NON-EXISTENCE OF FINITE APPROXIMATION RELATIVE TO MODEL-COMPLETENESS IN THE PROVABILITY-INTUITIONISTIC LOGIC

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Abstract: A. V. Kuznetsov [1] has introduced into consideration the provability-intuitionistic calculus and its interpretation in terms of $\Delta -$ pseudo-boolean algebras. Author has previously established an infinite number of model-pre-complete classes in provability-intuitionistic logic. In the present paper, author has strengthened this result, and proved that this logic is not finitely approximated relative to model-completeness. It is constructed an example of system of formulas which is not model-complete in LC^{Δ} logic, but it is model-complete in any tabular extension of this logic.

Keywords: intuitionistic calculus, provability-intuitionistic logic, $\Delta -$ pseudo-boolean algebra, model-complete system, approximated logic.

1. Introduction

Provability-intuitionistic logic formulas are defined usually on the alphabet that consists of propositional variables $p, q, r,$ eventually with indexes, symbols of operations

$$\&, \vee, \subset, \neg, \Delta \qquad (1)$$

and brackets) and (. The provability-intuitionistic propositional calculus I^{Δ} [1] is defined by the axioms of intuitionistic calculus I [2], three $\Delta -$ axioms

$$(p \supset \Delta p), \quad ((\Delta p \supset p) \supset p),$$
$$(((p \supset q) \supset p) \supset (\Delta q \supset p))$$

and two inference rules: modus ponens and substitution rule.

We agree to define the logic I^{Δ} as the set of deductible formulas in the calculus I^{Δ}. In general, any set of formulas in the signature (1), which contains the axioms of the calculus I^{Δ} and is closed relative to the rules for inference of this calculus, is called an extension of provability-intuitionistic logic. In this paper an important role will be played by the logic LC^{Δ} that is defined as

$$I^{\Delta} + ((p \supset q) \vee (q \supset p)).$$

As an algebraic interpretation of the calculus I^{Δ} and its extensions serve $\Delta -$ pseudo-boolean algebras, i.e. systems of the type $< E; \&, \vee, \subset, \neg, \Delta >$, where $< E; \&, \vee, \subset, \neg >$ is a pseudo-boolean algebra [3], and conditions:

$$x \leq \Delta x, \quad \Delta x \supset x = x, \quad \Delta x \leq y \vee (y \supset x),$$

are satisfied, where $x \leq y$ denote $x = x \,\&\, y$. A formula is valid in a $\Delta -$ pseudo-boolean algebra A, if it is identically equal to the unity 1_A of this algebra. The set of all formulas valid in the algebra A is an extension of the provability-intuitionistic logic. We call this logic as the logic of the algebra A, and denote it by $L\mathrm{A}$. In particular; the logic LC^{Δ} coincides with the logic of $\Delta -$ pseudo-boolean algebra

$$\mathrm{B} =< \{0 < \tau_1 < \tau_2 < \cdots < 1\}; \&, \vee, \subset, \neg, \Delta > .$$

So the equality $L\mathrm{B} = LC^{\Delta}$ is true.

We say that a formula F is expressible in the logic L through the system of formulas Σ, if F can be obtained from variables and formulas of the system Σ by applying a finite number of times the weak rule of substitution, which allows the passage from two formulas to the result of substitution of one of them into the other instead of all entries of some one of variables, and equivalent replacement rule in the logic L, which allows switching in L from one formula to a formula equivalent to her. However, if the formula F can be obtained from variables and formulas of the system Σ by applying a finite number of times only weak substitution rule, then F is direct expressible through Σ. The result of substitution the formulas $B_1, ..., B_n$ in the formula F instead of variables $\pi_1, ..., \pi_n$, respectively, we denote by the symbol $F[\pi_1 / B_1, ..., \pi_n / B_n]$ (or $F[B_1, ..., B_n]$).

Suppose the logic L satisfies conditions $I^{\Delta} \subseteq L \subseteq LC^{\Delta}$. A formula $F(p_1,...,p_n)$ of the logic L is named a model of a boolean function $f(p_1,...,p_n)$, if the equality

$$F(p_1,...,p_n) = f(p_1,...,p_n)$$

is identically true on the set $\{0,1\}$. The system Σ of formulas of logic L is called model-complete in L, if for any boolean function, at least one its model is expressible in the logic L through the system of formulas Σ. The system Σ of formulas is called model-pre-complete in L, if Σ is not model-complete in L, but for any formula F, which is not expressible in L through the system Σ, the system $\Sigma \cup \{F\}$ is model-complete in L.

The idea to research of a model-completeness belongs to A.V. Kuznetsov. A criterion for model-completeness in general 3-valent logic was obtained by Iu. N. Tolstova [4].

We say that a formula $F(p_1,...,p_n)$ preserves the predicate $R(x)$ on the algebra A, if for any $\alpha_1,...,\alpha_n$ of the algebra A the proposition $R(F[\alpha_1,...,\alpha_n])$ is true each time when propositions $R(\alpha_1),...,R(\alpha_n)$ are true. It is obvious, that a class of formulas that preserve on this algebra a given predicate is closed relative to the expressibility in the logic of this algebra. Therefore, if all formulas of a system Σ preserves on the $\Delta-$pseudo-Boolean algebra A some predicate, but the formula F does not preserve this predicate, we say that F is separated on this algebra A, by the system Σ through this predicate.

In connection with the following fact, previously established by the author [5], that there is an infinite set of model-pre-complete classes in the logic I^{Δ}, in this logic there is no criterion of model-completeness formulated in terms of a finite number of model-pre-complete classes. Theorem and the example below makes more complicated the problem of existence and construction a criterion for model-completeness in logic I^{Δ}.

It is known [6] that provability-intuitionistic logic I^{Δ} is finitely approximated in the following sense: logic L is called finitely approximated, if for any formula $B \notin L$, there is a tabular extension L^t of the logic L, such that $B \notin L^t$. Also, it is known, that the logic I^{Δ} is not finitely approximated relative to completeness in the sense of [7].

Logic L is called finitely approximated relative to model-completeness, if for any finite system of formulas Σ, which is not model-complete in L, there is a tabular extension of the logic L, which also is not model-complete.

Theorem. *Logic I^{Δ}, logic LC^{Δ}, and all logics intermediate between them are not finitely approximated relative to model completeness.*

Theorem statement follows from the following example.

Example. The system of formulas $\{\Delta p, \neg p \,\&\, q\}$ is not model-complete in LC^{Δ}, but it is model-complete in any tabular extension of the logic I^{Δ}.

Indeed, any formula F, that is a model of a boolean constant 1, is separated in algebra B by the system $\{\Delta p, \neg p \,\&\, q\}$ through the predicate $x \neq 1$. Therefore, the given system of formulas is not model-complete in logic LC^{Δ}.

2. Conclusion

We proved now that this system is model-complete in any tabular extension L of the logic I^{Δ}. The logic L coincides with the logic of some finite $\Delta-$pseudo-boolean algebra A. Since the system of boolean functions $\{1, \neg p \,\&\, q\}$ is complete in P_2, it is sufficient to show, that a formula, which is a model of a boolean function 1, is expressible in the logic L through the system of formulas $\{\Delta p, \neg p \,\&\, q\}$.

Obviously, each term of the following string:

$$(\neg p \,\&\, p),\quad \Delta(\neg p \,\&\, p),\quad \Delta^2(\neg p \,\&\, p),...,$$

$$\Delta^k(\neg p \,\&\, p),...$$

is direct expressible through the system of formulas $\{\Delta p, \neg p \,\&\, q\}$. Since the algebra A is finite, there is a finite number k $(k > 1)$, such that all terms of the above string, starting with $\Delta^k(\neg p \,\&\, p)$, are equal to the largest element of this algebra, i. e. to 1, and they are models of boolean function 1. So, the system $\{\Delta p, \neg p \,\&\, q\}$ is model-complete in L.

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