# ON FUNCTIONAL COMPLETENESS IN THE GR. MOISIL DUAL INTUITIONISTIC LOGIC

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**Abstract:** Gr. Moisil [1] introduced into consideration the Dual Intuitionistic Logic. A.V. Kuznetsov [2] discovered the connection of this logic with the theory of distributive lattices. Dual Intuitionistic Logic [1] (DIL) is obtained from Propositional Intuitionistic Logic by enriching it with dual logic operations to implication and negation. Among the possible approaches of a problem in DIL is natural to study at the beginning the concerned problem for the case of intermediate logics between DIL and classical logic that are defined by finite distributive lattices. The simplest of these is the chain logic of three elements, which represents a specific fragment of trivalent general logic and coincides with the Lukasiewici trivalent logic. In the present paper we formulate four criterions of functional completeness in the chain extensions of Dual Intuitionistic Logic and also in the simplest not chain extensions of this logic.

Keywords: Intuitionistic Logic, Dual Intuitionistic Logic, expressible formula, functional complete system, predicate.

## 1. Introduction. Theoretical aspects

DIL is based on formulas built from symbols for variables $p$, $q$, $r$, possibly indexed by means of three pairs of dual operators: & and V, ¬ and ⌋ (the weak negation), ⊃ and \ (subtraction), and parentheses. Formula $F$V¬$F$ we note by ⊥ $F$.

An $F$ formula is expressible in $L$ logic by Σ system of formulas if $F$ can be obtained from the variables and formulas from Σ with the help of weak rule of substitution and replacement rule of the equivalent in $L$. An Σ system of formulas is called (functional) complete in $L$ logic if all its language formulas are expressible in $L$ by Σ.

Consider the following algebra, explicitly stating its type

$$A = \langle M; \&, \mathrm{V}, \supset, \setminus, \neg, \rfloor \rangle \qquad (1)$$

which is lattice relative to the & and V, the operations ⊃ and ¬ are also relatively pseudo-complement and respectively pseudo-complement, but the operations \ and ⌋ are dual to ⊃ and ¬, accordingly.

Whether $\tau_0 = 1$, $E_m = \{0, \tau_0, \tau_1, \ldots, \tau_{m-2}\}$ if $m$ is finite, and $E_m = \{0, \tau_0, \tau_1, \tau_2, \ldots\}$ if $m$ is ∞. Let the multitude $E_m$ is linearly ordered: $\tau_0 > \tau_1 > \tau_2 > \ldots$ . Let's define the following operations on $E_m$:

$$p \,\&\, q = \min(p, q), \quad p \vee q = \max(p, q),$$

$$p \supset q = \begin{cases} 1, & \text{if } p \le q, \\ q, & \text{if } p > q, \end{cases}$$

$$p \setminus q = \begin{cases} 0, & \text{if } p \ge q, \\ q, & \text{if } p < q, \end{cases}$$

$$\neg p = p \supset 0, \quad \rfloor p = p \setminus 1.$$

Thus, we get the algebra

$$A_m = \langle E_m; \&, \mathrm{V}, \supset, \setminus, \neg, \rfloor \rangle.$$

In interpreting the formulas on this algebra we get a logic that we note with $LA_m$.

Let's observe that take place the relations of inclusion

$$LA_2 \supseteq LA_3 \supseteq \ \ldots \ \supseteq LA_i \supseteq \ldots \supseteq \mathrm{DIL}$$

and also $LA_2$ is the logic of algebra

$$A_2 = \langle \{0, 1\}; \&, \mathrm{V}, \supset, \setminus, \neg, \rfloor \rangle$$

and coincides with the classical propositional logic. The logic $LA_m$ will be $m$-valent dual chain logic.

We will note that a formula $F(p_1, \ldots, p_n)$ preserves on algebra $A$ of (1) type a predicate $R(x_1, \ldots, x_m)$, if, for any elements $\alpha_{ij} \in A$ ($i = 1, \ldots,$

$m$; $j$ = 1, …., $n$), from the fact that the sentences are true

$R(\alpha_{11}, \alpha_{21}, \ldots, \alpha_{m1},), \ldots, R(\alpha_{1n}, \alpha_{2n}, \ldots, \alpha_{mn})$

result that the statement

$R(F[p_1/\alpha_{11}, \ldots, p_n/\alpha_{1n}], \ldots, F[p_1/\alpha_{m1}, \ldots, p_n/\alpha_{mn}])$.

is true.

Let's note by $\varphi_0, \varphi_1, \ldots, \varphi_4$, classes of formulas that preserve on algebra $A_2$ the predicates

$$x = 0,\ x = 1,\ x \neq y,\ x \leq y,\ x \sim y = z \sim u.$$

**Lemma 1 [3].** *In order that a system of formulas $\sum$ to be functionally complete in $LA_2$ it is necessary and sufficient that, for each classes $\varphi_i$ (i = 0,…, 4), to exist in $\sum$ system a formula that do not belong to this class.*

Let's note by $R_5, \ldots, R_{14}$ the following predicates on algebra $A_3$, where $\tau = \tau_1$ :

$$x = \tau,\ x = \neg\neg y,\ x = \rfloor\rfloor y,\ (x \,\&\, \bot y) \neq \tau,$$
$$(x \vee (y \,\&\, \rfloor y)) \neq \tau,\ (x = \neg\neg y) \vee (x = \rfloor\rfloor y),$$
$$\bot x = \bot y,\ (x \neq \tau) \vee (y \neq \tau),$$
$$(x = z \neq \tau) \vee ((x \neq \tau) \,\&(y = \tau) \,\&(z \neq \tau)),$$
$$(x = z \neq \tau) \vee ((x \neq \tau) \,\&(y \neq \tau) \,\&(z \neq \tau)).$$

**Theorem 1 [4].** *In order that a system of formulas $\sum$ to be functionally complete in $LA_3$ logic it is necessary and sufficient that $\Sigma$ to be functionally complete in logic $LA_2$ and, for each of the predicates $R_5$, ..., $R_{14}$, to exist in $\Sigma$ a formula that does not preserve this predicate.*

Let's define additionally two predicates $R_{15}$ and $R_{16}$ on algebra $A_4$:

$$(\neg x = \neg y) \,\&\, (\rfloor x = \rfloor y), \qquad x = wy,$$

where the operation $w$ is defined by table

| | | | | |
|---|---|---|---|---|
| $p$ | 0 | $\tau_1$ | $\tau_2$ | 1 |
| $wp$ | 0 | $\tau_2$ | $\tau_1$ | 1 |

Let's note that the operation $w$ is not expressed by any formula.

**Theorem 2.** *In order that a system of formulas $\sum$ to be functionally complete in $LA_4$ logic it is necessary and sufficient that $\Sigma$ to be functionally complete in the logic $LA_3$ and for each of the predicates $R_{15}$ and $R_{16}$ to exist in $\Sigma$ a formula that does not preserve this predicate.*

**Theorem 3.** *In order that a system of formulas $\sum$ to be functionally complete in logic $LA_m$ where $m \geq 4$, it is necessary and sufficient that $\sum$ to be functionally complete in the logic $LA_4$.*

Let's consider the simplest non chain algebra

$$Z_5 = \langle \{0, \rho, \delta, \omega, 1\}; \&, \vee, \supset, \backslash, \neg, \rfloor \rangle$$

where the elements $\rho$ and $\delta$ are incomparable and satisfy the relations

$$0 < \rho < \omega < 1; \qquad 0 < \delta < \omega.$$

Let's introduce into the analysis, 20 predicates on algebra $Z_5$ using $\Leftrightarrow$ symbol that we will read as "mean", but the operators & and V used at predicate union are understood classically:

$$P_1(x) \Leftrightarrow x = \rho, \quad P_2(x) \Leftrightarrow x \in \{\rho, \delta\},$$
$$P_3(x) \Leftrightarrow x \in \{0, \rho, 1\}, \quad P_4(x) \Leftrightarrow x \neq \omega,$$
$$P_5(x) \Leftrightarrow x \neq \delta,$$
$$P_6(x, y) \Leftrightarrow (x = 0) \vee ((x = 1) \,\&\, (\bot\neg y = 1)),$$
$$P_7(x, y) \Leftrightarrow (x = 1) \vee ((x = 0) \,\&\, (\bot\neg y = 1)),$$
$$P_8(x, y) \Leftrightarrow (x = y) \vee ((\bot\neg x = \bot\neg y = \omega)),$$
$$P_9(x, y) \Leftrightarrow (x = y) \vee ((\bot\neg x = \bot\neg y = 1)),$$
$$P_{10}(x, y) \Leftrightarrow$$
$$\Leftrightarrow (x \in \{0, 1\}) \vee ((x = \omega) \,\&(y \in \{\rho, \delta\})),$$
$$P_{11}(x, y) \Leftrightarrow (x = y = \rfloor\rfloor y) \vee (x = \bot y = \omega),$$
$$P_{12}(x, y) \Leftrightarrow$$
$$\Leftrightarrow (x \in \{0, 1\}) \vee ((x = \omega) \,\&\, (y \in \{0, \omega, 1\})),$$
$$P_{13}(x, y) \Leftrightarrow$$
$$\Leftrightarrow (x = \omega) \vee ((x \in \{0, 1\}) \,\&\, (y \in \{0, \omega, 1\})),$$
$$P_{14}(x, y) \Leftrightarrow \bot\neg x = \bot\neg y,$$
$$P_{15}(x, y) \Leftrightarrow (\bot\neg x = 1) \vee (\bot\neg y = 1),$$
$$P_{16}(x, y, z) \Leftrightarrow$$
$$\Leftrightarrow ((\bot x \,\&\bot z) = 1) \,\&\, ((x = z) \vee (\bot\neg y = 1)),$$
$$P_{17}(x, y, z) \Leftrightarrow$$
$$\Leftrightarrow ((\bot x \,\&\bot z) = 1) \,\&\, ((x = z) \vee (y \in \{\rho, \delta\})),$$
$$P_{18}(x, y, z)) \Leftrightarrow$$
$$\Leftrightarrow ((\bot x \,\&\bot\neg z) = 1) \&((x = \neg z) \vee (\bot\neg y \vee \bot z) = 1)),$$
$$P_{19}(x, y, z)) \Leftrightarrow$$
$$\Leftrightarrow ((\bot x \,\&\bot\neg z) = 1) \&((x = \neg\neg z) \vee (\bot\neg y \vee \bot z) = 1)),$$
$$P_{20}(x, y, z, u)) \Leftrightarrow ((\bot x \& \bot\neg z \& \bot u) = 1) \&$$
$$\& ((x = u) \vee (\bot\neg y = 1) \vee (\bot z = 1)).$$

**Theorem 4 [5].** *In order that a system of formulas $\sum$ to be functionally complete in the logic $LZ_5$ it is necessary and sufficient that $\Sigma$ to be functionally complete in the logic $LA_3$ and, for each of the predicates $P_i$ ($i$ = 1,..., 20), to exist in $\Sigma$ a formula $F_i$ that does not preserve this predicate*

## 2. Conclusions

In conclusion, let remind that the problem of functional completeness in DIL and the problem of functional expressibility in DIL remain open.

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