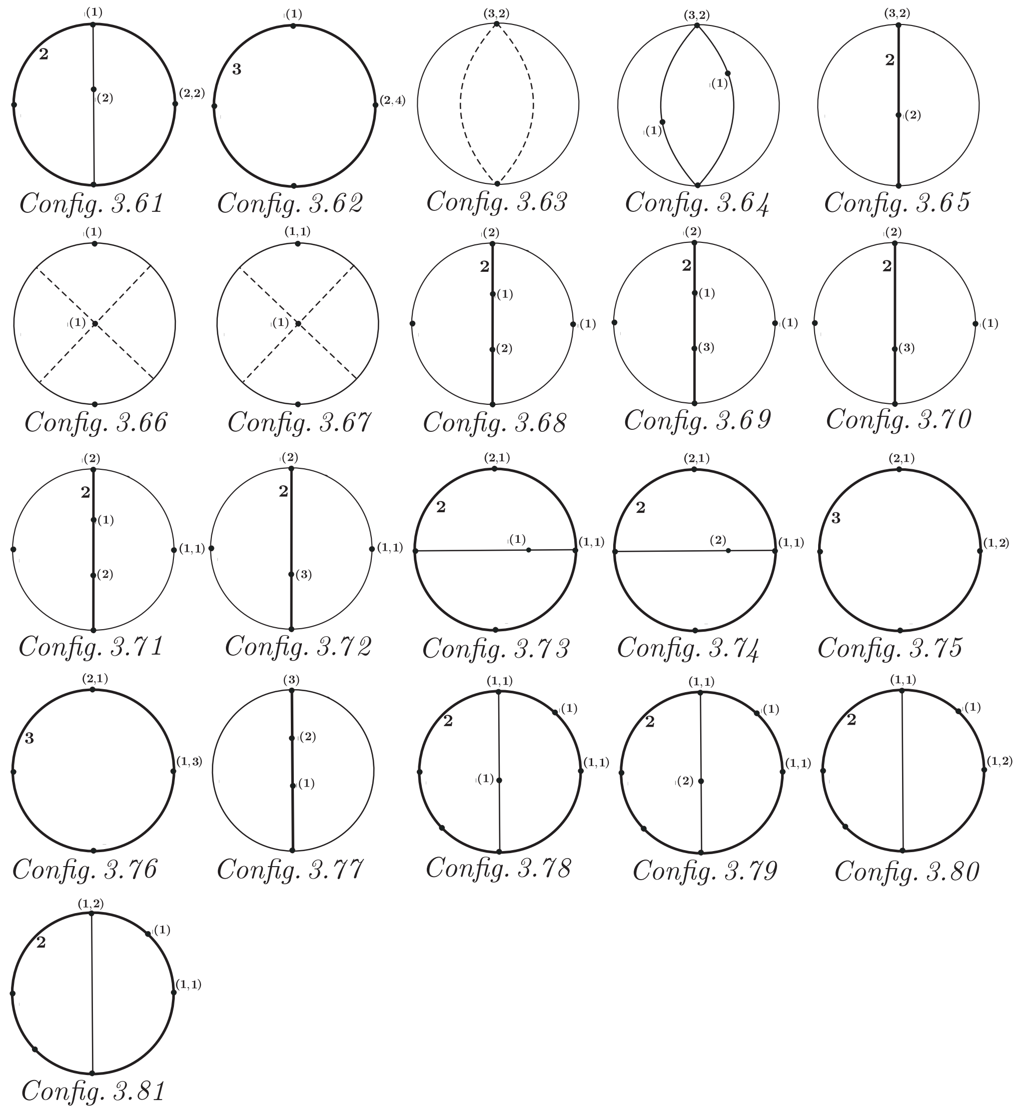

Figure 1 *(continuation)*. The configurations of quadratic systems in $\mathbf{QSL}_3$

of systems (2) with a subset of $\mathbb{R}^{12}$ via the map $\boldsymbol{QS} \longrightarrow \mathbb{R}^{12}$ which associates to each system (2) the 12–tuple $\tilde{a} = (a, c, d, g, h, k, b, e, f, l, m, n)$ of its coefficients. We associate to this group action polynomials in x, y and parameters which behave well with respect to this action, the GL–comitants (GL–invariants), the T–comitants (affine invariants) and the CT–comitants. For their definitions as well as their detailed constructions we refer the reader to the paper [10] (see also [1]).

According to [1] (see also [4]) we apply the differential operator $\mathcal{L} = x \cdot \mathbf{L}_2 - y \cdot \mathbf{L}_1$

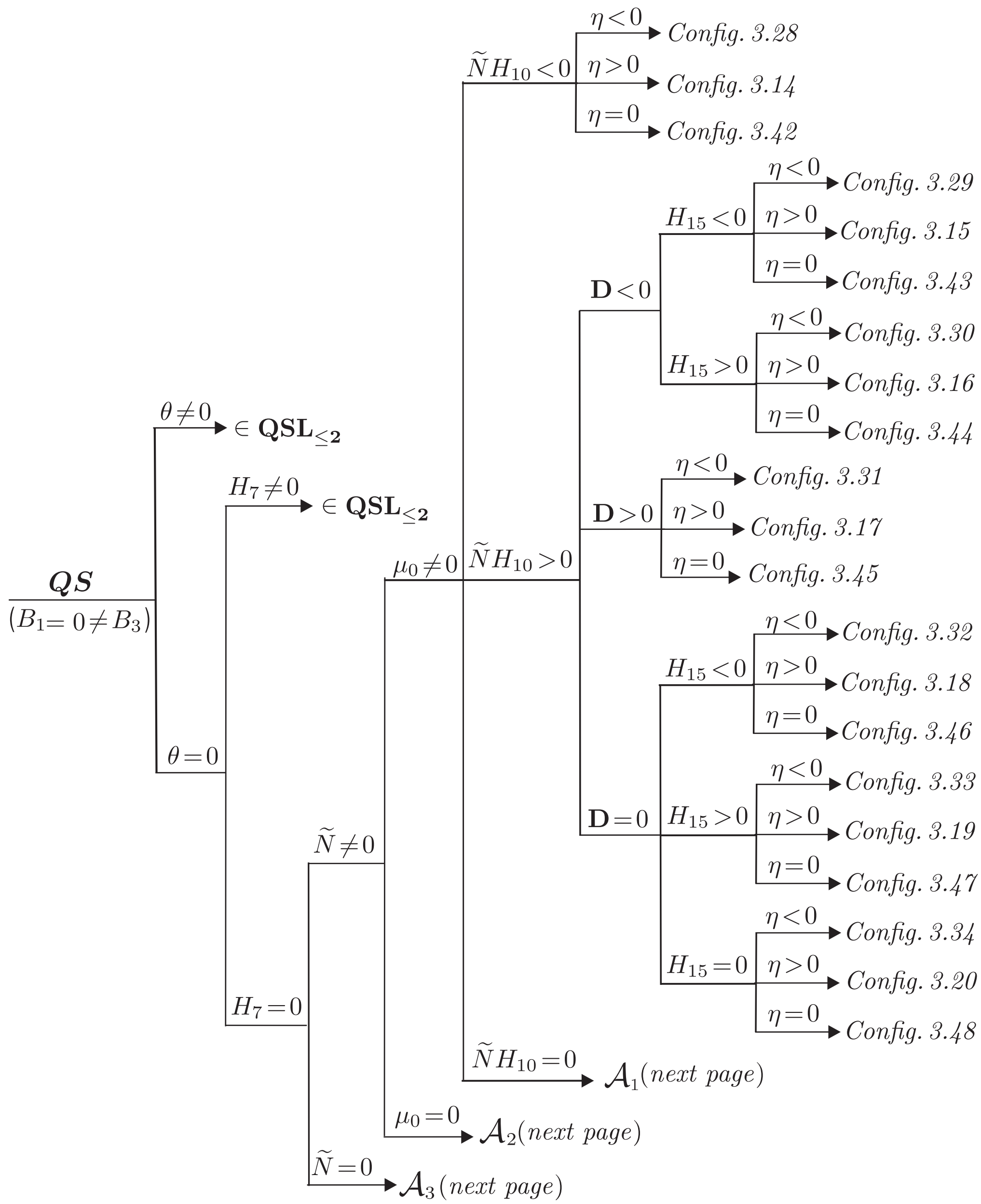

Diagram 1: The configurations of systems in **QSL** with $B_1 = 0$ and $B_2 \neq 0$

acting on $\mathbb{R}[\tilde{a}, x, y]$ with

$$\begin{aligned}\mathbf{L}_1 &= 2a\frac{\partial}{\partial c} + c\frac{\partial}{\partial g} + \frac{1}{2}d\frac{\partial}{\partial h} + 2b\frac{\partial}{\partial e} + e\frac{\partial}{\partial l} + \frac{1}{2}f\frac{\partial}{\partial m},\\ \mathbf{L}_2 &= 2a\frac{\partial}{\partial d} + d\frac{\partial}{\partial k} + \frac{1}{2}c\frac{\partial}{\partial h} + 2b\frac{\partial}{\partial f} + f\frac{\partial}{\partial n} + \frac{1}{2}e\frac{\partial}{\partial m},\end{aligned}$$

to construct several needed invariant polynomials. More precisely using this

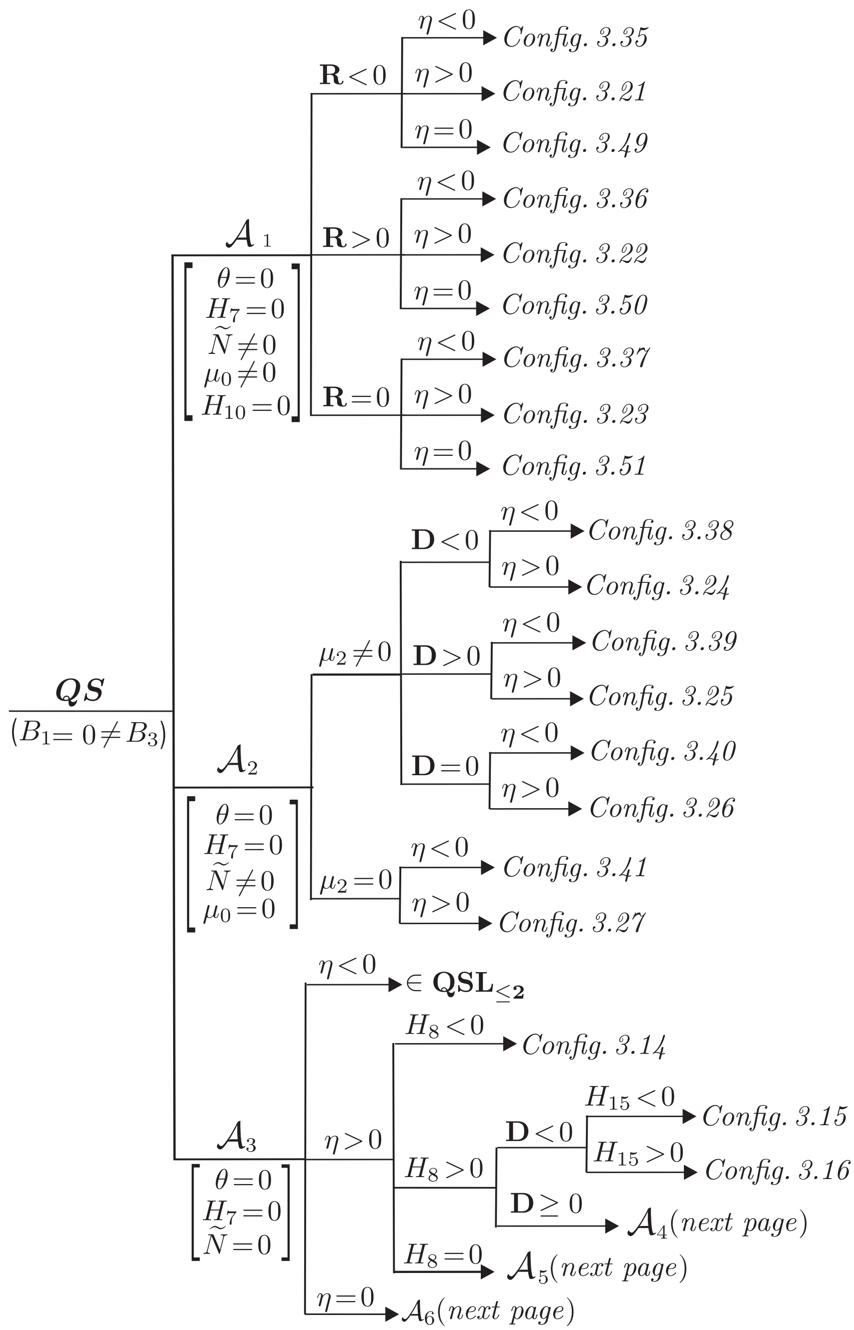

Diagram 1 *(continuation)*: The configurations of systems in **QSL** with $B_1 = 0$ and $B_2 \neq 0$

operator and the affine invariant $\mu_0 = \text{Res}_{\,x}\big(p_2(\tilde{a}, x, y), q_2(\tilde{a}, x, y)\big)/y^4$ we construct the following polynomials

$$\mu_i(\tilde{a}, x, y) = \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0), \;\; i = 1, .., 4, \quad \text{where} \;\; \mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0)).$$

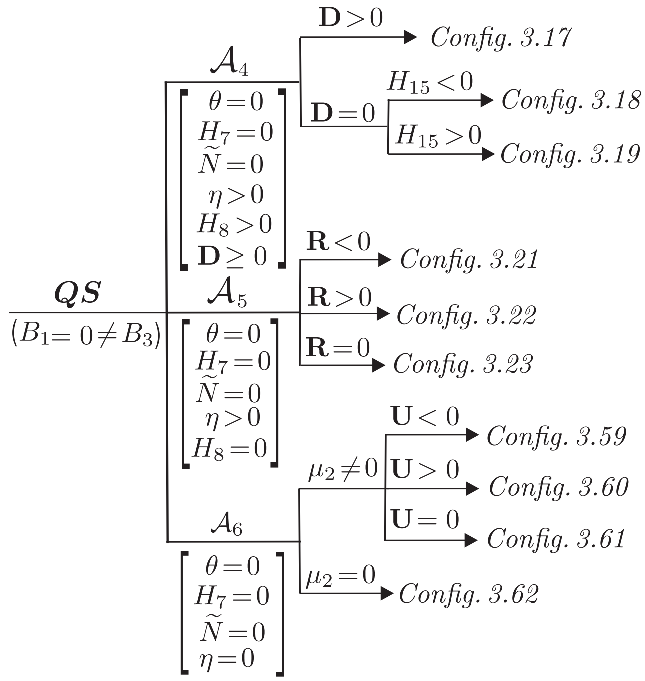

Diagram 1 *(continuation)*: The configurations of systems in **QSL** with $B_1=0$ and $B_2\neq 0$

Using these invariant polynomials we define some new ones, which according to [1] are responsible for the number and multiplicities of the finite singular points of (2):

$$\begin{aligned}
\mathbf{D} &= \Big[3\big((\mu_3,\mu_3)^{(2)},\ \mu_2\big)^{(2)} - \big(6\mu_0\mu_4-3\mu_1\mu_3+\mu_2^2,\ \mu_4\big)^{(4)}\Big]/48,\\
\mathbf{P} &= 12\mu_0\mu_4-3\mu_1\mu_3+\mu_2^2,\\
\mathbf{R} &= 3\mu_1^2-8\mu_0\mu_2,\\
\mathbf{S} &= \mathbf{R}^2-16\mu_0^2\mathbf{P},\\
\mathbf{T} &= 18\mu_0^2(3\mu_3^2-8\mu_2\mu_4)+2\mu_0(2\mu_2^3-9\mu_1\mu_2\mu_3+27\mu_1^2\mu_4)-\mathbf{PR},\\
\mathbf{U} &= \mu_3^2-4\mu_2\mu_4.
\end{aligned}$$

In what follows we also need the so-called *transvectant of order k* (see [7, 8]) of two polynomials $f,\ g\in\mathbb{R}[\tilde{a},x,y]$

$$(f,g)^{(k)} = \sum_{h=0}^{k}(-1)^h\binom{k}{h}\frac{\partial^k f}{\partial x^{k-h}\partial y^h}\ \frac{\partial^k g}{\partial x^h\partial y^{k-h}}.$$

In order to construct the invariant polynomials for the classification of this class of systems we first need to define some elementary bricks which help us to construct these elements of the set.

We remark that the following polynomials in $\mathbb{R}[\tilde{a},x,y]$ are the simplest invariant polynomials of degree one with respect to the coefficients of the differential

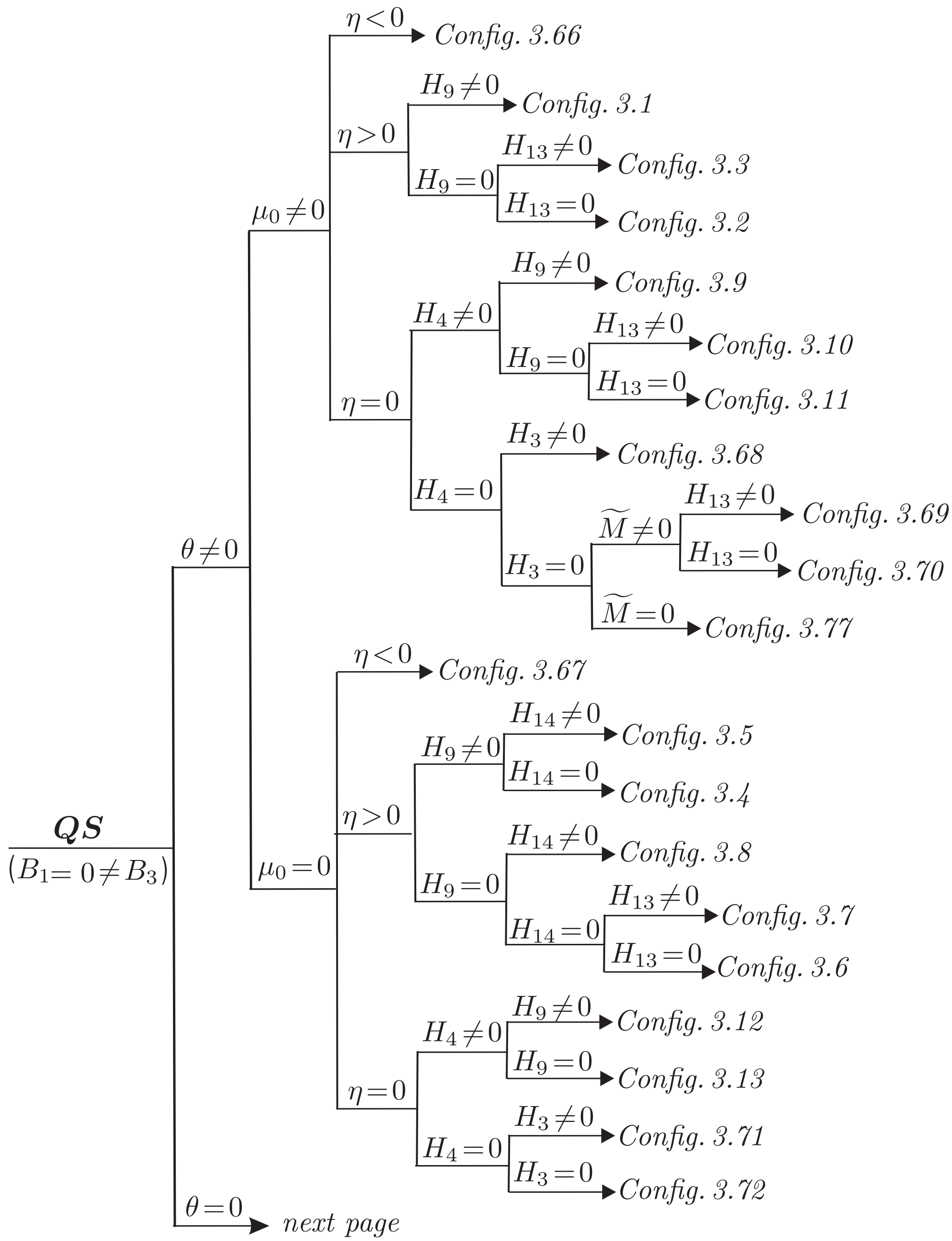

Diagram 2 : The configurations of systems in **QSL** with $B_2 = 0$ and $B_3 \neq 0$

systems (2) which are GL-comitants:

$$C_i(x,y) = yp_i(x,y) - xq_i(x,y), \ i = 0,1,2;$$
$$D_i(x,y) = \frac{\partial}{\partial x}p_i(x,y) + \frac{\partial}{\partial y}q_i(x,y), \ i = 1,2.$$

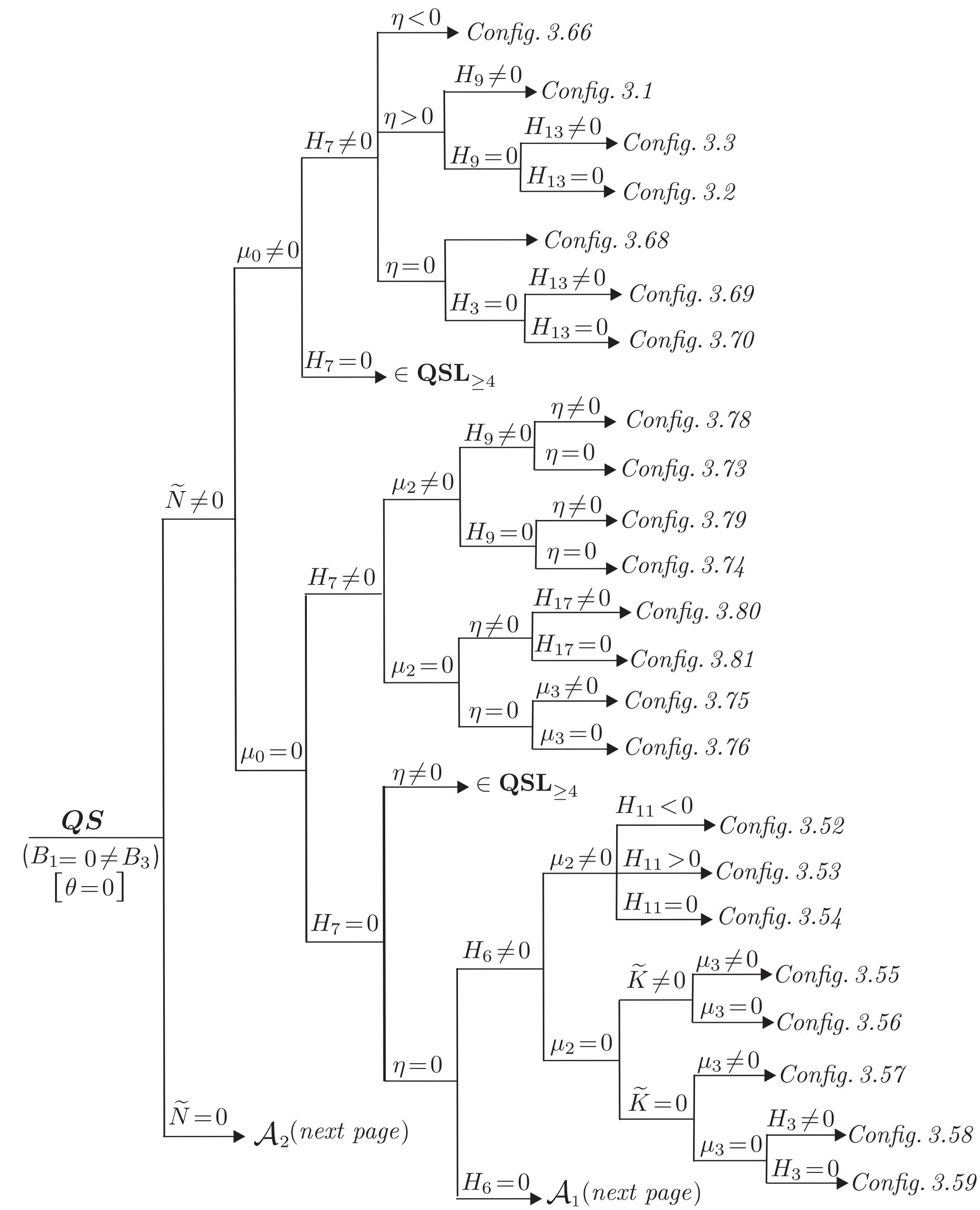

Diagram 2 *(continuation)*: The configurations of systems in **QSL** with $B_2 = 0$ and $B_3 \neq 0$

Apart from these simple invariant polynomials we shall also make use of the following nine GL-invariant polynomials:

$$
\begin{aligned}
&T_1 = (C_0, C_1)^{(1)}, \quad T_2 = (C_0, C_2)^{(1)}, \quad T_3 = (C_0, D_2)^{(1)}, \\
&T_4 = (C_1, C_1)^{(2)}, \quad T_5 = (C_1, C_2)^{(1)}, \quad T_6 = (C_1, C_2)^{(2)}, \\
&T_7 = (C_1, D_2)^{(1)}, \quad T_8 = (C_2, C_2)^{(2)}, \quad T_9 = (C_2, D_2)^{(1)}.
\end{aligned}
$$

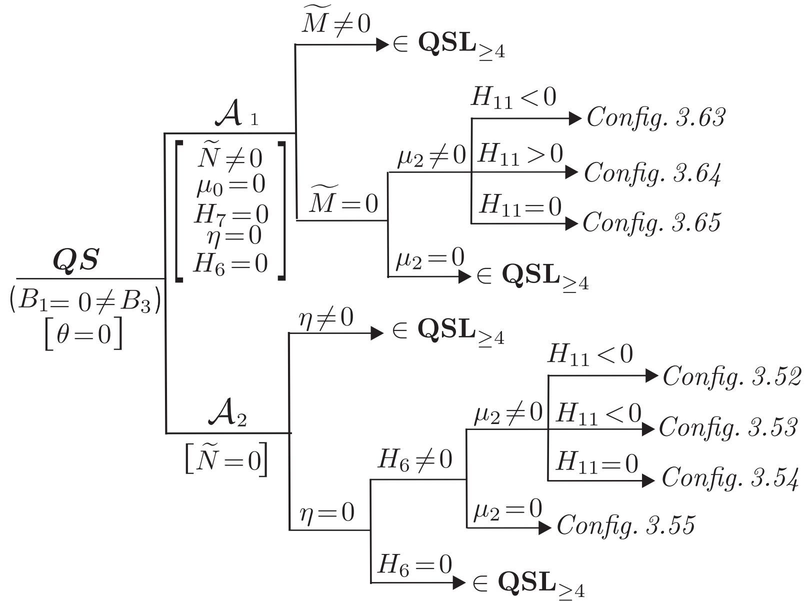

Diagram 2 *(continuation)*: The configurations of systems in **QSL** with $B_2 = 0$ and $B_3 \neq 0$

These are of degree two with respect to the coefficients of systems (2).

We next define a list of T-comitants:

$$
\begin{aligned}
\hat{A}(\tilde{a}) =\ & (C_1, T_8 - 2T_9 + D_2^2)^{(2)}/144,\\
\widehat{B}(\tilde{a},x,y) =\ & \Big\{16D_1(D_2,T_8)^{(1)}(3C_1D_1 - 2C_0D_2 + 4T_2) + 32C_0(D_2,T_9)^{(1)}(3D_1D_2\\
& - 5T_6 + 9T_7) + 2(D_2,T_9)^{(1)}\big(27C_1T_4 - 18C_1D_1^2 - 32D_1T_2 + 32(C_0,T_5)^{(1)}\big)\\
& + 6(D_2,T_7)^{(1)}\big[8C_0(T_8 - 12T_9) - 12C_1(D_1D_2 + T_7) + D_1(26C_2D_1 + 32T_5)\\
& + C_2(9T_4 + 96T_3)\big] + 6(D_2,T_6)^{(1)}\big[32C_0T_9 - C_1(12T_7 + 52D_1D_2)\\
& - 32C_2D_1^2\big] + 48D_2(D_2,T_1)^{(1)}(2D_2^2 - T_8) + 6D_1D_2T_4(T_8 - 7D_2^2 - 42T_9)\\
& - 32D_1T_8(D_2,T_2)^{(1)} + 9D_2^2T_4(T_6 - 2T_7) - 16D_1(C_2,T_8)^{(1)}(D_1^2 + 4T_3)\\
& + 12D_1(C_1,T_8)^{(2)}(C_1D_2 - 2C_2D_1) + 12D_1(C_1,T_8)^{(1)}(T_7 + 2D_1D_2)\\
& + 96D_2^2\big[D_1(C_1,T_6t)^{(1)} + D_2(C_0,T_6)^{(1)}\big] - 4D_1^3D_2(D_2^2 + 3T_8 + 6T_9)\\
& - 16D_1D_2T_3(2D_2^2 + 3T_8) + 6D_1^2D_2^2(7T_6 + 2T_7) - 252D_1D_2T_4T_9\Big\}/(2^83^3),\\
\widehat{E}(\tilde{a},x,y) =\ & \big[D_1(2T_9 - T_8) - 3(C_1,T_9)^{(1)} - D_2(3T_7 + D_1D_2)\big]/72,\\
\widehat{F}(\tilde{a},x,y) =\ & \big[6D_1^2(D_2^2 - 4T_9) + 4D_1D_2(T_6 + 6T_7) + 48C_0(D_2,T_9)^{(1)} - 9D_2^2T_4\\
& + 288D_1\widehat{E} - 24(C_2,\widehat{D})^{(2)} + 120(D_2,\widehat{D})^{(1)} - 36C_1(D_2,T_7)^{(1)}\\
& + 8D_1(D_2,T_5)^{(1)}\big]/144,
\end{aligned}
$$

$$\widehat{K}(\tilde{a},x,y) = (T_8+4T_9+4D_2^2)/72,$$
$$\widehat{H}(\tilde{a},x,y) = (-T_8+8T_9+2D_2^2)/72,$$

as well as the following affine invariants (which serve as bricks for constructing the needed invariant polynomials):

$$A_2(\tilde{a}) = (C_2,\widehat{D})^{(3)}/12, \qquad A_{17}(\tilde{a}) = \big(((\widehat{D},\widehat{D})^{(2)},D_2\big)^{(1)},D_2\big)^{(1)}/64,$$
$$A_{18}(\tilde{a}) = \big((\widehat{D},\widehat{F})^{(2)},D_2\big)^{(1)}/16, \qquad A_{19}(\tilde{a}) = \big((\widehat{D},\widehat{D})^{(2)},\widehat{H}\big)^{(2)}/16,$$
$$A_{20}(\tilde{a}) = \big((C_2,\widehat{D})^{(2)},\widehat{F}\big)^{(2)}/16.$$

Next we present here the list of invariant polynomials which are necessary for the classification of the configurations of invariant lines for the family $\mathbf{QSL}_3$:

$$\begin{aligned}
\widetilde{K}(\tilde{a},x,y) &= 4\widehat{K} \equiv \text{Jacob}\,\big(p_2(\tilde{a},x,y),q_2(\tilde{a},x,y)\big),\\
\widetilde{M}(\tilde{a},x,y) &= (C_2,C_2)^{(2)} \equiv 2\text{Hess}\,\big(C_2(\tilde{a},x,y)\big),\\
\widetilde{N}(\tilde{a},x,y) &= \widetilde{K}-4\widehat{H},\\
\widetilde{D}(\tilde{a},x,y) &= \widehat{D},\\
\eta(\tilde{a}) &= (\widetilde{M},\widetilde{M})^{(2)}/384 \equiv \text{Discrim}\,\big(C_2(\tilde{a},x,y)\big),\\
\theta(\tilde{a}) &= -\,(\widetilde{N},\widetilde{N})^{(2)}/2 \equiv \text{Discrim}\,\big(\widetilde{N}(\tilde{a},x,y)\big);\\
B_1(\tilde{a}) &= \text{Res}_{\,x}\left(C_2,\widetilde{D}\right)/y^9 = -2^{-9}3^{-8}\,(B_2,B_3)^{(4)}\,,\\
B_2(\tilde{a},x,y) &= (B_3,B_3)^{(2)} - 6B_3(C_2,\widetilde{D})^{(3)},\\
B_3(\tilde{a},x,y) &= (C_2,\widetilde{D})^{(1)} \equiv \text{Jacob}\left(C_2,\widetilde{D}\right),\\
H_1(\tilde{a}) &= -\,\big((C_2,C_2)^{(2)},C_2)^{(1)},\widetilde{D}\big)^{(3)},\\
H_3(\tilde{a},x,y) &= (C_2,\widetilde{D})^{(2)},\\
H_4(\tilde{a}) &= \big((C_2,\widetilde{D})^{(2)},(C_2,D_2)^{(1)}\big)^{(2)},\\
H_6(\tilde{a},x,y) &= 16N^2(C_2,\widetilde{D})^{(2)} + H_2^2(C_2,C_2)^{(2)},\\
H_7(\tilde{a}) &= (\widetilde{N},C_1)^{(2)},\\
H_8(\tilde{a}) &= 9\big((C_2,\widetilde{D})^{(2)},(\widetilde{D},D_2)^{(1)}\big)^{(2)} + 2\Big[(C_2,\widetilde{D})^{(3)}\Big]^2,\\
H_9(\tilde{a}) &= -\,[\![\widetilde{D},\widetilde{D})^{(2)},\widetilde{D},\big)^{(1)},\widetilde{D}\big)^{(3)},\\
H_{10}(\tilde{a}) &= \big((\widetilde{N},\widetilde{D})^{(2)},\ D_2\big)^{(1)},\\
H_{11}(\tilde{a},x,y) &= 8\widehat{H}\big[(C_2,\widetilde{D})^{(2)} + 8(\widetilde{D},D_2)^{(1)}\big] + 3\big[(C_1,\,2\widehat{H}-\widetilde{N})^{(1)} - 2D_1\widetilde{N}\big]^2,\\
H_{13}(\tilde{a},x,y) &= A_1A_2 - A_{14} - A_{15},\\
H_{14}(\tilde{a},x,y) &= A_2(156A_5 - 20A_3 - 33A_4) + 4(99A_1A_6 - 5A_{22} + 42A_{23} - 21A_{24}),\\
H_{15}(\tilde{a}) &= \big((\widetilde{D},\widetilde{D})^{(2)},\ \widetilde{H}\big)^{(1)},\\
H_{17}(\tilde{a}) &= 2A_2^2 - 16A_{17} - 16A_{18} + 12A_{19} - 2A_{20},
\end{aligned}$$

$$
\begin{aligned}
N_1(\tilde{a},x,y) =& C_1(C_2,C_2)^{(2)} - 2C_2(C_1,C_2)^{(2)},\\
N_2(\tilde{a},x,y) =& D_1(C_1,C_2)^{(2)} - \big((C_2,C_2)^{(2)},C_0\big)^{(1)},\\
N_3(\tilde{a},x,y) =& (C_2,C_1)^{(1)},\\
N_4(\tilde{a},x,y) =& 4\,(C_2,C_0)^{(1)} - 3C_1D_1,\\
N_5(\tilde{a},x,y) =& \big[(D_2,C_1)^{(1)} + D_1D_2\big]^2 - 4\big(C_2,C_2\big)^{(2)}\big(C_0,D_2\big)^{(1)},\\
N_6(\tilde{a},x,y) =& 8D + C_2\big[8(C_0,D_2)^{(1)} - 3(C_1,C_1)^{(2)} + 2D_1^2\big].
\end{aligned}
$$

3 Preliminary results involving the use of polynomial invariants

The following two lemmas reveal the geometrical meaning of the invariant polynomials B_1, B_2, B_3, θ and $\widetilde{N}$.

Lemma 1. [9] *For the existence of an invariant straight line in one (respectively 2 or 3 distinct) directions in the affine plane it is necessary that $B_1 = 0$ (respectively $B_2 = 0$ or $B_3 = 0$).*

Lemma 2. [9] *A necessary condition for the existence of one couple (respectively, two couples) of parallel invariant straight lines of a system* (2) *corresponding to $\boldsymbol{a} \in \mathbb{R}^{12}$ is the condition $\theta(\boldsymbol{a}) = 0$ (respectively, $\widetilde{N}(\boldsymbol{a},x,y) = 0$).*

We remark that the invariant polynomials $\mu_i(\tilde{a},x,y)$ $(i = 0,1,\ldots,4)$ defined earlier (see page 50) are responsible for the total multiplicity of the finite singularities of quadratic systems (2). Moreover they detect whether a quadratic system is degenerate or not as well as the coordinates of infinite singularities that result after the coalescence of finite singularities with an infinite one. More exactly according to [1, Lemma 5.2] we have the following lemma.

Lemma 3. *Consider a quadratic system (S) with coefficients $\boldsymbol{a} \in \mathbb{R}^{12}$. Then:*

(i) The total multiplicity of the finite singularities of this system is $4-k$ if and only if for every i such that $0 \le i \le k-1$ we have $\mu_i(\boldsymbol{a},x,y) = 0$ in $\mathbb{R}[x,y]$ and $\mu_k(\boldsymbol{a},x,y) \neq 0$.

In this case the factorization $\mu_k(\boldsymbol{a},x,y) = \prod_{i=1}^{k}(u_ix - v_iy) \neq 0$ over $\mathbb{C}$ yields the coordinates $[v_i : u_i : 0]$ of points at infinity that have multiplicity greater than one, this being the result of coalescence of finite and infinite singularities. Moreover the number of distinct expressions $u_ix - v_iy$ in this factorization is less than or equal to three (the maximum number of infinite singularities of a quadratic system), and the multiplicity of each one of the expressions $u_ix - v_iy$ gives us the number of the finite singularities of the system (S) that have coalesced with the infinite singular point $[v_i : u_i : 0]$.

(ii) Let the point $M_0(0,0)$ be a singular point for the quadratic system (S). Then the point $M_0(0,0)$ is a singular point of multiplicity k $(1 \le k \le 4)$ if and only if for every i such that $0 \le i \le k-1$ we have $\mu_{4-i}(\boldsymbol{a},x,y) = 0$ in $\mathbb{R}[x,y]$ and $\mu_{4-k}(\boldsymbol{a},x,y) \neq 0$.

(iii) *The system* (S) *is degenerate (i.e.* $\gcd(p,q) \neq$ *constant) if and only if* $\mu_i(\boldsymbol{a},x,y)=0$ *in* $\mathbb{R}[x,y]$ *for every* $i=0,1,2,3,4$.

On the other hand the invariant polynomials η, $\widetilde{M}$ and C_2 govern the number of real and complex infinite singularities. More precisely, according to [18] (see also [10]) we have the next result.

Lemma 4. *The number of infinite singularities (real and complex) of a quadratic system in* **QS** *is determined by the following conditions:*

(*i*) 3 *real if* $\eta > 0$;

(*ii*) 1 *real and* 2 *imaginary if* $\eta < 0$;

(*iii*) 2 *real if* $\eta = 0$ *and* $\widetilde{M} \neq 0$;

(*iv*) 1 *real if* $\eta = \widetilde{M} = 0$ *and* $C_2 \neq 0$;

(*v*) ∞ *if* $\eta = \widetilde{M} = C_2 = 0$.

Moreover, the quadratic systems (2)*, for each one of these cases, can be brought via a linear transformation to the corresponding case of the following canonical systems* $(\mathbf{S}_I) - (\mathbf{S}_V)$:

$$\begin{cases} \dot{x} &= a + cx + dy + gx^2 + (h-1)xy, \\ \dot{y} &= b + ex + fy + (g-1)xy + hy^2; \end{cases} \qquad (\mathbf{S}_I)$$

$$\begin{cases} \dot{x} &= a + cx + dy + gx^2 + (h+1)xy, \\ \dot{y} &= b + ex + fy - x^2 + gxy + hy^2; \end{cases} \qquad (\mathbf{S}_{II})$$

$$\begin{cases} \dot{x} &= a + cx + dy + gx^2 + hxy, \\ \dot{y} &= b + ex + fy + (g-1)xy + hy^2; \end{cases} \qquad (\mathbf{S}_{III})$$

$$\begin{cases} \dot{x} &= a + cx + dy + gx^2 + hxy, \\ \dot{y} &= b + ex + fy - x^2 + gxy + hy^2; \end{cases} \qquad (\mathbf{S}_{IV})$$

$$\begin{cases} \dot{x} &= a + cx + dy + x^2, \\ \dot{y} &= b + ex + fy + xy. \end{cases} \qquad (\mathbf{S}_V)$$

Remark 1. In order to describe the various kinds of multiplicity for infinite singular points we use the concepts and notations introduced in [9]. Thus we denote by "(a,b)" the ordered couple of a, respectively b, where a (respectively b) is the maximum number of infinite (respectively finite) singularities which can be obtained by perturbation of a multiple infinite singular point.

Now we define the affine comitants which are responsible for the existence of invariant lines for a non-degenerate quadratic system (2).

Let us apply a translation $x = x' + x_0$, $y = y' + y_0$ to the polynomials $p(\tilde{a},x,y)$ and $q(\tilde{a},x,y)$. We obtain $\hat{p}(\hat{a}(\tilde{a},x_0,y_0),x',y') = p(\tilde{a},x'+x_0,y'+y_0)$,

$\hat{q}(\hat{a}(\tilde{a}, x_0, y_0), x', y') = q(\tilde{a}, x' + x_0, y' + y_0)$. Let us construct the following polynomials

$$\Gamma_i(\tilde{a}, x_0, y_0) \equiv \operatorname{Res}_{x'}\Big(C_i\big(\hat{a}(\tilde{a}, x_0, y_0), x', y'\big), C_0\big(\hat{a}(\tilde{a}, x_0, y_0), x', y'\big)\Big)/(y')^{i+1},$$
$$\Gamma_i(\tilde{a}, x_0, y_0) \in \mathbb{R}[\tilde{a}, x_0, y_0],\ i = 1, 2.$$

Notation 1.

$$\tilde{\mathcal{E}}_i(\tilde{a}, x, y) = \Gamma_i(\tilde{a}, x_0, y_0)|_{\{x_0 = x,\ y_0 = y\}} \in \mathbb{R}[\tilde{a}, x, y] \quad (i = 1, 2).$$

Observation 1. We note that the constructed polynomials $\tilde{\mathcal{E}}_1(\tilde{a}, x, y)$ and $\tilde{\mathcal{E}}_2(\tilde{a}, x, y)$ are affine comitants of systems (2) and are homogeneous polynomials in the coefficients $a, \ldots, n$ and non-homogeneous in x, y and

$$\deg_{\tilde{a}} \tilde{\mathcal{E}}_1 = 3,\ \deg_{(x,y)} \tilde{\mathcal{E}}_1 = 5,\quad \deg_{\tilde{a}} \tilde{\mathcal{E}}_2 = 4,\ \deg_{(x,y)} \tilde{\mathcal{E}}_2 = 6.$$

Notation 2.
Let $\mathcal{E}_i(\tilde{a}, X, Y, Z)$ $(i = 1, 2)$ be the homogenization of $\tilde{\mathcal{E}}_i(\tilde{a}, x, y)$, i.e.

$$\mathcal{E}_1(\tilde{a}, X, Y, Z) = Z^5 \tilde{\mathcal{E}}_1(\tilde{a}, X/Z, Y/Z), \qquad \mathcal{E}_2(\tilde{a}, X, Y, Z) = Z^6 \tilde{\mathcal{E}}_2(\tilde{a}, X/Z, Y/Z)$$

and $\mathcal{H}(\tilde{a}, X, Y, Z) = \gcd\Big(\mathcal{E}_1(\tilde{a}, X, Y, Z),\ \mathcal{E}_2(\tilde{a}, X, Y, Z)\Big)$ in $\mathbb{R}[\tilde{a}, X, Y, Z]$.

The geometrical meaning of these affine comitants is given by the two following lemmas (see [9]):

Lemma 5. [9] *The straight line $\mathcal{L}(x, y) \equiv ux + vy + w = 0$, $u, v, w \in \mathbb{C}$, $(u, v) \neq (0, 0)$ is an invariant line for a quadratic system* (2) *if and only if the polynomial $\mathcal{L}(x, y)$ is a common factor of the polynomials $\tilde{\mathcal{E}}_1(\boldsymbol{a}, x, y)$ and $\tilde{\mathcal{E}}_2(\boldsymbol{a}, x, y)$ over $\mathbb{C}$, i.e.*

$$\tilde{\mathcal{E}}_i(\boldsymbol{a}, x, y) = (ux + vy + w)\widetilde{W}_i(x, y) \quad (i = 1, 2),$$

where $\widetilde{W}_i(x, y) \in \mathbb{C}[x, y]$.

Lemma 6. *1) If $\mathcal{L}(x, y) \equiv ux + vy + w = 0$, $u, v, w \in \mathbb{C}$, $(u, v) \neq (0, 0)$ is an invariant straight line of multiplicity k for a quadratic system* (2) *then $[\mathcal{L}(x, y)]^k \mid \gcd(\tilde{\mathcal{E}}_1, \tilde{\mathcal{E}}_2)$ in $\mathbb{C}[x, y]$, i.e. there exist $W_i(\boldsymbol{a}, x, y) \in \mathbb{C}[x, y]$ $(i = 1, 2)$ such that*

$$\tilde{\mathcal{E}}_i(\boldsymbol{a}, x, y) = (ux + vy + w)^k W_i(\boldsymbol{a}, x, y), \quad i = 1, 2.$$

2) If the line $l_\infty : Z = 0$ is of multiplicity $k > 1$ then $Z^{k-1} \mid \gcd(\mathcal{E}_1, \mathcal{E}_2)$, in other words we have $Z^{k-1} \mid \mathcal{H}(\boldsymbol{a}, X, Y, Z)$.

In what follows the following Lemma will be useful.

Lemma 7. *The non-singular invariant line at infinity for a non-degenerate quadratic system has the multiplicity greater than or equal to two if and only if the condition $\widetilde{K} = 0$ holds.*

Proof. Considering Lemma 6 (see statement 2) we deduce that the line at infinity of a quadratic system is of multiplicity > 1 if and only if $Z \mid \gcd(\mathcal{E}_1, \mathcal{E}_2)$. In other words Z is a common factor of the polynomials $\mathcal{E}_1(X,Y,Z)$ and $\mathcal{E}_2(X,Y,Z)$ (see Notation 2).

Taking into account the definition of the invariant polynomials $\mathcal{E}_1(X,Y,Z)$ and $\mathcal{E}_2(X,Y,Z)$ (see Notations 1 and 2) for systems (2) we calculate

$$\begin{aligned}
\mathcal{E}_1(X,Y,Z) =& \frac{1}{2}C_2(X,Y)\widetilde{K}(X,Y) + \phi_1(X,Y)Z + \phi_2(X,Y)Z^2 + \ldots + \phi_5(X,Y)Z^5,\\
\mathcal{E}_2(X,Y,Z) =& C_2(X,Y)\Psi(X,Y) + \psi_1(X,Y)Z + \psi_2(X,Y)Z^2 + \ldots + \psi_6(X,Y)Z^6,
\end{aligned}$$

where

$$\begin{aligned}
C_2(X,Y) =& -lX^3 + (g-2m)X^2Y + (2h-n)XY^2 + kY^3,\\
\widetilde{K}(X,Y) =& 4\big[(gm-hl)X^2 + (gn-kl)XY + (hn-km)Y^2\big] \equiv 4\big[\alpha X^2 + \beta XY + \gamma Y^2\big],\\
\Psi(X,Y) =& (2g\alpha + l\beta)X^3 + \big[(4h+2n)\alpha + g\beta + 4l\gamma\big]X^2Y\\
&+ \big[2k\alpha + (2h+n)\beta + 4m\gamma\big]XY^2 + (k\beta + 2n\gamma)Y^3.
\end{aligned}$$

Therefore we conclude that the invariant polynomials $\mathcal{E}_1(X,Y,Z)$ and $\mathcal{E}_2(X,Y,Z)$ have the common factor Z if and only if the conditions $C_2(X,Y)\widetilde{K}(X,Y) = C_2(X,Y)\Psi(X,Y) = 0$ hold. Since $C_2 = 0$ leads to systems with the line at infinity filled up with singularities (see Lemma 4) clearly the condition $C_2 \neq 0$ has to be satisfied.

On the other hand we observe that the condition $\widetilde{K}(X,Y) = 0$ implies $\alpha = \beta = \gamma = 0$ and then $\Psi(X,Y) = 0$. Therefore we obtain that the condition $\widetilde{K}(X,Y) = 0$ is necessary and sufficient for a quadratic system to have the invariant line at infinity of multiplicity at least 2. This completes the proof of Lemma 7. □

4 The quadratic systems belonging to the family $\mathbf{QSL}_3$

As it is mentioned in Introduction some of the configurations of the quadratic systems in the family $\mathbf{QSL}_3$ were determined earlier in other papers. More exactly in [16] the configurations *Config. 3.1–Config. 3.13* are constructed. In a recent published article [5] the family of systems possessing two parallel invariant lines is considered and the configurations *Config. 3.14–Config. 3.65* are determined.

In this section we complete the investigation of the family $\mathbf{QSL}_3$ and prove that there exist 16 possible new configurations *Config. 3.66–Config. 3.81*.

First of all we prove some necessary conditions for a quadratic system to belong to the family $\mathbf{QSL}_3$. We have the following lemma.

Lemma 8. *Assume that a non-degenerate quadratic system belongs to the class* $\mathbf{QSL}_3$. *Then for this system the conditions* $B_1 = 0$ *and* $B_3 \neq 0$ *have to be fulfilled.*

Proof. According to Lemma 1 if for a quadratic system the condition $B_1 \neq 0$ holds then this system could not have any invariant affine line going in some direction. On the other hand if a system belongs to the class $\mathbf{QSL}_3$ then either there exists at least one invariant affine line or the line at infinity is triple. However in the second case there must exist a perturbation such that the perturbed system necessarily possesses at least one invariant affine line and this means that for this system we must have $B_1 = 0$. So we deduce that this condition must be satisfied for the non-perturbed system, too. □

Therefore we obtain that for a system in $\mathbf{QSL}_3$ the condition $B_1 = 0$ have to be satisfied. In order to complete the proof of Lemma 8 we have to show that for a system in $\mathbf{QSL}_3$ the condition $B_3 \neq 0$ is also necessary. We prove the following lemma.

Lemma 9. *Assume that for a non-degenerate quadratic system the condition $B_3 = 0$ holds. Then this system belongs to the class $\mathbf{QSL}_{\geq 4}$. Moreover any system in this class could have a configuration of invariant lines given in Diagram 3 if and only if the corresponding conditions are satisfied, respectively.*

Proof. Assume that for a non-degenerate quadratic system the condition $B_3 = 0$ is fulfilled. In the articles [9] and [11] the families of quadratic systems possessing invariant line of total multiplicity at least four are investigated and the corresponding possible configurations of invariant lines are determined.

So considering Tables 2 and 4 from [9] as well as Table 2 from [11] it is not too difficult to convince ourselves that the conditions given in these tables for the corresponding configurations are equivalent to the respective conditions presented in Diagram 3.

We observe that this diagram gives us a complete partition of the whole set $\mathbf{QSL}_{\{B_3=0\}}$. This completes the proof of Lemma 9 as well as the proof of Lemma 8. □

4.1 Configurations of systems belonging to the subfamily $\mathbf{QSL}_3 \cap \mathbf{QS}_{2cIL}$

In paper [2] (see also [19]) the phase portraits of the family of quadratic systems possessing two complex invariant lines intersecting at a real finite point are considered. We denote this family by $\mathbf{QS}_{2cIL}$. A result in [2] determined 20 topologically distinct phase portraits. However the problem of how many configurations of invariant lines systems in the family $\mathbf{QS}_{2cIL}$ could have remains open.

Here we are interested in the configurations of the quadratic systems belonging to the subfamily $\mathbf{QSL}_3 \cap \mathbf{QS}_{2cIL}$. We prove the following theorem.

Theorem 1. *An arbitrary non-degenerate quadratic system belongs to the subfamily $\mathbf{QSL}_3 \cap \mathbf{QS}_{2cIL}$ if and only if the conditions $\eta < 0$, $B_2 = 0$ and $B_3\widetilde{N} \neq 0$ hold. Moreover this system possesses the configuration Config. 3.66 if $\mu_0 \neq 0$ and Config. 3.67 if $\mu_0 = 0$.*

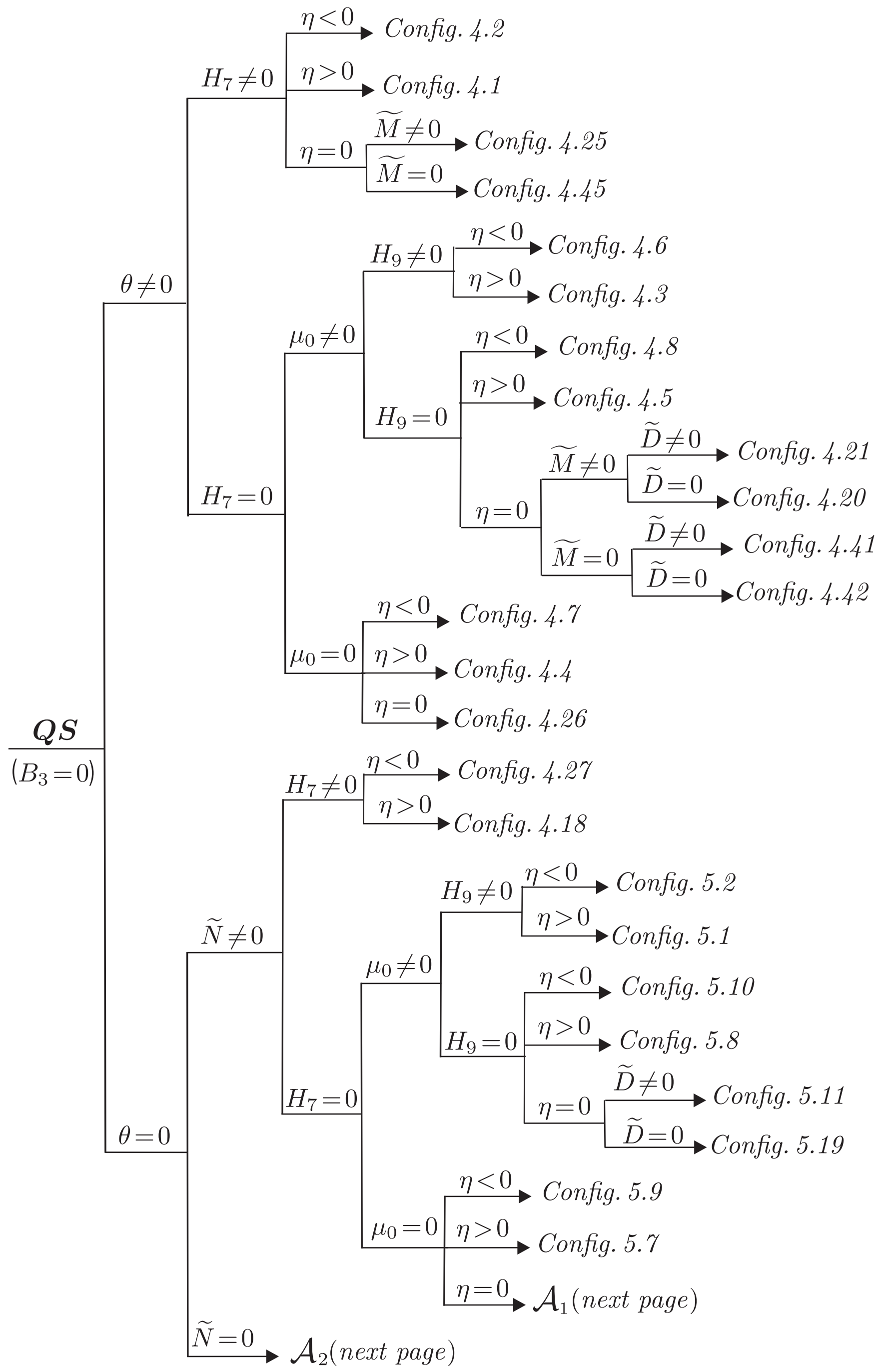

Diagram 3: The configurations of systems in **QSL** with $B_3 = 0$

Proof. According to [2, Theorem 1] a non-degenerate quadratic system possesses two complex invariant lines meeting at a finite real point if and only if one of the

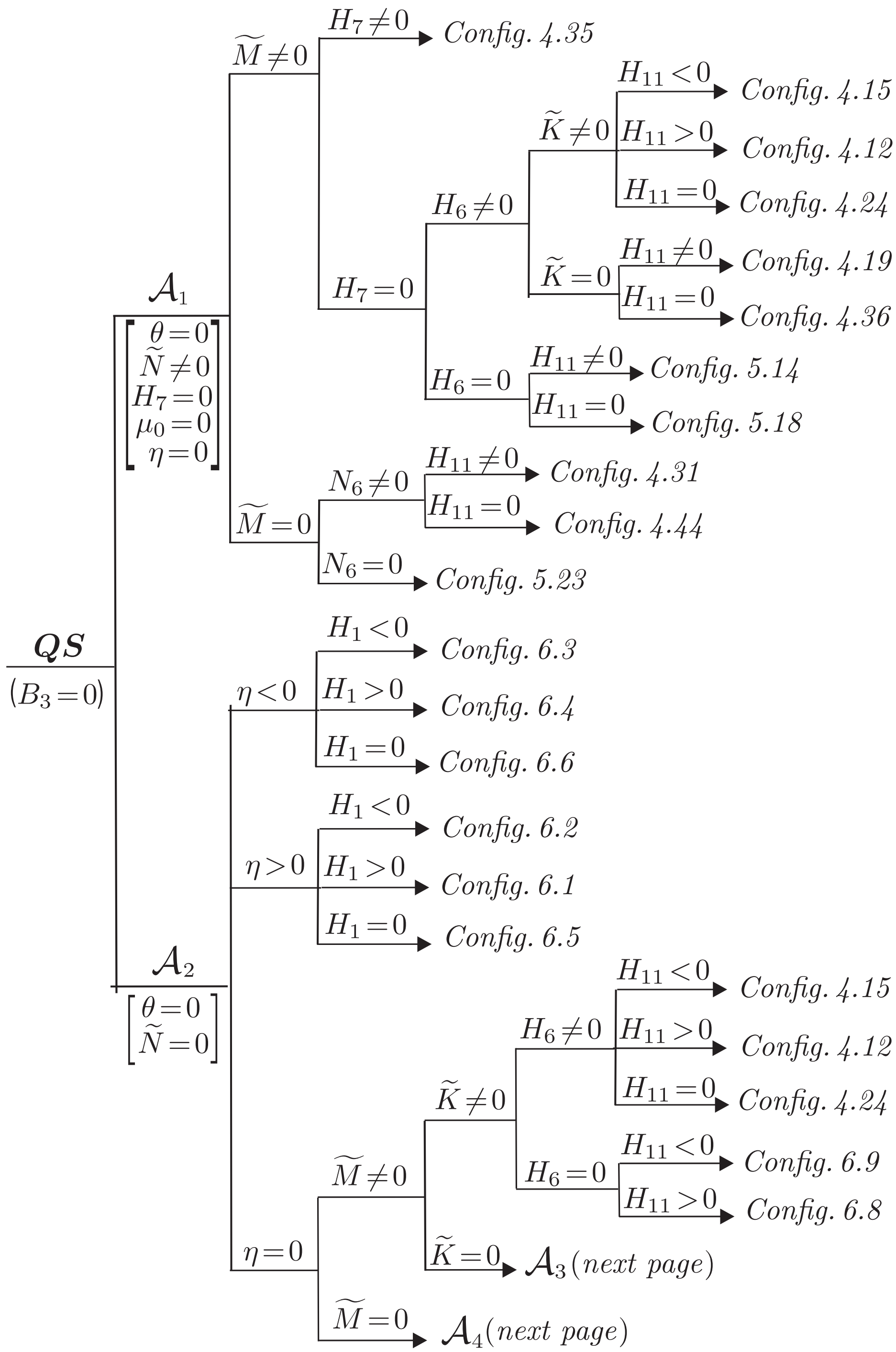

Diagram 3 *(continuation)*: The configurations of systems in **QSL** with $B_3 = 0$

following two sets of conditions are satisfied:

$$(i)\quad \eta < 0, \quad B_2 = 0; \qquad (ii)\quad C_2 = 0, \quad \mathbf{D} > 0.$$

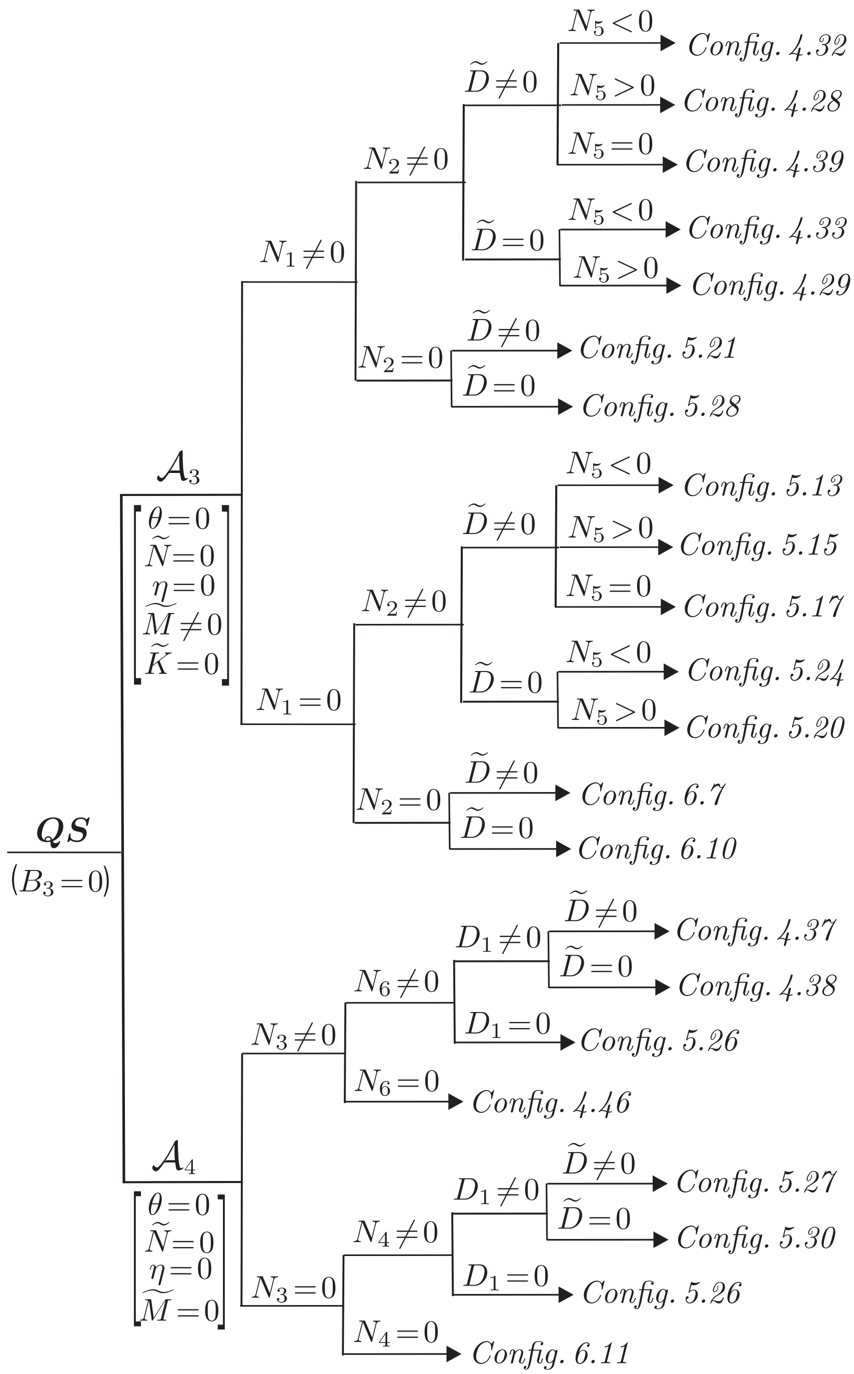

Diagram 3 *(continuation)*: The configurations of systems in **QSL** with $B_3=0$

By [15] quadratic systems with $C_2=0$ possess in the finite part of the phase plane invariant lines of total multiplicity three. Therefore we obtain that a system with $C_2=0$ could not belong to the class $\boldsymbol{QSL}_3$. Moreover we deduce that for

$C_2 \neq 0$ the conditions $\eta < 0$ and $B_2 = 0$ are necessary and sufficient for a system to belong to the family $\boldsymbol{QS}_{2cIL}$.

Since we are interested in the determinations of the configuration of the quadratic systems in the subclass $\boldsymbol{QSL}_3 \cap \boldsymbol{QS}_{2cIL}$ we consider that for a non-degenerate quadratic system the conditions $\eta < 0$ and $B_2 = 0$ are satisfied. Thus according to what is mentioned above we conclude that in order to complete the proof of Theorem 1 it is sufficient to prove that if for a quadratic system we have $\eta < 0$ and $B_2 = 0$ then the condition $B_3\widetilde{N} \neq 0$ guarantees that this system belongs to the class $\boldsymbol{QSL}_3$. Moreover we have also to determine the possible configurations of invariant lines of these systems.

According to [20] if a quadratic system possesses two complex invariant lines intersecting at a real finite singular point then via an affine transformation this system takes the following form:

$$\begin{aligned} \frac{dx}{dt} &= (\alpha x - \beta y)(ax + by + c) + k(x^2 + y^2) \equiv P(x, y), \\ \frac{dy}{dt} &= (\beta x + \alpha y)(ax + by + c) \equiv Q(x, y) \end{aligned} \tag{3}$$

where $\alpha, \beta, a, b, c, k$ are arbitrary real parameters. These systems possess the complex invariant lines $x \pm iy = 0$ and we calculate

$$\eta = -4\big[(k - b\beta)^2 + a^2\beta^2\big]^2 < 0, \quad B_2 = 0, \quad B_3 = 3ac^2k\beta(\alpha^2 + \beta^2)(x^2 + y^2)^2.$$

According to Lemma 8 for a system (3) to belong to the class $\boldsymbol{QSL}_3$ the condition $B_3 \neq 0$ is necessary. The question which appears is the following: which conditions must be added in order to get the necessary and sufficient ones?

Providing the conditions $\eta < 0$ and $B_3 \neq 0$ are fulfilled for a system (3) we examine what additional conditions could increase the total multiplicity of the invariant lines of this system.

Assume that a system (3) possesses invariant lines of total multiplicity exactly four. In [11] the family of systems belonging to $\boldsymbol{QSL}_4$ has been investigated and in Table 2 necessary and sufficient conditions for the realization of each one of the possible 46 configurations of invariant lines for this class are given. Considering Table 2 from [11] we detect that systems with the condition $\eta < 0$ (i.e. having 2 complex and one real infinite singularities) could possess only one of the following 4 configurations: *Config.4.2* and *Config.4.6–Config.4.8*. However for all these configurations the condition $B_3 = 0$ has to be satisfied and hence we get a contradiction to Lemma 8.

Thus we conclude that a system (3) could not belong to the class $\boldsymbol{QSL}_4$.

Suppose now that a system (3) possesses invariant lines of total multiplicity at least five. According to [9] (see Theorem 50, statement (ii)) for having invariant lines of total multiplicity 6 the condition $B_3 = 0$ is necessary for any quadratic system. So we conclude that a system (3) could not belong to the class $\boldsymbol{QSL}_6$.

It remains to consider the possibility when a system (3) with $\eta \neq 0$ (i.e. $\eta < 0$) and $B_3 \neq 0$ belongs to the class $\boldsymbol{QSL}_5$. In this case we consider Table 4 from [9] and we detect that subject to these conditions we could have the unique configuration *Config.5.6*. However to obtain this configuration the condition $\widetilde{N} = 0$ must be satisfied.

Thus we conclude that a system (3) with $\eta < 0$ and $B_3 \neq 0$ belongs to the class $\boldsymbol{QSL}_3$ if $\widetilde{N} \neq 0$ and to the class $\boldsymbol{QSL}_5$ if $\widetilde{N} = 0$. This means that the conditions provided by Theorem 1 for a quadratic systems to belong to the subclass $\boldsymbol{QSL}_3 \cap \boldsymbol{QS}_{2cIL}$ are satisfied.

Next we determine the configurations which a system (3) from the class $\boldsymbol{QSL}_3$ could possess. For this we have to determine the position of the singularities of this system with respect to the invariant lines.

A straightforward calculation gives us the following finite singularities of systems (3):

$$M_1(0,0), \quad M_2 = \Big(-\frac{c\alpha}{k+a\alpha-b\beta}, \frac{c\beta}{k+a\alpha-b\beta}\Big), \quad M_{3,4} = \Big(-\frac{c}{a\pm ib}, -\frac{c}{b\mp ia}\Big).$$

Since the condition $B_3 \neq 0$ implies $ack\beta \neq 0$ we conclude that the singular points M_2 and $M_{3,4}$ could not coalesce with M_1. Moreover the singular point M_2 exists if $k + a\alpha - b\beta \neq 0$, otherwise it goes to infinity coalescing with the real infinite singularity.

On the other hand for systems (3) we calculate

$$\mu_0 = (a^2+b^2)k(k+a\alpha-b\beta)(\alpha^2+\beta^2)$$

and hence for $\mu_0 \neq 0$ these systems possess two real and two complex finite singular points and we arrive at the configuration given by *Config.3.66*.

Assume now $\mu_0 = 0$. Due to the condition $B_3 \neq 0$ (i.e. $ack\beta \neq 0$) we get $k = b\beta - a\alpha \neq 0$ and hence we calculate

$$\mu_1 = c(a^2+b^2)(a\alpha-b\beta)(\alpha^2+\beta^2)(\beta x+\alpha y).$$

We observe that $\mu_1 \neq 0$ due to the condition $ack\beta(b\beta - a\alpha) \neq 0$. Since $\mu_0 = 0$, according to Lemma 3 one finite singular point went to infinity and coalesced with the infinite real singularity $N_1[\alpha, -\beta, 0]$ (see the factor of the invariant polynomial $\mu_1(x,y)$). In this case we arrive at the configuration given by *Config.3.67*.

As all the cases are examined we conclude that Theorem 1 is proved. □

4.2 Configurations of quadratic systems that are limit points of the family of Lotka-Volterra systems

It turned out that a quadratic system could have invariant lines of total multiplicity 3 which are not included in one of the following three classes: (*i*) Lotka-Volterra systems, or (*ii*) systems with two parallel invariant lines, or (*iii*) systems with two complex lines meeting at a finite singularity.

Indeed such kind of configurations could be obtained from an L-V system using the following two possibilities:

($\boldsymbol{\alpha}$) Two simple invariant affine lines of an L-V system belonging to the subclass $\mathbf{QSL}_3$ coalesced and we obtain a double invariant affine line and a multiple real singular point at infinity.

($\boldsymbol{\beta}$) One (or two) simple invariant affine lines of an L-V system in $\mathbf{QSL}_3$ coalesced with infinite line $Z=0$ giving a double (or a triple) infinite invariant line.

Since we are in the class of L-V systems by Lemma 1 it is clear that the condition $B_2=0$ must be satisfied in both these cases. Moreover in the case **($\boldsymbol{\alpha}$)** the condition $\eta=0$ has to be fulfilled, because we have a double (or triple) singular point at infinity.

On the other hand considering Lemma 7 we conclude that in the case **($\boldsymbol{\beta}$)** the condition $\widetilde{K}(a,x,y)=0$ is necessary.

In what follows assuming the condition $B_2=0$ should be fulfilled we examine each one of the cases we mentioned above and determine the possible configurations of invariant lines as well as the corresponding affine invariant conditions for their realization.

($\boldsymbol{\alpha}$) In this case for a quadratic system the condition $\eta=0$ has to be satisfied. We examine two cases: $\widetilde{M}\neq 0$ and $\widetilde{M}=0$.

Proof. 1. **The case $\widetilde{M}\neq 0$.** According to Lemma 4 a quadratic system could be brought via a linear transformation to the canonical form $(\mathbf{S}_{III})$, i.e. we have to examine the family systems

$$\begin{aligned}\dot{x}=&a+cx+dy+gx^2+hxy,\\ \dot{y}=&b+ex+fy+(g-1)xy+hy^2.\end{aligned}\tag{4}$$

For these systems calculations yield:

$$\theta=8h^2(1-g),\quad \mu_0=gh^2,\quad C_2=x^2y,\quad \widetilde{N}=(g^2-1)x^2+2h(g-1)xy+h^2y^2.\tag{5}$$

Since $C_2=x^2y$ we conclude that these systems possess two infinite singularities: $N_1[1:0:0]$ (simple) and $N_2[0:1:0]$ (double). We discuss two subcases: $\theta\neq 0$ and $\theta=0$.

1.1. **The subcase $\theta\neq 0$.** The condition $\theta\neq 0$ yields $h(g-1)\neq 0$ and we may consider $d=e=0$ due to a translation. Moreover, since $h\neq 0$ we may assume $h=1$ due to the rescaling $y\to y/h$. Thus we obtain the family of systems

$$\dot{x}=a+cx+gx^2+xy,\quad \dot{y}=b+fy+(g-1)xy+y^2,$$

for which we calculate Coefficient$[B_2,y^4]=-648a^2$. Hence the necessary condition $B_2=0$ yields $a=0$ and then

$$\begin{gathered}B_2=-648b(b+c^2-cf)(g-1)^2x^4,\quad H_4=48(b+c^2-cf),\quad \theta=8(1-g),\\ B_3=-3\big[b(g-1)^2x^2-(b+c^2-cf)y^2\big]x^2.\end{gathered}$$

We shall consider two possibilities: $H_4\neq 0$ and $H_4=0$.

1.1.1. **The possibility** $H_4 \neq 0$**.** In this case the condition $B_2 = 0$ yields $b = 0$ and we arrive at the family of systems

$$\dot{x} = x(c + gx + y), \quad \dot{y} = y[f + (g-1)x + y],$$

possessing the invariant lines $x = 0$ and $y = 0$. So we obtain LV-systems, i.e. no new configurations could de detected.

1.1.2. **The possibility** $H_4 = 0$**.** Then we have $b = c(f - c)$ and this leads to the family of systems

$$\dot{x} = x(c + gx + y), \quad \dot{y} = c(f - c) + fy + (g-1)xy + y^2, \tag{6}$$

possessing the invariant line $x = 0$ which is double because $\mathcal{H}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) = X^2$ (see Notation 2). So, these systems possess invariant lines of total multiplicity at least 3. However for these systems the condition $B_3 = 3c(c - f)(g - 1)^2 x^4 \neq 0$ is necessary and therefore by Lemma 1 we could not have an additional invariant line in the direction $y = 0$.

Thus we deduce that in the case $B_3 \neq 0$ systems (6) possess invariant lines of total multiplicity exactly 3. More exactly we have a double invariant affine line $x = 0$, on which there are located two finite singularities: $M_1(0, -c)$ and $M_2(0, c - f)$. The third finite singularity $M_3(x_3, y_3)$ of systems (6) has the coordinates

$$x_3 = -\frac{cg + c - fg}{g}, \quad y_3 = (c - f)g.$$

Since for systems (6) we have $\mu_0 = g$ we conclude that for $\mu_0 \neq 0$ all the finite singularities are on the plane and this means that one of the mentioned finite singularities is double. We claim that the double singularity is $M_1(0, -c)$. Indeed after translation of the origin of coordinates to the singular point M_1 we obtain the systems

$$\dot{x} = x(gx + y), \quad \dot{y} = c(1 - g)x + (f - 2c)y + (g - 1)xy + y^2 \tag{7}$$

possessing a double singular point at the origin because the determinant of the linear part equals zero. So these systems have the finite singular points

$$M_1(0, 0), \quad M_2(0, 2c - f), \quad M_3\big(-(c + cg - fg)/g, c + cg - fg\big)$$

and we observe that M_3 goes to infinity if $g = 0$. Moreover it is clear that M_2 coalesces with M_1 if $2c - f = 0$ and M_3 coalesces with M_1 if $c + cg - fg = 0$.

On the other hand for systems (7) calculations yield:

$$\mu_0 = g, \; H_3 = 8(2c - f)(c + cg - fg)x^2, \;\; H_{13} = -288c(2c - f)^2(g - 1),$$
$$B_3 = 3c(c - f)(g - 1)^2 x^4$$

and we observe that due to $B_3 \neq 0$ the condition $H_{13} = 0$ is equivalent to $f = 2c$. So we consider two cases: $\mu_0 \neq 0$ and $\mu_0 = 0$.

1.1.2.1. *The case* $\mu_0 \neq 0$. Then $g \neq 0$ and the finite singularity M_3 remains in the finite plane. So if $H_3 \neq 0$ none of the singular points could coalesced and we arrive at the configuration *Config. 3.68* (see Figure 1)

Assume now $H_3 = 0$, i.e. $(2c - f)(c + cg - fg) = 0$. Then evidently we obtain *Config. 3.69* if $H_{13} \neq 0$ and *Config. 3.70* if $H_{13} = 0$.

We point out that all three finite singularities could not coalesced due to $B_3 \neq 0$ (i.e. $c \neq 0$).

1.1.2.2. *The case* $\mu_0 = 0$. Then $g = 0$ and systems (7) become

$$\dot{x} = xy, \quad \dot{y} = cx + (f - 2c)y - xy + y^2$$

possessing the following two finite singularities: $M_1(0, 0)$ and $M_2(0, 2c - f)$. Since for the above systems we have $\mu_0 = \mu_1 = 0$ and $\mu_2 = -cy \neq 0$ (otherwise we get degenerate systems), according to Lemma 3 the singular point M_3 of systems (7) has gone to infinity and coalesced with the infinite singular point $N_1[1 : 0 : 0]$ which becomes of multiplicity 2 of the type $(1, 1)$.

On the other hand the finite singularity M_2 could coalesce with M_1 if the condition $f = 2c$ holds. For the above systems we calculate

$$B_3 = 3c(c - f)x^4 \neq 0, \quad H_3 = 8c(2c - f)x^2$$

and therefore we arrive at the configuration *Config. 3.71* if $H_3 \neq 0$ and *Config. 3.72* if $H_3 = 0$.

1.2. **The subcase** $\theta = 0$**.** Considering (5) this condition gives $h(g - 1) = 0$ and since $\mu_0 = gh^2$ we examine two possibilities: $\mu_0 \neq 0$ and $\mu_0 = 0$.

1.2.1. **The possibility** $\mu_0 \neq 0$**.** Then $h \neq 0$ and hence the condition $\theta = 0$ yields $g = 1$. Therefore we may consider $h = 1$ due to the rescaling $y \to y/h$ and $d = f = 0$ due to a translation. Thus we obtain the family of systems

$$\dot{x} = a + cx + x^2 + xy, \quad \dot{y} = b + ex + y^2,$$

for which we have Coefficient$[B_2, y^4] = -648a^2$ and therefore the condition $B_2 = 0$ implies $a = 0$. Then we calculate

$$B_2 = -648(b + c^2)e^2x^4, \quad H_7 = -4e.$$

and clearly if $e = 0$ (i.e. $H_7 = 0$) then the above systems with $a = e = 0$ possess three invariant affine lines $x = 0$ and $y^2 + b = 0$. Therefore we could not obtain new configurations apart from the ones already known.

Assuming $H_7 \neq 0$ we get the conditions $b = -c^2$ and this leads to the family of systems

$$\dot{x} = x(c + x + y), \quad \dot{y} = -c^2 + ex + y^2,$$

possessing the invariant line $x = 0$ which is double because $\mathcal{H}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) = X^2$ (see Notation 2). These systems have three finite singularities $M_1(0, -c)$, $M_2(0, c)$ and

$M_3(-2c-e, c+e)$. We observe that the singular point M_1 is double because after the translation $(x,y) \to (x, y+c)$ we arrive at the systems

$$\dot{x} = x(x+y), \quad \dot{y} = ex - 2cy + y^2, \tag{8}$$

possessing a double singularity $M_1(0,0)$ at the origin of coordinates (since the determinant of linear part vanishes) and two elemental singularities $M_2(0,2c)$ and $M_3(-2c-e, 2c+e)$. It is clear that in the case $e = -2c$ the singular point M_3 coalesces with the double point M_1 whereas for $c = 0$ the singularity M_2 coalesces with M_1.

On the other hand for the above systems we calculate

$$B_3 = -3e^2x^4, \quad H_3 = 16c(2c+e)x^2, \quad H_{13} = 2c^2e.$$

and due to $B_3 \neq 0$ (i.e. $e \neq 0$), by Lemma 1 systems (8) could not possess invariant lines in the direction $y = 0$. Therefore we deduce that in this case systems (8) possess invariant lines of total multiplicity 3.

Thus considering the condition $H_7 \neq 0$ (i.e. $e \neq 0$) it is not too difficult to determine that we get the configuration *Config. 3.68* if $H_3 \neq 0$, *Config. 3.69* if $H_3 = 0$ and $H_{13} \neq 0$, and *Config. 3.70* if $H_3 = H_{13} = 0$.

1.2.2. **The possibility** $\mu_0 = 0$**.** Considering (5) we get $h = 0$ and therefore for systems (4) we obtain $\widetilde{N} = (g^2-1)x^2$.

So we discuss two cases: $\widetilde{N} \neq 0$ and $\widetilde{N} = 0$.

1.2.2.1. *The case* $\widetilde{N} \neq 0$. Then $g - 1 \neq 0$ and assuming $e = f = 0$ (due to a translation) we arrive at the systems

$$\dot{x} = a + cx + dy + gx^2, \quad \dot{y} = b + (g-1)xy,$$

for which we calculate

$$H_7 = 4d(g^2-1), \quad \widetilde{N} = (g^2-1)x^2, \quad \text{Coefficient}[B_2, y^4] = -648d^4g^2.$$

We observe that for $d = 0$ the above systems possess two parallel invariant lines $a + cx + gx^2 = 0$ and hence no new configurations could be obtained in this case.

Since $\widetilde{N} \neq 0$ we obtain that the condition $d = 0$ is equivalent to $H_7 = 0$ and in what follows we assume $H_7 \neq 0$. Then the condition $B_2 = 0$ implies $g = 0$ and then we obtain

$$B_2 = -648bcdx^4, \quad H_7 = -4d, \quad \mu_0 = \mu_1 = 0, \quad \mu_2 = -cdxy$$

and we discuss two subcases: $\mu_2 \neq 0$ and $\mu_2 = 0$.

1.2.2.1.1. *The subcase* $\mu_2 \neq 0$. Then we have $c \neq 0$ and the condition $B_2 = 0$ gives $b = 0$ and we obtain the family of systems

$$\dot{x} = a + cx + dy, \quad \dot{y} = -xy, \tag{9}$$

possessing the invariant line $y = 0$. Moreover for these systems we calculate $\mathcal{H}(\mathcal{X},\mathcal{Y},\mathcal{Z}) = YZ$ and by Lemma 6 we deduce that the infinite invariant line is double. In other words we have invariant lines of total multiplicity 3.

Since $\mu_0 = \mu_1 = 0$ and $\mu_2 = -cdxy \neq 0$, according to Lemma 3 we deduce that two finite singular points have gone to infinity and coalesced with infinite singular points $N_1[1:0:0]$ and $N_2[0:1:0]$, respectively. So at infinity we get two multiple singularities of multiplicities $(1,1)$ and $(2,1)$ (see Remark 1), correspondingly.

On the other hand due to $\mu_2 \neq 0$ (i.e. $cd \neq 0$) systems (9) possess two finite singularities $M_1(0,-a/d)$ and $M_2(-a/c,0)$ both simple (i.e. of multiplicity one). We observe that M_2 is located on the invariant line $y = 0$ and these singularities coalesce if and only if $a = 0$.

Since this condition is captured by the invariant polynomial $H_9 = -576a^2c^2d^2$ we arrive at the configuration *Config. 3.73* if $H_9 \neq 0$ and *Config. 3.74* if $H_9 = 0$.

1.2.2.1.2. *The subcase $\mu_2 = 0$.* Since $d \neq 0$ (due to $H_7 \neq 0$) we obtain $c = 0$ and this leads to the systems

$$\dot{x} = a + dy, \quad \dot{y} = b - xy,$$

for which we have

$$B_2 = 0, \quad B_3 = -3bx^4, \quad H_7 = -4d \neq 0, \quad \mu_0 = \mu_1 = \mu_2 = 0, \quad \mu_3 = adxy^2.$$

For these systems we calculate $\mathcal{H}(\mathcal{X},\mathcal{Y},\mathcal{Z}) = Z^2$ and by Lemma 6 we deduce that the infinite invariant line is triple, i.e. we have invariant lines of total multiplicity 3. It is clear that we remain in this class due to the condition $B_3 \neq 0$.

It $\mu_3 = adxy^2 \neq 0$ then by Lemma 3 we deduce that two finite singular points have gone to infinity and coalesced with the infinite singularity $N_1[1:0:0]$ producing a triple point of the multiplicity $(1,2)$. At the same time one finite singularity has coalesced with $N_2[0:1:0]$ and we obtain a triple infinite singularity of multiplicity $(2,1)$. As a result we obtain the configuration *Config. 3.75*.

Assume now $\mu_3 = 0$. Then due to $H_7 \neq 0$ (i.e. $d \neq 0$) we get $a = 0$ and hence we arrive at the systems

$$\dot{x} = dy, \quad \dot{y} = b - xy,$$

for which we have

$$B_2 = 0, \ B_3 = -3bx^4 \neq 0, \ H_7 = -4d \neq 0, \ \mu_0 = \mu_1 = \mu_2 = \mu_3 = 0, \ \mu_4 = -bd^2xy^3.$$

We observe that $\mu_4 = -bd^2x \neq 0$ (due to $B_3H_7 \neq 0$) and therefore according to Lemma 3 in the same manner as it was described above these systems possess at infinity the singularities $N_1[1:0:0]$ and $N_2[0:1:0]$ of multiplicities $(2,1)$ and $(1,3)$, respectively. In this case we obtain the configuration *Config. 3.76*.

1.2.2.2. *The case $\widetilde{N} = 0$.* In this case $g^2 - 1 \neq 0$ and since for systems (4) with $h = 0$ we have $\widetilde{K} = 2g(g-1)x^2$ we consider two subcases: $\widetilde{K} \neq 0$ and $\widetilde{K} = 0$.

1.2.2.2.1. *The subcase $\widetilde{K} \neq 0$.* Then $g \neq 1$ and the condition $\widetilde{N} = 0$ gives $g = -1$. Then we may assume in systems (4) $e = f = 0$ and we arrive at the systems

$$\dot{x} = a + cx + dy - x^2, \quad \dot{y} = b - 2xy,$$

for which we have Coefficient$[B_2(\boldsymbol{a}, x, y), y^4] = -648d^4y^4$ and hence the condition $B_2 = 0$ implies $d = 0$. However in this case we obtain two parallel invariant lines $a + cx - x^2 = 0$ and this class of systems is already investigated in [5].

1.2.2.2.2. *The subcase* $\widetilde{K} = 0$. Then the condition $\widetilde{N} = 0$ gives $g = 1$ and we may assume $c = 0$ in systems (4) with $h = 0$ and $g = 1$. This leads to the family of systems

$$\dot{x} = a + dy + x^2, \quad \dot{y} = b + ex + fy,$$

for which we have $B_2 = -648d^4y^4$. Therefore the condition $B_2 = 0$ yields $d = 0$ giving two invariant affine lines $x^2 + a = 0$. So we get two parallel invariant lines and we conclude that in this case we also could not have new configurations.

2. **The case** $\widetilde{M} = 0$**.** According to Lemma 4 a quadratic system in this class could be brought via a linear transformation to the canonical form $(\mathbf{S}_{IV})$, i.e. we have to examine the family of systems

$$\begin{aligned} \dot{x} =& a + cx + dy + gx^2 + hxy, \\ \dot{y} =& b + ex + fy - x^2 + gxy + hy^2, \end{aligned} \tag{10}$$

for which calculations yield:

$$\theta = 8h^3, \quad \mu_0 = -h^3, \quad C_2 = x^3.$$

Since $C_2 = x^3$ we conclude that these systems possess only one infinite singularity $N_1[0:1:0]$ which is triple. We discuss two subcases: $\theta \neq 0$ and $\theta = 0$.

2.1. **The subcase** $\theta \neq 0$**.** Then $h \neq 0$ and we may assume $c = d = 0$ due to a translation. So we obtain the systems

$$\begin{aligned} \dot{x} =& a + gx^2 + hxy, \\ \dot{y} =& b + ex + fy - x^2 + gxy + hy^2, \end{aligned}$$

for which we calculate Coefficient$[B_2, y^4] = -3888a^2h^4x^2y^2$ and therefore the condition $B_2 = 0$ implies $a = 0$ due to $h \neq 0$. In this case we obtain $B_2 = -648b^2h^4x^4 = 0$ which implies $b = 0$ and we get the systems

$$\dot{x} = x(gx + hy), \quad \dot{y} = ex + fy - x^2 + gxy + hy^2. \tag{11}$$

For these systems following Notation 2 we calculate $\mathcal{H}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) = X^2$, i.e. by Lemma 6 the invariant line $x = 0$ of systems (11) has the multiplicity 2.

On the other hand due to $\theta \neq 0$ (i.e. $h \neq 0$) the above systems possess the following three finite singularities:

$$M_1(0, 0), \quad M_2(0, -f/h), \quad M_3((eh - fg)/h, g(fg - eh)/h^2).$$

It is clear that M_1 is double because the first equation of systems (11) does not have linear terms (nor constant one).

For systems (11) we have $B_2 = 0$ and by Lemma 8 in order to remain in the class $\mathbf{QSL}_3$ the condition $B_3 = -3f(fg - eh)x^4 \neq 0$ is necessary.

Then considering the information pointed out above about the multiplicity of finite and infinite singularities of systems (11) we arrive at the configuration *Config. 3.77*.

2.2. **The subcase $\theta = 0$.** This condition gives $h = 0$ and then for systems (10) we have

$$\text{Coefficient}[B_2, x^2y^2] = -3888d^4g^2, \quad \widetilde{N} = g^2x^2.$$

We observe that in the case $\widetilde{N} \neq 0$ (i.e. $g \neq 0$) the condition $B_2 = 0$ implies $d = 0$ and then systems (10) possess two parallel invariant lines $gx^2+cx+a = 0$. Since this family of systems was already investigated we have to impose the condition $\widetilde{N} = 0$ which yields $g = 0$. However in this case we get $B_2 = -648d^4x^4 = 0$, i.e. $d = 0$ and again we conclude that no new configurations could be obtained in this case.

Thus in the case $\widetilde{M} = 0$ and $B_2 = 0$ we have exactly one new configuration *Config. 3.77* and for this it is necessary $\theta \neq 0$.

($\boldsymbol{\beta}$) It was mentioned earlier (see page 66) that in this case for a quadratic system apart from the condition $B_2 = 0$ the condition $\widetilde{K} = 0$ has to be satisfied. According to Lemma 7 the infinite invariant line is of multiplicity at least two. This case contains two possibilities: either $Z = 0$ is double and we have an additional invariant affine line or $Z = 0$ is triple. Clearly in both cases we are in the class $\mathbf{QSL}_3$.

In the previous case **($\boldsymbol{\alpha}$)** when $\eta = 0$ we have examined all the possibilities when the invariant line $Z = 0$ is either simple or double or triple. So we have to investigate the cases $\eta < 0$ and $\eta > 0$ when in addition we have the multiple invariant line at infinity.

1. **The case $\eta < 0$.** We prove the following lemma.

Lemma 10. *If for a quadratic system the conditions $\eta < 0$ and $B_2 = \widetilde{K}(a, x, y) = 0$ hold, then this system possesses invariant lines of total multiplicity at least 4.*

Proof. Assume that for a quadratic system the condition $\eta < 0$ holds. Then according to Lemma 4 a quadratic system in this class could be brought via a linear transformation to the canonical form $(\mathbf{S}_{II})$, i.e. we have to examine the family of systems

$$\begin{aligned}\dot{x} =& a + cx + dy + gx^2 + (h+1)xy,\\ \dot{y} =& b + ex + fy - x^2 + gxy + hy^2,\end{aligned}$$

for which calculations yield:

$$C_2 = x(x^2 + y^2), \quad \widetilde{K} = 2(1 + g^2 + h)x^2 + 4ghxy + 2h(1 + h)y^2.$$

Evidently the condition $\widetilde{K} = 0$ is equivalent to $g = 0$ and $h = -1$ and therefore applying an additional translation which gives $e = f = 0$ we get the family of systems

$$\dot{x} = a + cx + dy, \quad \dot{y} = b - x^2 - y^2.$$

For these systems we have

$$B_2 = -648\big[16a^2 + (c^2 + d^2 - 4b)^2\big]x^4 = 0 \ \Rightarrow\ a = 0,\ b = (c^2 + d^2)/4$$

and we arrive at the systems

$$\dot{x} = cx + dy, \quad \dot{y} = (c^2 + d^2)/4 - x^2 - y^2$$

which possess the double invariant line $Z = 0$ and two complex invariant affine lines

$$(c \pm id \mp 2ix + 2y) = 0.$$

So the above systems have invariant lines of total multiplicity at least four and this completes the proof of Lemma 10. □

2. **The case** $\eta > 0$**.** By Lemma 4 a quadratic system in this class could be brought via a linear transformation to the canonical form $(\mathbf{S}_I)$, i.e. we have to examine the family of systems

$$\begin{aligned}\dot{x} =& a + cx + dy + gx^2 + (h-1)xy,\\ \dot{y} =& b + ex + fy + (g-1)xy + hy^2,\end{aligned} \tag{12}$$

for which we calculate:

$$C_2 = xy(x - y), \quad \widetilde{K} = 2g(g-1)x^2 + 4ghxy + 2h(h-1)y^2.$$

Therefore the condition $\widetilde{K} = 0$ implies $gh = g(g-1) = h(h-1) = 0$. Evidently we can assume $g = 0$, otherwise we apply the change

$$(x, y, a, b, c, d, e, f, g, h) \mapsto (y, x, b, a, f, e, d, c, h, g) \tag{13}$$

which conserves systems (12). In this case we have either $g = h = 0$ or $g = 0$ and $h = 1$. We claim that the second case can be reduced to the first one via a transformation. Indeed assuming $g = h = 0$ we get the family of systems

$$\dot{x} = a + cx + dy - xy, \quad \dot{y} = b + ex + fy - xy, \tag{14}$$

whereas for $g = 0$ in the case $h = 1$ we arrive at the systems

$$\dot{x} = a + cx + dy, \quad \dot{y} = b + ex + fy - xy + y^2. \tag{15}$$

Then applying the linear transformation $x_1 = y$, $y_1 = y - x$ to systems (15) we arrive at the family of systems

$$\dot{x}_1 = a' + c'x_1 + d'y_1 - x_1y_1, \quad \dot{y}_1 = b' + e'x_1 + f'y_1 - x_1y_1$$

where

$$a' = b,\ c' = e + f,\ d' = -e,\ b' = b - a,\ e' = e + f - c - d,\ f' = c - e.$$

Comparing the above system with (14) we deduce that our claim is proved.

Thus $g = h = 0$ and for systems (14) we calculate

$$B_2 = -648(a + cd)(b + ef)(x - y)^4 = 0.$$

Due to the change (13) we may assume $b = -ef$. Then we arrive at the systems

$$\dot{x} = a + cx + dy - xy, \quad \dot{y} = (f - x)(y - e), \tag{16}$$

which besides the double infinite invariant line $Z = 0$ possess the invariant affine line $y = e$. We point out that for these systems we have

$$B_3 = 3(a + cd)(x - y)^2 y^2, \quad H_7 = -4(c + d - e - f)$$

and by Lemma 8 the condition $B_3 \neq 0$ has to be satisfied. We claim that in order to be in the class $\mathbf{QSL}_3$ we must force also the condition $H_7 \neq 0$ to be fulfilled. Indeed supposing $H_7 = 0$ we obtain $f = c + d - e$ and we arrive at the family of systems

$$\dot{x} = a + cx + dy - xy, \quad \dot{y} = (c + d - e - x)(y - e)$$

possessing the following two invariant affine lines:

$$y = e, \quad a + ce + de - e^2 + (c - e)(x - y) = 0,$$

i.e. the above systems belong to the class $\mathbf{QSL}_{\geq 4}$ and this completes the proof of our claim.

Next we examine configurations of invariant lines for the family of systems (16) in the case $B_3 H_7 \neq 0$. We determine that these systems possess the singular points $M_i(x_i, y_i)$ with the coordinates:

$$x_1 = f, \ \ y_1 = -\frac{a + cf}{d - f}; \quad x_2 = -\frac{a + de}{c - e}, \ \ y_2 = e$$

and evidently these finite singularities exist if and only if $(c-e)(d-f) \neq 0$. Moreover the singularity M_2 is located on the invariant line $y = e$ of systems (16).

On the other hand for systems (16) we calculate

$$\mu_0 = \mu_1 = 0, \ \ \mu_2 = -(c-e)(d-f)xy, \ \ H_9 = -576(c-e)^2(d-f)^2(a+de+cf-ef)^2$$

and hence if $\mu_2 \neq 0$ we have two finite singularities M_1 and M_2, where M_2 is located on the invariant line $y = e$. Moreover we observe that in the case $a+de+cf-ef = 0$ the singular point M_1 coalesced with M_2 giving the double singularity $M_{1,2}(f, e)$ on the invariant line $y = e$. We examine two possibilities: $\mu_2 \neq 0$ and $\mu_2 = 0$.

2.1. *The possibility* $\mu_2 \neq 0$. In this case the condition $H_9 = 0$ is equivalent to $a + de + cf - ef = 0$. Then taking into account the factorization of the invariant polynomial $\mu_2(x, y)$ by Lemma 3 we obtain that at infinity both the singular points $N_1[1 : 0 : 0]$ and $N_2[0 : 1 : 0]$ are double of the type $(1, 1)$. Therefore we arrive at the configuration *Config. 3.78* if $H_9 \neq 0$ and at *Config. 3.79* if $H_9 = 0$.

2.2. *The possibility* $\mu_2 = 0$. This condition implies $(c-e)(d-f) = 0$ and since we have

$$\mu_3 = -(c-e)(a-de+cf+ef)x^2y + (d-f)(a+de-cf+ef)xy^2$$

by Lemma 3 we deduce that for $d = f$ the finite singularity M_1 coalesced with infinite singular point $N_1[1:0:0]$ which becomes of the multiplicity $(1,2)$. This leads to the configuration *Config. 3.80*.

In the case $c = e$ the finite singularity M_2 coalesced with infinite singular point $N_1[0:1:0]$ located at the "end" of the invariant affine line $y = e$ and we get the configuration *Config. 3.81*.

On the other hand for systems (16) we have

$$H_{17} = 9(a+cd)(c-e)^2$$

and therefore in the case $\mu_2 = 0$ (i.e. $(c-e)(d-f) = 0$) we get $d = f$ if $H_{17} \neq 0$ (*Config. 3.80*) and we obtain $c = e$ if $H_{17} = 0$ (*Config. 3.81*).

We point out that we could not have simultaneously $d = f$ and $c = e$ because in this case we get $H_7 = -4(c+d-e-f) = 0$ and this contradicts our assumption $H_7 \neq 0$.

On the other hand in the case $d = f$ we obtain

$$\mu_3 = (e-c)(a+cf)x^2y, \quad B_3 = 3(a+cf)(x-y)^2y^2, \quad H_7 = 4(e-c)$$

whereas for $c = e$ we have

$$\mu_3 = (d-f)(a+de)xy^2, \quad B_3 = 3(a+de)(x-y)^2y^2, \quad H_7 = -4(d-f).$$

We observe that in both cases the condition $B_3H_7 \neq 0$ implies $\mu_3 \neq 0$ and therefore we could not have other new configurations in the case under consideration. □

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Counting configurations of limit cycles and centers

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Abstract. We present several results on the determination of the number and distribution of limit cycles or centers for planar systems of differential equations. In most cases, the study of a recurrence is one of the key points of our approach. These results include the counting of the number of configurations of stabilities of nested limit cycles, the study of the number of different configurations of a given number of limit cycles, the proof of some quadratic lower bounds for Hilbert numbers and some questions about the number of centers for planar polynomial vector fields.

Mathematics subject classification: 34C05, 34C07, 37C27.
Keywords and phrases: Limit cycle; Configuration; Center; Phase portrait; Recurrence; Fibonacci numbers.

Dedicated to the memory of Professor Constantin Sibirschi

1 Introduction

In the qualitative study of differential systems in the plane, there are several questions about topological configurations that naturally lead to enumeration and combinatorial problems which, sometimes, can be approached by using recurrences.

This paper is devoted to study such type of questions for smooth planar differential systems,

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y). \tag{1}$$

In some parts of the work, the functions P and Q defining these differential equations will also be assumed to be polynomials.

After giving some definitions in Section 2, in Section 3 we will consider the growth of the number of stability configurations of n nested limit cycles of (1) in terms of their stability. We will show that this number of configurations can be explicitly determined using expressions involving the Fibonacci numbers, $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Although it may not be necessary to highlight the importance and ubiquity of the Fibonacci sequence both in arithmetic and geometry problems, and in some applied questions, we point out a couple of examples: their use in graph theory ([33]) and their earliest appearance in Indian mathematics to count the number of sums of 1 and 2 (taking into account the order of the addends) that add up to n. For instance, if $n = 4$, there appear $F_{n+1} = F_{4+1} = 5$ possibilities:

$$1 + 1 + 1 + 1, \ 2 + 1 + 1, \ 1 + 2 + 1, \ 1 + 1 + 2, \ 2 + 2.$$

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Our second result deals with planar systems having exactly n limit cycles. In Section 4 we study the number of different configurations that these limit cycles can exhibit regarding only its topological distribution. In Figure 1, we show a colored illustration of the 20 different configurations of 5 limit cycles.

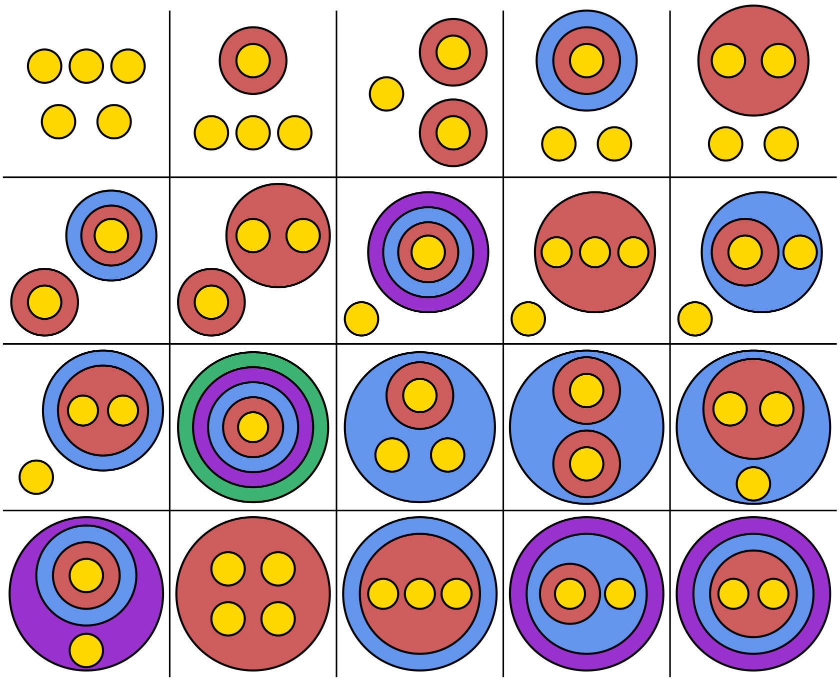

Figure 1. The 20 configurations in the case $n = 5$. The colors have a purely aesthetic function and mean nothing to the dynamics.

In our classification we do not take into account dynamical considerations such as the classification of the corresponding phase portraits. Notice that this point of view is totally different of that of Section 3 where the stability of the different periodic orbits is the key point to distinguish the different stability configurations. This problem can be seen as the pure geometrical problem of counting the number of ways of arranging n non-overlapping circles, which are in bijection with the different cases of unlabeled rooted trees with $n + 1$ nodes, [37]. The problem of finding this last number was studied by A. Cayley in his 1875's paper, [7], where a recurrence relation to compute this number is presented (see also [5, p. 43] or [15, p. 71]). In Proposition 3, we will give another recurrent expression. Our approach is self-contained and, to our knowledge, novel in the context of the study of limit cycles.

These two sections have no relation with existing literature where the number of possible phase portraits of some families of planar polynomial differential equations, modulo topological conjugacy (see for instance [3]) or modulo the existence of limit cycles (see for instance [14]), is studied in detail.

Section 5 deals with Hilbert numbers for polynomial systems. Recall that when P and Q are arbitrary polynomials of degree at most n, the *Hilbert number* $\mathcal{H}(n)$ is the maximum number of limit cycles that these differential equations (1) can have, or infinity if there is no upper bound for the number of limit cycles. It is well-known that $\mathcal{H}(0) = \mathcal{H}(1) = 0$ and $\mathcal{H}(2) \geq 4$, but even for $n = 2$ it is not known whether $\mathcal{H}(2)$ is finite. Following [17], we present a very simple proof that the Hilbert numbers increase at least quadratically with n; the proof is based on the ideas of [9] and uses a very basic background on ordinary differential equations (ODEs) and recurrences.

Finally, in Section 6 we collect several known results about the maximum number of centers of planar polynomials systems of degree n, see also [16]. Special attention is paid to Hamiltonian and holomorphic centers, following [2, 10, 11, 35].

2 Some definitions

In the theory of planar differential systems, *a limit cycle* is a periodic orbit such that at least there is another trajectory that either spirals into it or from it as time approaches infinity. In this paper we will deal with analytic systems, hence a limit cycle is an isolated periodic orbit. Notice that even for $\mathcal{C}^\infty$-systems limit cycles can be non-isolated periodic orbits.

Consider an analytic planar differential system. Let γ be a limit cycle of this differential system. We will say that it is *stable*, and denote it as γ^s, if it is locally asymptotically stable. We say that it is *unstable*, and denote it by γ^u, if either it is a repelle or semistable (in this last case, either the nearby external paths converge to it and the nearby internal ones diverge from it, or the reciprocal situation is given).

We introduce some more preliminary definitions. Let c be a closed curve embedded in $\mathbb{R}^2$. A *configuration of cycles* is a finite set $\mathcal{C} = \{c_1, ..., c_n\}$ of disjoint simple closed curves. Following [29, 31] we say that two configurations of cycles $\mathcal{C}$ and $\mathcal{C}'$ are *equivalent* if there exists a homeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\varphi(\mathcal{C}) = \mathcal{C}'$. Clearly the different configurations are characterized by the topology of the set $\mathbb{R}^2 \setminus \mathcal{C}$. It is also known that any configuration of cycles is homeomorphic to a set of disjoint circles, see again [29].

When the closed curves defining a configuration correspond to the limit cycles of a differential equation (1) we will say that it is a *limit cycles configuration.* If the stability of the limit cycles is also taken into account we will say that it is a *limit cycles stability configuration.*

If γ is a limit cycle, we will denote by C_γ the bounded open set with boundary γ. Consider now a differential equation (1) with finitely many periodic orbits (so, all them are limit cycles). We can introduce a natural partial order in the set $\mathcal{P}$ formed by the union of all limit cycles by defining $\gamma' \prec \gamma$ whenever $\gamma' \subset C_\gamma$ given $\gamma', \gamma \in \mathcal{P}$. With this order, $(\mathcal{P}, \prec)$ is a *partially ordered set (poset)* and the following concepts can be introduced:

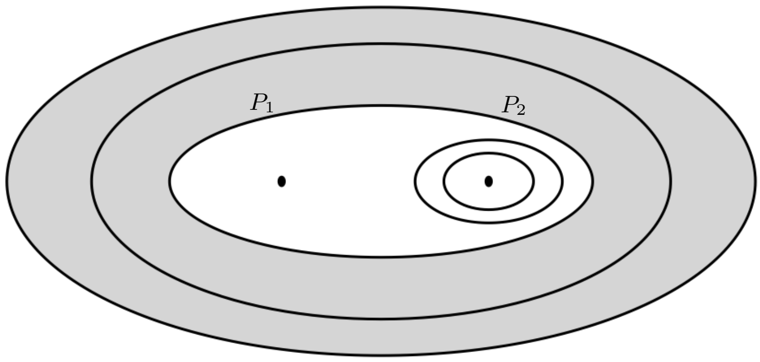

Figure 2. Two different nests of limit cycles. One of level 2, surrounding the singular point P_2, and another one, included in the shaded region, of level 3 surrounding the two singular points (and eventually others located in the white region). The level of the maximal limit cycle is 5.

- A maximal element γ of the poset $(\mathcal{P}, \prec)$ will be called *maximal limit cycle*. In this case, C_γ will be called the *domain* of γ and the number of cycles γ' such that $\gamma' \subset \overline{C_\gamma}$, the *level* of γ.
- We say that a subset $\mathcal{N} \subseteq \mathcal{P}$ is a *nest* if all periodic orbits $\gamma \in \mathcal{N}$ are such that all the corresponding C_γ sets contain the same singular points and moreover it is the biggest set in $\mathcal{P}$ with this property. The *level* of the nest $\mathcal{N}$ is its number of elements.

In Figure 2, we show a configuration of limit cycles forming two nests.

We also will use a common function in number theory, see [32]. The *partition function* $p : \mathbb{N} \longrightarrow \mathbb{N}$ assigns to each natural number n the number of combinations of positive natural numbers that add up to n. We define $p(0) = 1$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$p(n)$	1	2	3	5	7	11	15	22	30	42	56	77	101	135

Table 1. Some values of the partition function $p(n)$.

The partition function grows very fast with n; for example $p(50) = 204\,226$, $p(100) = 190\,569\,292, p(1000) \approx 2.4 \times 10^{31}$, $p(10000) \approx 3.6 \times 10^{106}, \ldots$ In fact, Hardy and Ramanujan, and, independently, Uspensky proved that

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\sqrt{\pi\frac{2n}{3}}\right) \quad \text{as} \quad n \sim \infty.$$

In Niven and Zuckerman's book [32] some recurring expressions for the calculation of $p(n)$ are shown. We highlight, for instance,

$$\begin{aligned} p(n) &= \sum_{j\in\mathbb{Z}\setminus\{0\}} (-1)^{j+1} p\Big(n - \frac{1}{2}(3j^2 - j)\Big) \\ &= p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + \cdots, \end{aligned}$$

where $p(k)$ is taken to be zero when it is evaluated at negative values.

Finally, we also will use *combinations with repetition.* A combination with repetition of k objects taken from a set with n elements is a way of selecting an object in the set, k times in succession, without taking into account the order of the k choices and "with replacement", so the same object can be selected several times. It is well known that there are $\binom{n+k-1}{k}$ combinations with repetition, where $\binom{m}{q}$ is the usual combinatorial number.

3 Fibonacci numbers and limit cycles

In this section we will show how Fibonacci numbers appear when counting the limit cycles stability configurations. Note that since the repelling limit cycles and the semistable ones are both considered unstable, we will not be counting the number of phase portraits when studying the stability configurations.

It is well known (in fact, it is an exercise that appears in some textbooks) that the number of stability configurations of nested limit cycles satisfies the Fibonacci sequence.

Lemma 1. *Let c_n denote the number of stability configurations of a nest of n limit cycles for an analytic planar system* (1). *Then*

$$c_n = c_{n-1} + c_{n-2}, \ \textit{with } c_0 = 1 \textit{ and } c_1 = 2. \tag{2}$$

Hence, $c_n = F_{n+2}$, being F_i the i-th Fibonacci number.

Proof. Suppose that we have n nested limit cycles $\{\gamma_1, \ldots, \gamma_n\}$ forming a nest $\mathcal{N} \subseteq C_{\gamma_n}$ with level n. We consider that $\gamma_1 \prec \gamma_2 \prec \cdots \prec \gamma_n$. These limit cycles can be stable or unstable. We write any stability configuration as $\gamma_1^{k_1} \prec \gamma_2^{k_2} \prec \cdots \prec \gamma_n^{k_n}$ where $k_i \in \{s, u\}$ (below, we also use this notation to indicate the ordering of partial configurations of 2 or 3 cycles).

Observe that if γ_n is unstable then γ_{n-1} can be either stable or unstable, so the following two partial configurations are possible

$$\gamma_{n-1}^u \prec \gamma_n^u \quad \text{and} \quad \gamma_{n-1}^s \prec \gamma_n^u.$$

As a consequence there are as many stability configurations with γ_n^u as stability configurations of $n-1$ limit cycles.

On the contrary, if γ_n is stable then γ_{n-1} can only be unstable, hence we only have the partial stability configuration $\gamma_{n-1}^u \prec \gamma_n^s$. By using the preceding argument we have that the only possible configurations are

$$\gamma_{n-2}^u \prec \gamma_{n-1}^u \prec \gamma_n^s \quad \text{and} \quad \gamma_{n-2}^s \prec \gamma_{n-1}^u \prec \gamma_n^s,$$

so there are as many configurations with γ_n^s as configurations of $n-2$ limit cycles, which proves the relation (2). The initial conditions are also clear: there is only one way to have no limit cycles, so $c_0 = 1$, and, if a unique limit cycle exists, then it can be either stable or unstable and so $c_1 = 2$. □

We consider now the number of possible stability configurations of two nests of limit cycles each of them surrounding only one of the two singular points P_1 and P_2 with index $+1$ (recall that any limit cycle of a smooth system surrounds at least one singular point). In this case, there can be a nest with i limit cycles surrounding P_1 and another one with j limit cycles surrounding P_2. We denote this class of stability configurations by (i, j). Observe that if there are only two singular points with index $+1$, there cannot exist a nest enveloping both points because, otherwise, there would exist an invariant disk containing only the two fixed points so that the sum of indices would be 2, which would contradict the Poincaré-Hopf theorem, see [13].

Because of the definition of configuration, we identify, and only count as one, the symmetric configurations in the (i, j) and the (j, i) classes, and we will write $(i, j) \sim (j, i)$. More explicitly, consider a case in the class (i, j) such that the stability configuration of the nest surrounding P_1 is $\gamma_1^{k_1} \prec \gamma_2^{k_2} \prec \cdots \prec \gamma_i^{k_i}$ and the stability configuration of the nest surrounding P_2 is $\bar{\gamma}_1^{\ell_1} \prec \bar{\gamma}_2^{\ell_2} \prec \cdots \prec \bar{\gamma}_j^{\ell_j}$, where $k_m, \ell_m \in \{s, u\}$. For reasons of economy, we will write this configuration as

$$(k_1, k_2, \ldots, k_i; \ell_1, \ell_2, \ldots, \ell_j). \tag{3}$$

We will only count as one case the symmetric configurations (3) and

$$(\ell_1, \ell_2, \ldots, \ell_j; k_1, k_2, \ldots, k_i),$$

which belong to the classes (i, j) and (j, i) respectively.

By using (2) we get the following result:

Proposition 1. *The number of stability configurations of n limit cycles surrounding only two singular points of index $+1$ for an analytic planar system* (1) *is*

$$c_{n,2}^* = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} F_{i+2} F_{n-i+2} + \delta(n) \binom{F_{\lfloor n/2 \rfloor + 2} + 1}{2}, \tag{4}$$

where $\delta(n) = (1 + (-1)^n)/2$ and $\lfloor\ \rfloor$ is the floor function.

In Table 2, we present some of the values of $c_{n,2}^*$:

n	1	2	3	4	5	6	7	8	9	10	11	12
$c^*_{n,2}$	2	6	11	24	44	86	155	287	510	916	1608	2833

Table 2. Some values of the number of stability configurations of limit cycles surrounding only two points of index +1.

Proof of Proposition 1. It is easy to see that, by Lemma 1, the number of configurations in the classes of the form $(i, n-i)$ with $i \neq n-i$, taking into account that $(i, n-i) \sim (n-i, i)$, is

$$\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} c_i c_{n-i} = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} F_{i+2} F_{n-i+2}.$$

If n is odd, there are no other cases and Equation (4) holds.

If n is even, we must add the configurations that belong to the class $(n/2, n/2)$. In this case, it is easy to see that if we did not take into account the identification of symmetric cases we would have $c^2_{n/2}$ configurations. But, taking into account the equivalence relation of symmetric cases, and by Lemma 1, we have only

$$\binom{c_{n/2}+1}{2} = \binom{F_{\lfloor n/2 \rfloor+2}+1}{2}$$

configurations, which is precisely the number of combinations with repetition of $c_{n/2}$ elements taken in groups of 2 elements.

Notice that, even though the term $F_{\lfloor n/2 \rfloor+2}$ in the formula (4) makes sense for n odd, it does not apply in this case because $\delta(n) = 0$. □

Now we consider the case in which, apart from the nests surrounding P_1 and P_2 with levels i and j, there is a third (outer) nest consisting of k cycles surrounding both P_1 and P_2. In this case we say that the system is of class $(i, j; k)$. For instance, the class of the configuration shown in Figure 2 is $(0, 2; 3)$. Again we must identify the symmetric cases in the classes $(i, j; k)$ and $(j, i; k)$. Observe that, from the Poincaré-Hopf Theorem, if $k > 0$ there must be other singular points enveloped by the limit cycles of the outer nest. Moreover, if there are finitely many singular points inside the outer nest, the total sum of their indices must be +1 or, equivalently, the sum of the indices out of the nests P_1 and P_2 must be −1.

Proposition 2. *The number of stability configurations of n limit cycles, with configuration $(i, j; k)$ such that $i + j + k = n$, is*

$$c_{n,2} = \sum_{k=0}^{n} \left(\sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} F_{i+2} F_{k-i+2} + \delta(k) \binom{F_{\lfloor k/2 \rfloor+2}+1}{2} \right) F_{n-k+2}.$$

n	1	2	3	4	5	6	7	8	9	10	11
$c_{n,2}$	4	13	34	82	184	396	821	1659	3277	6362	12163

Table 3. Some values of the number of stability configurations of limit cycles surrounding only two singular points of index $+1$ and some other singular points with total sum of their indices equal to -1.

Some of the values of $c_{n,2}$ are given in Table 3.

Proof of Proposition 2. If we have n limit cycles, we must study the stability configurations of all the classes of the form $(i, k-i; n-k)$ for each $k = 0, \dots, n$ and $i = 0, \dots, k$, taking into account that $(i, k-i; n-k) \sim (k-i, i; n-k)$.

By Proposition 1, for each $k \in \{0, \dots, n\}$, the contribution of the nests that surround P_1 and P_2 to the number of stability configurations is $c^*_{k,2}$. On the other hand, by using Lemma 1, the contribution of the outer nest is $c_{n-k} = F_{n-k+2}$. Therefore the number of configurations for each k is $c^*_{k,2} c_{n-k}$. Adding all the cases we get

$$c_{n,2} = \sum_{k=0}^{n} c^*_{k,2} c_{n-k},$$

so the result follows. □

Obviously, the stability configurations in other nest arrangements can be treated analogously.

4 Number of configurations of ODEs with n limit cycles

In contrast with Section 3, where the stability of the limit cycles plays a key role, in this section we study the maximum number of configurations of limit cycles regarding only its topological distribution. We consider a planar system with exactly n limit cycles. As mentioned in the introduction, the different configurations are in bijection with the different cases of unlabeled rooted trees with $n+1$ nodes which were studied by Cayley in 1875. Our approach is independent of those we have found in the literature and self-contained. The results that we present are extracted from [21].

We will use some of the notations introduced in Section 2. In particular, recall that if γ is a periodic orbit of our planar system, C_γ denotes the bounded open set with boundary γ. If γ is maximal, C_γ is the domain of γ and the number of limit cycles in $\overline{C_\gamma}$ is the level of γ.

We will first address a simple question such as the number of maximal limit cycles configurations (see thicker cycles in Figures 3 and 4). We can state our first result in terms of the partition function introduced in Section 2.

Lemma 2. *Consider a system of differential equations in the plane with exactly n limit cycles. Then, the maximal limit cycles of this system can be distributed in $p(n)$ different ways.*

Proof. The level of every maximal cycle γ is a positive integer given by the number of limit cycles contained in $\overline{C_\gamma}$. Since the domains of the maximal limit cycles are disjoint, the total number of ways to arrange them in order to have exactly n limit cycles coincides with the number of ways to get the sum of their levels to be n, i.e. $p(n)$. □

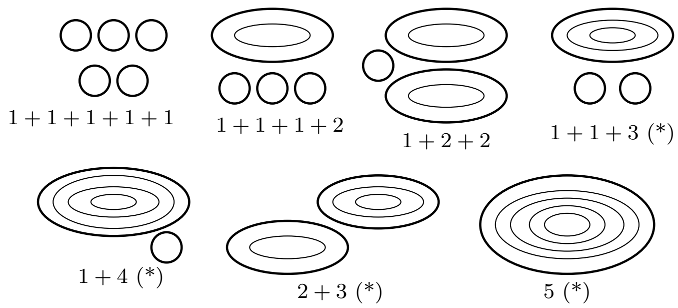

Figure 3. The $p(5) = 7$ different distributions of maximal limit cycles for $n = 5$. Thicker limit cycles are the maximal ones. Maximal limit cycle configurations with (*) give rise to different limit cycle configurations, see Figure 4.

A configuration of maximal limit cycles with α_j domains of level q_j, $j = 1, \ldots, s$ and $q_k \neq q_l$ if $k \neq l$, will be denoted by $(q_1^{\alpha_1}, q_2^{\alpha_2}, \ldots, q_s^{\alpha_s})$. Obviously, for a differential system with the above configuration and exactly n limit cycles it holds that $n = \alpha_1 q_1 + \cdots + \alpha_s q_s$. Notice that, given n there are $p(n)$ different ways of writing n as

$$n = \alpha_{1,k} q_{1,k} + \cdots + \alpha_{s_k,k} q_{s_k,k}, \quad k = 1, 2, \ldots, p(n),$$

with $q_{u,v} \geq 1$ and $\alpha_{u,v} \geq 1$ integer numbers.

Proposition 3. *Consider a system of differential equations in the plane with exactly n limit cycles. Then the maximum number of configurations of limit cycles is given by the recurrent formula*

$$C(n) = \sum_{k=1}^{p(n)} \prod_{i=1}^{s_k} \binom{C(q_{i,k} - 1) + \alpha_{i,k} - 1}{\alpha_{i,k}}, \tag{5}$$

with $C(0) = 1$ for the sake of notation, and $n = \sum_{i=1}^{s_k} \alpha_{i,k} q_{i,k}$ for each $k = 1, 2, \ldots, p(n)$.

The first values of $C(n)$ are given in Table 4. For instance, in Table 4, the value $C(5) = 20$ is obtained by using Proposition 3, by adding the 3 configurations in

Figure 3 without the (*) symbol with the 17 ones given in Figure 4. The whole list of cases is depicted in Figure 1.

n	1	2	3	4	5	6	7	8	9	10
$C(n)$	1	2	4	9	20	48	115	286	719	1842

Table 4. First values of $C(n)$, the number of different configurations of n limit cycles.

Note that, as a consequence of the fact that the sequence $C(n)$ also counts the number of unlabeled rooted trees with $n+1$ nodes, it is a shifted version of the sequence A000081 in OEIS, [38]. This sequence has been widely studied in connection with other problems (see the comments and references in the above citation). Among the results in the literature, we highlight that the asymptotic expression of this sequence and other properties of its generating function have been studied among others by Pólya in 1937 (see this and other references in [15, p. 72] and [19]).

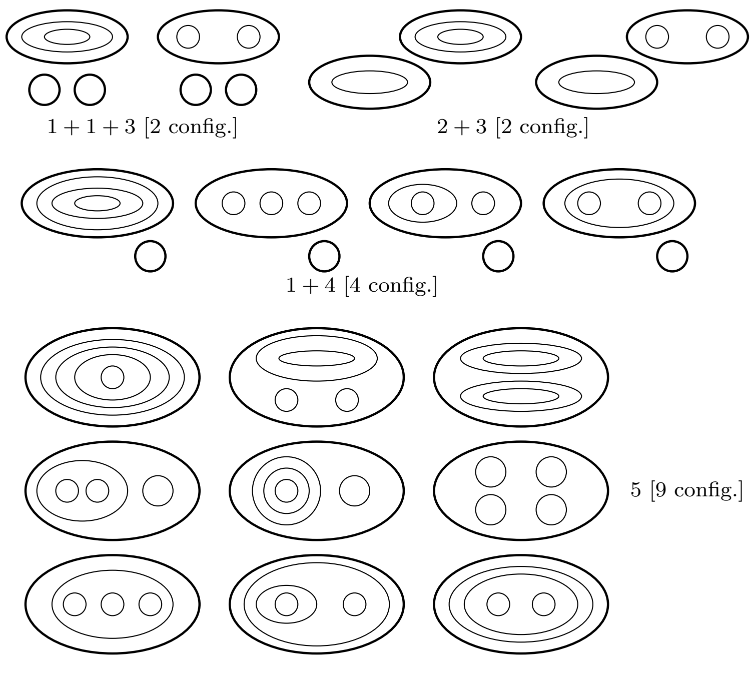

Figure 4. Unfolding of configurations in the case $n=5$ that have an $(*)$ in Figure 3.

Proof of Proposition 3. We will use the notations and results stated in Lemma 2. Let $(q_1^{\alpha_1}, q_2^{\alpha_2}, \ldots, q_s^{\alpha_s})$ be a partition of n. Since the formula of $C(n)$ is recurrent, it suffices to see how to calculate $C(n)$ knowing $C(j)$, $j = 1, \dots, n-1$.

We take a domain with level q. Let γ be its maximal limit cycle. In C_γ there will be exactly $q-1$ limit cycles and hence each domain of level q supports $C(q-1)$ different configurations.

To obtain the total number, we will study how many of them are produced when considering different domains simultaneously. We will distinguish two cases:

(a) Domains have different levels q_A and q_B. In this case, considering them simultaneously we will obtain $C(q_A - 1)C(q_B - 1)$ configurations.

(b) There are α domains of level q. In this case, similarly to Section 3, the total number of configurations cannot be computed simply as $(C(q-1))^\alpha$, since we would count many repetitions. For instance, if we had $\alpha = 2$ nests of level $q = 4$, see Figure 5, then $C(q-1) = C(3) = 4$. Counting all the possible pairs, we would obtain the $(C(3))^2 = 16$ configurations $\{AA, AB, AC, AD, \dots, DD\}$ while 6 of them are repeated. The correct number of configurations turns out to be $\binom{4+1}{2} = \binom{5}{2} = 10$.

In general, this is a purely combinatorial problem and the number of configurations obtained by joining α domains of level q corresponds to the number of combinations with repetition of $C(q-1)$ elements taken in groups of α elements:

$$\binom{C(q-1)+\alpha-1}{\alpha}.$$

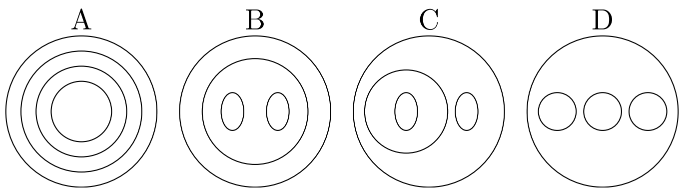

Figure 5. Possible configurations of a domain of level 4 ($q = 4$).

Cases (a) and (b) give us a complete final description: for each partition $(q_1^{\alpha_1}, q_2^{\alpha_2}, \dots, q_s^{\alpha_s})$, the number of configurations will be given by

$$\prod_{i=1}^{s} \binom{C(q_i-1)+\alpha_i-1}{\alpha_i};$$

if we now add all the partitions of n, we arrive to the formula of the statement. $\square$

To end this section we want to make a couple of remarks: (i) any configuration of limit cycles can be realized by a polynomial vector field (with a computable degree), see [29,31]. In [29] the n limit cycles are hyperbolic while in [31] the limit cycles can also be chosen with any given desired stability and multiplicity; (ii) the extended

Bendixson–Dulac Theorem ([8, 18, 30, 41]) gives, under some conditions, an upper bound of the number of limit cycles of a smooth planar differential system (1) and sometimes even the exact number, allowing to ensure that we are under our main hypothesis: *the system* (1) *has exactly a given number of limit cycles.*

5 Lower bounds for Hilbert numbers

Consider now real planar polynomial systems of ordinary differential equations (1) with P and Q polynomials of degree at most n. We are concerned about the maximum number of limit cycles, $\mathcal{H}(n)$. The knowledge of $\mathcal{H}(n)$ is one of the most elusive problems of the famous Hilbert's list and constitutes the second part of Hilbert's sixteenth problem.

As we have already explained, here we prove the existence of quadratic lower bounds for $\mathcal{H}(n)$ that are obtained following the ideas of [9] and by using very simple background on ODEs and recurrences. The results, as they are presented here, are also developed in [16].

Proposition 4. *There exists a sequence of values n_k tending to infinity and a constant $K > 0$ such that $\mathcal{H}(n_k) > Kn_k^2$.*

Proof. The construction of the ODE that gives this lower bound is a recurrent process. Let $X_0 = (P_0, Q_0)$ be a given polynomial vector field of degree n_0 with $c_0 > 0$ limit cycles. Since this number of limit cycles is finite, there exists a compact set containing all of them. Therefore, doing a translation if necessary, we can assume that all of them are in the first quadrant. By simplicity we continue calling X_0 this new translated vector field. From it, we construct a new vector field, by using the (non-invertible) transformation

$$x = u^2, \quad y = v^2.$$

The differential equation associated to X_0 is $\dot{x} = P_0(x, y)$, $\dot{y} = Q_0(x, y)$ and it writes in these new variables as

$$\dot{u} = \frac{P_0(u^2, v^2)}{2u}, \quad \dot{v} = \frac{Q_0(u^2, v^2)}{2v}.$$

By introducing a new time s, defined as $dt/ds = 2uv$, we get that this ODE is transformed into

$$u' = v\, P_0(u^2, v^2), \quad v' = u\, Q_0(u^2, v^2).$$

Since each point lying in the first quadrant (x, y) has four preimages $(\pm\sqrt{x}, \pm\sqrt{y})$, the new ODE has, at each quadrant, a diffeomorphic copy of the positive quadrant of the vector field X_0. Hence this new vector field, which we call X_1, has degree $n_1 = 2n_0 + 1$ and at least $c_1 = 4c_0$ limit cycles. By repeating this process, starting now with X_1 and so on, we get a sequence of vector fields X_k, with respective degrees n_k, having at least c_k limit cycles, where

$$n_{k+1} = 2n_k + 1, \qquad c_{k+1} = 4c_k,$$

see Figure 6.

Solving the above linear difference equation we get that

$$n_k = 2^k(n_0+1) - 1, \qquad c_k = 4^k c_0. \tag{6}$$

Hence, since

$$2^k = \frac{n_k+1}{n_0+1} \quad \text{and} \quad 4^k = \frac{c_k}{c_0},$$

we obtain that

$$\frac{c_k}{c_0} = \left(\frac{n_k+1}{n_0+1}\right)^2 > \frac{1}{(n_0+1)^2}\, n_k^2.$$

Consequently,

$$\mathcal{H}(n) > \frac{c_0}{(n_0+1)^2}\, n^2 \quad \text{and} \quad n = 2^k(n_0+1)-1, \quad k \in \mathbb{N},$$

as we wanted to prove. □

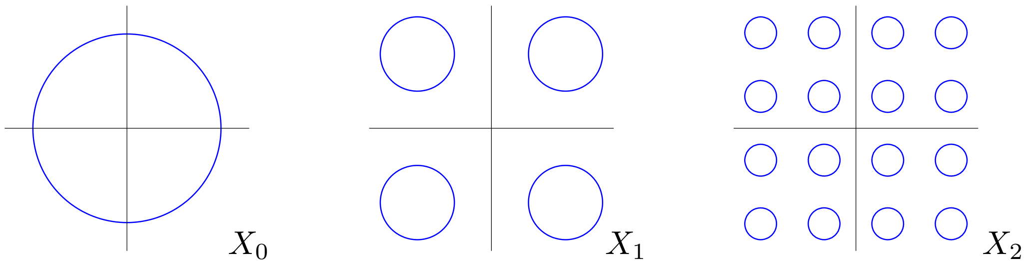

Figure 6. Two steps of the construction of the vector fields X_k, starting from a vector field X_0 with a unique limit cycle ($c_0 = 1$, $n_0 = 3$).

Notice that the proof of Proposition 4 shows that the constant K depends on the seed of the procedure. Let us give some examples:

1. Consider X_0 to be the trivial system of ODEs with one limit cycle that, in polar coordinates, writes as

$$\dot{r} = r(1-r^2), \quad \dot{\theta} = 1.$$

   Clearly, the limit cycle is $r = 1$ and, thus, $c_0 = 1$ and $n_0 = 3$. Hence, $K = 1/16$ and $n_k = 2^{k+2} - 1$.

2. Consider X_0 to be any quadratic system that reveals the lower bound $\mathcal{H}(2) \geq 4$. Then, $c_0 = 4$ and $n_0 = 2$ and, therefore, we get $K = 4/9$ and $n_k = 2^k \cdot 3 - 1$.

n	3	4	5	6	7	8	9	10
h_n	13	28	37	53	74	96	120	142
K	13/16	28/25	37/36	53/49	37/32	32/27	6/5	142/121

Table 5. Lower bounds h_n of $\mathcal{H}(n)$ and the corresponding values of K, for $n \in \{3, \dots, 10\}$.

3. Consider X_0 to be a system attaining any of the best lower bounds h_n of $\mathcal{H}(n)$ known in the literature, for $n \in \{3, \dots, 10\}$, see [22–24, 26, 28, 34, 40]. In the third row of Table 5, we get the values of K obtained by this procedure. We remark that the highest K value obtained is 6/5.

We notice that better bounds can be obtained by using finer arguments. Indeed, it is known that $\mathcal{H}(n) \geq L\, n^2 \log(n)$, which is currently the best general result on by lower bounds for $\mathcal{H}(n)$, see again [9] or [1, 26, 27]. In fact, our proof is totally inspired by that of [9], where the authors find additional limit cycles at each step that appear by perturbing the centers created by the method on the axes $uv = 0$. They obtain that, instead of (6), it holds that

$$n_{k+1} = 2n_k + 1, \quad c_{k+1} = 4c_k + (2^k - 2)^2 + (2^k - 1)^2.$$

By studying these new difference equations they get the improved lower bounds of type $L\, n^2 \log(n)$.

6 Maximum number of centers

In this section, following [17], we collect some known results about the maximum number of centers of planar polynomial systems of degree n and we propose an open problem.

The classification of centers for polynomial differential systems started with the quadratic ones with the works of Dulac, Kapteyn at the beginning of the 20th century. It continued with the works of Bautin [4] during the 50's and Sibirskii [36] in the 60's about the cubic systems with homogeneous nonlinearities, including also the study of the cyclicity of the centers.

Proposition 5. *Let $\mathcal{C}_n$ be the maximum number of centers of polynomial differential systems* (1) *of degree n. Then $\mathcal{C}_1 = 1$ and for $n \geq 2$ it holds that*

$$\Big\lfloor \frac{n^2+1}{2} \Big\rfloor \leq \mathcal{C}_n \leq \frac{n^2+n}{2} - 1.$$

Proof. It is clear that $\mathcal{C}_1 = 1$. Observe that, using Bezout's theorem, a planar polynomial differential system of degree $n > 0$, with finitely many equilibrium points, has at most n^2 equilibrium points. Moreover, at most $(n^2+n)/2$ can have index $+1$, see for instance [12, 25].

Therefore, since all centers have index $+1$, it holds that $\mathcal{C}_n \leq (n^2+n)/2$. The case with infinitely many critical points follows similarly and has a smaller upper bound.

We can refine this upper bound by using the beautiful Euler–Jacobi formula, which asserts, for the planar case, that if a polynomial system $P(x,y) = 0$, $Q(x,y) = 0$, with P and Q having degrees n and m, respectively, has exactly $n\,m$ solutions (hence all them are finite and simple), then

$$\sum_{\{(u,v)\,:\,P(u,v)=Q(u,v)=0\}} \frac{R(u,v)}{\det(\mathrm{D}(P,Q))(u,v)} = 0,$$

for any polynomial $R(x,y)$ of degree smaller than $n+m-2$, see for instance [20]. Here, $\mathrm{D}(P,Q)$ denotes the differential of the map (P,Q). As consequence of the Euler–Jacobi formula, one obtains that, for $n \geq 2$, if all points with index $+1$ lie on a single algebraic curve, then the curve must have degree strictly greater than $n-1$, see [10]. Having into account that all centers lie on the algebraic curve $\operatorname{div}(P,Q) = 0$, which has at most degree $n-1$, we get that $\mathcal{C}_n \leq (n^2+n)/2 - 1$.

On the other hand, in [11] it is also proved that planar polynomial *Hamiltonian* differential systems of degree n have at most $\lfloor (n^2+1)/2 \rfloor$ centers and, hence, $\mathcal{C}_n \geq \lfloor (n^2+1)/2 \rfloor$. Moreover, this upper bound is attained: consider, for instance,

$$\dot{x} = F(y), \quad \dot{y} = -F(x), \quad \text{with} \quad F(u) = \prod_{j=1}^{n} (u-j);$$

for these systems, centers and saddles are placed like white and black squares on a $n \times n$ chessboard. This concludes the proof. □

From Proposition 5, then, we know that $\mathcal{C}_2 = 2$, $\mathcal{C}_3 = 5$, and $8 \leq \mathcal{C}_4 \leq 9$. Therefore, it is natural to consider the following open problem:

> *Determine the maximum number of centers, $\mathcal{C}_n$, for planar polynomial differential systems of degree $n \geq 4$.*

Once the number $\mathcal{C}_n$ is obtained, it is also interesting to know the different possible phase portraits that systems having these maximal number of centers can have, see for instance [6], where the Hamiltonian case when $n = 3$ is studied. One of the reasons is that perturbations of these systems are good candidates to have different configurations of limit cycles.

We end this section with some comments for another family, the planar polynomial holomorphic systems of degree n, for which the full classification of the different phase portraits with the maximum number of centers is known. Recall that these systems are the ones that in complex coordinates $z = x + \mathrm{i}y$ write as $\dot{z} = p_n(z)$, being p_n a complex polynomial of degree n. For them, the maximum number of centers is n and, in this case, also the maximum number $N(n)$ of different topological phase portraits on the Poincaré disc can be computed, see [2, 35]. It turns out that $N(n)$ coincides with the number of labeled projective planar trees with n nodes and is given by the sequence A006082 in the OEIS' web page [39]. In particular, $N(4) = 2$, $N(5) = 3$ and $N(6) = 6$, see the 6 different classes of phase portraits for $n = 6$ in Figure 7. In these phase portraits each point is a center and the connected component where this point lies is filled of periodic orbits surrounding it.

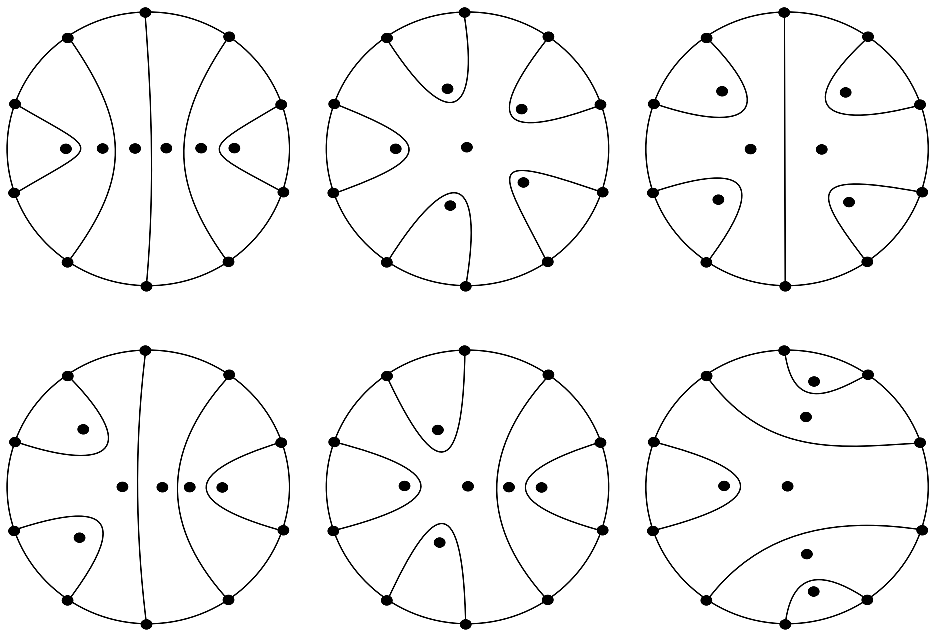

Figure 7. The six different phase portraits with six centers for holomorphic vector fields of degree 6.

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